Convergence and limits of linear representations of finite groups *

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November 11, 2015

Abstract

Motivated by the theory of graph limits, we introduce and study the convergence and limits of linear representations of finite groups over finite fields. The limit objects are infinite dimensional representations of free groups in continuous algebras. We show that under a certain integrality condition, the algebras above are skew fields. Our main result is the extension of Schramm's characterization of hyperfiniteness to linear representations.

Keywords. linear representations, amenability, soficity, continuous rings, skew fields

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*AMS Subject Classification: 20F99, 16E50 Research partly sponsored by MTA Renyi "Lendulet" Groups and Graphs Research Group

1 Introduction

Unitary representations. Our primary motivation and model example is the view of infinite dimensional unitary representations into tracial von Neumann algebras as limits of finite dimensional unitary representations. By a finite dimensional unitary representation (of degree r), we mean a homomorphism $\kappa : \mathbb{F}_r \to U(n)$ of the free group on r generators into the unitary group $U(n)$. Note that such representations can be given by the r-tuple $\{\kappa(\gamma_i\}_{i=1}^r$, where $\{\gamma_i\}_{i=1}^r$ are the standard generators of the free group $\mathbb{F}_r$. We say that a sequence $\{\kappa_k : \mathbb{F}_r \to U(n_k)\}_{k=1}^\infty$ of finite dimensional unitary represenations are *convergent* if for any $w \in \mathbb{F}_r$

$$\mathrm{Tr}(w) = \lim_{k\to\infty} \mathrm{Tr}_{n_k}(\kappa_k(w))$$

exists, where Tr_{n_k} stands for the normalized trace function on the complex matrix algebra $\mathrm{Mat}_{n_k \times n_k}(\mathbb{C})$. Note that from each sequence of representations one can choose a convergent subsequence. The limit of a convergent system $\{\kappa_k : \mathbb{F}_r \to U(n_k)\}_{k=1}^\infty$ is defined as a representation of $\mathbb{F}_r$ into the unitary group of a tracial von Neumann algebra via the GNS-construction, given below.

The function Tr extends to the group algebra $\mathbb{C}\mathbb{F}_r$ as a trace, that is a linear functional satisfying

$$\mathrm{Tr}(ab) = \mathrm{Tr}(ba)\,.$$

Let $I \subset \mathbb{C}\mathbb{F}_r$ be the set of elements a, for which $\mathrm{Tr}(a^*a) = 0$. It is not hard to see that I is an ideal of $\mathbb{C}\mathbb{F}_r$ and the trace function Tr vanishes on I. Let $A = \mathbb{C}\mathbb{F}_r/I$, then

$$< [p], [q] >= \mathrm{Tr}[pq^*]$$

is well defined and gives rise to an inner product structure on the algebra A. Let $\mathcal{H}$ be the Hilbert completion of A. Then the left multiplication $L_{[p]} : A \to A$ defines a bounded linear representation of A on $\mathcal{H}$. The weak closure of the image of A is a tracial von Neumann algebra $\mathcal{N}$ equipped with a trace (the extension of Tr). Also, we have a natural homomorphism $S : \mathbb{F}_r \to U(\mathcal{N})$ such that for any $w \in \mathbb{F}_r$,

$$\mathrm{Tr}(S(w)) = \lim_{k\to\infty} \mathrm{Tr}_{n_k}(\kappa_k(w)).$$

Thus S can be viewed as the limit object of the finite dimensional unitary representations $\{\kappa_k\}_{k=1}^\infty$. One can ask the following question. If $S : \mathbb{F}_r \to U(\mathcal{N})$ is a representation of the free group into the unitaries of a tracial von Neumann algebra, is it true that S is the limit of finite dimensional unitary representations. This question is called the Connes Embedding Problem [23].

Graph limits. We are also strongly motivated by the emerging theory of graph limits. The original definition of graph convergence is due to Benjamini and Schramm [3] (see also the monography of Lovász [19]). Here we consider the limit of Schreier graphs (see e.g. [8]). Let $\lambda : \mathbb{F}_r \to S(n)$ be a homomorphism of the free group into the symmetric group of permutations on n elements. These homomorphism are in bijective correspondence with Schreier graphs. The vertex set of the corresponding graph G_λ is the set $[n] = \{1, 2, \ldots, n\}$. Two vertices a and b are connected with an edge labeled by the generator γ_i, if $\lambda(\gamma_i)(a) = b$. A sequence of permutation representations (or Schreier graphs) $\{\lambda_k : \mathbb{F}_r \to S(n_k)\}_{k=1}^{\infty}$ is called *convergent* if for any $m \geq 1$ and m-tuple $\{w_1, w_2, \ldots, w_m\} \subset \mathbb{F}_r$

$$\lim_{k\to\infty} \frac{|\mathrm{Fix}_k(\lambda_k(w_1)) \cap \mathrm{Fix}_k(\lambda_k(w_2)) \cap \cdots \cap \mathrm{Fix}_k(\lambda_k(w_m))|}{n_k}$$

exists, where $\mathrm{Fix}_k(\lambda_k(w))$ is the fixed point set of the permutation $\lambda_k(w)$. Note that the original definition is somewhat different from the one above, nevertheless a simple inclusion-exclusion argument shows that the two definitions are equivalent. One can define various limit objects for such convergent sequences e.g. the invariant random subgroups (see [1]). A notion of limit, analogous to the unitary case, can be defined the following way. Let (X, μ) be a standard Borel probability measure space equipped with a countable measure preserving equivalence relation E (see [14]). An E-transformation is a measure preserving bijection $T : X \to X$ such that for almost all $p \in X$, $p \equiv_E T(p)$. A zero transformation is an E-transformation S such that $\mu(\mathrm{Fix}(S)) = 1$. The full group of E, $[E]$ is the quotient of the group of E-transformations by the normal subgroup of zero transformations. Note that if $Q \in [E]$, then $\mathrm{Fix}(Q)$ is well-defined up to a zero measure perturbation. We call a homomorphism $\lambda : \mathbb{F}_r \to [E]$ *generating* if for almost all $p \in X$: for any q such that $p \equiv_E q$, there exists $w \in \mathbb{F}_r$ such that $\lambda(w)(p) = q$. A generating representation $\lambda : \mathbb{F}_r \to [E]$ is a limit of the convergence system of permutation representations $\{\lambda_k\}_{k=1}^{\infty}$ if for any $m \geq 1$ and m-tuple $\{w_1, w_2, \ldots, w_m\} \subset \mathbb{F}_r$

$$\lim_{k\to\infty} \frac{|\mathrm{Fix}_k(\lambda_k(w_1)) \cap \mathrm{Fix}_k(\lambda_k(w_2)) \cap \cdots \cap \mathrm{Fix}_k(\lambda_k(w_m))|}{n_k} =$$

$$= \mu\left(\mathrm{Fix}(\lambda(w_1)) \cap \mathrm{Fix}(\lambda(w_2)) \cap \cdots \cap \mathrm{Fix}(\lambda(w_m))\right) .$$

Again, for each convergent sequence $\{\lambda_k\}_{k=1}^\infty$ one can find a limit representation into some full group. On the other hand, it is not known, whether any generating representation $\lambda : \mathbb{F}_r \to [E]$ can be obtained as a limit.

Schramm's Theorem. Let $\lambda : \mathbb{F}_r \to S(n)$ be a permutation representation with corresponding Schreier graph $G_\lambda([n], E_\lambda)$. We say that a convergence sequence of representations $\{\lambda_k : \mathbb{F}_r \to S(n_k)\}_{k=1}^\infty$ is hyperfinite if for any $\epsilon > 0$, there exists an integer $K_\epsilon > 0$ such that for any $k \geq 1$, one can remove εn_k edges from the graph G_{λ_k} in such a way, that all the components of the remaining graph have at most K_ϵ vertices. Schramm [22] proved that the hyperfiniteness of the sequence is equivalent to the amenability of its limit (see also [8]). A generating representation $\lambda : \mathbb{F}_r \to [E]$ is amenable if E is a hyperfinite (amenable) equivalence relation [14].

2 Definitions and results

In the course of our paper we fix a finite field $\mathcal{K}$. Our goal is to study the convergence and limits of finite dimensional representations $\theta : \mathbb{F}_r \to \mathrm{Mat}_{n\times n}(\mathcal{K})$. Note that such representations are in one to one correspondence with injective linear representations $\pi : \Gamma \to GL(n, \mathcal{K})$, where Γ is a finite group of r marked generators.

Definition 2.1. *A sequence of finite dimensional representations* $\{\theta_k : \mathbb{F}_r \to Mat_{n_k\times n_k}(\mathcal{K})\}_{k=1}^\infty$ *is convergent if for all* $n \geq 1$ *and matrix* $A \in Mat_{n\times n}(\mathcal{K}\mathbb{F}_r)$ *,*

$$\lim_{k\to\infty} rk^n_{n_k}(\theta_k(A))$$

exists, where $\theta_k(A)$ *is the image of* A *in*

$$Mat_{n_k n\times n_k n}(\mathcal{K}) \cong Mat_{n\times n}(Mat_{n_k\times n_k}(\mathcal{K}))$$

and

$$rk^n_{n_k}(\theta_k(A)) = \frac{Rank(\theta_k(A))}{n_k}\,.$$

Note that θ_k naturally extends to the group algebra $\mathcal{K}\mathbb{F}_r$ and we denote the extension by θ_k, as well. We will make clear at the end of Section 10, why we consider matrices instead of single elements of the group algebra $\mathcal{K}\mathbb{F}_r$. Now we define the limit objects for convergent sequences. The objects we need, continuous algebras, were defined by John von Neumann in the thirties [21]. Let R be a separable, continuous $\mathcal{K}$-algebra with rank function rk_R (see Section 4 for a brief survey on continuous algebras).

