ON C*-ALGEBRAS WHICH CANNOT BE DECOMPOSED INTO TENSOR PRODUCTS WITH BOTH FACTORS INFINITE-DIMENSIONAL

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ABSTRACT. We prove that C*-algebras which, as Banach spaces, are Grothendieck cannot be decomposed into a tensor product of two infinite-dimensional C*-algebras. By a result of Pfitzner, this class contains all von Neumann algebras and their norm-quotients. We thus complement a recent result of Ghasemi who established a similar conclusion for the class of SAW*-algebras.

1. Introduction and statement of the main result

During the London Mathematical Society Meeting held in Nottingham on 6th September 2010, Simon Wassermann asked a question of whether the Calkin algebra can be decomposed into a C*-tensor product of two infinite-dimensional C*-algebras. This question stems from the study of the elusive nature of the automorphism group of the Calkin algebra whose structure is independent of the usual axioms of Set Theory ([7, 18]). Ghasemi ([9]) studied tensorial decompositions of SAW*-algebras answering the above-mentioned question in the negative—one cannot thus expect to build automorphisms of such algebras out of automorphisms of non-trivial tensorial factors. Let us remark that the commutative version of Ghasemi's result was known to experts as it follows directly from the conjunction of [24, Theorem B] with the main theorem of [4].

The aim of this note is to prove that C*-algebras which satisfy a certain Banach-space property cannot be decomposed into a tensor product of C*-algebras. More specifically, we prove that C*-algebras which, as Banach spaces, are Grothendieck (*i.e.*, weak*-null sequences in the dual space converge weakly) do not allow such a tensorial decomposition. In particular, we give a new solution to the problem of Wassermann as the Calkin algebra falls into the class of Grothendieck spaces.

Theorem 1.1. *Let A be a C*-algebra which, as a Banach space, is a Grothendieck space. Suppose that E and F are C*-algebras such that*

$$A \cong E \otimes_\gamma F$$

for some C-norm γ. Then either E or F (or both) are finite-dimensional.*

In other words, a C-algebra, which is also a Grothendieck space, cannot be decomposed into a tensor product of two infinite-dimensional C*-algebras.*

Let us list examples of classes of C*-algebras which meet the assumptions of Theorem 1.1.

Proposition 1.2. *C*-algebras in each of the following classes are Grothendieck spaces:*
(i) *von Neumann algebras (or more generally, AW*-algebras) and their norm-quotients; in particular $\mathscr{B}(H)$ and the Calkin algebra,*
(ii) *ultraproducts of C*-algebras over countably incomplete ultrafilters,*
(iii) *unital C*-algebras with the countable Riesz interpolation property,*

Proof. By Corollary 2.4, von Neumann algebras and hence their continuous, linear images (such as the Calkin algebra) satisfy the hypothesis of Theorem 1.1. The assertion for AW*-algebras follows from Proposition 2.5 as every maximal abelian self-adjoint subalgebra of an AW*-algebra is Grothendieck by the main results of [22] and [23].

Avilés *et al.* ([2, Proposition 3.3]) proved that ultraproducts of Banach spaces over countably incomplete ultrafilters cannot contain complemented copies of c_0. This, combined with Theorem 2.2 and Proposition 2.1(iii), yields that ultraproducts of C*-algebras are Grothendieck spaces.

The assertion (iii) follows from applying [19, Theorem 9] to the real Banach space $A_{\rm sa}$ of all self-adjoint elements of a unital C*-algebra A with the countable Riesz interpolation property and noticing that the Grothendieck property passes from $A_{\rm sa}$ to the complex Banach space $A = A_{\rm sa} \oplus iA_{\rm sa}$. □

It is perhaps worthwhile to mention that even at the abelian level there exist many Grothendieck $C(X)$-spaces that are not SAW* (*i.e.*, for which X is not sub-Stonean); an example of such is a space constructed by Haydon ([12]).

Each unital C*-algebra with the countable Riesz interpolation property is an SAW*-algebra in the sense of Pedersen ([25, Proposition 2.7]); see [15] for the definition of an SAW*-algebra. Conjecturally all unital SAW*-algebras have the countable Riesz interpolation property ([25, p. 117]), hence our result covers all known examples of SAW*-algebras—thus, we extend the main result of [9] to the class of ultraproducts of C*-algebras and other C*-algebras created, for instance, out of Grothendieck abelian C*-algebras that are not SAW*.

To the best of our knowledge, it is not known whether a maximal abelian self-adjoint subalgebra of an SAW*-algebra is SAW* too. If this were the case, Proposition 2.5 would immediately imply that SAW*-algebras are Grothendieck spaces, because abelian SAW*-algebras are of the form $C_0(X)$ for some locally compact sub-Stonean space X, hence Grothendieck by [1] or [24].

