

Colombeau Algebra: A pedagogical introduction

Jonathan Gratus*

Physics Department, Lancaster University, Lancaster LA1 4YB
and the Cockcroft Institute.

August 2, 2013

Abstract

A simple pedagogical introduction to the Colombeau algebra of generalised functions is presented, leading the standard definition.

1 Introduction

This is a pedagogical introduction to the Colombeau algebra of generalised functions. I will limit myself to the Colombeau Algebra over \mathbb{R} . Rather than \mathbb{R}^n . This is mainly for clarity. Once the general idea has been understood the extension to \mathbb{R}^n is not too difficult. In addition I have limited the introduction to \mathbb{R} valued generalised functions. To replace with \mathbb{C} valued generalised functions is also rather trivial.

I hope that this guide is useful in your understanding of Colombeau Algebras. Please feel free to contact me.

There is much general literature on Colombeau Algebras but I found the books by Colombeau himself[1] and the Masters thesis by Tạ Ngọc Trí[2] useful.

2 Test functions and Distributions

The set of infinitely differentiable functions on \mathbb{R} is given by

$$\mathcal{F}(\mathbb{R}) = \{\phi : \mathbb{R} \rightarrow \mathbb{R} \mid \phi^{(n)} \text{ exists for all } n\} \quad (1)$$

Test functions are those function which in addition to being smooth are zero outside an interval, i.e.

$$\mathcal{F}_0(\mathbb{R}) = \{\phi \in \mathcal{F}(\mathbb{R}) \mid \text{there exists } a, b \in \mathbb{R} \text{ such that } f(x) = 0 \text{ for } x < a \text{ and } x > b\} \quad (2)$$

I will assume the reader is familiar with distributions, either in the notation of integrals or as linear functionals. Thus the most important distributions is the Dirac- δ . This is defined either as a “function” $\delta(x)$ such that

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0) \quad (3)$$

Or as a distribution $\Delta : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathbb{R}$,

$$\Delta[\phi] = \phi(0) \quad (4)$$

We will refer to (3) as the integral notation and (4) as the Schwartz notation. An arbitrary distribution will be written either as $\psi(x)$ for the integral notation or Ψ for the Schwartz notation.

*j.gratus@lancaster.ac.uk

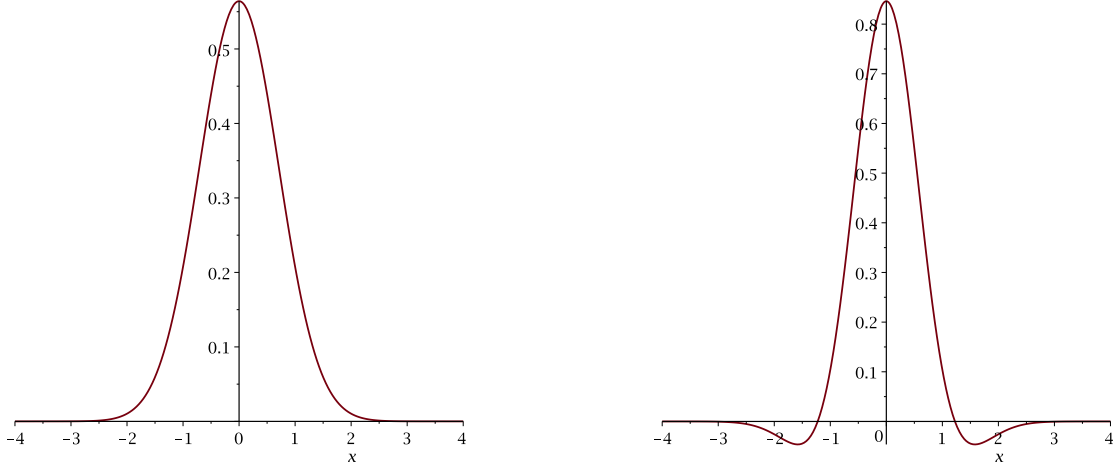


Figure 1: Plots of $\phi_1 \in \mathbb{A}_1$ and $\phi_3 \in \mathbb{A}_3$

3 Function valued distributions

The first step in understanding the Colombeau Algebra is to convert distributions into a new object which takes a test functions ϕ and gives a functions

$$\mathbf{A} : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$$

This is achieved by using translation of the test functions. Given $\phi \in \mathcal{F}_0$ then let

$$\phi^y \in \mathcal{F}_0(\mathbb{R}), \quad \phi^y(x) = \phi(x - y) \quad (5)$$