Definition 2.2. *Let* $\{\theta_k\}_{k=1}^\infty$ *be finite dimensional representations as above. A representation* $\theta : \mathbb{F}_r \to R$*, that is a homomorphism of the free group into the group of invertible elements of the continuous algebra* R *is a limit of* $\{\theta_k\}_{k=1}^\infty$*, if for any* $n \geq 1$ *and* $A \in Mat_{n\times n}(\mathcal{K}\mathbb{F}_r)$

$$\lim_{k\to\infty} rk^n_{n_k}(\theta_k(A)) = rk^n_R(\theta(A))\,,$$

where rk^n_R *is the matrix rank on* $Mat_{n\times n}(R)$*.*

Note, that if $\mathcal{N}$ is a tracial von Neumann algebra, then $\mathcal{N}$ is equipped with a natural rank function and its completion is a continuous rank regular ring; the algebra of affiliated operators [24]. Hence the limit of finite dimensional unitary representations can also be viewed as a homomorphism into the group of invertible elements of a continuous algebra. Our first theorem is about the existence of limits.

Theorem 1. *For any convergent sequence of finite dimensional representations* $\{\theta_k\}_{k=1}^\infty$*, there exists a separable, continuous* $\mathcal{K}$*-algebra* R *and a representation* $\theta : \mathbb{F}_r \to R$ *such that* θ *is the limit of* $\{\theta_k\}_{k=1}^\infty$*.*

It turns out that under a certain integrality condition the limit is unique. We say that the convergence sequence $\{\theta_k\}_{k=1}^\infty$ satisfies the Atiyah condition, if for any $n \geq 1$ and $A \in \mathrm{Mat}_{n\times n}(\mathcal{K}\mathbb{F}_r)$

$$\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\theta_k(A)) \in \mathbb{Z}\,.$$

The condition above is intimately related to Atiyah's Conjecture on the integrality of the L^2-betti numbers (see [17] and [18]).

Theorem 2. *If the convergence sequence of linear representations* $\{\theta_k\}_{k=1}^\infty$ *satisfies the Atiyah condition, then there exists a skew field* D *over the base field* $\mathcal{K}$ *and a homomorphism* $\theta : \mathbb{F}_r \to D$ *(that is a homomorphism into the multiplicative group of non-zero elements of* D*) such that* θ *is the limit of* $\{\theta_k\}_{k=1}^\infty$ *and* $Im(\theta)$ *generates* D*. Moreover, if* $\theta' : \mathbb{F}_r \to D'$ *is another generating limit homomorphism into a skew field* D'*, then there exists a skew field isomorphism* $\pi : D \to D'$ *such that* $\pi \circ \theta = \theta'$*.*

If the Atiyah condition is satisfied, we will be able to generalize Schramm's Theorem cited in the Introduction. It is worth to note that hyperfinite sequences of graphs are basically the opposites of expander sequences. The notion of expander sequences for linear representations were introduced by Lubotzky and Zelmanov (see also [6]). We say that a sequence of linear representations $\{\theta_k : \mathbb{F}_r \to \mathrm{Mat}_{n_k\times n_k}(\mathcal{K})\}_{k=1}^\infty$ is a *dimension expander* if there exists $\alpha > 0$ such that for all $k \geq 1$ and linear subspace $W \subset \mathcal{K}^{n_k}$ with $\dim_{\mathcal{K}}(W) \leq \frac{n_k}{2}$

$$\dim_{\mathcal{K}}(W + \sum_{i=1}^{r} \theta_k(\gamma_i)(W)) \geq (1+\alpha)\dim_{\mathcal{K}}(W),$$

where $\{\gamma_i\}_{i=1}^r$ are the standard generators of $\mathbb{F}_r$. Note that

$$\sup_{W, \dim_{\mathcal{K}}(W) \leq \frac{n_k}{2}} \frac{\dim_{\mathcal{K}}(W + \sum_{i=1}^r \theta_k(\gamma_i)(W))}{\dim_{\mathcal{K}}(W)}$$

is the linear analogue of the Cheeger constant of a graph. It was observed in [6] that a random choice of the r-tuple $\{\theta_k(\gamma_1), \theta_k(\gamma_2), \dots, \theta_k(\gamma_r)\}_{k=1}^\infty$ leads to a dimension expander with probability one, provided that r is large enough. Later, Bourgain and Yehudayoff [4] constructed explicit families of dimension expanders. Using the linear graph theory vocabulary: *subsets*$\to$*linear subspaces*, *disjoint*$\to$ *independent*, *cardinality*$\to$ *dimension*, we can define the hyperfiniteness for sequences of linear representations.

Definition 2.3. *The linear representations* $\{\theta_k : \mathbb{F}_r \to Mat_{n_k \times n_k}(\mathcal{K})\}_{k=1}^\infty$ *form a hyperfinite sequence if for any* $\epsilon > 0$ *there exists* $K_\epsilon > 0$ *such that for all* $k \geq 1$*, we have* $\mathcal{K}$*-linear subspaces* $V_1^k, V_2^k, \dots, V_{t_k}^k \subset \mathcal{K}^{n_k}$ *such that*

- *For any* $1 \leq j \leq t_k$*,* $\dim_{\mathcal{K}}(V_j^k) \leq K_\epsilon$*.*
- $\{V_j^k + \sum_{i=1}^r \theta_k(\gamma_i)V_j^k\}_{j=1}^{t_k}$ *are independent subspaces such that*

$$\dim_{\mathcal{K}}(V_j^k + \sum_{i=1}^r \theta_k(\gamma_i)V_j^k) < (1+\epsilon)\dim_{\mathcal{K}}(V_j^k).$$

- $\sum_{j=1}^{t_k} \dim_{\mathcal{K}}(V_j^k) \geq (1-\epsilon)n_k$ *.*

Our main result is the generalization of Schramm's Theorem for convergent sequences of linear representations.

Theorem 3. *Let* $\{\theta_k\}_{k=1}^\infty$ *be a convergent sequence of linear representations satisfying the Atiyah condition. Let* $\theta : \mathbb{F}_r \to D$ *be the unique limit representation of* $\{\theta_k\}_{k=1}^\infty$*. Then* $\{\theta_k\}_{k=1}^\infty$ *is hyperfinite if and only if* D *is an amenable skew field.*

3 Universal localizations

In this section we recall the notion of universal localization from the book of Cohn [5]. Let R be a unital ring, Σ be a set of matrices over R and $f : R \to S$ be a unital ring homomorphism. Let Σ^f be the image of Σ under f. If Σ is the set of matrices whose images under f are invertible , then $R^f(S)$ denotes the set of entries in the inverses M^{-1}, for $M \in \Sigma^f$. We call $R^f(S)$ the rational closure of R in S. According to Theorem 7.1.2 [5] $R^f(S)$ is a ring containing $\mathrm{Im}\,(f)$. An important tool for the understanding of the ring $R^f(S)$ is the following variant of Cramer's Rule.

Lemma 3.1 (Proposition 7.1.3 [5]). *For any element $p \in R^f(S)$, there exists $n \geq 1$ and $Q \in Mat_{n\times n}(Im\,(f))$, $A \in Mat_{n\times n}(Im\,(f))$ invertible in $Mat_{n\times n}(S)$ and $B \in Mat_{n\times n}(R^f(S))$ in the form of $B = \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix}$ such that*

$$Q = A \begin{pmatrix} I & 0 \\ 0 & p \end{pmatrix} B\,.$$

Note that I denotes the unit matrix of size $n-1$.

Recall that the division closure of $\mathrm{Im}\,(f)$ in S is the smallest ring in S containing $\mathrm{Im}\,(f)$ closed under taking inverses. The following lemma is given as an exercise in [5].

Lemma 3.2. *The division closure $D(f)$ is a subring of $R^f(S)$.*

Proof. It is enough to prove that if $p \in R^f(S)$ is invertible in S, then $p^{-1} \in R^f(S)$. By Lemma 3.1, we can write

$$Q = A \begin{pmatrix} I & 0 \\ 0 & p \end{pmatrix} B\,.$$

So $\begin{pmatrix} I & 0 \\ 0 & p \end{pmatrix} = A^{-1}QB^{-1}$ that is $\begin{pmatrix} I & 0 \\ 0 & p^{-1} \end{pmatrix} = BQ^{-1}A$. Since all the entries of B, Q^{-1} and A are, by definition, in the subring $R^f(S)$, we have that $p^{-1} \in R^f(S)$. ☐

Now, let Σ be a set of square matrices over R. The universal localization of R with respect to Σ is the unique ring R_Σ equipped with a homomorphism $\lambda : R \to R_\Sigma$ such that the elements of Σ^λ are all invertible matrices and if $f : R \to S$ is an arbitrary homomorphism and the elements of Σ^f are all invertible matrices, then there exists a unique homomorphism $\overline{f} : R_\Sigma \to S$ such that $\overline{f} \circ \lambda = f$. Let D be a skew field and $f : R \to D$ be a homomorphism. We call D epic if $\mathrm{Im}\,(f)$ generates D as a skew field.

Proposition 3.1 (Theorem 7.2.2 [5]). *If D is epic and Σ_f is the set of matrices over R whose images in D are invertible, then the universal localization R_{Σ_f} is a local ring with residue-class field isomorphic to D.*

We have the following corollary.

Corollary 3.1. *If $f_1 : R \to D_1$ and $f_2 : R \to D_2$ are two epic maps and for all $n \geq 1$ and for all matrices $A \in Mat_{n\times n}(R)$*

$$rk_{D_1}(f_1(A)) = rk_{D_2}(f_2(A)),$$

then $D_1 \cong D_2$ and there is an isomorphism $\iota : D_1 \to D_2$ such that

$$\iota \circ f_1 = f_2. \tag{1}$$

Proof. By our condition, $\Sigma_{f_1} = \Sigma_{f_2} = \Sigma$. Let $\lambda' : R \to R_\Sigma/M$ be the natural map, where M is the unique maximal ideal of R_Σ. Then there exist two skew field isomorphisms $\pi_1 : R_\Sigma/M \to D_1$ and $\pi_2 : R_\Sigma/M \to D_2$ such that $\pi_1\circ\lambda' = f_1$ and $\pi_2\circ\lambda' = f_2$. Then we can choose $\iota = \pi_2\circ\pi^{-1}$ to satisfy (1). □

4 Continuous algebras

In this section we recall the notion of a continuous algebra from the book of Goodearl [10] and present some important examples. A ring R is called von Neumann regular if for any $a \in R$ there exists $x \in R$ such that $axa = a$. In other words, R is von Neumann regular if any finitely generated left ideal is generated by a single idempotent. A rank regular ring is a unital regular ring R equipped with a rank function rk_R satisfying the conditions below.

