

Stochastic Resonance in Electrical Circuits—I: Conventional Stochastic Resonance

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Abstract—Stochastic resonance (SR), a phenomenon in which a periodic signal in a nonlinear system can be amplified by added noise, is introduced and discussed. Techniques for investigating SR using electronic circuits are described in practical terms. The physical nature of SR, and the explanation of weak-noise SR as a linear response phenomenon, are considered. Conventional SR, for systems characterized by static bistable potentials, is described together with examples of the data obtainable from the circuit models used to test the theory.

Index Terms—Fluctuations, noise, nonlinear.

I. INTRODUCTION

THE phenomenon of stochastic resonance (SR) has been in the news recently, partly on account of its wide occurrence in many areas of science. In this paper, we present a succinct introductory review of SR, discussing its physical nature and the insights that can be obtained by treating it as a linear response phenomenon. We describe how electronic experiments have been making major contributions to the understanding of SR in systems characterized by static bistable potentials, which we will describe as *conventional SR*. In a following companion paper [1], we extend the treatment to other forms of SR that arise in seemingly very different kinds of systems.

A. What Is SR?

SR is commonly said to occur when a weak periodic signal in a nonlinear system is enhanced by an increase of the ambient noise intensity; a stronger definition requires that the signal-to-noise *ratio* (SNR) should also increase. The usual observation is that the signal amplitude increases with increasing noise intensity, passes through a maximum, and then decreases again. Thus, the general behavior is somewhat similar to a conventional resonance curve, but plotting the response as a

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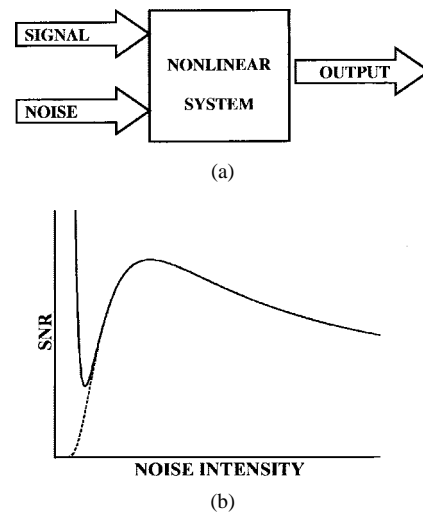


Fig. 1. Sketch showing basis of the SR phenomenon. A signal passing through a nonlinear system (a) and its SNR at the output (b) can sometimes be enhanced by the addition of noise at the input. The full and dashed curves show the behavior expected for continuous systems, and systems with discrete levels, respectively.

function of noise intensity instead of frequency. The SNR typically displays very similar behavior, except that it falls very rapidly before the onset of the counter-intuitive increase (full curve in Fig. 1). In the case of systems with thresholds, or 2-state systems (or systems with just a few discrete states, or where the signal is “filtered” through such a system prior to analysis), this initial decrease in the SNR is not seen (dashed line in Fig. 1). In the strongly nonlinear (large signal/weak noise) regime, the variation of the response with noise intensity can be more complicated.

The SR phenomenon appears to be widespread. After being introduced as a possible explanation of the earth’s ice-age cycle [2], [3], SR has subsequently been observed or invoked in contexts that include lasers [4]–[7], passive optical systems [8]–[11], tunnel diodes [12], a Brownian particle in an optical trap [13], a magnetoelastic ribbon [14], crayfish [15] and rat [16] mechanoreceptors, a bistable SQUID (superconducting quantum interference device) [17], arrays of SR elements [18], ion channels [19], magnetic systems [20], the El Nino phenomenon [21], social ills [22], a quantum 2-level system [23], an array of coupled bistable systems [24], [25], a system driven by quasi-monochromatic (harmonic) noise [26], excitable neurons [27], chemical systems [28], [29], and various types of bistable electronic system [30]–[35] modeling e.g., coexisting periodic attractors [36], [37], subcritical bifurcations [38], systems with

thresholds [39], and transient dynamics [40]. There have been two topical conferences [41], [42], several general scientific articles [43]–[46] and topical reviews [47]–[51], one of which [50] places SR in its historical context in physics.

