Planar aggregation and the Brownian web

Amanda Turner

Lancaster University

Department of Mathematics and Statistics, Lancaster University, Lancaster, UK

Abstract

Diffusion limited aggregation (DLA) is a random growth model which was originally introduced in 1981 by Witten and Sander. This model is prevalent in nature and has many applications in the physical sciences as well as industrial processes. We consider a simplified version of DLA known as the Hastings-Levitov HL(0) model, and show that under certain scaling conditions this model gives rise to a limit object known as the Brownian web.

Keywords: Brownian web, random growth model, diffusion-limited aggregation, conformal mapping, stochastic flow

AMS: 82B24

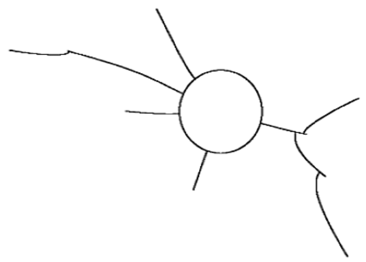
1. Introduction

The Brownian web can loosely be defined as a family of coalescing Brownian motions, starting at all points in continuous space-time. Arratia [1] first considered this object in 1979 as a limit for discrete coalescing random walks, and since then it has been studied by Tóth and Werner [7] and Fontes, Isopi, Newman and Ravishankar [3] amongst others.

Our motivation for looking at the Brownian web arises from a surprising connection with Hastings-Levitov diffusion limited aggregation (DLA). DLA is a random growth model which was originally introduced in 1981 by Witten and Sander [8]. In this model particles perform Brownian motions in the plane until they collide with a cluster at the origin, at which point they stick to the cluster. In 1998 Hastings and Levitov [4] formulated a family of models HL(α), parameterized by $\alpha \in [0, 2]$, of which DLA corresponds to the case HL(2). In these models the cluster is represented by a sequence of iterated conformal maps. We study the HL(0) model and show that the boundary values of the associated process of conformal mappings converge to the Brownian web.

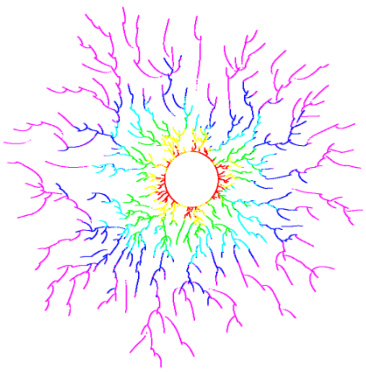
We consider the case of the HL(0) model where the incoming particles are slits of length N^{-1} sticking to the unit disc. If time is scaled in such a way that particles arrive as a Poisson process of rate proportional to N^3, the resulting flow map, restricted to points on the unit circle, converges to the Brownian web.

This paper consists largely of results from the 2008 paper of Norris and Turner [6]. The simplified Hastings-Levitov DLA model is discussed in Section 2. In Section 3 we establish scaling limits for the the boundary of the associated process and show that the finite dimensional distributions converge to those of coalescing Brownian motions.

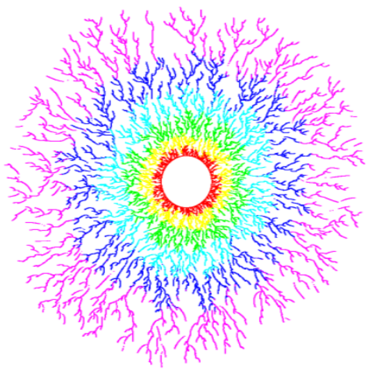


(a) The HL(0) cluster after a few arrivals with $N = 1$.

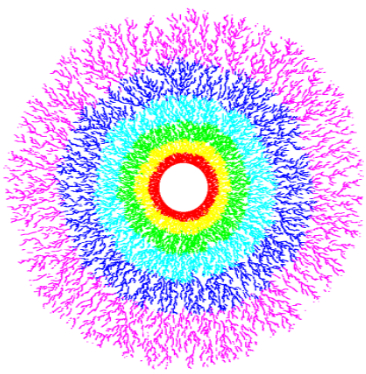
(b) The HL(0) cluster after 100 arrivals with $N = 1$.



