Propagation of Electromagnetic Waves in Spatially Dispersive Inhomogeneous Media

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Declaration

I hereby declare that the work presented in this thesis was done by myself in collaboration with Dr. Jonathan Gratus (Lancaster University), and has not been submitted in substantially the same form for the award of a higher degree at any institution.

The work presented in chapters 2 and 3 is an extension of work which has previously been published, [1]. A condensed form of the work in chapter 4 has been submitted to the Journal of Optics for publication and is awaiting response.

Abstract

Spatial dispersion is the effect where media respond not only to a signal at one particular point, but to signals in an area around that point. While temporal dispersion is a well studied topic, spatial dispersion is relatively unexplored. This thesis investigates the behaviour of electromagnetic waves in spatially dispersive, inhomogeneous media. In particular, two types of inhomogeneity are considered: media formed from two homogeneous regions with a common interface, and those with a periodic structure.

For a material made of two homogeneous regions joined together we establish a set of boundary conditions to describe the behaviour of waves at this interface. These boundary conditions are additional to the standard ones provided by Maxwell's equations. The conditions found are shown to reduce to those established previously by Pekar in the case of a boundary between a spatially dispersive region and a purely temporally dispersive region.

The polarisation is also found for a spatially dispersive medium with periodic structure. Numeric solutions are found and non-divergent modes are identified. Analytic solutions are also found for small magnitudes of the inhomogeneity. Most interestingly these results show that, for certain conditions, there exist coupled mode solutions. This is an unusual phenomena which arises as a result of the spatial dispersion in the system.

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List of Symbols

R	Positive integer at which the infinite difference equation is truncated	54
Ω_n	Value of ω which solves $f_n(\omega) = 0$	59
P_q^n	Spatial mode amplitude associated with the n^{th} mode	59
n, m	Integer indices used to label the spatial modes	59
$\mathcal{F}_q$	The function f_q evaluated at $\omega = \Omega_n$	60
$\mathcal{F}'_q, \mathcal{F}''_q$	First and second derivatives of f_q, with respect to ω, evaluated at $\omega = \Omega_n$	60
ω_n	Frequency solution for the n^{th} mode	60
R_q^n	Function defined by (4.25)	60
$\mathcal{Q}_q^n$	Remainder of the difference equation, (4.36)	67
$\omega_n^{\mathbf{o}}$, $P_n^{\mathbf{o}}$	Frequency and polarisation for odd modes	69
$\omega_n^{\mathbf{e}}$, $P_n^{\mathbf{e}}$	Frequency and polarisation for even modes	71
$\omega_n^{(r)}$	Coefficient of Λ^r in the expansion of ω_n	77
$P_q^{(r)}$	Coefficient of Λ^r in the expansion of P_q	78
α_1	Factor in the solution for $n - m = 3$ which can't be determined at order $O(\Lambda^3)$	80
ρ	Solution to the quadratic equation (4.89)	82
ρ_2	Factor in the solution for $n - m = 2$ which can't be determined at order $O(\Lambda^3)$	82
W	Function given by (4.91)	82
$\mathcal{G}$	Function given by (4.93)	83
k_p	Plasma frequency, given by (4.96)	84
b	Wire medium spacing	84
r	Wire radius in wire medium	84
k_0	Value of the plasma frequency at $r = r_0$	84

Chapter 1

Introduction

This thesis investigates two problems pertaining to the topic of the propagation of electromagnetic waves in spatially inhomogeneous media with consideration to the effects of both temporal and spatial dispersion. While the effects of temporal dispersion are widely known, spatial dispersion is a less studied field and as such presents the opportunity for new results to be found.

Typically spatial dispersion, or non-locality as it is also known, is considered only to be a small modification to local models and is usually ignored. Typically, only in the short wavelength limit is the effect considered to be significant enough to have a measurable effect. However, it has been shown that there exist structures, such as wire media, in which non-local, dispersive behaviour is observed for all frequencies, including the large wavelength limit [2]. Hence, this suggests that spatial dispersion can be a physically significant effect and in some cases an essential factor in modelling certain materials.

As mentioned, wire media are a major focus of research regarding spatial dispersion. These are artificial dielectrics formed by a rectangular array of thin, perfectly conducting wires (as shown in figure 1.1) and have been shown to exhibit spatial dispersion effects [3]. In the case where the spacing of these wires is much smaller

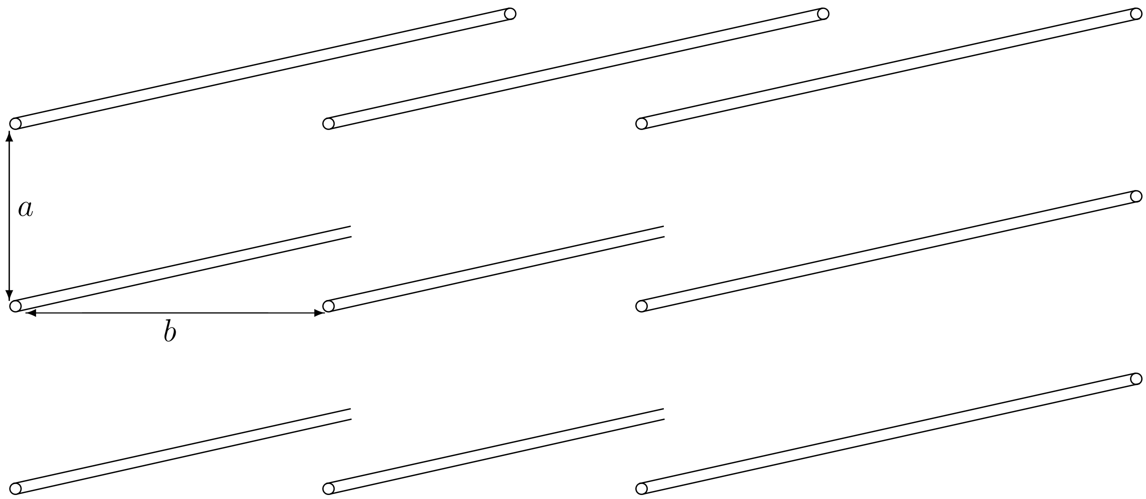

Figure 1.1: Artificial dielectric formed by a periodic array of thin, uniform wires. Spatially dispersive effects become stronger when the lattice constants a, b become significantly smaller than the wavelength [9].

than the wavelength then strong spatial dispersion is observed [2]. Such strong spatial dispersion can result in negative diffraction [4] which makes it possible to construct a perfect lens, potentially capable of imaging with unlimited resolution, from a wire medium [5, 6, 7, 8].

Recent research has also shown the importance of non-locality in graphene, a material made of a single atom layer of graphite and a substance of major interest in modern physics. Including spatial dispersion in the calculation of the conductivity of graphene can be used to explain the previously known dependence of the conductivity on temperature and pressure [10]. It has also been shown that an effective medium formed of a periodic lattice of graphene layers displays non-local effects. In particular, the emergence of additional waves, excitation of graphene plasmons, and the occurrence of negative refraction [11].

This thesis investigates two types of spatially dispersive inhomogeneous media: those composed of two homogeneous region (chapters 2 and 3) and those with a periodic structure (chapter 4).

In chapter 2 the response function is calculated for homogeneous media, and it is shown that this solution is damped in the direction of propagation. This result provides the foundation for the subsequent chapter.

In chapter 3 we construct an inhomogeneous medium by joining two spatially dispersive, homogeneous regions together. We then consider the boundary between these two regions and obtain a complete set of boundary conditions for the system. In the case where the boundary is between a spatially dispersive homogeneous medium and the vacuum these boundary conditions have been found previously by Pekar [12]. We obtain a set of boundary conditions through two alternative methods which, for appropriate choices, reduce to the Pekar result in the limiting case where one of the media is no longer spatially dispersive. This result has been published in [1], however a more exhaustive version is presented here.

In chapter 4 we investigate spatially dispersive media with periodic structure and present both numeric solutions as well as approximate analytic solutions to Maxwell's equations in the case where the magnitude of the inhomogeneity is small. This is a continuation and advancement on the work that was present in an early form in [1]. While there have been some studies into such materials there have been none that provide analytic solutions to Maxwell's equations for these media with spatial dispersion. However, some properties of spatially dispersive, periodic structures have been concluded previously, in particular a parabolic-like dispersion, and the potential for double eigenwaves [13]. This parabolic-like dispersion will also be seen in our approach to the problem.

1.1 Spatially Dispersive Media

Spatial dispersion refers to the phenomenon where the polarisation at a given point is determined not only by the value of the electric field at the point, but also the field values in the neighbourhood of the point [14]. The term 'spatial dispersion' was first used for this effect by Gertsenshtein in 1952 [15]. Mathematically, spatial dispersion refers to the inclusion of a wave vector dependence in the permittivity, as was demonstrated by Born's microscopic theory [16].

A simple illustration of spatial dispersion is shown in figure 1.2. These series of springs represent, in the infinite limit, dielectrics. The addition of springs connecting the masses in (b) provides a secondary method of energy transfer which causes the system to become non-local.

Belov [2] shows that spatial dispersion can be significant for all frequencies and shows that in wire media the physical results are very different for local and non-local models, hence spatial dispersion is an essential consideration.

The physical effect of spatial dispersion on reflectivity in crystals was shown by Hopfield [17], where a theoretical model for spatial dispersion was presented which predicted a number of anomalies not present in a classical medium. These were verified experimentally, giving strong support to the validity of considering spatial dispersion as a physical effect. This study primarily considered the effect of spatial dispersion as a second mechanism of energy transport within a crystal.

A comprehensive summary of early research into the electrodynamics of spatially dispersive media is given by Rukhadze [18]. In particular, this paper shows that, in a typical dielectric, the effects of spatial dispersion become more significant when the fields vary rapidly in space and can often be ignored for sufficiently smooth field variations. Additionally, Rukhadze shows that, when spatial dispersion is considered, there may exist a number of transverse waves propagating in the medium with the same frequency but with different indices of refraction.

A potential interesting application of spatial dispersion is that, for sufficiently strong spatial dispersion, any dielectric medium may exhibit negative group velocity [4]. This is an effect normally seen only when the permittivity and permeability are negative, $\varepsilon < 0$ and $\mu < 0$. These negative refraction meta-materials are the subject

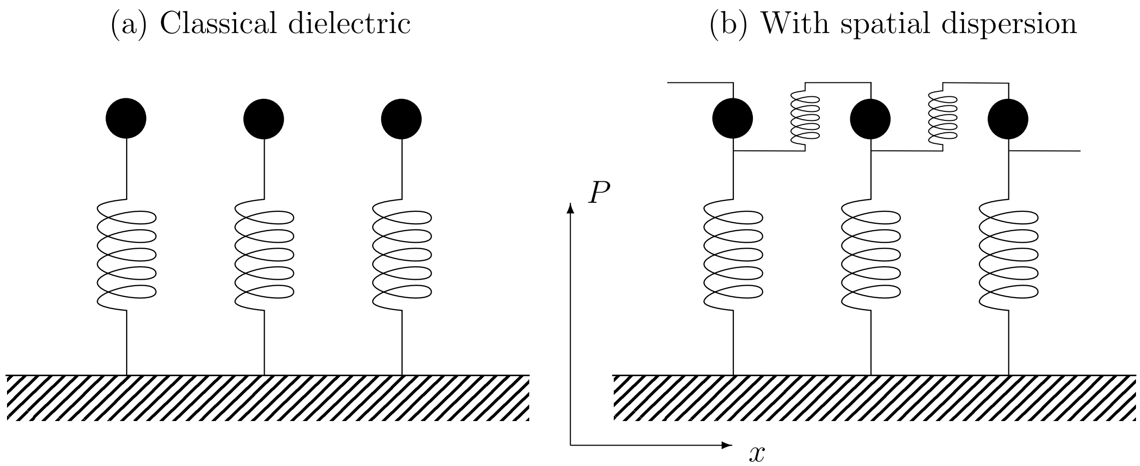

Figure 1.2: Spring and charged point-mass model representing a classical dielectric, (a); and a dielectric with spatial dispersion, (b) [17].

of much interest and many novel applications rely on this property.

While spatial dispersion can be beneficial in a number of materials, such as those mentioned above, this is not always the case. Such as negative-index metamaterials, in which these non-local effects are detrimental to the desired behaviour [19].

1.2 Maxwell's Equations

Maxwell's equations for electromagnetic fields in a medium, which were first derived in the 1860s, are

$$\nabla \cdot \mathbf{D} = \rho_c, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Here **E** and **H** are, respectively, the electric and magnetic fields, **D** the electric displacement, and **B** the magnetic induction. In this work we disregard the source terms, ρ_c and **J**, in the above and hence use Maxwell's equations in the source-free form

$$\nabla \cdot \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{1.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.2}$$

We also work with the polarisation density, $\mathbf{P}$, which is defined as the average electric dipole moment per unit volume. This is used in the mathematical definition of the electric displacement

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon_0 \mathbf{E}(t, \mathbf{x}) + \mathbf{P}(t, \mathbf{x}).$$

For the sake of simplifying the analysis we make the following assumptions which will hold throughout the work:

- There is no magnetisation, so we have that $\mathbf{H} = \mu_0^{-1} \mathbf{B}$.
- All fields are functions of time, t, and one spatial coordinate only, $x = x_1$, and independent of x_2, x_3. Likewise in the frequency domain $k_2 = k_3 = 0$ and we set $k = k_1$.
- We take the polarisation, electric and magnetic fields to be transverse fields, hence $E_1(t, x) = 0$, $P_1(t, x) = 0$, and $B_1(t, x) = 0$. This automatically satisfies the two non-dynamic Maxwell's equations, (1.1).
- Linearly polarised waves are chosen such that, in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ frame, we have $\mathbf{E}(t, x) = E(t, x)\mathbf{e}_2$, $\mathbf{P}(t, x) = P(t, x)\mathbf{e}_2$, and $\mathbf{B}(t, x) = B(t, x)\mathbf{e}_3$

From these assumptions we can rewrite the two remaining Maxwell's equations, (1.2), as

$$\frac{1}{\mu_0} \frac{\partial B}{\partial x} = -\frac{\partial}{\partial t} \left(\varepsilon_0 E + P \right) \qquad \text{and} \qquad \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

which, taking derivatives of t and x respectively, give

$$\frac{\partial^2 B}{\partial x \partial t} = -\frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}E + \mu_0 P\right) \qquad \text{and} \qquad \frac{\partial^2 E}{\partial x^2} = -\frac{\partial^2 B}{\partial t \partial x}.$$

These can be combined to give a single equation,

$$\frac{\partial^2 E(t,x)}{\partial x^2} = \mu_0 \frac{\partial^2 P(t,x)}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2 E(t,x)}{\partial t^2}. \tag{1.3}$$

1.3 Constitutive Relation

In this thesis we consider a linear constitutive relation[1] between $\mathbf{E}(t,\mathbf{x})$ and $\mathbf{P}(t,\mathbf{x})$. In other words P is a single-valued, linear functional of E. For an inhomogeneous medium, in general, this constitutive relation is given by

$$P_i(t,\mathbf{x}) = \sum_{j=1}^{3} \varepsilon_0 \iiiint_{-\infty}^{\infty} \chi_{ij}(t,t',\mathbf{x},\mathbf{x}')E_j(t',\mathbf{x}')dt'd\mathbf{x}' \tag{1.4}$$

where χ is the electric susceptibility which is related to the permittivity of the medium, ε, by the expression

$$\varepsilon = (1+\chi)\varepsilon_0.$$

In this thesis we will only consider media which are homogeneous in time and this simplification, along with the assumptions made above, reduces (1.4) to

$$P(t,\mathbf{x}) = \varepsilon_0 \iiiint_{-\infty}^{\infty} \psi(t-t',\mathbf{x},\mathbf{x}')E(t',\mathbf{x}')dt'd\mathbf{x}' \tag{1.5}$$

where $\psi(t-t',x,x') = \chi_{22}(t-t',x,x')$. Limiting our consideration to time homogeneous media allows us to take the Fourier transform with respect to t without difficulty.

[1] We will consider only a linear relation in this work, for a discussion of nonlinear constitutive relations see, for example, [20].

We will consider only one type of spatial dispersion here, where the spatial dispersion appears on a second derivative in x. While this is not the only type of dispersion which can be considered, it is the simplest. Making this choice ensures that parity is preserved as there is no first order derivative. In this model of spatial dispersion, which will be used throughout this thesis, the electric and polarisation fields are related by

$$L(\omega, x)\hat{P}(\omega, x) + \frac{\partial}{\partial x}\left(\frac{\beta(\omega, x)^2}{(2\pi)^2}\frac{\partial \hat{P}(\omega, x)}{\partial x}\right) = \varepsilon_0 \hat{E}(\omega, x) \tag{1.6}$$

where

$$\hat{P}(\omega, x) = \int_{-\infty}^{\infty} e^{-2\pi i\omega t} P(t, x)dt, \tag{1.7}$$

is our chosen definition for the Fourier transform of $P(t, x)$ with respect to t only. Note that in this work ω denotes the temporal frequency rather than the angular frequency for which it is commonly used.

The function $L(\omega, x)$ contains the properties of the particular medium, the x dependence representing an inhomogeneous medium, and β is the speed of wave propagation in the medium. The temporal dispersion is contained entirely within $L(\omega, x)$. Along with this, we also have a corresponding partial differential equation for the response function, ψ,

$$L(x)\psi + \frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \psi}{\partial x}\right) = \varepsilon_0 \delta(t - t', x, x'). \tag{1.8}$$

Note that when the x dependency is removed, that is $L(\omega, x) = L(\omega)$ and $\beta(\omega, x) = \beta(\omega)$, then taking the Fourier transform of (1.6) with respect to the spatial coordinate gives the permittivity relation[2]

$$\widetilde{P}(\omega, k) = \frac{\varepsilon_0 \widetilde{E}(\omega, k)}{L(\omega) - \beta(\omega)^2|k|^2} \tag{1.9}$$

[2]Since this system has damping then causality dictates that all poles will be found entirely in the upper-half plane (as will be seen later) and so there are no poles along the real axis. Hence this function is well defined for real ω, k.

where

$$\widetilde{P}(\omega, k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(\omega t + kx)} P(t, x) dt dx \tag{1.10}$$

is the Fourier transform of $P(t, x)$, and likewise for $\widetilde{E}$.

1.4 Floquet Theory

This section provides a brief introduction to Floquet theory, which will be necessary for the study of periodically structured media in chapter 4. Floquet theory refers to the study of a particular set of linear differential equations with periodic coefficients, and is often the preferred method in analysing periodic structures [21, 22]. The key result of this theory is Floquet's Theorem[23] which determines the solutions to linear homogeneous differential equations of the form

$$P(y) = \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + p_2 \frac{d^{m-2} y}{dx^{m-2}} + \ldots + p_m y = 0, \tag{1.11}$$

where the coefficients p_j are periodic in x, with a consistent period r. Floquet's Theorem then tells us that the solution y must be of the form

$$y = e^{\nu x} \mathcal{P}(x) \tag{1.12}$$

where ν is constant and $\mathcal{P}(x)$ is periodic with period r,

$$\mathcal{P}(x + r) = \mathcal{P}(x).$$

A similar result was established by Bloch in 1928 [24] to describe an electron in a crystal lattice. These solutions are known as Bloch waves and consist of a plane wave multiplied by a periodic function,

$$\psi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}u(\mathbf{x})$$

where $\mathbf{x}$ is the position, $\mathbf{k}$ is the crystal wave vector, and $u(x)$ is periodic. Bloch's theorem states that the energy eigenstates of an electron in a crystal structure can always be written as Bloch waves. This has important physical consequences, particularly in defining the electronic band structure of the crystal.

Other results regarding the solutions to second order linear differential equations with periodic coefficients were derived, independently, in the late 19th century by Hill [25], and Mathieu [26]. While these can be seen to be only specific cases of Floquet's Theorem they are, however, still used in many physical applications particularly in quantum systems [27, 28].

Recent work has shown that Floquet theory can be extended to more general systems including media with hysteresis and non-local potentials [29].

Chapter 2

Response in Homogeneous Media

2.1 Introduction

This chapter is concerned with the propagation of electromagnetic waves in a homogeneous medium. In addition to Maxwell's equations, (1.3), we need a constitutive relation between the polarisation and the electric field. For a homogeneous medium, and with the assumptions made in section 1.2, this constitutive relation is given by

$$P(t,x) = \varepsilon_0 \iint_{-\infty}^{\infty} \psi(t,x,x')E(t,x)dx' \tag{2.1}$$

where the response $\psi = \chi_{22}$ and x' is the location of a point source. In this chapter we will consider a permittivity relation that is a generalisation of the Lorentz single-resonance model [30], so that

$$\widetilde{P}(\omega,k) = \frac{\varepsilon_0 \widetilde{E}(\omega,k)}{(2\pi i\omega+\lambda)^2+\alpha^2+(2\pi)^2\beta^2|k|^2}. \tag{2.2}$$

The terms λ and α represent, respectively, the damping of the medium and the resonant frequency of the polarisation.

The aim of this chapter is to find solutions for the response function and show that such solutions decay appropriately as they propagate through the medium. This result provides the foundation for the subsequent chapter where the boundary between homogeneous media is considered.

2.2 Response Function

Taking the inverse Fourier transform of (2.2) gives the partial differential equation

$$\frac{\partial^2 P(t,x)}{\partial t^2} - \beta^2 \frac{\partial^2 P(t,x)}{\partial x^2} + 2\lambda \frac{\partial P(t,x)}{\partial t} + (\lambda^2 + \alpha^2) P(t,x) = \varepsilon_0 E(t,x). \tag{2.3}$$

We see from (2.1) that $\psi(t, x, x')$ is a Green's function hence (2.3) becomes a partial differential equation for ψ

$$\frac{\partial^2 \psi(t,x,x')}{\partial t^2} - \beta^2 \frac{\partial^2 \psi(t,x,x')}{\partial x^2} + 2\lambda \frac{\partial \psi(t,x,x')}{\partial t} + (\lambda^2 + \alpha^2) \psi(t,x,x') = \delta(t, x - x').$$

For now, however, we consider a source at the origin and so $x' = 0$. This gives us

$$\frac{\partial^2 \psi(t,x)}{\partial t^2} - \beta^2 \frac{\partial^2 \psi(t,x)}{\partial x^2} + 2\lambda \frac{\partial \psi(t,x)}{\partial t} + (\lambda^2 + \alpha^2) \psi(t,x) = \delta(t,x). \tag{2.4}$$

We solve (2.4) by Fourier methods. The Fourier transform of (2.4) is given by

$$-\omega^2 \hat{\psi}(\omega,x) - \beta^2 \frac{\partial^2 \hat{\psi}(\omega,x)}{\partial x^2} + 2i\lambda\omega \hat{\psi}(\omega,x) + (\lambda^2 + \alpha^2) \hat{\psi}(\omega,x) = \delta(x). \tag{2.5}$$

which is solved by (2.14) given below. To guarantee causality we present the inverse Fourier transform of this, $\psi(t, x)$, first.

Lemma 2.2.1. *The differential equation* (2.4) *is solved by the function*

$$\psi(t,x) = \begin{cases} \dfrac{1}{2\beta} e^{-\lambda t} J_0\left(\alpha\sqrt{t^2 - \frac{x^2}{\beta^2}}\right) & \text{for } t \geq \pm\dfrac{x}{\beta} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

where J_0 is the zero order Bessel function of the first kind.

Proof. To prove that this is indeed a solution to (2.4) we must first make the following substitution

$$\psi = e^{-\lambda t}\phi$$

then (2.4) becomes

$$\frac{\partial^2\phi}{\partial t^2} + \alpha^2\phi - \beta^2\frac{\partial^2\phi}{\partial x^2} = e^{\lambda t}\delta(t,x) = \delta(t,x). \tag{2.7}$$

The next step is to employ a change of variable: $u = t + x/\beta$ and $v = t - x/\beta$. This gives the transformed derivatives as

$$\frac{\partial}{\partial u} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}\right) \qquad \text{and} \qquad \frac{\partial}{\partial v} = \frac{1}{2}\left(\frac{\partial}{\partial t} - \beta\frac{\partial}{\partial x}\right)$$

hence

$$\frac{\partial^2}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2}\right).$$

Using this, along with the transform of $\delta(t,x)$ given in appendix A.1, (2.7) becomes

$$4\frac{\partial^2\phi}{\partial u\partial v} + \alpha^2\phi = \frac{2}{\beta}\delta(u)\delta(v). \tag{2.8}$$

Applying these transforms to the solution, (2.6), gives

$$\phi(u,v) = \begin{cases} \frac{1}{2\beta}J_0\left(\alpha\sqrt{uv}\right) & \text{for } u \geq 0 \text{ and } v \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

There are three cases to consider: away from the boundaries ($u > 0$, $v > 0$), the boundary $u = 0$, $v > 0$ (which, by symmetry, is identical to the boundary $u > 0$, $v = 0$), and the boundary point $u = 0$, $v = 0$.

Firstly, looking away from the boundaries ($u > 0$, $v > 0$), this result is shown by a trivial substitution of (2.9) into (2.8)

$$
\begin{aligned}
&4\frac{\partial^2}{\partial u \partial v}\left(\frac{1}{2\beta}J_0(\alpha\sqrt{uv})\right) + \alpha^2\frac{1}{2\beta}J_0(\alpha\sqrt{uv}) \\
&= 4\left(-\frac{1}{2\beta}\frac{\alpha}{4\sqrt{uv}}J_1(\alpha\sqrt{uv}) + \frac{1}{2\beta}\frac{\alpha}{4\sqrt{uv}}J_1(\alpha\sqrt{uv}) - \frac{1}{2\beta}\frac{\alpha^2}{4}J_0(\alpha\sqrt{uv})\right) \\
&\quad + \alpha^2\frac{1}{2\beta}J_0(\alpha\sqrt{uv}) \\
&= -\alpha^2\frac{1}{2\beta}J_0(\alpha\sqrt{uv}) + \alpha^2\frac{1}{2\beta}J_0(\alpha\sqrt{uv}) \\
&= 0
\end{aligned}
$$

as required.

Next, consider the boundary where $v > 0$, $u = 0$. As mentioned, due to the symmetry of u and v in this problem this also covers the boundary $u = 0$, $v > 0$. Take a test function $h(u, v)$ with support bounded by $0 < v_1 < v < v_2$. Since $v > 0$ we have

$$
\frac{\partial \phi}{\partial v} = \begin{cases} -\frac{1}{2\beta}\frac{\alpha}{2}\sqrt{\frac{u}{v}}J_1(\alpha\sqrt{uv}) & \text{for } u \geq 0 \\ 0 & \text{for } u < 0. \end{cases}
$$

In this case we are considering only the range $0 \leq u < \epsilon$, for any $\epsilon \in \mathbb{R}$, so there is no need to integrate beyond this scope. For small x $J_0(x) \approx 1$ and $J_1(x) \approx x$. So, for $0 < u < \epsilon$

$$
\begin{aligned}
\phi &\approx \frac{1}{2\beta} \\
\frac{\partial \phi}{\partial v} &\approx -\frac{1}{2\beta}\frac{\alpha^2}{2}u
\end{aligned}
$$

and, for $u < 0$,

$$
\phi = \frac{\partial \phi}{\partial v} = 0
$$

so

$$
\begin{aligned}
I[h] &= \int_{v_1}^{v_2} dv \int_{-\epsilon}^{\epsilon} du \left(4\frac{\partial^2\phi}{\partial u \partial v} + \alpha^2\phi\right) h(u,v) \\
&= \int_{v_1}^{v_2} \left(\int_{-\epsilon}^{\epsilon} 4\frac{\partial\phi}{\partial v}\frac{\partial h}{\partial u} du - \left[4h(u,v)\frac{\partial\phi}{\partial v}\right]_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} \alpha^2\phi h(u,v) du\right) dv \\
&\approx \int_{v_1}^{v_2} \Big(\underbrace{\int_{-\epsilon}^{\epsilon} -\frac{1}{\beta}\alpha^2 u\frac{\partial h}{\partial u} du}_{O(\epsilon^2)} + \left[\frac{1}{\beta}\alpha^2 u h(u,v)\right]_{-\epsilon}^{\epsilon} + \frac{1}{2\beta}\alpha^2 h(\epsilon,v)\int_{-\epsilon}^{\epsilon} du\Big) dv \\
&= \int_{v_1}^{v_2} \Big(\frac{1}{\beta}\alpha^2\epsilon + \frac{1}{\beta}\alpha^2\epsilon\Big) h(\epsilon,v) dv + O(\epsilon^2)
\end{aligned}
$$

hence the result as $\epsilon \to 0$.