In bistable systems, the underlying mechanism of SR is easily appreciated, and in fact has been known since the work of Debye [52] on reorienting polar molecules. In a static double-well potential with equally deep wells, the effect of an additive low-frequency periodic force is to tilt the wells first in one direction and then the other, so that one of the wells become deeper than the other, in turn. The effect of additive noise is, on average, to induce fluctuational transitions to the deeper well. For very weak noise intensity, there will be, on average, no transitions within a half cycle of the periodic force; for very strong noise intensity, the directions of transitions will be random, virtually unaffected by the periodic force; and for an optimum noise intensity, the probability of such transitions occurring coherently twice (once in each direction) per full cycle of the periodic force will be maximized. Thus, noise can effectively amplify small coordinate variations within one of the potential wells, caused directly by the rocking effect of the periodic force, to an amplitude corresponding to the coordinate separation of the potential minima—which can be made large.

B. Diverse Forms of Stochastic Resonance

SR was originally discussed [2], [3] for a bistable system and, for the next several years, it was widely assumed that bistability was an essential prerequisite for the phenomenon to occur. The perception of SR as a linear response phenomenon (see Section III-A below) led naturally, however, to the realization that SR can also occur without bistability [53] and to observation of the phenomenon in an underdamped, monostable, nonlinear oscillator [54]. In fact, it is well known that the response of a monostable system to signals in certain frequency ranges can be strongly increased by noise, e.g., just by raising the temperature. Examples range from currents in electron tubes to optical absorption near absorption edges in semiconductors. For underdamped oscillators, a temperature-induced shift and broadening of the absorption peaks, i.e., “tuning” by external driving due to the oscillator nonlinearity, was first discussed by Ivanov *et al.* [55]; complete classical and quantum theories of these effects were given by Dykman and Krivoglaz [56].

Some nonconventional forms of SR, including monostable SR, were reviewed in [57]. Since then, forms of SR without bistability have been identified in a system with a cyclic variable [58], a class of systems where the signal is applied as a multiplicative force [59], and thresholdless systems [60]. SR in a monostable SQUID model has recently been shown [61] to meet *a fortiori* even the stronger of the above definitions: the observed noise-induced SNR enhancements were comparable both in magnitude and form with those of conventional SR. We return to nonconventional SR in the companion paper [1], where we consider specific examples of such phenomena in detail.

In the next section, we describe the use of circuit models for the study of SR. In Section III, we outline the theory of

conventional SR. In Section IV, we discuss how the theory has been tested by circuit modeling, and in Section V, we draw conclusions.

II. CIRCUIT MODELLING OF SR

Analytic theories of stochastic phenomena often involve approximation, so it is important to find independent means of testing and validating them. One approach that has been found to be of great value in practice is based on analog electronic experiments [62]. The aim here is to build an electronic model of the system in which the occurrence of SR has been inferred theoretically, and then to study its properties under appropriate conditions, usually while being driven by random fluctuations (noise) and a periodic force. Quantitative measurements are made for comparison with the predictions of the (usually) approximate analytic theory. In this section, we briefly review the basic principles of circuit design, discuss the noise-generators used for producing the necessary driving noise, and outline the steps to be taken in analysing the signal(s) coming from the circuit model.

A. Circuit Design

The workhorse of non-VLSI analog circuit design still remains the operational amplifier. For present purposes, it enables most of the arithmetic operations needed to model the equations of interest to be effected in a very convenient and economical fashion [63]. Although addition, subtraction, multiplication/division by a constant factor, integration, and differentiation are easily implemented with operational amplifiers, it is usual to employ specialized integrated circuits (IC's) or devices for certain other operations. In particular:

- **Multiplication** of two voltages is most conveniently performed by use of an analog multiplier IC, e.g., the Analog Devices AD534 or the Burr-Brown MPY-100. The operation of these differential input IC's in multiplier mode can be as sketched in Fig. 2(a), for which

$$V_{\text{out}} = \frac{1}{\Phi}(V_{x1} - V_{x2})(V_{y1} - V_{y2}) + (V_{z1} - V_{z2}) \quad (1)$$

where the scale factor Φ is used to prevent the product voltage exceeding the specified maximum. The default value $\Phi = 10$ can be adjusted to suit particular situations. In what follows, multipliers will often be shown in circuit block-diagrams as though their inputs were single-ended (rather than differential). In such cases, the appropriate input terminals are chosen so as to provide the desired sign of product at the output, and the other terminal of each pair is earthed.