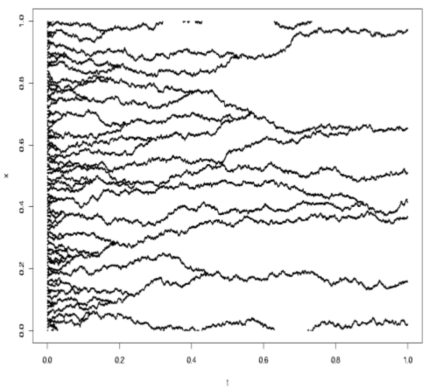
(c) The HL(0) cluster after 800 arrivals with $N = 10$.



(d) The HL(0) cluster after 5000 arrivals with $N = 25$.



(e) The HL(0) cluster after 20000 arrivals with $N = 50$.



(f) The stochastic flow $(X_{t0})_{t\in[0,1]}$ with $N = 50$.

Figure 1: *The slit model case of the Hastings-Levitov* HL(0) *model*

2. Hastings–Levitov growth models

Hastings and Levitov formulated a family of growth models in 1998 [4] in which the cluster is represented by a sequence of iterated conformal maps. We describe the construction of the special case of the HL(0) model below.

Suppose that $\mathbb{D}$ is the unit disc, and let A be a compact subset of $\mathbb{C}$, of diameter r_0 such that $K = A \cup \overline{\mathbb{D}}$ is simply connected. Set $D_0 = \mathbb{C} \setminus \overline{\mathbb{D}}$ and $D = \mathbb{C} \setminus K$. There is a unique conformal map $g : D \to D_0$ and constant $\kappa \in [0, \infty)$ such that $g(z) \sim e^{-\kappa} z$ as $z \to \infty$ [5].

For $\theta \in [0, 1)$ and $z \in e^{2\pi i\theta} D$, set $g_\theta(z) = e^{2\pi i\theta} g(e^{-2\pi i\theta} z)$. Let $(\Theta_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, distributed uniformly on $[0, 1)$ and suppose $0 = T_0 < T_1 < \cdots$ are jump times of a Poisson process of rate η (to be determined).

Define processes $(K_t)_{t\geq 0}$, $(D_t)_{t\geq 0}$ and, for each $z \in D_0$, $(X_t(z))_{t<\zeta(z)}$ as follows. Set $X_0(z) = z$. Recursively for $n \geq 0$, if $X_{T_n}(z) \in e^{2\pi i\Theta_{n+1}} K$, set $\zeta(z) = T_{n+1}$, otherwise, set $X_{T_{n+1}}(z) = g_{\Theta_{n+1}}(X_{T_n}(z))$. Then, for $T_n \leq t < T_{n+1} \wedge \zeta(z)$, let $X_t(z) = X_{T_n}(z)$. Define $D_t = \{z \in D_0 : \zeta(z) > t\}$ and $K_t = \mathbb{C} \setminus D_t$. Note that X_t is the unique conformal map $D_t \to D_0$ with $X_t(z) \sim e^{-\kappa\nu_t} z$ as $z \to \infty$. The set K_t represents the cluster formed by particles with shape $A \setminus \mathbb{D}$.

We are interested in the asymptotics of these flows X (and the corresponding clusters K) in the limit as r_0 (and hence κ) tends to 0. In particular, we study the case where A is the slit $\{z = iy : 1 \leq y \leq 1 + N^{-1}\}$ as $N \to \infty$ (see Figure 1).

