MAKING MATHEMATICS MORE ATTRACTIVE

REPORT OF GROUP MMA4

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1. Introduction

In addressing the brief of making mathematics more attractive, the group specifically focussed on A- and AS-level. To be more attractive, A- and AS-level mathematics courses must not be perceived to be 'hard', 'boring' or 'pointless'. The group interpreted this as saying that A- and AS-level syllabuses must not contain too much material, thus allowing flexibility of teaching. However, to understand how the words 'hard', 'boring' and 'pointless' may be used by post-GCSE students, we need to try to look at the world through their eyes. To get insight into their experience, and the nature of the foundations provided by GCSE, we started by considering the aims and objectives of these syllabuses and asked which we would wish to carry forward into A-level.

Having drawn up the list of aims, the group considered how these aims might be realised at A-level and how, in this context, the question of making mathematics more attractive might be answered. Rather than producing a syllabus for A-level mathematics, the group concentrated on producing guidelines, or a teaching syllabus, for a particular topic within A-level. These guidelines would consist of a route through the topic and would include methods of teaching, resources to be used and methods of assessment. The topic selected was 'calculus' as it might best typify A-level. Even at its most unambitious level it captures an essentially new aspect of mathematical development—whether as pure mathematics or as a tool in modelling and in other disciplines. Also, being aware of the role of mathematics as a service subject, we felt that, even in an A-level with reduced content Calculus will be a core subject.

Thus, after many excursions, the group arrived at a restricted, and we hope more meaningful, version of our brief:

- how should calculus be introduced and developed so as to appeal to post GCSE students, and to extend the mathematical vocabulary (an enjoyment) of the average present A-level pupil, while offering a stimulating view of more to come for the most able;
- how can mathematics 16–18 retain the interest of the pupils who thrived on the GCSE approach, and at the same time encourage them to pursue mathematically related studies beyond 18.

2. The Aims

To enable students

(a) to develop their mathematical knowledge and oral, written and practical skills in a way which encourages confidence and provides satisfaction and enjoyment;
(b) to read mathematics, and write and talk about the subject;
(c) to apply mathematics in various situations and develop an understanding of the part which mathematics plays in the world around them;
(d) to solve problems, present the solutions clearly, check and interpret the results;
(e) to develop an understanding of mathematical principles;
(f) to construct, analyse, interpret and, where necessary revise mathematical models;
(g) to use mathematics as a means of communication with emphasis on the use of clear expression;
(h) to develop the abilities to reason logically, to classify, to particularise, to generalise and to prove;
(i) to appreciate patterns and relationships in mathematics;
(j) to appreciate the interdependence of different branches of mathematics;
(k) to acquire a foundation appropriate to further study of mathematics and of other disciplines;
(l) to acquire relevant mathematical techniques and manipulative skills.

3. Our Approach to the Introduction of Calculus

As mentioned in the introductory section above, we found the task of writing a whole new A-level syllabus in the space of two days much too daunting and decided instead on the more restricted task of introducing calculus in an exciting and attractive way. We realise that when it comes to teaching this in the context of an A-level, thought must be given to prerequisites (e.g. functions) and to the amount of time that can be spent on the topic. It is likely that not all of the following suggestions could be included and those who construct the whole examination syllabus will have to make these decisions.

The approach we adopt is similar to that suggested by David Tall [1] in his paper *'A Versatile Approach to Calculus and Numerical Methods'*. This approach attempts to teach the concepts of function, differentiation and integration from three points of view, which taken together and properly integrated should help students gain useful insights and understanding. Pictures, numbers and algebraic formulae should all be used to teach calculus. While this may be common practice in some schools, we suggest that it is not universal.

The fundamental ideas with which we are dealing are 'change, rate of change and accumulation due to change' [1]. These ideas are commonplace in everyday life and we would expect that the teacher would arouse the students' curiosity to such an extent that they unconsciously learn and absorb the mathematical tools necessary to satisfy this curiosity. Questions relating to displacement, velocity and acceleration, processes of growth and decay, heating and cooling can all be asked, as can questions relating to economics, history, medicine and industry. A brief discussion of the history of calculus could be included, especially dealing with the problems which motivated Issac Newton.

