

# A REFLEXIVE BANACH SPACE WHOSE ALGEBRA OF OPERATORS IS NOT A GROTHENDIECK SPACE

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ABSTRACT. By a result of Johnson, the Banach space  $F = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_{\infty}}$  contains a complemented copy of  $\ell_1$ . We identify  $F$  with a complemented subspace of the space of (bounded, linear) operators on the reflexive space  $(\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_p}$  ( $p \in (1, \infty)$ ), thus solving negatively the problem posed in the monograph of Diestel and Uhl which asks whether the space of operators on a reflexive Banach space is Grothendieck.

## 1. INTRODUCTION

A Banach space  $E$  is *Grothendieck* if weak\* convergent sequences in  $E^*$  converge weakly. Certainly, every reflexive Banach space is Grothendieck. Notable examples of non-reflexive Grothendieck spaces are  $C(K)$ -spaces for externally disconnected compact spaces  $K$  ([4]) and the Hardy space  $H^{\infty}$  of bounded holomorphic functions on the unit disc ([1]). Diestel and Uhl wrote in their famous monograph [3, p. 180]:

*Finally, there is some evidence (Akemann [1967], [1968]) that the space  $\mathcal{L}(H; H)$  of bounded linear operators on a Hilbert space is a Grothendieck space and that more generally the space  $\mathcal{L}(X; X)$  is a Grothendieck space for any reflexive Banach space  $X$ .*

The question of whether the space of (bounded, linear) operators on a reflexive Banach space is Grothendieck was raised also by Soybaş ([7]). Pfitzner proved in [6] that  $C^*$ -algebras have the so-called *Pelczyński's property (V)* which for dual Banach spaces is equivalent to being a Grothendieck space (cf. [2, Exercise 12, p. 116]). In particular, von Neumann algebras are Grothendieck spaces which confirms that the space of operators on a Hilbert space is Grothendieck. It is known that duals of spaces with property (V) are weakly sequentially complete. We shall present an example of a reflexive Banach space  $E$  such that  $\mathcal{B}(E)$  fails to be Grothendieck, giving thus a negative answer to the above-mentioned problem. To do this, we require a result of Johnson which asserts that the Banach space  $F = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_{\infty}}$  contains a complemented copy of  $\ell_1$  (cf. the Remark after Theorem 1 in [5]), so it is not a Grothendieck space.

By an *operator* we understand a bounded, linear operator acting between Banach spaces. The space  $\mathcal{B}(E_1, E_2)$  of operators acting between spaces  $E_1$  and  $E_2$  is a Banach space when endowed with the operator norm. We write  $\mathcal{B}(E)$  for  $\mathcal{B}(E, E)$ . Let  $p \in [1, \infty]$ . We denote by  $(\bigoplus_{n=1}^{\infty} E_n)_{\ell_p}$  the  $\ell_p$ -sum of a sequence  $(E_n)_{n=1}^{\infty}$  of Banach spaces. We identify elements of  $\mathcal{B}((\bigoplus_{n=1}^{\infty} E_n)_{\ell_p})$  with *matrices*  $(T_{ij})_{i,j \in \mathbb{N}}$ , where  $T_{ij} \in \mathcal{B}(E_j, E_i)$  ( $i, j \in \mathbb{N}$ ). Let  $(e_n)_{n=1}^{\infty}$  be the canonical basis of  $\ell_1$ . For each  $n \in \mathbb{N}$  we define  $\ell_1^n = \text{span}\{e_1, \dots, e_n\}$ .

## 2. THE RESULT

**The main result.** *Let  $p \in (1, \infty)$  and consider the reflexive Banach space  $E = \left(\bigoplus_{n=1}^{\infty} \ell_1^n\right)_{\ell_p}$ . Then  $\mathcal{B}(E)$  is not a Grothendieck space.*

*Proof.* Recall that  $F = \left(\bigoplus_{n=1}^{\infty} \ell_1^n\right)_{\ell_{\infty}}$  contains a complemented copy of  $\ell_1$ . To complete the proof it is enough to embed  $F$  as a complemented subspace of  $\mathcal{B}(E)$ .

