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Abstract

In this paper we study a model of political competition where citizens vote sincerely and candidates may be either citizens or Downsian politicians. The model extends the citizen-candidate model proposed by Osborne and Slivinski [1996] by including Downsian politicians similar to those studied by Osborne [1993]. We give necessary and sufficient conditions for existence, together with complete characterisation, of one party and two party Nash equilibria in our model. An important feature, in view of the Duverger's Law, of the two-party equilibrium is that these equilibria cannot have any Downsian contestant. Moreover, we compare our model with that studied by Osborne and Slivinski [1996], showing that in both cases there exist political configurations that can appear in one of the models only. We show also that in our settings it is possible to have Nash equilibria with Downsian candidates, without requiring to have very restrictive constraints on the distribution function. We also argue that as the number of parties in equilibrium increases, the 'likelihood' of an ideology driven citizen-candidate winning the elections and running the government falls. Finally we argue that in any equilibrium extremist parties proposing their policies uniquely are typically ideology-driven as well.

Keywords: Citizen-candidates, Downsian Politicians, Plurality Rule.
JEL classification: C70, D70, D72

1 Introduction

The seminal Hotelling-Downs model (Hotelling [1929] and Downs [1957]) of electoral competition and its well known result of policy convergence to the median voter have remained central in the literature on formal political economics.[1] However, in a seminal work by Osborne [1993] it is shown that this result is not robust to the possibility of new entry in a model where politicians care only about winning the elections and voters vote sincerely. In particular it was shown that when there are $n > 2$ potential candidates, for almost all distributions over the political preferences of voters, a Nash equilibrium in pure strategies fails to exist.[2] The literature then naturally moved towards exploring models of sincere voting where politicians are ideology driven. In this respect, Osborne and Slivinski [1996] have proposed the citizen-candidate model to address the problem of entry. In that model, each voter (a citizen) with an ideal policy can become a candidate and implement his ideal policy upon winning the election.[3] The citizen-candidate model has become central to many on-going research today as it has been successful in overcoming the equilibrium existence problem with free entry.

In any large and matured democracy it is natural to have both Downsian politicians who are professionals and care only about winning elections along with ideology-driven citizen-candidates who care also about the policy that is finally implemented. Hence it is natural to study a model that incorporates both these features to be able to address issues relating to the nature of equilibrium policies and the likely profile of candidates who compete.

We study a model of political competition where citizen-candidates and Downsian players compete side-by-side to win the election. The model en-

[1] Osborne [1995] presses on the fact that the convergence result is strongly robust as long as there are only 2 competing politicians.

[2] Osborne [1995] also suggests that no easy extension of the basic Hotelling-Downs model exists that resolves this equilibrium non-existence problem. However in a beautiful recent work by Sengupta and Sengupta [2008], this problem is partially overcome by allowing Downsian politicians the choice of a cost-saving withdrawal option.

[3] Besley and Coate [1997] studies a similar model but assumes strategic voting.

riches the one studied in Osborne and Slivinski [1996] with Downsian players similar to those studied in Osborne [1993]. In our environment, there is a continuum of citizens with single peaked preferences over a one-dimensional ideology space. In addition, there are also a countably infinite number of Downsian politicians. The identity of each agent is common knowledge. Any agent can enter the electoral competition (governed by the plurality rule) that comes with an entry fee that is common to all. These costs reflect the expenses associated with mobilizing and administering a campaign.[4] An eventual winner also receives a personal benefit from holding the office. We assume that these benefits are larger for Downsian politicians, since we see them as seasoned office-users who are, therefore, more efficient in extracting the benefits.[5] Citizens in our model vote sincerely while votes of the Downsian politicians do not affect the outcome as they are countably many.

We focus on pure strategy Nash equilibrium and provide a set of necessary and sufficient conditions for one and two party equilibria. An interesting feature, central to Duverger's law, is that for that law to hold (that is for equilibria with exactly two parties) we show that no Downsian politician enters the competition. Since existence of equilibria with more than two parties is established in Osborne and Slivinski [1996], we address the issue of such equilibria by comparing across those in Osborne and Slivinski [1996] and ours. In that sense, we give complete characterizations on costs and benefits such that an equilibrium in Osborne and Slivinski [1996] can be supported as an equilibrium in our general framework, and vice versa. In doing so, we provide several examples to address some impossibility results in this respect. We also provide some other comparisons with respect to costs and benefits. We also argue that as the number of parties in equilibrium increases, the 'likelihood' of an ideology driven citizen-candidate winning the elections and running the government falls. Moreover we show that in any equilibrium,

[4]Our results go through even if we assume that costs vary between citizens and Downsians, though the proofs get more lengthy.

[5]This is an assumption we make as a first step towards studying this general framework. Some of our results depend on it.

unique (that is those who do not share their electoral platforms) extremist parties are ideology-driven as well.

1.1 Related Literature

The central theme of our work is competition amongst politicians with heterogeneous motivations. This has been raised as early as in Calvert [1985] where the author had informally concluded that competition amongst politicians with heterogeneous motivations, but with full post-election commitment to announced policies, should not affect equilibria in spatial models of elections. Heterogeneity within a political party is formally modelled in Roemer [1999] to analyse how this affects the ability of parties to compete in a model with full policy commitment. A recent work by Callander [2008] studies a model of electoral competition amongst two parties who can either be policy or office motivated with these types being private information. The paper shows that although office motivated parties are favoured in two-party elections, policy motivated ones do win quite often. It also shows that the very existence of this heterogeneity does affect the announced equilibrium policies. These results are driven by a novel aspect called 'policy development': policy motivated candidates care more about the actual implementation of the policy and hence they exert more effort once they win the elections. This fact is liked by the voters and hence even if the policy announced by a policy motivated party otherwise should earn less votes than that announced by an office motivated party (since such parties more carefully do the vote-bank calculations to announce their policies), such motivated parties are endogenously rewarded by the voters and so it is no longer the case that voters necessarily support a candidate closest to them in the ideology space. The paper then addresses the issue of 'free-entry' but restricting attention to the existence of the same two parties to show that a unique equilibrium always exists where both types of citizens enter the candidate pool. However, we learn little about equilibria with more than two parties or about robustness of two-party equilibria to entry at other policy positions. In this sense our work here is more general as it addresses the issue of free-entry more seriously. Of course we do not address here the otherwise important aspect of policy development (and the nice feature of incomplete information) that is both crucial and novel in Callander [2008]. The possibility that types of politicians is private information is an extension that we wish to address in the future.

The rest of the paper is structured as follows. In Section 2 the model is described formally. Section 3 studies Nash equilibrium with one and two parties. Section 4 deals with equilibria with more than two parties in general and makes formal comparisons of the sets of equilibria with and without Downsian politicians. This section also discuses issues concerning which type of politicians are more likely to contest plurality elections. The paper draws its conclusions in Section 5. Some proofs are collected in an Appendix at the end.

2 The Model

There are two sets of players: set $\mathcal{C}$ consists of a continuum of *citizens* and an infinite countable set $\mathcal{D}$ consists of *Downsian politicians*. These identities are common knowledge amongst all players.[6]

Each citizen from $\mathcal{C}$ has a single-peaked preference over the set of *policy positions*, which is assumed to be the real line $\mathbb{R}$. The ideal policies of the citizens are distributed on $\mathbb{R}$ with a distribution function F and density f. We assume that F is non atomic, that is the support of f is an interval or the whole line. Notice that this implies in particular that F is continuous and has unique median m.[7] Throughout the paper we will need to refer to points at which F attains particular values. Hence to facilitate our presentation we

[6] Relaxing this assumption leads to a much more involved game of incomplete information, and we reserve this for future research.

[7] The same assumption about distribution function is made in Osborne [1993], while in Osborne and Slivinski [1996] a weaker assumption is made, requiring F to be continuous and to have unique median.

will use a_x to denote $F^{-1}(x)$.

Each player (i. e. a citizen or a Downsian politician) chooses (simultaneously and independently) whether to enter an electoral competition or to stay out. If a citizen $i \in \mathcal{C}$ enters, he proposes his ideal position x_i. Since it is common knowledge that every $i \in \mathcal{C}$ is ideology driven, so any other proposal is not credible. On the other hand, if a Downsian player i enters, he has to announce any policy $x_i \in \mathbb{R}$ as his platform. A player who decides to enter the electoral competition is called a *candidate*. After all players have decided on whether to enter or not, they cast their votes. In case of citizens, voting is sincere, that is each citizen votes (with equal probability) for one of candidates proposing the policy closest to his own ideal positions.[8] In case of Downsian players, we assume that they either vote for their own proposed policy (if they are candidates) or they do not vote at all (or cast their vote randomly). In any case, votes of players from $\mathcal{D}$ do not affect the final outcome of elections. This is because the set $\mathcal{D}$ is countable and their support towards any policy cannot affect the frequency of voters supporting that policy as defined by f.

We study elections governed by the *plurality rule*, so that the potential winner of the elections is one of the candidates who receives the maximum mass of votes (from the citizens). As usual, the unique winner is selected from the set of such potential winners with equal probability.

Each citizen's payoff depends on the distance between his ideal position and the policy proposed by the winner of the election, as well as on the benefits from winning the election and the costs of entering them, in case the citizen chooses to be a candidate. Payoff of Downsian players is either 0 if they do not enter the election, or it depends on similar benefits and costs if they enter as a candidate. Thus, citizens are ideology driven, while Downsian politicians are not.

[8]Notice that this approach means that voters vote for policies rather than candidates, which is consistent with the fact that their preferences are over the set of policies rather then preferences over the set of pairs (policy, player).

Costs of entering the election are the same for both types of players and are represented by real number $c > 0$. Benefits from winning the election may vary between types of players and are represented by real numbers $b_{\mathcal{C}} > 0$ (for citizens) and $b_{\mathcal{D}} \geq b_{\mathcal{C}}$ (for Downsian politicians). As highlighted in the introduction, the motivation behind varying benefits between types of players is that Downsian players are presumed in this paper to be "professionals" who are at least as efficient as the citizens in appropriating personal (and non-ideological) gains from winning the elections. To keep the model interesting, we make the usual assumption that $b_{\mathcal{D}} > c$ so that Downsian players have the motivation to become candidates.

Let $y \in \mathbb{R}$ be the policy that is implemented. Then the payoff of a citizen $i \in \mathcal{C}$ with ideal position x_i is

$$\begin{cases} b_{\mathcal{C}} - c & \text{if } i \text{ enters and wins,} \\ -|x_i - y| - c & \text{if } i \text{ enters and loses,} \\ -|x_i - y| & \text{if } i \text{ does not enter.} \end{cases}$$

We assume that if no candidate contests the elections (so that no policy is implemented), each citizen $i \in \mathcal{C}$ gets $-\infty$.

Independent of the value of the implemented policy (including the case where no policy is implemented), the payoff of any Downsian player $i \in \mathcal{D}$ is

$$\begin{cases} b_{\mathcal{D}} - c & \text{if he enters and wins,} \\ -c & \text{if he enters and loses,} \\ 0 & \text{if he does not enter.} \end{cases}$$

In summary, a strategic game, denoted Γ_{CD}, is studied in which the set of players is $\mathcal{C} \cup \mathcal{D}$. For each player $i \in \mathcal{C}$, the set of pure strategies is $\{E, N\}$, where E stays for "entering the competition" and N stays for "not entering the competition". For each player $i \in \mathcal{D}$, the set of pure strategies is $\mathbb{R} \cup \{N\}$, where $x_i \in \mathbb{R}$ stays for "entering the competition and proposing policy x_i" and N stays for "not entering the competition".[9]

[9] Every player (that is a citizen-candidate or a Downsian politician) has some preferences

Given a strategy profile $\mathbf{s}$ of the game Γ_{CD}, we use $K(\mathbf{s})$ to denote the set of players who enter the competition under $\mathbf{s}$ (i. e. the set of candidates in $\mathbf{s}$), $W(\mathbf{s})$ to denote the set of candidates from $K(\mathbf{s})$ who gain maximum mass of votes (i. e. the set of candidates tying for the first place) and $L(\mathbf{s}) = K(\mathbf{s})\backslash W(\mathbf{s})$ to denote the set of candidates from $K(\mathbf{s})$ who lose with certainty. We will also use $X(\mathbf{s}) \subset \mathbb{R}$ to denote the set of policies proposed by candidates from $K(\mathbf{s})$.

All players are risk neutral so that $u_i(\mathbf{s})$, the expected payoff of player i with respect to strategy profile $\mathbf{s}$, is given as follows. If $i \in \mathcal{D}$ then

$$u_i(\mathbf{s}) = \begin{cases} -c & \text{if } i \in L(\mathbf{s}), \\ \frac{b_{\mathcal{D}}}{|W(\mathbf{s})|} - c & \text{if } i \in W(\mathbf{s}), \\ 0 & \text{otherwise,} \end{cases}$$

while if $i \in \mathcal{C}$ then

$$u_i(\mathbf{s}) = \begin{cases} -\infty & \text{if } K(\mathbf{s}) = \varnothing, \\ -c - \hat{u}_i(\mathbf{s}) & \text{if } i \in L(\mathbf{s}), \\ \frac{b_{\mathcal{C}}}{|W(\mathbf{s})|} - c - \hat{u}_i(\mathbf{s}) & \text{if } i \in W(\mathbf{s}), \\ -\hat{u}_i(\mathbf{s}) & \text{otherwise,} \end{cases}$$

where

$$\hat{u}_i(\mathbf{s}) = \frac{\sum_{j \in W(\mathbf{s})} |x_i - x_j|}{|W(\mathbf{s})|}$$

In addition, we assume that if indifferent, then each player prefers to stay out of the competition than to enter, that is for any two strategy profiles $\mathbf{s}$ and $\mathbf{s}'$ and for any $i \in \mathcal{C} \cup \mathcal{D}$, we have:

$$\mathbf{s} \precsim_i \mathbf{s}' \text{ iff } u_i(\mathbf{s}) < u_i(\mathbf{s}') \text{ or } u_i(\mathbf{s}) = u_i(\mathbf{s}') \text{ and either } s_i' = s_i \text{ or } s_i \neq N.$$

over the policy space; being indifferent to all policies is just a special case. But to call that preference ordering Downsian may look somewhat odd. In principle what is central is that while those we call Downsian are able to commit to any policy announcement, those who are citizens have their unique ideal policy as the only credible pre-election platform. We thank Kunal Sengupta for pointing this out.