Then in integral notation

$$\overline{\psi}[\phi](y) = \int_{-\infty}^{\infty} \psi(x) \phi(x - y) dx \quad (6)$$

and in Schwartz notation

$$\overline{\Psi}[\phi](y) = \Psi[\phi^y] \quad (7)$$

We will define the Colombeau Algebra in such a way that they include the elements $\overline{\psi}$ and $\overline{\Psi}$. The overline will be used to covert distributions into elements of the Colombeau algebra.

We label the set of all function valued functionals

$$\mathcal{H}(\mathbb{R}) = \{\mathbf{A} : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})\} \quad (8)$$

We see below that we need to restrict $\mathcal{H}(\mathbb{R})$ further in order to define the Colombeau algebra $\mathcal{G}(\mathbb{R})$.

Observe that we use a slightly non standard notation. Here $\mathbf{A}[\phi] : \mathbb{R} \rightarrow \mathbb{R}$ is a function, so that given a point $x \in \mathbb{R}$ then $\mathbf{A}[\phi](x) \in \mathbb{R}$. One can equally write $\mathbf{A}[\phi](x) = \mathbf{A}(\phi, x)$, which is the standard notation in the literature. However I claim that the notation $\mathbf{A}[\phi](x)$ does have advantages.

4 Three special examples.

For the Dirac- δ we see that

$$\overline{\delta} = \overline{\Delta} = \mathbf{R} \quad (9)$$

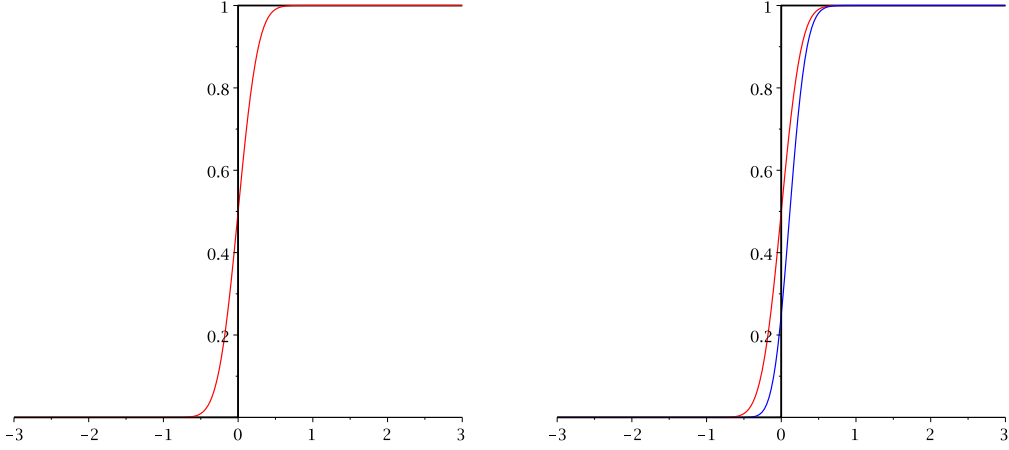


Figure 2: Heaviside (black) and $\bar{\theta}[\phi]$ (red) and $(\bar{\theta}[\phi])^2$ (blue)

where $\mathbf{R} \in \mathcal{H}(\mathbb{R})$ is the reflection map

$$\mathbf{R}[\phi](y) = \phi(-y) \quad (10)$$

This is because

$$\bar{\delta}[\phi](y) = \int_{-\infty}^{\infty} \delta(x) \phi(x - y) dx = \phi(-y)$$

and is Schwartz notation

$$\bar{\Delta}[\phi](y) = \Delta[\phi^y] = \phi^y(0) = \phi(-y)$$

Regular distribution: Given any function $f \in \mathcal{F}$ then there is a distribution f^D given by

$$f^D[\phi] = \int_{-\infty}^{\infty} f(x) \phi(x) dx \quad (11)$$

Thus we set $\bar{f} = \overline{f^D} \in \mathcal{H}(\mathbb{R})$ as

$$\bar{f}[\phi](y) = f^D[\phi^y] = \int_{-\infty}^{\infty} f(x) \phi(x - y) dx \quad (12)$$

The other important generalised functions are the regular functions. That is given $f \in \mathcal{F}$ we set

$$\tilde{f} \in \mathcal{H}(\mathbb{R}), \quad \tilde{f}[\phi] = f \quad \text{that is} \quad \tilde{f}[\phi](y) = f(y) \quad (13)$$

The effect of replacing $\bar{\psi}[\phi_\epsilon]$ is to smooth out ψ . Examples of ϕ are given in figure 1. The action $\bar{\theta}[\phi]$ where θ is the Heaviside function is given in figure 2.