- $0 \leq \mathrm{rk}_R(a) \leq 1$, for any $a \in R$.
- $\mathrm{rk}_R(a) = 0$ if and only if $a = 0$.
- $\mathrm{rk}_R(1) = 1$.
- $\mathrm{rk}_R(a + b) \leq \mathrm{rk}_R(a) + \mathrm{rk}_R(b)$.
- $\mathrm{rk}_R(ab) \leq \mathrm{rk}_R(a), \mathrm{rk}_R(b)$
- $\mathrm{rk}_R(e+f) = \mathrm{rk}_R(e)+\mathrm{rk}_R(f)$ if e and f are orthogonal idempotents.

Note, that in a rank regular ring an element is invertible if and only if it has rank one. Also, a rank regular ring is a metric space with respect to the distance function

$$d_R(a, b) := rk_R(a - b)\,.$$

If a rank regular ring R is complete with respect to the distance function, then R is called a *continuous ring*. We are particularly interested in continuous $\mathcal{K}$-algebras. The simplest examples are skew fields over $\mathcal{K}$ and matrix rings over such skew fields. For these continuous algebras the rank function may take only finitely many values. Another important example is due to John von Neumann. Let us consider the diagonal maps

$$d_n : \mathrm{Mat}_{2^n \times 2^n}(\mathcal{K}) \to \mathrm{Mat}_{2^{n+1} \times 2^{n+1}}(\mathcal{K})\,.$$

The maps preserve the normalized rank functions, hence the direct limit $\varinjlim \mathrm{Mat}_{2^n \times 2^n}(\mathcal{K})$ is a rank regular ring. Its metric completion $A_{\mathcal{K}}$ is a continuous $\mathcal{K}$-algebra, with a rank function rk_A taking all values in between zero and one.

Note that if R is a rank regular ring, the metric completion of R is always a continuous ring [11]. Also, if R is a rank regular ring, then for any $n \geq 1$ the matrix ring $\mathrm{Mat}_{n \times n}(R)$ can be equipped with a unique matrix rank function rk^n_R such that $\mathrm{rk}^n_R(\mathrm{Id}) = n$ and

$$\mathrm{rk}^n_R \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{rk}^k_R(A) + \mathrm{rk}^{n-k}_R(B) \tag{2}$$

if $A \in \mathrm{Mat}_{k \times k}(R), B \in \mathrm{Mat}_{(n-k)\times(n-k)}(R)$ [12]. Finally, let us recall the notion of the ultraproduct of finite dimensional matrix algebras. This construction will be crucial in our paper. Let $M = \{\mathrm{Mat}_{n_k \times n_k}(\mathcal{K})\}^\infty_{k=1}$ be a sequence of matrix algebras over our base field $\mathcal{K}$ equipped with the normalized rank functions $\{\mathrm{rk}_{n_k}\}^\infty_{k=1}$ such that $n_k \to \infty$. Let ω be an ultrafilter on the natural numbers and let $\lim_\omega$ be the associated ultralimit. The ultraproduct $\mathcal{M}_M$ of the algebras $\{\mathrm{Mat}_{n_k \times n_k}(\mathcal{K})\}^\infty_{k=1}$ can be defined the following way. Consider the elements

$$\{a_k\}^\infty_{k=1} \in \prod_{k=1}^\infty \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})\,,$$

for which $\lim_\omega \mathrm{rk}_{n_k}(a_k) = 0$. It is easy to see that these elements form an ideal I_M. The ultraproduct is defined as

$$\mathcal{M}_M := \prod_{k=1}^\infty \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})/I_M\,.$$

The $\mathcal{K}$-algebra $\mathcal{M}_M$ is a simple continuous algebra equipped with a rank function [9]

$$\mathrm{rk}_{\mathcal{M}}\left([\{a_k\}^\infty_{k=1}]\right) = \lim_\omega \mathrm{rk}_{n_k}(a_k)\,,$$

where $[\{a_k\}^\infty_{k=1}]$ denotes the class of $\{a_k\}^\infty_{k=1} \in \prod_{k=1}^\infty \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})$ in $\mathcal{M}_M$.

5 Sofic algebras

Let R be a countable $\mathcal{K}$-algebra over our finite base field $\mathcal{K}$, with $\mathcal{K}$-linear basis $\{1 = r_1, r_2, r_3, \dots\}$. Following Arzhantseva and Paunescu [2], we call R (linearly) sofic if there exists a function $j : R \backslash \{0\} \to \mathbb{R}^+$ and a sequence of positive numbers $s_i \to 0$ such that for any $i \geq 1$ there exists $n_i \geq 1$ and a $\mathcal{K}$-linear map $\phi_i : R \to \mathrm{Mat}_{n_i \times n_i}(\mathcal{K})$ satisfying the conditions below:

- $\phi_i(1) = \mathrm{Id}$
- $\mathrm{rk}_{n_i}(\phi_i(a)) \geq j(a)$ if $0 \neq a \in \mathrm{Span}\,\{r_1, r_2, \dots, r_i\}$
- $\mathrm{rk}_{n_i}(\phi_i(ab) - \phi_i(a)\phi_i(b)) < s_i$ if $a, b \in \mathrm{Span}\,\{r_1, r_2, \dots, r_i\}$.

Such a system is called a *sofic representation* of R. Clearly, the soficity of an algebra does not depend on the particular choice of the basis $\{r_1, r_2, \dots\}$. Using the maps above, we can define a map $\phi : R \to \mathcal{M}_M$, by

$$\phi(s) := [\{\phi_i(s)\}_{i=1}^{\infty}]\,,$$

where $M = \{\mathrm{Mat}_{n_i \times n_i}(\mathcal{K})\}_{i=1}^{\infty}$. By our assumptions, ϕ is a unital embedding. Conversely, we have the following proposition. Note that the proposition was already implicite in [2], the proof below was suggested by the referee.

Proposition 5.1. *Let R be a countable algebra over our base field $\mathcal{K}$. If R can be embedded into an ultraproduct $\mathcal{M}_M$, then R is sofic.*

Proof. Let $\{1 = r_1, r_2, \dots\}$ be a basis for R and let $j(a) := \frac{1}{2}\mathrm{rk}_{\mathcal{M}}(\phi(a))$. It is enough to prove that for any $\epsilon > 0$ and $i \geq 1$, there exists an integer $n \geq 1$ and a linear, unit preserving map $\sigma : R \to \mathrm{Mat}_{n \times n}(\mathcal{K})$ such that

- $\mathrm{rk}_n(\sigma(a)) \geq j(a)$ if $0 \neq a \in \mathrm{Span}\,\{r_1, r_2, \dots, r_i\}$.
- $\mathrm{rk}_n(\sigma(ab) - \sigma(a)\sigma(b)) < \varepsilon$ if $a, b \in \mathrm{Span}\,\{r_1, r_2, \dots, r_i\}$.

Let $\phi : R \to \mathcal{M}_M$ be the embedding. Lift ϕ to a unital linear map

$$\prod_{k=1}^{\infty} \phi_k : R \to \prod_{k}^{\infty} \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})$$

by lifting the basis of R first and then extending to R linearly. Then, for any $a, b \in R$ and $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} \mid \mathrm{rk}_{n_k}(\phi_k(ab) - \phi_k(a)\phi_k(b)) < \varepsilon\} \in \omega\,.$$

Also, for any $0 \neq a \in R$ the set

$$\{k \in \mathbb{N} \mid \mathrm{rk}_{n_k}(\phi_k(a)) > \mathrm{rk}_{\mathcal{M}}(\phi(a))/2\} \in \omega \,.$$

Hence, for any i and $\varepsilon > 0$, the set of all $k \in \mathbb{N}$ satisfying $\mathrm{rk}_{n_k}(\phi_k(ab) - \phi_k(a)\phi_k(b)) < \varepsilon$ for all $a, b \in \mathrm{Span}\,\{r_1, r_2, \ldots, r_i\}$ and $\mathrm{rk}_{n_k}(\phi_k(a)) > \mathrm{rk}_{\mathcal{M}}(\phi(a))/2$ for all $0 \neq a \in \mathrm{Span}\,\{r_1, r_2, \ldots, r_i\}$ is in the ultrafilter ω, in particular, it is not empty. □

6 Amenable skew fields are sofic

In this section we recall the notion of amenability for skew fields from [7]. Let D be a countable dimensional skew field over a finite base field K. We say that D is amenable if for any $\varepsilon > 0$ and finite dimensional K-subspace $1 \in E \subset D$, there exists a finite dimensional K-subspace $V \subset D$ such that

$$\dim_K EV < (1 + \varepsilon) \dim_K V \,.$$

All commutative fields and skew fields of finite Gelfand-Kirillov dimension are amenable. Also, if for a torsion-free amenable group G, the group algebra KG is a domain, then its classical field of fraction is an amenable skew field.

Proposition 6.1. *All amenable skew fields are sofic.*

Proof. It is enough to prove that for any finite dimensional subspace $1 \in F \subset D$ and $\varepsilon > 0$, there exists $n \geq 1$ and a K-linear map $\tau : D \to \mathrm{Mat}_{\,n \times n}(K)$ such that

- $\tau(1) = \mathrm{Id}$
- $\mathrm{rk}_n(\tau(d)) > 1 - \varepsilon$ if $0 \neq d \in F$
- $\mathrm{rk}_n(\tau(fg) - \tau(f)\tau(g)) < \varepsilon$ for any pair $f, g \in F$.

First we need a technical lemma.