Finally, we look at the boundary point where $u = v = 0$. In this case we integrate over the range $0 < u < \epsilon_1$ and $0 < v < \epsilon_2$, for any $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Again, taking a test function $h(u,v)$,

$$
\begin{aligned}
I[h] &= \int_{-\epsilon_1}^{\epsilon_1} dv \int_{-\epsilon_2}^{\epsilon_2} du \left(4\frac{\partial^2\phi}{\partial u \partial v} + \alpha^2\phi - \frac{2}{\beta}\delta(u)\delta(v)\right) h(u,v) \\
&= \left[[4\phi h(u,v)]_{u=-\epsilon_1}^{\epsilon_1}\right]_{v=-\epsilon_2}^{\epsilon_2} + \int_{-\epsilon_1}^{\epsilon_1}\int_{-\epsilon_2}^{\epsilon_2} \alpha^2\phi h(u,v) dv du - \frac{2}{\beta}h(0,0) \\
&\approx \frac{2}{\beta}h(\epsilon_1,\epsilon_2) + \frac{1}{2\beta}\alpha^2 h(0,0) \underbrace{\int_0^{\epsilon_1}\int_0^{\epsilon_2} dv du}_{O(\epsilon_1\epsilon_2)} - \frac{2}{\beta}h(0,0) \\
&= \frac{2}{\beta}h(\epsilon_1,\epsilon_2) - \frac{2}{\beta}h(0,0) + O(\epsilon_1\epsilon_2)
\end{aligned}
$$

hence as $\epsilon_1, \epsilon_2 \to 0$ we have the result as required. $\square$

Figure 2.1 shows a three-dimensional plot of the response function $\psi(t,x)$. From this the causal structure is clearly visible (since $\psi = 0$ outside of $t \geq \pm x/\beta \geq 0$), as well as the damping of the wave over time.

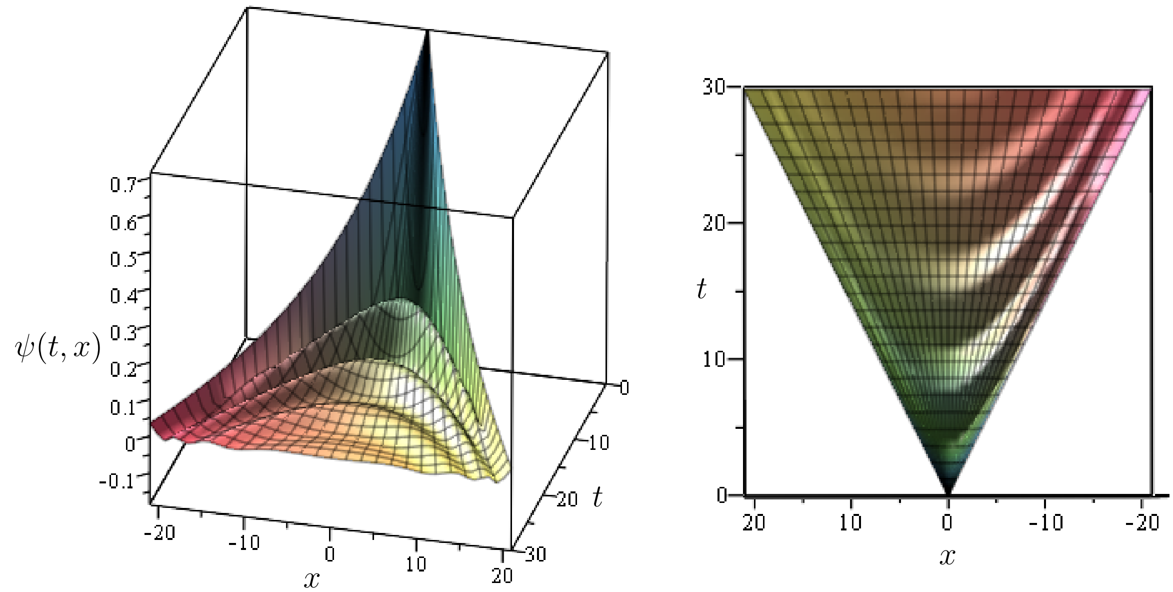

Figure 2.1: Plot of the response function $\psi(t, x)$ where $\alpha = 0.9$, $\beta = 0.7$, and $\lambda = 0.1$. The image on the right shows this plot from above, in which the causal structure is more clearly seen.

2.2.1 Fourier Transform of the Response Function

We will now work in the frequency domain, and so we take the Fourier transform of (2.6) with respect to t. This, however, is not a simple integral to calculate and thus some preliminary steps are required. It is also necessary to consider separately the cases where $x \geq 0$ and $x < 0$.

Firstly, looking at $x \geq 0$. The Fourier transform with respect to t is written

$$\hat{\psi}(\omega, x) = \int_{x/\beta}^{\infty} \exp\left(-(2\pi i\omega + \lambda)t\right) \frac{1}{2\beta} J_0\left(\alpha\sqrt{t^2 - \frac{x^2}{\beta^2}}\right) dt. \tag{2.10}$$

Now applying the variable transformation $\tau = t - x/\beta$, (2.10) then becomes

$$\hat{\psi}(\omega, x) = \frac{1}{2\beta} \exp\left(-(2\pi i\omega + \lambda)\frac{x}{\beta}\right) \int_{0}^{\infty} \exp\left(-(2\pi i\omega + \lambda)\tau\right) J_0\left(\alpha\sqrt{\tau^2 + 2\frac{x}{\beta}\tau}\right) d\tau. \tag{2.11}$$

The solution to the integral in (2.11) can be found in section 6.616 of Gradshteyn

[31],

$$\int_0^\infty e^{-ry} J_0(s\sqrt{y^2+2wy})dy = \frac{1}{\sqrt{r^2+s^2}} \exp\left(w(r-\sqrt{r^2+s^2})\right).$$

Using this we can solve (2.11) to give

$$\begin{aligned}
\hat{\psi}(\omega,x) &= \frac{1}{2\beta}\frac{\exp\left(-(2\pi i\omega+\lambda)\frac{x}{\beta}\right)}{\sqrt{-(2\pi\omega-i\lambda))^2+\alpha^2}}\exp\left[\frac{x}{\beta}\left((2\pi i\omega+\lambda)-\sqrt{-(2\pi\omega-i\lambda))^2+\alpha^2}\right)\right] \\
&= \frac{1}{2\beta}\frac{1}{i\sqrt{(2\pi\omega-i\lambda))^2-\alpha^2}}\exp\left[\frac{x}{\beta}\left(-i\sqrt{(2\pi\omega-i\lambda))^2-\alpha^2}\right)\right] \\
&= \frac{1}{2\beta}\frac{1}{i\beta L_H}\exp\left[\frac{x}{\beta}\left(-i\beta L_H\right)\right] \\
&= \frac{1}{2i\beta^2 L_H}e^{-iL_H x}
\end{aligned}$$

where

$$L_H = \frac{\sqrt{(2\pi\omega-i\lambda)^2-\alpha^2}}{\beta}. \tag{2.12}$$

Similarly, for $x < 0$ the Fourier transform of ψ is

$$\hat{\psi}(\omega,x) = \int_{-x/\beta}^{\infty}\exp\left(-(2\pi i\omega+\lambda)t\right)\frac{1}{2\beta}J_0\left(\alpha\sqrt{t^2-\frac{x^2}{\beta^2}}\right)dt. \tag{2.13}$$

This time we use the transformation $\tau = t + x/\beta$,

$$\begin{aligned}
\hat{\psi}(\omega,x) &= \int_0^\infty \exp\left(-(2\pi i\omega+\lambda)(\tau-x/\beta)\right)\frac{1}{2\beta}J_0\left(\alpha\sqrt{\tau^2-2\frac{x}{\beta}\tau}\right)d\tau \\
&= \frac{1}{2\beta}\exp\left((2\pi i\omega+\lambda)(x/\beta)\right)\int_0^\infty \exp\left(-(2\pi i\omega+\lambda)\tau\right)J_0\left(\alpha\sqrt{\tau^2-2\frac{x}{\beta}\tau}\right)d\tau \\
&= \frac{1}{2\beta}\frac{\exp\left((2\pi i\omega+\lambda)(x/\beta)\right)}{i\sqrt{(2\pi\omega-i\lambda))^2-\alpha^2}}\exp\left[-\frac{x}{\beta}\left((2\pi i\omega+\lambda)-\sqrt{-(2\pi\omega-i\lambda))^2+\alpha^2}\right)\right] \\
&= \frac{1}{2\beta}\frac{1}{i\sqrt{(2\pi\omega-i\lambda))^2-\alpha^2}}\exp\left[\frac{x}{\beta}\left(i\sqrt{(2\pi\omega-i\lambda))^2-\alpha^2}\right)\right] \\
&= \frac{1}{2i\beta^2 L_H}e^{iL_H x}.
\end{aligned}$$

Hence the complete result for the Fourier transformed response function

$$\hat{\psi}(\omega, x) = \begin{cases} \dfrac{e^{-iL_H x}}{2i\beta^2 L_H} & x \geq 0 \\ \dfrac{e^{iL_H x}}{2i\beta^2 L_H} & x < 0. \end{cases} \tag{2.14}$$

2.2.2 Damping of Solutions

In this section we show that $\hat{\psi}(\omega, x) \to 0$ as $x \to \pm\infty$, which we refer to as the solution being damped. Note that damping typically refers to a system dissipating energy, which is seen as the amplitude of the wave tending to zero as time increases. However, here we are looking at $\hat{\psi}(\omega, x)$ which has no t dependence, so cannot experience damping according to this definition. The behaviour we are looking for, an exponential decay, is the same as that seen for evanescent waves. Evanescent waves, though, are found when the wave number is purely imaginary which is not the situation here either. Since neither of these existing terms is an exact description of the behaviour of $\hat{\psi}(\omega, x)$ we choose to describe the response as being damped as this is the most appropriate term.

In order to guarantee that $\hat{\psi}$ is damped in the direction of propagation, we must consider the branch structure of $L_H(\omega)$. Observe that if ω is extended to the complex plane then L_H has branch points at $2\pi\omega = i\lambda \pm \alpha$. Since we are considering a damped system we have that $\lambda > 0$ and so both these branch points are located in the upper half plane.

Looking at the high frequency limit, $\omega \to \pm\infty$, then we have $L_H(\omega) \to \frac{2\pi}{\beta}(\omega - i\lambda)$, hence we must take the branch cut such that it runs parallel to the real axis. Further, we make the choice that the branch cut runs between the two branch points as shown in figure 2.2. This choice is ultimately arbitrary, however this is the simplest structure we can choose in which the branch cut does not cross the real axis, including the point at infinity.

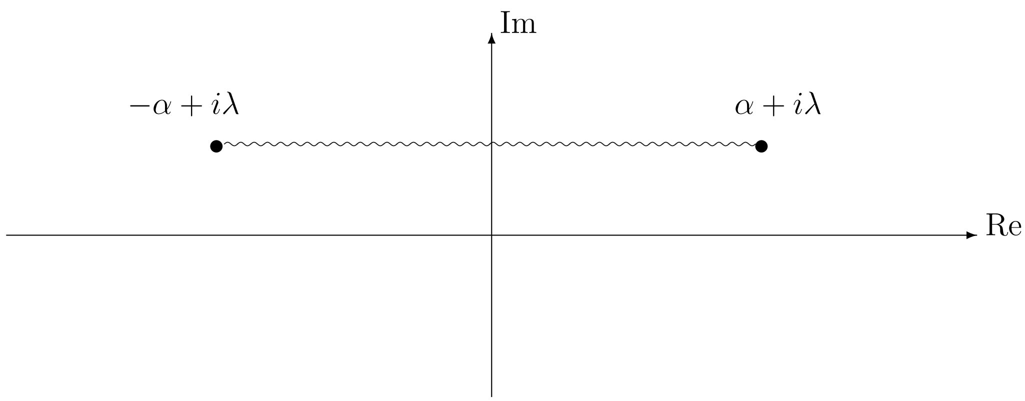

Figure 2.2: Diagram indicating the chosen branch cut and the branch points of $L_H(\omega)$.

Observe that both branch points, as well as our choice of branch cut, are located entirely in the upper-half plane and thus this fulfils the requirements of causality. To see this, consider a function $g(t)$ such that its Fourier transform, $\hat{g}(\omega)$, is analytic for $\text{Im}(\omega) \leq 0$. For causality to hold we require that $g(t) = 0$ for $t < 0$. Taking the inverse transform of $\hat{g}(\omega)$ gives, for $\omega \in \mathbb{R}$,

$$g(t) = \int_{-\infty}^{\infty} \hat{g}(\omega) e^{2\pi i \omega t} d\omega. \tag{2.15}$$

Extending ω to the complex plane, we can construct the path C with two parts, C_0 and C_1, as shown in figure 2.3. The integral along C_0 is the inverse Fourier transform given by (2.15). Having assumed that $\hat{g}(\omega)$ is analytic for $\text{Im}(\omega) \leq 0$ then, from Cauchy's Integral Theorem, we have that the integral over the entire path C is

$$\int_C \hat{g}(\omega) e^{2\pi i \omega t} d\omega = 0.$$

To show that the integral along C_1 is also zero, observe that $e^{2\pi i \omega t} \to 0$ as $\omega \to -i\infty$ for $t < 0$. Hence, for $t < 0$,

$$\int_{C_0} \hat{g}(\omega) e^{2\pi i \omega t} d\omega = g(t) = 0.$$

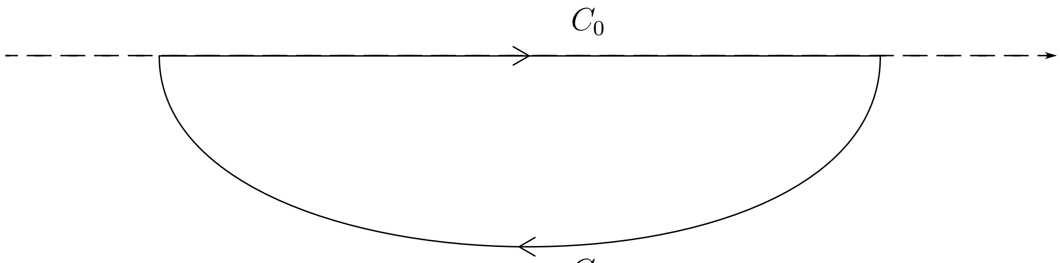

Figure 2.3: Integral path over ω on complex plane.

Having made the choice of branch cut we now need to make a choice regarding which branch we are on. Considering $L(0) = \pm i\beta^{-1}\sqrt{\lambda^2 + \alpha^2}$ we see that the correct choice is the negative square root.

To progress we require the following lemma,

Lemma 2.2.2. *Given $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, $\alpha > 0$, then*

$$\text{Sign}\left(\text{Im}\left(\sqrt{z^2 - \alpha^2}\right)\right) = \text{Sign}\left(\text{Im}\left(z\right)\right) \tag{2.16}$$

if the branch cut for $\sqrt{z^2 - \alpha^2}$ lies entirely along the real axis between $z = -\alpha$ and $z = \alpha$ and the branch is chosen such that $\sqrt{z^2 - \alpha^2} \to z$ as $z \to \infty$.

Proof. Let $z = t + iy$, where $t, y \in \mathbb{R}$ and $y > 0$, and then consider the function $g(t) = \sqrt{(t + iy)^2 - \alpha^2}$. The branch structure, as given in the statement of the lemma, is shown in figure 2.4. This choice of branch cut means we must take the positive square root, hence

$$\text{Im}(g(t)) \to y > 0 \qquad \text{as} \quad t \to \infty.$$

Observe that $\sqrt{z^2 - \alpha^2} \in \mathbb{R}$ if $z \in \mathbb{R}$ and $|z| > \alpha$, hence $g(t) \notin \mathbb{R}\ \forall\ t$. Since $g(t)$ is a continuous function of t then, for all t,

$$\text{Im}(g(t)) > 0.$$

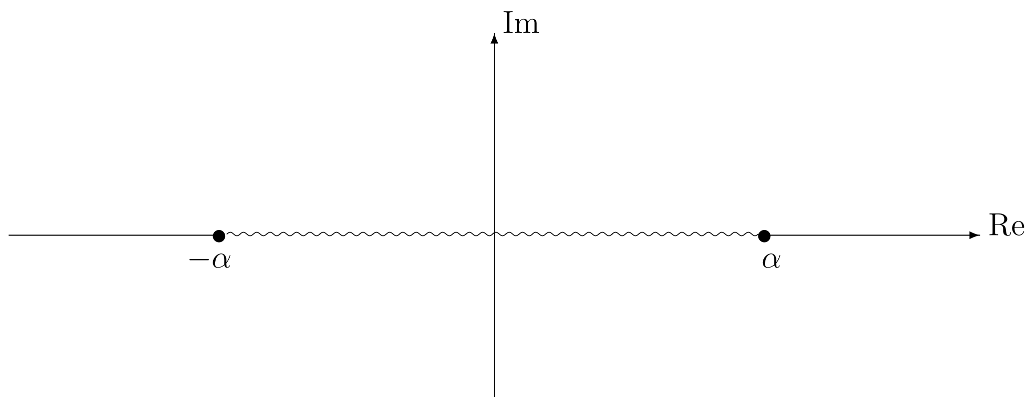

Figure 2.4: Branch structure of $g(t)$.

□

By setting $z = (2\pi\omega - i\lambda)$ in lemma 2.2.2 we can apply this result to the definition of L_H,

$$\begin{aligned}\operatorname{Sign}\left(\operatorname{Im}\left(L_H(\omega)\right)\right) &= \operatorname{Sign}\left(\operatorname{Im}\left(\frac{\sqrt{(2\pi\omega - i\lambda)^2 - \alpha^2}}{\beta}\right)\right) \\ &= \operatorname{Sign}\left(\operatorname{Im}\left(2\pi\omega - i\lambda\right)\right).\end{aligned}$$

Note that $\beta > 0$ and so has no effect on the overall sign. Hence, for real ω, $\operatorname{Im}\left(L_H(\omega)\right) < 0$.

Returning to our solutions, (2.14), we see that there are two limits to consider:

$$e^{iL_H x} \qquad x \to -\infty \qquad \text{and} \qquad e^{-iL_H x} \qquad x \to \infty.$$

Since $\operatorname{Im}\left(L_H(\omega)\right) < 0$ then we see that $iL_H(\omega) = -\operatorname{Im}\left(L_H(\omega)\right) > 0$ and as such the result of both of these limits is a convergence to zero, giving the damped solutions we require.

2.3 Dispersion Relation

Before we calculate the polarisation induced by the above response it is first necessary to look at the dispersion relation for our system. Taking (2.2) and the Fourier transform of (1.3) we have

$$\omega^2 \mu_0 \widetilde{P}(\omega, k) = \left(k^2 - \frac{\omega^2}{c^2}\right) \widetilde{E}(\omega, k)$$

and

$$\widetilde{P}(\omega, k) = \frac{\varepsilon_0 \widetilde{E}(\omega, k)}{(2\pi i\omega + \lambda)^2 + \alpha^2 + \beta^2 k^2} = \frac{\varepsilon_0 \widetilde{E}(\omega, k)}{-L_H^2 \beta^2 + \beta^2 k^2},$$

hence from these we can obtain the following dispersion relation

$$\frac{\omega^2}{c^2} = \beta^2 \left(k^2 - L_H^2\right) \left(k^2 - \frac{\omega^2}{c^2}\right). \tag{2.17}$$

This relation can be solved to find k, which will have four values for each ω These are given by

$$\left\{k^i, i = 1, \ldots, 4\right\} = \left\{k^+, -k^+, k^-, -k^-\right\} \tag{2.18}$$

where

$$k^{\pm} = \frac{1}{\sqrt{2}} \sqrt{L_H^2 + \frac{\omega^2}{c^2} \pm \sqrt{\left(L_H^2 - \frac{\omega^2}{c^2}\right)^2 + 4\frac{\omega^2}{\beta^2 c^2}}}. \tag{2.19}$$

From this we can thus write the electric field for the medium

$$\hat{E}(\omega, x) = A^+(\omega) e^{2\pi i k^+ x} + A^-(\omega) e^{-2\pi i k^+ x} + B^+(\omega) e^{2\pi i k^- x} + B^-(\omega) e^{-2\pi i k^- x}. \tag{2.20}$$

This expansion of the electric field in homogeneous, spatially dispersive media was first derived by Agarwal [32].

We will show the following,

Theorem 2.3.1. *The modes given by* (2.20) *are damped in the direction of propagation.*

In order to do this, however, we first need to establish the following result

Lemma 2.3.2. *Given numbers $u \in \mathbb{R}$ and $v \in \mathbb{C}$, where $u > 0$, then*

$$\Big(\mathrm{Im}(-iu + \sqrt{v}) < 0 \quad \& \quad \mathrm{Im}(-iu - \sqrt{v}) < 0\Big) \qquad \Longleftrightarrow \qquad \mathrm{Re}(v) - \frac{\mathrm{Im}(v)^2}{4u^2} + u^2 > 0. \tag{2.21}$$

Proof. set $v = (a + bi)^2 = a^2 - b^2 + 2abi$, where a and b are real. This gives us

$$\begin{aligned} \mathrm{Re}(v) - \frac{\mathrm{Im}(v)^2}{4u^2} + u^2 &= a^2 - b^2 - \frac{a^2b^2}{u^2} + u^2 \\ &= \frac{1}{u^2}\left(a^2u^2 - b^2u^2 - a^2b^2 + u^4\right) \\ &= \frac{1}{u^2}\left(u^2 - b^2\right)\left(a^2 + u^2\right) \end{aligned}$$

which, since a and u are both real, tells us that

$$\mathrm{Re}(v) - \frac{\mathrm{Im}(v)^2}{4u^2} + u^2 > 0 \qquad \Longleftrightarrow \qquad u^2 - b^2 > 0.$$

Since $u > 0$ then $u^2 - b^2 > 0$ is equivalent to $u > b$ & $u > -b$. Hence

$$\begin{aligned} 0 > -u + b &= \mathrm{Im}(-iu + bi) \\ &= \mathrm{Im}(-iu + \sqrt{v}) \end{aligned} \qquad \& \qquad \begin{aligned} 0 > -u - b &= \mathrm{Im}(-iu - bi) \\ &= \mathrm{Im}(-iu - \sqrt{v}) \end{aligned}$$

as required. □

From this we can present a further lemma,

Lemma 2.3.3.

$$\mathrm{Im}((k^{\pm})^2) < 0. \tag{2.22}$$

Proof. Using lemma 2.3.2 where

$$u = \frac{4\pi\omega\lambda}{\beta^2} \qquad \text{and} \qquad v = \left(L_H^2 - \frac{\omega^2}{c^2}\right)^2 + 4\frac{\omega^2}{\beta^2 c^2}$$

with $\omega > 0$. To make use of the above lemma we need to calculate the real and imaginary parts of v, this is done by writing

$$\begin{aligned} v &= (\mathrm{Re}(Z) + i\mathrm{Im}(Z))^2 + 4\frac{\omega^2}{\beta^2 c^2} \\ &= (\mathrm{Re}(Z))^2 - (\mathrm{Im}(Z))^2 + 2i\mathrm{Re}(Z)\mathrm{Im}(Z) + 4\frac{\omega^2}{\beta^2 c^2} \end{aligned}$$

where

$$Z = L_H^2 - \frac{\omega^2}{c^2}.$$

From the definition of L_H we see that

$$\mathrm{Im}(Z) = -\frac{4\pi\omega\lambda}{\beta^2} = -u$$

and so

$$\begin{aligned} \mathrm{Re}(v) - \frac{\mathrm{Im}(v)^2}{4u^2} + u^2 &= (\mathrm{Re}(Z))^2 - u^2 + 4\frac{\omega^2}{\beta^2 c^2} - \left(-\frac{2\mathrm{Re}(Z)u}{2u}\right)^2 + u^2 \\ &= (\mathrm{Re}(Z))^2 + 4\frac{\omega^2}{\beta^2 c^2} - (\mathrm{Re}(Z))^2 \\ &= 4\frac{\omega^2}{\beta^2 c^2} > 0 \end{aligned}$$

hence from lemma 2.3.2 and (3.14) we have that

$$\mathrm{Im}((k^{\pm})^2) < 0.$$

For $\omega < 0$ we repeat the above calculations but instead use

$$u = -\frac{4\pi\omega\lambda}{\beta^2}.$$

Observe that in the proof we only use u^2 terms, and so this change in sign will have no effect on the result. □

This tells us that $k^{\pm}$ lies either in the upper left or lower right quadrants of the complex plane, that is either

$$\mathrm{Re}(k^{\pm}) > 0 \qquad \text{and} \qquad \mathrm{Im}(k^{\pm}) < 0 \tag{2.23}$$

or

$$\mathrm{Re}(k^{\pm}) < 0 \qquad \text{and} \qquad \mathrm{Im}(k^{\pm}) > 0. \tag{2.24}$$

For the case where we have (2.23) we see that the corresponding Fourier modes, $e^{2\pi i(\omega t + k^{\pm}x)}$, are left moving and that

$$e^{2\pi i k^{\pm} x} \to 0 \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad e^{2\pi i k^{\pm} x} \to \infty \quad \text{as} \quad x \to \infty$$

hence the mode is damped in the direction of motion. Likewise for the right moving modes, $e^{2\pi i(\omega t - k^{\pm}x)}$.

In the case of (2.24) the limits of the modes are

$$e^{2\pi i k^{\pm} x} \to \infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad e^{2\pi i k^{\pm} x} \to 0 \quad \text{as} \quad x \to \infty.$$

Now $e^{2\pi i(\omega t + k^{\pm}x)}$ are right moving modes and $e^{2\pi i(\omega t - k^{\pm}x)}$ are left moving, so again the modes are damped in the direction of propagation. Hence this completely shows theorem 2.3.1.

Chapter 3

Boundary Between Spatially Dispersive Homogeneous Media

3.1 Introduction

In the study of inhomogeneous media a common approach is to build these materials by combining a number of homogeneous regions into a single block [33, 34, 35, 36]. While this is common practice in general, little work has been done with media where these two regions are both spatially dispersive. The goal of this work is to investigate such a medium.

To understand the behaviour of such constructed materials, it is necessary to have a full set of boundary conditions for the interface between these two regions. Suitable boundary conditions have been found previously using the energy theorem [37],

$$P \quad \text{is continuous} \qquad \text{and} \qquad \beta^2 \frac{\partial P}{\partial x} \quad \text{is continuous,} \tag{3.1}$$

as well as the continuity of E and its first spatial derivative which are standard for Maxwell's equations. We seek to find equivalent boundary conditions through different methods. Firstly, this is done by applying the response function calculated in chapter

2 on either side of a boundary, and calculating boundary conditions between these two regions. Secondly, a Lagrangian formulation is presented and it is seen that taking appropriate variations gives the same set of boundary conditions.

It has been shown that discrepancies in the additional boundary conditions obtained [17, 32, 38] for spatially dispersive media is due to the choice of model used [39]. As such there is benefit in deriving boundary conditions using different methods as we have done here.

In the case of the boundary between a homogeneous region and the vacuum a complete set of boundary conditions has been established [12]. This can be considered as a special, limiting case of the boundary between two spatially dispersive homogeneous media.

It is worth highlighting here that our use of the term 'boundary' is not necessarily adhering to the exact definition. Strictly speaking, a boundary is the interface at the outer edge of a medium, whereas an interface within a medium is referred to as a junction. In this work we use the term boundary to describe the interface between any two differing regions.

Combining (2.2) and the Fourier transform of (1.3), and taking the inverse Fourier transform, we have a fourth order partial differential equation for $\hat{P}$,

$$\frac{\beta^2}{(2\pi)^2}\frac{\partial^4 \hat{P}}{\partial x^4} + \left(\frac{\beta^2\omega^2}{c^2} - \frac{L(\omega)}{(2\pi)^2}\right)\frac{\partial^2 \hat{P}}{\partial x^2} - \frac{\omega^2}{c^2}(L(\omega)+1) = 0. \tag{3.2}$$

While we have a fourth order differential equation for the system, the standard boundary conditions for Maxwell's equations only give us two conditions, that the electric field and its first spatial derivative are continuous. As such, two additional constraints are required for a full specification.