2. Preliminaries

Grothendieck spaces. A series $\sum_{n=1}^\infty y_n$ in a Banach space E is *weakly unconditionally convergent* if the scalar series $\sum_{n=1}^\infty |\langle f, y_n\rangle|$ converges for each $f \in E^*$. An operator between Banach spaces is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent series. A Banach space E has *property (V)* if for each Banach space F the class of unconditionally converging operators $T\colon E \to F$ coincides with the class of weakly compact operators. It is a result of Pełczyński that $C(X)$-spaces (abelian C*-algebras) have property (V) ([16]).

A Banach space E is *Grothendieck* if every weak*-null sequence in E^* converges weakly. The name *Grothendieck space* stems from a result of Grothendieck ([11]) who identified ℓ_∞ as a space having this property. It is a trivial remark that reflexive spaces have this property too. By the Hahn–Banach theorem, the class of Grothendieck spaces is closed under surjective linear images, *i.e.*, whenever E and F are Banach spaces, $T\colon E \to F$ is a surjective bounded linear operator, then if E is Grothendieck, so is F.

Let us record a proposition which links property (V) with the Grothendieck property.

Proposition 2.1. *Let X be a Banach space. Then the following are equivalent.*
(i) *X is a Grothendieck space,*
(ii) *each bounded linear operator $T\colon X \to c_0$ is weakly compact.*
(iii) *X has property (V) and no subspace of X isomorphic to c_0 is complemented.*

Proof. For the proof of equivalences (i) $\iff$ (ii) see [5, Corollary 5 on p. 150]. Equivalence (i) $\iff$ (iii) is due to Räbiger ([20]); see also [10, Theorem 28] (this argument is also implicit in the proof of [4, Corollary 2]). □

We require the following theorem of Pfitzner ([17, Theorem 1]), which can be thought as a non-commutative generalisation of the above-mentioned result of Pełczyński.

Theorem 2.2 (Pfitzner)**.** *Let A be a C*-algebra and let $\mathcal{K} \subset A^*$ be a bounded set. Then $\mathcal{K}$ is not relatively weakly compact if and only if there are a sequence $(x_n)_{n=1}^\infty$ of pairwise orthogonal, norm-one self-adjoint elements in A and $\delta > 0$ such that*

$$\sup_{f\in\mathcal{K}} |\langle f, x_n\rangle| > \delta.$$

In particular, C-algebras have property (V).*

The original proof was highly sophisticated and relied on numerous deep facts from Banach space theory. Fortunately, Fernández-Polo and Peralta ([8]) supplied a short and elementary proof of Pfitzner's theorem.

By virtue of Proposition 2.1(iii), we arrive at the following corollary.

Corollary 2.3. *A C*-algebra is a Grothendieck space if and only if it does not contain complemented subspaces isomorphic to c_0.*

The special case of Corollary 2.3 where the C*-algebra is also a von Neumann algebra was noted by Pfitzner ([17, Corollary 7]):

Corollary 2.4. *Von Neumann algebras are Grothendieck spaces.*

Let us take this opportunity to record the following easy corollary to Theorem 2.2.

Proposition 2.5. *Let A be a C*-algebra with the property that each maximal abelian self-adjoint subalgebra B of A is a Grothendieck space. Then A is a Grothendieck space.*

Proof. Let $T\colon A \to c_0$ be a bounded linear operator. By Proposition 2.1(ii) it is enough to show that T is weakly compact.

Assume contrapositively that T is not weakly compact. By Gantmacher's theorem, T is weakly compact if and only if T^* is, so the set $\mathcal{K} = T^*(B)$ is not relatively weakly compact, where B is the unit ball of c_0^*. By Theorem 2.2, there exist $\delta > 0$ and a sequence $(x_n)_{n=1}^\infty$ of pairwise orthogonal, norm-one self adjoint elements in A such that

$$\sup_{f\in\mathcal{K}} |\langle f, x_n\rangle| = \sup_{y\in B} |\langle T^*y, x_n\rangle| = \sup_{y\in B} |\langle y, Tx_n\rangle| > \delta. \tag{2.1}$$

Let $B_0 \subseteq A$ be the C*-algebra generated by $\{x_n\colon n \in \mathbb{N}\}$. Since the x_n ($n \in \mathbb{N}$) are pairwise orthogonal, B_0 is abelian. Let B be a maximal abelian subalgebra of A containing B_0. Consider the restriction $T|_B\colon B \to c_0$. It is not weakly compact by (2.1), so B is not a Grothendieck space. □

3. Proof of Theorem 1.1

We are now in a position to prove our main result. The general strategy of the proof was inspired by the path taken by Cembranos in [4].