This approach unashamedly makes use of graphics calculators and computers which we recognise are not universally available. They may well be by the time an A-level like this comes to fruition.

Our aspiration is that this approach—curiosity driven and technologically supported—will make calculus a more attractive and understandable subject which pupils will want to go on studying after they leave school.

· Background

Candidates who have taken Intermediate or High Level GCSE Mathematics will have studied functions. They will have constructed tables of value for given functions of the form $ax+b$, ax^2, $a/x \quad (x \neq 0)$, where a and b are integer constants; they will have drawn and interpreted the related graphs and will have met the idea of gradient. Those who have taken a High level course may have met functions of the form ax^2+bx+c, $a/x^2 \quad (x \neq 0)$ and exponential functions like $y = a^x$ and $y = a^{-x}$ where a is a small positive integer.They may have drawn the graphs of these functions and met the idea of exponential growth and decay. They may have defined and drawn the graphs of $\sin x$ and $\cos x$ for x in the range $0°$ to $360°$ and they may have derived relationships between variables of the forms $y = ax+b$, $y = a/x$, $y = a/x + b \quad (x \neq 0)$, $y = ax^2 + b$, by interpreting appropriate straight line graphs. So called 'travel graphs' and their interpretation may have been met. Candidates will also have used at least the arithmetic functions of an electronic calculator.

When it comes to A-level, students will advance their study of functions to include low order polynomial functions, rational functions, the exponential and logarithmic functions, and the function $|x|$. They will study the inverses of one to one onto functions, and composition of functions. They might also explore the reverse mappings of simple many-one functions like $y = x^2$.

As stated above, we advocate a curiosity driven, technologically supported approach. Experiments could be carried out to find for example the relationship between:

1) temperature and time as water is heated by electricity,
2) temperature and time as water cools,
3) displacement and time as a ball rolls down an inclined plane.

Libraries could be raided to find for example the relationship between

) population and time (humans, foxes and rabbits, bacteria),
) drug concentration in the bloodstream and time,
) radioactivity and time.

Building Societies and Banks could be consulted to find the relationship between

capital plus interest and time.

And so on.

Each of these relationships would be illustrated graphically, computed numerically and described symbolically. Students should be able to transfer readily between these different representations and they should be able to talk about the data and their interpretation. They should be able to deal as easily with inverse functions (and associated problems) as they can with the functions themselves (and the situations they model).

5. Differentiation

Functions have to do with 'change'; differentiation has to do with 'rate of change'.

Travel graphs for example can be used to motivate a discussion on rates of change. Gradients of graphs can be compared visually, and a discussion of evaluating the gradient as a point will lead to the ideas of drawing tangents or evaluating

$$\frac{f(x+h)-f(x)}{h} \quad \text{or} \quad \frac{f(x+h)-f(x-h)}{2h}.$$

Having found a method of calculating an approximate numerical value of the gradient at any point on a curve the idea of the gradient function can be developed.

For example to find the gradient function (derivative) of $f(x) = x^2$ the following sequence could be followed. By working in groups students can do this quite quickly.

(a) Draw an accurate graph of $y = f(x) = x^2$.

(b) Draw tangents at a sequence of points on the graph and hence estimate the gradient at each point. Plot the graph of the gradient function.

(c) Having discovered the inaccuracy of method (b) calculate $\big(f(x+h) - f(x)\big)/h$ for a chosen small value of h and a sequence of values of x. Plot the graph of $\big(f(x+h) - f(x)\big)/h$ using these values.

(d) Discuss the effect of using different values of h

(e) On a graphical calculator plot $y = \big(f(x+h) - f(x)\big)/h$ for a sequence of decreasing values of h.

