# THE CONVERGENCE OF UNITARY QUANTUM RANDOM WALKS

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Abstract. We give a simple and direct treatment of the convergence of quantum random walks to quantum stochastic operator cocycles, using the semigroup method. The pointwise product of two such quantum random walks is shown to converge to the quantum stochastic Trotter product of the respective limit cocycles. Since such Trotter products themselves reduce to pointwise products when the cocycles inhabit commuting subspaces of the system algebra, this yields an elementary approach to the quantum random walk approximation of the 'tensorisation' of cocycles with common noise dimension space. The repeated quantum interactions model is shown to fit nicely into the convergence scheme described.

## Contents



## Introduction

Quantum random walks have been a feature of quantum probability for at least twenty-five years; as emphasised by Bouten and van Handel [BvH], "the convergence of discrete quantum Markov chains to continuous ones is a fundamental problem in quantum probability". In Meyer's book [$\mathrm{Me}_2$], Journé is credited as the first to use discrete approximations to Fock space and to quantum stochastic processes; around the same time, Accardi and Bach ([AcB], [$\mathrm{Me}_1$]) proved a central-limit theorem which yields the quantum harmonic oscillator as a limit of quantum Bernoulli processes. Further early work, by Parthasarathy ([Par]) and by Lindsay and Parthasarathy ([LiP]), showed that certain quantum stochastic flows, which are generalisations of classical diffusions, may be approximated by so-called spin random walks.

As well as their probabilistic interpretation as noncommutative Markov chains, quantum random walks may also be seen as models for the dynamics of a quantum-mechanical system undergoing repeated interactions with an environment composed of an infinite number of identical particles. Attal and Pautrat adopted this point of view in [AtP], with a repeated-interactions model of quantum random walks; in [Gou], Gough showed the links between the repeated-interactions model and Holevo's time-ordered exponentials ([Hol]). Work by Belton ([$\mathrm{B}_{1–3}$]) produced a theory of quantum random walks generated by completely bounded maps on operator spaces, admitting the treatment of particle algebras in an arbitrary normal state. Das and Lindsay extended the theory to quantum random walks in Banach algebras ([DaL]). These convergence theorems may

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be considered to be analogues of Donsker's invariance theorem, with the limit process being a quantum stochastic cocycle rather than a classical Wiener process.

There have been many applications of quantum random walks: to quantum filtering and quantum feedback control ([BvHJ], [GoS]); to the approximation of quantum Lévy processes ([FrS], [LiS]); to the construction of dilations of quantum dynamical semigroups ([B$_1$], [Sah]). Repeated-interactions models for the one-atom maser, an important system in quantum optics [GaZ], have been investigated by Bruneau, Joye and Merkli ([BJM]) and by Bruneau and Pillet ([BrP]); in contrast to the results we prove below, the convergence theorems they obtained give only the reduced dynamics of the limit system and disregard the limit behaviour of the environment. Gohm has found ([Ghm]) interesting connections between noncommutative Markov chains and multivariate operator theory.

Here we use the semigroup decomposition of quantum stochastic cocycles and the notion of associated semigroups, introduced by Lindsay and Wills (see [L$_1$]), to give a new short and direct proof of the convergence of suitably scaled quantum random walks to quantum stochastic cocycles.

Our main convergence theorem, Theorem 3.3, allows us to provide short and transparent demonstrations of results on repeated-interactions models previously proved by Attal and Pautrat ([AtP]), Attal and Joye ([AtJ]) and Attal, Deschamps and Pellegrini ([ADP]); see Examples 3.6 and 4.3.

In a sister paper ([BGL]), we consider embeddings of toy Fock space appropriate to faithful states on the particle algebra, and obtain quasifree stochastic cocycles, in the sense of [LiM], as limits of scaled random walks in that setting.

*Notation.* We make extensive use of the following extension to the Dirac bra-ket notation. For a vector $u$ in a Hilbert space $\mathsf{h}$, the operators $\mathsf{H} \to \mathsf{h} \otimes \mathsf{H}$ and $\mathsf{H} \to \mathsf{H} \otimes \mathsf{h}$ given by $\xi \mapsto u \otimes \xi$ and $\xi \mapsto \xi \otimes u$, are denoted $E_u$; their adjoints are denoted $E^u$. Both the Hilbert space $\mathsf{H}$ and the appropriate order is always clear from the context. The ultraweak tensor product is denoted $\overline{\otimes}$. We use the following notation for the symmetric Fock space over a Hilbert space $\mathsf{h}$ and exponential vectors: $\Gamma(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}^{\vee n}$, where $\mathsf{h}^{\vee n}$ denotes the $n$-fold symmetric tensor power of $\mathsf{h}$ for $n \geq 1$ and $\mathsf{h}^{\vee 0} := \mathbb{C}$, and

$$\varepsilon(u) := \big((n!)^{-1/2} u^{\otimes n}\big)_{n \geq 0} \qquad (u \in \mathsf{h}).$$

*Fix a Hilbert space* $\mathfrak{h}$ which we refer to as the 'initial space' or 'system space'.

## 1. Quantum stochastic cocycles

In this short section we recall briefly the basic facts that are needed concerning quantum stochastic (QS) analysis, and specifically operator cocycles and their generation via QS differential equations. For further detail, see [L$_1$].

*Fix a second Hilbert space* $\mathsf{k}$, which we refer to as the 'noise dimension space', and set

$$\widehat{c} := \begin{pmatrix} 1 \\ c \end{pmatrix} \in \widehat{\mathsf{k}} \quad (c \in \mathsf{k}).$$

Identifying the Hilbert space $\mathfrak{h} \otimes \widehat{\mathsf{k}}$ with $\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k})$, as we frequently do, each operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ has a block matrix form $\left[\begin{smallmatrix} K & M \\ L & N \end{smallmatrix}\right]$. In particular, the *quantum Itô projection* is given by

$$\Delta := 0_{\mathfrak{h}} \oplus I_{\mathfrak{h} \otimes \mathsf{k}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}} \end{bmatrix}$$

For any subinterval $I$ of $\mathbb{R}_+$, set

$$\mathcal{F}^{\mathsf{k}}_I := \Gamma(L^2(I; \mathsf{k})) \ \text{ and } \ \Omega^{\mathsf{k}}_I := (1, 0, 0, \cdots) \in \mathcal{F}^{\mathsf{k}}_I,$$

abbreviating to $\mathcal{F}^{\mathsf{k}}$ and $\Omega^{\mathsf{k}}$ when $I = \mathbb{R}_+$. Letting $\mathbb{S}_{\mathsf{T}}$ and $\mathbb{S}_{\mathsf{T},\mathrm{loc}}$ denote the subspaces of $L^2(\mathbb{R}_+; \mathsf{k})$ and $L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathsf{k})$ consisting of $\mathsf{T}$-valued step functions, whose right-continuous versions we always take, set $\mathcal{E}_{\mathsf{T}} := \mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{S}_{\mathsf{T}}\}$. (When $\mathsf{T} = \mathsf{k}$ we abbreviate to $\mathbb{S}$, $\mathbb{S}_{\mathrm{loc}}$ and $\mathcal{E}$.) The subspace $\mathcal{E}_{\mathsf{T}}$ is dense in $\mathcal{F}^{\mathsf{k}}$ if and only if the set $\mathsf{T}$ is total ([Ske]; see [L$_1$]). A typical example of $\mathsf{T}$ is an orthonormal basis augmented by the vector 0. The natural identification