Lemma 6.1. *If D is an amenable skew field, then for any $\delta > 0$ and finite dimensional subspace $1 \in E \subset D$, there exists a pair of linear subspaces $V_1 \subset V \subset D$ such that $EV_1 \subset V$ and*

$$\dim_K(V_1) \geq (1 - \delta) \dim_K(V) \,.$$

Proof. Let $\varepsilon > 0$ such that $\frac{1}{1+\varepsilon} \geq 1 - \delta$. By the definition of amenability, there exists a finite dimensional K-subspace $V_1 \subset D$ such that $\dim_K(EV_1) < (1 + \varepsilon) \dim_K(V_1)$. Now set $V = EV_1$. Then the pair $V_1 \subset V$ satisfies the condition of the lemma. □

Now let $H \subset D$ be the linear subspace spanned by $F \cdot F$. Let $V \subset D$ be an n-dimensional linear subspace such that for some linear subspace $V_1 \subset V$, $HV_1 \subset V$ and $\dim_K(V_1) \geq (1-\varepsilon)\dim_K(V)$ hold. Let $W \subset D$ be a linear subspace complementing V and let $P : D \to V$ be the K-linear projection onto V such that

$$P_{|V} = \mathrm{Id} \quad \text{and} \quad P_{|W} = 0\,.$$

Now we define $\tau(d)$ as $P \circ M_d$, where M_d is the left-multiplication by d, Clearly, $\tau : D \to \mathrm{End}_K(V) \cong \mathrm{Mat}_{n\times n}(K)$ is a K-linear map satisfying $\tau(1) = \mathrm{Id}$. If $d \in F$, then $\mathrm{Ker}\,(\tau(d)) \cap V_1 = 0$, that is $\mathrm{rk}_n(\tau(d)) \geq 1 - \varepsilon$. Also, if $f, g \in F$, then $\tau(fg) = \tau(f)\tau(g)$ restricted on V_1. Therefore, $\mathrm{rk}_n(\tau(fg) - \tau(f)\tau(g)) < \varepsilon$. □

7 The free skew field is sofic

By Theorem 1 of [7], if $K \subset E$ is a sub-skew field of D and E is non-amenable, then D is non-amenable as well. If $K = \mathbb{C}$ then the free skew field on r generators over K is a non-amenable skew field. We conjecture that this is the case for all base fields. There are many ways to define the free skew fields, in this paper we regard these objects as the skew fields of noncommutative rational functions. As it follows, we use the approach of [15] and [16]. Let $\mathcal{K}[z_1, z_2, \dots, z_r]$ be the free algebra over r noncommutative indeterminates, where $\mathcal{K}$ be a finite field. Formal expressions over $\{z_1, z_2, \dots, z_r\}$ can be defined inductively the following way.

- The elements of $\mathcal{K}[z_1, z_2, \dots, z_r]$ are formal expressions.
- If R_1 and R_2 are formal expressions, so are R_1R_2, $R_1 + R_2$ and $(R_1)^{-1}$.

Let $M_1, M_2, \dots, M_r \in \mathrm{Mat}_{n\times n}(\mathcal{K})$, for some $n \geq 1$. We say that the formal expression R can be evaluated on $\{M_1, M_2, \dots, M_r\}$, if all the inverses involved in the inductive calculation of $R(M_1, M_2, \dots, M_r)$ exist. Then we say that $\{M_1, M_2, \dots, M_r\} \in \mathrm{dom}\, R$. We call a formal expression R a noncommutative rational expression (nre) if $\mathrm{dom}\, R \neq \emptyset$. Two nre's R and S are equivalent if $\mathrm{dom}\, R \cap \mathrm{dom}\, S \neq \emptyset$ and if $\{M_1, M_2, \dots, M_r\} \in \mathrm{dom}\, R \cap \mathrm{dom}\, S$ then

$$R(M_1, M_2, \dots, M_r) = S(M_1, M_2, \dots, M_r)\,.$$

The equivalence classes of nre's are called noncommutative rational functions and they form the free skew field $\mathcal{D}_r(\mathcal{K})$ over $\mathcal{K}$ on r generators

(Proposition 2.2 [15]). Let $T = (T_n^1, T_n^2, \dots, T_n^r)$ be a r-tuple of $n \times n$-matrices with entries $\{T_{jk}^i\}_{1\le i\le r, 1\le j,k\le n}$ that are commuting indeterminates. That is, each matrix T_n^i can be viewed as an element of the ring $\mathrm{Mat}_{n\times n}(P_{(n)})$, where $P_{(n)}$ is the commutative polynomial algebra over $\mathcal{K}$ with rn^2 variables. The evaluation

$$p \to p(T_n^1, T_n^2, \dots, T_n^r)$$

defines a homomorphism $\rho_n : \mathcal{K}[z_1, z_2, \dots, z_r] \to \mathrm{Mat}_{n\times n}(P_{(n)})$. The algebra $\mathrm{Im}(\rho_n) = G_n$ is called the algebra of generic matrices. Let $Q_{(n)}$ be the field of fraction of the algebra $P_{(n)}$. The skew field D_n is defined as division closure of G_n in $\mathrm{Mat}_{n\times n}(Q_{(n)})$ (Proposition 2.1 [15]). If R is a nre, then $\mathcal{N}_R$ is defined the following way. The natural number n is an element of $\mathcal{N}_R$ if

$$R_n := R(T_n^1, T_n^2, \dots, T_n^r)$$

can be evaluated inductively as an element of D_n. For any noncommutative rational expression R, there exists n_R such that

- If $n \ge n_R$ then $n \in \mathcal{N}_R$. [Remark 2.3,[16]].
- If R and S are equivalent rational expressions and $n \in \mathcal{N}_R \cap \mathcal{N}_S$, then $R_n = S_n$. [Remark 2.6 and Definition 2.8,[16]].
- For any pair of expressions R, S if $n \in \mathcal{N}_R \cap \mathcal{N}_S$, then $\mathcal{N}_{RS} = \mathcal{N}_R \cap \mathcal{N}_S$, $\mathcal{N}_{R+S} = \mathcal{N}_R \cap \mathcal{N}_S$ then $(RS)_n = R_n S_n$, $(R+S)_n = R_n + R_m$, whenever $n \in \mathcal{N}_R \cap \mathcal{N}_S$. Also, if $n \in \mathcal{N}_R \cap \mathcal{N}_{R^{-1}}, R_n^{-1} = (R_n)^{-1}$. [Definition 2.1,[16]].

Lemma 7.1. *If $1 \in E \subset \mathcal{D}_r(\mathcal{K})$ is a finite dimensional $\mathcal{K}$-subspace and let F be the subspace spanned by $E \cdot E$. Then there exists $n \ge 1$ and a $\mathcal{K}$-linear map $\phi : F \to D_n$ such that $\phi(ab) = \phi(a)\phi(b)$, if $a, b \in E$.*

Proof. For any element $a \in F$, let us pick a nre $\hat{a}$ that represents a. Choose an integer large enough so that

$$n \ge \max\left(\max_{a\in F}(n_{\hat{a}}) , \max_{a,b\in F}(n_{\hat{ab}}) , \max_{a,b\in F}(n_{a\hat{+}b}) \right) .$$

Then, $(a \hat{+} b)_n = \hat{a}_n + \hat{b}_n$. Indeed, $a \hat{+} b$ is equivalent to $\hat{a} + \hat{b}$, hence by the basic properties above,

$$(a \hat{+} b)_n = (\hat{a} + \hat{b})_n = \hat{a}_n + \hat{b}_n .$$

Similarly, $(\hat{ab})_n = \hat{a}_n \hat{b}_n$. Therefore, $\phi(a) := \hat{a}_n$ defined a linear map from F into D_n such that $\phi(ab) = \phi(a)\phi(b)$, whenever $a, b \in E$. $\square$

Lemma 7.2. *D_n is an amenable skew field.*

Proof. First note that G_n is an amenable domain. Indeed, it is a subalgebra of the matrix algebra $\mathrm{Mat}_{n\times n}(P_{(n)})$, hence it has polynomial growth. Therefore, by Proposition 2.2 [7] its classical field of fractions is an amenable skew field. However, since D_n is the division closure of G_n in a ring, D_n must be the classical field of fractions of G_n. $\square$

Since amenable skew fields are sofic, by Lemma 7.1 we have the following proposition.

Proposition 7.1. *The free skew field on r generators over our finite base field $\mathcal{K}$ is sofic.*

Proof. It is enough to prove that for any $\mathcal{K}$-linear subspace $1 \in E \subset \mathcal{D}_r(\mathcal{K})$ and $\varepsilon > 0$, there exists $s \geq 1$ and a unital linear map $\Omega : \mathcal{D}_r(\mathcal{K}) \to \mathrm{Mat}_{s\times s}(\mathcal{K})$ such that

$$\text{For any}\, a, b \in E, \mathrm{rk}_s(\Omega(ab) - \Omega(a)\Omega(b)) < \varepsilon. \tag{3}$$

$$\text{For any}\, a \in E, \mathrm{rk}_s(\Omega(a)) \geq 1/2\,. \tag{4}$$

Let $\phi : F \to D_n$ be the map defined in Lemma 7.1. Since D_n is an amenable skew field, there exists $s \geq 1$, and a unital linear map $\tau : D_n \to \mathrm{Mat}_{s\times s}(\mathcal{K})$ such that

- For any $c, d \in \phi(F)$, $\mathrm{rk}_s(\tau(cd) - \tau(c)\tau(d)) < \varepsilon$.
- For any $c \in \phi(F)$, $\mathrm{rk}_s(\tau(c)) > 1/2$.