Proof of Theorem 1.1. Let A be a C*-algebra and suppose that it is a Grothendieck space. Assume towards a contradiction that $A \cong E\otimes_\gamma F$ for some infinite-dimensional C*-algebras E, F and a C*-norm γ.

Since *-homomorphisms between C*-algebras have closed range, there is a surjective *-homomorphism $Q\colon E \otimes_\gamma F \to E \otimes_{\min} F$ that extends the identity map on $E \odot F$. (Here $E \otimes_{\min} F$ denotes the minimal C*-tensor product of E and F.)

Let $B_1 \subset E$ and $B_2 \subset F$ be infinite-dimensional abelian C*-algebras. (Such subalgebras exist because infinite-dimensional C*-algebras contain self-adjoint elements with infinite spectrum ([13, Ex. 4.6.12]), hence the assertion follows from the spectral theorem.) We may thus identify $B_1 \otimes_{\min} B_2$ with a subalgebra of $E \otimes_{\min} F$ (*cf.* [3, II.9.6.2]). However, the (minimal) tensor product of abelian C*-algebras is the same as the Banach-space injective tensor product, *i.e.*,

$$B_1 \otimes_{\min} B_2 = B_1 \check{\otimes} B_2.$$

Let $(e_n)_{n=1}^\infty$ be a sequence of pairwise orthogonal, positive, norm-one elements in B_1. In particular, $E_0 = \overline{\operatorname{span}}\{e_n\colon n \in \mathbb{N}\}$ is isometric to c_0 and $(e_n)_{n=1}^\infty$ is equivalent to the canonical basis for c_0. Choose norm-one functionals $e_n^* \in E^*$ such that $\langle e_n^*, e_m\rangle = \delta_{n,m}$ ($n, m \in \mathbb{N}$). Moreover, let $(x_n^*)_{n=1}^\infty$ be a sequence of unit vectors in F^* which converges to 0 in the weak* topology. (Such a sequence exists by the Josefson–Nissenzweig theorem (see [6, Chapter XII].) Let $(x_n)_{n=1}^\infty \subset F$ be a sequence such that $\langle x_n, x_n^*\rangle = 1$. Without loss of generality we may suppose that $\|x_n\| \leqslant 2$ for all n.

Define a map $T\colon E \odot_{\min} F \to \ell_\infty$ by the formula

$$T\xi = (\langle e_n^* \otimes x_n^*, \xi\rangle)_{n=1}^\infty \qquad (\xi \in E \odot_{\min} F).$$

This is a well-defined bounded linear operator because $(e_n^*\otimes x_n^*)_{n=1}^\infty$ is a bounded sequence of functionals on $E\odot_{\min} F$. We can thus extend T to the whole of $E\otimes_{\min} F$. Moreover, for all $f\in E$ and $x\in F$ we have

$$|\langle e_n^*, f\rangle\cdot\langle x, x_n^*\rangle| \leqslant \|f\|\cdot|\langle x, x_n^*\rangle|$$

so T takes values in c_0 as $(x_n^*)_{n=1}^\infty$ is a weak*-null sequence. Since the injective tensor product 'respects subspaces' (see [21, p. 49]), $E_0\,\check{\otimes}\,B_2$ can be identified with a subspace of $B_1\,\check{\otimes}\,B_2$ and the latter is a subspace of $E\otimes_{\min} F$.

As E_0 and c_0 are isometrically isomorphic, so are $E_0\,\check{\otimes}\,B_2$ and $c_0\,\check{\otimes}\,B_2\cong c_0(B_2)$ (*cf.* [21, Example 3.3]). Let $n\in\mathbb{N}$ and let $(a_k)_{k=1}^\infty$ be a scalar sequence with only finitely many non-zero entries. We have

$$\Big\|\sum_{k=1}^n a_k e_k\otimes x_k\Big\| = \Big\|\sum_{k=1}^n e_k\otimes(a_k x_k)\Big\| \leqslant \sup_{1\leqslant k\leqslant n}\|a_k x_k\| \leqslant 2\max\{|a_k|\colon 1\leqslant k\leqslant n\}.$$

By [14, Proposition 4.3.9], $\sum_{n=1}^\infty e_n\otimes x_n$ is a weakly unconditionally convergent series in $E\otimes_{\min} F$. On the other hand, for all $k,n\in\mathbb{N}$ we have $(T(e_n\otimes x_n))(k)=\delta_{k,n}$ so

$$\sum_{n=1}^\infty T(e_n\otimes x_n)$$

fails to converge in c_0. Consequently, $E\otimes_{\min} F$ is not a Grothendieck space as we proved that the c_0-valued operator T is not unconditionally converging. Indeed, if $E\otimes_{\min} F$ were Grothendieck, T would be weakly compact (Proposition 2.1(ii)), hence also unconditionally converging (Proposition 2.1(iii)). □

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