$$\mathcal{F}^{\mathsf{k}} = \mathcal{F}^{\mathsf{k}}_{[0,r[} \otimes \mathcal{F}^{\mathsf{k}}_{[r,t[} \otimes \mathcal{F}^{\mathsf{k}}_{[t,\infty[} \qquad (r, t \in \mathbb{R}_+, r \leqslant t). \tag{1.1}$$

witnessed by exponential vectors: $\varepsilon(f) = \varepsilon(f|_{[0,r[}) \otimes \varepsilon(f|_{[r,t[}) \otimes \varepsilon(f|_{[t,\infty[})$, is frequently invoked. We use the notation ${}^{\mathcal{F}}\!I^{\mathsf{k}}_{[r,t[}$ for the corresponding identity operator. The *CCR flow of index* $\mathsf{k}$ is the semigroup of endomorphisms ${}^{\mathcal{F}}\!\sigma^{\mathsf{k}} = ({}^{\mathcal{F}}\!\sigma^{\mathsf{k}}_t)_{t\in\mathbb{R}_+}$ on $B(\mathcal{F}^{\mathsf{k}})$ defined by

$$ {}^{\mathcal{F}}\!\sigma^{\mathsf{k}}_t(T) := {}^{\mathcal{F}}\!I^{\mathsf{k}}_{[0,t[} \otimes S_t T S_t^* $$

where $S_t$ here denotes the unitary shift operator $\mathcal{F}^{\mathsf{k}} \to \mathcal{F}^{\mathsf{k}}_{[t,\infty[}$, again witnessed by exponential vectors: $\varepsilon(f) \mapsto \varepsilon(s_t f)$ where $(s_t f)(s) = f(s-t)$ for $s \in [t,\infty[$.

**Definition.** A *left QS bounded-operator cocycle on* $\mathfrak{h}$ *with noise dimension space* $\mathsf{k}$ is a family of operators $X = (X_t)_{t\geqslant 0}$ in $B(\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ satisfying the following adaptedness, continuity and cocycle conditions:

$$ X_t \in B(\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}_{[0,t[}) \otimes {}^{\mathcal{F}}\!I^{\mathsf{k}}_{[t,\infty[} \qquad (t \in \mathbb{R}_+); $$
$s \mapsto X_s$ is strongly continuous;
$$ X_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}}} \text{ and } X_{r+t} = X_r(\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes}\, {}^{\mathcal{F}}\!\sigma^{\mathsf{k}}_r)(X_t) \qquad (r,t \in \mathbb{R}_+). $$

It is called *Markov regular* if also

$$ s \mapsto X^{f,g}_s \text{ is continuous} \qquad (f,g \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathsf{k})). $$

The notation here is as follows:

$$ X^{f,g}_s := E^{\varepsilon(f_{[0,s[})} X_s E_{\varepsilon(g_{[0,s[})} \ \text{ and } \ f_{[0,s[} := \mathbf{1}_{[0,s[} f. \tag{1.2} $$

A QS cocycle $X$ is called *contractive*, *isometric* or *unitary* if each operator $X_t$ has that property; it is called *quasicontractive* if, for some $\beta \in \mathbb{R}_+$, the QS cocycle $(e^{-\beta t}X_t)_{t\geqslant 0}$ is contractive.

If $X$ is a QS cocycle then, for each $c$, $d \in \mathsf{k}$,

$$ P^{c,d} := (X^{c_{[0,t[},d_{[0,t[}}_t)_{t\geqslant 0} $$

defines a $C_0$-semigroup on $\mathfrak{h}$, and $X$ is Markov regular if and only if each of these *associated semigroups* is norm continuous. Moreover, QS cocycles enjoy the semigroup-decomposition property

$$ X^{f,g}_t = P^{f(t_0),g(t_0)}_{t_1-t_0} \cdots P^{f(t_n),g(t_n)}_{t_{n+1}-t_n} \qquad (f,g \in \mathbb{S}, t \in \mathbb{R}_+) $$

in which the set $\{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t\}$ contains the points of discontinuity of $f_{[0,t[}$ and $g_{[0,t[}$. The semigroup-decomposition property characterises QS cocycles among adapted, strongly continuous QS processes.

The *series product* on $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ is the composition defined by

$$ F_1 \lhd F_2 := F_1 + F_2 + F_1 \Delta F_2. \tag{1.3} $$

See [L$_2$] for the 'quantum Itô algebra' associated with this product. For us here, the following two properties are key: setting $F = F_1 \lhd F_2$,

$$ \text{if } F_i^* \lhd F_i \leqslant \beta_i \Delta^{\perp} \text{ for } i = 1,2, \text{ then } F^* \lhd F \leqslant (\beta_1+\beta_2)\Delta^{\perp}; $$
$$ \text{if } F_i^* \lhd F_i = 0 \text{ for } i = 1,2, \text{ then } F^* \lhd F = 0. $$

The structure relation $F^* \lhd F = 0$ is equivalent to $F$ enjoying the block matrix form

$$ F = \begin{bmatrix} iH - \frac{1}{2}L^*L & -L^*W \\ L & W - I_{\mathfrak{h}\otimes\mathsf{k}} \end{bmatrix}, \text{ with } H \text{ selfadjoint and } W \text{ isometric }, $$

and in this case the further structure relation $F \lhd F^* = 0$ is then equivalent to $W$ being unitary.

**Theorem 1.1.** *Let $X$ be a Markov-regular QS quasicontractive cocycle on $\mathfrak{h}$ with noise dimension space $\mathsf{k}$. Then there is a unique operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that $X$ satisfies the QS differential equation*

$$ X_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}}}, \quad \mathrm{d}X_t = X_t\, \mathrm{d}\Lambda_F(t). \tag{1.4} $$

*Moreover, for* $\beta \in \mathbb{R}$,

$$F^* \lhd F \leqslant \beta\Delta^\perp \textit{ if and only if } (e^{-\beta t}X_t)_{t\geqslant 0} \textit{ is contractive;}$$
$$F^* \lhd F = 0 \textit{ if and only if } X \textit{ is isometric;}$$
$$F \lhd F^* = 0 \textit{ if and only if } X \textit{ is co-isometric.}$$

*Conversely, let* $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$. *Then the* (1.4) *has a unique weakly regular, weak solution, denoted* $X^F$. *Moreover, if* $F$ *satisfies* $F^* \lhd F \leqslant \beta\Delta^\perp$ *for some* $\beta \in \mathbb{R}$, *then* $X^F$ *is a Markov-regular QS quasicontractive cocycle.*

*Remark.* Suppose that, for $i = 1, 2$, $F_i \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ satisfies $F_i^* \lhd F_i \leqslant \beta_i\Delta^\perp$ for some $\beta_i \in \mathbb{R}$. Then the QS quasicontractive cocycle $X^{F_1 \lhd F_2}$ is expressible in terms of *QS Trotter products* of the cocycles $X^{F_1}$ and $X^{F_2}$ ([L$_2$]).