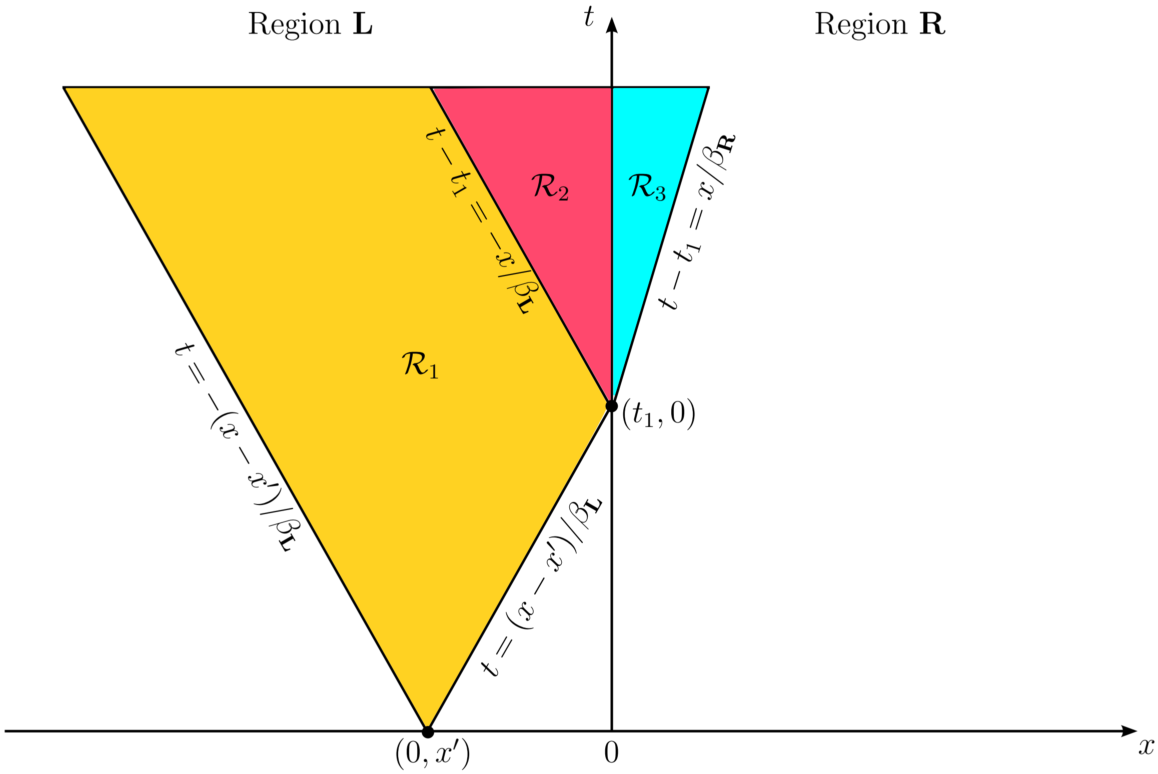

Figure 3.1: The response at the boundary of two homogeneous media, with a delta source at the point $(0, x')$ where $x' < 0$.

3.2 Response Function Across Boundary

In this section we consider a medium formed of two semi-infinite homogeneous media, each as described in chapter 2, which share a common interface at $x = 0$. We assign the parameters for each medium as

$$(\alpha, \beta, \lambda) = \begin{cases} (a_{\mathbf{L}}, \beta_{\mathbf{L}}, \lambda_{\mathbf{L}}) & x < 0 \\ (a_{\mathbf{R}}, \beta_{\mathbf{R}}, \lambda_{\mathbf{R}}) & x > 0. \end{cases} \tag{3.3}$$

Define also the quantity L_μ, as an extension of (2.12),

$$L_\mu = L_\mu(\omega) = \frac{\sqrt{(2\pi\omega - i\lambda_\mu)^2 - \alpha_\mu^2}}{\beta_\mu} \tag{3.4}$$

where $\mu = \mathbf{L}, \mathbf{R}$. Figure 3.1 shows that there are three regions where the response is non-zero, $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$. Outside of these regions would be non-causal and as such no response is present. Observe that for $\mathcal{R}_1$ the response is $\hat{\psi} = \hat{\psi}_0$ where $\hat{\psi}_0$ is the homogeneous case given by (2.14). For regions $\mathcal{R}_2$ and $\mathcal{R}_3$ the response is given by $\hat{\psi} = \hat{\psi}_0 + \hat{\psi}_{\mathbf{L}}$ and $\hat{\psi} = \hat{\psi}_{\mathbf{R}}$ respectively, where

$$\hat{\psi}_{\mathbf{L}} = a_{\mathbf{L}}(\omega)e^{iL_{\mathbf{L}}x} \qquad \text{and} \qquad \hat{\psi}_{\mathbf{R}} = a_{\mathbf{R}}(\omega)e^{-iL_{\mathbf{R}}x}.$$

The unknown functions $a_{\mathbf{L}}(\omega)$ and $a_{\mathbf{R}}(\omega)$ are determined by imposing boundary conditions for $\hat{\psi}$.

Taking the above we can write the complete response function for $x' < 0$

$$\hat{\psi}(\omega, x, x') = \begin{cases} a_{\mathbf{L}}(\omega)e^{iL_{\mathbf{L}}x} + \dfrac{e^{iL_{\mathbf{L}}(x-x')}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} & x \le x' \le 0 \\[2ex] a_{\mathbf{L}}(\omega)e^{iL_{\mathbf{L}}x} + \dfrac{e^{-iL_{\mathbf{L}}(x-x')}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} & x' \le x \le 0 \\[2ex] a_{\mathbf{R}}(\omega)e^{-iL_{\mathbf{R}}x} & x' \le 0 \le x. \end{cases} \tag{3.5}$$

In order to determine the unknown functions in the above it is necessary to make assumptions about the boundary conditions for $\hat{\psi}$,

$$\left[\hat{\psi}\right] = 0 \qquad \text{and} \qquad \left[\beta(x)^2 \frac{\partial \hat{\psi}}{\partial x}\right] = 0 \tag{3.6}$$

where $[\hat{\psi}]$ is the discontinuity $[\hat{\psi}] = \lim_{x\to 0^+} \hat{\psi}(x) - \lim_{x\to 0^-} \hat{\psi}(x)$. We choose these assumptions for $\hat{\psi}$ so that the resulting boundary conditions for $\hat{P}$ and $\hat{E}$ are consistent with those that are already known, (3.1).

These assumptions give us the conditions

$$\hat{\psi}(\omega, 0, x') = a_{\mathbf{L}}(\omega) + \frac{e^{iL_{\mathbf{L}}x'}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} = a_{\mathbf{R}}(\omega) \tag{3.7}$$

and

$$\beta(x)^2 \frac{\partial \hat{\psi}}{\partial x}(\omega, 0, x') = i\beta_{\mathbf{L}}^2 L_{\mathbf{L}} a_{\mathbf{L}}(\omega) - \frac{e^{iL_{\mathbf{L}}x'}}{2} = -i\beta_{\mathbf{R}}^2 L_{\mathbf{R}} a_{\mathbf{R}}(\omega). \tag{3.8}$$

Hence

$$a_{\mathbf{L}} = \gamma \frac{e^{iL_{\mathbf{L}}x'}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \qquad \text{and} \qquad a_{\mathbf{R}} = (1+\gamma) \frac{e^{iL_{\mathbf{L}}x'}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \tag{3.9}$$

where

$$\gamma = \frac{L_{\mathbf{L}}\beta_{\mathbf{L}}^2 - L_{\mathbf{R}}\beta_{\mathbf{R}}^2}{L_{\mathbf{L}}\beta_{\mathbf{L}}^2 + L_{\mathbf{R}}\beta_{\mathbf{R}}^2}. \tag{3.10}$$

Substituting these solutions back into (3.5) gives us

$$\hat{\psi}(\omega, x, x') = \begin{cases} \dfrac{1}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \left(\gamma e^{iL_{\mathbf{L}}x'} + e^{-iL_{\mathbf{L}}x'}\right) e^{iL_{\mathbf{L}}x} & x \leq x' \leq 0 \\ \dfrac{1}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \left(\gamma e^{iL_{\mathbf{L}}x} + e^{-iL_{\mathbf{L}}x}\right) e^{iL_{\mathbf{L}}x'} & x' \leq x \leq 0 \\ \dfrac{1+\gamma}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} e^{iL_{\mathbf{L}}x'} e^{-iL_{\mathbf{R}}x} & x' \leq 0 \leq x. \end{cases} \tag{3.11}$$

This completes the result when the source is in the left hand medium, it now remains to calculate the response for a source located in the right hand medium, that is $x' \geq 0$.

Figure 3.2 shows the regions in this situation. Again $\mathcal{R}_1$ is the same as the homogeneous result. Region $\mathcal{R}_3$ is a purely left moving solution and $\mathcal{R}_2$ is a superposition of the homogeneous solution ψ_0 and a purely right moving wave. Hence, for $x' > 0$, the response is written

$$\hat{\psi}(\omega, x, x') = \begin{cases} b_{\mathbf{L}}(\omega) e^{iL_{\mathbf{L}}x} & x \leq 0 \leq x' \\ b_{\mathbf{R}}(\omega) e^{-iL_{\mathbf{R}}x} + \dfrac{e^{iL_{\mathbf{R}}(x-x')}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} & 0 \leq x \leq x' \\ b_{\mathbf{R}}(\omega) e^{-iL_{\mathbf{R}}x} + \dfrac{e^{-iL_{\mathbf{R}}(x-x')}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} & 0 \leq x' \leq x \end{cases}$$

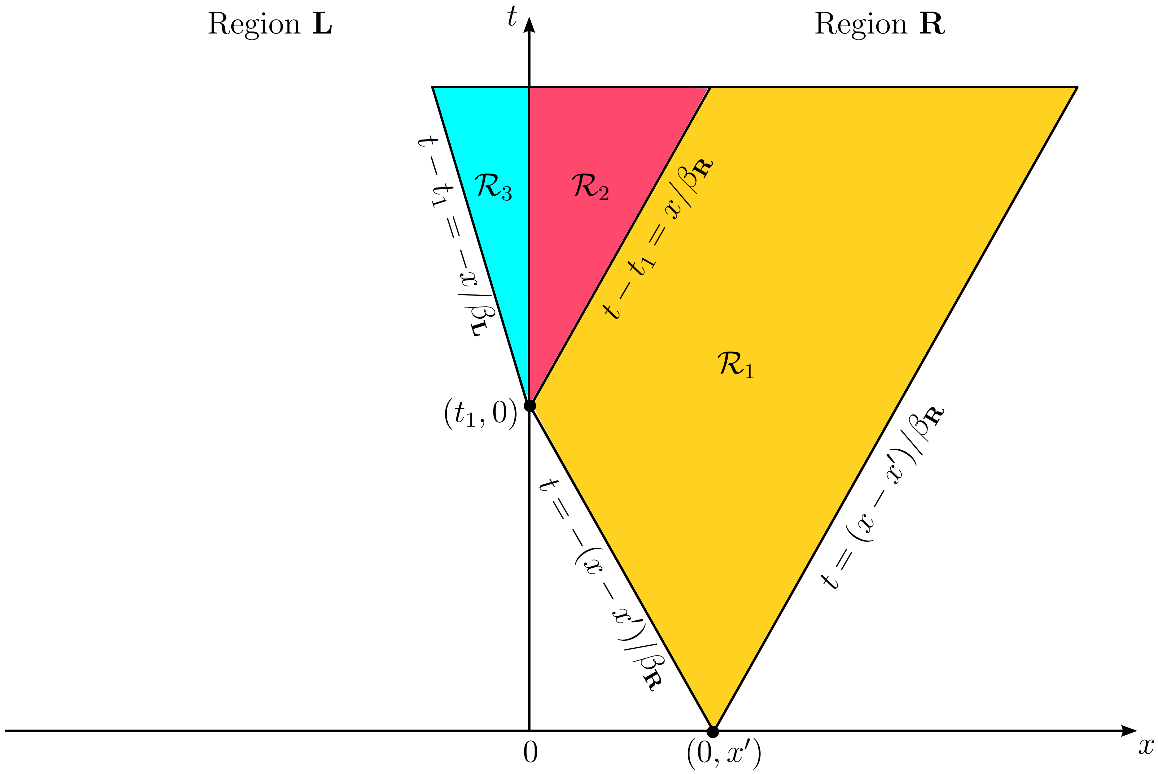

Figure 3.2: The response at the boundary of two homogeneous media, with a delta source at the point $(0, x')$ where $x' > 0$.

where $b_{\mathbf{L}}(\omega)$ and $b_{\mathbf{R}}(\omega)$ are to be determined. Using the same continuity assumptions as in (3.6) gives the expressions

$$\hat{\psi}(\omega, 0, x') = b_{\mathbf{L}}(\omega) = b_{\mathbf{R}}(\omega) + \frac{e^{-iL_{\mathbf{R}}x'}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}$$

and

$$\beta(x)^2 \frac{\partial \hat{\psi}}{\partial x}(\omega, 0, x') = i\beta_{\mathbf{L}}^2 L_{\mathbf{L}} b_{\mathbf{L}}(\omega) = -i\beta_{\mathbf{R}}^2 L_{\mathbf{R}} b_{\mathbf{R}}(\omega) + \frac{e^{-iL_{\mathbf{R}}x'}}{2}$$

hence

$$b_{\mathbf{L}}(\omega) = (1 - \gamma)\frac{e^{-iL_{\mathbf{R}}x'}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \qquad \text{and} \qquad b_{\mathbf{R}}(\omega) = -\gamma\frac{e^{-iL_{\mathbf{R}}x'}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}.$$

These solutions, along with the previous part of the result (3.11), gives us the response function for the full range of x and x'

$$\hat{\psi}(\omega, x, x') = \begin{cases} \frac{1}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}}\left(\gamma e^{iL_{\mathbf{L}}x'} + e^{-iL_{\mathbf{L}}x'}\right) e^{iL_{\mathbf{L}}x} & x \leq x' \leq 0 \\ \frac{1}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}}\left(\gamma e^{iL_{\mathbf{L}}x} + e^{-iL_{\mathbf{L}}x}\right) e^{iL_{\mathbf{L}}x'} & x' \leq x \leq 0 \\ \frac{1+\gamma}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} e^{iL_{\mathbf{L}}x'} e^{-iL_{\mathbf{R}}x} & x' \leq 0 \leq x \\ \frac{1-\gamma}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} e^{-iL_{\mathbf{R}}x'} e^{iL_{\mathbf{L}}x} & x \leq 0 \leq x' \\ \frac{1}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}\left(e^{iL_{\mathbf{R}}x} - \gamma e^{-iL_{\mathbf{R}}x}\right) e^{-iL_{\mathbf{R}}x'} & 0 \leq x \leq x' \\ \frac{1}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}\left(e^{iL_{\mathbf{R}}x'} - \gamma e^{-iL_{\mathbf{R}}x'}\right) e^{-iL_{\mathbf{R}}x} & 0 \leq x' \leq x. \end{cases} \tag{3.12}$$

3.2.1 Dispersion Relation

As in (2.17), the dispersion relation for each medium is given by

$$\frac{\omega^2}{c^2} = \beta_\mu^2 \left(k^2 - L_\mu^2\right)\left(k^2 - \frac{\omega^2}{c^2}\right) \tag{3.13}$$

solving gives, for each of the media,

$$\left\{k_\mu^i, i = 1, \ldots, 4\right\} = \left\{k_\mu^+, -k_\mu^+, k_\mu^-, -k_\mu^-\right\}$$

where

$$k_\mu^\pm = \frac{1}{\sqrt{2}}\sqrt{L_\mu^2 + \frac{\omega^2}{c^2} \pm \sqrt{\left(L_\mu^2 - \frac{\omega^2}{c^2}\right)^2 + 4\frac{\omega^2}{\beta_\mu^2 c^2}}}. \tag{3.14}$$

Hence we can write the expansion of the electric field

$$\begin{aligned}\hat{E}(x) &= A_\mu^+ e^{2\pi i k_\mu^+ x} + A_\mu^- e^{-2\pi i k_\mu^+ x} + B_\mu^+ e^{2\pi i k_\mu^- x} + B_\mu^- e^{-2\pi i k_\mu^- x} \\ &= \sum_{a=1}^{4} \mathcal{A}_\mu^a e^{i k_\mu^a x'} \end{aligned} \tag{3.15}$$

and hence,

$$\hat{P}(\omega, x) = \frac{A_\mu^+ e^{2\pi i k_\mu^+ x} + A_\mu^- e^{-2\pi i k_\mu^+ x}}{\beta_\mu^2 L_\mu^2 + (2\pi)^2 (k_\mu^+)^2 \beta_\mu^2} + \frac{B_\mu^+ e^{2\pi i k_\mu^- x} + B_\mu^- e^{-2\pi i k_\mu^- x}}{\beta_\mu^2 L_\mu^2 + (2\pi)^2 (k_\mu^-)^2 \beta_\mu^2}. \tag{3.16}$$

These modes, in each of the two media, are shown in figure 3.3. That these solutions are damped is detailed in section 2.2.2. The standard scattering problem is to assume that the incoming modes, $\{A_{\mathbf{L}}^-, B_{\mathbf{L}}^-, A_{\mathbf{R}}^+, B_{\mathbf{R}}^+\}$, are known and to then find the four outgoing modes, $\{A_{\mathbf{L}}^+, B_{\mathbf{L}}^+, A_{\mathbf{R}}^-, B_{\mathbf{R}}^-\}$. As such four boundary conditions are required for a full description.

3.2.2 Boundary Conditions for the Polarisation

Having calculated the response, (3.12), in the previous section the next step is to use this, along with the electric field given by (2.20), to calculate the polarisation density according to the definition

$$\hat{P}(\omega, x) = \int_{-\infty}^{\infty} \hat{\psi}(\omega, x, x') \hat{E}(\omega, x') dx'. \tag{3.17}$$

Since the response given in (3.12) has six components it is necessary to calculate the integral in (3.17) separately for each of these six cases. First, we take the three

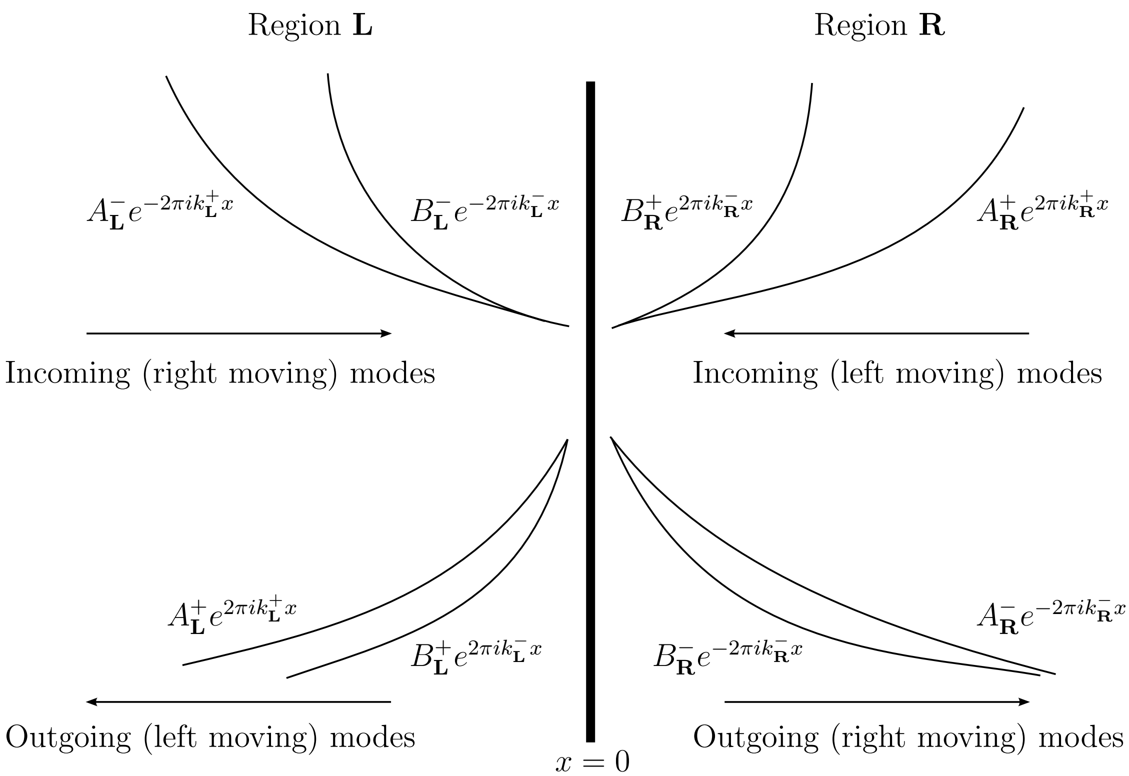

Figure 3.3: Incoming and outgoing solutions to the dispersion relation.

cases for which $x < 0$. For $x \leq x' \leq 0$,

$$\begin{aligned}
\int_x^0 \hat{\psi}(\omega, x, x')\hat{E}(\omega, x')dx' &= \int_x^0 \frac{1}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \left(\gamma e^{iL_{\mathbf{L}}x'} + e^{-iL_{\mathbf{L}}x'}\right) e^{iL_{\mathbf{L}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) e^{ik_{\mathbf{L}}^a x'} dx' \\
&= \frac{e^{iL_{\mathbf{L}}x}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \int_x^0 \gamma e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'} + e^{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})x'} dx' \\
&= \frac{e^{iL_{\mathbf{L}}x}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \left[\frac{\gamma e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'}}{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)} + \frac{e^{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})x'}}{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})}\right]_x^0 \\
&= \frac{e^{iL_{\mathbf{L}}x}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \left(\frac{\gamma}{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)} + \frac{1}{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})} - \frac{\gamma e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x}}{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)} - \frac{e^{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})x}}{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})}\right) \\
&= \frac{e^{iL_{\mathbf{L}}x}}{2\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \left(\frac{\gamma e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x}}{L_{\mathbf{L}}+k_{\mathbf{L}}^a} + \frac{e^{i(k_{\mathbf{L}}^a - L_{\mathbf{L}})x}}{k_{\mathbf{L}}^a - L_{\mathbf{L}}} - \frac{(\gamma+1)k_{\mathbf{L}}^a + (1-\gamma)L_{\mathbf{L}}}{(k_{\mathbf{L}}^a)^2 - L_{\mathbf{L}}^2}\right) \\
&= \sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{L}}^a(\omega)}{2\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \left(\frac{\gamma e^{i(2L_{\mathbf{L}}+k_{\mathbf{L}}^a)x}}{L_{\mathbf{L}}+k_{\mathbf{L}}^a} + \frac{e^{ik_{\mathbf{L}}^a x}}{k_{\mathbf{L}}^a - L_{\mathbf{L}}} - \frac{(\gamma+1)k_{\mathbf{L}}^a + (1-\gamma)L_{\mathbf{L}}}{(k_{\mathbf{L}}^a)^2 - L_{\mathbf{L}}^2} e^{iL_{\mathbf{L}}x}\right). \quad (3.18)
\end{aligned}$$

For $x' < x < 0$,

$$\int_{-\infty}^{x} \hat{\psi}(\omega, x, x')\hat{E}(\omega, x')dx' = \int_{-\infty}^{x} \frac{1}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \left(\gamma e^{iL_{\mathbf{L}}x} + e^{-iL_{\mathbf{L}}x}\right) e^{iL_{\mathbf{L}}x'} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) e^{ik_{\mathbf{L}}^a x'} dx'$$

$$= \frac{\gamma e^{iL_{\mathbf{L}}x} + e^{-iL_{\mathbf{L}}x}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \int_{-\infty}^{x} e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'} dx'$$

$$= \frac{\gamma e^{iL_{\mathbf{L}}x} + e^{-iL_{\mathbf{L}}x}}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \left[\frac{e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'}}{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)}\right]_{-\infty}^{x}$$

$$= -\sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{L}}^a(\omega)}{2\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \frac{\gamma e^{i(2L_{\mathbf{L}}+k_{\mathbf{L}}^a)x} + e^{ik_{\mathbf{L}}^a x}}{L_{\mathbf{L}}+k_{\mathbf{L}}^a}. \tag{3.19}$$

The term $e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'} \to 0$ as $x \to -\infty$ due to the results presented in sections 2.2.2 and 2.3. This applies also to the infinite limits that appear in several other of the following integrals. For $x < 0 < x'$,

$$\int_{0}^{\infty} \hat{\psi}(\omega, x, x')\hat{E}(\omega, x')dx' = \int_{0}^{\infty} \frac{1-\gamma}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} e^{-iL_{\mathbf{R}}x'} e^{iL_{\mathbf{L}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) e^{ik_{\mathbf{R}}^a x'} dx'$$

$$= \frac{1-\gamma}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} e^{iL_{\mathbf{L}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \int_{0}^{\infty} e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x'} dx'$$

$$= \frac{1-\gamma}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} e^{iL_{\mathbf{L}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \left[\frac{e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x'}}{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})}\right]_{0}^{\infty}$$

$$= \sum_{a=1}^{4} \frac{1-\gamma}{2\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \frac{\mathcal{A}_{\mathbf{R}}^a(\omega) e^{iL_{\mathbf{L}}x}}{k_{\mathbf{R}}^a - L_{\mathbf{R}}}. \tag{3.20}$$

Hence, for $x < 0$ we take the sum of (3.18), (3.19) and (3.20) to give

$$\hat{P}(\omega, x) = \int_{-\infty}^{\infty} \hat{\psi}(\omega, x, x')\hat{E}(\omega, x')dx'$$

$$= \sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{L}}^a(\omega)}{\beta_{\mathbf{L}}^2\left((k_{\mathbf{L}}^a)^2 - L_{\mathbf{L}}^2\right)} \left(e^{ik_{\mathbf{L}}^a x} - \frac{k_{\mathbf{L}}^a \beta_{\mathbf{L}}^2 + L_{\mathbf{R}}\beta_{\mathbf{R}}^2}{L_{\mathbf{L}}\beta_{\mathbf{L}}^2 + L_{\mathbf{R}}\beta_{\mathbf{R}}^2} e^{iL_{\mathbf{L}}x}\right) + \frac{\mathcal{A}_{\mathbf{R}}^a(\omega) e^{iL_{\mathbf{L}}x}}{\left(L_{\mathbf{L}}\beta_{\mathbf{L}}^2 + L_{\mathbf{R}}\beta_{\mathbf{R}}^2\right)\left(k_{\mathbf{R}}^a - L_{\mathbf{R}}\right)}. \tag{3.21}$$

Likewise for the case where $x > 0$ we must calculate another three integrals. For

$x' < 0 < x$,

$$\int_{-\infty}^{0} \hat{\psi}(\omega, x, x')\hat{E}(\omega, x')dx' = \int_{-\infty}^{0} \frac{1+\gamma}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} e^{iL_{\mathbf{L}}x'} e^{-iL_{\mathbf{R}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) e^{ik_{\mathbf{L}}^a x'} dx'$$
$$= \frac{1+\gamma}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} e^{-iL_{\mathbf{R}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \int_{-\infty}^{0} e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'} dx'$$
$$= \frac{1+\gamma}{2i\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} e^{-iL_{\mathbf{R}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{L}}^a(\omega) \left[\frac{e^{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)x'}}{i(L_{\mathbf{L}}+k_{\mathbf{L}}^a)} \right]_{-\infty}^{0}$$
$$= -\sum_{a=1}^{4} \frac{1+\gamma}{2\beta_{\mathbf{L}}^2 L_{\mathbf{L}}} \frac{\mathcal{A}_{\mathbf{L}}^a(\omega) e^{-iL_{\mathbf{R}}x}}{L_{\mathbf{L}}+k_{\mathbf{L}}^a}. \tag{3.22}$$

For $0 < x' < x$,

$$\int_{0}^{x} \hat{\psi}(\omega, x, x')\hat{E}(\omega, x')dx'$$
$$= \int_{0}^{x} \frac{1}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \left(e^{iL_{\mathbf{R}}x'} - \gamma e^{-iL_{\mathbf{R}}x'} \right) e^{-iL_{\mathbf{R}}x} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) e^{ik_{\mathbf{R}}^a x'} dx'$$
$$= \frac{e^{-iL_{\mathbf{R}}x}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \int_{0}^{x} e^{i(L_{\mathbf{R}}+k_{\mathbf{R}}^a)x'} - \gamma e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x'} dx'$$
$$= \frac{e^{-iL_{\mathbf{R}}x}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \left[\frac{e^{i(L_{\mathbf{R}}+k_{\mathbf{R}}^a)x'}}{i(L_{\mathbf{R}}+k_{\mathbf{R}}^a)} - \frac{\gamma e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x'}}{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})} \right]_{0}^{x}$$
$$= \frac{e^{-iL_{\mathbf{R}}x}}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \left(\frac{e^{i(L_{\mathbf{R}}+k_{\mathbf{R}}^a)x}}{i(L_{\mathbf{R}}+k_{\mathbf{R}}^a)} - \frac{\gamma e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x}}{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})} - \frac{1}{i(L_{\mathbf{R}}+k_{\mathbf{R}}^a)} + \frac{\gamma}{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})} \right)$$
$$= \sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{R}}^a(\omega)}{2\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \left(\frac{\gamma e^{i(k_{\mathbf{R}}^a - 2L_{\mathbf{R}})x}}{k_{\mathbf{R}}^a - L_{\mathbf{R}}} - \frac{e^{ik_{\mathbf{R}}^a x}}{L_{\mathbf{R}}+k_{\mathbf{R}}^a} + \frac{(1-\gamma)k_{\mathbf{R}}^a - (\gamma+1)L_{\mathbf{R}}}{(k_{\mathbf{R}}^a)^2 - L_{\mathbf{R}}^2} e^{-iL_{\mathbf{R}}x} \right). \tag{3.23}$$