The game Γ_{CD} is an extension of the citizen-candidate model studied in Osborne and Slivinski [1996] (the case where $\mathcal{D} = \varnothing$) and we shall refer to it as the game Γ_C.

In what follows we shall focus our attention on issues concerning existence and characteristics of pure strategy Nash equilibria of the game Γ_{CD}. But before we begin our analysis, we provide a discussion on the nature of the Downsian politicians that we study in relation to those studied in Osborne [1993].

2.1 A discussion on Downsian players

Downsian politicians studied in our model are similar to those studied in Osborne [1993]. However, there are some differences which we would like to point out here. Firstly, we assume that there are infinitely many Downsian politicians, while Osborne [1993] has a fixed number of Downsian politicians. Secondly, we define the preferences of players over strategy profiles on the basis of costs and benefits associated with the elections, while Osborne [1993] abstracts from costs and benefits and defines these preferences directly. The properties of our preferences and those from Osborne [1993] are in most cases the same: each Downsian player prefers to win outright than to tie, each Downsian player prefers to tie for the first place than to lose, each Downsian player prefers to stay out of the competition than to enter and lose and each Downsian player is indifferent between any two outcomes in which he wins outright. The difference is that in Osborne [1993] it is assumed that each Downsian player prefers to tie for the first place than to stay out from the competition, while in our model this preference depends on $b_{\mathcal{D}}$ and c. However, in our model, each Downsian player always prefers to enter the competition and win outright (that is win with probability 1) than to stay out. This is due to the assumption that $b_{\mathcal{D}} > c$.[10] We will refer to the model with n Downsian players with preferences as studied in Osborne [1993] as

[10] In Osborne [1993] one more variant of preferences is studied, where Downsian politicians always prefer to enter the competition. This variant is not relevant to our studies.

the game Γ_{D_n}.

3 Equilibria of Γ_{CD}

Before proving results concerning existence of Nash equilibria in pure strategies in the game Γ_{CD}, it will be helpful to first identify some characteristics of such equilibria. For this we will need the following notions.

Let $\mathbf{s}$ be a strategy profile in Γ_{CD} with $K(\mathbf{s})$ being finite and non empty. We will use $l(\mathbf{s})$ to denote a candidate proposing the *extreme left policy* under $\mathbf{s}$, that is a candidate such that $x_{l(\mathbf{s})} = \min X(\mathbf{s})$. Similarly we will use $r(\mathbf{s})$ to denote a candidate proposing *extreme right policy*, that is a candidate such that $x_{r(\mathbf{s})} = \max X(\mathbf{s})$.

Also, given a candidate $i \in K(\mathbf{s})$, we define a *left neighbour* $l(\mathbf{s}, i)$ and a *right neighbour* $r(\mathbf{s}, i)$ of i w. r. t. $\mathbf{s}$ as follows:

$$l(\mathbf{s}, i) = \begin{cases} -\infty & \text{if } i = l(\mathbf{s}), \\ j, \text{ such that } x_j = \max\{x \in X(\mathbf{s}) : x < x_i\} & \text{otherwise,} \end{cases}$$

and

$$r(\mathbf{s}, i) = \begin{cases} -\infty & \text{if } i = r(\mathbf{s}), \\ j, \text{ such that } x_j = \min\{x \in X(\mathbf{s}) : x > x_i\} & \text{otherwise.} \end{cases}$$

Notice that player i may have several neighbours proposing the same policy, in which case $l(\mathbf{s}, i)$ (or $r(\mathbf{s}, i)$) refers to any one such neighbour.

As in Osborne [1993], the *constituency* of x_i is the fraction of the population that votes for one of the candidates proposing x_i, that is it is equal to $F((x_i + x_{r(\mathbf{s},i)})/2) - F((x_i + x_{l(\mathbf{s},i)})/2)$. A constituency consists of two *semi-constituencies*, the *left constituency*, which is the fraction of the population that votes for a candidate proposing x_i and has an ideal point $< x_i$, that is it is equal to $F(x_i) - F((x_i + x_{l(\mathbf{s},i)})/2)$, and the *right constituency*, which is the fraction of the population that votes for a candidate proposing x_i and has an ideal point $> x_i$, that is it is equal to $F((x_i + x_{r(\mathbf{s},i)})/2) - F(x_i)$.

The following lemma is an extension of [Osborne, 1993, Lemma 1] that studied a model with only Downsian politicians. Point (i) of the lemma was shown in [Osborne and Slivinski, 1996, Lemma 2] while points (v) and (vi) were shown in [Osborne and Slivinski, 1996, Lemma 1] that studied the game Γ_C. The proof of the lemma can be found in the Appendix.

Lemma 1. *Let $\mathbf{s}^*$ be a Nash equilibrium in Γ_{CD}. Then the following hold*

(i). every policy $x \in X(\mathbf{s}^*)$ is proposed by at most two candidates,

(ii). for any $i \in \mathcal{D}$, if $|K(\mathbf{s}^*)| \geq 2$ and $i = l(\mathbf{s}^*)$ or $i = r(\mathbf{s}^*)$ (that is i is a Downsian candidate proposing an extreme policy), then x_i is proposed by more than one candidates,

(iii). for any $x \in X(\mathbf{s}^*)$ proposed by more than one candidates, the left and right constituencies of x are equal,

(iv). $L(\mathbf{s}^*) \cap \mathcal{D} = \varnothing$, that is no Downsian candidate loses with certainty,

(v). no candidate sharing his proposed policy with another candidate loses with certainty, and

(vi). $\{l(\mathbf{s}^*), r(\mathbf{s}^*)\} \subseteq W(\mathbf{s}^*)$, that is no candidate proposing extreme policy loses with certainty,

Remark 1. *Notice that points (iv), (v) and (vi) do not require any assumptions on the distribution function F.*

The next question we ask is as follows: suppose in the game Γ_{CD} there is an equilibrium where a Downsian politician (amongst other candidates) proposes a policy alone. What can we say about the announced policies which are his immediate neighbours? It turns out that it is impossible to have an equilibrium configuration with a Downsian candidate proposing his policy alone while his two neighbouring policies are both proposed by more than one candidate. The lemma is not vacuous as Example 2 (provided in Section 4) suggests. Moreover, this lemma is useful in proving some non-existence results in the game Γ_{D_n} studied by Osborne [1993] with only Downsian politicians.

Lemma 2. *Let $\mathbf{s}^*$ be a Nash equilibrium of Γ_{CD} with the distribution function F being non atomic. If $i \in K(\mathbf{s}^*) \cap \mathcal{D}$ is a Downsian candidate and x_i is proposed by exactly one candidate, then either $x_{l(\mathbf{s}^*,i)}$ or $x_{r(\mathbf{s}^*,i)}$ is proposed by exactly one candidate.*

Proof. Let $[a, b]$ be the support of f and let S be the mass of support that each player who does not lose with certainty gets (obviously $S > 0$). By point (ii) of Lemma 1 we know that $l(\mathbf{s}^*, i) > -\infty$ and $r(\mathbf{s}^*, i) < +\infty$. Suppose that $x_{l(\mathbf{s}^*,i)}$ and $x_{r(\mathbf{s}^*,i)}$ are proposed by more than one candidate, each. By point (i) of Lemma 1 each of these policies is proposed by two candidates if $\mathbf{s}^*$ is a Nash equilibrium. Moreover, by point (iii) of Lemma 1 it holds that $\left[x_{l(\mathbf{s}^*,i)}, x_{r(\mathbf{s}^*,i)}\right] \subseteq [a, b]$.

Also, by point (iii) of Lemma 1 it holds that $F\left(x_{r(\mathbf{s}^*,i)}\right) - F\left(x_{l(\mathbf{s}^*,i)}\right) = 3S$. Hence either

$$F\left((x_{l(\mathbf{s}^*,i)} + x_{r(\mathbf{s}^*,i)})/2\right) - F\left(x_{l(\mathbf{s}^*,i)}\right) \geq 3S/2$$

or

$$F\left(x_{r(\mathbf{s}^*,i)}\right) - F\left((x_{l(\mathbf{s}^*,i)} + x_{r(\mathbf{s}^*,i)})/2\right) \geq 3S/2.$$

Without loss of generality suppose that the first case holds.

Since F is non atomic and $\left[x_{l(\mathbf{s}^*,i)}, x_{r(\mathbf{s}^*,i)}\right] \subseteq [a, b]$ and

$$F\left((x_{l(\mathbf{s}^*,i)} + x_i)/2\right) - F\left(x_{l(\mathbf{s}^*,i)}\right) = S,$$

so there must exist $t \in \left(x_{l(\mathbf{s}^*,i)}, x_i\right)$, such that

$$F\left((x_{l(\mathbf{s}^*,i)} + x_i)/2\right) - F\left((x_{l(\mathbf{s}^*,i)} + t)/2\right) > 3S/4.$$

We will show that if player i proposed t instead of x_i, he would win outright, so $\mathbf{s}^*$ cannot be a Nash equilibrium.

Take any t satisfying conditions above and suppose that player i proposed t instead of x_i. Let

$$S_1 = F\left((x_i + x_{r(\mathbf{s}^*,i)})/2\right) - F\left((t + x_{r(\mathbf{s}^*,i)})/2\right)$$

and

$$S_2 = F\left((x_{l(\mathbf{s}^*,i)} + t)/2\right) - F\left((x_{l(\mathbf{s}^*,i)} + x_i)/2\right).$$

By the construction above it holds that $S_2 > 3S/4$. Moreover, since

$$\left(x_{l(\mathbf{s}^*,i)} + x_{r(\mathbf{s}^*,i)}\right)/2 < \left(t + x_{r(\mathbf{s}^*,i)}\right)/2 < \left(x_i + x_{r(\mathbf{s}^*,i)}\right)/2,$$

so

$$F\left(x_{r(\mathbf{s}^*,i)}\right) - F\left((x_{l(\mathbf{s}^*,i)} + x_{r(\mathbf{s}^*,i)})/2\right) \leq 3S/2$$

and

$$F\left(x_{r(\mathbf{s}^*,i)}\right) - F\left((x_i + x_{r(\mathbf{s}^*,i)})/2\right) = S,$$

thus $S_1 < S/2$.

The mass of support player i gets after repositioning himself to t is $S - S_1 + S_2 > 5S/4$, while the mass of support candidates proposing $x_{r(\mathbf{s}^*,i)}$ get is $(2S + S_1)/2 < 5S/4$. Also, the mass of support candidates proposing $x_{l(\mathbf{s}^*,i)}$ get is $< S$. Hence player i wins outright after repositioning himself to t.□

The following corollary is immediate from the above proof.

Corollary 1 (Non-existence of Nash equilibria in Downsian Competition)**.** *Consider the game* Γ_{D_n} *as in Osborne [1993]. If the distribution function* F *is non atomic then the game* Γ_{D_5} *has no Nash equilibrium. Moreover for any game* $\Gamma_{D_{2n}}$*, a Nash equilibrium where* $n+1$ *policies are proposed does not exist and for any game* $\Gamma_{D_{2n+1}}$ *a Nash equilibrium with* $\leq n+1$ *policies does not exist.*[11]

[11] As was shown in Osborne [1993], existence of Nash equilibria of game Γ_{D_n} with $n > 2$ puts very restrictive constraints on the distribution function F, so such equilibria hardly ever exist.

Remark 2. *It is of course possible to have a Nash equilibrium of* Γ_{CD} *where a Downsian candidate neighbours two candidates proposing the same policy on one side and another candidate proposing his policy alone on the other side, as illustrated by Example 3.*

Having established some properties of the Nash equilibria of Γ_{CD}, we can look closer at existence of equilibria with different numbers of candidates, in particular, with Downsian candidates.

3.1 One-party equilibrium

We start with equilibria where exactly one candidate stands for the election. The following proposition corresponds to [Osborne and Slivinski, 1996, Proposition 1], where conditions of one candidate Nash equilibria of Γ_C were characterized. Properties of such Nash equilibria in Γ_{CD} are slightly different, as explained in the remarks following the proposition. Most importantly, no one-party equilibrium with the party announcing a policy different from the median policy can exist in our settings. This is in contrast to Γ_C, where there are costs and benefits such that a non-median one-party equilibrium exists.

Proposition 1. *Let* $\mathbf{s}^*$ *be a Nash equilibrium of* Γ_{CD}*. Then* $|K(\mathbf{s}^*)| = 1$ *iff* $b_{\mathcal{D}} \leq 2c$*. Moreover, if* $K(\mathbf{s}^*) = \{i\}$*, then* $x_i = m$*.*

Proof. The proof is analogical to the proof of Proposition 1 from Osborne and Slivinski [1996]. So suppose that $\mathbf{s}^*$ is a Nash equilibrium and that $K(\mathbf{s}^*) = \{i\}$. If $b_{\mathcal{D}} > 2c$, then a Downsian player would enter and propose x_i, hence it must be that $b_{\mathcal{D}} \leq 2c$. Moreover, if $x_i \neq m$, then at least a Downsian player would enter proposing m and would win outright. Hence it must be $x_i = m$.