5 Sums and Products

Given two Generalised functions $\mathbf{A}, \mathbf{B} \in \mathcal{H}(\mathbb{R})$ then we can define there sum and product in the natural way

$$\mathbf{A} + \mathbf{B} \in \mathcal{H}(\mathbb{R}) \quad \text{via} \quad (\mathbf{A} + \mathbf{B})[\phi] = \mathbf{A}[\phi] + \mathbf{B}[\phi] \quad \text{i.e.} \quad (\mathbf{A} + \mathbf{B})[\phi](y) = \mathbf{A}[\phi](y) + \mathbf{B}[\phi](y) \quad (14)$$

and

$$\mathbf{A}\mathbf{B} \in \mathcal{H}(\mathbb{R}) \quad \text{via} \quad (\mathbf{A}\mathbf{B})[\phi] = \mathbf{A}[\phi]\mathbf{B}[\phi] \quad \text{i.e.} \quad (\mathbf{A}\mathbf{B})[\phi](y) = \mathbf{A}[\phi](y)\mathbf{B}[\phi](y) \quad (15)$$

We see that the product of delta functions $\bar{\delta}^2 \in \mathcal{H}(\mathbb{R})$ is clearly defined. That is

$$\bar{\delta}^2[\phi](y) = (\bar{\delta}[\phi]\bar{\delta}[\phi])(y) = \bar{\delta}[\phi](y)\bar{\delta}[\phi](y) = (\phi(-y))^2$$

Although this is a generalised function, it does not correspond to a distribution, via (7). That is there is no distribution Ψ such that $\bar{\Psi} = (\bar{\delta})^2$.

Likewise we can see from figure 2 that $(\bar{\theta})^2[\phi] = (\bar{\theta}[\phi])^2 \neq \bar{\theta}[\phi]$.

6 Making \bar{f} and \tilde{f} equivalent

Now compare the generalised function \bar{f} and \tilde{f} (12),(13). We would like these two generalised functions to be equivalent, so that we can write $\bar{f} \sim \tilde{f}$. One of the results of making $\bar{f} \sim \tilde{f}$ is that if $f, g \in \mathcal{F}$ then

$$\overline{(fg)} \sim \widetilde{(fg)} = \tilde{f} \tilde{g} \sim \bar{f} \bar{g}$$

In the Colombeau algebra, which is a quotient of equivalent generalised functions, we say that \bar{f} and \tilde{f} are the same generalised function.

The goal therefore is to restrict the set of possible ϕ so that when they are acted upon by $(\bar{f} - \tilde{f})$ the difference is *small*, where *small* will be made technically precise. When we think of quantities being small, we need a 1-parameter family of such quantities such that in the limit the difference vanishes. Here we label the parameter ϵ and we are interested in the limit $\epsilon \rightarrow 0$ from above, i.e. with $\epsilon > 0$. Given a one parameter set of functions $g_\epsilon \in \mathcal{F}$ then one meaning to say g_ϵ is small is if $g_\epsilon(y) \rightarrow 0$ for all y . However we would like a whole hierarchy of smallness. That is for any $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ then we can say

$$g_\epsilon = \mathcal{O}(\epsilon^q) \quad (16)$$

if $\epsilon^{-q}g_\epsilon(y)$ is bounded as $\epsilon \rightarrow 0$. Note that we use bounded, rather than tends to zero. However, clearly, if $g_\epsilon = \mathcal{O}(\epsilon^q)$ then $\epsilon^{-q+1}g_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$.

We will also need the notion of $g_\epsilon = \mathcal{O}(\epsilon^q)$ where $q < 0$. Thus we wish to consider functions which blow up as $\epsilon \rightarrow 0$, but not too quickly. Such functions will be called *moderate*.