Finally, for $0 < x < x'$,

$$\int_x^\infty \hat{\psi}(\omega,x,x')\hat{E}(\omega,x')dx' = \int_x^\infty \frac{1}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}(e^{iL_{\mathbf{R}}x} - \gamma e^{-iL_{\mathbf{R}}x})e^{-iL_{\mathbf{R}}x'} \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) e^{ik_{\mathbf{R}}^a x'} dx'$$

$$= \frac{1}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}(e^{iL_{\mathbf{R}}x} - \gamma e^{-iL_{\mathbf{R}}x}) \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \int_x^\infty e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x'} dx'$$

$$= \frac{1}{2i\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}(e^{iL_{\mathbf{R}}x} - \gamma e^{-iL_{\mathbf{R}}x}) \sum_{a=1}^{4} \mathcal{A}_{\mathbf{R}}^a(\omega) \left[e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x'}\right]_x^\infty$$

$$= \sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{R}}^a(\omega)}{2\beta_{\mathbf{R}}^2 L_{\mathbf{R}}}(e^{iL_{\mathbf{R}}x} - \gamma e^{-iL_{\mathbf{R}}x}) \frac{e^{i(k_{\mathbf{R}}^a - L_{\mathbf{R}})x}}{k_{\mathbf{R}}^a - L_{\mathbf{R}}}$$

$$= \sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{R}}^a(\omega)}{2\beta_{\mathbf{R}}^2 L_{\mathbf{R}}} \frac{e^{ik_{\mathbf{R}}^a x} - \gamma e^{i(k_{\mathbf{R}}^a - 2L_{\mathbf{R}})x}}{k_{\mathbf{R}}^a - L_{\mathbf{R}}}. \tag{3.24}$$

Summing (3.22), (3.23) and (3.24) gives $P(\omega, x)$ for $x > 0$

$$\hat{P}(\omega,x) = \int_{-\infty}^\infty \hat{\psi}(\omega,x,x')\hat{E}(\omega,x')dx'$$

$$= \sum_{a=1}^{4} \frac{\mathcal{A}_{\mathbf{R}}^a(\omega)}{\beta_{\mathbf{R}}^2\left((k_{\mathbf{R}}^a)^2 - L_{\mathbf{R}}^2\right)} \left(e^{ik_{\mathbf{R}}^a x} + \frac{k_{\mathbf{R}}^a \beta_{\mathbf{R}}^2 - L_{\mathbf{L}}\beta_{\mathbf{L}}^2}{L_{\mathbf{L}}\beta_{\mathbf{L}}^2 + L_{\mathbf{R}}\beta_{\mathbf{R}}^2} e^{-iL_{\mathbf{R}}x}\right)$$

$$- \frac{\mathcal{A}_{\mathbf{L}}^a(\omega)e^{-iL_{\mathbf{R}}x}}{(L_{\mathbf{L}}\beta_{\mathbf{L}}^2 + L_{\mathbf{R}}\beta_{\mathbf{R}}^2)(L_{\mathbf{L}} + k_{\mathbf{L}}^a)}. \tag{3.25}$$

From (3.21) and (3.25) it is trivial to see that $\hat{P}$ is continuous at the $x = 0$ boundary. We know already from the standard boundary conditions of Maxwell's equations that $\hat{E}$ and its first derivative are continuous at this boundary, hence we have now have a set of three boundary conditions. For a complete description we need one more condition and so we take the first spatial derivatives of (3.21) and (3.25),

$$\frac{\partial \hat{P}}{\partial x} = \sum_{a=1}^{4} \frac{i\mathcal{A}^a_{\mathbf{L}}(\omega)}{\beta^2_{\mathbf{L}}\left((k^a_{\mathbf{L}})^2 - L^2_{\mathbf{L}}\right)} \left(k^a_{\mathbf{L}} e^{ik^a_{\mathbf{L}}x} - L_{\mathbf{L}} \frac{k^a_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}}}{L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}}} e^{iL_{\mathbf{L}}x} \right) + \frac{iL_{\mathbf{L}}\mathcal{A}^a_{\mathbf{R}}(\omega) e^{iL_{\mathbf{L}}x}}{\left(L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}}\right)\left(k^a_{\mathbf{R}} - L_{\mathbf{R}}\right)}$$

$$\frac{\partial \hat{P}}{\partial x} = \sum_{a=1}^{4} \frac{i\mathcal{A}^a_{\mathbf{R}}(\omega)}{\beta^2_{\mathbf{R}}\left((k^a_{\mathbf{R}})^2 - L^2_{\mathbf{R}}\right)} \left(k^a_{\mathbf{R}} e^{ik^a_{\mathbf{R}}x} - L_{\mathbf{R}} \frac{k^a_{\mathbf{R}}\beta^2_{\mathbf{R}} - L_{\mathbf{L}}\beta^2_{\mathbf{L}}}{L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}}} e^{-iL_{\mathbf{R}}x} \right) + \frac{iL_{\mathbf{R}}\mathcal{A}^a_{\mathbf{L}}(\omega) e^{-iL_{\mathbf{R}}x}}{(L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}})(L_{\mathbf{L}} + k^a_{\mathbf{L}})}$$

for $x < 0$ and $x > 0$ respectively. Evaluating at $x = 0$ these two derivatives reduce to

$$\left.\frac{\partial \hat{P}}{\partial x}\right|_{x=0^-} = \sum_{a=1}^{4} \frac{i\beta^2_{\mathbf{R}} L_{\mathbf{R}}\mathcal{A}^a_{\mathbf{L}}(\omega)}{\beta^2_{\mathbf{L}}(L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}})(k^a_{\mathbf{L}} + L_{\mathbf{L}})} + \frac{iL_{\mathbf{L}}\mathcal{A}^a_{\mathbf{R}}(\omega)}{(L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}})(k^a_{\mathbf{R}} - L_{\mathbf{R}})}$$

$$\left.\frac{\partial \hat{P}}{\partial x}\right|_{x=0^+} = \sum_{a=1}^{4} \frac{iL_{\mathbf{R}}\mathcal{A}^a_{\mathbf{L}}(\omega)}{(L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}})(k^a_{\mathbf{L}} + L_{\mathbf{L}})} + \frac{i\beta^2_{\mathbf{L}} L_{\mathbf{L}}\mathcal{A}^a_{\mathbf{R}}(\omega)}{\beta^2_{\mathbf{R}}(L_{\mathbf{L}}\beta^2_{\mathbf{L}} + L_{\mathbf{R}}\beta^2_{\mathbf{R}})(k^a_{\mathbf{R}} - L_{\mathbf{R}})}$$

and so, clearly, we have

$$\beta^2_{\mathbf{L}} \left.\frac{\partial \hat{P}}{\partial x}\right|_{x=0^-} = \beta^2_{\mathbf{R}} \left.\frac{\partial \hat{P}}{\partial x}\right|_{x=0^+}.$$

Hence our complete boundary conditions are

$$\left[\hat{E}\right] = 0 \qquad \left[\frac{\partial \hat{E}}{\partial x}\right] = 0 \tag{3.26}$$

$$\left[\hat{P}\right] = 0 \qquad \left[\beta(x)^2 \frac{\partial \hat{P}}{\partial x}\right] = 0. \tag{3.27}$$

In terms of the coefficients $\{A^{\pm}_{\mu}, B^{\pm}_{\mu}\}$ these boundary conditions are found by substituting (3.15) and (3.16) into (3.26) and (3.27),

$$A_{\mathbf{L}}^{+} + A_{\mathbf{L}}^{-} + B_{\mathbf{L}}^{+} + B_{\mathbf{L}}^{-} = A_{\mathbf{R}}^{+} + A_{\mathbf{R}}^{-} + B_{\mathbf{R}}^{+} + B_{\mathbf{R}}^{-} \tag{3.28}$$

$$k_{\mathbf{L}}^{+}A_{\mathbf{L}}^{+} - k_{\mathbf{L}}^{+}A_{\mathbf{L}}^{-} + k_{\mathbf{L}}^{-}B_{\mathbf{L}}^{+} - k_{\mathbf{L}}^{-}B_{\mathbf{L}}^{-} = k_{\mathbf{R}}^{+}A_{\mathbf{R}}^{+} - k_{\mathbf{R}}^{+}A_{\mathbf{R}}^{-} + k_{\mathbf{R}}^{-}B_{\mathbf{R}}^{+} - k_{\mathbf{R}}^{-}B_{\mathbf{R}}^{-} \tag{3.29}$$

$$\frac{A_{\mathbf{L}}^{+} + A_{\mathbf{L}}^{-}}{\beta_{\mathbf{L}}^2 L_{\mathbf{L}}^2 + (k_{\mathbf{L}}^{+})^2\beta_{\mathbf{L}}^2} + \frac{B_{\mathbf{L}}^{+} + B_{\mathbf{L}}^{-}}{\beta_{\mathbf{L}}^2 L_{\mathbf{L}}^2 + (k_{\mathbf{L}}^{-})^2\beta_{\mathbf{L}}^2} = \frac{A_{\mathbf{R}}^{+} + A_{\mathbf{R}}^{-}}{\beta_{\mathbf{R}}^2 L_{\mathbf{R}}^2 + (k_{\mathbf{R}}^{+})^2\beta_{\mathbf{R}}^2} + \frac{B_{\mathbf{R}}^{+} + B_{\mathbf{R}}^{-}}{\beta_{\mathbf{R}}^2 L_{\mathbf{R}}^2 + (k_{\mathbf{R}}^{-})^2\beta_{\mathbf{R}}^2} \tag{3.30}$$

$$\frac{k_{\mathbf{L}}^{+}(A_{\mathbf{L}}^{+} - A_{\mathbf{L}}^{-})}{L_{\mathbf{L}}^2 + (k_{\mathbf{L}}^{+})^2} + \frac{k_{\mathbf{L}}^{-}(B_{\mathbf{L}}^{+} - B_{\mathbf{L}}^{-})}{L_{\mathbf{L}}^2 + (k_{\mathbf{L}}^{-})^2} = \frac{k_{\mathbf{R}}^{+}(A_{\mathbf{R}}^{+} - A_{\mathbf{R}}^{-})}{L_{\mathbf{R}}^2 + (k_{\mathbf{R}}^{+})^2} + \frac{k_{\mathbf{R}}^{-}(B_{\mathbf{R}}^{+} - B_{\mathbf{R}}^{-})}{L_{\mathbf{R}}^2 + (k_{\mathbf{R}}^{-})^2}. \tag{3.31}$$

The continuity of the electric field and its first derivative are standard boundary conditions. The condition $[\hat{P}] = 0$ tells us that the polarisation is a wave with continuity across the boundary. This, along with $[\hat{E}] = 0$, also means that the displacement field D is continuous across the boundary. From (3.14) we see that $k_{\mathbf{L}} \neq k_{\mathbf{R}}$ are both fixed and so, in order for the polarisation to be continuous, the derivative of P must be discontinuous as seen in this result.

3.3 Lagrangian Formulation

Another approach to this problem is to use a Lagrangian formulation. This work has been presented in a more condensed form in [1]. Due to the damping in the system it is non-trivial to find a Lagrangian which satisfies both Maxwell's equations, (1.3), as well as the constitutive relation (2.2). Fortunately, since we are interested only in finding boundary conditions we are able to use instead the Fourier transformed equations,

$$\frac{1}{(2\pi)^2}\frac{\partial^2 \hat{E}}{\partial x^2} = -\omega^2\left(\hat{E}+\hat{P}\right)$$

and

$$\frac{1}{(2\pi)^2}\frac{\partial}{\partial x}\left(\beta(x)^2\frac{\partial \hat{P}}{\partial x}\right) + L(x)\hat{P} = \hat{E} \qquad \text{where} \quad L(x) = (2\pi i\omega + \lambda(x))^2 + \alpha(x)^2.$$

The parameters $\alpha(x)$, $\beta(x)$, $\lambda(x)$ are defined to be piecewise constant, that is

$$\alpha(x) = \alpha_{\mathbf{L}}\theta(-x) + \alpha_{\mathbf{R}}\theta(x),\ \beta(x) = \beta_{\mathbf{L}}\theta(-x) + \beta_{\mathbf{R}}\theta(x),\ \lambda(x) = \lambda_{\mathbf{L}}\theta(-x) + \lambda_{\mathbf{R}}\theta(x)$$

where $\theta(x)$ is the Heaviside function

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

These equations can be derived by varying the action

$$\mathcal{S}[\hat{E},\hat{P}] = \int \mathcal{L}\left(\hat{E}, \frac{\partial \hat{E}}{\partial x}, \hat{P}, \frac{\partial \hat{P}}{\partial x}, x\right) dx \tag{3.32}$$

with

$$\mathcal{L}\left(\hat{E}, \frac{\partial \hat{E}}{\partial x}, \hat{P}, \frac{\partial \hat{P}}{\partial x}, x\right) = \frac{1}{2}\left(\frac{1}{(2\pi)^2\omega^2}\left(\frac{\partial \hat{E}}{\partial x}\right)^2 - \hat{E}^2 - \frac{\beta(x)^2}{(2\pi)^2}\left(\frac{\partial \hat{P}}{\partial x}\right)^2 + L(\omega)\hat{P}^2\right) - \hat{E}\hat{P}. \tag{3.33}$$

In order for the definition of this Lagrangian to be valid we required that E and P are both continuous for all x, in particular $[E] = 0$ and $[P] = 0$.

Varying this with respect to $\hat{E}$ (away from the boundary)

$$\frac{d}{d\epsilon}\int \mathcal{L}\left(\hat{E}+\epsilon\delta E, \frac{\partial}{\partial x}\left(\hat{E}+\epsilon\delta E\right), \hat{P}, \frac{\partial \hat{P}}{\partial x}\right) dx\bigg|_{\epsilon=0} = 0$$

and write $\zeta(\hat{P})$ for any terms that depend only on $\hat{P}$ and not $\hat{E}$

$$\begin{aligned}
&\mathcal{L}\left(\hat{E}+\epsilon\delta E, \frac{\partial}{\partial x}\left(\hat{E}+\epsilon\delta E\right), \hat{P}, \frac{\partial \hat{P}}{\partial x}\right) \\
&= \frac{1}{2}\left(\left(\frac{\partial}{\partial x}\left(\frac{\hat{E}+\epsilon\delta E}{2\pi\omega}\right)\right)^2 - (\hat{E}+\epsilon\delta E)^2 + \zeta(\hat{P})\right) - (\hat{E}+\epsilon\delta E)\hat{P} \\
&= \frac{1}{2}\left(\frac{1}{(2\pi\omega)^2}\left(\left(\frac{\partial \hat{E}}{\partial x}\right)^2 + 2\frac{\partial \hat{E}}{\partial x}\frac{\partial}{\partial x}(\epsilon\delta E)\right) - \hat{E}^2 - 2\epsilon\hat{E}\delta E + \zeta(\hat{P})\right) \\
&\quad - (\hat{E}+\epsilon\delta E)\hat{P} + O(\epsilon^2).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{d\epsilon}\mathcal{S}[\hat{E}+\epsilon\delta E, \hat{P}]\bigg|_{\epsilon=0} = \int \frac{1}{(2\pi\omega)^2}\frac{\partial \hat{E}}{\partial x}\frac{\partial(\delta E)}{\partial x} - \hat{E}\delta E - \hat{P}\delta E dx \\
&= \int \frac{1}{(2\pi\omega)^2}\left(\frac{\partial}{\partial x}\left(\frac{\partial \hat{E}}{\partial x}\delta E\right) - \frac{\partial^2 \hat{E}}{\partial x^2}\delta E\right) - \hat{E}\delta E - \hat{P}\delta E dx \\
&= \frac{1}{(2\pi\omega)^2}\int \frac{\partial}{\partial x}\left(\frac{\partial \hat{E}}{\partial x}\delta E\right) dx - \int \frac{1}{(2\pi\omega)^2}\left(\frac{\partial^2 \hat{E}}{\partial x^2}\delta E\right) + \hat{E}\delta E + \hat{P}\delta E dx.
\end{aligned}$$

However since δE has compact support then the first integral in the above vanishes leaving us with

$$\frac{d}{d\epsilon}\mathcal{S}[\hat{E}+\epsilon\delta E, \hat{P}]\bigg|_{\epsilon=0} = -\int \left(\frac{1}{(2\pi\omega)^2}\frac{\partial^2 \hat{E}}{\partial x^2} + \hat{E} + \hat{P}\right)\delta E dx = 0$$

for all δE, hence

$$\frac{1}{(2\pi)^2}\frac{\partial^2 \hat{E}}{\partial x^2} = -\omega^2(\hat{E}+\hat{P}) \tag{3.34}$$

as required.

Varying with respect to $\hat{P}$ (away from the boundary)

$$\frac{d}{d\epsilon}\int \mathcal{L}\left(\hat{E}, \frac{\partial \hat{E}}{\partial x}, \hat{P}+\epsilon\delta P, \frac{\partial}{\partial x}\left(\hat{P}+\epsilon\delta P\right)\right) dx\bigg|_{\epsilon=0} = 0$$

and denote by $\zeta(\hat{E})$ terms that depend only on $\hat{E}$ and not $\hat{P}$,

$$\begin{aligned}
&\mathcal{L}\left(\hat{E}, \frac{\partial \hat{E}}{\partial x}, \hat{P}+\epsilon\delta P, \frac{\partial}{\partial x}\left(\hat{P}+\epsilon\delta P\right)\right)\\
&= \frac{1}{2}\left(\zeta(\hat{E}) - \frac{\beta(x)^2}{(2\pi)^2}\left(\frac{\partial \hat{P}}{\partial x}+\epsilon\frac{\partial(\delta P)}{\partial x}\right)^2 + L(\omega)\left(\hat{P}+\epsilon\delta P\right)^2\right) - \hat{E}(\hat{P}+\epsilon\delta P)\\
&= \frac{1}{2}\left(\zeta(\hat{E}) - \frac{\beta(x)^2}{(2\pi)^2}\left(\left(\frac{\partial \hat{P}}{\partial x}\right)^2 + 2\epsilon\frac{\partial \hat{P}}{\partial x}\frac{\partial(\delta P)}{\partial x}\right) + L(\omega)\left(\hat{P}^2+2\epsilon\hat{P}\delta P\right)\right)\\
&\qquad - \hat{E}(\hat{P}+\epsilon\delta P) + O(\epsilon^2).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{S}[\hat{E},\hat{P}+\epsilon\delta P]\bigg|_{\epsilon=0} &= \int -\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\frac{\partial(\delta P)}{\partial x} + L(\omega)\hat{P}\delta P - \hat{E}\delta P dx\\
&= \int \frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\right)\delta P - \frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\delta P\right) + L(\omega)\hat{P}\delta P - \hat{E}\delta P dx\\
&= -\int \frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\delta P\right) dx + \int\left(\frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\right) + L(\omega)\hat{P} - \hat{E}\right)\delta P dx.
\end{aligned}$$

The first integral in the above vanishes due to the compact support of δP leaving us with

$$\frac{d}{d\epsilon}\mathcal{S}[\hat{E},\hat{P}+\epsilon\delta P]\bigg|_{\epsilon=0} = \int\left(\frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\right) + L(\omega)\hat{P} - \hat{E}\right)\delta P dx$$

for all δP, hence

$$\frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\right) + L(\omega)\hat{P} = \hat{E} \tag{3.35}$$

as required.

3.3.1 Boundary Conditions

In order to find boundary conditions at the boundary $x = 0$ we again look at variations with respect to $\hat{E}$ and $\hat{P}$ but this time consider that δE and δP have support that includes $x = 0$. As mentioned earlier, our definition of the Lagrangian requires that

$$[\hat{E}] = 0 \qquad \text{and} \qquad [\hat{P}] = 0 \tag{3.36}$$

hence giving two boundary conditions immediately.

Varying (3.32) with respect to to $\hat{E}$

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{S}[\hat{E}+\epsilon\delta E,\hat{P}]\bigg|_{\epsilon=0} &= \int_{-\infty}^{\infty}\frac{1}{(2\pi\omega)^2}\frac{\partial \hat{E}}{\partial x}\frac{\partial(\delta E)}{\partial x} - \hat{E}\delta E - \hat{P}\delta E dx \\ &= \int_{-\infty}^{\infty}\frac{1}{(2\pi\omega)^2}\left(\frac{\partial}{\partial x}\left(\frac{\partial \hat{E}}{\partial x}\delta E\right) - \frac{\partial^2 \hat{E}}{\partial x^2}\delta E\right) - \hat{E}\delta E - \hat{P}\delta E dx\end{aligned}$$

using (3.34) we have that the last three terms in this integral vanish, thus we have

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{S}[\hat{E}+\epsilon\delta E,\hat{P}]\bigg|_{\epsilon=0} &= \int_{-\infty}^{\infty}\frac{1}{(2\pi\omega)^2}\frac{\partial}{\partial x}\left(\frac{\partial \hat{E}}{\partial x}\delta E\right)dx \\ &= \frac{1}{(2\pi\omega)^2}\left(\int_{-\infty}^{0}\frac{\partial}{\partial x}\left(\frac{\partial \hat{E}}{\partial x}\delta E\right)dx + \int_{0}^{\infty}\frac{\partial}{\partial x}\left(\frac{\partial \hat{E}}{\partial x}\delta E\right)dx\right) \\ &= \left[\frac{\partial \hat{E}}{\partial x}\right]\delta E\end{aligned}$$

for all δE therefore

$$\left[\frac{\partial \hat{E}}{\partial x}\right] = 0. \tag{3.37}$$

Now varying (3.32) with respect to $\hat{P}$, where the support of δP includes $x = 0$,

$$\frac{d}{d\epsilon}\mathcal{S}[\hat{E}, \hat{P} + \epsilon\delta P]\bigg|_{\epsilon=0} = \int_{-\infty}^{\infty} -\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\frac{\partial(\delta P)}{\partial x} + L(\omega)\hat{P}\delta P - \hat{E}\delta P dx$$
$$= -\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial \hat{P}}{\partial x}\delta P\right) + \left(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial^2 \hat{P}}{\partial x^2} + L(\omega)\hat{P} - \hat{E}\right)\delta P dx.$$

Using (3.35) gives that the second part of this integral is zero, leaving

$$\frac{d}{d\epsilon}\mathcal{S}[\hat{E}, \hat{P} + \epsilon\delta P]\bigg|_{\epsilon=0} = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\beta(x)^2\frac{\partial \hat{P}}{\partial x}\delta P\right) dx$$
$$= -\left[\beta(x)^2\frac{\partial \hat{P}}{\partial x}\right]\delta P$$

for all δP, hence

$$\left[\beta(x)^2\frac{\partial \hat{P}}{\partial x}\right] = 0. \tag{3.38}$$

We have now arrived at a complete set of boundary conditions with (3.36), (3.37), (3.38).

3.4 Limiting Case

In the limit where $\beta_\mu \to 0$ that region becomes only temporally dispersive, and is no longer spatially dispersive. In the case of a boundary between a purely temporally dispersive region and a region which also has spatial dispersion the corresponding boundary conditions are those given by Pekar, which are

$$\left[\hat{E}\right] = 0, \qquad \left[\frac{\partial \hat{E}}{\partial x}\right] = 0, \qquad \left[\hat{P}\right] = 0. \tag{3.39}$$

Observe, from (3.2), that $\beta_\mu \to 0$ is a singular limit because the highest order derivative vanishes. As such, care must be taken when taking this limit.

Lemma 3.4.1. *Given the conditions in* (3.39) *and the electric and polarisation fields,*

$\hat{E}$ and $\hat{P}$, are bounded for all x, then we can choose the coefficients $\{A_\mu^\pm, B_\mu^\pm\}$ such that

$$\lim_{\beta_{\mathbf{L}}\to 0}\lim_{x\to 0^-}\beta_{\mathbf{L}}^2\frac{\partial\hat{P}(\omega,x)}{\partial x}=\lim_{x\to 0^+}\beta_{\mathbf{R}}^2\frac{\partial\hat{P}(\omega,x)}{\partial x} \tag{3.40}$$

even though in general

$$\lim_{x\to 0^-}\lim_{\beta_{\mathbf{L}}\to 0}\beta_{\mathbf{L}}^2\frac{\partial\hat{P}(\omega,x)}{\partial x}\neq\lim_{x\to 0^+}\beta_{\mathbf{R}}^2\frac{\partial\hat{P}(\omega,x)}{\partial x}. \tag{3.41}$$

Proof. For this proof redefine L_μ so that it no longer has a dependency on β_μ, that is

$$\mathbb{L}_\mu=\beta_\mu L_\mu. \tag{3.42}$$

The solutions to the dispersion relation, (3.13), are then

$$k_\mu^\pm=\frac{1}{\sqrt{2}\beta_\mu}\sqrt{\mathbb{L}_\mu^2+\frac{\beta_\mu^2\omega^2}{c^2}\pm\sqrt{\left(\mathbb{L}_\mu^2-\frac{\beta_\mu^2\omega^2}{c^2}\right)^2+4\frac{\beta_\mu^2\omega^2}{c^2}}}. \tag{3.43}$$

Observe that there are two solutions which diverge and two which converge as $\beta_\mu\to 0$. Choose the branch structure of the inner square root in (3.43) so that the expansion of $k_\mu^\pm$ gives

$$k_{\mathbf{L}}^+\sim\frac{\mathbb{L}_{\mathbf{L}}}{\beta_{\mathbf{L}}}\qquad\text{and}\qquad k_{\mathbf{L}}^-\sim\frac{\omega}{c}\sqrt{1-\frac{1}{\mathbb{L}_{\mathbf{L}}^2}} \tag{3.44}$$

and to higher order

$$\mathbb{L}_{\mathbf{L}}^2-(k_{\mathbf{L}}^+)^2\beta_{\mathbf{L}}^2\sim-\frac{\omega^2}{c^2\mathbb{L}_{\mathbf{L}}}\beta_{\mathbf{L}}^2\qquad\text{and}\qquad\mathbb{L}_{\mathbf{L}}^2-(k_{\mathbf{L}}^-)^2\beta_{\mathbf{L}}^2\sim\mathbb{L}_{\mathbf{L}}^2. \tag{3.45}$$

From (3.15) this gives

$$\hat{E}(\omega,x)\approx A_{\mathbf{L}}^+e^{2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}}+A_{\mathbf{L}}^-e^{-2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}}+B_{\mathbf{L}}^+e^{2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}}+B_{\mathbf{L}}^-e^{-2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}}. \tag{3.46}$$

Although the coefficients $\{A_{\mathbf{L}}^{\pm}, B_{\mathbf{L}}^{\pm}\}$ are constant with respect to x, in general they will depend on $\beta_{\mathbf{L}}$. Since $\mathrm{Im}(\mathbb{L}_{\mathbf{L}}) < 0$ then for $x < 0$ we have $\mathrm{Re}(i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}) < 0$ hence $e^{2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}} \to 0$ and $\left|e^{-2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}}\right| \to \infty$ as $\beta_{\mathbf{L}} \to 0$. Hence for physical solutions $A_{\mathbf{L}}^{-} \to 0$. Additionally, since we require $A_{\mathbf{L}}^{-}e^{-2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}}$ to be bounded for all $x < 0$ as $\beta_{\mathbf{L}} \to 0$ then we must have $A_{\mathbf{L}}^{-}e^{-2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}} \to 0$ for all $x < 0$ and hence $A_{\mathbf{L}}^{-} = 0$. Alternatively, in order to ensure $A_{\mathbf{L}}^{+}e^{2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}}$ is bounded in the $\beta_{\mathbf{L}} \to 0$ limit, for all $x < 0$, we again require $A_{\mathbf{L}}^{+}e^{2\pi i\mathbb{L}_{\mathbf{L}}x/\beta_{\mathbf{L}}} \to 0$. This, however, does not determine the particular value of $A_{\mathbf{L}}^{+}$.