On the other hand, suppose that $b_{\mathcal{D}} \leq 2c$. Than it also holds that $b_{\mathcal{C}} \leq 2c$. Then a strategy profile $\mathbf{s}^*$ with $K(\mathbf{s}^*) = \{i\}$ and $x_i = m$ is a Nash equilibrium. This is because no player j can enter and get a support larger than i, and if j gets the same support as i, then her expected payoff is not positive since

$b_{\mathcal{D}} \leq 2c$. Thus if $j \in \mathcal{D}$, then he prefers to stay out and if $j \in \mathcal{C}$, then he also prefers to stay out, as $\hat{u}_i(\mathbf{s}^*) = \hat{u}_i(\mathbf{s}^*_{-i}, E)$ and $u_i(\mathbf{s}^*_{-i}, E) - u_i(\mathbf{s}^*) = b_{\mathcal{C}}/2 - c \leq 0$.

It is also not profitable for i to stay out, as either $i \in \mathcal{D}$ and he prefers to enter and win outright than to stay out, or $i \in \mathcal{C}$, in which case he would get $-\infty$ if he stayed out instead of entering and would be worse off.

Notice that it is possible to have a one candidate Nash equilibrium with $i \in \mathcal{C}$ and $b_{\mathcal{C}} < c$. □

Remark 3. *Notice that Proposition 1 holds for a game Γ_{D_n} studied in Osborne [1993], if and only if $n = 1$. This is because it is assumed there that each Downsian player prefers to tie for the first place than to stay out of the competition, while in the model we study this is not the case when $b_{\mathcal{D}} \leq 2c$.*

Remark 4. *As the above proposition suggests, it is impossible to have a one candidate Nash equilibrium in Γ_{CD} with the candidate i proposing $x_i \neq m$, which is possible in Γ_C, when $b_{\mathcal{C}} < c$. This is due to the existence of Downsian players with the fact that they always prefer to enter the competition and win outright than to stay out (i. e. due to assumption that $b_{\mathcal{D}} > c$).*

3.2 Two-party equilibrium

Since our policy space is 1-dimensional, the classical conjecture of Duverger implies that in such scenarios two-party equilibria become important. We now look at Nash equilibria where exactly two candidates stand for the election. Again, the following proposition corresponds to [Osborne and Slivinski, 1996, Proposition 2], where Nash equilibria of Γ_C with two candidates are characterized.

We adopt the following notation from Osborne and Slivinski [1996]. Let $\varepsilon > 0$ and consider a configuration where there are exactly two candidates standing for the election, one of them proposing $m-\varepsilon$ and the other proposing $m + \varepsilon$. Let $s(\varepsilon, F) \in (m - \varepsilon, m + \varepsilon)$ be a point such that

$$F((m - \varepsilon + s(\varepsilon, F))/2) = 1 - F((m + \varepsilon - s(\varepsilon, F))/2),$$

that is if a player enters proposing $s(\varepsilon, F)$, then constituencies of candidates proposing $m - \varepsilon$ and $m + \varepsilon$ will be equal.

Also, let $e_p(F)$ be the critical value of ε, such that if $\varepsilon < e_p(F)$, then any entry by a third player gives him a mass of votes $< 1/3$ and if $\varepsilon > e_p(F)$, then there exists an entry by a third player that gives him a mass of votes $> 1/3$.

The following proposition gives a set of necessary and sufficient conditions for the existence of a 2-party equilibrium in the game Γ_{CD}. The most important feature of any such equilibrium is that both candidates must be citizens, that is *no two-party equilibrium can have a Downsian contestant.*

Proposition 2. *A Nash equilibrium of* Γ_{CD} *with two candidates standing for the election exists iff* $b_{\mathcal{C}} > 2(c - e_p(F))$. *Moreover,* $\mathbf{s}^*$ *is a Nash equilibrium of* Γ_{CD} *with* $K(\mathbf{s}^*) = \{i, j\}$ *such that* $x_i \leq x_j$ *iff*

(i). $\{i, j\} \subseteq \mathcal{C}$,

(ii). $m - x_i = x_j - m = \varepsilon$,

(iii). $0 < \varepsilon \leq e_p(F)$,

(iv). $|m - s(\varepsilon, F)| \leq c < \varepsilon + b/2$,

(v). $\varepsilon = e_p(F)$ *implies* $b_{\mathcal{D}} \leq 3c$,

Proof. The proof is analogical to the proof of Proposition 2 from Osborne and Slivinski [1996], where conditions for existence of two candidate Nash equilibria of Γ_C and their properties were characterized.

We start by showing that conditions (i) – (v) are necessary for a strategy profile $\mathbf{s}^*$ with $K(\mathbf{s}^*) = \{i, j\}$ and $x_i \leq x_j$ to be a Nash equilibrium. So suppose that $\mathbf{s}^*$ is a Nash equilibrium. Point (ii) follows from the fact that the constituencies of i and j must be equal, so m must be equidistant from x_i and x_j. For point (iii) notice that by definition of $e_p(F)$ it must be $\varepsilon < e_p(F)$. It must be also that $\varepsilon > 0$, as otherwise entry by a Downsian player at $t \in (a_{1/3}, m)$ provides the entrant with mass of support $> 1/3$, while i and

j would both have the mass of support $< 1/3$, so the entrant would win outright. Also, if $\varepsilon = e_p(F)$, then it must be that $b_{\mathcal{D}} \leq 3c$, as otherwise there exists an entry so that the entrant would not be losing with certainty, and so a Downsian player would enter as the entry would result in at most three candidates who do not lose with certainty and $b_{\mathcal{D}}/3 - c > 0$.

Point (i) follows immediately from points (ii) and (iii) shown above and point (ii) of Lemma 1.

For point (iv), as we have shown already, it must be that $c < \varepsilon + b/2$. For the second part of this point assume that $c < |m - s(\varepsilon, F)|$ and suppose that $s(\varepsilon, F) < m$ (arguments in case of $s(\varepsilon, F) < m$ are analogical and it is impossible to have $s(\varepsilon, F) = m$ under the assumption $c < |m - s(\varepsilon, F)|$, as $c > 0$). Then for any $t \in (s(\varepsilon, F), m)$ a citizen k whose ideal position is $x_k = t$ could enter and lose, making i (the proponent of $m - \varepsilon$) the unique winner. The expected payoff of k when he enters is $-c - t + m - \varepsilon$ and when he does not enter is $-\varepsilon$, so k would prefer to enter if $c < m - t$. Taking $t \in (s(\varepsilon, F), m - c)$ would make any citizen k enter, contradicting the fact that $\mathbf{s}^*$ is a Nash equilibrium. Thus it must be that $c \geq |m - s(\varepsilon, F)|$.

It follows from the discussion above, that $b_{\mathcal{C}} > 2(c - e_p(F))$ is a necessary condition for existence of two party Nash equilibrium in Γ_{CD}. For assume that $b_{\mathcal{C}} \leq 2(c - e_p(F))$ and suppose that $\mathbf{s}^*$ is a Nash equilibrium of Γ_{CD}. Then $b_{\mathcal{C}}/2 - c \leq -e_p(F)$ and so, by point (iii), $b_{\mathcal{C}}/2 - c \leq -\varepsilon$. Hence any of the players from $K(\mathbf{s}^*)$, being a citizen candidate, would prefer to stay out of the competition than to enter.

Next we show that if $b_{\mathcal{C}} > 2(c - e_p(F))$ is satisfied, then a strategy profile $\mathbf{s}^*$ with $K(\mathbf{s}^*) = \{i, j\}$ and $x_i \leq x_j$ and such, that conditions (i) – (v) are satisfied for it is a Nash equilibrium of Γ_{CD}. By point (ii) both players get the same mass of supporter, so i and j tie for the first place. Moreover, by point (i), they are both citizens, so we only need two consider them staying out of the competition instead of entering. This is prevented by point (iv). Also, no other player can enter and not lose with certainty, as guaranteed by points (ii), (iii) and (v). The possibility of entry by a player who loses with certainty but prefers to enter is ruled out by point (iv). Thus $\mathbf{s}^*$ is a Nash equilibrium and $b_{\mathcal{C}} > 2(c - e_p(F))$ is a sufficient condition for existence of such equilibrium. □

Remark 5. *The only difference between the sets of conditions for the existence of two candidate Nash equilibria in Γ_{CD} and Γ_C is in point (v). In case of Γ_C if $\varepsilon = e_p(F)$, then it is enough that $b_{\mathcal{C}} \leq 3c - \varepsilon$ (c. f. [Osborne and Slivinski, 1996, Proposition 2]). In case of Γ_{CD}, where there is the possibility of entry by a Downsian player, the condition has to be stronger to prevent such an entry.*

The following remark follows directly from the above proposition.

Remark 6 (Sufficient conditions for non-existence of 2-party equilibria)**.** *Notice that if $b_{\mathcal{C}} \leq 2(c - e_p(F))$ and $2c < b_{\mathcal{D}} \leq 3c$, then a Nash equilibrium of Γ_{CD} does not exist. Notice also that if $2c < b_{\mathcal{C}} \leq b_{\mathcal{D}} \leq 3c$, then a Nash equilibrium of Γ_{CD} with Downsian candidates does not exist.*

We do not give any direct propositions on Nash equilibria of Γ_{CD} with more than two candidates standing for the election. Instead, we take the following strategy is addressing multi-party equilibria: since existence of multi-party equilibria is well established for the game Γ_C in Osborne and Slivinski [1996], in the next section we study the relation between Nash equilibria in Γ_{CD} and Nash equilibria in Γ_C. The results we establish allow us to give some examples of Nash equilibria with larger sets of candidates, including Downsian players.

4 Relation between equilibrium configurations of Γ_{CD} and Γ_C

Since Γ_{CD} is an extension of Γ_C, it is natural to ask how equilibria in one game relate to those in another. In this section we show that it is possible to "transfer" Nash equilibria of Γ_{CD} to Nash equilibria of Γ_C and vice versa, under certain conditions. The transfer is made by replacing Downsian candidates by citizen candidates, or the other way around and then modifying, if necessary, costs and benefits characterising the models, so that the obtained strategy profile is a Nash equilibrium of the new model. We also have an interesting impossibility result, showing that there are configurations for which such transfers are impossible.

4.1 From Γ_{CD} to Γ_C

We start by studying possible transfers from Γ_{CD} to Γ_C. Let us define what we mean by saying that a Nash equilibrium $\mathbf{s}^*$ can be transferred to Γ_C. For this definition, we will need the following notion of correspondence:

Definition 1. *Given a strategy profile* $\mathbf{s}$ *of the game* $\Gamma_{\mathcal{M}}$*, we say that a strategy profile* $\mathbf{s}'$ *of the game* $\Gamma_{\mathcal{M}'}$ corresponds to $\mathbf{s}$ *if* $X(\mathbf{s}) = X(\mathbf{s}')$ *and each policy in* $x \in X(\mathbf{s})$ *is proposed by the same number of candidates under both* $\mathbf{s}$ *and* $\mathbf{s}'$*.*

Definition 2 (Nash equilibrium transferable from Γ_{CD} to Γ_C)**.** *Let* $\mathbf{s}^*$ *be a Nash equilibrium in* Γ_{CD} *with non atomic distribution* F*, benefits* $b_{\mathcal{C}}$*,* $b_{\mathcal{D}}$ *and costs* c*. We say that* $\mathbf{s}^*$ *is* transferable *to* Γ_C *iff there exists benefits* $b'(b_{\mathcal{C}}, b_{\mathcal{D}}, c, \mathbf{s}^*)$ *and costs* $c'(b_{\mathcal{C}}, b_{\mathcal{D}}, c, \mathbf{s}^*)$*, such that every strategy profile* $\mathbf{s}'$ *that corresponds to* $\mathbf{s}^*$ *is a Nash equilibrium in* Γ_C *with benefits* $b'(b_{\mathcal{C}}, b_{\mathcal{D}}, c, \mathbf{s}^*)$ *and cost* $c'(b_{\mathcal{C}}, b_{\mathcal{D}}, c, \mathbf{s}^*)$*.*

The following example gives an impossibility results as it proves that there are configurations which are not transferable.