Now, let $\Omega(a) := \tau \circ \phi(a)$, whenever $a \in F$ (and extend Ω onto $\mathcal{D}_r(\mathcal{K})$ linearly). Clearly, (3) and (4) will be satisfied. $\square$

8 Limits of linear representations

Let $\{\theta_k : \mathbb{F}_r \to \mathrm{Mat}_{n_k\times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ be a convergent sequence of linear representations. Then we can consider the ultraproduct representation $\theta : \mathbb{F}_r \to \mathcal{M}_M$ and the extended algebra homomorphism (we denote it with the same letter) $\theta : \mathcal{K}\mathbb{F}_r \to \mathcal{M}_M$. One should notice that $\mathcal{K}\mathbb{F}_r/\mathrm{Ker}\,(\theta)$ is a sofic algebra. By definition, for any $n \geq 1$ and $A \in \mathrm{Mat}_{n\times n}(\mathcal{K}\mathbb{F}_r)$

$$\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\theta_k(A)) = \mathrm{rk}^n_{\mathcal{M}}(\theta(A))\,.$$

Note that $\mathrm{Mat}_{n\times n}(\mathcal{M}_M)$ is the algebraic ultraproduct of the matrix algebras $\{\mathrm{Mat}_{n\times n}(\mathrm{Mat}_{n_k\times n_k}(\mathcal{K}))\}_{k=1}^{\infty}$ and the unique extended rank function [12] on $\mathrm{Mat}_{n\times n}(\mathcal{M}_M)$ is exactly the ultralimit of the matrix ranks

$\mathrm{rk}^n_{n_k}$. Now, let us prove Theorem 1. It is enough to see that there exists a von Neumann regular countable subalgebra $S \subset \mathcal{M}_M$ containing $\mathrm{Im}\,(\theta)$. Indeed, the limit object sought after in the theorem is the metric closure of S in $\mathcal{M}_M$ (recall that the completion of a rank regular algebra is a continuous algebra [11]). If T is a countable subset of $\mathcal{M}_M$, then let $R(T)$ be the $\mathcal{K}$-algebra generated by T. It is easy to see that $R(T)$ is still countable. Let $X : \mathcal{M}_M \to \mathcal{M}_M$ be a function such that for any $a \in \mathcal{M}_M$,

$$aX(a)a = a\,.$$

Let $R_1 = R(\mathrm{Im}\,(\theta) \cup X(\mathrm{Im}\,(\theta))$ and inductively, let $R_{n+1} = R(R_n \cup X(R_n)$. Then the ring $S = \cup_{n=1}^{\infty} R_n$ is a countable von Neumann regular algebra containing $\mathrm{Im}\,(\theta)$. This finishes the proof of Theorem 1. □

Now let us suppose that the convergent sequence of representations $\{\theta_k\}_{k=1}^{\infty}$ satisfies the Atiyah condition, that is for any $n \geq 1$ and $A \in \mathrm{Mat}_{\,n\times n}(\mathcal{K}\mathbb{F}_r)$

$$\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\theta_k(A)) \in \mathbb{Z}\,.$$

Proposition 8.1. *The division closure D of $\mathrm{Im}\,(\theta)$ in the algebra $\mathcal{M}_M$ is a skew field. That is, $\theta : \mathbb{F}_r \to D$ is the limit of the sequence $\{\theta_k\}_{k=1}^{\infty}$.*

Proof. We use an idea of Linnell [17]. Let $p \neq 0$ be an element of the rational closure of $\mathrm{Im}\,(\theta)$ in $\mathcal{M}_M$. By Lemma 3.1, we have matrices Q, A, B such that all the entries of Q are from $\mathrm{Im}\,(\theta)$ and

$$Q = A \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} B\,,$$

where A and B are invertible, and $\mathrm{rk}^n_{\mathcal{M}}(Q)$ is an integer. Since the matrix rank on $\mathrm{Mat}_{\,n\times n}(\mathcal{M}_M)$ is the ultralimit of the matrix ranks $\mathrm{rk}^n_{n_k}$ (or by [12])

$$n - 1 < \mathrm{rk}^n_{\mathcal{M}} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \leq n\,.$$

Since

$$\mathrm{rk}^n_{\mathcal{M}} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \mathrm{rk}^n_{\mathcal{M}}(Q)$$

by the integrality condition, $\mathrm{rk}^n_{\mathcal{M}} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = n$. Thus, p is invertible. By Lemma 3.2, the division closure of $\mathrm{Im}\,(\theta)$ in $\mathcal{M}_M$ is contained by the rational closure. Therefore, each nonzero element of the division closure is invertible. Hence, the division closure D is a skew field. □

Now suppose that for some skew field D', $\theta' : \mathbb{F}_r \to D'$ is another limit for the sequence $\{\theta_k\}_{k=1}^{\infty}$. Then for any $n \geq 1$ and $A \in \mathrm{Mat}_{n\times n}(\mathcal{K}\mathbb{F}_r)$

$$\mathrm{rk}_D^n(\theta(A)) = \mathrm{rk}_{D'}^n(\theta(A))\,.$$

Therefore, by Corollary 3.1, there exists an isomorphism $\pi : D \to D'$ such that $\pi \circ \theta = \theta'$. This finishes the proof of Theorem 2. □

Remark. Note that the proof of Proposition 8.1 shows that if $\theta' : \mathbb{F}_r \to S$ is a limit of the sequence $\{\theta_k\}_{k=1}^{\infty}$, where S is a continuous algebra, then the division closure of $\mathrm{Im}\,(\theta')$ in S is still isomorphic to D.

9 Linear tilings

In this section we prove a key technical result of our paper. Let D be a countable skew field over $\mathcal{K}$ with $\mathcal{K}$-basis $\{1 = r_1, r_2, \dots\}$ and let $\phi : D \to \mathrm{Mat}_{n\times n}(\mathcal{K}) \cong \mathrm{End}_{\mathcal{K}}(\mathcal{K}^n)$ be a unit preserving linear map. We say that $x \in \mathcal{K}^n$ is i-good with respect to ϕ if

$$x \in \mathrm{Ker}\,(\phi(ab) - \phi(a)\phi(b)),$$

whenever $a, b \in \mathrm{Span}\,\{r_1, r_2, \dots, r_i\}$. The i-good elements form the $\mathcal{K}$-subspace $G_n^{i,\phi} \subset \mathcal{K}^n$. By definition, if $\{\phi_k : D \to \mathrm{Mat}_{n\times n}(\mathcal{K})\}_{k=1}^{\infty}$ is a sofic representation, then for any fixed $i \geq 1$,

$$\lim_{k\to\infty} \frac{\dim_{\mathcal{K}} G_{n_k}^{i,\phi}}{n_k} = 1\,.$$

Definition 9.1. *A unit preserving linear map $\phi : D \to Mat_{n\times n}(\mathcal{K})$ is an i-good map if*

$$\frac{\dim_{\mathcal{K}} G_n^{i,\phi}}{n} \geq 1 - \frac{1}{i}\,.$$

Let $\phi : D \to \mathrm{Mat}_{n\times n}(\mathcal{K})$ and let $1 \in F \subset D$ and $H \subset \mathcal{K}^n$ be finite dimensional linear subspaces. We call a subset $T \subset \mathcal{K}^n$ a set of (i, F, H)-centers with respect to ϕ if

- For any $x \in T$, $\phi(F)(x) \subset H$ is a $\dim_{\mathcal{K}}(F)$-dimensional $\mathcal{K}$-subspace.
- If $x \in T$, then for any $0 \neq f \in F$, $\phi(f)(x)$ is i-good.
- The subspaces $\{\phi(F)(x)\}_{x\in T}$ are independent.

We say that ϕ has an (F, H, i, δ)-tiling if there exists a set of (i, F, H)-centers T for ϕ such that

$$|T| \dim_{\mathcal{K}}(F) \geq (1 - \delta)n\,.$$

Theorem 4. *Let D be a countable skew field over the base field $\mathcal{K}$, with basis $\{1 = r_1, r_2 \dots\}$. Then for any finite dimensional subspace $1 \in F \subset D$ and $\delta > 0$, we have a positive constant $N_{F,\delta}$ such that if $i, n \geq N_{F,\delta}$, $\phi : D \to Mat_{n\times n}(\mathcal{K})$ is an i-good unit preserving linear map and* $\dim_{\mathcal{K}} H \geq (1 - \frac{1}{i})n$*, then ϕ has an (F, H, i, δ)-tiling.*

Proof. Let $\phi : D \to \mathrm{Mat}_{n\times n}(\mathcal{K})$ be an i-good unit preserving linear map, the exact values of i and n will be given later. First note, that if i is larger than some constant N^1_F, then $(F\backslash\{0\})^{-1}, F \subset \mathrm{Span}\{r_1, r_2, \dots, r_i\}$. Note that if

$$0 \neq x \in \mathrm{Ker}\,(\phi(f^{-1})\phi(f) - 1)$$

for any $0 \neq f \in F$, then $\dim_{\mathcal{K}} \phi(F)(x) = \dim_{\mathcal{K}}(F)$. Thus, if $i \geq N^1_F$ and $0 \neq x \in G^{i,\phi}_n$, then $\dim_{\mathcal{K}} \phi(F)(x) = \dim_{\mathcal{K}}(F)$. Let

$$A_{F,i} = \{x \in \mathcal{K}^n \mid \phi(f)(x) \in G^{i,\phi}_n \cap H\,, \text{for any } f \in F\}\,.$$

It is easy to see that there exists some constant $N^2_{F,\delta}$ such that if $i \geq N^2_{F,\delta}$ then

$$\dim_{\mathcal{K}}(A_{F,i}) \geq (1 - \frac{\delta}{4})n\,. \tag{5}$$

and for any $0 \neq f \in F$,

$$\dim_{\mathcal{K}} \mathrm{Ker}\,\phi(f) = n - \dim_{\mathcal{K}} \mathrm{Im}\,\phi(f) \leq \frac{\delta}{3}n\,. \tag{6}$$

Finally, let $N^3_{F,\delta} > 0$ such that if $n > N^3_{F,\delta}$, then $|F| \leq |\mathcal{K}|^{\frac{\delta}{3}n}$ and let $N_{F,\delta} = \max(N^1_F, N^2_{F,\delta}, N^3_{F,\delta})$. For $0 \neq f \in F$ and $v \in \mathcal{K}^n$, let $L(f, v)$ denote the set of points y such that $\phi(f)(y) = v$. Then we have the estimate

$$|L(f, v)| \leq |\mathrm{Ker}\,\phi(f)| \leq |\mathcal{K}|^{\frac{\delta}{3}n}\,. \tag{7}$$

Let T be a maximal set of (i, F, H)-centers for ϕ. We need to prove that

$$|T| \dim_{\mathcal{K}}(F) \geq (1 - \delta)n\,.$$

Let V be the span of the subspace $\cup_{x\in T}\phi(F)(x)$. Then,

$$|V| = |\mathcal{K}|^{\dim_{\mathcal{K}}(V)} = |\mathcal{K}|^{|T| \dim_{\mathcal{K}}(F)}\,.$$

Assume that $|V| < |\mathcal{K}|^{(1-\delta)n}$. Now, suppose that $i, n \geq N_{F,\delta}$. Then

$$|\bigcup_{v\in V} \bigcup_{f\in F\backslash\{0\}} L(f, v)| < |V||F||\mathcal{K}|^{\frac{\delta}{3}n} \leq |\mathcal{K}|^{(1-\frac{\delta}{3})n} \leq |A_{F,i}|\,.$$

Therefore, there exists $x \in A_{F,i}$ such that $\phi(F)(x) \cap V = 0$. Hence, $\{x\} \cup T$ is a set of (i, F, H)-centers for ϕ, leading to a contradiction. □

10 Convergent sequences and sofic approximations

The goal of this section is to prove the following theorem.

