

Geodesic completeness and homogeneity condition for cosmic inflation

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There are two disjointed problems in cosmology within General Relativity (GR), which can be addressed simultaneously by studying the nature of geodesics around $t \rightarrow 0$, where t is the physical time. One is related to the past geodesic completeness of the inflationary trajectory due to the presence of a cosmological singularity, and the other one is related to the homogeneity condition required to inflate a local space-time patch of the universe. We will show that both the problems have a common origin, arising from how the causal structure of null and timelike geodesics are structured within GR. In particular, we will show how a non-local extension of GR can address both problems, while satisfying the *null* energy condition for the matter sources.

Primordial inflation is extremely successful in explaining the current observed universe [1]. However, there are many fundamental issues with inflation. Two of the most important ones are related to its embedding within General Relativity (GR) ¹.

- *Geodesic incompleteness*: Due to the inevitability of a cosmological singularity within GR, inflationary trajectories are past-incomplete [4]. One can see that this warrants a better theory of gravity in the ultraviolet (UV), which would ameliorate the UV divergences as well as make the theory *singularity-free* in the UV, for instance [5–8]. Such a singularity-free universe would yield a non-singular bouncing cosmology, and possibly this would leave some falsifiable imprints in the sky [9].
- *Homogeneity condition*: Slow roll inflation requires a patch of the universe to be sufficiently homogeneous on super-Hubble scales, see [10, 11], see also [12, 13]. In this respect, inflation within GR does not solve the homogeneity problem - it assumes homogeneity to begin with. Even if inflation begins at the Planckian epoch, one requires the spatial gradient terms in the action of the inflaton field (whose slow roll leads to inflation) to be sufficiently negligible compared to the homogeneous, time dependent terms.

A priori, these two problems seem to be unrelated. However, they have a common origin and if the first one is addressed, then the second one can also be understood, which would lead to a better understanding of inflation within a UV complete theory of gravity [14]. They are both related to the causal structure of the spacetime within GR, assuming the weak energy condition (WEC) for the matter field, i.e. $\rho \geq 0$ and $\rho + p \geq 0$, which necessarily implies the null energy condition (NEC), $\rho + p \geq 0$,

where ρ is the energy density and p is the pressure component.

The main aim of this paper is to build this connection and show how a geodesically past-complete universe would naturally evade the constraints of the homogeneity condition for slow roll models of inflation. We will illustrate this problem by modifying the UV aspects of gravity and therefore modifying the causal structure of the spacetime. In particular, we will invoke a non-local modification of GR, which can be made ghost-free in the UV, while also recovering GR and its predictions in the infrared (IR) [5, 6].

Causal structure of spacetime and the Raychaudhuri Equation: The structure of a singularity can be understood in a *model independent* way by studying the Raychaudhuri Equation (RE) for timelike and/or null geodesic congruences. For simplicity, we consider only null geodesic congruences such that $k^\mu k_\mu = 0$, where k_μ is a four vector tangential to the null geodesic congruence, defined by mostly positive convention, i.e. $(-, +, +, +)$, and the expansion parameter, θ , defined by $\theta = \nabla_\mu k^\mu$. Let us concentrate on the simplest possible scenario where the *twist* tensors vanish, which is true if we take the congruence of null rays to be orthogonal to the hypersurface. Furthermore, the shear tensor is purely spatial and thus makes a positive contribution. Taking all of this into account the RE can be simplified greatly, see [15]

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq -R_{\mu\nu}k^\mu k^\nu \quad (1)$$

where τ is the affine parameter, and $R_{\mu\nu}$ is the Ricci tensor.

We know from the Einstein equation that $G_{\mu\nu} = \kappa T_{\mu\nu}$, where $\kappa = 8\pi G = M_p^{-2}$, which in turn implies $R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$. Now, contracting with the vector field k^μ , we find $R_{\mu\nu}k^\mu k^\nu = \kappa T_{\mu\nu}k^\mu k^\nu$. Finally, imposing the NEC, $T_{\mu\nu}k^\mu k^\nu \geq 0$, we obtain the *null convergence condition* (null CC) expressed in two equivalent ways:

$$R_{\mu\nu}k^\mu k^\nu \geq 0, \quad \frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq 0 \quad (2)$$

¹ Inflation has many other challenges, see [2, 3].