Technically we say g_ϵ satisfies (16) if for any interval (a, b) there exists $C > 0$ and $\eta > 0$ such that

$$\epsilon^{-q}|g_\epsilon(x)| < C \quad \text{for all} \quad a \leq y \leq b \quad \text{and} \quad 0 < \epsilon < \eta \quad (17)$$

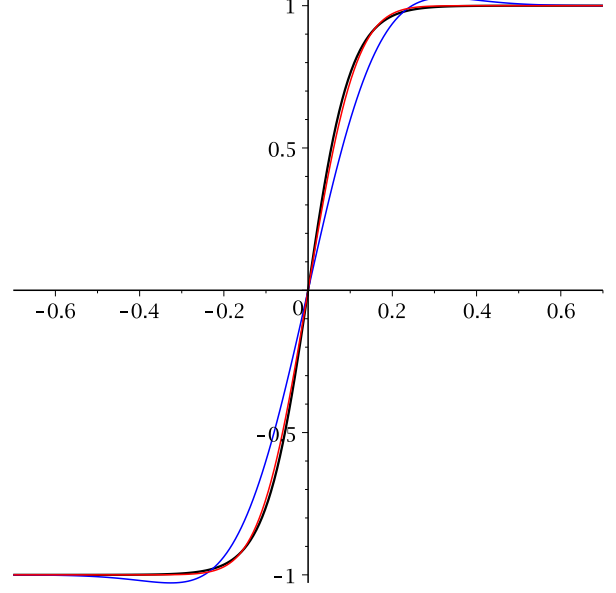
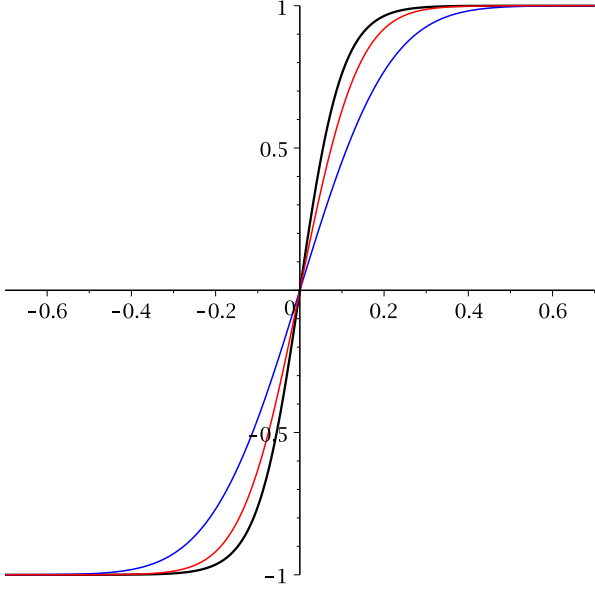
We introduce the parameter ϵ via the test functions, replacing $\phi \in \mathcal{F}_0$ with $\phi_\epsilon \in \mathcal{F}_0$ where

$$\phi_\epsilon(x) = \frac{1}{\epsilon} \phi\left(\frac{x}{\epsilon}\right) \quad (18)$$