Theorem 5. *Let D be a countable skew field over $\mathcal{K}$ and $\phi : \mathbb{F}_r \to D$ be a generating homomorphism. Then ϕ is the limit of a convergent sequence of finite dimensional representations satisfying the Atiyah condition if and only if D is sofic.*

Proposition 10.1. *Let $\phi : \mathbb{F}_r \to D$ as above, where D is sofic. Then ϕ is the limit of a convergent sequence of finite dimensional representations.*

Proof. Let $\{\psi_k : D \to \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ be a sofic approximation sequence for the skew field D.

Lemma 10.1. *For any $B \in Mat_{n\times n}(D)$*

$$\lim_{k\to\infty} rk^n_{n_k}(\psi_k(B)) = rk^n_D(B)\,,$$

where $rk^n_{n_k}$ is the matrix rank on $Mat_{n\times n}(Mat_{n_k\times n_k}(\mathcal{K}))$.

Proof. By taking a subsequence, we may suppose that $\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\psi_k(B))$ exists for all matrices B. Then,

$$\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\psi_k(B)) = \mathrm{rk}^n_{\mathcal{M}}(\psi(B))\,,$$

where $\psi : D \to M_{\mathcal{M}}$ is the ultraproduct embedding. Thus, $\hat{\mathrm{rk}}(B) := \mathrm{rk}^n_{\mathcal{M}}(\psi(B))$ defines a rank function on $\mathrm{Mat}_{n\times n}(D)$. Since there exists only one rank function on matrix rings,

$$\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\psi_k(B)) = \mathrm{rk}^n_D(B)\,. \qquad \square$$

Let $\hat{\phi}_k := \psi_k \circ \phi$ be a linear map. By the lemma above, for any $n \geq 1$ and matrix $A \in \mathrm{Mat}_{n\times n}(\mathcal{K}\mathbb{F}_r)$.

$$\lim_{k\to\infty} \mathrm{rk}^n_{n_k}(\hat{\phi}_k(A)) = \mathrm{rk}^n_D(\phi(A))\,.$$

Note however, that $\hat{\phi}_k$ does not necessarily define a linear representation of $\mathbb{F}_r$. However, we have the following lemma.

Lemma 10.2. *Let* $\{\hat{\phi}_k\}_{k=1}^{\infty}$ *be the maps as above.*
Let $\{\phi_k : \mathbb{F}_r \to Mat_{n_k \times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ *be linear representations such that for any generator* γ_i *of the free group*

$$\lim_{k\to\infty} rk_{n_k}(\phi_k(\gamma_i) - \hat{\phi}_k(\gamma_i)) = 0\,.$$

Then for any $n \geq 1$ *and* $A \in Mat_{n\times n}(\mathcal{K}\mathbb{F}_r)$

$$\lim_{k\to\infty} rk^n_{n_k}(\phi_k(A) - \hat{\phi}_k(A)) = 0\,.$$

Proof. Clearly, it is enough to show that for any $a \in \mathcal{K}\mathbb{F}_r$

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\phi_k(a) - \hat{\phi}_k(a)) = 0 \tag{8}$$

First we prove (8) in a special case.

Lemma 10.3.

$$\lim_{k\to\infty} rk_{n_k}(\phi_k(\gamma_i^{-1}) - \hat{\phi}_k(\gamma_i^{-1})) = 0\,.$$

Proof. By soficity,

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\hat{\phi}_k(\gamma_i^{-1})\hat{\phi}_k(\gamma_i) - 1) = 0\,.$$

Hence by our assumption,

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\hat{\phi}_k(\gamma_i^{-1})\phi_k(\gamma_i) - 1) = 0\,.$$

Thus

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}\Big((\hat{\phi}_k(\gamma_i^{-1}) - \phi_k(\gamma_i^{-1}))\phi_k(\gamma_i)\Big) = 0\,.$$

Since $\phi_k(\gamma_i)$ is an invertible element for all k, the lemma follows. □

Now suppose that for some $w_1, w_2 \in \mathbb{F}_r$

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\phi_k(w_1) - \hat{\phi}_k(w_1)) = 0 \quad \text{and} \quad \lim_{k\to\infty} \mathrm{rk}_{n_k}(\phi_k(w_2) - \hat{\phi}_k(w_2)) = 0\,.$$

By soficity,

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\hat{\phi}_k(w_1 w_2) - \hat{\phi}_k(w_1)\hat{\phi}_k(w_2)) = 0\,.$$

Since

$$\phi_k(a)\phi_k(b) - \hat{\phi}_k(a)\hat{\phi}_k(b) = (\phi_k(a) - \hat{\phi}_k(a))\phi_k(b) - \hat{\phi}_k(a)(\hat{\phi}_k(b) - \phi_k(b))$$

we have that

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\phi_k(w_1w_2) - \hat{\phi}_k(w_1w_2)) = 0\,.$$

Therefore by induction, for any $w \in \mathbb{F}_r$

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\phi_k(w) - \hat{\phi}_k(w)) = 0\,.$$

Now (8) follows easily. □

We finish the proof of Proposition 10.1 . Observe that $\lim_{k\to\infty} \mathrm{rk}_{n_k}\hat{\phi}_k(\gamma_i) = 1$ for all the generators, hence we have invertible elements $a_i^k \in \mathrm{Mat}_{\,n_k\times n_k}(\mathcal{K})$ such that

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\hat{\phi}_k(\gamma_i) - a_i^k) = 0\,.$$

Now, let us define $\phi_k : \mathbb{F}_r \to \mathrm{Mat}_{\,n_k\times n_k}(\mathcal{K})$ by setting $\phi_k(\gamma_i) = a_i^k$. Then the proposition immediately follows from Lemma 10.2 . □

Proposition 10.2. *Let $\{\theta_k : \mathbb{F}_r \to Mat_{n_k\times n_k}(\mathcal{K})\}_{k=1}^\infty$ be a convergent sequence of linear representations satisfying the Atiyah condition. Suppose that for some skew field D and generating map $\theta : \mathbb{F}_r \to D$, θ is the limit of $\{\theta_k\}_{k=1}^\infty$. Then D is sofic.*

We will prove a stronger version of Proposition 10.2 that will be used in the next section. We call $a \in k\mathbb{F}_r$ an element of length at most l, if all the non-vanishing terms of $a = \sum k_i w_i$ have length (as reduced words) at most l. A matrix $A \in \mathrm{Mat}_{\,s\times s}(\mathcal{K}\mathbb{F}_r)$ is of length at most l, if all the entries of A have length at most l. Now, let $\theta : \mathbb{F}_r \to D$ be a generating homomorphism, where D is a skew field and $\{1 = r_1, r_2, \dots\}$ is a $\mathcal{K}$-basis for D. Let $\rho : \mathbb{F}_r \to \mathrm{Mat}_{\,s\times s}(\mathcal{K})$ be a linear representation. We say that a linear map $\phi : D \to \mathrm{Mat}_{\,s\times s}(\mathcal{K})$ is an (m, δ)-approximate extension of ρ if

- ϕ is an m-good map (see Definition 9.1).
- For any element $a \in \mathcal{K}\mathbb{F}_r$ of length at most m

$$\mathrm{rk}_s(\phi(\theta(a)) - \rho(a)) < \delta\,.$$

Proposition 10.3. *Let $\theta : \mathbb{F}_r \to D$ be a linear representation into a countable skew field D such that $Im(\theta)$ generates D. Let $\{1 = r_1, r_2, \dots\}$ be a $\mathcal{K}$-basis of D. Then for any $m \geq 1$ and $\delta > 0$ there exists a constant $l_{m,\delta}$ such that if for a linear representation $\rho : \mathbb{F}_r \to Mat_{s\times s}(\mathcal{K})$,*

$$|rk_s^n(\rho(A)) - rk_D^n(\theta(A))| < \frac{1}{l_{m,\delta}} \qquad (9)$$

whenever $A \in Mat_{n\times n}(\mathcal{K}\mathbb{F}_r), n \leq l_{m,\delta}$ *is a matrix of length at most* $l_{m,\delta}$*, then there exists a* $\mathcal{K}$*-linear unit preserving map* $\phi : D \to Mat_{s\times s}(\mathcal{K})$ *that is an* (m,δ)*-approximate extension of* ρ*.*

Proof. Suppose that the Proposition does not hold for some pair m, δ. Then there exists a sequence $\{\theta_k : \mathbb{F}_r \to \mathrm{Mat}_{n_k\times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ of linear representations converging to θ such that none of the θ_k's have (m,δ)-approximate extension onto D. Consider that ultraproduct map $\hat{\theta} : \mathbb{F}_r \to \mathcal{M}_M$. By Proposition 8.1 and Theorem 2, we can extend $\hat{\theta}$ onto an embedding $\phi : D \to \mathcal{M}_M$ (that is $\phi \circ \theta = \hat{\theta}$). From now on, we follow the proof and the notation of Proposition 5.1. For $d \in D$, let

$$\phi(d) = [\{\phi_k(d)\}_{j=1}^{\infty}]\,,$$

where $\{\phi_k : D \to \mathrm{Mat}_{n_k\times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ are unital linear maps. As observed in the Proposition 5.1,

$$\{k \in \mathbb{N} \mid \ \phi_k \text{ is } m\text{-good}\} \in \omega$$

Observe that for any $a \in \mathcal{K}\mathbb{F}_r$

$$\lim_{\omega} \mathrm{rk}_{n_k}(\phi_k(\theta(a)) - \theta_k(a)) = 0\,.$$

Therefore,

$$\{k \in \mathbb{N} \mid \ \mathrm{rk}_{n_k}(\phi_k(\theta(a)) - \theta_k(a)) < \delta,$$
$$\text{for any } a \in \mathcal{K}\mathbb{F}_r \text{ of length at most } m\} \in \omega.$$

Hence,

$$\{k \in \mathbb{N} \mid \ \phi_k \text{ is a } (m,\delta)\text{-extension}\} \in \omega$$

leading to a contradiction. □

Remark. One should note that for a domain R (provided it is not an Ore domain) it is possible to have many non-isomorphic skew fields with epic embeddings $\phi : R \to D$. In fact, according to our knowledge, there is no finitely generated skew field D countable dimensional over $\mathcal{K}$ for which epic embeddings $\phi : \mathcal{K}\mathbb{F}_r \to D$ known not to exist. In [13], infinitely many examples of different epic embeddings of $\theta : \mathcal{K}\mathbb{F}_r \to Q$ are given, where the skew fields Q are the quotient fields of certain amenable domains. Since all these skew fields Q are amenable, they are sofic, hence by our result above these θ's are limits of finite dimensional representations. We cannot make the difference between these embeddings using only the ranks of group algebra elements. This observation shows why the use of matrices in Definition 2.1 is crucial.