## 2. Quantum random walk embedding

*For this section and the next, fix a Hilbert space* $\mathsf{K}$ which we refer to as the 'particle space'. In this section we describe the *standard embedding* of quantum random walks (QRW) as QS processes. This requires fixing a unit vector $e_0$ of $\mathsf{K}$; let $\omega_0$ be the corresponding vector state on $B(\mathsf{K})$. Set $\mathsf{k} := \mathsf{K} \ominus \mathbb{C}e_0$ and $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, and let $\pi_0$ denote the resulting unitarily implemented isomorphism from $B(\mathsf{K})$ to $B(\widehat{\mathsf{k}})$. Thus

$$\pi_0(T) = \begin{bmatrix} \langle e_0| \\ V^* \end{bmatrix} T \begin{bmatrix} |e_0\rangle & V \end{bmatrix} \qquad (T \in B(\mathsf{K}),$$

where $V$ is the inclusion $\mathsf{k} \to \mathsf{K}$. (In the next section we shall identify $\mathsf{K}$ with $\widehat{\mathsf{k}}$ so that $e_0$ is identified with $\binom{1}{0}$.) Set

$$\Upsilon^{\mathsf{k}}_{[M,N[} := \widehat{\mathsf{k}}_{(M)} \otimes \cdots \otimes \widehat{\mathsf{k}}_{(N-1)} \text{ and}$$
$$\Upsilon^{\mathsf{k}}_{[N,\infty[} := (\widehat{\mathsf{k}}_{(N)}, e_0) \otimes (\widehat{\mathsf{k}}_{(N+1)}, e_0) \otimes \cdots \qquad (M, N \in \mathbb{Z}_+, M \leqslant N),$$

where $\widehat{\mathsf{k}}_{(N)} = \widehat{\mathsf{k}}$ for each $N \in \mathbb{Z}_+$ and also set

$$\Upsilon^{\mathsf{k}} := \Upsilon^{\mathsf{k}}_{[0,\infty[}.$$

(Whether intervals are discrete or continuous will always be clear from context.) The toy Fock space identifications

$$\Upsilon^{\mathsf{k}} = \Upsilon^{\mathsf{k}}_{[0,M[} \otimes \Upsilon^{\mathsf{k}}_{[M,N[} \otimes \Upsilon^{\mathsf{k}}_{[N,\infty[} \qquad (M, N \in \mathbb{Z}_+, M \leqslant N)$$

are discrete analogues of the continuous tensor decompositions (1.1) of $\mathcal{F}^{\mathsf{k}}$.

We use the notation ${}^{\Upsilon}\!I^{\mathsf{k}}_{[M,N[}$ for the corresponding identity operators. The following notation, for *truncated exponential vectors*, proves to be very convenient:

$$\widetilde{\varepsilon}(g) := (1, g, 0, \cdots) \quad (g \in L^2(I; \mathsf{k}), I \text{ a subinterval of } \mathbb{R}_+).$$

**Definition.** *The QRW, with respect to the unit vector* $e_0 \in \mathsf{K}$, *generated by* $G \in B(\mathfrak{h} \otimes \mathsf{K})$ is the discrete QS process $(R_N := R_{0,N})_{N\in\mathbb{Z}_+}$ where

$$R_{M,N} := \overrightarrow{\prod_{M\leqslant n<N}} R_{n,n+1}$$

and

$$R_{n,n+1} := \big(\mathrm{id}_{B(\mathfrak{h})} \,\overline{\otimes}\, ({}^{\Upsilon}\!\sigma^{\mathsf{k}}_n \circ {}^{\Upsilon}\!\iota^{\mathsf{k}} \circ \pi_{e_0})\big)(G) \quad (n \in \mathbb{Z}_+)$$

through the right-shift endomorphism semigroup $\big({}^{\Upsilon}\!\sigma^{\mathsf{k}}_N\big)_{N\in\mathbb{Z}_+}$ on $B(\Upsilon^{\mathsf{k}})$

$${}^{\Upsilon}\!\sigma^{\mathsf{k}}_N : B(\Upsilon^{\mathsf{k}}) \to B(\Upsilon^{\mathsf{k}}), \quad T \mapsto {}^{\Upsilon}\!I^{\mathsf{k}}_{[0,N[} \otimes S_N T S_N^*$$

where $S_N$ here denotes the unitary shift operator $\Upsilon^{\mathsf{k}} \to \Upsilon^{\mathsf{k}}_{[N,\infty[}$, and the embedding

$${}^{\Upsilon}\!\iota^{\mathsf{k}} : B(\widehat{\mathsf{k}}) \to B(\Upsilon^{\mathsf{k}}), \quad T \mapsto T \otimes {}^{\Upsilon}\!I^{\mathsf{k}}_{[1,\infty[}.$$

*Remark.* The family $(R_{M,N})_{0\leqslant M\leqslant N}$ forms a discrete evolution:

$$R_{N,N} = I_{\mathfrak{h}\otimes\Upsilon^{\mathsf{k}}}, \quad R_{L,N} = R_{L,M}R_{M,N} \qquad (0\leqslant L\leqslant M\leqslant N),$$

which is *bi-adapted*:

$$R_{M,N} \in B(\mathfrak{h})\otimes {}^{\Upsilon}\!I^{\mathsf{k}}_{[0,M[}\,\overline{\otimes}\, B(\Upsilon^{\mathsf{k}}_{[M,N[})\otimes {}^{\Upsilon}\!I^{\mathsf{k}}_{[N,\infty[},$$

and *covariant*:

$$R_{M,N} = \big(\mathrm{id}_{B(\mathfrak{h})}\;\overline{\otimes}\; {}^{\Upsilon}\!\sigma^{\mathsf{k}}_M\big)(R_{N-M}).$$

Suitably scaled QRWs converge to QS cocycles in the sense made precise in Theorem 3.3 below. This entails embedding walks into the territory of cocycles, for which the relevant definition follows.

**Definition.** Let $h>0$. *The $h$-scale $e_0$-embedded QRW generated by $G\in B(\mathfrak{h}\otimes\mathsf{K})$*, denoted ${}^{\langle h\rangle}X^{e_0,G}$, is the bounded-operator QS process $X$ on $\mathfrak{h}$, with noise dimension space $\mathsf{k}:=\mathsf{K}\ominus\mathbb{C}e_0$, defined by $X_t := X_{0,h\lfloor t/h\rfloor}$ where

$$X_{hM,hN} := \overrightarrow{\prod_{M\leqslant n<N}} X_{hn,h(n+1)} \qquad (M,N\in\mathbb{Z}_+)$$

and

$$X_{hn,h(n+1)} := \big(\mathrm{id}_{B(\mathfrak{h})}\,\overline{\otimes}\,({}^{\mathcal{F}}\!\sigma^{\mathsf{k}}_{hn}\circ j^{\mathsf{k}}_h\circ\pi_{e_0})\big)(G) \qquad (n\in\mathbb{Z}_+),$$

through the embedding

$$j^{\mathsf{k}}_h : B(\widehat{\mathsf{k}})\to B(\mathcal{F}^{\mathsf{k}}), \quad T\mapsto J^{\mathsf{k}}_h\,T\,(J^{\mathsf{k}}_h)^*\otimes {}^{\mathcal{F}}\!I^{\mathsf{k}}_{[h,\infty[}$$

in which $J^{\mathsf{k}}_h:\widehat{\mathsf{k}}\to\mathcal{F}^{\mathsf{k}}_{[0,h[}$ denotes the isometry determined by the prescription $\widehat{c}\mapsto\widehat{\varepsilon}(h^{-1/2}c)$. Here the vector $h^{-1/2}c$ is considered as the corresponding constant function in $L^2([0,h[;\mathsf{k})$.

*Remark.* For future reference, we note the following elementary estimate on embedded quantum random walks:

$$\|{}^{\langle h\rangle}X^{e_0,G}_t\| \leqslant \|G\|^{\lfloor t/h\rfloor} \qquad (t\in\mathbb{R}_+). \tag{2.1}$$

In particular, the process ${}^{\langle h\rangle}X^{e_0,G}$ is contractive if the QRW generator $G$ is.