- **Division** of two voltages can be performed using the same multiplier IC's, but connecting the output back into one of the inputs as shown in Fig. 2(b), in which case

$$V_{\text{out}} = \frac{\Phi(V_{z2} - V_{z1})}{(V_{x1} - V_{x2})} + V_{y1}. \quad (2)$$

- **Trigonometric functions** can be implemented by use of e.g., the Analog Devices AD639 universal trigonometric converter IC, which can be connected to transform the input voltage into its sine, cosine, tangent, cosecant, secant, cotangent, or their inverse functions; e.g., it can

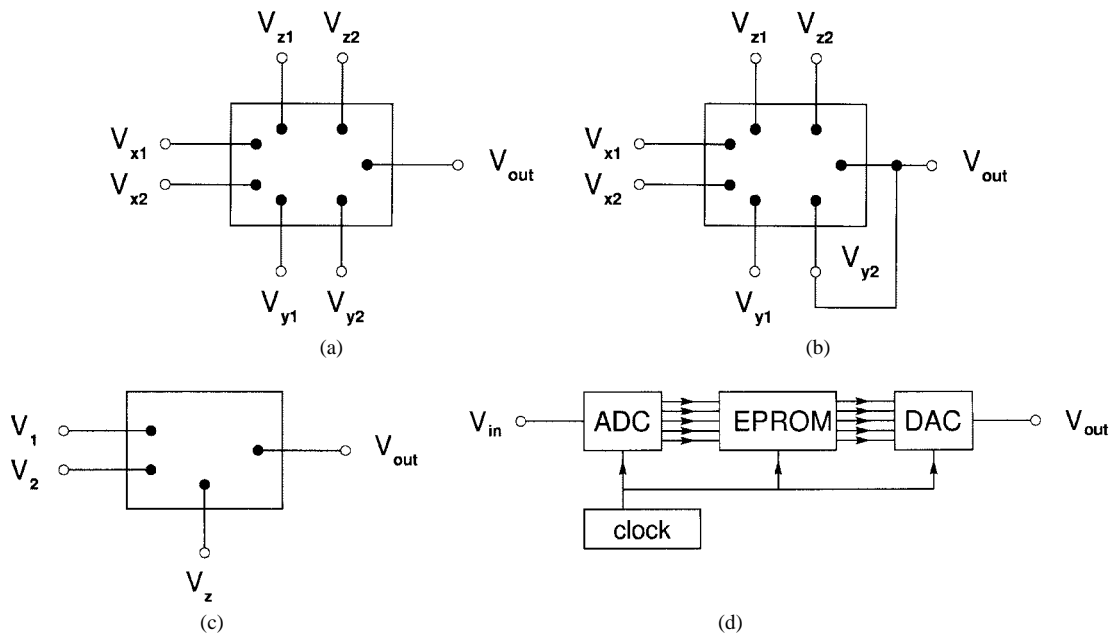


Fig. 2. Some mathematical operations with analog electronic circuits. (a) Multiplication of two voltages. (b) Division of two voltages. (c) Generation of trigonometric functions. (d) Generation of general functions. The respective transformation equations are given in the text.

be connected as indicated in Fig. 2(c), where the output voltage is

$$V_{out} = 10 \sin[50(V_1 - V_2)] + V_3 \quad (3)$$

Here, V_1 , V_2 , and V_3 are in volts and the argument of the sine is in degrees. Obviously, a cosine can be generated by setting either $50V_1$ or $50V_2$ to 90° . A range of $\pm 500^\circ$ is accommodated, but can effectively be extended where necessary by use of an analog multiplier IC to generate the double angle (see [1, Section II-B]).