One may identify  $\ell_1^n$  with a 1-complemented subspace of  $\mathcal{B}(\ell_1^n)$  via the mapping

$$e_k \mapsto e_k \otimes e_1^* \quad (k \leq n, n \in \mathbb{N}),$$

where  $e_1^*$  stands for the coordinate functional associated with  $e_1$ . Consequently, the space  $D = \left(\bigoplus_{n=1}^{\infty} \mathcal{B}(\ell_1^n)\right)_{\ell_{\infty}}$  contains a complemented subspace isomorphic to  $F$ . Let  $\Delta: D \rightarrow \mathcal{B}(E)$  be the *diagonal embedding*, that is,  $\Delta((T_n)_{n=1}^{\infty}) = \text{diag}(T_1, T_2, \dots)$  ( $(T_n)_{n=1}^{\infty} \in D$ ); this map is well-defined since the decomposition of  $E$  into the subspaces  $\ell_1^1, \ell_1^2, \dots$  is unconditional.

It is enough to notice that  $\Delta$  has a left-inverse  $\Xi: \mathcal{B}(E) \rightarrow D$  given by

$$\Xi(T_{ij})_{i,j \in \mathbb{N}} = (T_{ii})_{i=1}^{\infty} \quad ((T_{ij})_{i,j \in \mathbb{N}} \in \mathcal{B}(E)),$$

which is bounded. To this end, we shall perform a construction inspired by a trick of Tong (cf. [8, Theorem 2.3] and its proof). With each operator  $T = (T_{ij})_{i,j \in \mathbb{N}} \in \mathcal{B}(E)$  we shall associate a sequence  $(S^{(n)})_{n=1}^{\infty}$  of finite-rank perturbations of  $T$  such that for each  $n \in \mathbb{N}$  we have  $\|S^{(n)}\| \leq \|T\|$  and the matrix of  $S^{(n)}$  agrees with the matrix of the diagonal operator  $\text{diag}(-T_{11}, \dots, -T_{nn}, 0, 0, \dots)$  at entries  $(i, j)$  with  $i \leq n$  or  $j \leq n$ . This will immediately yield that

$$\|\Xi(T)\| = \sup_{n \in \mathbb{N}} \|T_{nn}\| = \sup_{n \in \mathbb{N}} \|-S_{nn}^{(n)}\| \leq \sup_{n \in \mathbb{N}} \|S^{(n)}\| \leq \|T\|.$$

Define operators  $T_k, T_r$  which have the same columns and rows as  $T$  respectively, except the first ones, where we instead set  $(T_k)_{i1} = -T_{i1}$  and  $(T_r)_{1j} = -T_{1j}$  for  $i, j \in \mathbb{N}$  (these are indeed elements of  $\mathcal{B}(E)$  as rank-1 perturbations of  $T$ ). Certainly,  $\|T\| = \|T_k\| = \|T_r\|$  and the norm of  $S = (T_k + T_r)/2$  does not exceed the norm of  $T$ . Arguing similarly, we observe that  $\|(S_k^{(n)} + S_r^{(n)})/2\| \leq \|T\|$ , where  $S^{(1)} = S$  and  $S^{(n+1)} = (S_k^{(n)} + S_r^{(n)})/2$  ( $n \in \mathbb{N}$ ). Consequently,  $(S^{(n)})_{n=1}^{\infty}$  is the desired sequence.  $\square$

**Remark.** The space  $\mathcal{B}(E)$  shares with the space of operators on a Hilbert space a number of common properties. For instance, since  $E$  has a Schauder basis,  $\mathcal{B}(E)$  can be identified with the bidual of  $\mathcal{K}(E)$ , the space of compact operators on  $E$ . Nonetheless,  $E$  is plainly not superreflexive and  $\mathcal{B}(E)$  fails to have weakly sequentially dual for the obvious reason that  $\ell_{\infty}$  embeds into  $\mathcal{B}(E)^*$ . We conjecture that the space of operators on a superreflexive space is Grothendieck (or at least it has weakly sequentially complete dual).

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