Thus, in the limit $\beta_{\mathbf{L}} \to 0$, (3.46) becomes

$$\hat{E}(\omega, x)\Big|_{\beta_{\mathbf{L}}=0} = B_{\mathbf{L}}^{+}e^{2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}} + B_{\mathbf{L}}^{-}e^{-2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}}$$

and so

$$\hat{P}(\omega, x)\Big|_{\beta_{\mathbf{L}}=0} = \frac{B_{\mathbf{L}}^{+}e^{2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}} + B_{\mathbf{L}}^{-}e^{-2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}}}{\mathbb{L}_{\mathbf{L}}^2}.$$

Taking the derivative of this gives

$$\frac{\partial\hat{P}(\omega, x)}{\partial x}\Bigg|_{\beta_{\mathbf{L}}=0} = \frac{2\pi i\omega}{c\mathbb{L}_{\mathbf{L}}^2}\sqrt{1-\frac{1}{\mathbb{L}_{\mathbf{L}}^2}}\left(B_{\mathbf{L}}^{+}e^{2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}} - B_{\mathbf{L}}^{-}e^{-2\pi i\omega x/c\sqrt{1-1/\mathbb{L}_{\mathbf{L}}^2}}\right)$$

hence

$$\lim_{\beta_{\mathbf{L}}\to 0}\beta_{\mathbf{L}}^2\frac{\partial\hat{P}(\omega, x)}{\partial x} = 0. \tag{3.47}$$

This gives the left hand side of (3.41), however the right hand side is still

$$\lim_{x\to 0^+}\beta_{\mathbf{R}}^2\frac{\partial\hat{P}(\omega, x)}{\partial x} = \frac{2\pi i\beta_{\mathbf{R}}^2 k_{\mathbf{R}}^{+}(A_{\mathbf{R}}^{+} - A_{\mathbf{R}}^{-})}{\mathbb{L}_{\mathbf{R}}^2 + (k_{\mathbf{R}}^{+})^2\beta_{\mathbf{R}}^2} + \frac{2\pi i\beta_{\mathbf{R}}^2 k_{\mathbf{R}}^{-}(B_{\mathbf{R}}^{+} - B_{\mathbf{R}}^{-})}{\mathbb{L}_{\mathbf{R}}^2 + (k_{\mathbf{R}}^{-})^2\beta_{\mathbf{R}}^2}. \tag{3.48}$$

Clearly, in general, (3.47) and (3.48) are not equal hence (3.41).

To show (3.40), look at the limit as $x \to 0^-$,

$$
\begin{aligned}
&\lim_{x\to 0^-} \beta_{\mathbf{L}}^2 \frac{\partial \hat{P}(\omega, x)}{\partial x} \\
&= \lim_{x\to 0^-} 2\pi i \beta_{\mathbf{L}}^2 \left(\frac{k_{\mathbf{L}}^+ A_{\mathbf{L}}^+ e^{2\pi i k_{\mathbf{L}}^+ x} - k_{\mathbf{L}}^+ A_{\mathbf{L}}^- e^{-2\pi i k_{\mathbf{L}}^+ x}}{\mathbb{L}_{\mathbf{L}}^2 + (k_{\mathbf{L}}^+)^2 \beta_{\mathbf{L}}^2} + \frac{k_{\mathbf{L}}^- B_{\mathbf{L}}^+ e^{2\pi i k_{\mathbf{L}}^- x} - k_{\mathbf{L}}^- B_{\mathbf{L}}^- e^{2\pi i k_{\mathbf{L}}^- x}}{\mathbb{L}_{\mathbf{L}}^2 + (k_{\mathbf{L}}^-)^2 \beta_{\mathbf{L}}^2} \right) \\
&= 2\pi i \beta_{\mathbf{L}}^2 \left(\frac{k_{\mathbf{L}}^+ (A_{\mathbf{L}}^+ - A_{\mathbf{L}}^-)}{\mathbb{L}_{\mathbf{L}}^2 + (k_{\mathbf{L}}^+)^2 \beta_{\mathbf{L}}^2} + \frac{k_{\mathbf{L}}^- (B_{\mathbf{L}}^+ - B_{\mathbf{L}}^-)}{\mathbb{L}_{\mathbf{L}}^2 + (k_{\mathbf{L}}^-)^2 \beta_{\mathbf{L}}^2} \right)
\end{aligned}
$$

and so, using the expansions (3.44) and (3.45), to order $O(\beta_{\mathbf{L}}^0)$ we have

$$
\begin{aligned}
\lim_{x\to 0^-} \beta_{\mathbf{L}}^2 \frac{\partial \hat{P}(\omega, x)}{\partial x} &= 2\pi i \beta_{\mathbf{L}}^2 \left(\frac{\mathbb{L}_{\mathbf{L}}^2 c^2 (A_{\mathbf{L}}^- - A_{\mathbf{L}}^+)}{\omega^2 \beta_{\mathbf{L}}^3} + \frac{\omega \sqrt{1 - 1/\mathbb{L}_{\mathbf{L}}^2} (B_{\mathbf{L}}^+ - B_{\mathbf{L}}^-)}{c \mathbb{L}_{\mathbf{L}}^2} \right) + O(\beta_{\mathbf{L}}^0) \\
&= -\frac{2\pi i \mathbb{L}_{\mathbf{L}}^2 c^2 A_{\mathbf{L}}^+}{\omega^2 \beta_{\mathbf{L}}} + O(\beta_{\mathbf{L}}^0)
\end{aligned}
$$

since $A_{\mathbf{L}}^- = 0$. Hence (3.40) holds if we set

$$
A_{\mathbf{L}}^+ = -\frac{\omega^2 \beta_{\mathbf{L}}}{\mathbb{L}_{\mathbf{L}}^2 c^2} \beta_{\mathbf{R}}^2 \left(\frac{k_{\mathbf{R}}^+ (A_{\mathbf{R}}^+ - A_{\mathbf{R}}^-)}{\mathbb{L}_{\mathbf{R}}^2 + (k_{\mathbf{R}}^+)^2 \beta_{\mathbf{R}}^2} + \frac{k_{\mathbf{R}}^- (B_{\mathbf{R}}^+ - B_{\mathbf{R}}^-)}{\mathbb{L}_{\mathbf{R}}^2 + (k_{\mathbf{R}}^-)^2 \beta_{\mathbf{R}}^2} \right).
$$

□

The result of this lemma is that in the general case, as given by (3.41), there are only three remaining boundary conditions, given by (3.39), which correspond to those given by Pekar. In the particular limiting case which results in (3.40) the full set of four boundary conditions remain.

3.5 Conclusion

In this chapter we have derived a complete set of boundary conditions for media composed of two semi-infinite, spatially dispersive, homogeneous regions. Two methods were presented for determining these conditions, from the response function and by varying a Lagrangian. Both of these methods produced the same results which are consistent with those found previously [37], however the approaches employed here are believed to be original.

Additionally it was shown that these boundary conditions reduce to the well known Pekar conditions in the limiting case where one of the regions is no longer spatially dispersive. However, we see that it is possible to take this limit in such a way that the boundary conditions for two spatially dispersive regions are still applicable. Hence, while our result is consistent with the Pekar boundary conditions, they also offer additional constraints for the boundary between a region which is both spatially and temporally dispersive and one which is purely temporally dispersive.

Chapter 4

Spatially Dispersive Periodic Media

4.1 Introduction

Spatial dispersion in periodically structured media is a topic of increasing interest as many metamaterials are significantly affected by nonlocal effects [4, 11, 13, 19]. The wire medium is another key example of a periodic material which exhibits strong spatially dispersive effects [2, 3, 9].

However, the electromagnetic behaviour of such systems are still not fully understood. Most numerical studies of these media consider the bulk permittivity and permeability, $\varepsilon_{\text{Bulk}}$ and μ_{Bulk}, to be spatially dispersive. However, the permittivity and permeability of the materials used to construct the unit cell are considered to be temporally dispersive only. This thesis investigates a system where the material within the unit cell is both spatially and temporally dispersive, i.e. $\varepsilon(\omega, k)$ and $\mu(\omega, k)$, which has been the subject of very little theoretical research.

First, we write the permittivity relation as the partial differential equation

$$L(\omega,x)\hat{P}(\omega,x)+\frac{\partial}{\partial x}\left(\frac{\beta(\omega,x)^2}{(2\pi)^2}\frac{\partial\hat{P}(\omega,x)}{\partial x}\right)=\varepsilon_0\hat{E}(\omega,x) \tag{4.1}$$

where $L(\omega,x)$ is the periodic function

$$L(\omega,x)=L_0(\omega)+2\Lambda(\omega)\cos\left(\frac{2\pi x}{a}\right), \tag{4.2}$$

with period a. The quantity Λ represents the magnitude of the inhomogeneity. This function has the Fourier transform

$$\widetilde{L}(\omega,k)=L_0(\omega)\delta(k)+\Lambda(\omega)\left(\delta\left(k-\frac{1}{a}\right)+\delta\left(k+\frac{1}{a}\right)\right). \tag{4.3}$$

For the most part, in the following work, the explicit ω dependence won't be written for $\widetilde{E}$, $\widetilde{P}$, L, and β. We will also only consider β to be constant with respect to x here.

4.2 The Difference Equation for Spatial Mode Amplitudes P_q

We can establish a difference equation for the spatial mode amplitudes, P_q, from the permittivity relation and Maxwell's equations. Taking the Fourier transform of (4.1) and (1.3) we have

$$\left(\frac{\omega^2}{c^2}-k^2\right)\widetilde{E}(k)=-\omega^2\mu_0\widetilde{P}(k)\quad\text{and}\quad -k^2\beta^2\widetilde{P}(k)+\left(\widetilde{L}*\widetilde{P}\right)(k)=\varepsilon_0\widetilde{E}(k) \tag{4.4}$$

where $(\widetilde{L} * \widetilde{P})(k)$, the Fourier transform of $L(x)P(x)$, is the convolution defined by

$$\left(\widetilde{L} * \widetilde{P}\right)(k) = \int_{-\infty}^{\infty} \widetilde{L}(k-k')\widetilde{P}(k')dk'. \tag{4.5}$$

In fact, the two equations in (4.4) can be combined into a single equation

$$\left(\widetilde{L} * \widetilde{P}\right)(k) = \left(\beta^2 k^2 - \frac{\omega^2}{\omega^2 - k^2c^2}\right)\widetilde{P}(k). \tag{4.6}$$

We can take the inverse Fourier transform of (4.6) with respect to k and see that, in (ω, x) space, this becomes

$$\begin{aligned} \frac{\beta^2}{(2\pi)^2}\frac{\partial^4 \hat{P}(x)}{\partial x^4} + \left(L(x) + \frac{\beta^2\omega^2}{c^2}\right)\frac{\partial^2 \hat{P}(x)}{\partial x^2} + 2\frac{\partial L(x)}{\partial x}\frac{\partial \hat{P}(x)}{\partial x} \\ + \left(\frac{\partial^2 L(x)}{\partial x^2} + \frac{2\pi\omega^2}{c^2}L(x) + \frac{2\pi\omega^2}{c^2}\right)\hat{P}(x) = 0. \end{aligned} \tag{4.7}$$

Since $L(x)$ is periodic we see that (4.7) fits the requirements of Floquet's Theorem, thus we know to look for solutions of the form

$$\hat{P}(x) = \exp\left(\frac{2\pi i\kappa x}{a}\right)\mathcal{P}(x) \tag{4.8}$$

where $0 \leq \kappa < 1 \in \mathbb{R}$ is the phase (which must be real for the solutions to be bounded) and $\mathcal{P}(x)$ is a periodic function with period a, $\mathcal{P}(x) = \mathcal{P}(x+a)$. The Fourier series for $\mathcal{P}(x)$ is

$$\mathcal{P}(x) = \sum_{q=-\infty}^{\infty} P_q \exp\left(\frac{2\pi iqx}{a}\right) \tag{4.9}$$

and so, with (4.8),

$$\hat{P}(x) = \sum_{q=-\infty}^{\infty} P_q \exp\left(\frac{2\pi i(q+\kappa)x}{a}\right) \tag{4.10}$$

the Fourier transform of which gives a series of delta functions

$$\widetilde{P}(k) = \sum_{q=-\infty}^{\infty} P_q \, \delta \left(k - \frac{q+\kappa}{a} \right). \tag{4.11}$$

Lemma 4.2.1. *From* (4.6) *and* (4.11)*, it can be shown that the coefficients* P_q *satisfy the difference equation*

$$\Lambda P_{q-1} + f_q(\omega) P_q + \Lambda P_{q+1} = 0 \tag{4.12}$$

where

$$f_q(\omega) = L_0(\omega) - \frac{\beta(\omega)^2 (q+\kappa)^2}{a^2} + \frac{a^2 \omega^2}{a^2\omega^2 - c^2 (q+\kappa)^2}. \tag{4.13}$$

Proof. Substituting (4.3) and (4.11) into (4.6),

$$\begin{aligned}
&\left(\widetilde{L} * \widetilde{P}\right)(k) = \int_{-\infty}^{\infty} \widetilde{L}(k-k')\widetilde{P}(k')dk' \\
&= \int_{-\infty}^{\infty} \left(L_0 \delta(k-k') + \Lambda \delta \left(k - k' - \frac{1}{a} \right) + \Lambda \delta \left(k - k' + \frac{1}{a} \right) \right) \\
&\qquad \sum_{q=-\infty}^{\infty} P_q \delta \left(k' - \frac{q+\kappa}{a} \right) dk' \\
&= \sum_{q=-\infty}^{\infty} P_q \left(L_0 \delta \left(k - \frac{q+\kappa}{a} \right) + \Lambda \delta \left(k - \frac{q+\kappa+1}{a} \right) + \Lambda \delta \left(k - \frac{q+\kappa-1}{a} \right) \right).
\end{aligned}$$

Now set $q' = q - 1$ and $q'' = q + 1$,

$$\begin{aligned}
\left(\widetilde{L} * \widetilde{P}\right)(k) = \sum_{q=-\infty}^{\infty} P_q L_0 \delta \left(k - \frac{q+\kappa}{a} \right) &+ \sum_{q''=-\infty}^{\infty} P_{q''-1} \Lambda \delta \left(k - \frac{q''+\kappa}{a} \right) \\
&+ \sum_{q'=-\infty}^{\infty} P_{q'+1} \Lambda \delta \left(k - \frac{q'+\kappa}{a} \right)
\end{aligned}$$

and then relabelling q' and q'' with q gives

$$\left(\widetilde{L} * \widetilde{P}\right)(k) = \sum_{q=-\infty}^{\infty} \left(\Lambda P_{q-1} + L_0 P_q + \Lambda P_{q+1} \right) \delta \left(k - \frac{q+\kappa}{a} \right).$$

Equating this to the right hand side of (4.6),

$$\sum_{q=-\infty}^{\infty} (\Lambda P_{q-1} + L_0 P_q + \Lambda P_{q+1})\, \delta\left(k - \frac{q+\kappa}{a}\right) = \left(\beta^2 k^2 - \frac{\omega^2}{\omega^2 - c^2 k^2}\right) \widetilde{P}(k)$$
$$= \sum_{q=-\infty}^{\infty} \left(\beta^2 k^2 - \frac{\omega^2}{\omega^2 - c^2 k^2}\right) P_q \delta\left(k - \frac{q+\kappa}{a}\right)$$

hence

$$\Lambda P_{q-1} + \left(L_0 - \beta^2 K^2 + \frac{\omega^2}{\omega^2 - K^2 c^2}\right) P_q + \Lambda P_{q+1} = 0$$

where

$$K = \frac{q+\kappa}{a}.$$

□

It is worth noting that including higher order harmonics, for example $\cos(4\pi a^{-1} x)$, in (4.2) will result in (4.12) being replaced by a higher order difference equation. Similarly, allowing β to have a periodic dependence on x will also increase the order of the difference equation.

It would be trivial to solve (4.12) by choosing some arbitrary value for two consecutive P_q, say P_0 and P_1, and then using the difference equation to determine all remaining values. However, in general these would give divergent solutions, i.e. $|P_q| \to \infty$ as $q \to \pm\infty$. Such solutions would likely not give a continuous function for the polarisation and so would be non-physical. Therefore, to ensure our solutions are physical, we must impose the condition

$$|P_q| \to 0 \qquad \text{as} \qquad q \to \pm\infty. \tag{4.14}$$

We will solve (4.12) to find supported frequencies and the corresponding spatial modes P_q. In the following sections we show two approaches to finding these solutions. In §4.3 we use a numerical approximation scheme which is valid generally. In

§4.4 we find a set of analytic solutions which are valid for small magnitudes of the inhomogeneity ($\Lambda < L_0$) and consider a number of special cases for which different solutions are required.

4.3 Numerical Method

In this section we present a method for solving (4.12) for P_q numerically and give a number of solutions. Before we can find values for P_q it is necessary to determine the corresponding value of ω.

To begin, using the limiting condition (4.14) allows us to truncate the infinite set of P_q by setting $P_r = 0$ for $|r| > R$. Since we are now working with a finite set of values it is possible to write (4.12) as a matrix equation. For example, for $R = 2$

$$\begin{pmatrix} f_{-2}(\omega) & \Lambda & 0 & 0 & 0 \\ \Lambda & f_{-1}(\omega) & \Lambda & 0 & 0 \\ 0 & \Lambda & f_0(\omega) & \Lambda & 0 \\ 0 & 0 & \Lambda & f_1(\omega) & \Lambda \\ 0 & 0 & 0 & \Lambda & f_2(\omega) \end{pmatrix} \begin{pmatrix} P_{-2} \\ P_{-1} \\ P_0 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.15}$$

For general R we can write this as

$$M\vec{P} = \vec{0} \tag{4.16}$$

where M is a $(2R+1) \times (2R+1)$ matrix with $f_q(\omega)$, $q = -R \dots R$, on the leading diagonal and Λ on the adjacent diagonals. $\vec{P}$ is the vector of P_q where $q = -R \dots R$ and $\vec{0}$ is the corresponding zero vector.

In order to find non-trivial solutions for P_q we require that the matrix M is non-invertible. This requirement allows us to find values for ω for which solutions exist.

This is done by calculating the determinant and then setting

$$|M| = 0. \tag{4.17}$$

Taking a particular value of ω calculated through this method, this can be substituted back into (4.16) to calculate the corresponding values of P_q. We normalise our solution by setting $P_1 = 1$ and all other values are then found in relation to this [1]. However, (4.16) now gives us an overdetermined system and so it is necessary to reduce the order of this equation. This is done by removing the column from M which contains $f_1(\omega)$ as this corresponds to the already assigned P_1. For example, for $R = 2$

$$\mathcal{M} = \begin{pmatrix} f_{-2}(\omega) & \Lambda & 0 & 0 \\ \Lambda & f_{-1}(\omega) & \Lambda & 0 \\ 0 & \Lambda & f_0(\omega) & 0 \\ 0 & 0 & \Lambda & \Lambda \\ 0 & 0 & 0 & f_2(\omega) \end{pmatrix}. \tag{4.18}$$

The column which was removed is now treated as a vector m, i.e.

$$m = \begin{pmatrix} 0 \\ 0 \\ \Lambda \\ f_1(\omega) \\ \Lambda \end{pmatrix}. \tag{4.19}$$

It is simple to see that this approach can be applied to a higher, general R. Given this, the P_q are given by

$$V = -\left(\mathcal{M}^T\mathcal{M}\right)^{-1}\mathcal{M}^T m \tag{4.20}$$

[1] We choose to prescribe P_1, as opposed to some other P_q, because, in the final solution, it is seen that $P_{\pm 1}$ is the highest amplitude mode for the convergent case.

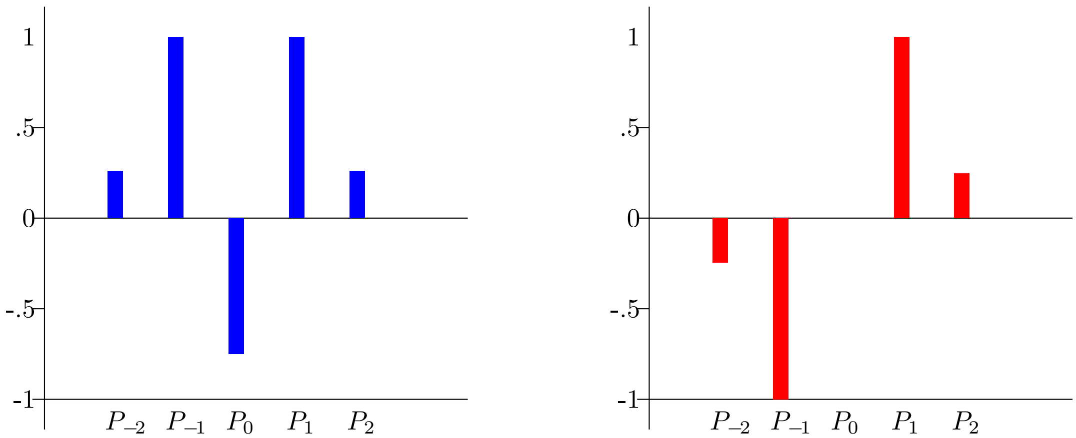

Figure 4.1: The P_q modes for $\omega = \pm 0.7610i$ (blue) and $\omega = \pm 0.3951$ (red) calculated numerically when $R = 2$. In this case we have set $a = 1$, $c = 1$, $\beta = 1$, $\Lambda = 0.75$, $L_0 = 1$, and $\kappa = 0$.

where

$$V = \begin{pmatrix} P_{-R} \\ \cdots \\ P_0 \\ P_2 \\ \cdots \\ P_R \end{pmatrix}.$$

4.3.1 Numerical Solutions

Exact solutions to (4.12) are given in the limit $R \to \infty$, however due to the nature of the matrix equation (4.20) the time taken to compute P_q rapidly becomes impractical as R increases. Here we present example solutions for $R = 2$ and $R = 5$, and use these to demonstrate the limiting behaviour for larger R. All the numerical results presented in this section are obtained by setting $a = 1$, $c = 1$, $\beta = 1$, $\Lambda = 0.75$, $L_0 = 1$, and $\kappa = 0$.

For $R = 2$ there are eight values of ω calculated using the above method. Note, however, that $f_q(\omega)$ contains only even powers of ω therefore we do not need to

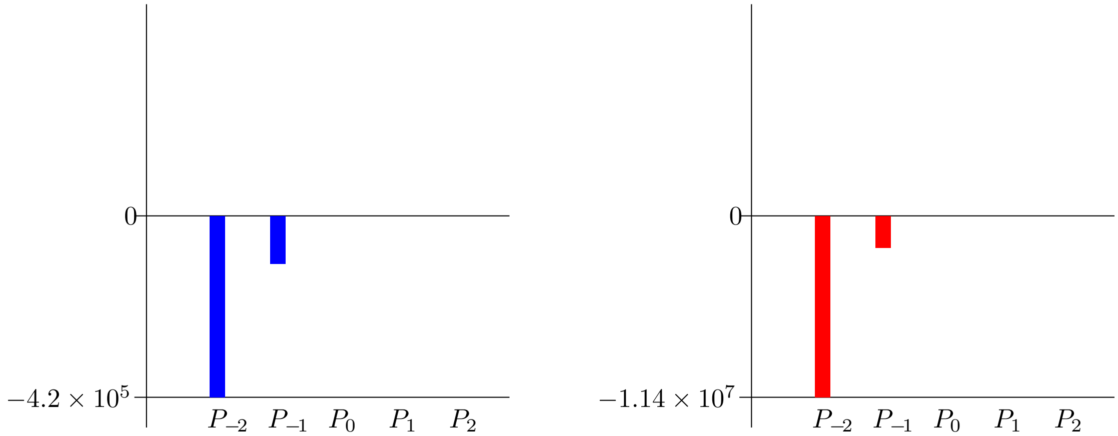

Figure 4.2: The P_q modes for $\omega = \pm 2.3249$ (blue) and $\omega = \pm 2.3716$ (red) calculated numerically when $R = 2$. In this case we have set $a = 1$, $c = 1$, $\beta = 1$, $\Lambda = 0.75$, $L_0 = 1$, and $\kappa = 0$.

consider separately ω solutions that vary only by sign. Hence this leaves us with four solutions for the frequency, given here to 4 decimal places

$$\omega \approx \pm 0.7610i, \quad \pm 0.3951, \quad \pm 2.3249, \quad \pm 2.3716.$$

The first two of these give modes which appear convergent, as shown in figure 4.1. The two remaining solutions appear to be potentially divergent, as seen is figure 4.2. Observe that, for these particular values, there are two values of ω which are purely imaginary. For these frequencies the solution will be damped and so will be unable to propagate through the medium.

For $R = 5$ we find, as would be expected, a larger number of values for the frequency given (to 4 decimal places) by

$$\omega \approx \pm 0.7532i \pm 0.3990, \ \pm 2.3311, \ \pm 2.3798, \ \pm 3.2151, \ \pm 3.2171, \ \pm 4.1416, \ \pm 5.1076.$$

We see that the first four of these are close to those calculated in the $R = 2$ case. In fact, these values approximately appear in the calculations for higher R as well and are convergent. Likewise, the additional solutions that appear also reoccur in

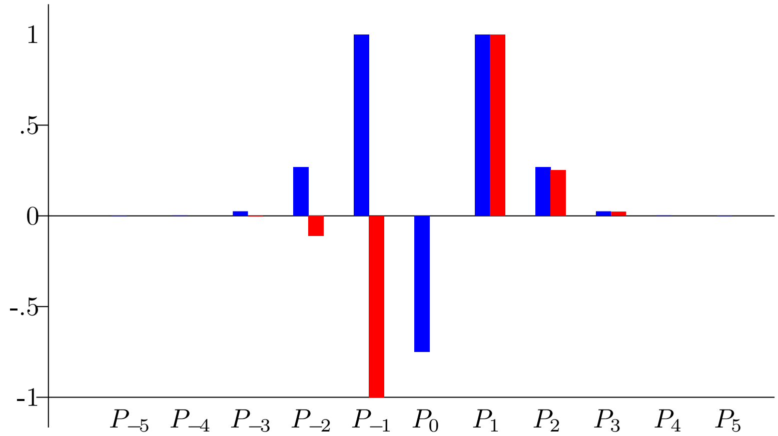

Figure 4.3: The P_q modes for $\omega = \pm 0.7532i$ (blue) and $\omega = \pm 0.3990$ (red) calculated numerically when $R = 5$. In this case we have set $a = 1$, $c = 1$, $\beta = 1$, $\Lambda = 0.75$, $L_0 = 1$, and $\kappa = 0$.

the higher order calculations. This is a pattern that continues as R increases, where additional values of ω appear, and then persist, for all higher R.

It becomes obvious at this order that many of the solutions are in fact divergent, as suspected at $R = 2$ with the modes given in figure 4.2. The only solutions that obey the condition (4.14) are those given by $\omega \approx \pm 0.7532i, \ \pm 0.3990$ which are shown in figure 4.3, which shows clearly the required convergence.

Having calculated these modes we then can insert them into a truncated approximation of the series in (4.10) to find an expression for $\hat{P}(\omega, x)$. Figure 4.4 shows a plot of this function using the values for ω and P_q calculated above which are displayed in figure 4.3.

4.4 Analytic Solutions

In this section we work under the assumption that the magnitude of the inhomogeneity is small, that is $\Lambda < L_0(\omega)$, and obtain approximate analytic solutions to (4.12) which are valid up to a given order in Λ. We use the notation $O(\Lambda^r)$ to indicate that the result will not be affected by the addition of correction terms of this, or higher, order

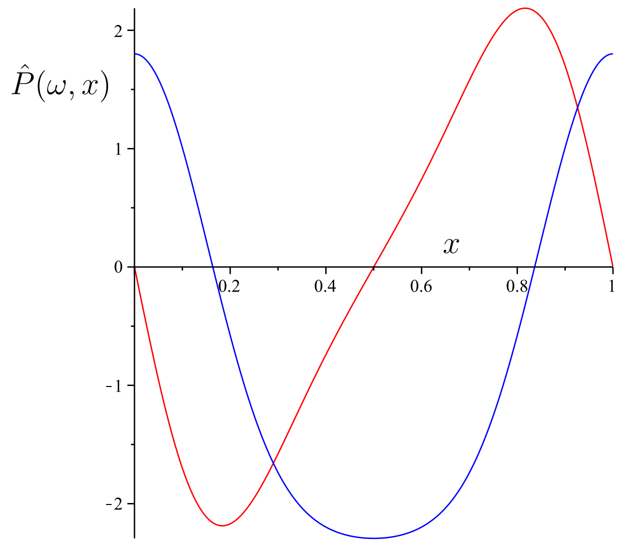

Figure 4.4: $\hat{P}(\omega, x)$ where $\omega = \pm 0.7532i$ (blue) and $\omega = \pm 0.3990$ (red). Here we have set $a = 1$, $c = 1$, $\beta = 1$, $\Lambda = 0.75$, $L_0 = 1$, and $\kappa = 0$.

in Λ. All solutions given here solve (4.12) to at least order $O(\Lambda^3)$.

Let Ω_n be a solution to

$$f_n(\Omega_n) = 0. \tag{4.21}$$

For each Ω_n there is a set of P_q that solve (4.12) to a particular order of Λ. We will use the notation P_q^n to identify the P_q with the corresponding Ω_n. Initially we consider *uncoupled modes* in which there are no other integers m such that $f_m(\Omega_n) = 0$. Cases where there exist $m \neq n \in \mathbb{Z}$ such that $f_n(\Omega_n) = f_m(\Omega_n) = 0$ give us *coupled modes*. The solutions for these coupled modes are dependent on the difference between the integers n and m and, when solving to $O(\Lambda^3)$, we see that we have four sets of solutions when $n - m \geq 4$, $n - m = 3$, $n - m = 2$, and $n - m = 1$. Due to the symmetry of f_q there is also a special case when $\kappa = 0$. These solutions will all be given in detail in section 4.5.