Example 1. *Let* $\mathbf{s}$ *be a strategy profile in* Γ_{CD}*, such that* $K(\mathbf{s}) = \{i_1, i_2, j_1, j_2, l, r\}$*, where* $\{i_1, i_2, j_1, j_2\} \subseteq \mathcal{D}$ *and* $\{l, r\} \subseteq \mathcal{C}$*. Moreover, let* $x_{i_1} = x_{i_2} = a_{11/60}$*,* $x_{j_1} = x_{j_2} = a_{49/60}$*,* $x_l = 2a_{22/60} - a_{11/60}$ *and* $x_r = 2a_{38/60} - a_{49/60}$*. Let* F *such that* $(x_l + x_r)/2 = m$*,* $b_{ir} = (a_{11/60} + x_r)/2 = a_{101/240}$*,* $b_{lj} = (x_l + a_{49/60})/2 = a_{139/240}$*,* $b_{lm} = (x_l + m)/2 = a_{49/120}$*,* $b_{mr} = (x_r + m)/2 = a_{71/120}$*,* $x_l - a_{11/60} = a_{49/60} - x_r$*,* $x_r - x_l > 2(x_l - a_{11/60})$*. Such a distribution exists*

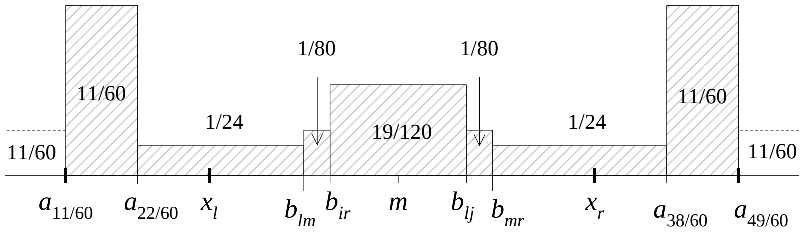

Figure 1: Configuration in Γ_{CD} that cannot be transferred to Γ_C.

and an example is presented in Figure 1, together with positions of players. Also, let costs be such that $0 < c < x_r + a_{11/60} - 3x_l$ *and benefits be such, that* $4c < b_{\mathcal{D}} \leq 5c$*,* $c \leq b_{\mathcal{C}} \leq b_{\mathcal{D}}$ *and* $b_{\mathcal{C}} \leq 5c - (a_{49/60} - a_{11/60})/4$*.*

We now show that $\mathbf{s}$ *is a Nash equilibrium of* Γ_{CD} *but is not transferable to* Γ_C*.*

It is easy to check that under the distribution F *and strategy profile* $\mathbf{s}$ *the set of candidates that do not lose with certainty* $W(\mathbf{s}) = \{i_1, i_2, j_1, j_2\}$*. Moreover candidates* l *and* r *lose with certainty, and each of them stands for the election to block another one from winning outright, that is* l *is the only winner under strategy profile* $(\mathbf{s}_{-r}, N)$ *and* r *is the only winner under strategy profile* $(\mathbf{s}_{-l}, N)$*.*

Strategy profile $\mathbf{s}$ *is a Nash equilibrium of* Γ_{CD}*. It is easy to check that none of the candidates is better off by withdrawing. Moreover it is not possible for any player to enter and win outright. There is a possibility of entering at* m *and joining the set of players that do not lose with certainty, but both citizen and Downsian players prefer to stay out than to join the set of winners.*

Consider a strategy profile $\mathbf{s}'$ *of* Γ_C *that corresponds to* $\mathbf{s}$*. There do not exist* $c' > 0$ *and* $b' \geq 0$ *such, that strategy profile* $\mathbf{s}'$ *is a Nash equilibrium of* Γ_C *with costs* c' *and benefits* b'*. This is because if entering the competition is to be preferred to staying out by citizen candidates taking external positions, then it must hold that* $b' > 4c' + 2(a_{49/60} - a_{11/60})$*. Moreover, if staying out is to be preferred to entering the competition by a citizen with ideal position* *m it must be that* $b' \leq 5c' - (a_{49/60} - a_{11/60})/4$*. This implies that it must be that* $c' > 7(a_{49/60} - a_{11/60})/4$*. But then we know that* $c' > x_r + a_{11/60} - 3x_l$ *and so candidates* l *and* r *prefer to stay out of the competition. Hence it is not possible to have* $c' > 0$ *and* $b' \geq 0$ *such, that* $\mathbf{s}'$ *is a Nash equilibrium.*

As Example 1 suggests, transferring a Nash equilibrium from Γ_{CD} to Γ_C may require changing costs and benefits. Before giving a proposition providing a complete characterization of Nash equilibria transferable from Γ_{CD} to Γ_C we will need the following two notions.

Let $\mathbf{s}^*$ be a Nash equilibrium of Γ_{CD}. Given a candidate $i \in K(\mathbf{s}^*)$, define

$$D_i(\mathbf{s}^*) = \hat{u}_i(\mathbf{s}^*) - \hat{u}_i((\mathbf{s}^*_{-i}, N)).$$

Moreover, let $\bar{c}(\mathbf{s}^*)$ denote the upper limit for the maximal costs level for which there exist $b'_{\mathcal{C}}$ and $b'_{\mathcal{D}}$, such that $\mathbf{s}^*$ is a Nash equilibrium of Γ_{CD} with costs $c' < \bar{c}(\mathbf{s}^*)$ and benefits $b'_{\mathcal{C}}$ and $b'_{\mathcal{D}}$. Notice that $\bar{c}(\mathbf{s}^*)$ exists if and only if there are candidates losing with certainty under $\mathbf{s}^*$. If there are no such candidates under $\mathbf{s}^*$, then for any costs there are benefits under which $\mathbf{s}^*$ is a Nash equilibrium. On the other hand, if any candidates i losing with certainty it must be that $c < D_i(\mathbf{s}^*)$. Hence it must be true that

$$\bar{c}(\mathbf{s}^*) = \min_{i \in L(\mathbf{s}^*)} D_i(\mathbf{s}^*).$$

Also, let $Z(\mathbf{s}^*)$ denote the set of all citizen players that are not candidates under $\mathbf{s}^*$ and such that $W((\mathbf{s}^*_{-i}, E)) = W(\mathbf{s}^*) \cup \{i\}$ for all $i \in Z(\mathbf{s}^*)$. That is, $Z(\mathbf{s}^*)$ is a set of citizens that could enter and join the set of winners under $\mathbf{s}^*$. Observe that either each citizens from $Z(\mathbf{s}^*)$ have its ideal positions placed between some two candidates from $\mathbf{s}^*$ that lose with certainty, there are exactly two policies proposed under $\mathbf{s}^*$, all candidates tie for the first place and $|Z(\mathbf{s}^*)| = 1$. Let

$$\bar{u}(\mathbf{s}^*) = \max_{i \in Z(\mathbf{s}^*)} \hat{u}_i(\mathbf{s}^*).$$

Now we are ready to give a complete characterization of Nash equilibria transferable from Γ_{CD} to Γ_C.

Proposition 3. *Let* $\mathbf{s}^*$ *be a Nash equilibrium in* Γ_{CD} *with non atomic distribution* F*, benefits* $b_{\mathcal{C}}$*,* $b_{\mathcal{D}}$ *and costs* c*. Then* $\mathbf{s}^*$ *is transferable to* Γ_C *iff*

(1). there is no pair of neighbouring candidates l *and* r*, such that* $\{l, r\} \subseteq L(\mathbf{s}^*)$ *for which there exists* $x_i \in (x_l, x_r)$*, such that* $W((\mathbf{s}^*_{-i}, E)) = W(\mathbf{s}^*) \cup \{i\}$*, or*

(2). there exists such a pair and for any Downsian candidate $j \in K(\mathbf{s}^*) \cap \mathcal{D}$ *it holds that* $|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) + \bar{u}(s^*) < \bar{c}(\mathbf{s}^*)$*.*

Proof. Suppose that $\mathbf{s}^*$ is a Nash equilibrium in Γ_{CD}.

For the left to right implication, suppose that $\mathbf{s}^*$ is transferable to CC and let $\mathbf{s}'$ be a strategy profile corresponding to $\mathbf{s}^*$. Moreover, let c' and b' be such that $\mathbf{s}'$ is a Nash equilibrium of Γ_C with costs c' and benefits b'.

Suppose that both conditions (1) and (2) are not satisfied. Hence there are two neighbouring candidates l and r, such that $\{l, r\} \subseteq L(\mathbf{s}^*)$ for which exists $x_i \in (x_l, x_r)$, such that $W((\mathbf{s}^*_{-i}, E)) = W(\mathbf{s}^*) \cup \{i\}$ (and the same holds for $\mathbf{s}'$). This means that $Z(\mathbf{s}^*) \neq \varnothing$.

Also, there must exist a Downsian candidate $j \in K(\mathbf{s}^*) \cap \mathcal{D}$ such that

$$|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) + \bar{u}(s^*) \geq \bar{c}(\mathbf{s}^*).$$

Let j' be a citizen candidate under $\mathbf{s}'$ such, that $x_j = x_{j'}$. Then j' must be a candidate not losing with certainty under $\mathbf{s}'$ and if j' prefers to stay in the competition than to stay out, it must hold that

$$\frac{b'}{|W(\mathbf{s}^*)|} - c' - \hat{u}_{j'}(\mathbf{s}^*) > -\hat{u}_{j'}((\mathbf{s}^*_{-j'}, N)).$$

This implies that $b' > |W(\mathbf{s}^*)|(c' + D_{j'}(\mathbf{s}^*))$ and since it must be that $c' < \bar{c}(\mathbf{s}^*)$ (as otherwise one of the candidates losing with certainty would prefer to stay out of the competition), so it must be also $b' > (|W(\mathbf{s}^*)| + 1)c' - \bar{u}(\mathbf{s}^*)$. To see this observe that from $|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) + \bar{u}(\mathbf{s}^*) \geq \bar{c}(\mathbf{s}^*)$ and $c' < \bar{c}(\mathbf{s}^*)$ it follows that $|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) > c' - \bar{u}((\mathbf{s}^*))$. Substituting this in $b' > |W(\mathbf{s}^*)|(c' + D_{j'}(\mathbf{s}^*))$, we get the result.

On the other hand, if $\mathbf{s}'$ is a Nash equilibrium, then none of the players from $Z(\mathbf{s}^*)$ prefers to enter, so it must be that for all $j \in Z(\mathbf{s}^*)$ we have

$$\frac{b'}{|W(\mathbf{s}^*)|} - c' - \hat{u}_j((\mathbf{s}^*_{-j}, E)) \leq -\hat{u}_j(\mathbf{s}^*).$$

Notice that

$$\hat{u}_j((\mathbf{s}^*_{-j}, E)) = \frac{|W(\mathbf{s}^*)|}{|W(\mathbf{s}^*)| + 1}\hat{u}_j(\mathbf{s}^*),$$

and therefore, the first inequality is equivalent to $b' \leq (|W(\mathbf{s}^*)|+1)c' - \hat{u}_j(\mathbf{s}^*)$. This implies that $b' \leq (|W(\mathbf{s}^*)|+1)c' - \bar{u}(\mathbf{s}^*)$, which contradicts the inequality above. Thus one of conditions (1) or (2) must be satisfied if $\mathbf{s}^*$ is a Nash equilibrium of Γ_{CD} and is transferable to Γ_C.

For the right to left implication suppose that one of the conditions (1), (2) is satisfied and let $\mathbf{s}'$ be a strategy profile of Γ_C corresponding to $\mathbf{s}^*$. We will show that there are c' and b', depending on $\mathbf{s}^*$, c and $b_{\mathcal{C}}$ only, such, that $\mathbf{s}'$ is a Nash equilibrium of Γ_C with costs c' and benefits b'.

Consider a citizen candidate $i \in K(\mathbf{s}')$ such, that $i \notin K(\mathbf{s}^*) \cap \mathcal{C}$. Hence there is $i' \in K(\mathbf{s}^*) \cap \mathcal{D}$ such, that $x_i = x_{i'}$ and i is tying for the first place under $\mathbf{s}'$ (as Downsian players prefer to stay out of the competition than to lose with certainty). Thus the difference between the payoffs of i when he enters and when he does not enter with benefits b' and costs c is

$$\frac{b'}{|W(\mathbf{s}^*)|} - c - D_i(\mathbf{s}^*).$$

Hence taking any b', such that $b' \geq b_{\mathcal{C}}$ and for any Downsian candidate $k \in K(\mathbf{s}^*)$, $b' > |W(\mathbf{s}^*)|(c + D_k(\mathbf{s}^*))$ makes entering more profitable then not entering for any citizen candidate replacing a Downsian candidate. Hence with b' satisfying the conditions above it is not profitable for i to withdraw. Notice that b' depends on $b_{\mathcal{C}}$, c and $\mathbf{s}^*$ only.

Notice also that it is not profitable for any citizen candidate under $\mathbf{s}'$ who is a candidate under $\mathbf{s}^*$ to stay out of the competition under $\mathbf{s}'$. This is because $b \geq b_{\mathcal{C}}$ and so it would be profitable for him to stay out under $\mathbf{s}^*$ as well, which is impossible, as $\mathbf{s}^*$ is a Nash equilibrium.

Now consider the possibility of entry by a player who is not a candidate under strategy profile $\mathbf{s}'$ in CC. Notice first that it is not profitable for any citizen player to enter and lose with certainty under $\mathbf{s}'$, as then it would have been profitable for him to enter under $\mathbf{s}^*$ in Γ_{CD} as well, which would contradict the fact that $\mathbf{s}^*$ is a Nash equilibrium (such decision does not depend on benefits).

Suppose now that there are no Downsian candidates in $K(\mathbf{s}^*)$. In this case $b = b_{\mathcal{C}}$ and if it was profitable to enter for any citizen under strategy profile $\mathbf{s}'$, then it would be also profitable for this citizen to enter under strategy profile $\mathbf{s}^*$. Hence no entry is possible in this case, as $\mathbf{s}^*$ is a Nash equilibrium.

Suppose then, that there are Downsian candidates in $K(\mathbf{s}^*)$. Suppose that there is only one candidate standing for the election, that is $|K(\mathbf{s}^*)| = 1$. This time $b' \geq b_{\mathcal{D}}$ and, by the same arguments as those used above, no entry is possible under strategy profile $\mathbf{s}'$ in CC.

If $|K(\mathbf{s}^*)| > 1$ and there is a Downsian candidate in $K(\mathbf{s}^*)$ then, by point (ii) of Lemma 1 and Proposition 2, it must be that $|K(\mathbf{s}^*)| \geq 3$ (and consequently $|K(\mathbf{s}')| \geq 3$). Notice that it is not possible for any citizen to enter under $\mathbf{s}'$ and win outright, as it would be possible for a Downsian player to enter and win outright under $\mathbf{s}^*$, which would contradict the fact that $\mathbf{s}^*$ is a Nash equilibrium.

Note: In all the cases considered so far it was enough to increase benefits (when needed) leaving costs level unchanged to make $\mathbf{s}'$ a Nash equilibrium of Γ_C. The remaining cases studied below require a change in costs as well.

The only kind of entry that remains to be analysed is the one where some citizen i enters and joins the set of candidates who do not lose with certainty under $\mathbf{s}'$. Observe, that if $b_{\mathcal{D}} > (|W(\mathbf{s}^*)| + 1)c$, then this situation is not possible, as it would be possible for a Downsian player to enter with positive expected payoff under $\mathbf{s}^*$, which would contradict the assumption that $\mathbf{s}^*$ is a Nash equilibrium. Hence suppose that $|W(\mathbf{s}^*)|c < b_{\mathcal{D}} \leq (|W(\mathbf{s}^*)| + 1)c$ (Notice that $|W(\mathbf{s}^*)|c < b_{\mathcal{D}}$ follows from the fact that $\mathbf{s}^*$ is a Nash equilibrium with at least one Downsian candidate and $W(\mathbf{s}^*)$ winning candidates).