- **General functions** can be created by use of a hybrid analog/digital device, as sketched in Fig. 2(d). The required function is held in the erasable programmable read-only memory (EPROM) as a look-up table. The input voltage is digitized by the analog-to-digital converter (ADC) and used to look up the corresponding number in the EPROM, which is then converted to a voltage at the output by the digital-to-analog converter (DAC). Devices of this kind have been used e.g., to create trigonometric functions [64] (before the corresponding converter IC's had become available), and for creating the potential corresponding to a one-dimensional (1-D) quantum-mechanical binary alloy [65]. The period of the clock used to drive the device must, of course, be very much shorter than all characteristic times in the system under study.

In assembling these circuit elements to model a given equation or system of equations, there are several points to be borne in mind.

- 1) To minimize extraneous noise introduced by the circuit itself, the design should minimize the number of active components [66].
- 2) Care must, of course, be taken to ensure that the voltage limits for the various components are not exceeded, even when external forces are driving the circuit far from its stationary state.

- 3) At the same time, care must also be taken to ensure that the signal is at all points in the circuit larger (preferably much larger) than the background noise and drift produced by the circuit itself. This requirement, taken with the preceding one, limits the dynamic range of the analog technique to $\sim 2 \times 10^4$. With careful circuit design, this is more than sufficient for most purposes.
- 4) To speed up data acquisition, it is normal to scale time in such a way that time in the model effectively runs much faster than real time. In doing so, care must of course be taken to ensure all components remain within their allowed frequency limits or slew-rates.
- 5) As usual, connections should be made as short and direct as possible to reduce the effect of stray capacitance. Where a circuit model is likely to be needed for many different applications, or where particular stability is needed, it is worth designing and fabricating a printed circuit board (PCB). In most cases, however, a simple mounting board into which the components can be inserted (with pressure contacts) is sufficient.
- 6) It is usually best to choose resistor values within (or not too far outside) the range 10–100 k Ω , and capacitance values of at least 100 pF in view of the typical stray capacitance of a few picofarads.

Optimization of the design requires that all of the above criteria should be taken into account, but some degree of compromise between them is usually necessary in practice. Minimization of the number of active components can often be achieved by making judicious combinations (in single circuit elements) of the separate arithmetic operations described above.

To see how a circuit model can be designed in practice, we now consider a particular example: the underdamped single-well Duffing oscillator [67] used [37] for experiments on supernarrow spectral peaks and high frequency stochastic resonance in a system with co-existing periodic attractors (see

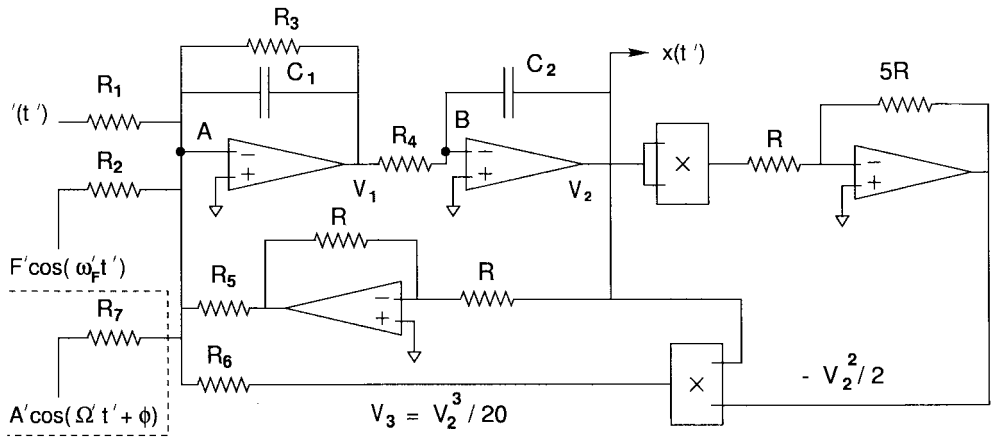


Fig. 3. Block diagram of analog electronic circuit modeling an underdamped single-well Duffing oscillator (4) [37].