At this point we introduce notation that is useful in writing the solutions throughout this section,

$$\mathcal{F}_q = f_q(\Omega_n) \qquad \mathcal{F}'_q = \left.\frac{df_q(\omega)}{d\omega}\right|_{\omega=\Omega_n} \qquad \mathcal{F}''_q = \left.\frac{d^2 f_q(\omega)}{d\omega^2}\right|_{\omega=\Omega_n}. \tag{4.22}$$

4.4.1 Uncoupled Modes

Here we present an analytic solution to (4.12) in the case where there is a unique integer n such that (4.21) is satisfied. Consider

$$\omega_n = \Omega_n + \frac{\Lambda^2}{\mathcal{F}'_n}\left(\frac{1}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}}\right) + O(\Lambda^4) \tag{4.23}$$

and

$$P_q^n = R_q^n \tag{4.24}$$

where R_q^n is defined as[2]

$$R_q^n = \begin{cases} \dfrac{(-\Lambda)^{|q-n|}}{\prod_{k=1}^{|q-n|}\mathcal{F}_{n+k}} + O(\Lambda^{|q-n|+2}) & q > n \\ 1 & q = n \\ \dfrac{(-\Lambda)^{|q-n|}}{\prod_{k=1}^{|q-n|}\mathcal{F}_{n-k}} + O(\Lambda^{|q-n|+2}) & \text{otherwise.} \end{cases} \tag{4.25}$$

Theorem 4.4.1. *The frequency ω_n and modes P_q^n given by* (4.23) *and* (4.24) *solve the difference equation* (4.12) *to an order of Λ which depends on n and q as follows*

$$\Lambda P_{q-1}^n + f_q(\omega_n)P_q^n + \Lambda P_{q+1}^n = O\left(\Lambda^{|q-n|+2}\right) \tag{4.26}$$

and

$$\Lambda P_{n-1}^n + f_n(\omega_n)P_n^n + \Lambda P_{n+2}^n = O\left(\Lambda^4\right). \tag{4.27}$$

[2]Note we define R_q^n here as they are useful when writing the solutions to the special cases later in this chapter.

Proof. We will first show (4.27). Take the Taylor expansion of $f_n(\omega_n)$ around Ω_n

$$f_n(\omega_n) = \mathcal{F}_n + (\omega_n - \Omega_n)\mathcal{F}'_n + \frac{1}{2}(\omega_n - \Omega_n)^2\mathcal{F}''_n + \dots \tag{4.28}$$

using (4.23) and (4.21) gives

$$f_n(\omega_n) = \Lambda^2\left(\frac{1}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}}\right) + O(\Lambda^4).$$

Taking this along with (4.24) then the left hand side of (4.27) becomes

$$\begin{aligned}
&\Lambda P^n_{n-1} + f_n(\omega_n)P^n_n + \Lambda P^n_{n+1} \\
&\qquad = \Lambda\left(\frac{-\Lambda}{\mathcal{F}_{n-1}} + O(\Lambda^3)\right) + \Lambda^2\left(\frac{1}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}}\right) + O(\Lambda^4) \\
&\qquad\quad + \Lambda\left(\frac{-\Lambda}{\mathcal{F}_{n+1}} + O(\Lambda^3)\right) \\
&\qquad = -\Lambda^2\left(\frac{1}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}}\right) + \Lambda^2\left(\frac{1}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}}\right) + O(\Lambda^4) \\
&\qquad = O(\Lambda^4)
\end{aligned}$$

as required.

To show (4.26) there are four cases that need to be considered: $q = n+1$, $q = n-1$, $q > n+1$, $q < n-1$. Note that for $q \neq n$ it is enough to use the expansion $f_q(\omega_n) = \mathcal{F}_q + O(\Lambda^2)$. Firstly, for $q = n+1$

$$\begin{aligned}
&\Lambda P^n_n + f_{n+1}(\omega_n)P^n_{n+1} + \Lambda P^n_{n+2} \\
&\qquad = \Lambda - \mathcal{F}_{n+1}\frac{\Lambda}{\mathcal{F}_{n+1}} + O(\Lambda)^3 + \Lambda\left(\frac{\Lambda^2}{\mathcal{F}_{n+1}\mathcal{F}_{n+2}} + O(\Lambda^4)\right) \\
&\qquad = \Lambda - \Lambda + O(\Lambda^3) = O(\Lambda^3)
\end{aligned}$$

as required. Next look at $q = n-1$,

$$
\begin{aligned}
&\Lambda P^n_{n-2} + f_{n-1}(\omega_n) P^n_{n-1} + \Lambda P^n_n \\
&\qquad = \Lambda \left(\frac{\Lambda^2}{\mathcal{F}_{n-1}\mathcal{F}_{n-2}} + O(\Lambda^4) \right) - \mathcal{F}_{n-1} \frac{\Lambda}{\mathcal{F}_{n-1}} + O(\Lambda)^3 + \Lambda \\
&\qquad = -\Lambda + \Lambda + O(\Lambda^3) = O(\Lambda^3).
\end{aligned}
$$

Thirdly, take $q > n+1$

$$
\begin{aligned}
&\Lambda P^n_{q-1} + f_q(\omega_n) P^n_q + \Lambda P^n_{q+1} \\
&= \Lambda \left(\frac{(-\Lambda)^{q-1-n}}{\prod_{k=1}^{q-1-n} \mathcal{F}_{n+k}} + O(\Lambda^{q-n+1}) \right) + f_q(\omega_n) \left(\frac{(-\Lambda^{q-n})}{\prod_{k=1}^{q-n} \mathcal{F}_{n+k}} + O(\Lambda^{q-n+2}) \right) \\
&\quad + \Lambda \left(\frac{(-\Lambda^{q+1-n})}{\prod_{k=1}^{q+1-n} \mathcal{F}_{n+k}} + O(\Lambda^{q-n+3}) \right) \\
&= \Lambda \frac{(-\Lambda)^{q-1-n}}{\prod_{k=1}^{q-1-n} \mathcal{F}_{n+k}} + \mathcal{F}_q \frac{(-\Lambda^{q-n})}{\prod_{k=1}^{q-n} \mathcal{F}_{n+k}} + O(\Lambda^{q-n+2}) \\
&= \Lambda \frac{(-\Lambda)^{q-1-n}}{\prod_{k=1}^{q-1-n} \mathcal{F}_{n+k}} + \mathcal{F}_q \frac{-\Lambda(-\Lambda^{q-n-1})}{\mathcal{F}_q \prod_{k=1}^{q-n-1} \mathcal{F}_{n+k}} + O(\Lambda^{q-n+2}) \\
&= (\Lambda - \Lambda) \frac{(-\Lambda)^{q-1-n}}{\prod_{k=1}^{q-1-n} \mathcal{F}_{n+k}} + O(\Lambda^{q-n+2}) \\
&= O(\Lambda^{q-n+2})
\end{aligned}
$$

as desired. Lastly, look at $q < n-1$

$$
\begin{aligned}
&\Lambda P_{q-1}^n + f_q(\omega_n) P_q^n + \Lambda P_{q+1}^n \\
&= \Lambda\left(\frac{(-\Lambda)^{n-q+1}}{\prod_{k=1}^{n-q+1}\mathcal{F}_{n-k}} + O(\Lambda^{n-q+3})\right) + f_q(\omega_n)\left(\frac{(-\Lambda^{n-q})}{\prod_{k=1}^{n-q}\mathcal{F}_{n-k}} + O(\Lambda^{n-q+2})\right) \\
&\qquad + \Lambda\left(\frac{(-\Lambda^{n-q-1})}{\prod_{k=1}^{n-q-1}\mathcal{F}_{n-k}} + O(\Lambda^{n-q+1})\right) \\
&= \mathcal{F}_q \frac{(-\Lambda^{n-q})}{\prod_{k=1}^{n-q}\mathcal{F}_{n-k}} + \Lambda \frac{(-\Lambda)^{n-q-1}}{\prod_{k=1}^{n-q-1}\mathcal{F}_{n-k}} + O(\Lambda^{n-q+2}) \\
&= \mathcal{F}_q \frac{-\Lambda(-\Lambda^{n-q-1})}{\mathcal{F}_q\prod_{k=1}^{n-q-1}\mathcal{F}_{n-k}} + \Lambda \frac{(-\Lambda)^{n-q-1}}{\prod_{k=1}^{n-q-1}\mathcal{F}_{n-k}} + O(\Lambda^{n-q+2}) \\
&= (-\Lambda + \Lambda)\frac{(-\Lambda)^{n-q-1}}{\prod_{k=1}^{n-q-1}\mathcal{F}_{n-k}} + O(\Lambda^{n-q+2}) \\
&= O(\Lambda^{|q-n|+2})
\end{aligned}
$$

which completes the proof. □

More accurate solutions could, as in standard perturbation theory, be obtained by including higher order corrections in (4.23) and (4.24). However, since we are considering only the approximation where $\Lambda < L_0$ it is sufficient to work to order $O(\Lambda^3)$.

Observe that the equations relating ω_n and κ, (4.23) and (4.21), can be considered as a dispersion relation. This dispersion relation is shown in figure 4.5 for a single pole resonance,

$$L_0(\omega) = \frac{(\omega - i\lambda)^2 + \omega_p^2}{\omega_p^2},$$

where ω_p is the natural frequency. Notice that the dispersion relation appears to "wrap around" when $\kappa = 1$. Looking at (4.13) we see that the only dependence of n and κ in $f_n(\omega)$ is in the terms $(n+\kappa)^2$. Hence the function takes the same value at, for example, $(n = 0, \kappa = 1)$ and $(n = 1, \kappa = 0)$, thus the observed shape.

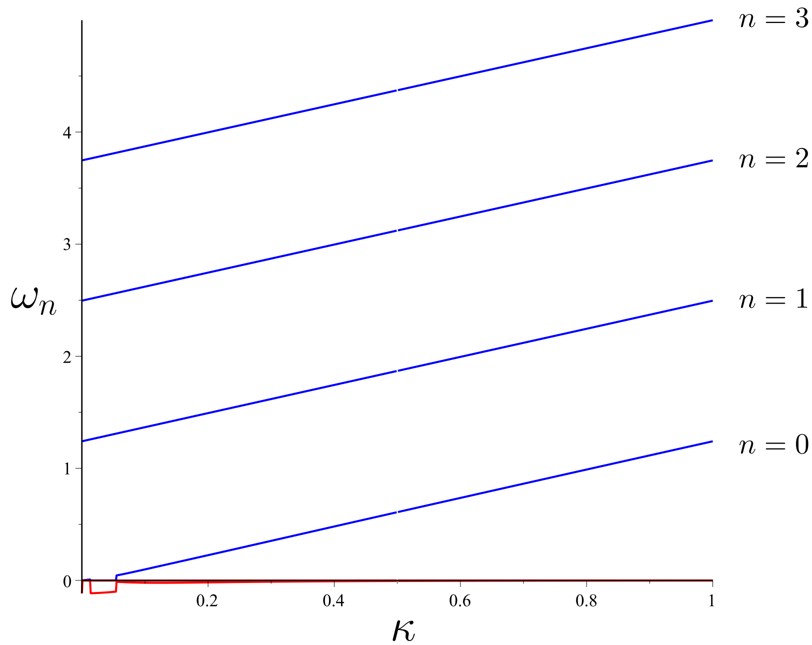

Figure 4.5: The real (blue) and imaginary (red) parts of the dispersion relation given by (4.23) and (4.21) for a single pole resonance, $L_0(\omega) = \dfrac{(\omega - i\lambda)^2 + \omega_p^2}{\omega_p^2}$. Here we have set $a = 0.8$, $\beta = 0.4$, $\omega_p = 0.15$, $c = 1$, $\lambda = 0.1$, and $\Lambda = 0.2$.

4.5 Coupled Modes

In this section consider that there are two distinct integers $n \neq m \in \mathbb{Z}$ and frequency Ω such that

$$f_n(\Omega) = f_m(\Omega) = 0. \tag{4.29}$$

Returning to the definition of f_q, (4.13), we can write this as a quadratic for $(q+\kappa)^2$ when $f_q(\Omega) = 0$,

$$(q+\kappa)^4 - \left(\frac{a^2\Omega^2}{c^2} + \frac{L_0 a^2}{\beta^2}\right)(q+\kappa)^2 + \frac{a^4\Omega^2}{\beta^2 c^2}(L_0+1) = 0. \tag{4.30}$$

Clearly there are two solutions to (4.30),

$$(q+\kappa)^2 = Q^+(\Omega)^2 \qquad \text{and} \qquad (q+\kappa)^2 = Q^-(\Omega)^2 \tag{4.31}$$

where

$$Q^{\pm}(\Omega)^2 = \frac{a^2}{2}\left(\frac{\Omega^2}{c^2} + \frac{L_0}{\beta^2} \pm \sqrt{\left(\frac{\Omega^2}{c^2} + \frac{L_0}{\beta^2}\right)^2 - \frac{4\Omega^2(L_0+1)}{\beta^2 c^2}}\right). \tag{4.32}$$

Given this, there are two possibilities that may occur. Firstly, both n and m correspond to the same root, i.e.

$$(n+\kappa)^2 = (m+\kappa)^2 = Q^+(\Omega)^2 \qquad \text{or} \qquad (n+\kappa)^2 = (m+\kappa)^2 = Q^-(\Omega)^2$$

so

$$\begin{aligned}(n+\kappa)^2 &= (m+\kappa)^2 \\ n^2 + 2n\kappa + \kappa^2 &= m^2 + 2m\kappa + \kappa^2 \\ -2\kappa(n-m) &= n^2 - m^2 \\ 2\kappa &= -n - m\end{aligned}$$

thus, since $n, m \in \mathbb{Z}$, we must have $\kappa = 0$ or $\kappa = 1/2$. We see that, for $\kappa = 0$, the above gives $m = -n$. Because of this symmetry this case has special solutions, which are given in section 4.5.1. For $\kappa = 1/2$ there is no significant improvement in the solutions and so this case is not considered separately. The solutions when $\kappa = 1/2$ can be found as a special case of the solutions for general κ calculated in section 4.5.2

Alternatively, n and m correspond to different roots

$$(n+\kappa)^2 = Q^+(\Omega)^2 \qquad \text{and} \qquad (m+\kappa)^2 = Q^-(\Omega)^2$$

hence

$$\kappa = -n \pm Q^+(\Omega) \qquad \text{and} \qquad \kappa = -m \pm Q^-(\Omega).$$

Eliminating κ we obtain the conditions

$$Q^-(\Omega) + Q^+(\Omega) \in \mathbb{Z} \qquad \text{or} \qquad Q^-(\Omega) - Q^+(\Omega) \in \mathbb{Z}. \tag{4.33}$$

Typically $Q^+(\Omega)$ and $Q^-(\Omega)$ will be continuous functions and so, in general, we can expect to see coupled modes for most ω.

4.5.1 Special Case: $\kappa = 0$

Now we will look at the particular situation in which $\kappa = 0$. Immediately we see from (4.13) that $f_n(\omega) = f_{-n}(\omega)$. Hence, given Ω_n as before, we have

$$\mathcal{F}_n = \mathcal{F}_{-n} = 0.$$

Because of this we must modify the solution given in §4.4.1. Note that if $n = 0$ then, trivially, $n = -n$ and there is no need to modify the previous solution as there is still only a single $n \in \mathbb{Z}$ such that $\mathcal{F}_n = 0$, thus the general solution given above is valid. For $n \geq 2$ the frequency ω_n is given, as before, by (4.23). The polarisation modes, however, are given by

$$P_q^n = \begin{cases} R_q^n & q > -n \\ O(\Lambda^{2n}) & q = -n \\ O(\Lambda^{n-q-2}) & q < -n \end{cases} \tag{4.34}$$

where R_q^n is as defined in (4.25). It is simple to see that this solution satisfies the requirement (4.14) as needed when $\Lambda < 1$. Note also that for $q \leq -n$ we can only give P_q^n to an order of Λ however this is sufficient to solve (4.12) to the required accuracy.

We can obtain the solution for $n \leq -2$ by setting

$$P_q^n = P_{-q}^{-n}. \tag{4.35}$$

The proof of this is a trivial modification to that for the $n \geq 2$ solution. For $n = \pm 1$ it is necessary to specify a different solution in order to solve (4.12) to at least $O(\Lambda^3)$. This solution is given later in this chapter.

In order to improve the clarity of the notation in the following, introduce $\mathcal{Q}_q^n$ as the remainder of the difference equation (4.12),

$$\mathcal{Q}_q^n = \Lambda P_{q-1}^n + f_q(\omega_n) P_q^n + \Lambda P_{q+1}^n. \tag{4.36}$$

Theorem 4.5.1. *The frequency and polarisation given by* (4.23) *and* (4.34) *solve the difference equation* (4.12) *to an order of* Λ *which depends on* n *and* q *according to the following*

$$\mathcal{Q}_q^n = O\left(\Lambda^{|q-n|+2}\right) \qquad q > -n, q \neq n \tag{4.37}$$

$$\mathcal{Q}_q^n = O\left(\Lambda^4\right) \qquad q = n \tag{4.38}$$

$$\mathcal{Q}_q^n = O\left(\Lambda^{2n}\right) \qquad q = -n \tag{4.39}$$

$$\mathcal{Q}_q^n = O\left(\Lambda^{n-q-2}\right) \qquad q < -n. \tag{4.40}$$

Proof. For $q > -n + 1$ this is identical to the situation given in theorem 4.4.1, and so the proof of (4.37) and (4.38) is mostly given there. However, the case where $q = -n + 1$ must be considered separately, as follows

$$\begin{aligned}
\mathcal{Q}_{-n+1}^n &= \Lambda P_{-n}^n + f_{-n+1}(\omega_n) P_{-n+1}^n + \Lambda P_{-n+2}^n \\
&= \Lambda O(\Lambda^{2n}) + \mathcal{F}_{-n+1} R_{-n+1}^n + \Lambda R_{-n+2}^n \\
&= O(\Lambda^{2n+1}) + \mathcal{F}_{-n+1} \left(\frac{(-\Lambda)^{2n-1}}{\prod_{k=1}^{2n-1} \mathcal{F}_{n-k}} + O(\Lambda^{2n+1}) \right) + \Lambda \left(\frac{(-\Lambda)^{2n-2}}{\prod_{k=1}^{2n-2} \mathcal{F}_{n-k}} + O(\Lambda^{2n}) \right) \\
&= -\Lambda \frac{(-\Lambda)^{2n-2}}{\prod_{k=1}^{2n-2} \mathcal{F}_{n-k}} + \Lambda \frac{(-\Lambda)^{2n-2}}{\prod_{k=1}^{2n-2} \mathcal{F}_{n-k}} + O(\Lambda^{2n+1}) \\
&= O(\Lambda^{2n+1})
\end{aligned}$$

as required. Thus it remains to show the result for $q = -n$ and $q < -n$.

Taking $q = -n$ in (4.36) then substitution of (4.34) gives

$$
\begin{aligned}
\mathcal{Q}^n_{-n} &= \Lambda P^n_{-n-1} + f_{-n}(\omega_n) P^n_{-n} + \Lambda P^n_{-n+1} \\
&= \Lambda O(\Lambda^{2n-1}) + f_n(\omega_n) O(\Lambda^{2n}) + \Lambda \left(\frac{(-\Lambda)^{2n-1}}{\prod_{k=1}^{2n-1} \mathcal{F}_{n-k}} + O(\Lambda^{2n+1}) \right) \\
&= O(\Lambda^{2n}) + O(\Lambda^2) O(\Lambda^{2n}) + O(\Lambda^{2n}) + O(\Lambda^{2n+2}) \\
&= O(\Lambda^{2n})
\end{aligned}
$$

hence (4.39).

Due to the definition of P^n_q in (4.34) the proof of (4.40) requires the cases $q = -n-1$ and $q < -n-1$ to be considered separately. First take $q = -n-1$.

$$
\begin{aligned}
\mathcal{Q}^n_{-n-1} &= \Lambda P^n_{-n-2} + f_{-n-1}(\omega_n) P^n_{-n-1} + \Lambda P^n_{-n} \\
&= \Lambda O(\Lambda^{2n}) + \mathcal{F}_{-n-1} O(\Lambda^{2n-1}) + \Lambda O(\Lambda^{2n}) \\
&= O(\Lambda^{2n+1}) + O(\Lambda^{2n-1}) + O(\Lambda^{2n+1}) \\
&= O(\Lambda^{2n-1})
\end{aligned}
$$

as required. Finally, consider $q < -n-1$,

$$
\begin{aligned}
\mathcal{Q}^n_q &= \Lambda P^n_{q-1} + f_q(\omega_n) P^n_q + \Lambda P^n_{q+1} \\
&= \Lambda O(\Lambda^{n-q-1}) + \mathcal{F}_q O(\Lambda^{n-q-2}) + \Lambda O(\Lambda^{n-q-3}) \\
&= O(\Lambda^{n-q}) + O(\Lambda^{n-q-2}) + O(\Lambda^{n-q-2}) \\
&= O(\Lambda^{n-q-2})
\end{aligned}
$$

hence (4.40). □

Now consider $n = 1$. In this case there is a pair of independent modes which can alternatively be written as odd and even modes. The superscript **o** is used to label the odd mode and **e** for the even mode. The odd mode is given by (4.41) and (4.42) below, and the even mode by (4.67) and (4.68).

Looking first at the odd solution, the frequency is given by

$$\omega_1^{\mathbf{o}} = \Omega_1 + \frac{\Lambda^2}{\mathcal{F}_1'\mathcal{F}_2} + O(\Lambda^4) \tag{4.41}$$

and the corresponding P_q^n are given by

$$P_q^{1,\mathbf{o}} = \begin{cases} R_q^1 & q > 0 \\ 0 & q = 0 \\ -R_{-q}^1 & q < 0. \end{cases} \tag{4.42}$$

Theorem 4.5.2. *The frequency and polarisation given by* (4.41) *and* (4.42) *solve the difference equation* (4.12) *to an order of* Λ *as follows*

$$\mathcal{Q}_q^1 = O(\Lambda^{q+1}) \qquad q > 1 \tag{4.43}$$

$$\mathcal{Q}_q^1 = O(\Lambda^4) \qquad q = \pm 1 \tag{4.44}$$

$$\mathcal{Q}_q^1 = 0 \qquad q = 0 \tag{4.45}$$

$$\mathcal{Q}_q^1 = O(\Lambda^{-q+1}) \qquad q < 1. \tag{4.46}$$

Proof. There are five cases here that must be considered: $q = 0$, $q = 1$, $q = -1$, $q > 1$, and $q < -1$. For this proof we will use the expansion of $f_q(\omega_1^{\mathbf{o}})$ which is given by

$$f_q(\omega_1^{\mathbf{o}}) = \begin{cases} \mathcal{F}_q & q \neq \pm 1 \\ \dfrac{\Lambda^2}{\mathcal{F}_2} + O(\Lambda^4) & q = 1 \\ -\dfrac{\Lambda^2}{\mathcal{F}_2} + O(\Lambda^4) & q = -1. \end{cases}$$

First look at $q = 0$,

$$\begin{aligned}\mathcal{Q}_0^1 &= \Lambda P_{-1}^{1,\mathbf{o}} + f_0(\omega_1^{\mathbf{o}}) P_0^{1,\mathbf{o}} + \Lambda P_1^{1,\mathbf{o}} \\ &= -\Lambda R_1^1 + \Lambda R_1^1 \\ &= 0.\end{aligned}$$

Next consider $q = 1$,

$$\mathcal{Q}_1^1 = \Lambda P_0^{1,\mathbf{o}} + f_1(\omega_1^{\mathbf{o}}) P_1^{1,\mathbf{o}} + \Lambda P_2^{1,\mathbf{o}} \tag{4.47}$$

$$= f_1(\omega_1^{\mathbf{o}}) R_1^1 + \Lambda R_2^1 \tag{4.48}$$

$$= \frac{\Lambda^2}{\mathcal{F}_2} + O(\Lambda^4) + \Lambda \left(\frac{-\Lambda}{\mathcal{F}_2} + O(\Lambda^3) \right) \tag{4.49}$$

$$= O(\Lambda^4) \tag{4.50}$$

and similarly $q = -1$

$$\mathcal{Q}_{-1}^1 = \Lambda P_{-2}^{1,\mathbf{o}} + f_{-1}(\omega_1^{\mathbf{o}}) P_{-1}^{1,\mathbf{o}} + \Lambda P_0^{1,\mathbf{o}} \tag{4.51}$$

$$= -\Lambda R_2^1 - f_1(\omega_1^{\mathbf{o}}) R_1^1 \tag{4.52}$$

$$= -\Lambda \left(\frac{-\Lambda}{\mathcal{F}_2} + O(\Lambda^3) \right) - \frac{\Lambda^2}{\mathcal{F}_2} + O(\Lambda^4) \tag{4.53}$$

$$= O(\Lambda^4). \tag{4.54}$$

Finally, we have for $q > 1$

$$\begin{aligned}
\mathcal{Q}_q^1 &= \Lambda P_{q-1}^{1,\mathbf{o}} + f_q(\omega_1^{\mathbf{o}}) P_q^{1,\mathbf{o}} + \Lambda P_{q+1}^{1,\mathbf{o}} && (4.55)\\
&= \Lambda R_{q-1}^1 + \mathcal{F}_q R_q^1 + \Lambda R_{q+1}^1 && (4.56)\\
&= \Lambda \left(\frac{(-\Lambda)^{q-2}}{\prod_{k=1}^{q-2} \mathcal{F}_{1+k}} + O(\Lambda^q) \right) + \mathcal{F}_q \left(\frac{(-\Lambda)^{q-1}}{\prod_{k=1}^{q-1} \mathcal{F}_{1+k}} + O(\Lambda^{q+1}) \right) && (4.57)\\
&\quad + \Lambda \left(\frac{(-\Lambda)^{q}}{\prod_{k=1}^{q} \mathcal{F}_{1+k}} + O(\Lambda^{q+2}) \right) && (4.58)\\
&= \Lambda \frac{(-\Lambda)^{q-2}}{\prod_{k=1}^{q-2} \mathcal{F}_{1+k}} - \Lambda \frac{(-\Lambda)^{q-2}}{\prod_{k=1}^{q-2} \mathcal{F}_{1+k}} + O(\Lambda^{q+1}) && (4.59)\\
&= O(\Lambda^{q+1}) && (4.60)
\end{aligned}$$

and $q < -1$

$$\begin{aligned}
\mathcal{Q}_q^1 &= \Lambda P_{q-1}^{1,\mathbf{o}} + f_q(\omega_1^{\mathbf{o}}) P_q^{1,\mathbf{o}} + \Lambda P_{q+1}^{1,\mathbf{o}} && (4.61)\\
&= -\Lambda R_{-q+1}^1 - \mathcal{F}_q R_{-q}^1 - \Lambda R_{-q-1}^1 && (4.62)\\
&= -\Lambda \left(\frac{(-\Lambda)^{-q}}{\prod_{k=1}^{-q} \mathcal{F}_{1+k}} + O(\Lambda^{-q+2}) \right) - \mathcal{F}_q \frac{(-\Lambda)^{-q-1}}{\prod_{k=1}^{-q-1} \mathcal{F}_{1+k}} + O(\Lambda^{-q+1}) && (4.63)\\
&\quad - \Lambda \left(\frac{(-\Lambda)^{-q-2}}{\prod_{k=1}^{-q-2} \mathcal{F}_{1+k}} + O(\Lambda^{-q}) \right) && (4.64)\\
&= \Lambda \frac{(-\Lambda)^{-q-2}}{\prod_{k=1}^{-q-2} \mathcal{F}_{1+k}} - \Lambda \frac{(-\Lambda)^{-q-2}}{\prod_{k=1}^{-q-2} \mathcal{F}_{1+k}} + O(\Lambda^{-q+1}) && (4.65)\\
&= O(\Lambda^{-q+1}) && (4.66)
\end{aligned}$$

completing the proof. □

Now we present the even solution, with frequency given by

$$\omega_1^{\mathbf{e}} = \Omega_1 + \frac{\Lambda^2}{\mathcal{F}_1'} \left(\frac{1}{\mathcal{F}_2} + \frac{2}{\mathcal{F}_0} \right) + O(\Lambda^4) \tag{4.67}$$

and polarisation

$$P_q^{1,\mathbf{e}} = \begin{cases} R_q^1 & q > 0 \\ -\dfrac{2\Lambda}{\mathcal{F}_0} & q = 0 \\ R_{-q}^1 & q < 0. \end{cases} \tag{4.68}$$