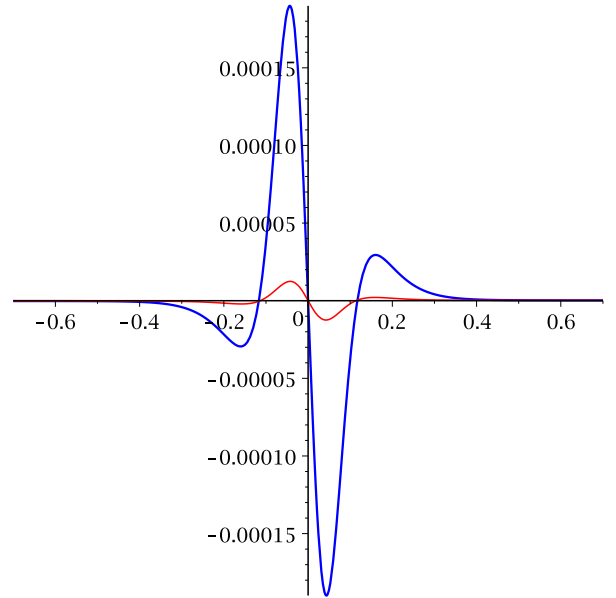
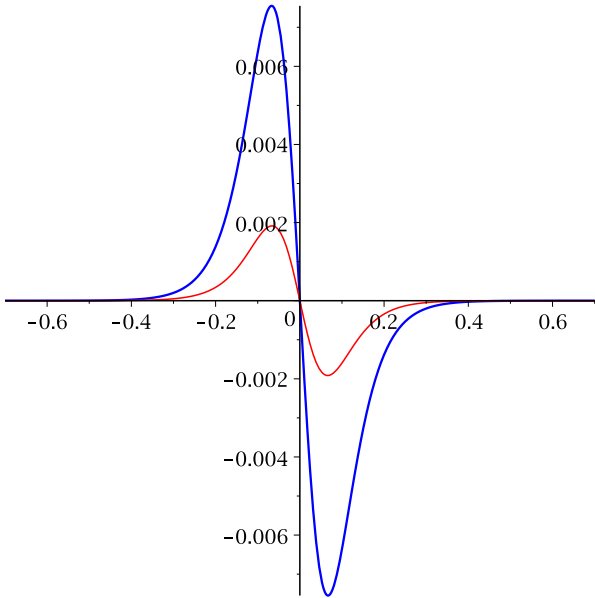
Observe that as $\epsilon \rightarrow 0$ then ϕ_ϵ becomes narrower and taller, in a definite sense more like a δ -function. Thus we consider a generalised function \mathbf{A} to be small, if for some appropriate set of test functions $\phi \in \mathcal{F}_0$ and for some $q \in \mathbb{Z}$, $\mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^q)$.

Let us first restrict $\phi \in \mathcal{F}_0$ to be test function which integrate to 1. That is we define $\mathbb{A}_0 \subset \mathcal{F}_0$,

$$\mathbb{A}_0 = \left\{ \phi \in \mathcal{F}_0 \mid \int_{-\infty}^{\infty} \phi(x) dx = 1 \right\} \quad (19)$$



f (back), $\bar{f}[\phi_1|_{\epsilon=0.2}]$ (blue) and $\bar{f}[\phi_1|_{\epsilon=0.1}]$ (red). f (back), $\bar{f}[\phi_3|_{\epsilon=0.2}]$ (blue) and $\bar{f}[\phi_3|_{\epsilon=0.1}]$ (red).



$(\bar{f} - \tilde{f})[\phi_1|_{\epsilon=0.02}]$ (blue) and
 $(\bar{f} - \tilde{f})[\phi_1|_{\epsilon=0.01}]$ (red).

$(\bar{f} - \tilde{f})[\phi_3|_{\epsilon=0.02}]$ (blue) and
 $(\bar{f} - \tilde{f})[\phi_3|_{\epsilon=0.01}]$ (red).

Figure 3: Plots of $\bar{f}[\phi_\epsilon]$ with $f(x) = \tanh(10x)$

Given $\phi \in \mathbb{A}_0$ and setting $z = (x - y)/\epsilon$ so that $x = y + \epsilon z$

$$\begin{aligned}\bar{f}[\phi_\epsilon](y) &= f^D[\phi_\epsilon^y] = \int_{-\infty}^{\infty} f(x) \phi_\epsilon(x - y) dx = \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(x) \phi\left(\frac{x - y}{\epsilon}\right) dx \\ &= \int_{-\infty}^{\infty} f(y + \epsilon z) \phi(z) dz\end{aligned}\tag{20}$$

Thus as $\epsilon \rightarrow 0$ then $f(y + \epsilon z) \approx f(y)$ so that, since $\phi \in \mathbb{A}_0$,

$$\bar{f}[\phi_\epsilon](y) = \int_{-\infty}^{\infty} f(y + \epsilon z) \phi(z) dz \approx \int_{-\infty}^{\infty} f(y) \phi(z) dz = f(y) \int_{-\infty}^{\infty} \phi(z) dz = f(y) = \tilde{f}[\phi_\epsilon](y)$$

In fact since $f(y + \epsilon z) - f(y) = \mathcal{O}(\epsilon)$ we can show using (17) that

$$\text{if } \phi \in \mathbb{A}_0 \quad \text{then} \quad (\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon)\tag{21}$$

This is good so far, but we want to further restrict the set ϕ so that we can satisfy

$$(\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^q)\tag{22}$$

to any order of $\mathcal{O}(\epsilon^q)$.