Situation where some citizen i enters and joins the set of candidates who do not lose with certainty is possible only if either (a) i's entry does not affect any of the candidates from $W(\mathbf{s}')$ or (b) i's entry affects all the candidates from $W(\mathbf{s}')$. In both cases $Z(\mathbf{s}^*) \neq \varnothing$.

Consider case (a) first. This means that there exists a pair of neighbouring candidates l and r such that $\{l, r\} \subseteq L(\mathbf{s}^*)$ for which there exists $x_i \in (x_l, x_r)$ such that $x_i \notin X(\mathbf{s}^*)$ and $W((\mathbf{s}^*_{-i}, E)) = W(\mathbf{s}^*) \cup \{i\}$. Moreover, by condition (2), it must be that for any Downsian candidate $j \in K(\mathbf{s}^*) \cap \mathcal{D}$ it holds that $|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) + \bar{u}(\mathbf{s}^*) < \bar{c}(\mathbf{s}^*)$.

Hence there exists $c \leq c' < \bar{c}(\mathbf{s}^*)$ such that for any Downsian candidate $j \in K(\mathbf{s}^*) \cap \mathcal{D}$ it holds that $|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) + \bar{u}(\mathbf{s}^*) < c'$. This c' guarantees that no candidate losing with certainty under $\mathbf{s}'$ will prefer to withdraw. Moreover, it guarantees that there exists $b' \geq b_{\mathcal{C}}$, such that for any Downsian candidate $k \in K(\mathbf{s}^*) \cap \mathcal{D}$, $b' > |W(\mathbf{s}^*)|(c' + D_k(\mathbf{s}^*)$ and $b' \leq (|W(\mathbf{s}^*)| + 1)c' + \bar{u}(\mathbf{s}^*)$.

Thus with costs c' and benefits b' no candidate tying for the first place prefers staying out of the competition to entering it. Moreover, no player from $Z(\mathbf{s}^*)$ prefers to enter the competition joining the set of candidates tying for the first place to staying out. Hence $\mathbf{s}'$ is a Nash equilibrium of Γ_C with costs c' and benefits b'. Notice that b' and c' depend on $b_{\mathcal{C}}$ and $\mathbf{s}^*$ only.

Secondly, consider case (b). Since $|K(\mathbf{s}')| \geq 3$, so this situation is possible only if $X(\mathbf{s}') = 2$, that is two policies are proposed under $\mathbf{s}'$, and x_i lies between the two proposed policies. By points (i) and (vi) of Lemma 1, there are either 4 or 3 candidates under $\mathbf{s}'$ (and $\mathbf{s}^*$), all of them tying for the first place.

By arguments similar to those used in Lemma 2, it can be shown that the situation with 4 candidates is impossible. If it was possible to enter between the two policies proposed by the four candidates and join the set of players that do not lose with certainty, then there would have to exist a policy between the two policies, such that a player that would enter proposing it would win outright. Hence such a situation is not possible if $\mathbf{s}^*$ is a Nash equilibrium.

The situation with 3 candidates is possible (c.f. Example 3). Suppose then, that there are 3 winning candidates $\{i, j, k\}$ under $\mathbf{s}^*$ in CCD. Without loss of generality, suppose that $x_i < x_j$ and $x_j = x_k$. By point (ii) of Lemma 1 it must be that $i \in \mathcal{C}$. Moreover, if $\{j, k\} \cap \mathcal{C} neq$, then $\mathbf{s}'$ is a Nash equilibrium of Γ_C with benefits $b_{\mathcal{C}}$ and costs c. Hence assume that $\{j, k\} \subseteq \mathcal{D}$. Take any c', such that $c' \geq c$ and $c' > 5|x_i - x_j|/3$. Take any b' such that $b' \geq b_{\mathcal{C}}$ and $3c' + |x_i - x_j| < b' \leq 4c' - 2|x_i - x_j|/3$. Notice that such b' exists, as $3c' + |x_i - x_j| < 4c' - 2|x_i - x_j|/3$ when $c' > 5|x_i - x_j|/3$.

Strategy profile $\mathbf{s}'$ is then a Nash equilibrium of Γ_C with benefits b' and costs c'. It can be easily checked that it is not profitable for any of the candidates to withdraw. It cannot be profitable for any player to enter and lose with certainty under $\mathbf{s}'$, as $c' \geq c$ and it cannot be profitable for any player to enter and lose under $\mathbf{s}^*$ with costs c. Moreover, as we argued above, it is not possible for any player to enter and win outright. However it is possible for a player to enter and join the set of winners. Entering in this case is not profitable as the condition $b' \leq 4c - 2|x_i - x_j|/3$ prevents it. Notice that b' and c' depend on $b_{\mathcal{C}}$ and $\mathbf{s}^*$ only.

Thus we have shown that there are $b'(b_{\mathcal{C}}, c, \mathbf{s}^*)$ and $c'(b_{\mathcal{C}}, c, \mathbf{s}^*)$, such that $\mathbf{s}'$ is a Nash equilibrium in Γ_C with benefits $b'(b_{\mathcal{C}}, c, \mathbf{s}^*)$ and costs $c'(b_{\mathcal{C}}, c, \mathbf{s}^*)$.
□

Remark 7. *Notice that any Nash equilibrium $\mathbf{s}^*$ of Γ_{CD} with $K(\mathbf{s}^*) \subseteq \mathcal{C}$ is a Nash equilibrium of Γ_C with benefits $b_{\mathcal{C}}$ and costs c.*

Remark 8. *Notice that any Nash equilibrium $\mathbf{s}^*$ of Γ_{CD} which is robust to increases in benefits, i. e. there is no increase in benefits that would make entry by any player profitable, can always be transferred to Γ_C without requiring any change n costs. Observe also that any Nash equilibrium of Γ_{CD} with $b_{\mathcal{D}} > (|W(\mathbf{s}^*)| + 1)c$ is robust to increases in benefits.*

We now prove our next result on transferability from Γ_{CD} to Γ_C.

Proposition 4. *Let $\mathbf{s}^*$ be a Nash equilibrium of the game Γ_{CD} with cost c and benefits $b_{\mathcal{C}}$ and $b_{\mathcal{D}}$, such that extremist candidates are citizens earning a* *non-negative expected payoff. Then* $\mathbf{s}^*$ *is a Nash equilibrium of the game* Γ_C *with cost* c *and benefit* $b_{\mathcal{C}}$.

To prove the above proposition we first prove the following lemma.

Lemma 3. *Let* $\mathbf{s}$ *be a strategy profile of* Γ_{CD} *with the set of candidates* $K(\mathbf{s})$ *being finite and non empty. Then* $\max_{i\in W(\mathbf{s})} \hat{u}_i(\mathbf{s}) = \max\{\hat{u}_{l(\mathbf{s})}(\mathbf{s}), \hat{u}_{r(\mathbf{s})}(\mathbf{s})\}$.

Proof. Let $W(\mathbf{s}) = \{i_0, \ldots, i_M\}$ and suppose that for all $0 \leq j, k \leq M$, $j \leq k$ implies $x_{i_j} \leq x_{i_k}$. Then

$$(M+1)\hat{u}_{i_n}(\mathbf{s}) = \sum_{j=1}^{n} j|x_{i_j} - x_{i_{j-1}}| + \sum_{j=n+1}^{M} (M-j+1)|x_{i_j} - x_{i_{j-1}}|.$$

Hence, for $1 \leq n \leq M$, the following holds

$$\begin{aligned}(M+1)(\hat{u}_{i_n}(\mathbf{s}) - \hat{u}_{i_{n-1}}(\mathbf{s})) &= n|x_{i_n} - x_{i_{n-1}}| - (M-n+1)|x_{i_n} - x_{i_{n-1}}| \\ &= (2n - M - 1)|x_{i_n} - x_{i_{n-1}}|.\end{aligned}$$

If $n \leq (M+1)/2$, then $(M+1))\hat{u}_{i_n}(\mathbf{s}) - \hat{u}_{i_{n-1}}(\mathbf{s})) \leq 0$ and so $\hat{u}_{i_n}(\mathbf{s}) \leq \hat{u}_{i_{n-1}}(\mathbf{s})$. On the other hand, if $n > (M+1)/2$, then $\hat{u}_{i_n}(\mathbf{s}) \geq \hat{u}_{i_{n-1}}(\mathbf{s})$.

Thus it follows, that $\max_{i\in W(\mathbf{s})} \hat{u}_i(\mathbf{s}) = \max\{\hat{u}_{l(\mathbf{s})}(\mathbf{s}), \hat{u}_{r(\mathbf{s})}(\mathbf{s})\}$. □

The proof of the above proposition is now straightforward.

Remark 9. *Notice that a Nash equilibrium* $\mathbf{s}^*$ *of* Γ_{CD} *is not transferable iff*

(1). there is a pair of neighbouring candidates l *and* r *losing with certainty and* $Z(\mathbf{s}^*) \neq \varnothing$, *that is there exists a player that could enter an join the set of candidates tying for the first place (which means that* $|W(\mathbf{s}^*)|c < b_{\mathcal{D}} \leq (|W(\mathbf{s}^*)| + 1)c$*), and*

(2). there exists a Downsian candidate $j \in K(\mathbf{s}^*) \cap \mathcal{D}$, *such that* $|W(\mathbf{s}^*)|D_j(\mathbf{s}^*) + \bar{u}(s^*) \geq \bar{c}(\mathbf{s}^*)$.

4.2 From Γ_C to Γ_{CD}

Having established properties of Nash equilibria of Γ_{CD} that can be transferred to Γ_C, we switch our attention to properties of Nash equilibria of Γ_C that can be transferred to Γ_{CD}. We start by defining what we mean by saying that a Nash equilibrium of Γ_C can be transferred to Γ_{CD}.

Definition 3 (Nash equilibrium transferable from Γ_C to Γ_{CD})**.** *Let* $\mathbf{s}^*$ *be a Nash equilibrium in* Γ_C *with non atomic distribution* F*, benefits* b *and costs* c*. We say that* $\mathbf{s}^*$ *is* transferable *to* Γ_C *if there exists a strategy profile* $\mathbf{s}'$ *in* Γ_{CD} *corresponding to* $\mathbf{s}^*$*,* $b_{\mathcal{C}}(b, c, \mathbf{s}^*)$*,* $b_{\mathcal{D}}(b, c, \mathbf{s}^*)$ *and* $c'(b, c, \mathbf{s}^*)$ *such that* $\mathbf{s}'$ *is a Nash equilibrium of* Γ_C *with benefits* $b_{\mathcal{C}}(b, c, \mathbf{s}^*)$*,* $b_{\mathcal{D}}(b, c, \mathbf{s}^*)$ *and cost* $c'(b, c, \mathbf{s}^*)$*.*

It turns out that almost every Nash equilibrium of Γ_C is transferable to Γ_{CD} with the same costs and benefits of citizen players. The transfer is done by not changing the strategies of citizen players and assigning strategy N to all Downsian players.

The only kind of Nash equilibria of Γ_C that are an exception to the above observation are those with a single candidate whose ideal position is not m (the fact that such equilibria are not transferable to Γ_{CD} follows immediately from Proposition 1). The following proposition states two important things therefore: (1) Any multi-party Nash equilibria in Γ_C can be transferred to Γ_{CD} and almost all of them without requiring any change in benefits and costs, and (b) One-party Nash equilibria of Γ_C can be transferred to Γ_{CD} if and only if in all such equilibria, the median candidate stands.

Proposition 5. *Let* $\mathbf{s}^*$ *be a Nash equilibrium in* Γ_C *with non atomic distribution* F*, benefits* b *and costs* c*. Then* $\mathbf{s}^*$ *is transferable to* Γ_{CD} *iff the following is true in* Γ_C*:* $|K(\mathbf{s}^*)| = 1$ *implies that* $X(\mathbf{s}^*) = \{m\}$*.*

If $\mathbf{s}^*$ *is transferable, then there exists* $b_{\mathcal{D}}$*, such that* $b_{\mathcal{D}} > c$*,* $b_{\mathcal{D}} \geq b$ *and* $\mathbf{s}' = (\mathbf{s}^*, (N)_{i\in\mathcal{D}})$ *is a Nash equilibrium of* Γ_{CD} *with benefits* $b_{\mathcal{C}} = b$*,* $b_{\mathcal{D}}$ *and costs* c*. Moreover, if* $|W(\mathbf{s}^*)| \neq 2$*, then there exists* $b_{\mathcal{D}}$ *which additionally* *satisfies* $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$ *such that* $\mathbf{s}'$ *is a Nash equilibrium of* Γ_{CD} *with benefits* $b_{\mathcal{C}} = b$, $b_{\mathcal{D}}$ *and costs* c.

Proof. Let $\mathbf{s}^*$ be a Nash equilibrium of Γ_C, like stated in the proposition. By Proposition 1, if $|K(\mathbf{s}^*)| = 1$ and $X(\mathbf{s}^*) \neq \{m\}$, then it is easy to see that such equilibrium is not transferable to Γ_{CD}. Hence $|K(\mathbf{s}^*)| = 1$ implying $X(\mathbf{s}^*) = \{m\}$ is a necessary condition for such transferability to be possible.

Suppose then, that $|K(\mathbf{s}^*)| = 1$ implies $X(\mathbf{s}^*) = \{m\}$.

Let $\mathbf{s}' = (\mathbf{s}^*_{-\mathcal{D}}, (N)_{i\in\mathcal{D}})$ be a strategy profile of Γ_{CD}.