(f) Guess a rule for the gradient function $g(x)$. Superimpose the graph of $y = g(x)$ on top of the graphs drawn in (e). If they do not coincide when h is very small try a different form for $g(x)$

(g) Define

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

and prove that in this case $f'(x) = 2x$.

Once students are familiar with the use of a graphical calculator the derivatives of other functions can be found by starting at step (e) and omitting drawing graphs and measuring gradients of tangents. It is then a very quick and effective method of arriving at the derivative of a function by numerical and graphical processes before doing so analytically. In particular it distinguishes clearly between the derivatives of, for instance, $\sin x$ with x in both degrees and radians and provides a good introduction to the derivative of a^x and a definition of e.

By approaching differentiation in this way students should develop a much clearer understanding of the limiting process of differentiation and should also be able to tackle the problem of finding the derivative of any new function in a systematic way.

It will continue to be important for students to be able to differentiate standard functions accurately and quickly and to know and be able to use the rules for differentiating sums, products, quotients and functions of functions.

The applications of differentiation should always be stressed, including realistic maximising problems and related rates of change.

The inverse problem to finding derivatives, i.e. solving (simple) differential equations could be considered next. The following forms could readily be discussed:

$$\frac{dy}{dt} = k\,; \qquad \frac{dy}{dt} = kt\,; \qquad \frac{dy}{dt} = ky$$

—the last of these could be solved by numerical and graphical methods.

6. Integration

Integration has to do with 'accumulation due to change'. Visually this can be represented by the area under a curve and is readily amenable to graphical and numerical approaches.

Following Tall's approach [1], numerical investigation can be used to explore successively the mid-ordinate rule and the trapezium rule and these can be taken to a suitable degree of accuracy. The early introduction of Simpson's rule may cause confusion, but it could also be given as a more efficient algorithm (without details of its derivation) after students are familiar with the earlier ideas. Successive refinements of the intervals to spot patterns for standard functional form (i.e. simple powers) can precede introduction of standard notations and tabulation of results.

An approximation to the area function $A(x) = \int_a^x f(t)\,dt$ can be drawn on a graphical calculator using the programming facility to evaluate $A(x)$ by rectangular approximation for a sequence of values of x and, once it has been drawn, its analytical form can be guessed and checked as was done for the gradient function. This provides a very convincing demonstration of the fundamental theorem of calculus. (See Tall [1]).

Again, travel graphs are a useful motivation for finding the area under a curve and volumes of revolution strengthen the concept of integration as a summation rather than the reverse of differentiation.

Techniques of integration can be confined to manipulation of standard forms. [We accept the omission of substitution, partial fractions and integration by parts. In the near future symbolic calculators will allow integrals to be done.] The omission of further techniques of integrating need not inhibit the study of models leading to ordinary differential equations with separable variables, since the necessary standard integrals can be given.

7. Assessment

We envisage including a number of different forms of assessment in order to give students every opportunity to demonstrate what they can do, and to encourage a variety of mathematical activities in the classroom. It is our view that existing Advanced Level examinations are too difficult; we would wish to reduce the difficulty to the point where any candidate who passes will have demonstrated confidence and competence across a broad area of the syllabus. We believe that this can be achieved without diminishing the discrimination of the assessment.

In detail we suggest the following components of the assessment.

(1) Coursework

This would take the form of four projects, one major and three minor. It is expected that students will do other coursework throughout the course but that only four pieces would be assessed. Coursework would represent about 25% of the whole assessment. The major project would attract 15% of the marks and the minor projects together 10% of the marks.

The three minor projects would be supervised and administered by the teacher. The topic for the major project would be chosen from a list provided by the examining board and based on topics within the syllabus. The length of the project would not normally exceed about 10 sides of A4 paper. There should be an oral examination on the project, conducted by an external examiner, and the mark for the major project would be based both on the written report and on the oral examination. A project forming part of the Northern Ireland Further Mathematics Mode 2 A-level [2] was based on the item on Drug Therapy appearing in Applying Mathematics [3]. We refer to this as an example of a project, but point out that, as part of a Further Mathematics course, it is of a higher standard than that expected for Mathematics A-level.