## 3. Quantum random walk approximation

In this section we show that suitably scaled families of QRWs converge to QS cocycles, in good analogy with the Donsker invariance principle. In the current form, this result is deducible from [B$_1$] and was proved in [DaL]. For an early version of the result, see [Par]. We now suppress the map $\pi_0$ and identify $\mathsf{K}$ with $\widehat{\mathsf{k}}$, so that $e_0=\binom{1}{0}$. We speak of the *$h$-scale embedded QRW generated by $G\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$*, and denote it simply by ${}^{\langle h\rangle}X^G$.

For $n\in\mathbb{Z}_+$ and $g\in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathsf{k})$, or $g\in L^2([hn,h(n+1)[;\mathsf{k})$, define $g[n,h]$ to be the average of $g$ over the interval $[hn,h(n+1)[$:

$$g[n,h] := h^{-1}\int_{hn}^{h(n+1)} g. \tag{3.1}$$

Thus, for $g\in L^2([0,h[;\mathsf{k})$, $(J^{\mathsf{k}}_h)^*\varepsilon(g)=\widehat{\sqrt{h}\,g[0,h]}$.

*Remark.* Observe that, in the notation

$$X^{f,g}_{mh,nh} := E^{\varepsilon(f_{[hm,hn[})}X_{mh,nh}E_{\varepsilon(g_{[hm,hn[})} \quad (f,g\in\mathbb{S}_{\mathrm{loc}}, m,n\in\mathbb{Z}_+, m\leqslant n),$$

where $X={}^{\langle h\rangle}X^G$, we have discrete evolutions for each $f,g\in\mathbb{S}_{\mathrm{loc}}$:

$$X^{f,g}_{hn,hn}=I_{\mathfrak{h}}, \quad X^{f,g}_{hm,hn}X^{f,g}_{hn,hp}=X^{f,g}_{hm,hp} \quad (m,n,p\in\mathbb{Z}_+, m\leqslant n\leqslant p).$$

For $h>0$, define the standard 'scaling matrix' (*cf.* [LiP]):

$$\mathcal{S}_h := \begin{bmatrix} h^{-1/2} & 0\\ 0 & I_{\mathsf{k}}\end{bmatrix}\in B(\widehat{\mathsf{k}}), \tag{3.2}$$

and let $s_h$ denote conjugation by $I_{\mathfrak{h}}\otimes\mathcal{S}_h$ on $B(\mathfrak{h}\otimes\widehat{\mathsf{k}})$.

**Lemma 3.1.** *Set $X = {}^{\langle h\rangle}X^G$ where $G \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ and $h > 0$. Let $f$, $g \in \mathbb{S}_{\mathrm{loc}}$ and $m$, $n \in \mathbb{Z}_+$ with $m \leqslant n$.*

(a) *Then*

$$X^{f,g}_{hn,h(n+1)} = I_{\mathfrak{h}} + h\, E^{\widehat{f[n,h]}}\, s_h(G - \Delta^{\perp})\, E_{\widehat{g[n,h]}} \tag{3.3}$$

*and*

$$\big\| X^{f,g}_{hn,h(n+1)} - I_{\mathfrak{h}} \big\| \leqslant h \max_{c \in \operatorname{Ran} f,\, d \in \operatorname{Ran} g} \big\| E^{\widehat{c}}\, s_h(G - \Delta^{\perp})\, E_{\widehat{d}} \big\|. \tag{3.4}$$

(b) *Suppose that $f$ and $g$ are constant, say $c$ and $d$ respectively, on the interval $[hm, hn[$. Then*

$$X^{f,g}_{hm,hn} = \big( I_{\mathfrak{h}} + h\, E^{\widehat{c}}\, s_h(G - \Delta^{\perp})\, E_{\widehat{d}} \big)^{n-m}. \tag{3.5}$$

*Proof.* (a) Since $\widehat{\sqrt{h}c} = \sqrt{h}\, \mathcal{S}_h \widehat{c}$ for $c \in \mathsf{k}$, the first identity follows from the definition:

$$\begin{aligned} X^{f,g}_{hn,h(n+1)} - I_{\mathfrak{h}} &= E^{\widehat{\sqrt{h}\, f[n,h]}}\, (G - \Delta^{\perp})\, E_{\widehat{\sqrt{h}\, g[n,h]}}, \\ &= h\, E^{\widehat{f[n,h]}}\, s_h(G - \Delta^{\perp})\, E_{\widehat{g[n,h]}}, \end{aligned}$$

Since

$$\widehat{f[n,h]} = h^{-1} \int_{hn}^{h(n+1)} \widehat{f} \in \operatorname{Conv} \operatorname{Ran} \widehat{f},$$

and similarly for $g$, (3.4) follows from (3.3).

(b) Since $\widehat{f[p,h]} = \widehat{c}$ and $\widehat{g[p,h]} = \widehat{d}$ for $p = m, \ldots, n-1$, the factorisation

$$X^{f,g}_{hm,hn} := X^{f,g}_{hm,h(m+1)} \cdots X^{f,g}_{h(n-1),hn}$$

implies that (3.5) follows from (3.3). □

*Remark.* QRWs are the discrete-time analogues of QS cocycles.

For the full power of the approximation result below, we need a lemma.

**Lemma 3.2.** *For a Hilbert space $\mathsf{H}$ and compact subinterval $I$ of $\mathbb{R}_+$, let $(a^h)_{h>0}$ be a family of contraction-valued maps $I \to B(\mathsf{H})$, let $a : I \to B(\mathsf{H})$ be isometry valued and strongly continuous, and suppose that $\langle \zeta, a^h(\cdot)\eta \rangle \to \langle \zeta, a(\cdot)\eta \rangle$ uniformly as $h \to 0$, for all $\zeta$, $\eta \in \mathsf{H}$. Then $a^h(\cdot)\eta \to a(\cdot)\eta$ uniformly as $h \to 0$, for all $\eta \in \mathsf{H}$.*

*Proof.* Let $\eta \in \mathsf{H}$ and $\epsilon > 0$. Since $a$ is strongly continuous and $I$ is compact, there is an $\mathsf{H}$-valued step function $\varphi = \sum_{j=1}^N \zeta_j \mathbf{1}_{I_j}$ such that $\sup_{t \in I} \|a(t)\eta - \varphi(t)\| < \epsilon$. Therefore, for all $t \in I$,

$$\begin{aligned} &\|(a^h(t) - a(t))\eta\|^2 \\ &\quad \leqslant 2\operatorname{Re}\langle a(t)\eta, (a(t) - a^h(t))\eta \rangle \\ &\quad = 2\operatorname{Re}\langle a(t)\eta - \varphi(t), (a(t) - a^h(t))\eta \rangle + \sum_{j=1}^N \mathbf{1}_{I_j}(t) \langle \zeta_j, (a(t) - a^h(t))\eta \rangle \\ &\quad = 4\|\eta\|\epsilon + \max_{j=1}^N |\langle \zeta_j, (a(t) - a^h(t))\eta \rangle|. \end{aligned}$$

Since the second term tends to zero uniformly, the result follows. □

In the proof of Theorem 3.3 below, we use Euler's exponential formula in the following form. Let $a$, $a(h) \in B(\mathfrak{h})$, for $h > 0$, and let $I$ be a compact subinterval of $\mathbb{R}_+$; if $a(h) \to a$ as $h \to 0$ then

$$\sup_{[r,t] \subset I} \big\| (I_{\mathfrak{h}} + h a(h))^{\lfloor t/h \rfloor - \lfloor r/h \rfloor} - e^{(t-r)a} \big\| \to 0 \quad \text{as} \quad h \to 0.$$