11 Amenable limit fields

The goal of this section is to prove the first part of Theorem 3.

Proposition 11.1. *Let $\{\theta_k : \mathbb{F}_r \to Mat_{n_k \times n_k}(\mathcal{K})\}_{k=1}^{\infty}$, $n_k \to \infty$ be a convergent sequence of representations satisfying the Atiyah condition. Let $\theta : \mathbb{F}_r \to D$ be a limit representation of $\{\theta_k\}_{k=1}^{\infty}$, where D is an amenable skew field and $Im(\theta)$ generates D. Then $\{\theta_k\}_{k=1}^{\infty}$ is a hyperfinite sequence.*

Proof. Let $\{1 = r_1, r_2, \dots\}$ be a $\mathcal{K}$-basis for D. By Proposition 10.3, we have a sofic approximation sequence $\{\phi_k : D \to \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ such that for any $a \in \mathcal{K}\mathbb{F}_r$

$$\lim_{k\to\infty} \mathrm{rk}_{n_k}(\phi_k(\theta(a)) - \theta_k(a)) = 0\,.$$

Let $\epsilon > 0$ and choose $\delta > 0$ in such a way that $(1-\delta)^2 > 1-\varepsilon$ and $(1-\delta)^{-1} \leq 1+\varepsilon$. By Lemma 6.1, we have finite dimensional $\mathcal{K}$-subspaces $F_1 \subset F \subset D$ such that $\theta(\gamma_s)F_1 \subset F$ holds for any generator γ_s and

$$\dim_{\mathcal{K}}(F_1) > (1-\delta)\dim_{\mathcal{K}}(F)\,.$$

Now let $N_{F,\delta} > 0$ be the constant in Theorem 4. Let $i \geq N_{F,\delta}$ be an integer such that

$$\cup_{s=1}^{r}\theta(\gamma_s) \cup F \subset \mathrm{Span}\,\{r_1, r_2, \dots, r_i\}\,.$$

By definition, there exists $q \geq 1$ such that if $k \geq q$, then

- ϕ_k is i-good.
- $\dim_{\mathcal{K}} H_k > (1-\frac{1}{i})n_k$, where

$$H_k = \cap_{s=1}^{r}\{x \in \mathcal{K}^{n_k} \mid \phi_k(\theta(\gamma_s))(x) = \theta_k(\gamma_s)(x)\}\,.$$

By Theorem 4, if $k > q$, then ϕ_k has an (F, H_k, i, δ)-tiling. Let T_k be the set of centers of the tiling above. For $x \in T_k$, let

$$V_x = \{\phi_k(F_1)(x)\}\,.$$

By our assumptions,

- For any $x \in T_k$, $\dim_{\mathcal{K}}(V_x + \sum_{s=1}^{r}\theta_k(\gamma_s)V_x) \leq \dim_{\mathcal{K}}(F)\,.$
- $\sum_{x\in T_k}\dim_{\mathcal{K}}(V_x) \geq (1-\delta)^2 n_k\,.$
- The subspaces $\{W_x = V_x + \sum_{s=1}^{r}\theta_k(\gamma_s)V_x\}_{x\in T_k}$ are independent.

Hence, $\{\theta_k\}_{k=1}^{\infty}$ is a hyperfinite sequence. Indeed, K_ε can be chosen as $\dim_{\mathcal{K}}(F)\,.$ □

12 Non-amenable limit fields

The goal of this section is to finish the proof of Theorem 3, by proving the following proposition.

Proposition 12.1. *Let $\{\theta_k : \mathbb{F}_r \to Mat_{n_k\times n_k}(\mathcal{K})\}_{k=1}^\infty$ be a convergent sequence of finite dimensional representations satisfying the Atiyah condition. Suppose that the generating map $\theta : \mathbb{F}_r \to D$ is a limit of $\{\theta_k\}_{k=1}^\infty$, where D is a non-amenable skew field. Then $\{\theta_k\}_{k=1}^\infty$ is not hyperfinite.*

The proof of the proposition will be given in several steps. Let $\theta : \mathbb{F}_r \to D$ be a generating map, where D is a $\mathcal{K}$-skew field with basis $\{1 = r_1, r_2, \dots\}$. Let $\{\rho_k : D \to \mathrm{Mat}_{n_k\times n_k}(\mathcal{K}) \cong \mathrm{End}_{\mathcal{K}}(\mathcal{K}^{n_k})\}_{k=1}^\infty$ be a sequence of unit preserving linear maps such that some non-trivial $\mathcal{K}$-subspaces $L_k \subset \mathcal{K}^{n_k}$ are fixed with uniform bound

$$\dim_{\mathcal{K}} L_k \le C \in \mathbb{N} \qquad \text{for any } k \ge 1.$$

Proposition 12.2. *Suppose that $D, \theta, \{\rho_k\}_{k=1}^\infty$ are as above, $\delta > 0$, and for any $k \ge 1$*

- $\dim_{\mathcal{K}}(L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))L_k) \le (1+\delta)\dim_{\mathcal{K}} L_k\,.$
- $\rho_k(ab)(x) = \rho_k(a)\rho_k(b)(x)$, *if $x \in L_k$ and* $a, b \in Span\{r_1, r_2, \dots, r_k\}\,.$

Then there exists some integer $m \ge 1$ and a finite dimensional $\mathcal{K}$-linear subspace $L \subset D^m$ such that

$$\dim_{\mathcal{K}}(L + \sum_{i=1}^r \theta(\gamma_i)L) \le (1+\delta)\dim_{\mathcal{K}}(L)\,.$$

Proof. Again, let ω be a nonprincipal ultrafilter on the natural numbers. If $\{V_k\}_{k=1}^\infty$ are finite dimensional $\mathcal{K}$-linear vectorspaces, then their ultraproduct $V = \prod_\omega V_k$ is defined the following way. Let $Z \subset \prod_{k=1}^\infty V_k$ be the subspace of sequences $\{x_k\}_{k=1}^\infty$ such that

$$\{k \mid x_k = 0\} \in \omega\,.$$

Then $V = \prod_\omega V_k := \prod_{k=1}^\infty V_k/Z$. Observe, that if $\{W_k \subset V_k\}_{k=1}^\infty$ is a sequence of subspaces, then $\prod_\omega W_k \subset \prod_\omega V_k$. Also, if $\{\zeta_k : D \to \mathrm{End}_{\mathcal{K}}(V_k)\}_{k=1}^\infty$ is a sequence of linear maps, the ultraproduct map $\zeta : D \to \mathrm{End}_{\mathcal{K}}(V)$ is defined as $\zeta(d)(x) = [\{\zeta_k(x_k)\}_{k=1}^\infty]$, where $x = [\{x_k\}_{k=1}^\infty] \in V$.

Lemma 12.1. *$L = \prod_\omega L_k$ is a non-trivial finite dimensional subspace of $K = \prod_\omega \mathcal{K}^{n_k}$. Also, $\dim_{\mathcal{K}}(L) = t$, where $\{k \mid \dim_{\mathcal{K}}(L_k) = t\} \in \omega$.*

Proof. Let $\{a_1^k, a_2^k, \dots, a_C^k\}$ be a $\mathcal{K}$-generator system for L_k. Let $x = [\{x_k\}_{k=1}^\infty] \in L$. Then by finiteness, there exist elements $\{\lambda_i \in \mathcal{K}\}_{i=1}^C$ such that

$$\{k \mid \sum_{i=1}^C \lambda_i a_i^k = x_k\} \in \omega\,.$$

Therefore, $x = \sum_{i=1}^C \lambda_i a_i$, where $a_i = [\{a_i^k\}_{k=1}^\infty\}] \in L$. $\square$

Lemma 12.2. *The ultraproduct of the finite dimensional spaces* $\{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\}$ *is* $L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L)$.

Proof. All the elements of $L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L)$ can be written in the form of

$$x_0 + \sum_{i=1}^r \rho(\theta(\gamma_i))(x_i)\,,$$

where $\{x_0, x_1, x_2, \dots, x_r\} \subset V$. Hence

$$\prod_\omega \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\} \supset L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L)\,.$$

On the other hand, all the elements of $\prod_\omega \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\}$ can be written as

$$[\{x_0^k + \sum_{i=1}^r \rho_k(\theta)(\gamma_i)) x_i^k\}_{k=1}^\infty] = [\{x_0^k\}_{k=1}^\infty] + \sum_{i=1}^r \rho(\theta)(\gamma_i))[\{x_i^k\}_{k=1}^\infty]\,.$$

Therefore

$$\prod_\omega \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\} \subset L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L)\,.$$

By our conditions, if k is large enough, then

- $\rho_k(b)\rho_k(c)(x) = \rho_k(bc)(x)$
- $\rho_k(a)\rho_k(bc)(x) = \rho_k(abc)(x)$
- $\rho_k(ab)\rho_k(c)(x) = \rho_k(abc)(x)$

whenever $x \in L_k$ and $a, b, c \in D$. Hence for the ultraproduct map ρ,

$$\rho(ab)\rho(c)(x) = \rho(a)\rho(b)\rho(c)(x)\,, \tag{10}$$

whenever $x \in L$.