Section II-A of [1]). The equation to be modeled is

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2x + \gamma x^3 = F \cos \omega_F t + f(t) \quad (4)$$

where the oscillator is driven by a periodic force of amplitude F , frequency ω_F , and $f(t)$ is zero-mean white Gaussian noise of intensity D such that

$$\langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 4\Gamma D \delta(t - t'). \quad (5)$$

Here, $\langle f(t) \rangle$ represents the average of $f(t)$, and $\langle f(t)f(t') \rangle$ its correlation function.

The circuit used to model (4) is shown in Fig. 3; two integrators are needed because of the inertial term. The periodic force in the dashed box is set to zero for present purposes (but it will be needed in [1, Section II-A] in the discussion of high-frequency stochastic resonance in the same system). We use superscript primes to distinguish times and frequencies in the circuit (in units of seconds and hertz) from the corresponding dimensionless times and frequencies that appear in (4). To understand the relationships between quantities in the circuit and in (4), we sum the currents at point A, and those at point B, and we equate them to zero in each case (using Kirchhoff's law and the assumption of infinite input impedance of an operational amplifier). For point A

$$\frac{V_1}{R_3} + \frac{f'(t')}{R_1} + \frac{F'}{R_2} \cos(\omega'_F t') - \frac{V_2^3}{20R_6} - \frac{V_2}{R_5} + C_1 \frac{dV_1}{dt'} = 0 \quad (6)$$

and at point B

$$\frac{V_1}{R_4} + C_2 \frac{dV_2}{dt'} = 0. \quad (7)$$

Using (7) to substitute for V_1 in (6), we obtain

$$C_1 R_4 C_2 \frac{d^2 V_2}{dt'^2} + \frac{R_4 C_2}{R_3} \frac{dV_2}{dt'} + \frac{V_2}{R_5} + \frac{V_2^3}{20R_6} - \frac{f'(t')}{R_1} - \frac{F'}{R_2} \cos(\omega'_F t') = 0. \quad (8)$$

The component values used in the circuit were

$$R_1 = R_4 = R_5 = 2R_6 = \frac{R_2}{10} = \frac{R_7}{10} = 100 \text{ k}\Omega$$

$$R_3 = 2.5 \text{ M}\Omega$$

$$C_1 = C_2 = 1 \text{ nF.}$$

Thus, (8) may be written

$$\tau^2 \frac{d^2 V_2}{dt'^2} + 2\Gamma \tau \frac{dV_2}{dt'} + V_2 + \frac{V_2^3}{10} = \frac{F'}{10} \cos(\omega'_F t') + f'(t') \quad (9)$$

where

$$\tau = R_1 C_1 = R_4 C_2, \quad \Gamma = \frac{R_1}{2R_3}. \quad (10)$$

It can be seen, therefore, that with the transformations

$$V_2 \rightarrow x, \quad t' \rightarrow \tau t, \quad \omega'_F \rightarrow \omega_F / \tau$$

$$F' \rightarrow 10F, \quad f'(t') \rightarrow f(t) \quad (11)$$

(9) goes over into (4) with $\omega_0 = 1$ and $\gamma = 0.1$. Note that the multiplication of V_2^2 by $\times 5$ (Fig. 3) is used to prevent the V_2^3 term being too small compared to background noise and voltage offsets in the circuit, following the additional scaling by 0.1 in the second multiplier. The relatively large value of $R_3 = 2.5 \text{ M}\Omega$ was used to obtain a small value of the damping constant Γ . Under these circumstances, it is better to *measure* quantities like Γ and ω_0 for the completed circuit [68], treating it as an experimental object in its own right, rather than just calculating them from component values; the measured and calculated values typically agree to within $\pm 10\%$.

B. Noise Generators

Experiments are often undertaken to model systems driven by white noise. In reality, of course, white noise—with zero correlation time and thus a power spectrum that remains flat up to infinite frequency—is an idealization. Noise in real physical systems always has a finite correlation time, and correspondingly a roll-off in the power spectrum above some characteristic frequency. Provided that the correlation time of the noise is much smaller (e.g., by a factor of 30) than all characteristic times (the vibration period(s) for an underdamped system, or the relaxation time(s) for an overdamped one), the noise can be considered quasi-white and its effects will be indistinguishable from those of white noise with the same intensity.