**Theorem 3.3** (*cf.* [B$_1$], [DaL]). *Let $\mathsf{T}'$ and $\mathsf{T}$ be total subsets of $\mathsf{k}$ containing* 0*, let $F$, $G(h) \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ $(h > 0)$ satisfy*

$$E^{\widehat{c}}\big(s_h(G(h) - I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}}) - F)\big)E_{\widehat{d}} \to 0 \quad as \quad h \to 0 \qquad (c \in \mathsf{T}', d \in \mathsf{T}), \tag{3.6}$$

*and let $I$ be a compact subinterval of $\mathbb{R}_+$. Then*

$$\sup_{t\in I} \big\|E^{\varepsilon'}\big({}^{\langle h\rangle}X_t^{G(h)} - X_t^F\big)E_\varepsilon\big\| \to 0 \ as \ h \to 0 \qquad (\varepsilon' \in \mathcal{E}_{\mathsf{T}'}, \varepsilon \in \mathcal{E}_{\mathsf{T}}). \tag{3.7}$$

*Moreover, the following refinements hold.*

(a) *If, for some $\beta \in \mathbb{R}$,*

$$\limsup_{h\to 0} \sup_{t\in I} \|G(h)\|^{\lfloor t/h\rfloor} < \infty \quad and \quad F^* \lhd F \leqslant \beta\Delta^\perp$$

*then*

$$\sup_{t\in I} \big\|(\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes}\, \varphi)\big({}^{\langle h\rangle}X_t^{G(h)} - X_t^F\big)\big\| \to 0 \ as \ h \to 0 \qquad (\varphi \in B(\mathcal{F}^{\mathsf{k}})_*).$$

(b) *If each $G(h)$ is a contraction and $F$ satisfies $F^* \lhd F = 0$ then*

$$\sup_{t\in I} \big\|\big({}^{\langle h\rangle}X_t^{G(h)} - X_t^F\big)\xi\big\| \to 0 \ as \ h \to 0 \qquad (\xi \in \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}).$$

*Proof.* Set $X^{(h)} := {}^{\langle h\rangle}X^{G(h)}$ and $X := X^F$. Let $f \in \mathbb{S}_{\mathsf{T}'}$ and $g \in \mathbb{S}_{\mathsf{T}}$, and write

$$\{0\} \cup D \cup \{T\} = \{t_0 < \cdots < t_{p+1}\}$$

where $D$ is the set of points of discontinuity of $f$ and $g$, and $T \in \mathbb{R}_+$ is both larger than all of these and such that $[0, T[ \supset I$; let $h > 0$ be smaller than mesh $D$. Exploiting the semigroup decomposition of QS cocycles and grouping like terms together in the product which is $X_t^{(h)}$, we may express $\|E^{\varepsilon(f)}(X^{(h)} - X_t)E_{\varepsilon(g)}\|$ as

$$\big|\langle\varepsilon(f_{[t,\infty[}), \varepsilon(g_{[t,\infty[})\rangle\big|$$
$$\times \sum_{k=0}^{p} \mathbf{1}_{[t_k,t_{k+1}[}(t)\, \big\|A_1(h) \cdots A_{k-1}(h)b(h,t)B_k(h,t) - P^{(0)}_{t_1-t_0} \cdots P^{(k-1)}_{t_k-t_{k-1}} P^{(k)}_{t-t_k}\big\|,$$

where $b(h,t) := \langle\varepsilon(f_{[h\lfloor t/h\rfloor,t[}), \varepsilon(g_{[h\lfloor t/h\rfloor,t[})\rangle I_{\mathfrak{h}}$,

$$A_j(h) = X^{(h)f,g}_{h\lfloor t_j/h\rfloor, h(1+\lfloor t_j/h\rfloor)} X^{(h)f,g}_{h(1+\lfloor t_j/h\rfloor)h, h\lfloor t_{j+1}/h\rfloor} \qquad (j = 0, \cdots, p-1),$$

$$B_k(h,t) = \begin{cases} X^{(h)f,g}_{h\lfloor t_k/h\rfloor, h(1+\lfloor t_k/h\rfloor)} X^{(h)f,g}_{h(1+\lfloor t_k/h\rfloor), h\lfloor t/h\rfloor} & \text{if } \lfloor t_k/h\rfloor < \lfloor t/h\rfloor, \\ I_{\mathfrak{h}} & \text{otherwise,} \end{cases}$$

and $P^{(i)}$ denotes the $(f(t_i), g(t_i))$-associated semigroup of the QS cocycle $X$ for $i = 0, \ldots, p$. Now the generator of the semigroup $P^{(i)}$ is $E^{\widehat{f}(t_i)}(F + \Delta)E_{\widehat{g}(t_i)}$ so, by Lemma 3.1 and Euler's exponential formula,

$$\big\|A_j(h) - P^{(j)}_{t_{j+1}-t_j}\big\| \to 0 \ \text{ and } \sup_{t\in[t_k,t_{k+1}[} \big\|B_k(h,t) - P^{(k)}_{t-t_k}\big\| \to 0 \ \text{ as } \ h \to 0.$$

Thus (3.7) holds for $\varepsilon' = \varepsilon(f)$ and $\varepsilon = \varepsilon(g)$; it therefore holds for all $\varepsilon' \in \mathcal{E}_{\mathsf{T}'}$ and $\varepsilon \in \mathcal{E}_{\mathsf{T}}$, and so the first part is proved.

(a) In this case, by the basic estimate (2.1), $\{X_t^{(h)} : h \in ]0, H], t \in I\}$ is uniformly bounded and, by the characterisation of quasicontractivity of Markov-regular QS cocycles recalled in Theorem 1.1, $\|X_t\| \leqslant e^{\beta t}$ $(t \in \mathbb{R}_+)$ so $\{X_t : t \in I\}$ is uniformly bounded too. The result therefore follows from the first part, by the norm totality of the family $\{\omega_{\varepsilon',\varepsilon} : \varepsilon' \in \mathcal{E}'_{\mathsf{T}'}, \varepsilon \in \mathcal{E}_{\mathsf{T}}\}$ in $B(\mathcal{F}^{\mathsf{k}})_*$ and the well-known fact (see *e.g.* [EfR]) that $\|\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes}\, \varphi\| = \|\varphi\|$ for any $\varphi \in B(\mathcal{F})_*$.

(b) In this case it follows from Part (a) that, for all $\zeta, \eta \in \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}$,

$$\sup_{t\in I} \big|\langle\zeta, (X_t^{(h)} - X_t)\eta\rangle\big| \to 0 \text{ as } h \to 0.$$

Since $X$ is strongly continuous, the result follows from Lemma 3.2. □

*Remarks.* (i) The sets $\mathsf{T}'$ and $\mathsf{T}$ typically each consist of vectors from an orthonormal basis for $\mathsf{k}$ augmented by the vector 0.

(ii) In (a) the limit QS cocycle $X^F$ is quasicontractive, with $\|X_t^F\| \leqslant e^{\beta t}$; in (b) the cocycle $X^F$ is isometric, and is unitary if also $F \lhd F^* = 0$. These follow from the characterisations of (quasi)contractivity, isometry and unitarity of Markov-regular QS cocycles listed in Theorem 1.1.