Theorem 4.5.3. *The frequency and polarisation given by* (4.67) *and* (4.68) *solve the difference equation* (4.12) *to an order of* Λ *as follows*

$$\mathcal{Q}_q^1 = O(\Lambda^{q+1}) \qquad q > 1 \tag{4.69}$$

$$\mathcal{Q}_q^1 = O(\Lambda^4) \qquad q = \pm 1 \tag{4.70}$$

$$\mathcal{Q}_q^1 = O(\Lambda^3) \qquad q = 0 \tag{4.71}$$

$$\mathcal{Q}_q^1 = O(\Lambda^{-q+1}) \qquad q < 1. \tag{4.72}$$

Proof. First consider $q = 0$

$$\begin{aligned} \mathcal{Q}_0^1 &= \Lambda P_{-1}^{1,\mathbf{e}} + f_0(\omega_1^{\mathbf{e}}) P_0^{1,\mathbf{e}} + \Lambda P_1^{1,\mathbf{e}} \\ &= \Lambda R_1^1 - \left(\mathcal{F}_0 + O(\Lambda^2)\right) \frac{2\Lambda}{\mathcal{F}_0} + \Lambda R_1^1 \\ &= 2\Lambda - 2\Lambda + O(\Lambda^3) \\ &= O(\Lambda^3). \end{aligned}$$

Next we have $q = 1$

$$\begin{aligned} \mathcal{Q}_1^1 &= \Lambda P_0^{1,\mathbf{e}} + f_1(\omega_1^{\mathbf{e}}) P_1^{1,\mathbf{e}} + \Lambda P_2^{1,\mathbf{e}} \\ &= -\frac{2\Lambda^2}{\mathcal{F}_0} + \left(\Lambda^2 \left(\frac{1}{\mathcal{F}_2} + \frac{2}{\mathcal{F}_0}\right) + O(\Lambda^4)\right) R_1^1 + \Lambda R_2^1 \\ &= -\frac{2\Lambda^2}{\mathcal{F}_0} + \Lambda^2 \left(\frac{1}{\mathcal{F}_2} + \frac{2}{\mathcal{F}_0}\right) + O(\Lambda^4) + \Lambda \left(\frac{-\Lambda}{\mathcal{F}_2} + O(\Lambda^3)\right) \\ &= O(\Lambda^4) \end{aligned}$$

and $q = -1$

$$\begin{aligned}
\mathcal{Q}_1^1 &= \Lambda P_{-2}^{1,\mathbf{e}} + f_{-1}(\omega_1^{\mathbf{e}}) P_{-1}^{1,\mathbf{e}} + \Lambda P_0^{1,\mathbf{e}} \\
&= \Lambda R_2^1 + f_1(\omega_1^{\mathbf{e}}) R_1^1 - \frac{2\Lambda^2}{\mathcal{F}_0} \\
&= \Lambda \left(\frac{-\Lambda}{\mathcal{F}_2} + O(\Lambda^3) \right) + \Lambda^2 \left(\frac{1}{\mathcal{F}_2} + \frac{2}{\mathcal{F}_0} \right) + O(\Lambda^4) - \frac{2\Lambda^2}{\mathcal{F}_0} \\
&= O(\Lambda^4).
\end{aligned}$$

The remaining two cases are $q > 1$

$$\begin{aligned}
\mathcal{Q}_q^1 &= \Lambda P_{q-1}^{1,\mathbf{e}} + f_q(\omega_1^{\mathbf{e}}) P_q^{1,\mathbf{e}} + \Lambda P_{q+1}^{1,\mathbf{e}} \\
&= \Lambda R_{q-1}^1 + \left(\mathcal{F}_q + O(\Lambda^2)\right) R_q^1 + \Lambda R_{q+1}^1 \\
&= \Lambda \left(\frac{(-\Lambda)^{q-2}}{\prod_{k=1}^{q-2} \mathcal{F}_{k+1}} + O(\Lambda^q) \right) + \left(\mathcal{F}_q + O(\Lambda^2)\right) \left(\frac{(-\Lambda)^{q-1}}{\prod_{k=1}^{q-1} \mathcal{F}_{k+1}} + O(\Lambda^{q+1}) \right) \\
&\quad + \Lambda \left(\frac{(-\Lambda)^q}{\prod_{k=1}^{q} \mathcal{F}_{k+1}} + O(\Lambda^{q+2}) \right) \\
&= \Lambda \frac{(-\Lambda)^{q-2}}{\prod_{k=1}^{q-2} \mathcal{F}_{k+1}} - \Lambda \frac{(-\Lambda)^{q-2}}{\prod_{k=1}^{q-2} \mathcal{F}_{k+1}} + O(\Lambda^{q+1}) \\
&= O(\Lambda^{q+1})
\end{aligned}$$

and $q < -1$

$$\begin{aligned}
\mathcal{Q}_q^1 &= \Lambda P_{q-1}^{1,\mathbf{e}} + f_q(\omega_1^{\mathbf{e}}) P_q^{1,\mathbf{e}} + \Lambda P_{q+1}^{1,\mathbf{e}} \\
&= \Lambda R_{-q+1}^1 + f_q(\omega_1^{\mathbf{e}}) R_{-q}^1 + \Lambda R_{-q-1}^1 \\
&= \Lambda \left(\frac{(-\Lambda)^{-q}}{\prod_{k=1}^{-q} \mathcal{F}_{k+1}} + O(\Lambda^{-q+2}) \right) + \left(\mathcal{F}_q + O(\Lambda^2)\right) \left(\frac{(-\Lambda)^{-q-1}}{\prod_{k=1}^{-q-1} \mathcal{F}_{k+1}} + O(\Lambda^{-q+1}) \right) \\
&\quad + \Lambda \left(\frac{(-\Lambda)^{-q-2}}{\prod_{k=1}^{-q-2} \mathcal{F}_{k+1}} + O(\Lambda^{-q}) \right) \\
&= -\Lambda \frac{(-\Lambda)^{-q-2}}{\prod_{k=1}^{-q-2} \mathcal{F}_{k+1}} + \Lambda \frac{(-\Lambda)^{-q-2}}{\prod_{k=1}^{-q-2} \mathcal{F}_{k+1}} + O(\Lambda^{-q+1}) \\
&= O(\Lambda^{-q+1})
\end{aligned}$$

which completes the proof. □

4.5.2 Special Cases: General κ

In this section we present the solution to (4.12) for coupled modes but this time without specifying a value for κ. We take two integers $n > m$ where, as in (4.29), $\mathcal{F}_n = \mathcal{F}_m = 0$. We observe that the solutions vary significantly depending on the value of the spacing of n and m, thus we need to consider different values of $n - m$ as individual cases.

4.5.2.1 Case $n - m \geq 4$

When $n - m \geq 4$ we see that, to order $O(\Lambda^3)$, the coupled modes effectively become decoupled. Because of this, this case is the simplest to solve. Take the frequency, as in (4.23),

$$\omega_n = \Omega_n + \frac{\Lambda^2}{\mathcal{F}_n'} \left(\frac{1}{\mathcal{F}_{n+1}} + \frac{1}{\mathcal{F}_{n-1}} \right) + O(\Lambda^4)$$

then the corresponding modes are given by

$$P_q^n = \begin{cases} R_q^n & q > m \\ O(\Lambda^{n-m}) & q = m \\ O(\Lambda^{n-q-2}) & q < m \end{cases} \tag{4.73}$$

and

$$P_q^m = P_{q'}^n \qquad \text{where} \quad q' = n + m - q \tag{4.74}$$

and R_q^n is as defined in (4.25).

Theorem 4.5.4. *The values of ω_n and P_q^n given by* (4.23) *and* (4.73) *solve the difference equation* (4.12) *to the following orders of Λ:*

$$\mathcal{Q}_q^n = O(\Lambda^{|q-n|+2}) \qquad q > m, q \neq n \tag{4.75}$$

$$\mathcal{Q}_q^n = O(\Lambda^4) \qquad q = n \tag{4.76}$$

$$\mathcal{Q}_q^n = O(\Lambda^{n-m}) \qquad q = m \tag{4.77}$$

$$\mathcal{Q}_q^n = O(\Lambda^{n-m+1}) \qquad q = m - 1 \tag{4.78}$$

$$\mathcal{Q}_q^n = O(\Lambda^{n-q-2}) \qquad q < m - 1. \tag{4.79}$$

Proof. To show (4.75) for $q > m + 1$ the proof is identical to that given for Theorem 4.4.1 and so will not be repeated here. There remain four other cases to consider: $q = m + 1$, $q = m$, $q = m - 1$, and $q < m - 1$. Firstly $q = m + 1$, substituting (4.73) and (4.23) into (4.12)

$$\begin{aligned}
&\Lambda P_m^n + f_{m+1}(\omega_n)P_{m+1}^n + \Lambda P_{m+2}^n = \Lambda O(\Lambda^{n-m}) + f_{m+1}(\omega_n)R_{m+1}^n + \Lambda R_{m+2}^n \\
&= O(\Lambda^{n-m+1}) + \mathcal{F}_{m+1}\left(\frac{(-\Lambda)^{n-m-1}}{\prod_{k=1}^{n-m-1}\mathcal{F}_{n-k}} + O(\Lambda^{n-m+1})\right) \\
&\qquad + \Lambda\left(\frac{(-\Lambda)^{n-m-2}}{\prod_{k=1}^{n-m-2}\mathcal{F}_{n-k}} + O(\Lambda^{n-m})\right) \\
&= \mathcal{F}_{m+1}\left(\frac{-\Lambda(-\Lambda)^{n-m-2}}{\mathcal{F}_{m+1}\prod_{k=1}^{n-m-2}\mathcal{F}_{n-k}}\right) + \Lambda\left(\frac{(-\Lambda)^{n-m-2}}{\prod_{k=1}^{n-m-2}\mathcal{F}_{n-k}}\right) + O(\Lambda^{n-m+1}) \\
&= O(\Lambda^{n-m+1}) = O(\Lambda^{|(m+1)-n|+2})
\end{aligned}$$

hence (4.75) holds for all $q > m$. Next consider $q = m$

$$\begin{aligned}
&\Lambda P_{m-1}^n + f_m(\omega_n)P_m^n + \Lambda P_{m+1}^n \\
&= \Lambda O(\Lambda^{n-m-1}) + f_m(\omega_n)O(\Lambda^{n-m}) + \Lambda R_{m+1}^n \\
&= O(\Lambda^{n-m}) + \Lambda\left(\frac{(-\Lambda)^{n-m-1}}{\prod_{k=1}^{n-m-1}\mathcal{F}_{n-k}} + O(\Lambda^{n-m+1})\right) \\
&= O(\Lambda^{n-m})
\end{aligned}$$

as in (4.77). Now the case where $q = m-1$

$$\begin{aligned}
\Lambda P_{m-2}^n + f_{m-1}(\omega_n)P_{m-1}^n + \Lambda P_m^n &= \Lambda O(\Lambda^{n-m}) + \mathcal{F}_{m-1}O(\Lambda^{n-m-1}) + \Lambda O(\Lambda^{n-m}) \\
&= O(\Lambda^{n-m+1})
\end{aligned}$$

as desired. Finally, for $q < m-1$

$$\begin{aligned}
\Lambda P_{q-1}^n + f_q(\omega_n)P_q^n + \Lambda P_{q+1}^n &= \Lambda O(\Lambda^{n-q-1}) + \mathcal{F}_q O(\Lambda^{n-q-2}) + \Lambda O(\Lambda^{n-q-3}) \\
&= O(\Lambda^{n-q}) + O(\Lambda^{n-q-2}) + O(\Lambda^{n-q-2}) \\
&= O(\Lambda^{n-q-2})
\end{aligned}$$

hence (4.79). □

The corresponding theorem and proof for P_q^m can be obtained trivially by using the substitution given in (4.74).

4.5.2.2 Case $n - m = 3$

Take, as before, integers $n, m \in \mathbb{Z}$ but now set $n - m = 3$. In this scenario the modes are not yet decoupled, as is true for $n - m \geq 4$, and as such the solution is more difficult to obtain. The solutions for this case must be obtained using a perturbative method.

Firstly, we will write the frequency as

$$\omega_n = \Omega_n + \Lambda\omega_n^{(1)} + \Lambda^2\omega_n^{(2)} + O(\Lambda^3) \tag{4.80}$$

where $\omega_n^{(1)}$ and $\omega_n^{(2)}$ are to be determined and, as before, Ω_n is a solution to

$$f_n(\Omega_n) = \mathcal{F}_n = 0.$$

In this case we have that

$$\mathcal{F}_n = \mathcal{F}_{n-3} = 0$$

and so the Taylor expansions of $f_n(\omega_n)$, $f_{n-3}(\omega_n)$ and $f_q(\omega_n)$ (where $q \neq n, n-3$) are

$$\begin{aligned} f_n(\omega_n) &= \Lambda\omega_n^{(1)}\mathcal{F}_n' + \Lambda^2\left(\omega_n^{(2)}\mathcal{F}_n' + \frac{1}{2}\mathcal{F}_n''\left(\omega_n^{(1)}\right)^2\right) + O(\Lambda^3) \\ f_{n-3}(\omega_n) &= \Lambda\omega_n^{(1)}\mathcal{F}_{n-3}' + \Lambda^2\left(\omega_n^{(2)}\mathcal{F}_{n-3}' + \frac{1}{2}\mathcal{F}_{n-3}''\left(\omega_n^{(1)}\right)^2\right) + O(\Lambda^3) \\ f_q(\omega_n) &= \mathcal{F}_q + \Lambda\omega_n^{(1)}\mathcal{F}_q' + O(\Lambda^2). \end{aligned} \tag{4.81}$$

We assume, based on the structure of previous solutions, the following orders for the modes

$$P^n_{n+2} = O(\Lambda^2) \qquad P^n_{n+1} = O(\Lambda^1) \qquad P^n_n = 1$$
$$P^n_{n-1} = O(\Lambda^0) \qquad P^n_{n-2} = O(\Lambda^0) \qquad P^n_{n-3} = O(\Lambda^0)$$
$$P^n_{n-4} = O(\Lambda^1) \qquad P^n_{n-5} = O(\Lambda^2).$$

Note that all modes other than those listed are assumed to be of at least order $O(\Lambda^3)$ and so are not necessary for this calculation. We now write the expansions for these modes up to order $O(\Lambda^3)$ as

$$\begin{aligned}
P^n_{n+2} &= \Lambda^2 P^{(2)}_{n+2} + O(\Lambda^3) \\
P^n_{n+1} &= \Lambda P^{(1)}_{n+1} + \Lambda^2 P^{(2)}_{n+1} + O(\Lambda^3) \\
P^n_{n} &= 1 \\
P^n_{n-1} &= P^{(0)}_{n-1} + \Lambda P^{(1)}_{n-1} + \Lambda^2 P^{(2)}_{n-1} + O(\Lambda^3) \\
P^n_{n-2} &= P^{(0)}_{n-2} + \Lambda P^{(1)}_{n-2} + \Lambda^2 P^{(2)}_{n-2} + O(\Lambda^3) \\
P^n_{n-3} &= P^{(0)}_{n-3} + \Lambda P^{(1)}_{n-3} + \Lambda^2 P^{(2)}_{n-3} + O(\Lambda^3) \\
P^n_{n-4} &= \Lambda P^{(1)}_{n-4} + \Lambda^2 P^{(2)}_{n-4} + O(\Lambda^3) \\
P^n_{n-5} &= \Lambda^2 P^{(2)}_{n-5} + O(\Lambda^3),
\end{aligned} \tag{4.82}$$

where the coefficients $P^{(r)}_q$ are to be determined.

In order to find the unknown factors in the above expansions, we will substitute the expansions of $f_q(\omega_n)$ and P^n_q into the difference equation (4.12). Initially we will work to order $O(\Lambda^2)$ to find the coefficients of the lower order terms, and then repeat the process to order $O(\Lambda^3)$ for the complete solution.

Working to order $O(\Lambda^2)$, we substitute (4.81) and (4.82) into the difference equation (4.12) to give

$$\begin{aligned}
\Lambda + \left(\mathcal{F}_{n+1} + \omega_n^{(1)}\mathcal{F}'_{n+1}\right)\Lambda P_{n+1}^{(1)} &= O(\Lambda^2) \\
\Lambda P_{n-1}^{(0)} + \Lambda\omega_n^{(1)}\mathcal{F}'_n &= O(\Lambda^2) \\
\Lambda P_{n-2}^{(0)} + \left(\mathcal{F}_{n-1} + \Lambda\omega_n^{(1)}\mathcal{F}'_{n-1}\right) P_{n-1}^{(0)} + \mathcal{F}_{n-1}\Lambda P_{n-1}^{(1)} + \Lambda &= O(\Lambda^2) \\
\Lambda P_{n-3}^{(0)} + \left(\mathcal{F}_{n-2} + \Lambda\omega_n^{(1)}\mathcal{F}'_{n-2}\right) P_{n-2}^{(0)} + \mathcal{F}_{n-2}\Lambda P_{n-2}^{(1)} + \Lambda P_{n-1}^{(0)} &= O(\Lambda^2) \\
\Lambda\omega_n^{(1)}\mathcal{F}'_{n-3}P_{n-3}^{(0)} + \Lambda P_{n-2}^{(0)} &= O(\Lambda^2) \\
\Lambda\mathcal{F}_{n-4}P_{n-4}^{(1)} + \Lambda P_{n-3}^{(0)} &= O(\Lambda^2).
\end{aligned}$$

Solving this set of equations gives

$$\begin{array}{llll}
\omega_n^{(1)} = 0 & P_{n-1}^{(0)} = 0 & P_{n-2}^{(0)} = 0 & P_{n-1}^{(1)} = -\dfrac{1}{\mathcal{F}_{n-1}} \\
P_{n-3}^{(0)} = \alpha_0 & P_{n-2}^{(1)} = -\dfrac{\alpha_0}{\mathcal{F}_{n-2}} & P_{n-4}^{(1)} = -\dfrac{\alpha_0}{\mathcal{F}_{n-4}} & P_{n+1}^{(1)} = -\dfrac{1}{\mathcal{F}_{n+1}}
\end{array} \tag{4.83}$$

where the value of α_0 cannot be determined at this order.

Next we repeat the process but this time working to order $O(\Lambda^3)$. Again substituting (4.81) and (4.82) into (4.12) and using the coefficients calculated in (4.83) we obtain

$$\begin{aligned}
-\Lambda^2\frac{1}{\mathcal{F}_{n+1}}+\Lambda^2\mathcal{F}_{n+2}P^{(2)}_{n+2} &= O(\Lambda^3)\\
\Lambda^2P^{(2)}_{n+1} &= O(\Lambda^3)\\
-\Lambda^2\frac{1}{\mathcal{F}_{n-1}}+\Lambda^2\omega^{(2)}_n\mathcal{F}'_n-\Lambda^2\frac{1}{\mathcal{F}_{n+1}} &= O(\Lambda^3)\\
-\Lambda^2\frac{\alpha_0}{\mathcal{F}_{n-2}}+\Lambda^2\mathcal{F}_{n-1}P^{(2)}_{n-1} &= O(\Lambda^3)\\
\Lambda^2P^{(1)}_{n-3}+\Lambda^2\mathcal{F}_{n-2}P^{(2)}_{n-2}-\Lambda^2\frac{1}{\mathcal{F}_{n-1}} &= O(\Lambda^3)\\
-\Lambda^2\frac{\alpha_0}{\mathcal{F}_{n-4}}+\Lambda^2\omega^{(2)}_n\mathcal{F}'_{n-3}\alpha_0-\Lambda^2\frac{\alpha_0}{\mathcal{F}_{n-2}} &= O(\Lambda^3)\\
\Lambda^2\mathcal{F}_{n-4}P^{(2)}_{n-4}+\Lambda^2P^{(1)}_{n-3} &= O(\Lambda^3)\\
\Lambda^2\mathcal{F}_{n-5}P^{(2)}_{n-5}-\Lambda^2\frac{\alpha_0}{\mathcal{F}_{n-4}} &= O(\Lambda^3),
\end{aligned}$$

which are solved simply to give

$$\begin{array}{lllll}
\alpha_0=0 & P^{(2)}_{n+2}=\dfrac{1}{\mathcal{F}_{n+1}\mathcal{F}_{n+2}} & P^{(2)}_{n+1}=0 & \omega^{(2)}_n=\dfrac{1}{\mathcal{F}'_n}\left(\dfrac{1}{\mathcal{F}_{n-1}}+\dfrac{1}{\mathcal{F}_{n+1}}\right) & \\
P^{(2)}_{n-1}=0 & P^{(2)}_{n-2}=\dfrac{1}{\mathcal{F}_{n-2}}\left(\dfrac{1}{\mathcal{F}_{n-1}}-\alpha_1\right) & P^{(1)}_{n-3}=\alpha_1 & P^{(2)}_{n-4}=-\dfrac{\alpha_1}{\mathcal{F}_{n-4}} & P^{(2)}_{n-5}=0
\end{array} \tag{4.84}$$

where, as before, α_1 cannot be determined at this order. Hence we can substitute the coefficients found in (4.83) and (4.84) into (4.80) and (4.82) to give the solution

$$\omega_n=\Omega_n+\frac{\Lambda^2}{\mathcal{F}'_n}\left(\frac{1}{\mathcal{F}_{n-1}}+\frac{1}{\mathcal{F}_{n+1}}\right)+O(\Lambda^3) \tag{4.85}$$

and

$$P^n_q = \begin{cases} \Lambda^2 \dfrac{1}{\mathcal{F}_{n+1}\mathcal{F}_{n+2}} + O(\Lambda^3) & q = n+2 \\ -\dfrac{\Lambda}{\mathcal{F}_{n+1}} + O(\Lambda^3) & q = n+1 \\ 1 & q = n \\ -\dfrac{\Lambda}{\mathcal{F}_{n-1}} + O(\Lambda^3) & q = n-1 \\ \Lambda^2 \left(\dfrac{1}{\mathcal{F}_{n-1}\mathcal{F}_{n-2}} - \dfrac{\alpha_1}{\mathcal{F}_{n-2}} \right) + O(\Lambda^3) & q = n-2 \\ \Lambda\alpha_1 + O(\Lambda^2) & q = n-3 \\ -\Lambda^2 \dfrac{\alpha_1}{\mathcal{F}_{n-4}} + O(\Lambda^3) & q = n-4. \end{cases} \tag{4.86}$$

The remaining terms are given by $P^n_q = O(\Lambda^3)$ for $q > n+2$ or $q < n-4$.

Theorem 4.5.5. *The difference equation* (4.12) *is solved by the frequency and polarisation given by* (4.85) *and* (4.86) *to order* $O(\Lambda^3)$*.*

Proof. This is trivially shown by substitution of (4.85) and (4.86) into (4.12). □

The other corresponding mode, i.e. $m - n = 3$, is found by exchanging n and m in (4.85) and (4.86). This again solves (4.12) to order $O(\Lambda^3)$ and is shown by direct substitution.

4.5.2.3 Case $n - m = 2$

We now take two integers n, m such that $n - m = 2$ and where $\mathcal{F}_n = \mathcal{F}_m = 0$. For this case the solutions are given by[3]

$$\omega_n = \Omega_n + \frac{\Lambda^2}{\mathcal{F}'_n} \left(\frac{\rho + 1}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}} \right) + O(\Lambda^3) \tag{4.87}$$

[3]Note that the derivation of this solution is similar to that given in section 4.5.2.2 and so will not be shown here, and can instead by found in appendix B.1.

and

$$P_q^n = \begin{cases} \Lambda^2 \dfrac{1}{\mathcal{F}_{n+1}\mathcal{F}_{n+2}} + O(\Lambda^3) & q = n+2 \\ -\dfrac{\Lambda}{\mathcal{F}_{n+1}} + O(\Lambda^3) & q = n+1 \\ 1 & q = n \\ -\Lambda \dfrac{\rho+1}{\mathcal{F}_{n-1}} - \Lambda^2 \dfrac{\rho_2}{\mathcal{F}_{n-1}} + O(\Lambda^3) & q = n-1 \\ \rho + \Lambda\rho_2 + O(\Lambda^2) & q = n-2 \\ -\Lambda \dfrac{\rho}{\mathcal{F}_{n-3}} - \Lambda^2 \dfrac{\rho_2}{\mathcal{F}_{n-3}} & q = n-3 \\ \Lambda^2 \dfrac{\rho}{\mathcal{F}_{n-3}\mathcal{F}_{n-4}} + O(\Lambda^3) & q = n-4 \end{cases} \tag{4.88}$$

where ρ_2 cannot be determined at this order of Λ and ρ is the solution to the quadratic equation

$$\mathcal{F}'_{n-2}\rho^2 + \left(\frac{\mathcal{F}_{n-1}\mathcal{F}'_{n-2}}{\mathcal{F}_{n+1}} + \mathcal{F}'_{n-2} - \frac{\mathcal{F}_{n-1}\mathcal{F}'_n}{\mathcal{F}_{n-3}} - \mathcal{F}'_n \right) \rho - \mathcal{F}'_n = 0. \tag{4.89}$$

Solving this quadratic will give two values for ρ and hence we have a pair of solutions for ω_n and P_q^n as expected.

Theorem 4.5.6. *The difference equation* (4.12) *is solved, to order* $O(\Lambda^3)$*, by the frequency and polarisation given by* (4.87) *and* (4.88)*.*

Proof. Substitution of (4.87) and (4.88) into (4.12) gives the result. □

4.5.2.4 Case $n - m = 1$

In this case there are two solutions given by

$$\omega_n = \Omega_n + \Lambda W + \Lambda^2 \frac{1}{\mathcal{F}'_n\mathcal{F}'_{n-1}} \left(\frac{\mathcal{F}'_n}{2\mathcal{F}_{n-2}} - \frac{\mathcal{F}''_{n-1}}{4\mathcal{F}'_{n-1}} + \frac{\mathcal{F}'_{n-1}}{2\mathcal{F}_{n+1}} - \frac{\mathcal{F}''_n}{4\mathcal{F}'_n} \right) + O(\Lambda^3) \tag{4.90}$$

where

$$W = \pm \left(\mathcal{F}'_n\mathcal{F}'_{n-1}\right)^{-1/2} \tag{4.91}$$

and

$$
P_q^n = \begin{cases} \dfrac{\Lambda^2}{\mathcal{F}_{n+2}\mathcal{F}_{n+1}} + O(\Lambda^3) & q = n+2 \\ -\dfrac{\Lambda}{\mathcal{F}_{n+1}} + \dfrac{\Lambda^2 \mathcal{F}'_{n+1} W}{(\mathcal{F}_{n+1})^2} + O(\Lambda^3) & q = n+1 \\ 1 & q = n \\ -\mathcal{F}'_n W + \dfrac{\Lambda}{\mathcal{F}'_{n-1}}\mathcal{G} + O(\Lambda^3) & q = n-1 \\ \dfrac{\Lambda \mathcal{F}'_n W}{\mathcal{F}_{n-2}} - \dfrac{\Lambda^2}{\mathcal{F}'_{n-1}\mathcal{F}_{n-2}}\left(\mathcal{G} + \dfrac{\mathcal{F}'_{n-2}}{\mathcal{F}_{n-2}}\right) + O(\Lambda^3) & q = n-2 \\ -\dfrac{\Lambda^2 \mathcal{F}'_n W}{\mathcal{F}_{n-2}\mathcal{F}_{n-3}} + O(\Lambda^3) & q = n-3 \end{cases} \tag{4.92}
$$

where

$$
\mathcal{G} = \frac{\mathcal{F}''_{n-1}}{4\mathcal{F}'_{n-1}} - \frac{\mathcal{F}''_n}{4\mathcal{F}'_n} + \frac{\mathcal{F}'_{n-1}}{2\mathcal{F}_{n+1}} - \frac{\mathcal{F}'_n}{2\mathcal{F}_{n-2}}. \tag{4.93}
$$

Observe that the split into two solutions is caused by the choice of sign when taking the square root in (4.91). The derivation of this result is again similar to those shown previously and is given in full in appendix B.2.

Theorem 4.5.7. *The difference equation* (4.12) *is solved by* (4.90) *and* (4.92) *to order* $O(\Lambda^3)$.

Proof. The result is given by substitution of (4.90) and (4.92) into (4.12). □

4.6 Longitudinal Modes in Wire Media

So far in this chapter we have considered fields which are transverse, as described in section 1.2. In this section we assume instead that the electric and polarisation fields are longitudinal and that the magnetic field vanishes. This allows us to investigate a wire medium in which the radius, r, of the wires varies periodically along their length, as illustrated in figure 4.6.