Taylor expanding $f(y + \epsilon z)$ to order $\mathcal{O}(\epsilon^{q+1})$ we have

$$f(y + \epsilon z) = \sum_{r=0}^q \frac{\epsilon^r z^r f^{(r)}(y)}{r!} + \mathcal{O}(\epsilon^{q+1})$$

Thus

$$\begin{aligned}(\bar{f} - \tilde{f})[\phi_\epsilon](y) &= \int_{-\infty}^{\infty} (f(y + \epsilon z) - f(y)) \phi(z) dz = \int_{-\infty}^{\infty} \left(\sum_{n=1}^q \frac{\epsilon^n z^n f^{(n)}(y)}{n!} + \mathcal{O}(\epsilon^{q+1}) \right) \phi(z) dz \\ &= \sum_{n=1}^q \frac{\epsilon^n f^{(n)}(y)}{n!} \int_{-\infty}^{\infty} z^n \phi(z) dz + \mathcal{O}(\epsilon^{q+1})\end{aligned}\tag{23}$$

Thus we can satisfy (16) to order $\mathcal{O}(\epsilon^{q+1})$ if the first q moments of $\phi(z)$ vanish:

$$\int_{-\infty}^{\infty} z^r \phi(z) dz = 0 \quad \text{for } 1 \leq r \leq q$$

We now define all the elements with vanishing moments.

$$\mathbb{A}_q = \left\{ \phi \in \mathcal{F}_0(\mathbb{R}) \mid \int_{-\infty}^{\infty} \phi(z) dz = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} z^r \phi(z) dz = 0 \quad \text{for } 1 \leq r \leq q \right\}\tag{24}$$

So clearly $\mathbb{A}_{q+1} \subset \mathbb{A}_q$. We can show that these functions exist. Thus from (23) we have

$$\phi \in \mathbb{A}_q \quad \text{implies} \quad (\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+1})\tag{25}$$

Two example test functions $\phi_1 \in \mathbb{A}_1$ and $\phi_3 \in \mathbb{A}_3$ are given in figure 1. The result $\bar{f}[\phi_\epsilon]$, (12), (20) is given in fig 3.

The easiest way to construct $\phi \in \mathbb{A}_q$ is to choose a test function ψ and then set

$$\phi(z) = \lambda_0 \psi(z) + \lambda_1 \psi'(z) + \cdots + \lambda_{q-1} \psi^{(q-1)}(z)$$

where $\lambda_0, \dots, \lambda_{q-1} \in \mathbb{R}$ are constants determined by (24).

7 Null and moderate generalised functions.

As we stated we wanted \bar{f} and \tilde{f} to be considered equivalent. From (25) we have $\phi \in \mathbb{A}_q$ then $(\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+1})$. We generalise this notion. We say that $\mathbf{A}, \mathbf{B} \in \mathcal{H}(\mathbb{R})$ are equivalent, $\mathbf{A} \sim \mathbf{B}$, if for all $q \in \mathbb{N}$ there is a $p \in \mathbb{N}$ such that

$$\phi \in \mathbb{A}_p \quad \text{implies} \quad \mathbf{A}[\phi_\epsilon] - \mathbf{B}[\phi_\epsilon] = \mathcal{O}(\epsilon^q) \quad (26)$$

We label $\mathcal{N}^{(0)}(\mathbb{R}) \subset \mathcal{H}(\mathbb{R})$ the set of all elements which are *null*, that is equivalent to the zero element $\mathbf{0} \in \mathcal{H}(\mathbb{R})$ that is

$$\mathcal{N}^{(0)}(\mathbb{R}) = \{\mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \mathbf{A} \sim \mathbf{0}\}$$

I.e.

$$\mathcal{N}^{(0)}(\mathbb{R}) = \{\mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{for all } p \in \mathbb{N} \text{ there exists } q \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_q, \mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^p)\} \quad (27)$$