(iii) The condition (3.6) is usefully expressed in the following equivalent form:

$$E^{\widehat{c}}\big(s_h(G(h) - \Delta^{\perp}) - (F + \Delta)\big)E_{\widehat{d}} \to 0 \text{ as } h \to 0 \qquad (c \in \mathsf{T}', d \in \mathsf{T}).$$

Then, writing in block matrix form,

$$G(h) = \begin{bmatrix} I_{\mathfrak{h}} + hK(h) & \sqrt{h}M(h) \\ \sqrt{h}L(h) & C(h) \end{bmatrix} \text{ and } F = \begin{bmatrix} K & M \\ L & C - I \end{bmatrix}, \tag{3.8}$$

it follows that

$$s_h(G(h) - \Delta^{\perp}) = \begin{bmatrix} K(h) & M(h) \\ L(h) & C(h) \end{bmatrix} \text{ and } F + \Delta = \begin{bmatrix} K & M \\ L & C \end{bmatrix}.$$

In these terms (3.6) amounts to the following, more transparent condition:

$$E^{\widehat{c}}\begin{bmatrix} K(h) - K & M(h) - M \\ L(h) - L & C(h) - C \end{bmatrix} E_{\widehat{d}} \to 0 \text{ as } h \to 0 \qquad (c \in \mathsf{T}', d \in \mathsf{T}). \tag{3.9}$$

When $\dim \mathsf{k} < \infty$, this is equivalent to the simple norm-convergence conditions

$$K(h) \to K,\ L(h) \to L,\ M(h) \to M \text{ and } C(h) \to C \text{ as } h \to 0.$$

However, when $\mathsf{k}$ is infinite dimensional, it is only the *components* of $L(h)$, $M(h)$ and $C(h)$ with respect to some orthonormal basis of $\mathsf{k}$ that need to converge to the corresponding components of $L$, $M$ and $C$.

We next consider families of QRWs which are of exponential form. To this end, let $e_1$, $e_2$ and $e_3$ denote the entire functions whose values at $z \neq 0$ are given respectively by

$$\frac{e^z - 1}{z}, \quad \frac{e^z - 1 - z}{z^2} \text{ and } \frac{\sinh z - z}{z^2}, \tag{3.10}$$

and note the identities

$$e_1(z) = e_1(-z)e^z,\ e_3(z) = e_2(z) - \frac{1}{2}e_1(-z)e_1(z) \text{ and } e_3(-z) = -e_3(z). \tag{3.11}$$

**Proposition 3.4.** *Let the family* $(E(h))_{h>0} \subseteq B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) = B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}))$ *be such that*

$$s_h(E(h)) \to \begin{bmatrix} A & -B^* \\ B & D \end{bmatrix} \quad \textit{as } h \to 0,$$

*in which $D$ is skewadjoint. Then the family* $(G(h) := e^{E(h)})_{h>0}$ *satisfies*

$$s_h(G(h) - I) \to F \quad \textit{as} \quad h \to 0, \quad \textit{where} \quad F = \begin{bmatrix} K & -L^*W \\ L & W - I \end{bmatrix}$$

*in which* $L = e_1(D)B$, $W$ *is the unitary operator* $e^D$ *and* $K = A - \frac{1}{2}L^*L - B^*e_3(D)B$.

*Proof.* It is straightforward to verify that, as $h \to 0$, $s_h(G(h) - I)$ converges to the operator

$$F = \begin{bmatrix} A - B^*e_2(D)B & -B^*e_1(D) \\ e_1(D)B & e^D - I \end{bmatrix}.$$

In view of the skewadjointness of $D$, the identities (3.11) imply that

$$e_1(D) = e_1(D)^*e^D,\ e_2(D) = e_3(D) + \frac{1}{2}e_1(D)^*e_1(D) \text{ and } e_3(D)^* = -e_3(D).$$

In turn, these imply that $F$ is as claimed. ◻

*Remarks.* (i) Since the operator $e_3(D)$ is skewadjoint, $F \lhd F^* = F^* \lhd F = \left[\begin{smallmatrix} 2\,\mathrm{Re}\,A & 0 \\ 0 & 0 \end{smallmatrix}\right] \leqslant 2\|\mathrm{Re}\,A\|\Delta^\perp$ so the QS cocycle $X^F$ is quasicontractive, and is unitary provided that the operator $A$ is skewadjoint.

(ii) This result should be compared with (22) in [Gou] and Theorem 19 in [AtP].

**Notation.** The following two notations are germane to QRW approximation. For $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$, set

$$R_L := \begin{bmatrix} 0 & -L^* \\ L & 0 \end{bmatrix}, \tag{3.12}$$

and, provided that $\|L\| < 1$, set

$$U_L := \begin{bmatrix} (1 - L^*L)^{1/2} & -L^* \\ L & (1 - LL^*)^{1/2} \end{bmatrix}. \tag{3.13}$$

Thus $e^{R_L}$ and $U_L$ are unitary operators, and

$$(R_L)^2 = -\begin{bmatrix} L^*L & 0 \\ 0 & LL^* \end{bmatrix}.$$

We use the following abbreviation for Hilbert space operators:

$$T_+ := (\mathrm{Re}\,T)_+ \quad \text{for} \quad T \in B(\mathsf{h}). \tag{3.14}$$

**Proposition 3.5.** *Let $G(h)$ be of the form*

$$e^{(\sqrt{h}R_L + hZ)}\big(I_\mathfrak{h} \oplus C\big) \quad \textit{or}$$
$$e^{hZ}e^{\sqrt{h}R_L}\big(I_\mathfrak{h} \oplus C\big), \quad \textit{or}$$
$$U_{\sqrt{h}L}\big(e^{hA} \oplus C\big), \quad \textit{for } 0 < h < \|L\|^{-2},$$

*where $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$, $Z \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$, $C \in B(\mathfrak{h} \otimes \mathsf{k})$ is a contraction operator, and $A := (\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes}\, \omega_0)(Z) \in B(\mathfrak{h})$.*

(a) *Then*

$$\sup_{t \in I} \big\|(\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes}\, \varphi)\big({}^{\langle h \rangle}X_t^{G(h)} - X_t^F\big)\big\| \to 0 \textit{ as } h \to 0 \qquad (\varphi \in B(\mathcal{F}^\mathsf{k})_*),$$

*where $F = \left[\begin{smallmatrix} K & -L^*C \\ L & C - I_\mathsf{k} \end{smallmatrix}\right]$ and $K = A - \frac{1}{2}L^*L$.*

(b) *Moreover if $C$ is isometric, $Z$ is dissipative and $A$ is skewadjoint then*

$$\sup_{t \in I} \big\|\big({}^{\langle h \rangle}X_t^{G(h)} - X_t^F\big)\xi\big\| \to 0 \textit{ as } h \to 0 \qquad (\xi \in \mathfrak{h} \otimes \mathcal{F}^\mathsf{k}).$$

*Proof.* In each case, it is easily verified that

$$s_h(G(h) - \Delta^\perp) = F + \Delta + O(h^{1/2}).$$

Since

$$\|G(h)\|^{\lfloor t/h \rfloor} \leqslant e^{h\|Z_+\|\lfloor t/h \rfloor} \leqslant e^{t\|Z_+\|} \quad \text{and}$$
$$F^* \lhd F = \begin{bmatrix} A^* + A & 0 \\ 0 & C^*C - I_{\mathfrak{h} \otimes \mathsf{k}} \end{bmatrix} \leqslant 2\|A_+\|\Delta^\perp,$$

the first part follows from part (a) of Theorem 3.3.

If $Z$ is dissipative then $\|G(h)\| \leqslant 1$, and if $C$ is isometric and $A$ is skewadjoint then $F^* \lhd F = 0$ so the second part follows from part (b) of the Theorem. □

*Remark.* From the proof we see that, in (a) the embedded processes ${}^{\langle h \rangle}X^{G(h)}$ and limit QS cocycle $X^F$ satisfy $\|{}^{\langle h \rangle}X_t^{G(h)}\| \leqslant e^{t\|Z_+\|}$ and $\|X_t^F\| \leqslant e^{t\|A_+\|}$; in (b), each process ${}^{\langle h \rangle}X^{G(h)}$ is contractive and the limit QS cocycle $X^F$ is isometric.