The process X exhibits some interesting and unexpected behaviour. In particular, let us consider how the boundary of the disc evolves under appropriate scaling. There exists some non-decreasing continuous function $f : (0, 1) \to (0, 1)$ for which $g(e^{2\pi ix}) = \exp(2\pi i f(x))$ for all $x \in (0, 1)$. Define a right continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(x + n) = f(x) - (x + n)$ for all $x \in (0, 1)$ and $n \in \mathbb{N}$. Let $X_{ts}(x)$ be the position at time t of the point on the boundary that was at x at time s, under the identification of the boundary with the interval $[0, 1)$. Then, if μ is a Poisson random measure with intensity $\eta dr dz$, $X_{ts}(x)$ satisfies

$$X_{ts}(x) = x + \int_s^t \int_0^1 \tilde{f}(X_{r-,s}(x) - z)\mu(dr, dz).$$

Furthermore, the map $x \mapsto X_{ts}(x)$ expresses how the harmonic measure on ∂K_s is transformed by the arrival of new particles up to time t.

3. Convergence to the Brownian web

The Brownian web is the collection of graphs of coalescing one-dimensional Brownian motions $(B_{ts}(x))_{t\geq s}$ starting from all possible points (s, x) in continuous space-time.

In this section we show that for an appropriately chosen scaling factor η, the finite dimensional distributions of the flow X converge to those of the Brownian web. It is possible to prove the stronger result that the flow X converges to the Brownian web in a space of flows, but for this we refer the reader to the paper of Norris and Turner [6].

Let $\eta = \eta(f)$ be the positive real number that satisfies

$$\eta \int_0^1 \tilde{f}(x)^2 dx = 1. \tag{1}$$

It can be shown that, for the slit, $\eta(f) = 6N^3\pi^3 + o(N^3) \to \infty$ as $N \to \infty$.

Theorem. Let X be constructed as above. Then the following hold.

(i) As $N \to \infty$, $(X_{ts}(x))_{t \geq s} \Rightarrow (B_{ts}(x))_{t \geq s}$, where $(B_{ts}(x))_{t \geq s}$ is a standard Brownian motion, starting from x.

(ii) Given any $n \in \mathbb{N}$ and $(s_1, x_1), \ldots, (s_n, x_n) \in \mathbb{R}^2$, $((X_{ts_1}(x_1))_{t \geq s_1}, \ldots, (X_{ts_n}(x_n))_{t \geq s_n}) \Rightarrow ((B_{ts_1}(x_1))_{t \geq s_1}, \ldots, (B_{ts_n}(x_n))_{t \geq s_n})$ as $N \to \infty$, where B is a Brownian web.

Proof.

(i) Denote $X_t = X_{ts}(x)$, $t \geq s$. By Itô's formula,

$$\exp\left(i\theta X_t - \left(\int_{(s,t]\times[0,1)} \left(e^{i\theta \tilde{f}(X_r - u)} - 1 - i\theta \tilde{f}(X_r - u) \right) \eta dr du \right) \right)$$

is a martingale and hence, for any $\theta \in \mathbb{R}$ and $t_2 \geq t_1 \geq s$, by Taylor's theorem, Fubini's theorem and substituting u for $X_r - u$,

$$\mathbb{E}(e^{i\theta(X_{t_2} - X_{t_1})}) = \exp\left(-\theta^2 (t_2 - t_1) \int_0^1 (1-z)\eta \int_{[0,1)} \tilde{f}(u)^2 e^{iz\theta \tilde{f}(u)} du dz \right).$$

By (1)

$$\left| \eta \int_{[0,1)} \tilde{f}^2(u) e^{iz\theta \tilde{f}(u)} du - 1 \right| \leq z|\theta|\eta \int_{[0,1)} |\tilde{f}(u)|^3 du \leq z|\theta| \|\tilde{f}\| \to 0$$

as $N \to \infty$. Hence $\mathbb{E}(e^{i\theta(X_{t_2} - X_{t_1})}) \to \exp\left(-\frac{1}{2}\theta^2(t_2 - t_1)\right)$.

As X_t has independent increments, the finite dimensional distributions converge to those of Brownian motion. By a standard argument (see [2], page 143, for example) the family of laws $(X_{ts}(x))_{t \geq s}$ is tight and hence $(X_{ts}(x))_{t \geq s} \Rightarrow (B_{ts}(x))_{t \geq s}$ as $N \to \infty$.