This suggests that the converging null geodesics cannot start to diverge before meeting the *origin of coordinates* or, in other words, the converging null geodesic must meet the space-like singularity in a finite time within GR, where the NEC is satisfied [16, 17].

Trapped, antitrapped and normal surfaces: In an asymptotically flat spacetime, *trapped* surfaces and an apparent horizon are formed when both the ingoing and outgoing expansions are negative, i.e. $\theta_{IN,OUT} < 0$. A period of cosmic acceleration with positive ingoing and outgoing expansion, $\theta_{IN,OUT} > 0$, gives rise to *antitrapped* surfaces and *normal* regions are defined by the behaviour $\theta_{IN} < 0$ and $\theta_{OUT} > 0$. Any surface with physical size greater or equal to the *minimally antitrapped surface* (MAS) has *vanishing expansion* and is, by definition, antitrapped. Within the Friedmann-Robertson-Walker (FRW) metric, $ds^2 = dt^2 - a^2(t)dr^2$, where $a(t)$ is the scale factor and r is the coordinate of 3 spatial directions, $x_{MAS} = H^{-1}(t)$, where $H(t) \equiv \dot{a}(t)/a(t)$ and the physical size of the MAS is represented by x_{MAS} . The inner boundary of such a surface is known as the *cosmological apparent horizon* and is defined as the inverse of the physical distance of the MAS of the background cosmology such that

$$x_{FRW} = x_{MAS} = H_{FRW}^{-1} \quad (3)$$

Similarly, in Fig.1 the line OQ denotes the inflationary patch, with the segment OP equal to the inverse of the inflationary apparent horizon which is necessarily of smaller physical size to the MAS. Now, in the usual FRW universe - complete with an initial singularity by virtue of the NEC condition within GR, see Eq. (2) - the ingoing null ray cannot go from a normal region to an antitrapped region, as depicted by the arrow. As argued in [11], the inflationary patch must be *embedded* already within an antitrapped region of spacetime in order to trigger inflation without violating the null CC. The conclusion was that late inflation requires a prior phase of inflation. However with an impending singularity in the past, one would be left with inflation occurring already at the Planckian epoch [10, 11].

Non-singular bouncing cosmology: As we shall see below, a non-singular bouncing cosmology naturally leads to an accelerated expansion near the bounce, $\ddot{a}(t) > 0$. The challenge is to realise a non-singular bounce which requires modification to GR. In particular, a reversal of the inequality in the null CC without violating the NEC, would allow the converging null rays to be made past complete, thus resolving *not only* the cosmological singularity problem, but also allowing the arrow shown in Fig.1 to go from a normal region of spacetime to an antitrapped region of spacetime. Therefore, the *homogeneity* condition for inflation is ameliorated, especially at later stages.

In a word, a non-singular bouncing cosmology naturally provides all the necessary conditions for successful

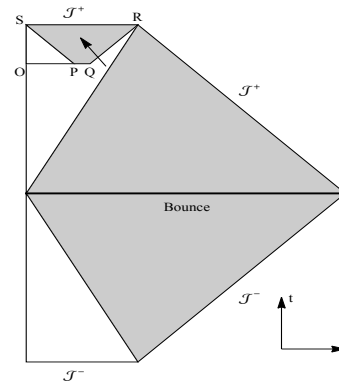


FIG. 1: A conformal diagram depicting a bouncing cosmology in an isotropic and homogeneous FRW metric. Shaded regions are antitrapped and unshaded regions are normal. A patch begins to inflate at later times, t , from O to Q with inflationary size x_{inf} , where the line OP borders the apparent inflationary horizon. The arrow, pictured, describes an ingoing null ray entering an antitrapped surface from a normal surface. All notations are as in [11].

inflation, which must occur at later stages in order to produce the large scale structures present in the universe.

Modifying GR in the UV: We may now ask the question: what modification of GR would yield a reversal of the inequality contained within the null CC, such that

$$R_{\mu\nu}k^\mu k^\nu \leq 0, \quad \frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \geq 0 \quad (4)$$

and thus describe a singularity-free theory of gravity, whilst retaining the NEC? There are two generic ways in which this may be satisfied.