Thus, given a Markov-regular QS cocycle which is isometric or unitary, then from its QS generator, we may easily construct QRWs which are isometric or unitary, respectively, and converge to the cocycle.

**Exercise.** Let $X$ be a Markov-regular QS cocycle. Then $X$ is quasicontractive if and only if its stochastic generator has the form $F_1 = \begin{bmatrix} A-\frac{1}{2}L^*L & -L^*C-D(I-C^*C)^{1/2} \\ L & C-I \end{bmatrix}$ for arbitrary operators $A \in B(\mathfrak{h})$, $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$, $D \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$ and a contraction operator $C \in B(\mathfrak{h} \otimes \mathsf{k})$.

On the other hand, $X$ is contractive if and only if its stochastic generator has the form $F_2 = \begin{bmatrix} -iH-\frac{1}{2}(L^*L+B^2) & -L^*C-BV(I-C^*C)^{1/2} \\ L & C-I \end{bmatrix}$ for $L$ and $C$ as above, $H \in B(\mathfrak{h})_{\mathrm{sa}}$, $B \in B(\mathfrak{h})_+$ and $V \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$ a contraction operator.

(1) Find $(G(h))_{0<h\leqslant H}$ in $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that, in the abbreviated notation (3.14),

$$\sup_{h\in]0,H]} \|G(h)\|^{\lfloor t/h \rfloor} < e^{t\|A_+\|} \quad \text{and} \quad s_h(G(h) - \Delta^\perp) \to F_1 + \Delta \quad \text{as} \quad h \to 0.$$

(2) Find contractive $(G(h))_{0<h\leqslant H}$ in $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that $s_h(G(h) - \Delta^\perp) \to F_2 + \Delta$ as $h \to 0$.

The cases that remain to be proved are where $D(I - C^*C)^{1/2} \neq 0$ and $BV(I - C^*C)^{1/2} \neq 0$, respectively.

*Hint.* For (2), dilate $F_2$ to a generator o a QS unitary cocycle (see [L$_2$]). For (1), use the solution to (2) in conjunction with the results of Section 4.

**Example 3.6.** In the repeated quantum interactions model developed by Attal–Pautrat and Attal–Joye ([AtP], [AtJ]),

$$G(h) = e^{-ihH_{\mathsf{T}}},$$

where the *total Hamiltonian* decomposes as

$$H_{\mathsf{T}} = H_{\mathsf{S}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{P}} + H_{\mathsf{I}}(h)$$

for a *system Hamiltonian* $H_{\mathsf{S}} \in B(\mathfrak{h})_{\mathrm{sa}}$, a *particle Hamiltonian* $H_{\mathsf{P}} \in B(\widehat{\mathsf{k}})_{\mathrm{sa}}$ and an *interaction Hamiltonian* taking the form

$$H_{\mathsf{I}}(h) = \frac{1}{\sqrt{h}}\, iR_B + \frac{1}{h}\, 0_{\mathfrak{h}} \oplus H_{\mathsf{Sc}}$$

for operators $B = V_{\mathsf{Di}} \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and $H_{\mathsf{Sc}} \in B(\mathfrak{h} \otimes \mathsf{k})_{\mathrm{sa}}$,

This fits perfectly into the general scheme described here. Indeed, setting $E(h) = -ihH_{\mathsf{T}}(h)$, we have

$$E(h) = \begin{bmatrix} -ihH_{\mathsf{S}} & -\sqrt{h}V_{\mathsf{Di}}^* \\ \sqrt{h}V_{\mathsf{Di}} & -iH_{\mathsf{Sc}} \end{bmatrix} - ih(I_{\mathfrak{h}} \otimes H_{\mathsf{P}} + 0_{\mathfrak{h}} \oplus (H_{\mathsf{S}} \otimes I_{\mathsf{k}}))$$

so

$$s_h(E(h)) \to \begin{bmatrix} -i(H_{\mathsf{S}} + \omega_0(H_{\mathsf{P}})I_{\mathfrak{h}}) & -V_{\mathsf{Di}}^* \\ V_{\mathsf{Di}} & -iH_{\mathsf{Sc}} \end{bmatrix} \quad \text{as} \quad h \to 0,$$

where $\omega_0$ is the vector state corresponding to the vector $\binom{1}{0} \in \widehat{\mathsf{k}}$. Therefore, by Proposition 3.4 and Theorem 3.3, we have the following strong convergence of scaled unitary quantum random walks to a QS unitary cocycle:

$$\sup_{t\in I} \big\| \big({}^{\langle h \rangle}X_t^{G(h)} - X_t^F\big)\xi \big\| \to 0 \text{ as } h \to 0 \qquad (\xi \in \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$$

for any compact subinterval of $\mathbb{R}_+$, where

$$F = \begin{bmatrix} -iH - \frac{1}{2}L^*L & -L^*W \\ L & W - I_{\mathfrak{h}\otimes\mathsf{k}} \end{bmatrix},$$

in which

$$L = e_1(-iH_{\mathsf{Sc}})V_{\mathsf{Di}}, \; W = e^{-iH_{\mathsf{Sc}}} \; \text{ and } \; H = H_{\mathsf{S}} + \omega_0(H_{\mathsf{P}})I_{\mathfrak{h}} - V_{\mathsf{Di}}^*\, e_4(H_{\mathsf{Sc}})V_{\mathsf{Di}}$$

for the entire functions $e_1$ and $e_4$ whose values at $z \neq 0$ are respectively $(e^z-1)/z$ and $(\sin z - z)/z^2$.

*Remarks.* (i) The Hilbert–Schmidt type assumptions on the coefficients of $F$, employed in the main theorem of [AtP] (namely Theorem 13), play no role in Theorem 3.3 and Proposition 3.4, as was pointed out in [B$_1$]. These results therefore extend the validity of that paper, licensing free use of infinite-dimensional noise.

(ii) For a discussion of the physical origins of the components of the interaction Hamiltonian see [BrP]. In brief, the scaling order $\sqrt{h}$ corresponds to a *weak coupling limit*, or van Hove limit ([vHo], [Dav]), whereas the scaling order $h$ corresponds to a *low density limit.*

## 4. Products of quantum random walks

In this section we show how, under the convergence scheme of the previous section, pointwise products of QRWs converge to QS Trotter products of QS cocycles ([L₂]). This specialises nicely to the case where the initial space $\mathfrak{h}$ is a tensor product and the two cocycles live on separate tensor components.

Recall the series-product notation (1.3).