Suppose first that $K(\mathbf{s}^*) = \{i\}$ and $x_i = m$. As was shown in [Osborne and Slivinski, 1996, Proposition 1], it must hold that $b \leq 2c$. Hence there exists $b_{\mathcal{D}}$, such that $b \leq b_{\mathcal{D}}$ and $c < b_{\mathcal{D}} \leq 2c$, and for any such $b_{\mathcal{D}}$, the strategy profile $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$, as no entry is possible under these conditions and candidate i prefers to stay for the competition than to stay out (c.f. Proposition 1).

Secondly, suppose that $W(\mathbf{s}^*) = \{l, r\}$ with $x_l < x_r$. We will show first that in Γ_C it is not possible for any player to enter and win outright under $\mathbf{s}^*$.[12] For suppose the opposite and let i be a player that could enter proposing x_i and win outright. It must be that $x_l < x_i < x_r$. Since $\mathbf{s}^*$ is a Nash equilibrium, it must be that i prefers to stay out than to enter, so $b \leq c - (x_r - x_l)$. On the other hand, it must be that both l and r prefer to enter than to stay out, so $b > 2c - (x_r - x_l)$, which contradicts the previous inequality. Thus it is not possible that such player i exists.

Suppose now that there exists a player i that could enter proposing x_i and join the set of candidates that tie for the first place, making one of them losing with certainty (such situation is possible when i enters between a candidate that loses with certainty and a candidate that ties for the first place).

Since $\mathbf{s}^*$ is a Nash equilibrium of Γ_C, so it must be that i prefers to stay

[12] To be sure, note that this does not necessarily mean that such a player would then enter.

out then to enter, hence

$$\frac{b}{2} - c - \frac{x_r - x_l - d}{2} \leq -\frac{x_r - x_l}{2},$$

where d is the distance between x_i and the policy proposed by a candidate who becomes a certain loser after i's entry. Thus the first inequality is equivalent to $b \leq 2c - d$. Hence there exists $b_{\mathcal{D}}$, such that $b \leq b_{\mathcal{D}}$ and $c < b_{\mathcal{D}} \leq 2c$, and for any such $b_{\mathcal{D}}$, the strategy profile $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$, as no entry is possible under these conditions and player i prefers to enter the competition than to stay out.

Suppose now that there exists a player i that could enter proposing x_i and join the set of candidates that tie for the first place. Since $\mathbf{s}^*$ is a Nash equilibrium of Γ_C, it must be that i prefers to stay out than to enter, so $b \leq 3c - (x_r - x_l)/2$. Hence there exists $b_{\mathcal{D}}$, such that $b \leq b_{\mathcal{D}}$ and $c < b_{\mathcal{D}} \leq 3c$, and for any such $b_{\mathcal{D}}$, the strategy profile $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$, as no entry is possible under these conditions and player i prefers to enter the competition than to stay out.

If any player entering under $\mathbf{s}^*$ loses with certainty, then for any $b_{\mathcal{D}}$, such that $b_{\mathcal{D}} > c$ and $b_{\mathcal{D}} \geq b$, the strategy profile $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$, as no entry is possible under these conditions and each of candidates prefers to stay for the competition than to stay out.

For the last case, suppose that $|W(\mathbf{s}^*)| \geq 3$. We will show first that it is not possible for any player to enter and win outright under $\mathbf{s}^*$. For suppose the opposite and let i be a player that could enter and win outright. Since $\mathbf{s}^*$ is a Nash equilibrium of Γ_C, it must be that i prefers to stay out then to enter, so $b \leq c - \hat{u}_i(\mathbf{s}^*)$. On the other hand, as it was shown in [Osborne and Slivinski, 1996, Proposition 4], it must be that $b \geq |W(\mathbf{s}^*)|c$, which contradicts the previous inequality. Thus it is not possible that such a player i exists.

Secondly, we will show that it is not possible for any player i to enter proposing x_i and join the set of candidates that tie for the first place, making one of them lose with certainty (such situation could be possible when i enters between a candidate that loses with certainty and a candidate that ties for the first place).

Since $\mathbf{s}^*$ is a Nash equilibrium of Γ_C, so it must be that i prefers to stay out then to enter, hence

$$\frac{b}{|W(\mathbf{s}^*)|} - c - \hat{u}_i((\mathbf{s}^*_{-i}, E)) \leq -\hat{u}_i(\mathbf{s}^*).$$

Notice that

$$\hat{u}_i((\mathbf{s}^*_{-i}, E)) = \hat{u}_i(\mathbf{s}^*) - \frac{d}{|W(\mathbf{s}^*)|},$$

where d is a distance between x_i and the policy proposed by a candidate who becomes a certain loser after i's entry. Thus the first inequality is equivalent to $b \leq |W(\mathbf{s}^*)|c - d/|W(\mathbf{s}^*)|$. But then $b < |W(\mathbf{s}^*)|c$, which contradicts the fact that it must be $b \geq |W(\mathbf{s}^*)|c$. Hence such situation is not possible.

Suppose now that there exists a player i that could enter proposing x_i and join the set of candidates that tie for the first place. Since $\mathbf{s}^*$ is a Nash equilibrium of Γ_C, so it must be that i prefers to stay out then to enter, and so $b \leq (|W(\mathbf{s}^*)| + 1)c - \hat{u}_i(\mathbf{s}^*)$ (c.f. proof of Proposition 3).

Hence there exists $b_{\mathcal{D}}$, such that $b \leq b_{\mathcal{D}}$ and $c < b_{\mathcal{D}} \leq (|W(\mathbf{s}^*)| + 1)c$, and for any such $b_{\mathcal{D}}$ strategy profile $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$, as no entry is possible under these conditions and player i prefers to stay in than to stay out of competition.

If any player entering under $\mathbf{s}^*$ loses with certainty, then for any $b_{\mathcal{D}}$, such that $b_{\mathcal{D}} > c$ and $b_{\mathcal{D}} \geq b$, the strategy profile $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$, as no entry is possible under these conditions and each of candidates prefers to stay for the competition than to stay out.

Notice that in all the cases studied above, apart from the one with $|W(\mathbf{s}^*)| = 2$, it was possible to have $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$ for which $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with costs c and benefits $b_{\mathcal{C}} = b$ and $b_{\mathcal{D}}$. □

We are particularly interested in Nash equilibria of Γ_{CD} with Downsian candidates. Hence we ask the following question: *under what conditions is* *it possible to transfer Nash equilibria of Γ_C to Γ_{CD} when some of the citizen candidates are replaced by Downsian candidates.* Obviously, the necessary conditions for such replacements follow from properties of Nash equilibria of Γ_{CD} given in Lemma 1 and Lemma 2. Additional constraints follow from properties of the distribution F: for each Downsian player that replaces a citizen candidate it cannot be profitable to move to a new position.

To prove our result, we nee the following lemma that gives necessary and sufficient conditions for impossibility of profitable repositioning by a Downsian player replacing a citizen candidate within the interval defined by its left and right neighbours. The proof of the lemma is moved to the Appendix.

Lemma 4. *Let* $\mathbf{s}$ *be a strategy profile of* Γ_{CD} *satisfying properties given in Lemma 1 and Lemma 2. Let* $\{l, i, r\} \subseteq K(\mathbf{s})$ *be candidates such that* $l = l(\mathbf{s}, i)$, $r = r(\mathbf{s}, i)$, $i \in W(\mathbf{s}) \cap \mathcal{D}$ *and* x_i *is proposed by exactly one candidate. Then there is no* $x \in (x_{l(\mathbf{s},i)}, x_{r(\mathbf{s},i)})$, *such that* i *strictly prefers* $(\mathbf{s}_{-i}, x)$ *to* $\mathbf{s}$ *iff the following conditions are satisfied:*

NMR(i,$\mathbf{s}$)

(1). if $l \in W(\mathbf{s})$, *then for all* $t \in (0, (x_r - x_i)/2)$,
$F(b_{il} + t) - F(b_{il}) > L(F(b_{lr} + t) - F(b_{lr}))/(L + 1)$,

(2). if $l \in L(\mathbf{s})$ *and the mass of support that* l *gets is* S_l, *then for all* $t \in (0, (x_r - x_i)/2)$, *if* $F(b_{ir}) - F(b_{il} + t) \geq S_l$, *then*
$F(b_{il} + t) - F(b_{il}) \geq F(b_{lr} + t) - F(b_{lr}))$ *and*
if $F(b_{ir}) - F(b_{il} + t) < S_l$, *then*
$F(b_{il} + t) - F(b_{il}) \geq L(F(b_{lr} + t) - F(b_{lr}) + S - S_l)/(L + 1)$,

NML(i,$\mathbf{s}$)

(1). if $r \in W(\mathbf{s})$, *then for all* $t \in (0, (x_i - x_l)/2)$,
$F(b_{ir}) - F(b_{ir} - t) > R(F(b_{lr}) - F(b_{lr} - t))/(R + 1)$,

(2). if $r \in L(\mathbf{s})$ *and the mass of support that* r *gets is* S_r, *then for all* $t \in (0, (x_r - x_i)/2)$, *if* $F(b_{ir} - t) - F(b_{il}) \geq S_r$, *then*

$F(b_{ir}) - F(b_{ir} - t) \geq F(b_{lr}) - F(b_{lr} - t)$ and
if $F(b_{ir} - t) - F(b_{il}) \geq S_r$, then
$F(b_{ir}) - F(b_{ir} - t) > R(F(b_{lr}) - F(b_{lr} - t) + S - S_r)/(R + 1)$.

where $b_{li} = (x_l + x_i)/2$, $b_{ir} = (x_i + x_r)/2$, $b_{lr} = (x_l + x_r)/2$, R is the number of right neighbours of i, L is the number of left neighbours of i and $S = F(b_{ir}) - F(b_{lr})$ is the mass of support that i gets.

We are unable to give any general properties of the function f that would be associated with conditions **NMR(i,s)** and **NML(i,s)** given in Lemma 4. However we can give the following property which is necessary if the conditions are satisfied, given that f is continuous. The proof can be found in the appendix.

Fact 1. *Let $\mathbf{s}$ and i be like in Lemma 4 and suppose that f is continuous on (x_l, x_r). If conditions **NMR(i,s)** and **NML(i,s)** are satisfied, then*

1. *if $l \in L(\mathbf{s})$, then $f(b_{li}) \geq f(b_{ir})$.*
2. *if $l \in W(\mathbf{s})$, then $f(b_{li}) \geq Lf(b_{ir})/(L + 1)$.*
3. *if $r \in L(\mathbf{s})$, then $f(b_{li}) \leq f(b_{ir})$.*
4. *if $r \in W(\mathbf{s})$, then $f(b_{li}) \leq (R + 1)f(b_{ir})/R$.*

Having established necessary and sufficient conditions for impossibility of profitable repositioning by Downsian players within the interval defined by the left and right neighbour, we are ready to give a proposition that gives necessary and sufficient conditions for strategy profiles in Γ_{CD} constructed on the basis of Nash equilibria of Γ_C to be Nash equilibria of Γ_{CD}.

Proposition 6. *Let $\mathbf{s}^*$ be a Nash equilibrium in Γ_C with a non atomic distribution F, benefits b and costs c and let $\mathbf{s}'$ be a strategy profile of Γ_{CD} corresponding to $\mathbf{s}^*$, such that there exists $i' \in \mathcal{D}$ that replaces some $i \in \mathcal{C}$ in $\mathbf{s}^*$. Then there exist $b_{\mathcal{C}}$, $b_{\mathcal{D}}$ and c' such that $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with benefits $b_{\mathcal{C}}$, $b_{\mathcal{D}}$ and costs c' iff $|K(\mathbf{s}^*)| = 1$ implies that $X(\mathbf{s}^*) = \{m\}$ and* *the following conditions are satisfied for each Downsian player i' replacing citizen candidate i not sharing his proposed policy with another candidate:*

(i). $i \notin L(\mathbf{s}^*)$,

(ii). if $l(\mathbf{s}^*, i) = -\infty$ *or* $r(\mathbf{s}^*, i) = +\infty$*, then* x_i *is proposed by two candidates,*

(iii). if $x_{l(\mathbf{s}^*,i)}$ *and* $x_{r(\mathbf{s}^*,i)}$ *are proposed by two candidates each, then* x_i *is proposed by two candidates,*

(iv). conditions ***NML(***i'***,***$\mathbf{s}'$***)*** *and* ***NMR(***i'***,***$\mathbf{s}'$***)*** *are satisfied.*

Proof. Let $\mathbf{s}^*$ be a Nash equilibrium in Γ_C and let $\mathbf{s}'$ be a strategy profile in a Γ_{CD} obtained from $\mathbf{s}^*$ as stated in the proposition.

The left to right implication follows immediately from Lemma 1, Lemma 2 and Lemma 4.

For the right to left implication, suppose that conditions (i) – (iv) are satisfied for any Downsian candidate replacing a citizen candidate proposing his policy alone. By condition (ii), this means that $|W(\mathbf{s}^*)| \neq 2$.

Let $b_{\mathcal{D}}$ be such that $(\mathbf{s}^*, (N)_{j\in\mathcal{D}})$ is a Nash equilibrium of Γ_{CD} with benefits $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$, $b_{\mathcal{C}} = b$ and costs c (by Proposition 5 such $b_{\mathcal{D}}$ exists). We will show that $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with exactly the same costs and benefits.

To show this, it is enough to show that none of the Downsian candidates under $\mathbf{s}'$ strictly prefers to stay out of the competition or propose a different policy (by the fact that $(\mathbf{s}^*, (N)_{j\in\mathcal{D}})$ is a Nash equilibrium of Γ_{CD}, it follows that no player staying out of the competition prefers to enter the competition and no citizen candidate strictly prefers to stay out of the competition).

Notice first that since $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$, so it is not profitable for any of the Downsian candidates to stay out of the competition.