A number of commercial noise generators have been used for experiments in stochastic nonlinear dynamics. These have included the Quan-Tech model 420 (no longer manufactured) and, more recently, the Wandel and Goltermann¹ model

¹Wandel u. Goltermann, 7410 Reutlingen, Germany.

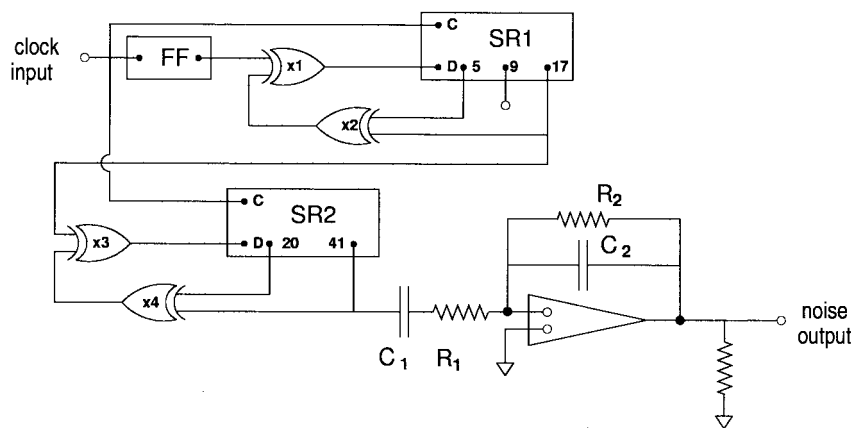


Fig. 4. Block diagram of the linear-feedback shift-register digital noise-generators used at Lancaster [74], based on a design developed at the University of Pisa [66], [69], [70]. Pseudorandom dichotomous (2-level) pulse sequences are generated by the 41-stage feedback shift-register SR2. The high-cut filter converts this dichotomous noise to an output that is Gaussian and exponentially correlated, with a correlation time R_2C_2 . The first feedback shift-register SR1 pseudorandomly inverts the sign of the feedback in SR2, thereby eliminating skewness of the distribution. An additional 41-stage feedback shift-register SR3 (not shown), taking its inverting input from stage-9 of SR1 but otherwise connected exactly like SR2, provides a second independent noise source. FF is a flip-flop that divides the 4-MHz clock frequency by a factor of 32, and X1–X4 are exclusive-OR gates. The low-frequency cutoff, determined by R_1C_1 , is fixed at ~ 1 Hz, and the high-frequency cutoff is adjustable. The output noise finally passes through a variable-gain operational amplifier (not shown) before being applied to an experiment.

RG1. The latter produces Gaussian noise with a power spectrum that is flat to within ± 0.5 dB up to 108 kHz; above this frequency, there is a very fast roll-off. It possesses the advantage that the output is genuinely random, with essentially infinite repetition period.

For many purposes, however, a much cheaper and simpler “home-made” noise generator will suffice which, in some respects, is actually superior. This is the device used and described by Faetti *et al.* [69], [70] and Fronzoni [66]. It is based on the generation of a pseudorandom sequence by a linear-feedback shift-register [71]; the dichotomous (two-state) output is then filtered through a Miller integrator to produce [72] Ornstein–Uhlenbeck (i.e., exponentially correlated) noise. In its original form, the dichotomous noise suffered from asymmetry in that the times spent in the upper and lower states were systematically different on account of the OR-feedback being used [73]. Consequently, the distribution function of the approximately Gaussian noise created after passage through the filter suffered from significant skewness. Faetti *et al.* showed [70] that this problem could be overcome by randomly inverting the sign of the feedback voltage between EX-OR and EX-NOR. A version of this device (Fig. 4) used at Lancaster [74] employs the output from two different stages of the same 17-stage feedback shift register (SR1) as feedback inverters for two separate 41-stage [75] shift registers (SR2 and SR3). In this way, two independent pseudorandom pulse sequences are obtained which, after filtration, provide uncorrelated pseudo-white noise sources. With a clock frequency of ~ 4 MHz, and the filter time constants set to give a cut-off above 40 kHz, the distribution functions at the output are Gaussian to more than ± 4 standard deviations and the repetition time of the pseudorandom time sequences is ~ 6.5 days. It is essential to remember that these noise generators should not be used for experiments whose duration exceeds the repetition time; in all other respects, however, they have been found entirely satisfactory.