**Proposition 4.1.** *Let $c$, $d \in \mathsf{k}$ and, for $i = 1, 2$, let $F_i$, $G_i(h) \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ $(h > 0)$. Set $G(h) := G_1(h)G_2(h)$ and $F := F_1 \lhd F_2$, and suppose that*

$$E^{\widehat{c}}\big(s_h(G_1(h) - I) - F_1\big) \to 0 \quad \text{and} \quad \big(s_h(G_2(h) - I - F_2\big)E_{\widehat{d}} \to 0 \text{ as } h \to 0. \tag{4.1}$$

*Then*

$$E^{\widehat{c}}\big(s_h(G(h) - I) - F_1 \lhd F_2\big)E_{\widehat{d}} \to 0 \text{ as } h \to 0.$$

*Proof.* Let $h > 0$ and set

$$F_1(h) := s_h(G_1(h) - I),\ F_2(h) := s_h(G_2(h) - I) \ \text{ and } \ F(h) := s_h(G(h) - I).$$

Then, from the identity

$$G(h) - I = (G_1(h) - I)(G_2(h) - I) + (G_1(h) - I) + (G_2(h) - I),$$

we see that

$$\begin{aligned} F(h) - F_1(h) - F_2(h) =&(h^{-1/2}\Delta^\perp + \Delta)(G_1(h) - I)(G_2(h) - I)(h^{-1/2}\Delta^\perp + \Delta) \\ =&F_1(h)\Delta F_2(h) + h\, F_1(h)\Delta^\perp F_2(h). \end{aligned}$$

Thus, as $h \to 0$,

$$\begin{aligned} &E^{\widehat{c}}F(h)E_{\widehat{d}} \\ &\quad = E^{\widehat{c}}F_1(h)E_{\widehat{d}} + E^{\widehat{c}}F_2(h)E_{\widehat{d}} + E^{\widehat{c}}F_1(h)\Delta F_2(h)E_{\widehat{d}} + hE^{\widehat{c}}F_1(h)E_{\widehat{0}}E^{\widehat{0}}F_2(h)E_{\widehat{d}} \\ &\quad \to E^{\widehat{c}}F_1E_{\widehat{d}} + E^{\widehat{c}}F_2E_{\widehat{d}} + E^{\widehat{c}}F_1\Delta F_2E_{\widehat{d}} = E^{\widehat{c}}FE_{\widehat{d}}, \end{aligned}$$

as claimed. □

As an immediate consequence we have the following result.

**Theorem 4.2.** *Let $F_1, F_2 \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$, let $\mathsf{T}'$ and $\mathsf{T}$ be total subsets of $\mathsf{k}$ containing $0$, and suppose that $(G_1(h))_{h>0}$ and $(G_2(h))_{h>0}$ are families in $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that, for all $c \in \mathsf{T}'$ and $d \in \mathsf{T}$, (4.1) holds. Then, setting $G(h) = G_1(h)G_2(h)$ and $F = F_1 \lhd F_2$, the conclusion (3.7) and refinements $(a)$ and $(b)$ of Theorem 3.3 hold.*

*Remarks.* (i) If $F_i^* \lhd F_i \leqslant \beta_i\Delta^\perp$, where $\beta_i \in \mathbb{R}$ (respectively $F_i^* \lhd F_i = 0$), for $i = 1, 2$, then the QS cocycle $X^{F_1 \lhd F_2}$ may be realised as a QS Trotter product of the quasicontractive(respectively isometric) QS cocycles $X^{F_1}$ and $X^{F_2}$ (see [L₂]).

(ii) The following observation in [JuL] is relevant here. Let $X^1$ and $X^2$ be QS quasicontractive cocycles on $\mathfrak{h}$, each with noise dimension space $\mathsf{k}$. If $X^1$ and $X^2$ 'commute on $\mathfrak{h}$', in other words each $\mathcal{F}^{\mathsf{k}}$-slice of $X^1_s$ commutes with each $\mathcal{F}^{\mathsf{k}}$-slice of $X^2_t$, for all $s, t \in \mathbb{R}_+$, then the QS process $X^1X^2 := (X^1_tX^2_t)_{t\geqslant 0}$ is also a QS cocycle. Moreover, if $X^1$ and $X^2$ are both Markov regular then $X^1X^2 = X^{F_1 \lhd F_2}$, where $F_1$ and $F_2$ are the stochastic generators of $X^1$ and $X^2$.

**Example 4.3.** Let $X^{(i)}$ be a QS quasicontractive cocycle on $\mathfrak{h}_i$ with noise dimension space $\mathsf{k}$, for $i = 1, 2$. These ampliate to QS cocycles on $\mathfrak{h} := \mathfrak{h}_1 \otimes \mathfrak{h}_2$, by setting $I_1 := I_{\mathfrak{h}_1}$, $I_2 := I_{\mathfrak{h}_2}$,

$$X^2_t := I_1 \otimes X^{(2)}_t \quad \text{and} \quad X^1_t := I_2 \mathbin{\widetilde{\otimes}} X^{(1)}_t,$$

where $B(\mathfrak{h}_1)\overline{\otimes}\,B(\mathfrak{h}_2 \otimes \mathcal{F}^{\mathsf{k}})$ is identified with $B(\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ and the notation $\widetilde{\otimes}$ incorporates the tensor flip from $B(\mathfrak{h}_2)\overline{\otimes}\,B(\mathfrak{h}_1 \otimes \mathcal{F}^{\mathsf{k}})$ to $B(\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$. Since the $\mathcal{F}^{\mathsf{k}}$-slices of $X^1_s$ and $X^2_t$ belong to $B(\mathfrak{h}_1) \otimes I_2$ and $I_1 \otimes B(\mathfrak{h}_2)$ respectively, the cocycles manifestly commute on $\mathfrak{h}$. Therefore $X^1X^2$ is a QS cocycle and, if $X^{(i)}$ is Markov regular with stochastic generator $F_{(i)}$ $(i = 1, 2)$, then $X^1X^2 = X^{F_1 \triangleleft F_2}$, where

$$F_2 := I_1 \otimes F_{(2)} \quad \text{and} \quad F_1 := I_2 \,\widetilde{\otimes}\, F_{(1)},$$

in which the tilde now denotes the tensor flip from $B(\mathfrak{h}_2)\overline{\otimes}\,B(\mathfrak{h}_1 \otimes \widehat{\mathsf{k}})$ to $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$.

In terms of the block matrix decompositions $F_{(i)} = \left[\begin{smallmatrix} K_i & M_i \\ L_i & C_i - I \end{smallmatrix}\right]$,

$$F_1 \triangleleft F_2 = \begin{bmatrix} K_1 \otimes I_2 + I_1 \otimes K_2 + (I_2 \widetilde{\otimes} M_1)(I_1 \otimes L_2) & (I_2 \widetilde{\otimes} M_1)(I_1 \otimes C_2) + I_1 \otimes M_2 \\ I_2 \widetilde{\otimes} L_1 + (I_1 \widetilde{\otimes} C_1)(I_1 \otimes L_2) & (I_2 \widetilde{\otimes} C_1)(I_1 \otimes C_2) - I \end{bmatrix}$$

In the case of one-dimensional noise this simplifies to

$$\begin{bmatrix} K_1 \otimes I_2 + I_1 \otimes K_2 + M_1 \otimes L_2 & M_1 \otimes C_2 + I_1 \otimes M_2 \\ L_1 \otimes I_2 + C_1 \otimes L_2 & C_1 \otimes C_2 - I \end{bmatrix},$$

whereas the scaled quantum random walk generator takes the form

$$\begin{bmatrix} I + hK(h) & \sqrt{h}M(h) \\ \sqrt{h}L(h) & C(h) \end{bmatrix} + h\,O(h)$$

where $\left[\begin{smallmatrix} K(h) & M(h) \\ L(h) & C(h) \end{smallmatrix}\right]$ equals

$$\begin{bmatrix} K_1(h) \otimes I_2 + I_1 \otimes K_2(h) + M_1(h) \otimes L_2(h) & M_1(h) \otimes C_2(h) + I_1 \otimes M_2(h) \\ L_1(h) \otimes I_2 + C_1(h) \otimes L_2(h) & C_1(h) \otimes C_2(h) \end{bmatrix}$$

and

$$O(h) = \begin{bmatrix} h\big(K_1(h) \otimes K_2(h)\big) & \sqrt{h}\big(K_1(h) \otimes M_2(h)\big) \\ \sqrt{h}\big(L_1(h) \otimes K_2(h)\big) & L_1(h) \otimes M_2(h) \end{bmatrix}$$

*Remark.* This example specialises to the entanglement of bipartite systems under repeated interactions as considered in [ADP], by taking the setup of Example 3.6.

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