In order to gain a passing grade at A-level, candidates must submit a major project and obtain a threshold mark.

(2) Written Examinations

We suggest two 3 hour papers so that students have time to think and to show what they know. The questions should be more straightforward than those currently appearing in A-level papers.

The questions in both papers should be graded: increasing in length and difficulty, with the less structured questions at the end. The easier elements of the papers should themselves cover the whole syllabus and candidates should be able to obtain a pass by answering only these elements of the paper. Full marks would be awarded to candidates answering correctly all the questions on the papers.

One of the papers should include a comprehension question, expected to take one hour, based on an article given to the candidates about a month before the examination. No 'hard questions' should be included on the paper that contains the comprehension question. An example of a comprehension question used in the Northern Ireland Further Mathematics Mode 2 A-level is given in the Appendix. We are grateful to NISEC for permission to reproduce it.

8. Conclusions

We see this course in calculus as contributing to the achievement of many of the aims listed in the first two sections above. Students will have developed their mathematical knowledge, etc, and by attempting problems such as those described in the section on assessment, will have read, written and talked about mathematics. They will have applied mathematics to various situations in the world around them and will have solved problems with suitable checks. They will have developed an understanding of calculus and will have engaged to some extent in mathematical modelling. They will have produced extended arguments using mathematics to express ideas. They will have found patterns and relationships and will see calculus as a whole. They will have developed manipulation skills and mathematical techniques, and all of this should have whetted their appetite for more.

References

[1] TALL, D., 1989. *A versatile approach to calculus and numerical methods*, Math. Education Research Centre, University of Warwick, Coventry.

See also papers by SWANN, M. and WILLIAMS, J. in these proceedings.

For a detailed discussion of the Northern Ireland Further Mathematics Mode 2, see

[2] HOUSTON, S.K. 1989 *Teaching mathematics and its applications*, Vol 2 (Ellis Horwood).

[3] BURGHES, D.N., HUNTLEY, I. and MCDONALD, J. 1982 *Applying mathematics: a course in mathematical modelling* (Ellis Horwood) pp.124-129.,

NORTHERN IRELAND SCHOOLS EXAMINATIONS COUNCIL

ADVANCED LEVEL

General Certificate of Education

1989

FURTHER MATHEMATICS (MODE 2)

PAPER 3

1. Enclosed is a copy of the article, "What the eye doesn't see . . .", which was sent to you in advance of the examination. Anwer the following questions relating to the article.

(i) What simplifying modelling assumptions are implicit in Figure 1? [6]

(ii) Indicate in a sketch the angle which PQ bisects. [2]

(iii) Prove the statements in (2), (3) and (4). [8]

(iv) Derive the formula for x in (6) from that in (5). [3]

(v) What does the formula in (6) predict will happen to x in each of the following cases:

(a) w increases;
(b) b increases?

In each case interpret the predicted change in terms of the practical situation. [6]

(vi) Use (6) to find a formula for $\frac{dx}{d\theta}$.

What statement in the article is justified by the content of this formula? [4]

(vii) Write down the formula which the author is using as a "first approximation" for x in (7). [1]

(viii) Prove that the identity in (8) is true. [2]

(ix) Practical considerations determine a permissible range of values for the angle θ in (1). State the range of values so determined. [3]

(x) Use 'standard calculus methods' to show that the value of θ required to maximise

$$2(\sin\frac{1}{2}\theta - \sin^3\frac{1}{2}\theta)$$

must satisfy the equation in (9). [9]

(xi) Justify the author's claim in (10) that a change in the value of θ from 70.5° to 62.3° "increases the estimate for the best distance of P in front of E from 14 cm to 21 cm." [2]

(xii) Having had the opportunity to reflect on this article, what advice would you give to:

(a) someone buying a car;
(b) someone driving along a motorway? [6]

Appendix

What the eye doesn't see, . . .