Take any Downsian candidate $i \in K(\mathbf{s}^*)$ proposing policy x_i. If $|K(\mathbf{s}^*)| = 1$, then it is easy to see that changing the proposed policy does not change i's expected payoff. On the other hand, if $|K(\mathbf{s}^*)| > 1$, then two cases are possible: (a) x_i is proposed by some other candidate or (b) x_i is proposed by i only.

For case (a) observe that since $(\mathbf{s}^*, (N)_{j\in\mathcal{D}})$ is a Nash equilibrium of Γ_{CD}, so it is not possible for any player to enter and obtain a mass of support greater of equal to the mass of support obtained by candidates tying for the first place. Hence if i repositioned himself to any new policy $x \neq x_i$, he would obtain smaller mass of support that the other candidate proposing x_i and so i would lose with certainty. Thus it is not profitable for i to change his proposed policy.

For case (b) observe that, by condition (iv) and Lemma 4, it is not profitable for i to propose a different policy x within the interval $\left(x_{l(\mathbf{s}^*,i)}, x_{r(\mathbf{s}^*,i)}\right)$. Also, since $(\mathbf{s}^*, (N)_{i\in\mathcal{D}})$ is a Nash equilibrium of Γ_{CD}, so there cannot be x outside $\left(x_{l(\mathbf{s}^*,i)}, x_{r(\mathbf{s}^*,i)}\right)$, such that i would strictly prefer to propose x instead of x_i (as otherwise it would be possible for some other player to enter at x and win outright).

Hence $\mathbf{s}'$ is a Nash equilibrium of Γ_{CD} with benefits $b_{\mathcal{C}}$, $b_{\mathcal{D}}$ and costs c. □

As Proposition 6 suggests, there can exist many equilibrium configurations involving both citizen and Downsian candidates, provided that the distribution F satisfies additional properties and conditions from Lemma 1 and Lemma 2. We are unable give any general characterisation of such equilibria, however we give the following examples to show how constructing Nash equilibria of Γ_{CD} from Nash equilibria of Γ_C can be done.

The following example demonstrates a case where a Nash equilibrium with 3 candidates in Γ_C can be replicated in Γ_{CD} with one new Downsian player as a candidate.

Example 2. *Let $\mathbf{s}^*$ be a strategy profile of Γ_C with candidates $K(\mathbf{s}^*) = \{i, j, k\}$, such that $x_i < x_j < x_k$. As was shown in Osborne and Slivinski [1996], if $(x_i + x_j)/2 = a_{1/3}$, $(x_k + x_j)/2 = a_{2/3}$, then $\mathbf{s}^*$ is a Nash equilibrium of Γ_C, for any non atomic distribution F. Benefits b and costs c must satisfy the following conditions: $b > 3(c + x_i - x_j) + x_k - x_i$ and $b > 3(c + x_j - x_k) + x_k - x_i$. Notice that this implies that $b > 3c$.*

Consider a strategy profile $\mathbf{s}' = \big((\mathbf{s}^*{}_{-j}, N), x_j, (N)_{l\in\mathcal{D}\setminus\{j'\}}\big)$ *of* Γ_{CD}*, obtained from* $\mathbf{s}^*$ *by replacing citizen candidate* j *with a Downsian candidate* j'*, who enters the competition proposing policy* x_j*. It follows from Proposition 6,* $\mathbf{s}'$ *is a Nash equilibrium of* Γ_{CD} *with costs* c *and benefits* $b_{\mathcal{C}} = b_{\mathcal{D}} = b$ *iff* F *satisfies conditions* ***NMR(***j',$\mathbf{s}'$***)*** *and* ***NML(***j',$\mathbf{s}'$***)****.*

For example, for any F*, such that* f *is a single peaked function, with its maximum within the interval* $(a_{1/3}, a_{2/3})$ *and satisfying* $f(a_{1/3}) \geq f(a_{2/3})/2$ *and* $f(a_{2/3}) \geq f(a_{1/3})/2$*, conditions* ***NMR(***j',$\mathbf{s}'$***)*** *and* ***NML(***j',$\mathbf{s}'$***)*** *are satisfied. Hence, for any such distribution a Nash equilibrium with three candidates proposing different policies and tying for the first place, with Downsian candidate proposing the middle policy, exists if benefits and costs satisfy conditions given above. Moreover the policy proposed by a Downsian candidate lies within interval* $\big(a_{1/3}, a_{2/3}\big)$*, while policies proposed by citizen candidates lie outside this interval.*

Notice that if either $f(a_{1/3}) < f(a_{2/3})/2$ *or* $f(a_{2/3}) < f(a_{1/3})/2$*, then it is impossible to have a Nash equilibrium in* Γ_{CD} *with three candidates, each taking a different position and one of the a Downsian player.*

The next example shows that it is possible to have a Nash equilibrium in Γ_{CD} with four winning candidates and up to three Downsian players. It shows also that it is possible to have an equilibrium configuration where a Downsian player proposing his policy alone neighbours two players proposing the same policy.

Example 3. *Let* $\mathbf{s}$ *be a strategy profile in* Γ_{CD}*, such that* $K(\mathbf{s}) = \{i, j_1, j_2, k\}$*, where* $i \in \mathcal{C}$*,* $k \in \mathcal{D}$ *and players* j_1 *and* j_2 *could be either Downsians or citizens. Moreover, let* $x_{j_1} = x_{j_2} = a_{3/4}$*,* $x_k = 2a_{1/2} - a_{3/4}$ *and* $x_i = 2a_{1/4} + a_{3/4} - 2a_{1/2}$*. Let costs* $c > 0$ *be such that* $c \geq a_{1/2} - (a_{1/4} + 3a_{3/4})/2$ *and* $c \geq (3a_{1/4} + 5a_{3/4})/2 - 4a_{1/2}$ *and let benefits* $b_{\mathcal{C}} > 0$ *and* $b_{\mathcal{D}} \geq b_{\mathcal{C}}$ *be such that* $b_{\mathcal{C}} > 4c + 2(a_{3/4} - a_{1/4})$*. Suppose also that the distribution* F *is such that* $2a_{1/2} > a_{1/4} + a_{3/4}$*.*

It is easy to check that if conditions given above are satisfied, it holds that $x_i < x_k < x_{j_1} = x_{j_2}$ *and all candidates receive the same mass of support* $1/4$*.*

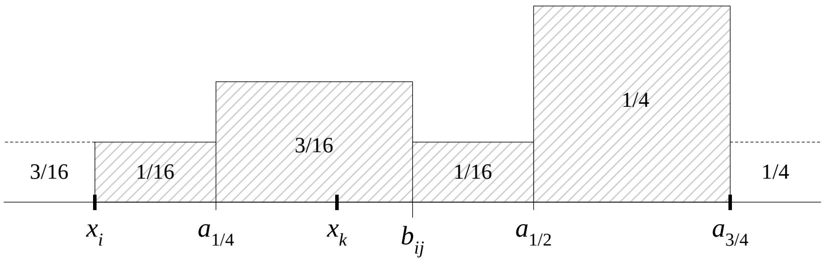

Figure 2: Four party equilibrium with a Downsian candidate having two neighbours proposing the same policy.

Moreover none of the candidates is better off by withdrawing independent of whether any of the candidates j_1 and j_2 is a citizen, as guaranteed by high enough level of benefits. It is also impossible for any player to enter and win outright or to join the set of winners. Entering and losing with certainty by a citizen player is also impossible as costs are high enough to prevent it. It is also impossible for any of the players j_1 and j_2 to reposition himself to a position which would allow him to be one of the players that do not lose with certainty (if any of these players is a Downsian player). Hence the only thing that needs to be guaranteed for $\mathbf{s}$ *to be a Nash equilibrium of* Γ_{CD} *is impossibility of profitable repositioning by a Downsian player* k*. Not every distribution satisfying conditions given above guarantees that, however there are distributions that do, as illustrated in Figure 2. In the figure,* $b_{ij} = (x_i + a_{3/4})/2 = a_{1/4} + a_{3/4} - a_{1/2}$.

Thus under the distribution F *presented in Figure 2, the strategy profile* $\mathbf{s}$ *is a Nash equilibrium in* Γ_{CD}.

4.3 Benefits of The Citizens

As we can see from the proof of Proposition 3, the constraints required on citizens' benefits for a configuration to remain a Nash equilibria of Γ_C are more strict than those for the same citizens in Γ_{CD}. Hence it is possible to have equilibrium configurations in Γ_{CD} which, if to be supported as equilibria in Γ_C would in turn require higher benefits for citizens in Γ_C. In particular, it is not true for Γ_{CD} that the condition $b_{\mathcal{C}} \geq |W(\mathbf{s}^*)|$ is necessary to have $\mathbf{s}^*$ as a Nash equilibrium, as it was in Γ_C (c. f. [Osborne and Slivinski, 1996, Proposition 4]). The following two examples illustrate this.

Example 4. *Let $\mathbf{s}^*$ be a strategy profile of Γ_{CD} with candidates $K(\mathbf{s}^*) = \{l_1, l_2, i, r_1, r_2\}$, such, that $x_{l_1} = x_{l_2} = a_{1/5}$, $x_{r_1} = x_{r_2} = a_{4/5}$, $x_i = 2a_{2/5} - a_{1/5}$, $\{l_1, l_2, r_1, r_2\} \subseteq \mathcal{D}$ and $i \in \mathcal{C}$. Let the distribution F be such that $2a_{2/5} - a_{1/5} = 2a_{3/5} - a_{4/5}$. Then $a_{1/5} < x_i < a_{4/5}$ and each candidate receives the same mass of support. Suppose also that $(a_{1/5} + a_{4/5})/2 < m$. Then after i withdraws, x_{r_1} and x_{r_2} would be the only candidates tying for the first place.*

If we take $b_{\mathcal{D}} > 5c$ and $b_{\mathcal{C}} > 5(c + (3a_{4/5} - a_{1/5} - 2a_{3/5}))$, then it is not profitable for any candidate to stay out of the competition. Moreover, if $c > 2(a_{2/5} - a_{1/5} - 2(a_{4/5} - a_{3/5}))$ and $c > 2(a_{4/5} - a_{3/5} - 2(a_{2/5} - a_{1/5}))$, then it is not profitable for any citizen to enter and lose with certainty. It is also not possible for any player to enter and win outright or join the set of candidates tying for the first place. Similarly, it is impossible for any Downsian candidate to reposition himself and win outright or be a candidate tying for the first place. Hence $\mathbf{s}^*$ is a Nash equilibrium of Γ_{CD}.

Notice that if $2a_{2/5} - a_{1/5} = 2a_{3/5} - a_{4/5} < (a_{1/5} + a_{4/5})/2$, so that x_i is closer to $a_{1/5}$ than to $a_{4/5}$, then constraints on citizens benefits allow for benefits below $5c$, as this condition is equivalent to $3a_{4/5} - a_{1/5} - 2a_{3/5} < 0$. Hence it is possible to have a Nash equilibrium in Γ_{CD} with $b_{\mathcal{C}} < |W(\mathbf{s}^*)|c$.

Example 5. *Let $\mathbf{s}^*$ be a strategy profile of Γ_{CD} with candidates $K(\mathbf{s}^*) = \{i, j, r_1, r_2\}$, such, that $x_{r_1} = x_{r_2} = a_{3/4}$, $x_j = 2m - a_{3/4}$, $x_i = 2a_{1/4} + a_{3/4} - 2m$, $\{r_1, r_2\} \subseteq \mathcal{D}$ and $\{i, j\} \subseteq \mathcal{C}$. Then $x_i < x_j < a_{3/4}$ and each candidate receives the same mass of support. Suppose also that $a_{1/4} + a_{3/4} - m < a_{1/3}$, so that after j would withdraw, candidate i would win outright.*

If we take $b_{\mathcal{D}} > 4c$, $b_{\mathcal{C}} > 4(c + 2(3m + a_{1/4} - 2a_{3/4}))$ and $b_{\mathcal{C}} > 4(c + 2(8m + 3a_{1/4} - 5a_{3/4}))$, then it is not profitable for any candidate to stay out of the competition (the first condition on $b_{\mathcal{C}}$ prevents withdrawal by i and the second one prevents withdrawal by j). Moreover, if $c > 4a_{3/4} - a_{1/4} - 5m$ and

$c > 8m + 3a_{1/4} - 5a_{3/4}$, then it is not profitable for any citizen to enter and lose with certainty. It is also not possible for any player to enter and win outright or join the set of candidates tying for the first place. Similarly, it is impossible for any Downsian candidate to reposition himself and win outright or be a candidate tying for the first place. Hence $\mathbf{s}^*$ is a Nash equilibrium of Γ_{CD}.

Notice that if $2a_{3/4} > 3m + a_{1/4}$, so that x_j is closer to $a_{3/4}$ than to x_i, then constraints on benefits of citizen players allow for benefits below $4c$, as this condition is equivalent to $3m + a_{1/4} - 2a_{3/4} < 0$ and, moreover, it implies that $8m + 3a_{1/4} - 5a_{3/4} < 0$.

Hence it is possible to have a Nash equilibrium in Γ_{CD} with $b_{\mathcal{C}} < |W(\mathbf{s}^*)|c$.

The following proposition gives necessary conditions relating benefits and costs of players and possible equilibrium configurations with at least three candidates tying for the first place.