- *Local modification of GR:* Higher order corrections such as $\mathcal{L} \sim c_2 R^2 + c_3 R^3 + \dots + d_2 R_{\mu\nu}^2 + \dots + e_2 R_{\mu\nu\lambda\sigma}^2 + \dots$, with appropriate coefficients would modify the UV behaviour of gravity. The higher derivatives help to ameliorate the UV aspects of gravity in 4 dimensions but they typically contain *ghosts*. This has been known from the days of Stelle's theory of 4th order gravity, which is renormalizable but contains massive ghosts [18].
- *Non-local modification of GR:* The ghost problem can be addressed in the case of *infinite* higher order derivatives. Let us concentrate on quadratic curvature with $\mathcal{L} \sim R\mathcal{F}(\square)R + R_{\mu\nu}\mathcal{G}(\square)R^{\mu\nu}$, where $\mathcal{F}(\square)$, $\mathcal{G}(\square)$ are *analytic functions* containing higher derivatives up to infinite order, where $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian operator. In the Minkowski background, these comprise the most generalised action of gravity with non-local contributions, yielding a *ghost-free* condition for certain analytic choices of \mathcal{F} , \mathcal{G} , constructed, necessarily, from an *entire function* [5–8].

Explicit Example: In order to illustrate and for the remainder of the paper, let us concentrate on a non-local generalisation of GR in the UV, up to quadratic order in curvature,

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R + \frac{R \mathcal{F}(\square) R}{2} \right) \quad (5)$$

where $\mathcal{F}(\square) \equiv \sum_{n=0}^{\infty} \frac{f_n}{M_P^{2n}} \square^n$ and, without loss of generality, we have assumed that non-renormalisable operators are suppressed by the 4 dimensional scale of gravity, i.e. M_P .

It is important to add, that the RE and its convergence conditions hold independently of the background action. However, we must introduce an action in order to compute the Ricci tensor. We compute the equations of motion as in [19], and impose the FRW metric.

In order to understand the nature of these null geodesic congruences, let us concentrate on the simplest regime with a *homogeneous* metric near $t = 0$. To this end, we may describe the scale factor expanding around $t = 0$, as follows

$$a(t) = 1 + a_2 t^2 + \mathcal{O}(t^4) + \dots \quad (6)$$

Note that we may consider *even* powers of t to understand the solution around $t = 0$, with coefficient $a_2 > 0$ as a consequence of $\ddot{a}(t) > 0$ at the bounce point, since we are interested in seeking a non-singular solution. This implies that $H(t)$ is an *odd* function of physical time t , while $R(t) = 12H^2(t) + 6\dot{H}(t)$ is an even function of t such that

$$R(t) = R_0 + R_2 t^2 + \mathcal{O}(t^4) + \dots \quad (7)$$

with $R_0 > 0$ at the bounce, as $\lim_{t \rightarrow 0} R(t) = 12a_2$ and, as we have already stated, $\ddot{a} > 0$ implies $a_2 > 0$ at the bounce point.

By solving for the equations of motion of the action given by Eq. (5) (for details, see [19]), one can extract the Ricci tensor $R_{\mu\nu}$ and contract with the vector field k^μ to find, at the bounce point $t = 0$:

$$R_{\mu\nu} k^\mu k^\nu = (k^0)^2 \frac{(\rho + p) + 2\partial_t^2(\mathcal{F}(\square)R)}{M_P^2 + 2\mathcal{F}(\square)R}. \quad (8)$$

Next, we must compute the non-local terms $\mathcal{F}(\square)R$ and $\partial_t^2 \mathcal{F}(\square)R$, Eqs. (15,16). For detailed steps, see Appendix.

Crucially, we are now in a position to deduce that, in order for the l.h.s. of Eq. (8) to be negative, i.e. $R_{\mu\nu} k^\mu k^\nu \leq 0$, thus determining the conditions for which null geodesics can be made past-complete, the following inequalities must hold for either upper or lower signs,

$$\frac{(\rho + p)}{2R_0} \leq y \mathcal{F}(y), \quad \frac{M_P^2}{2R_0} \geq -\mathcal{F}(y), \quad (9)$$

where $y \equiv -2R_2/R_0$ is defined in the Appendix, and we have assumed the NEC to hold true always.

Ghost-Free choice: Following Ref. [5, 6], a particular class of $\mathcal{F}(\square)$ can be chosen in order to make a non-local theory of gravity ghost-free without violating general covariance. Here, we choose [5, 6]:

$$\mathcal{F}(\square) = \frac{e^{-\square/M_P^2} - 1}{\square/M_P^2} \quad (10)$$

Note that due to the particular nature of Eq. (10), we have $\mathcal{F}(y < 0) < -1$, and $-1 \leq \mathcal{F}(y \geq 0) < 0$.