Now, we finish the proof of Proposition 12.2. Define the $\mathcal{K}$-vectorspace T, by

$$T := \rho(D)(L) \subset K\,.$$

Then by (10), we have an embedding

$$\psi : D \to \mathrm{End}_{\mathcal{K}}(T)$$

defined by $\psi(d)(z) = \sum_{i=1}^{t} \rho(dd_i) l_i$, where $\{l_1, l_2, \ldots, l_t\}$ is a $\mathcal{K}$-basis of L and $z = \sum_{i=1}^{t} \rho(d_i) l_i$. Hence, T is a left D-vectorspace, with generating system $\{l_1, l_2, \ldots, l_i\}$. Also, by Lemma 12.1 and Lemma 12.2, L is finite dimensional and for $L \subset T \cong D^m$,

$$\dim_{\mathcal{K}}(L + \sum_{i=1}^{r} \theta(\gamma_i) L) \leq (1+\delta) \dim_{\mathcal{K}}(L)\,.$$

This finishes the proof of Proposition 12.2. □

Recall [7](Proposition 3.1), that if D is a countable non-amenable skew field over $\mathcal{K}$, then there exist elements $d_1, d_2, \ldots, d_l$ and $\varepsilon > 0$ such that for any $m \geq 1$ and finite dimensional $\mathcal{K}$-subspace $W \subset D^m$,

$$\frac{\dim_{\mathcal{K}}(W + \sum_{i=1}^{l} d_i W)}{\dim_{\mathcal{K}} W} > 1 + \varepsilon\,. \tag{11}$$

Now let $\theta : \mathbb{F}_r \to D$ be a generating map.

Lemma 12.3. *Let $\theta : \mathbb{F}_r \to D$ be a generating map, where D is non-amenable. Then there exists $\delta > 0$ such that for any $m \geq 1$ and finite dimensional $\mathcal{K}$-subspace $W \subset D^m$,*

$$\frac{\dim_{\mathcal{K}}(W + \sum_{i=1}^{r} \theta(\gamma_i) W)}{\dim_{\mathcal{K}} W} > 1 + \delta\,,$$

where $\{\gamma_i\}_{i=1}^{r}$ is the standard generator system for $\mathbb{F}_r$.

Proof. Suppose that such $\delta > 0$ does not exist. Then, we have a sequence of finite dimensional $\mathcal{K}$-subspaces $\{W_j \subset D^{n_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \frac{\dim_{\mathcal{K}}(W_j + \sum_{i=1}^{r} \theta(\gamma_i) W_j)}{\dim_{\mathcal{K}} W_j} = 1\,.$$

Let $S \subset D$ be the set of elements s in D such that

$$\lim_{j\to\infty} \frac{\dim_{\mathcal{K}}(W_j + s W_j)}{\dim_{\mathcal{K}} W_j} = 1\,.$$

Lemma 12.4. *S is the division closure of $Im(\mathcal{K}\mathbb{F}_r)$, that is $S = D$.*

Proof. Clearly, if $a, b \in S$, then $a + b \in S$ and $ab \in S$. We need to show that if $0 \neq a \in S$ then $a^{-1} \in S$. Let $a \in S$. Since $a(a^{-1}W_j + W_j) = (W_j + aW_j)$, we get that

$$\lim_{j\to\infty} \frac{\dim_{\mathcal{K}}(W_j + a^{-1}W_j)}{\dim_{\mathcal{K}} W_j} = 1\,,$$

therefore $a^{-1} \in S$. □

Thus by (11), Lemma 12.3 follows. □

Now we finish the proof of Proposition 12.1. Let $\{\theta_k : \mathbb{F}_r \to \text{Mat}_{n_k\times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ be a convergent, hyperfinite sequence of finite dimensional representations satisfying the Atiyah condition. Let $\theta : \mathbb{F}_r \to D$ be the limit map, where D is a non-amenable skew field and θ is a generating map. Let $\delta > 0$ be the constant in Lemma 12.3. Let $V_1^k, V_2^k, \dots, V_{t_k}^k \subset \mathcal{K}^{n_k}$ be $\mathcal{K}$-linear subspaces such that

- For any $1 \leq j \leq t_k$, $\dim_{\mathcal{K}}(V_j^k) \leq K_\delta$.
- $\{V_j^k + \sum_{i=1}^r \theta_k(\gamma_i)V_j^k\}_{j=1}^{t_k}$ are independent subspaces such that

$$\dim_{\mathcal{K}}(V_j^k + \sum_{i=1}^r \theta_k(\gamma_i)V_j^k) < (1+\delta)\dim_{\mathcal{K}}(V_j^k)\,.$$

- $\sum_{j=1}^{t_k} \dim_{\mathcal{K}}(V_j^k) \geq (1-\delta)n_k\,.$

We say that the above subspaces V_m^k and V_n^l are equivalent if there exists a linear isomorphism

$$\zeta : (V_m^k + \sum_{i=1}^r \theta_k(\gamma_i)V_m^k) \to (V_n^l + \sum_{i=1}^r \theta_k(\gamma_i)V_n^l)$$

such that $\zeta(V_m^k) = V_n^l$ and

$$\zeta(\theta_k(\gamma_i))(x) = \theta_j(\gamma_i)(\zeta(x))$$

for any generator γ_i and $x \in V_m^k$. By the finiteness of the base field and the uniform dimension bound, there are only finitely many equivalence classes. Hence, by taking a subsequence we can assume that there exists a constant $\tau > 0$ such that for each $k \geq 1$ there are elements $V_1^k, V_2^k, \dots, V_{s_k}^k$ of the above subspaces with the following properties:

- All the V_j^k's are equivalent.
- For each $k \geq 1$, $\sum_{j=1}^{s_k} \dim_{\mathcal{K}}(V_j^k) \geq \tau n_k$.

For $1 \leq j \leq s_k$, let $\zeta_j^k : (V_1^k + \sum_{i=1}^r \theta_k(\gamma_i) V_1^k) \to (V_j^k + \sum_{i=1}^r \theta_k(\gamma_i) V_j^k)$ be the isomorphism showing the equivalence. If $0 \neq \underline{\lambda} = \{\lambda_1, \lambda_2, \dots, \lambda_{s_k}\} \subset \mathcal{K}^{s_k}$ is a s_k-tuple of elements of $\mathcal{K}$, then $\zeta_{\underline{\lambda}} : \sum_{j=1}^{s_k} \lambda_j \zeta_j^k$ defines an isomorphism from $(V_1^k + \sum_{i=1}^r \theta_k(\gamma_i) V_1^k)$ to $(W_{\underline{\lambda}}^k + \sum_{i=1}^r \theta_k(\gamma_i) W_{\underline{\lambda}}^k)$, where

- $W_{\underline{\lambda}}^k \subset \mathcal{K}^{n_k}$ is a $\mathcal{K}$-subspace of dimension $\dim_{\mathcal{K}}(V_1^k)$
- $\zeta_{\underline{\lambda}}(\theta_k(\gamma_i)(x)) = \theta_k(\gamma_i)\zeta_{\underline{\lambda}}(x)$, for all $x \in V_1^k$.

Lemma 12.5. *Let $\{H_k \subset \mathcal{K}^{n_k}\}_{k=1}^\infty$ be a sequence of subspaces such that $\lim_{k\to\infty} \frac{\dim_{\mathcal{K}}(H_k)}{n_k} = 1$. Then if k is large enough, there exists $\underline{\lambda}$ such that $W_{\underline{\lambda}}^k \subset H_k$ (Note that we cannot assume that for large enough k, $V_j^k \subset H_k$ for some $j \geq 1$).*

Proof. Let $y_1^k, y_2^k, \dots, y_t^k, t \leq K_\varepsilon$ be a $\mathcal{K}$-basis for V_1^k. We define the linear map $\iota_j^k : \mathcal{K}^{s_k} \to \mathcal{K}^{n_k}$ by

$$\iota_j^k(\underline{\lambda}) = \zeta_{\underline{\lambda}}(y_j^k)\,.$$

Since $\{V_1^k, V_2^k, \dots, V_{s_k}^k\}$ are independent subspaces, ι_j^k is always an embedding. Let

$$M_j^k = \{\underline{\lambda} \in \mathcal{K}^{s_k} \mid \iota_j^k(\underline{\lambda}) \in H_k\}\,.$$

By our assumption, for any $j \geq 1$,

$$\lim_{k\to\infty} \frac{\dim_{\mathcal{K}}(M_j^k)}{s_k} = 1\,.$$

Hence, if k is large enough, then $\cap_{j=1}^t M_j^k \neq \emptyset$. Therefore, there exists $\underline{\lambda} \in \mathcal{K}^{s_k}$ such that $\zeta_{\underline{\lambda}}(V_1^k) \subset H_k$. □

By Proposition 10.3, there exists a sequence of maps $\{\rho_k : D \to \mathrm{Mat}_{n_k \times n_k}(\mathcal{K})\}_{k=1}^\infty$ and subspaces $\{H_k \subset \mathcal{K}^{n_k}\}_{k=1}^\infty$ such that

- For any $j \geq 1$, there exists k_j such that

$$\rho_k(ab)(x) = \rho_k(a)\rho_k(b)(x)\,,$$

  if $k \geq k_j$, $a, b \in \mathrm{Span}\{r_1, r_2, \dots, r_j\}$ and $x \in H_k$.
- $\rho_k(\theta(\gamma_i))(x) = \theta_k(\gamma_i)(x)$, if $x \in H_k$ and γ_i is a generator of $\mathbb{F}_r$.
- $\lim_{k\to\infty} \frac{\dim_{\mathcal{K}}(H_k)}{n_k} = 1$.

Hence, the sequence of maps $\{\rho_k\}_{k=1}^{\infty}$ and subspaces $L_k = W_{\lambda}^k$ satisfy the condition of Proposition 12.2. Therefore, there exists $m \geq 1$ and a finite $\mathcal{K}$-dimensional subspace $L \subset D^m$ such that

$$\frac{\dim_{\mathcal{K}}(L + \sum_{i=1}^{r} \theta(\gamma_i)L)}{\dim_{\mathcal{K}} L} \leq 1 + \delta\,,$$

in contradiction with the statement of Lemma 12.3. This finishes the proof of Proposition 12.1. □

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