The root-mean-square (rms) amplitude of the Ornstein–Uhlenbeck noise being applied to a circuit model can conveniently be measured with a true rms-to-dc converter such as the Analog Devices AD536A IC. It is then necessary to relate this value (in volts) to the intensity D of white noise appearing in model equations such as (4) and (5). To do so, we note that Ornstein–Uhlenbeck noise (e.g., see [76]) is exponentially correlated, with

$$\langle f(t)f(s) \rangle = \frac{2\Gamma D}{\tau_N} e^{-|t-s|/\tau_N} \quad (12)$$

where τ_N is the correlation time. Thus, the mean-square noise amplitude is

$$\langle f^2(t) \rangle = \frac{2\Gamma D}{\tau_N}. \quad (13)$$

But the circuit operates in scaled time, so that

$$\tau_N = \frac{\tau_{ou}}{\tau} \quad (14)$$

where τ_{ou} is correlation time of the noise (in seconds), which is given by the resistance–capacitance (RC) time constant of the Miller integrator used to create it (see above), or can be obtained from a measurement of the correlation function of the noise; τ is the time scaling of the model, given by (11) in the example in Section I-C. Hence

$$D = \left(\frac{\tau_{ou}}{2\Gamma\tau} \right) \langle f^2(t) \rangle. \quad (15)$$

Here, $\langle f^2(t) \rangle$ can be measured directly. Values of all the other parameters are also known, because they can be calculated from the component values or measured directly. Thus, (15) provides the required connection between D and the rms noise voltage measured by the true rms-to-dc converter IC.

C. Signal Acquisition and Analysis

Analysis of the behavior of the circuit model usually involves two main stages: digitization of the analog signal $x(t)$ ($V_2(t)$ in the circuit of Fig. 3), and then processing

of the resultant digital time series to extract the particular information required, which is often in the form of a statistical distribution.

There is now a very wide range of ADC cards available for operation in PC's; reviews are published from time to time, e.g., [77], [78], helping in selection for a particular application. Sample intervals below 1 μ s are not normally needed for modeling of this kind, and 10 μ s is short enough for most purposes. If fast data acquisition without gaps—coupled with intensive processing—is needed, a co-processor is likely to be essential: most of the data processing can then take place on the card itself, with relatively infrequent transfers of input data and ensemble-averaged distribution to the main memory. The PC's central processing unit (CPU) thus remains free to operate displays of the input and averaged data, interact with the user, and service the “housekeeping” needs of the operating system. The systems currently in use at Lancaster are based on the Microstar² model DAP 3200a/415 (12-bit) and DAP 3216/415 (16-bit) ADC's, each of which incorporates an on-board 100-MHz Intel 486DX co-processor. A MatLab-based PC software system developed by I. Kh. Kaufman is used to provide real-time data acquisition with simultaneous digital signal processing within the MatLab environment, for both Win32 and Linux operating systems. It is designed as a configurable and extendable system with additional drivers, service tools, and software templates.

III. THEORY OF STOCHASTIC RESONANCE

A. Stochastic Resonance as a Linear Response Phenomenon

When it was first discovered, and for some time afterwards, SR seemed a rather mysterious phenomenon and a number of highly sophisticated theoretical approaches were proposed (see citations in e.g., the reviews [47]–[49], [51]). Only some years later, it was appreciated [32], [33] that a much simpler formalism—*linear response theory* (LRT)—would suffice to describe what was often the most interesting limit in practice, where the signal was relatively small and the noise was relatively strong; an analytic theory of the more complicated effects that occur for stronger signal strengths [33], [79], [80], [81] was also developed. Both limits have been explored in considerable detail through analog electronic experiments, validating and clarifying the theory. LRT places SR in perspective, enables it to be understood in the context of other more familiar phenomena in physics [50], and therefore provides the best starting point for understanding SR.