We are just contemplating the purchase of a new car, and have been test-driving a number of competing models. A mirror fixed to the driver's door is nowadays standard equipment, and on most of the models this is attached near the front of the door, more or less in line with the hinges. But in the VW Golf it is set further back; a good feature, claimed the salesman, since it reduces the driver's "blind spot".

I confess that it was not obvious to me that this is so. If true, then perhaps it would compensate for the inconvenience of having to turn one's head further round to look in the mirror. But mathematical investigation seemed to be called for.

The problem is, of course, three-dimensional; but the geometry is quite hard enough in two dimensions, so I chose to work with a plan view. Fig. 1 shows the driver's side of the car, facing to the right, and E indicates the driver's eye, a distance a from the off-side of the car. The mirror PQ, attached to the door at P, has width w. The line l, pointing to the right, is the path of the near-side of an overtaking car. The gap between the vehicles has width b.

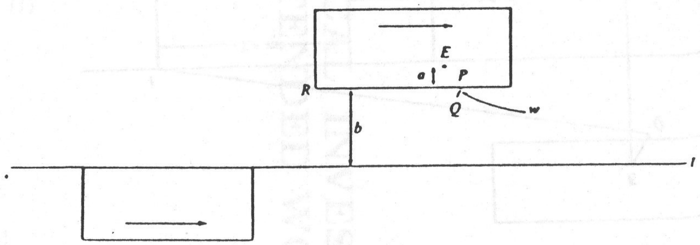

FIGURE 1.

How should the mirror be set? Since there can be little point in looking at one's own car in the mirror, the ideal angle would seem to be such that the side of the car should be reflected along the line PE. This implies that the mirror PQ lies along the external bisector of the angle EPR, where R is the off-side rear of the car. The question then is, where does the reflection of the ray EQ intersect l? For, once the near-side tail passes this point, the overtaking car is invisible to the driver until he sees it (out of his side window) passing him.

In Fig. 2 this point of intersection is denoted by I. We need to find how far I

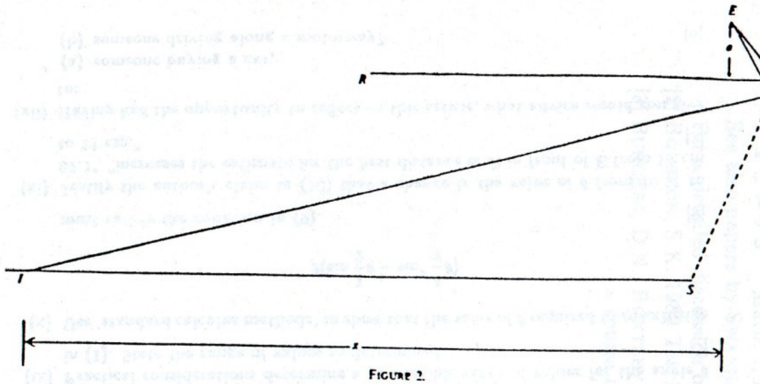

FIGURE 2.

is behind E; this distance is denoted by x. Consider first the angles in Fig. 2. Clearly, from the definition of reflection, angle PQE is equal to angle SQI, where S is the point where PQ produced meets I. Also, the angle between the rays IQ and RP is equal to the angle between the reflected rays QE and PE; and since RP is parallel to I, this means that angle QIS is equal to angle QEP. It follows that the triangles QIS and QEP are similar. Finally, if the line EP makes an angle θ with the side of the car, then the bisector property shows that (1)
angle QPR has magnitude $90° - \frac{1}{2}\theta$.