In order to extract the physics, we may entertain the simplest interesting scenario when the *curvature* of the universe is evolving *adiabatically* near the Planck scale, in such a way that R_2 is small, i.e. $R_2 \ll R_0$ in Eq. (7). This is justifiable since we are ignoring the higher order terms in both our expressions, Eqs. (6, 7). We will be interested then in a limit when $y \rightarrow 0$. Note that at this point R_2 could be either positive or negative.

- $R_2 \geq 0$: when $y \rightarrow 0$, the *lower signs* in Eq. (9) yield:

$$\frac{(\rho + p)}{2R_0} > 0, \quad \frac{M_P^2}{2R_0} < 1. \quad (11)$$

This tells us that the NEC must be satisfied, irrespective of R_0 , and $R_0 > M_P^2/2$ at the bounce. Note that one would naturally expect $R_0 \leq M_P^2$ at the time of the bounce.

The *upper signs* do not yield any physically motivated solution when $R_2 \geq 0$.

- $R_2 \leq 0$: The first inequality in Eq. (9) holds true with the lower sign as long as $\rho + p > 0$, yielding:

$$\frac{M_P^2}{2R_0} < -\mathcal{F}(y) \leq 1 \quad (12)$$

which is analogous to Eq. (11)

For either $R_2 \geq 0$ or $R_2 \leq 0$, one ensures that the null geodesics are past-complete and a non-singular bouncing cosmology can be constructed near $t \sim 0$, at the limit of $y \rightarrow 0$, without violating the NEC.

Past-completeness of null geodesics implies the past-completeness of timelike geodesics, independent of any choice of $\mathcal{F}(\square)$, as can be shown by a straightforward computation.

Discussion: In this paper, we pointed out a neat solution for two disjointed problems of inflationary cosmology. We argued that a non-singular bouncing cosmology is a must for a successful inflationary paradigm, where one can make the inflationary trajectory past-complete, while also explaining the pre-requisite homogeneity condition for inflation by modifying the Raychaudhuri equation. Thus, a reversal of the inequality in the null CC,

see Eqs. (2, 4), allows both a non-singular bounce and the movement of null rays from a normal surface to anti-trapped surface, without violating the NEC in the matter sector.

We showed that a non-local modification of GR would modify the Raychaudhuri equation in such a way that a bouncing non-singular cosmology can be constructed. In particular, we have shown that for a slowly varying curvature with a scale, at bounce, comparable to the Planck scale within a homogeneous and isotropic metric, one can avoid a cosmological singularity. The matter at the same time enjoys the NEC.

Our results have implications for completing the past inflationary geodesics and also ameliorating the *homogeneity* conditions for inflation at later epochs. Note that a non-singular bounce naturally provides a platform for an inflationary cosmology, by virtue of $\ddot{a}(t) > 0$, at the bounce point, thus providing the initial conditions for primordial inflation. Similar analyses for avoiding cosmological singularity may be performed for a variety of modifications to GR in the UV.

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APPENDIX

Here we compute the non-local terms $\mathcal{F}(\square)R$ and $\partial_t^2 \mathcal{F}(\square)R$. We do this by first calculating $e^{s\square}$, where s is a constant. Using the diffusion equation method [20], we find for $t \rightarrow 0$

$$(e^{s\square}R)(0) = R_0 e^{sy} \quad (13)$$

where we have defined $y \equiv -2R_2/R_0$. We then represent the operator $\mathcal{F}(\square)$, using the inverse Laplace integral transform, with the integration contour such that all poles are on one side of the contour with α , real

$$\mathcal{F}(\square) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \tilde{\mathcal{F}}(s) e^{s\square} ds, \quad (14)$$

The next step is to solve for the relevant non-local terms at the bounce. In the first instance, we find $\mathcal{F}(\square)R$ at

the bounce to be

$$(\mathcal{F}(\square)R)(0) = R_0 \mathcal{F}(y) \quad (15)$$

In order to compute, the second time derivative, we note that we are effectively computing $(-\square \mathcal{F}(\square)R)(0)$. The properties of the Laplace transform yield

$$(\partial_t^2 \mathcal{F}(\square)R)(0) = 2R_2 \mathcal{F}(y). \quad (16)$$

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