Mathematically, we now have $\mathbf{E} = E(t,x)\mathbf{e}_1$, $\mathbf{P} = P(t,x)\mathbf{e}_1$, and $\mathbf{B} = \mathbf{0}$. Maxwell's

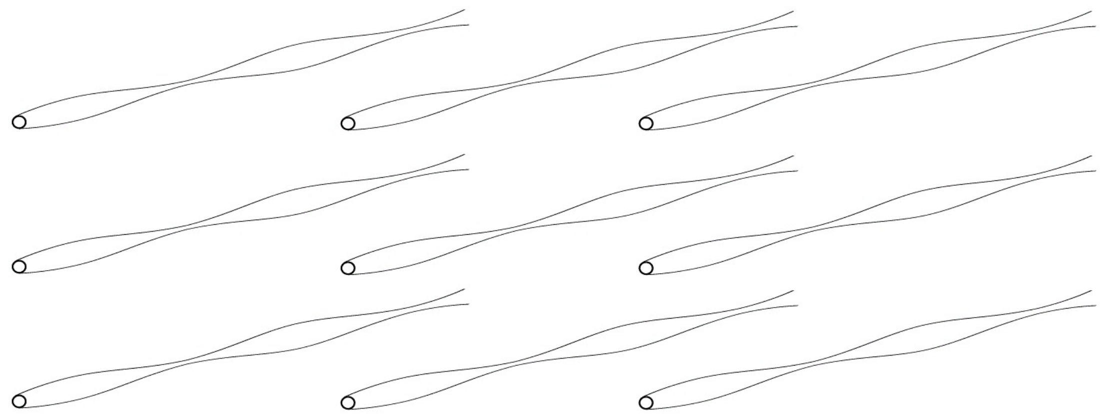

Figure 4.6: Wire medium with periodic variation of the radius along the wire length.

equations are then automatically satisfied if

$$\varepsilon_0 E + P = 0. \tag{4.94}$$

For a wire medium we use the longitudinal component of the permittivity given by [9, eqn. 4b]

$$\widetilde{P}(\omega, k) = \frac{-k_p^2 \varepsilon_0 \widetilde{E}(\omega, k)}{\omega^2 - k^2} \tag{4.95}$$

where the grid spacing, b, is the same in both the y and z directions, and

$$k_p^2 = \frac{2\pi}{b^2(\log b - \log 2\pi r + 0.5275)}. \tag{4.96}$$

Since we are looking at wires with periodically varying radius, k_p becomes periodic in x. By substituting

$$r(x) = \frac{b}{2\pi} \exp\left(\frac{-2\pi}{b^2(k_0^2 + 2\Lambda\cos(2\pi x/a))} + 0.5275\right), \tag{4.97}$$

where $k_0^2 = k_p^2|_{r=r_0}$ is constant, into (4.96) we get

$$k_p^2 = k_0^2 + 2\Lambda \cos\left(\frac{2\pi x}{a}\right). \tag{4.98}$$

We can then find a difference equation for the modes in the same way detailed in section 4.2. Again we get the difference equation,

$$\Lambda P_{q-1} + f_q(\omega) P_q + \Lambda P_{q+1} = 0, \tag{4.12}$$

however the function $f_q(\omega)$ is now given by

$$f_q(\omega) = \omega^2 - k_0^2 - \frac{(q+\kappa)^2}{a^2}. \tag{4.99}$$

Looking at $f_q(\omega) = 0$, we see that

$$q + \kappa = \pm a\sqrt{k_0^2 - \omega^2}. \tag{4.100}$$

This dispersion relation, for ω and κ, is plotted in figure 4.7. Observe that the perturbed modes are dramatically different near $n = 0$ and $\kappa = 0.5$. This is due to being near the point where the modes couple. Detailed analysis of such behaviour is potentially a subject of future work.

Finding solutions to the difference equation is the same as for the transverse modes given earlier in this chapter, differing only by the definition of $f_q(\omega)$. As the analytic solutions are all given in terms of f_q then clearly the form of these solutions is the same.

From (4.100) we see that, for two distinct integers $n \neq m$, we must have

$$(n+\kappa)^2 = (m+\kappa)^2$$

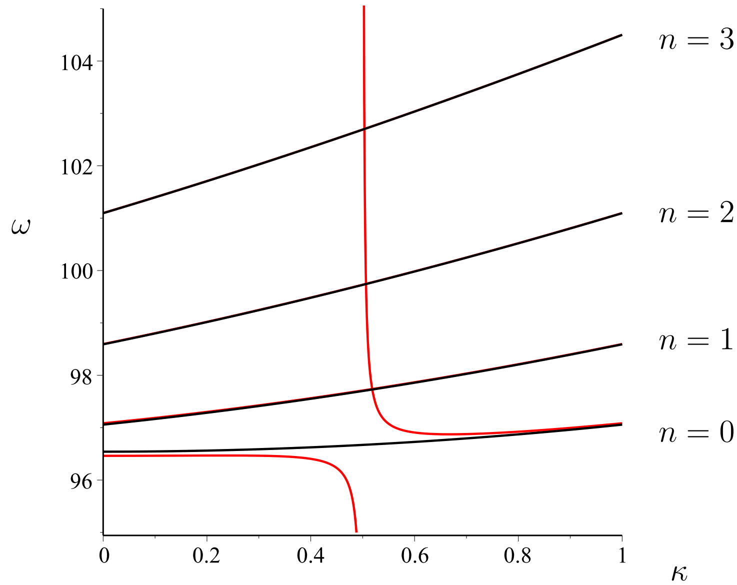

Figure 4.7: Dispersion curve for unperturbed (black) and perturbed (red) longitudinal modes in a wire medium. Here $a = 0.1$, $b = 0.02$, $r_0 = 0.001$, and $\Lambda = r_0/100$. Only with the $n = 0$ modes can we see the difference between the perturbed and unperturbed modes.

and so the only possibilities are for $\kappa = 0$ or $\kappa = 1/2$. As such, coupled modes are only observed in these two cases and so only the coupled solutions previously calculated (for transverse modes) for these two cases are applicable in the longitudinal case.

For uncoupled modes, the solution is again given by (4.25). For $a = 0.1$, $b = 0.02$, $r_0 = 0.001$, and $\Lambda = r_0/5$ we can calculate $\hat{P}(x)$, as shown in figure 4.8. The corresponding mode amplitudes in this example are $P^0_{-2} = 0.027$, $P^0_{-1} = 0.174$, $P^0_0 = 1$, $P^0_1 = 0.154$ and $P^0_2 = 0.019$.

4.7 Conclusion

In this chapter we have presented solutions to Maxwell's equations for a periodically structured medium with spatial dispersion. A difference equation for the amplitudes of the spatial modes was derived and solved numerically. We see that many of these modes, P_q, are divergent as $q \to \pm\infty$. However, from those modes that do converge we are able to calculate $\hat{P}(\omega, x)$ and see that this is strongly distorted from what

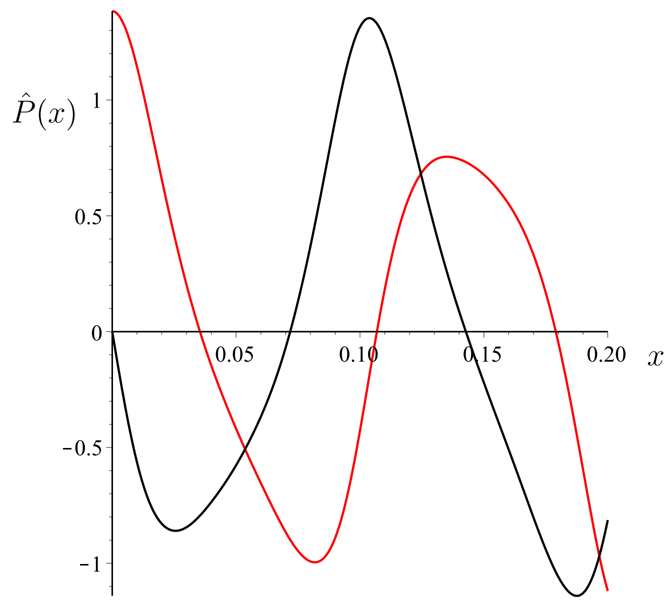

Figure 4.8: Real (black) and imaginary (red) longitudinal mode shape for $n = 0$ and $\kappa = 0.3$. Here $a = 0.1$, $b = 0.02$, $r_0 = 0.001$, and $\Lambda = r_0/5$.

would be seen in the non-dispersive case.

Under the assumption that the magnitude of this periodic inhomogeneity is small we were able to find analytic solutions to the spatial modes from our difference equation. Two types of solutions emerged, coupled and uncoupled modes. While the uncoupled modes are unsurprising, the existence of the coupled modes is a new feature. This shows that the spatial dispersion present in the medium has a significant effect on the physical behaviour. It can also be seen that some of the modes which appear as uncoupled in this work, such as when $n-m = 4$, would actually be coupled for higher orders of Λ. However, since these results required the assumption that $\Lambda < L_0$ this coupling would still be too weak to become significant.

It is also important to note that including higher order harmonics in the definition of $L(\omega, x)$, such as $\cos(4\pi x/a)$ etc, would give additional terms in the difference equation. While this would increase the complexity of solving it would be possible to obtain more accurate solutions. This would, potentially, make it possible to specify particular electric field profiles. This would have applications such as in the design of cavities in accelerators or the manipulation of light in cloaking metamaterials. For

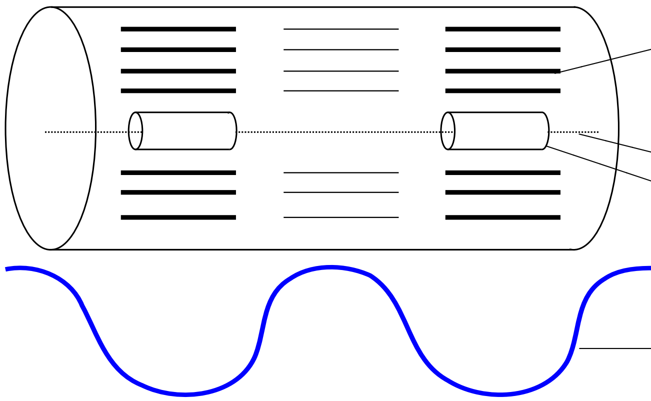

Figure 4.9: Drift tube accelerator with a periodic, spatially dispersive medium added in the cavity. This allows the electric field experienced by the protons to be optimised.

example, adding a periodic, spatially dispersive medium, such as a wire medium, to the cavity of a drift tube accelerator the electric field could be optimised to improve acceleration effects, as shown in figure 4.9.

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Appendix A

Response in Homogeneous Media

A.1 Delta Function Transformation

Given the coordinate transformation $u = t + x/\beta$ and $v = t - x/\beta$, we seek to obtain the transformation of $\delta(t)\delta(x)$. Take $\alpha = \varphi dt \wedge dx$ for some test function φ, then

$$\begin{aligned}
\alpha &= \varphi dt \wedge dx \\
&= \varphi \left(\frac{du + dv}{2} \right) \wedge \left(\frac{\beta(du - dv)}{2} \right) \\
&= \varphi \frac{\beta}{4} \left(du \wedge du + dv \wedge du - du \wedge dv - dv \wedge dv \right) \\
&= \varphi \frac{\beta}{2} dv \wedge du.
\end{aligned}$$

Now consider

$$\begin{aligned}
\delta(u)\delta(v)\alpha &= \delta(u)\delta(v) \left[\varphi dt \wedge dx \right] \\
&= \frac{\beta}{2} \delta(u)\delta(v) \left[\varphi dv \wedge du \right] \\
&= \frac{\beta}{2} \varphi(0, 0).
\end{aligned}$$

Since we know already that

$$\delta(t)\delta(x)[\varphi dt \wedge dx] = \int \varphi\delta(t)\delta(x)dt \wedge dx = \varphi(0,0)$$

then we can write

$$\delta(u)\delta(v)\alpha = \frac{\beta}{2}\delta(t)\delta(x)[\varphi dt \wedge dx] = \frac{\beta}{2}\delta(t)\delta(x)\alpha$$

for all α, therefore

$$\delta(t)\delta(x) = \frac{2}{\beta}\delta(u)\delta(v). \tag{A.1}$$

Appendix B

Coupled Mode Solutions

B.1 $n - m = 2$

Here we have $n - m = 2$, hence

$$\mathcal{F}_n = \mathcal{F}_{n-2} = 0$$

and so, taking the expansion of f_q for each q,

$$\begin{aligned} f_n(\omega_n) &= \Lambda\omega_n^{(1)}\mathcal{F}'_n + \Lambda^2\left(\omega_n^{(2)}\mathcal{F}'_n + \frac{1}{2}(\omega_n^{(1)})^2\mathcal{F}''_n\right) + O(\Lambda^3) \\ f_{n-2}(\omega_n) &= \Lambda\omega_n^{(1)}\mathcal{F}'_{n-2} + \Lambda^2\left(\omega_n^{(2)}\mathcal{F}'_{n-2} + \frac{1}{2}(\omega_n^{(1)})^2\mathcal{F}''_{n-2}\right) + O(\Lambda^3) \\ f_q(\omega_n) &= \mathcal{F}_q + \Lambda\omega_n^{(1)}\mathcal{F}'_q + O(\Lambda^2) \end{aligned} \tag{B.1}$$

where ω_n is as given in (4.80). We make the following assumptions about the orders of P_q^n

$$\begin{array}{llll} P_{n+2}^n = O(\Lambda^2) & P_{n+1}^n = O(\Lambda) & P_n^n = 1 & P_{n-1}^n = O(\Lambda^0) \\ P_{n-2}^n = O(\Lambda^0) & P_{n-3}^n = O(\Lambda) & P_{n-4}^n = O(\Lambda^2) & \end{array}$$

where all modes other than these are considered to be of at least order $O(\Lambda^3)$. Expanding these modes up to order $O(\Lambda^3)$ gives

$$\begin{aligned}
P^n_{n+2} &= \Lambda^2 P^{(2)}_{n+2} + O(\Lambda^3) \\
P^n_{n+1} &= \Lambda P^{(1)}_{n+1} + \Lambda^2 P^{(2)}_{n+1} + O(\Lambda^3) \\
P^n_{n} &= 1 \\
P^n_{n-1} &= P^{(0)}_{n-1} + \Lambda P^{(1)}_{n-1} + \Lambda^2 P^{(2)}_{n-1} + O(\Lambda^3) \\
P^n_{n-2} &= P^{(0)}_{n-2} + \Lambda P^{(1)}_{n-2} + \Lambda^2 P^{(2)}_{n-2} + O(\Lambda^3) \\
P^n_{n-3} &= \Lambda P^{(1)}_{n-3} + \Lambda^2 P^{(2)}_{n-3} + O(\Lambda^3) \\
P^n_{n-4} &= \Lambda^2 P^{(2)}_{n-4} + O(\Lambda^3).
\end{aligned} \tag{B.2}$$

Now substituting (B.2) and (B.1) into (4.12), and working to order $O(\Lambda^2)$, gives

$$\begin{aligned}
\Lambda + \Lambda \mathcal{F}_{n+1} P^{(1)}_{n+1} &= O(\Lambda^2) \\
\Lambda P^{(0)}_{n-1} + \Lambda \omega^{(1)}_n \mathcal{F}'_n &= O(\Lambda^2) \\
\Lambda P^{(0)}_{n-2} + \left(\mathcal{F}_{n-1} + \Lambda \omega^{(1)}_n \mathcal{F}'_{n-1}\right) P^{(0)}_{n-1} + \Lambda \mathcal{F}_{n-1} P^{(1)}_{n-1} + \Lambda &= O(\Lambda^2) \\
\Lambda \omega^{(1)}_n \mathcal{F}'_{n-2} P^{(0)}_{n-2} + \Lambda P^{(0)}_{n-1} &= O(\Lambda^2) \\
\Lambda \mathcal{F}_{n-3} P^{(1)}_{n-3} + \Lambda P^{(0)}_{n-2} &= O(\Lambda^2).
\end{aligned}$$

Solving this set of equations gives

$$\begin{array}{lll}
\omega^{(1)}_n = 0 & P^{(1)}_{n+1} = -\dfrac{1}{\mathcal{F}_{n+1}} & P^{(0)}_{n-1} = 0 \\
P^{(1)}_{n-1} = -\dfrac{1+\rho}{\mathcal{F}_{n-1}} & P^{(0)}_{n-2} = \rho & P^{(1)}_{n-3} = -\dfrac{\rho}{\mathcal{F}_{n-3}}
\end{array}$$

where the quantity ρ cannot be determined to this order of Λ. To get the remaining coefficients we repeat the calculation at $O(\Lambda^3)$. Substituting (B.2) and (B.1) into

(4.12) again

$$-\frac{\Lambda^2}{\mathcal{F}_{n+1}}+\Lambda^2\mathcal{F}_{n+2}P^{(2)}_{n+2}=O(\Lambda^3) \tag{B.3}$$

$$\Lambda^2\mathcal{F}_{n+1}P^{(2)}_{n+1}=O(\Lambda^3) \tag{B.4}$$

$$-\Lambda^2\frac{1+\rho}{\mathcal{F}_{n-1}}+\Lambda^2\omega^{(2)}_n\mathcal{F}'_n-\frac{\Lambda^2}{\mathcal{F}_{n+1}}=O(\Lambda^3) \tag{B.5}$$

$$\Lambda^2P^{(1)}_{n-2}+\Lambda^2\mathcal{F}_{n-1}P^{(2)}_{n-1}=O(\Lambda^3) \tag{B.6}$$

$$-\Lambda^2\frac{\rho}{\mathcal{F}_{n-3}}+\Lambda^2\omega^{(2)}_n\mathcal{F}'_{n-2}\rho-\Lambda^2\frac{1+\rho}{\mathcal{F}_{n-1}}=O(\Lambda^3) \tag{B.7}$$

$$\Lambda^2\mathcal{F}_{n-3}P^{(2)}_{n-3}+\Lambda^2P^{(1)}_{n-2}=O(\Lambda^3) \tag{B.8}$$

$$\Lambda^2\mathcal{F}_{n-4}P^{(2)}_{n-4}-\Lambda^2\frac{\rho}{\mathcal{F}_{n-3}}=O(\Lambda^3). \tag{B.9}$$

Setting $P^{(1)}_{n-2}=\rho_2$ we can immediately solve (B.3), (B.4), (B.6), (B.8) and (B.9) to obtain

$$P^{(2)}_{n+2}=\frac{1}{\mathcal{F}_{n+1}\mathcal{F}_{n+2}} \qquad P^{(2)}_{n+1}=0 \qquad P^{(2)}_{n-1}=-\frac{\rho_2}{\mathcal{F}_{n-1}}$$
$$P^{(2)}_{n-3}=-\frac{\rho_2}{\mathcal{F}_{n-3}} \qquad P^{(2)}_{n-4}=\frac{\rho}{\mathcal{F}_{n-3}\mathcal{F}_{n-4}}.$$

Note that we cannot determine ρ_2 without looking at $O(\Lambda^4)$. However, to find ρ and $\omega^{(2)}_n$ we need to solve (B.5) and (B.7) together. Eliminating $\omega^{(2)}_n$ from these two equations gives

$$\frac{1+\rho}{\mathcal{F}'_n\mathcal{F}_{n-1}}+\frac{1}{\mathcal{F}'_n\mathcal{F}_{n+1}}-\frac{1}{\mathcal{F}'_{n-2}\mathcal{F}_{n-3}}-\frac{1}{\rho}\frac{1+\rho}{\mathcal{F}'_{n-2}\mathcal{F}_{n-1}}=0$$

which, when rearranged, becomes

$$\rho^2+\left(1+\frac{\mathcal{F}_{n-1}}{\mathcal{F}_{n+1}}-\frac{\mathcal{F}'_n\mathcal{F}_{n-1}}{\mathcal{F}'_{n-2}\mathcal{F}_{n-3}}-\frac{\mathcal{F}'_n}{\mathcal{F}'_{n-2}}\right)\rho-\frac{\mathcal{F}'_n}{\mathcal{F}'_{n-2}}=0.$$

We can then simply rearrange (B.5) to obtain an expression for $\omega_n^{(2)}$

$$\omega_n^{(2)} = \frac{1}{\mathcal{F}'_n}\left(\frac{1+\rho}{\mathcal{F}_{n-1}} + \frac{1}{\mathcal{F}_{n+1}}\right).$$

This completes the set of required coefficients, which can then be substituted into (B.2) to give the solution.

B.2 $n - m = 1$

Here we have $n - m = 1$, hence

$$\mathcal{F}_n = \mathcal{F}_{n-1} = 0$$

and so, taking the expansion of f_q for each q,

$$\begin{aligned} f_n(\omega_n) &= \Lambda\omega_n^{(1)}\mathcal{F}'_n + \Lambda^2\left(\omega_n^{(2)}\mathcal{F}'_n + \frac{1}{2}(\omega_n^{(1)})^2\mathcal{F}''_n\right) + O(\Lambda^3) \\ f_{n-1}(\omega_n) &= \Lambda\omega_n^{(1)}\mathcal{F}'_{n-1} + \Lambda^2\left(\omega_n^{(2)}\mathcal{F}'_{n-1} + \frac{1}{2}(\omega_n^{(1)})^2\mathcal{F}''_{n-1}\right) + O(\Lambda^3) \\ f_q(\omega_n) &= \mathcal{F}_q + \Lambda\omega_n^{(1)}\mathcal{F}'_q + O(\Lambda^2) \end{aligned} \tag{B.10}$$

where the expansion of ω_n is the same as in (4.80). We assume the following orders of P_q^n

$$\begin{array}{lll} P_{n+2}^n = O(\Lambda^2) & P_{n+1}^n = O(\Lambda) & P_n^n = 1 \\ P_{n-1}^n = O(\Lambda^0) & P_{n-2}^n = O(\Lambda) & P_{n-3}^n = O(\Lambda^2) \end{array}$$

and all modes other than these are taken to be of at least order $O(\Lambda^3)$. Expanding these modes up to order $O(\Lambda^3)$ gives

$$\begin{aligned}
P^n_{n+2} &= \Lambda^2 P^{(2)}_{n+2} + O(\Lambda^3) \\
P^n_{n+1} &= \Lambda P^{(1)}_{n+1} + \Lambda^2 P^{(2)}_{n+1} + O(\Lambda^3) \\
P^n_{n} &= 1 \\
P^n_{n-1} &= P^{(0)}_{n-1} + \Lambda P^{(1)}_{n-1} + \Lambda^2 P^{(2)}_{n-1} + O(\Lambda^3) \\
P^n_{n-2} &= \Lambda P^{(1)}_{n-2} + \Lambda^2 P^{(2)}_{n-2} + O(\Lambda^3) \\
P^n_{n-3} &= \Lambda^2 P^{(2)}_{n-3} + O(\Lambda^3).
\end{aligned} \tag{B.11}$$

Now substitute (B.11) and (B.10) into (4.12) gives, to order $O(\Lambda^2)$

$$\begin{aligned}
\Lambda + \Lambda \mathcal{F}_{n+1} P^{(1)}_{n+1} &= O(\Lambda^2) \\
\Lambda P^{(0)}_{n-1} + \Lambda \omega^{(1)}_n \mathcal{F}'_n &= O(\Lambda^2) \\
\Lambda \omega^{(1)}_n \mathcal{F}'_{n-1} P^{(0)}_{n-1} + \Lambda &= O(\Lambda^2) \\
\Lambda \mathcal{F}_{n-2} P^{(1)}_{n-2} + \Lambda P^{(0)}_{n-1} &= O(\Lambda^2).
\end{aligned}$$

This system of equations can be solved completely to give

$$\begin{aligned}
P^{(1)}_{n+1} &= -\frac{1}{\mathcal{F}_{n+1}} \qquad & \omega^{(1)}_n &= \pm \left(\mathcal{F}'_n \mathcal{F}'_{n-1} \right)^{-1/2} \\
P^{(0)}_{n-1} &= -W \mathcal{F}'_n \qquad & P^{(1)}_{n-2} &= \frac{W \mathcal{F}'_n}{\mathcal{F}_{n-2}}
\end{aligned} \tag{B.12}$$

where $W = \omega^{(1)}_n$.

Next we look at (4.12) again but this time work to order $O(\Lambda^3)$, substituting

(B.12)

$$-\frac{\Lambda^2}{\mathcal{F}_{n+1}} + \Lambda^2 \mathcal{F}_{n+2} P^{(2)}_{n+2} = O(\Lambda^3) \tag{B.13}$$

$$-\frac{\Lambda^2 W \mathcal{F}'_{n+1}}{\mathcal{F}_{n+1}} + \Lambda^2 \mathcal{F}_{n+1} P^{(2)}_{n+1} = O(\Lambda^3) \tag{B.14}$$

$$\Lambda^2 P^{(1)}_{n-1} + \Lambda^2 \left(\omega^{(2)}_n \mathcal{F}'_n + \frac{1}{2} \frac{\mathcal{F}''_n}{\mathcal{F}'_n \mathcal{F}'_{n-1}} \right) - \frac{\Lambda^2}{\mathcal{F}_{n+1}} = O(\Lambda^3) \tag{B.15}$$

$$\Lambda^2 \frac{W \mathcal{F}'_n}{\mathcal{F}_{n-2}} - \Lambda^2 \left(\omega^{(2)}_n \mathcal{F}'_{n-1} + \frac{1}{2} \frac{\mathcal{F}''_{n-1}}{\mathcal{F}'_n \mathcal{F}'_{n-1}} \right) W \mathcal{F}'_n + \Lambda^2 W \mathcal{F}'_{n-1} P^{(1)}_{n-1} = O(\Lambda^3) \tag{B.16}$$

$$\Lambda^2 \mathcal{F}_{n-2} P^{(2)}_{n-2} + \Lambda^2 \frac{\mathcal{F}'_{n-2}}{\mathcal{F}_{n-2} \mathcal{F}'_{n-1}} + \Lambda^2 P^{(1)}_{n-1} = O(\Lambda^3) \tag{B.17}$$

$$\Lambda^2 \mathcal{F}_{n-3} P^{(2)}_{n-3} + \Lambda \frac{W \mathcal{F}'_n}{\mathcal{F}_{n-2}} = O(\Lambda^3). \tag{B.18}$$

Equations (B.13),(B.14), and (B.18) can be solved immediately to give

$$P^{(2)}_{n+2} = \frac{1}{\mathcal{F}_{n+1} \mathcal{F}_{n+2}} \qquad P^{(2)}_{n+1} = \frac{W \mathcal{F}'_{n+1}}{(\mathcal{F}_{n+1})^2} \qquad P^{(2)}_{n-3} = -\frac{W \mathcal{F}'_n}{\mathcal{F}_{n-2} \mathcal{F}_{n-3}}. \tag{B.19}$$

The remaining equations, however, are less straight forward. We first need to look at the pair (B.15) and (B.16) in order to find $P^{(1)}_{n-1}$ and $\omega^{(2)}_n$. Firstly, eliminating $P^{(1)}_{n-1}$ gives

$$\omega^{(2)}_n = \frac{1}{\mathcal{F}'_n \mathcal{F}'_{n-1}} \left(\frac{\mathcal{F}'_n}{2\mathcal{F}_{n-2}} - \frac{\mathcal{F}''_{n-1}}{4\mathcal{F}'_{n-1}} + \frac{\mathcal{F}'_{n-1}}{2\mathcal{F}_{n+1}} - \frac{\mathcal{F}''_n}{4\mathcal{F}'_n} \right) \tag{B.20}$$

then substituting this into (B.15) gives us

$$P^{(1)}_{n-1} = \frac{1}{\mathcal{F}'_{n-1}} \left(\frac{\mathcal{F}'_{n-1}}{2\mathcal{F}_{n+1}} - \frac{\mathcal{F}'_n}{2\mathcal{F}_{n-2}} + \frac{\mathcal{F}''_{n-1}}{4\mathcal{F}'_{n-1}} - \frac{\mathcal{F}''_n}{4\mathcal{F}'_n} \right) = \frac{1}{\mathcal{F}'_{n-1}} \mathcal{G} \tag{B.21}$$

where we have introduced the symbol $\mathcal{G}$ for the sake of clarity. Now that we have

$P^{(1)}_{n-1}$ we can substitute this into (B.17) to complete the solution

$$P^{(2)}_{n-2} = -\frac{P^{(1)}_{n-1}}{\mathcal{F}_{n-2}} - \frac{\mathcal{F}'_{n-2}}{(\mathcal{F}_{n-2})^2\mathcal{F}'_{n-1}} = -\frac{1}{\mathcal{F}'_{n-1}\mathcal{F}_{n-2}}\left(\frac{\mathcal{F}'_{n-2}}{\mathcal{F}_{n-2}} + \mathcal{G}\right). \tag{B.22}$$

Substituting the coefficients (B.12), (B.19), (B.20), (B.21) and (B.22) into (B.11) and (4.80) completes the solution.