From this it is easy to calculate that

$$PE = a \operatorname{cosec} \theta, \qquad (2)$$

$$QS = b \sec \tfrac{1}{2}\theta - w, \qquad (3)$$

$$SI = x + a \cot \theta - b \tan \tfrac{1}{2}\theta. \qquad (4)$$

And, since $SI/PE = QS/QP$,

$$\frac{x + a \cot\theta - b \tan\frac{1}{2}\theta}{a \operatorname{cosec}\theta} = \frac{b \sec\frac{1}{2}\theta - w}{w}.$$

This gives

$$x = \frac{ab}{w} \operatorname{cosec} \theta \sec \tfrac{1}{2}\theta + b \tan \tfrac{1}{2}\theta - a(\operatorname{cosec} \theta + \cot \theta) \qquad (5)$$

$$= \frac{ab}{w} \operatorname{cosec} \theta \sec \tfrac{1}{2}\theta + b \tan \tfrac{1}{2}\theta - a \cot \tfrac{1}{2}\theta \qquad (6)$$

To reduce the size of the blind spot, x has to be made as small as possible; and the variable within the designer's control is θ, since the mirror is attached to the door at a distance $a \cot \theta$ in front of the driver's eye. (Ideally, since the driver's seat can be adjusted fore-and-aft, the mirror should move with it; but in practice it must be positioned for the "average driver".)

The expression for x is too complicated for a theoretical minimum to be found exactly. However, in practice w will be quite small compared with a, and (hopefully) smaller still compared with b; so that ab/w is substantially larger than either b or a. The first term is therefore the dominant one, and a first approximation can be found by choosing θ so as to minimise (7)
$\operatorname{cosec}\theta \sec\frac{1}{2}\theta$; or, more simply, to maximise

$$\sin \theta \cos \tfrac{1}{2}\theta = 2(\sin \tfrac{1}{2}\theta - \sin^3 \tfrac{1}{2}\theta). \qquad (8)$$

Standard calculus methods show that the optimum value of θ must then satisfy the equation

$$\sin^2 \tfrac{1}{2}\theta = \tfrac{1}{3}, \qquad (9)$$

so that $\theta \approx 70{\cdot}5°$. This seems to support Volkswagen's practice of setting the mirror well back on the door.

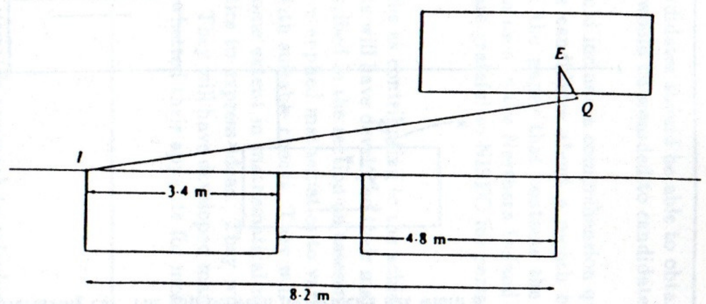

FIGURE 3.

When the other terms are taken into account, the optimum value of θ turns out to be somewhat smaller than this. For example, taking numerical values $a = 40$ cm, $w = 10$ cm and $b = 150$ cm, we get (in cm)

$$x = 600 \operatorname{cosec} \theta \sec \tfrac{1}{2}\theta + 150 \tan \tfrac{1}{2}\theta - 40 \cot \tfrac{1}{2}\theta.$$

A numerical search gives a minimum when $\theta \approx 62{\cdot}3°$. This increases the (10)
estimate for the best distance of P in front of E from 14 cm to 21 cm; but this is still closer to the position of the mirror on the VW than on the other models.

There is one further sobering thought. The minimum value of x with these figures is 816 cm, or about 8·2 m. If the overtaking car is a Metro 3·4 m long, and we suppose that it does not again become visible until its front is level with the driver's eye, then the blind spot extends for a distance of 4·8 m (see Fig. 3). On a motorway, where most vehicles travel at about the maximum permitted speed and the relative velocity of overtaking is often only about 3 km/h, this implies that a small overtaking car may be invisible for $\frac{4{\cdot}8}{3000} \times 3600$ s, or about 6 seconds: *caveat auriga*.