Proposition 7. *Let $\mathbf{s}^*$ be a Nash equilibrium of Γ_{CD} with $|W(\mathbf{s}^*)| \geq 3$. Then it holds that $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$. Moreover if each of the extreme policies if proposed by a citizen candidate, then $b_{\mathcal{C}} > |W(\mathbf{s}^*)|c$.*

Proof. Let $\mathbf{s}^*$ be a Nash equilibrium of Γ_{CD} as stated in the proposition. Suppose that there is a Downsian candidate under $\mathbf{s}^*$. Then, obviously, it must hold that $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$. On the other hand, suppose that each of the extreme policies is proposed by a citizen candidate. Let a citizen candidate proposing the left extreme policy be denoted by l and a citizen candidate proposing the right extreme policy be denoted by r. Since it is not profitable for neither l nor r to stay out of the competition, so it must hold that:

$$\frac{b_{\mathcal{C}}}{|W(\mathbf{s}^*)|} - c - \hat{u}_l(\mathbf{s}^*) > -d_l$$

and

$$\frac{b_{\mathcal{C}}}{|W(\mathbf{s}^*)|} - c - \hat{u}_r(\mathbf{s}^*) > -d_r,$$

where d_l is the distance between l and his right neighbour (if x_l is proposed by exactly one candidate) or 0 (otherwise), and d_r is defined analogically.

It can be easily seen that $\hat{u}_l(\mathbf{s}^*) + \hat{u}_r(\mathbf{s}^*) = x_r - x_l$ (see proof of Lemma 3). Hence, adding sides of the two inequalities we get

$$\frac{2b_{\mathcal{C}}}{|W(\mathbf{s}^*)|} - 2c - (x_r - x_l) > -(d_l + d_r)$$

and consequently $b_{\mathcal{C}} > |W(\mathbf{s}^*)|(c + (x_r - x_l - (d_l + d_r))/2)$. Since $x_r - x_l > d_l + d_r$, so it follows that $b_{\mathcal{C}} > |W(\mathbf{s}^*)|c$.

Notice, in particular, that if there are no Downsian candidates under $\mathbf{s}^*$, then it must be that $b_{\mathcal{C}} > |W(\mathbf{s}^*)|c$ and since $b_{\mathcal{D}} > b_{\mathcal{C}}$, so it must also be that $b_{\mathcal{D}} > |W(\mathbf{s}^*)|c$. □

4.4 Who runs office: a discussion

As mentioned in Section 1, an important question that arises when a model of political competition is studied where ideology motivated politicians compete with those motivated by benefits from running the office only is which type of candidates is more likely to win the election. This question is directly posed in a model of incomplete information and two competing parties by Callander [2008]. We can ask a similar question in our model of complete information but with free-entry and the answer would rely solely on the number of candidates of different types that are tying for the first place in equilibrium configurations.

The possible number of Downsian candidates depends on two factors: (a) the configuration of candidates under the equilibrium (i. e. the policies proposed, the numbers of candidates proposing each policy and candidates tying for the first place) and (b) the distribution of voters' ideal policies, F. The way in which factor (a) affects this number follows immediately from conditions (i) – (iii) given in Proposition 6 and by the fact that a citizen candidate sharing his proposed policy can be always replaced by Downsian candidate. The way in which factor (b) affects this number follows from condition (iv).

We have already shown explicitly that if there are two candidates in equilibrium, then all of them must be citizens (that is ideology motivated politicians) (c. f. Proposition 2). This, in light of Callander [2008] is new – it shows that if identity of a politician is common knowledge and there is free entry, then in cases where there are exactly two competing parties (a case that is empirically well supported in view of Duverger's Law), none of them can be only office motivated.

If there are three candidates under an equilibrium configuration, then, as was shown in [Osborne and Slivinski, 1996, Proposition 3] three configurations are possible:

1. If two candidates are tying for the first place and one is losing with certainty, than all candidates must be citizens (that is ideology motivated);

2. If all three candidates are tying for the first place, but exactly two different policies are proposed, then it is possible to have at most one or two Downsian candidates sharing their proposed policy with other candidate. Notice that if there are exactly two Downsian candidates in this case, then the ideology motivated candidate is less likely to win the elections.

3. Lastly, if all three candidates are tying for the first place and each of the proposes a different policy, then there can be at most one Downsian candidate in this configuration, depending on the distribution (c. f. Example 2). Hence it is more likely to have an ideology motivated candidate winning the election under this configuration.

Such analysis of different cases could be continued for equilibrium configurations with more than three candidates (e. g. it is possible to have a four party equilibrium where only one candidate is ideology motivated, c. f. Example 3). However, one thing comes out clearly from our analysis: the more candidates in equilibrium, the more possibilities there are for ideology motivated candidates to be in minority, and hence less likely to win the elections.

5 Conclusions

The paper studies electoral competition amongst citizen-candidates and Downsian politicians with one dimensional ideology space and provides existence and characterization of Nash equilibria in pure strategies. In this, an interesting result is that in any 2-party equilibrium, all contesting players must necessarily be ideology driven. The paper also sheds some light on how the set of multi-party equilibria compare between a model with only ideology-driven players with that where there are Downsian players. Throughout the analysis it is assumed that Downsian and citizen identities of players are common knowledge. Relaxing this assumption, in line with Callander [2008] is our next research agenda. Also some of the results depend upon the assumption that Downsian politicians earn a weakly higher personal reward from holding office. We hope to relax this assumption in a future study as well.

Appendix

Lemma 1. The proof is very similar to the proof of Lemma 1 from Osborne [1993]. Point (iv) holds immediately because each Downsian player prefers to stay out of the competition, if he loses with certainty. For Point (v), assume that some policy $x \in X(\mathbf{s}^*)$ is proposed by more than one candidate. Take any such candidate. If he is a Downsian candidate, then by point (iv), point (v) is satisfied. If this player is a citizen candidate, then it is more profitable for him to stay out of the competition, if he loses with certainty, as his ideal point will still be proposed by some other player.

For point (vi), the only case that is not covered by points (iv) and (v) is the one where an extreme policy is proposed by uniquely by a single citizen candidate i. Without loss of generality suppose that x_i is the leftmost extreme policy. Notice that if i was losing with certainty, then he would be better of by staying out of the competition. This is because if he is losing with certainty, then $|K(\mathbf{s}^*)| \geq 2$ and there must be some other player

$j \in W(\mathbf{s}^*)$. Moreover, if i does not enter, then the expected payoff he gets can only increase, as the cost of entry is removed and negative payoff following from distances to winning candidates can only increase, because it might only make a candidate proposing $r(\mathbf{s}^*, i)$ to win outright.

For point (iii), suppose that $x_i \in X(\mathbf{s}^*)$ is a policy proposed by $n \geq 2$ candidates (among them player i) and let S be the constituency of x_i. By point (v), none of the candidates proposing x_i loses with certainty. Hence the mass of votes each of them gets is S/n. Assume that one of the semi-constituencies of x_i is greater than another. Without loss of generality assume this is the left constituency. Hence $F(x_i) - F\left((x_{l(\mathbf{s}^*,i)} + x_i)/2\right) > S/2$. By the fact that F is continuous and non decreasing, there exists $t \in \left(x_{l(\mathbf{s}^*,i)}, x_i\right)$, such that $F(t) - F\left((x_{l(\mathbf{s}^*,i)} + x_i)/2\right) > S/2$. Entry by a player proposing t gives him the support $> S/2 \geq S/n$, so that the entering player would win outright. But this contradicts the assumption that $\mathbf{s}^*$ is a Nash equilibrium. Analogical argument can be used to show that point (i) holds as well.

For point (ii), suppose, without loss of generality, that $x_i \in X(\mathbf{s}^*)$ is the extreme left policy proposed by some unique Downsian candidate $i \in K(\mathbf{s}^*) \cap \mathcal{D}$. Since we have $|K(\mathbf{s}^*)| \geq 2$, so $r(\mathbf{s}^*, i) \neq +\infty$ and, by point (vi), there is a candidate proposing a policy different to x_i, who does not lose with certainty. Also, by the assumption that F is non atomic, if i proposed some different policy $x_i' \in \left(x_i, x_{r(\mathbf{s}^*,i)}\right)$, then he would win outright. But this contradicts the assumption that $\mathbf{s}^*$ is a Nash equilibrium. Hence point (ii) must hold. □

Lemma 4. Let $\mathbf{s}$ be a strategy profile of Γ_{CD} with $\{l, i, r\} \subseteq K(\mathbf{s})$ like stated in the lemma.

For the left to right implication, suppose that there does not exist x such that i prefers to propose x instead of x_i. Suppose also that one of the conditions **NMR(i,s)** or **NML(i,s)** is not satisfied. Without loss of generality suppose that **NMR(i,s)** is not satisfied (arguments in case of **NML(i,s)** are analogical).

Suppose first that $l \in W(\mathbf{s})$. In this case, if condition **NMR(i,s)** is not satisfied, then there exists $t \in (0, (x_r - x_i)/2)$, such that $F(b_{il} + t) - F(b_{il}) \leq (F(b_{lr} + t) - F(b_{lr}))/R$. Suppose that i proposes $x = x_i + 2t$. Then the mass of support of all right neighbours (one or two) of i will decrease. On the other hand, the mass of support of i will increase by $F(b_{lr} + t) - F(b_{lr}) - (F(b_{il} + t) - F(b_{il}))$, and the mass of support of each of the left neighbours (one or two) of i will increase by $(F(b_{il} + t) - F(b_{il}))/L$. Hence if $F(b_{il} + t) - F(b_{il}) \leq L(F(b_{lr} + t) - F(b_{lr}))/(L + 1)$, then either i gets the largest support and wins outright (if the inequality is strict) or he ties for the first place with his left neighbour (otherwise). In both cases i strictly prefers to propose x instead of x_i, hence we get a contradiction with our assumptions and so the condition **NMR(*i*,s)** must be satisfied.

Secondly, suppose that $l \in L(\mathbf{s})$ (by point (v) of Lemma 1, x_l is proposed by l only). In this case, if condition **NMR(*i*,s)** is not satisfied, then there exists $t \in (0, (x_r - x_i)/2)$, such that either $F(b_{ir}) - F(b_{il} + t) \geq S_l$ and $F(b_{il} + t) - F(b_{il}) < F(b_{lr} + t) - F(b_{lr}))$, or $F(b_{ir}) - F(b_{il} + t) < S_l$ and $F(b_{il} + t) - F(b_{il}) \geq L(F(b_{lr} + t) - F(b_{lr}) + S - S_l)/(L + 1)$. Again, suppose that i proposes $x = x_i + 2t$.

Suppose, that $F(b_{ir}) - F(b_{il} + t) \geq S_l$ and $F(b_{il} + t) - F(b_{il}) < F(b_{lr} + t) - F(b_{lr}))$. Then the mass of support of all right neighbours (one or two) of i will decrease. On the other hand, the mass of support of i will increase to $S + F(b_{lr} + t) - F(b_{il} + t)$, and the mass of support of will increase to $S_l + F(b_{il} + t) - F(b_{il}) \leq S$. Hence i will gain the largest mass of support and will win outright.

Now suppose, that $F(b_{ir}) - F(b_{il} + t) < S_l$ and $F(b_{il} + t) - F(b_{il}) \geq L(F(b_{lr} + t) - F(b_{lr}) + S - S_l)/(L + 1)$. Again, the mass of support of all right neighbours (one or two) of i will decrease, the mass of support of i will increase to $S + F(b_{lr} + t) - F(b_{il} + t))$ and the mass of support of l will increase to $S_l + F(b_{il} + t) - F(b_{il})$. Hence if $F(b_{il} + t) - F(b_{il}) \leq L(F(b_{lr} + t) - F(b_{lr}) + S - S_l)/(L + 1)$, then either i gets the largest support and wins outright (if the inequality is strict) or he ties for the first place with his left neighbour (otherwise). In both cases i strictly prefers to propose x

instead of x_i.

In all the cases studied above we get a contradiction with our assumptions and so the condition **NMR(i,s)** must be satisfied. Showing that condition **NML(i,s)** must be satisfied as well can be done by similar arguments.

For the right to left implication, suppose that conditions **NMR(i,s)** and **NML(i,s)** are satisfied. By analysis similar to the one above it can be seen that condition **NMR(i,s)** denies profitable repositioning to $x \in (x_i, x_r)$ and condition **NMR(i,s)** denies profitable repositioning to $x \in (x_l, x_i)$. □

Fact 1. Suppose that $\mathbf{s}$, i and f are like stated in the lemma and that conditions **NMR(i,s)** and **NML(i,s)** are satisfied.

Assume first, that $l \in L(\mathbf{s})$. Since $S_l < S$, so there exists $0 < \varepsilon \leq (x_r - x_i)/2$, such that for all $t \in (0, \varepsilon)$, $S_l + F(b_{il} + t) - F(b_{il}) \leq S$, that is $F(b_{ir}) - F(b_{il} + t) \geq S_l$. By condition **NMR($i$,s)** it holds that

$$\int_{b_{li}}^{b_{li}+t} f(x)\ \mathrm{d}x \geq \int_{b_{ri}}^{b_{ri}+t} f(x)\ \mathrm{d}x,$$

for all $t \in (0, \varepsilon)$. Suppose that $f(b_{li}) < f(b_{ir})$. Since f is continuous, so there exists $\delta > 0$, such that $f(b_{li} + t) < f(b_{ir} + t)$, for all $t \in (0, \delta)$. But then we get a contradiction with condition **NMR(i,s)**. Hence it must be that $f(b_{li}) \geq f(b_{ir})$. Similarly, it can be shown that if $r \in L(\mathbf{s})$, then it must be that $f(b_{ir}) \geq f(b_{li})$.

Secondly, assume that $l \in W(\mathbf{s})$. By condition **NMR(i,s)** it holds that

$$\int_{b_{li}}^{b_{li}+t} f(x)\ \mathrm{d}x \geq \frac{L}{L+1} \int_{b_{ri}}^{b_{ri}+t} f(x)\ \mathrm{d}x,$$

for all $t \in (0, (x_r - x_i)/2)$. Using similar arguments as above, we can show that it must be then, that $f(b_{li}) \geq Lf(b_{ir})/(L+1)$. Similarly, it can be shown that if $r \in W(\mathbf{s})$, then it must be that $f(b_{ir}) \geq Rf(b_{li})/(R+1)$. □

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