Examples of null elements are of course $\bar{f} - \tilde{f} \in \mathcal{N}^{(0)}(\mathbb{R})$, which is true by construction. Another example is $\mathbf{N} \in \mathcal{N}^{(0)}(\mathbb{R})$ which is given by

$$\mathbf{N}[\phi](y) = \phi(1) \quad (28)$$

Since for any $\phi \in \mathbb{A}_0$ there exists $\eta > 0$ such that $1/\eta$ is outside the support of ϕ . Thus $\phi_\epsilon(1) = 0$ for all $\epsilon < \eta$ and hence $\mathbf{N}[\phi_\epsilon] = 0$ so $\mathbf{N} \in \mathcal{N}^{(0)}(\mathbb{R})$. However, although $\mathbf{N} \in \mathcal{N}^{(0)}$, we can choose ϕ so that $\mathbf{N}[\phi](y) = \phi(1)$ is any value we choose. Thus knowing that a generalised function \mathbf{A} is null says nothing about the value of $\mathbf{A}[\phi]$ but only the limit of $\mathbf{A}[\phi_\epsilon]$ as $\epsilon \rightarrow 0$.

We would like $\mathcal{N}^{(0)}(\mathbb{R})$ to form an ideal in $\mathcal{H}(\mathbb{R})$, that is that if $\mathbf{A}, \mathbf{B} \in \mathcal{N}^{(0)}(\mathbb{R})$ and $\mathbf{C} \in \mathcal{H}(\mathbb{R})$ then

- $\mathbf{A} + \mathbf{B} \in \mathcal{N}^{(0)}(\mathbb{R})$ and
- $\mathbf{AC} \in \mathcal{N}^{(0)}(\mathbb{R})$.

It is easy to see that the first of these is automatically satisfied. However the second requires one additional requirement. We need

$$\mathbf{C}[\phi_\epsilon] = \mathcal{O}(\epsilon^{-N}) \quad (29)$$

for some $N \in \mathbb{Z}$. Thus although $\mathbf{C}[\phi_\epsilon] \rightarrow \infty$ as $\epsilon \rightarrow 0$ we don't want it to blow up too quickly. Now we have the following:

Given $\mathbf{A} \in \mathcal{N}^{(0)}(\mathbb{R})$ and \mathbf{C} satisfying (29) and given $q \in \mathbb{N}_0$ then there exists $p \in \mathbb{Z}$ such that $\phi \in \mathbb{A}_p$ implies $\mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+N})$. Hence

$$(\mathbf{AC})[\phi_\epsilon] = \mathbf{A}[\phi_\epsilon]\mathbf{C}[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+N})\mathcal{O}(\epsilon^{-N}) = \mathcal{O}(\epsilon^q)$$

hence $\mathbf{AC} \in \mathcal{N}^{(0)}(\mathbb{R})$. We call the set of elements $\mathbf{C} \in \mathcal{H}(\mathbb{R})$ satisfying (29), *moderate* and set of moderate functions

$$\mathcal{E}^{(0)}(\mathbb{R}) = \{\mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{there exists } N \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_0, \mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^{-N})\} \quad (30)$$

Examples of moderate functions include

$$\bar{\Delta}[\phi_\epsilon](y) = \phi_\epsilon(-y) = \frac{1}{\epsilon} \phi\left(-\frac{y}{\epsilon}\right) = \mathcal{O}(\epsilon^{-1}), \quad (\bar{\Delta})^n[\phi_\epsilon] = \mathcal{O}(\epsilon^{-n})$$

and

$$\tilde{f}[\phi_\epsilon](y) = f(y) = \mathcal{O}(\epsilon^0)$$

8 Derivatives

The last part in the construction of the Colombeau Algebra is to extend all the definitions so that they also apply to the derivatives $\frac{d\mathbf{A}[\phi]}{dy}$, $\frac{d^2\mathbf{A}[\phi]}{dy^2}$, etc. We require that not only does a moderate function not blow up too quickly, but neither do its derivatives, i.e.

$$(\mathbf{A}[\phi])^{(n)} = \frac{d^n}{dy^n}(\mathbf{A}[\phi]) \in \mathcal{E}^{(0)}(\mathbb{R}) \quad (31)$$

Thus we define the set of moderate function as

$$\mathcal{E}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{E}^{(0)}(\mathbb{R}) \mid (\mathbf{A}[\phi])^{(n)} \in \mathcal{E}^{(0)}(\mathbb{R}) \text{ for all } n \in \mathbb{N}, \phi \in \mathbb{A}_0 \right\} \quad (32)$$