(ii) Denote $X_t^k = X_{ts_k}(x_k)$. Convergence of the finite dimensional distributions and tightness of $(X_t^1, \ldots, X_t^n)$ can be shown by similar arguments to (i). Let $(Z_t^1, \ldots, Z_t^n)$ be some limit process. For j, k distinct, let $T^{jk} = \inf\{t \geq s_j \vee s_k : Z_t^j - Z_t^k \in \mathbb{N}\}$.

The process

$$X_t^j X_t^k - \int_{s_j \vee s_k}^t b(X_s^j, X_s^k) ds, \quad t \geq s_j \vee s_k,$$

is a martingale, where

$$b(x, x') = \eta \int_0^1 \tilde{f}(x - z) \tilde{f}(x' - z) dz.$$

It can be shown that there exists some $\delta = O(N^{-\frac{1}{4}})$ such that $|b(x,x')| \leq \delta$ whenever $\delta \leq |x - x'| \leq 1 - \delta$. Hence, the process $(Z_t^j Z_t^k : s_j \vee s_k \leq t < T^{jk})$ is a local martingale. From (i), the processes $(Z_t^j : t \geq s_j)$, $((Z_t^j)^2 - t : t \geq s_j)$ and $(Z_t^k : t \geq s_k)$ are continuous local martingales. But Z inherits from the X the property that, almost surely, for all $n \in \mathbb{Z}$, the process $(Z_t^j - Z_t^k + n : t \geq s_j \vee s_k)$ does not change sign. Hence, by an optional stopping argument, $Z_t^j - Z_t^k$ is constant for $t \geq T^{jk}$. It follows that $(Z_t^j Z_t^k - (t - T^{jk})^+)_{t \geq s_j \vee s_k}$ is a continuous local martingale, and so the limit process is that of coalescing Brownian motions.

Corollary. Let $x_1, \ldots, x_n$ be a positively oriented set of points in $\mathbb{R}/\mathbb{Z}$ and set $x_0 = x_n$. For $k = 1, \ldots, n$, write H_t^k for the harmonic measure in K_t of the boundary segment of all fingers in K_t attached between x_{k-1} and x_k. Let $(B_t^1, \ldots, B_t^n)_{t \geq 0}$ be a family of coalescing Brownian motions in $\mathbb{R}/\mathbb{Z}$ starting from $(x_1, \ldots, x_n)$. Then, in the limit $N \to \infty$, $(H_t^1, \ldots, H_t^n)_{t \geq 0}$ converges weakly to $(B_t^1 - B_t^0, \ldots, B_t^n - B_t^{n-1})_{t \geq 0}$.

Acknowledgements: This research was carried out in collaboration with James Norris, University of Cambridge, UK.

REFERENCES

[1] Arratia, R. (1979), *Coalescing Brownian motions on the line*, Ph.D. dissertation, University of Wisconsin, Madison.

[2] Billingsley, P. (1999), *Convergence of Probability Measures*, Second Edition, John Wiley and Sons, New York.

[3] Fontes, L.R.G., Isopi, M., Newman, C.M., Ravishankar, K. (2004), The Brownian Web: Characterization and Convergence, *Annals of Probability* 32, 2857-2883.

[4] Hastings, M.B. and Levitov, S.L. (1998), Laplacian growth as one-dimensional turbulence, *Physica D* 116 (1-2), 244.

[5] Lawler, G.F. (2005), *Conformally Invariant Processes in the Plane*, American Mathematical Society.

[6] Norris, J. and Turner,A. (2008), Planar aggregation and the coalescing Brownian flow, *arXiv preprint.*

[7] Tóth, B. and Werner, W. (1998), The true self-repelling motion,*Probability Theory and Related Fields* 111, 375-452.

[8] Witten, T.A. and Sander, L.M.(1981), Diffusion-Limited Aggregation, a Kinetic Critical Phenomenon, *Physical Review Letters* 47, 1400-1403.