That is

$$\mathcal{E}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{for all } n \in \mathbb{N}_0 \text{ there exists } N \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_0, \right. \\ \left. (\mathbf{A}[\phi_\epsilon])^{(n)} = \mathcal{O}(\epsilon^{-N}) \right\} \quad (33)$$

Likewise we require that for two generalised functions to be equivalent then we require that all the derivatives are small

$$\mathcal{N}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{N}^{(0)}(\mathbb{R}) \mid (\mathbf{A}[\phi])^{(n)} \in \mathcal{N}^{(0)}(\mathbb{R}) \text{ for all } n \in \mathbb{N} \right\} \quad (34)$$

That is

$$\mathcal{N}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{for all } n \in \mathbb{N}_0 \text{ and } q \in \mathbb{N} \text{ there exists } p \in \mathbb{N} \text{ such that} \right. \\ \left. \text{for all } \phi \in \mathbb{A}_p, (\mathbf{A}[\phi_\epsilon])^{(n)} = \mathcal{O}(\epsilon^q) \right\} \quad (35)$$

9 Quotient Algebra

We write the Colombeau Algebra as a quotient algebra,

$$\mathcal{G}(\mathbb{R}) = \mathcal{E}(\mathbb{R})/\mathcal{N}(\mathbb{R}) \quad (36)$$

This means that, with regard to elements $\mathbf{A}, \mathbf{B} \in \mathcal{E}(\mathbb{R})$ we say $\mathbf{A} \sim \mathbf{B}$ if $\mathbf{A} - \mathbf{B} \in \mathcal{N}(\mathbb{R})$. For elements in $\mathbf{A}, \mathbf{B} \in \mathcal{G}(\mathbb{R})$ we simply write $\mathbf{A} = \mathbf{B}$.

Given $\mathbf{A} \in \mathcal{G}(\mathbb{R})$, then in order to get an actual number we must first choose a representative $\mathbf{B} \in \mathcal{E}(\mathbb{R})$ of $\mathbf{A} \in \mathcal{G}(\mathbb{R})$, then we must choose $\phi \in \mathbb{A}_0$ and $y \in \mathbb{R}$ then the quantity $\mathbf{B}[\phi](y) \in \mathbb{R}$.

10 Summary

We can summarise the steps needed to go from distributions to Colombeau functions:

- Convert distributions which give a number $\Psi[\phi]$ as an answer to functionals $\mathbf{A}[\phi]$ which give a function as an answer.
- Construct the sets of test functions \mathbb{A}_q , so that $\bar{f} \sim \tilde{f}$, i.e. $\bar{f} - \tilde{f} \in \mathcal{N}^{(0)}(\mathbb{R})$
- Limit the generalised functions to elements of $\mathcal{E}^{(0)}(\mathbb{R})$ so that the set $\mathcal{N}^{(0)}(\mathbb{R}) \subset \mathcal{E}^{(0)}(\mathbb{R})$ is an ideal.
- Extend the definitions of $\mathcal{E}^{(0)}(\mathbb{R})$ and $\mathcal{N}^{(0)}(\mathbb{R})$ to $\mathcal{E}(\mathbb{R})$ and $\mathcal{N}(\mathbb{R})$ so that they also apply to derivatives.
- Define the Colombeau Algebra as the quotient $\mathcal{G}(\mathbb{R}) = \mathcal{E}(\mathbb{R})/\mathcal{N}(\mathbb{R})$.

The formal definition, we define $\mathcal{E}(\mathbb{R})$ via (33), then $\mathcal{N}(\mathbb{R})$ via (35) and (24). Then define the Colombeau Algebra $\mathcal{G}(\mathbb{R})$ as the quotient (36).

Acknowledgement

The author is grateful for the support provided by STFC (the Cockcroft Institute ST/G008248/1) and EPSRC (the Alpha-X project EP/J018171/1).

References

- [1] Jean François Colombeau. *Elementary introduction to new generalized functions*. Elsevier, 2011.
- [2] Tạ Ngọc Trí and Tom H Koornwinder. The Colombeau Theory of Generalized Functions. Master's thesis, KdV Institute, Faculty of Science, University of Amsterdam, The Netherlands, 2005.