If a system with a coordinate x is driven by a weak force $A \cos \Omega t$ then, according to LRT [82], a small periodic term will appear in the ensemble-averaged value of the coordinate $\delta \langle x(t) \rangle$, oscillating at the same frequency Ω and with amplitude a proportional to that of the force

$$\begin{aligned} \delta \langle x(t) \rangle &= a \cos(\Omega t + \phi) \equiv \text{Re}[\chi(\Omega) A e^{-i\Omega t}] \\ a &= A |\chi(\Omega)|, \quad \phi = -\arctan[\text{Im} \chi(\Omega) / \text{Re} \chi(\Omega)]. \end{aligned} \quad (16)$$

²Microstar Laboratories Inc., 2265 116th Avenue NE, Bellevue, WA 98004 USA; and see <http://www.mstarlabs.com/>. The U.K. distributor is: Amplicon Liveline Ltd., Centenary Industrial Estate, Hollinlean Road, Brighton BN2 4AW U.K.

In the limit $A \rightarrow 0$, this equation holds for dissipative and fluctuating systems that do not display persistent periodic oscillations in the absence of the force $A \cos \Omega t$ and where the correlations of fluctuations decay in time. The susceptibility $\chi(\Omega)$ contains all information on the response of the system to a weak driving force. It gives both the *amplitude* a of the signal and its *phase lag* ϕ with respect to the force (and the partial amplitudes and phase lags for vibrations at the combination frequencies). The theory is readily generalized to the case of nonsinusoidal forces, and forces acting under nonequilibrium conditions. Provided that the amplitude of the modulation is weak enough, the response of the system will always be linear and described by (16).

The periodic term (16) induced by the force gives rise to a δ -shaped spike in the power spectrum of the coordinate, i.e., in the *spectral density of fluctuations* (SDF)

$$Q(\omega) = \lim_{\tau \rightarrow \infty} (4\pi\tau)^{-1} \left| \int_{-\tau}^{\tau} dt x(t) \exp(i\omega t) \right|^2 \quad (17)$$

at the frequency Ω of the force. The *intensity* (i.e., the area) of this spike is equal to one fourth of the squared amplitude of the corresponding vibrations, i.e., to $1/4A^2 |\chi(\Omega)|^2$. The SNR R can thus be written

$$R = \frac{1}{4} A^2 |\chi(\Omega)|^2 / Q^{(0)}(\Omega) \quad (A \rightarrow 0) \quad (18)$$

where $Q^{(0)}(\omega)$ is the SDF in the absence of the periodic driving. The evolutions of the susceptibility and of $Q^{(0)}(\omega)$ with varying noise intensity D therefore show immediately whether or not SR (in terms of the strong definition, as an increase in SNR with increasing D in a certain range of D) is to be expected at a given frequency.

Describing SR in terms of a susceptibility in this way is particularly advantageous for systems that are in thermal equilibrium, or in quasi-equilibrium. In such cases, the Kramers–Kronig and fluctuation-dissipation relations [82] can be used to express the susceptibility in terms of $Q^{(0)}(\omega)$

$$\text{Im} \chi(\omega) = \frac{\pi\omega}{D} Q^{(0)}(\omega) \quad (19)$$

$$\text{Re} \chi(\omega) = \frac{2}{D} P \int_0^{\infty} d\omega_1 Q^{(0)}(\omega_1) \frac{\omega_1^2}{\omega_1^2 - \omega^2} \quad (20)$$

where P denotes the Cauchy principal value and D corresponds to the temperature in energy units. It follows from (18)–(20) that it should be possible to predict the onset of SR in a given system purely from the evolution of its SDF $Q^{(0)}(\omega)$ with noise intensity (temperature), without knowing or assuming anything at all about the equations that describe its dynamics, i.e., for a system treated as a “black box.”

Note that the linear response is obtained as a *consequence* of the noise. The paradigmatic systems that we consider below are all highly nonlinear, but their response to a periodic force—which need not necessarily be an extremely weak one—is linearized in each case by the addition of noise [83] (see also [1, Section III-B]) of sufficient intensity.

B. What Can SR Do and Not Do?

We now address the vexed question of whether an SR-displaying system (a “stochastic resonator”) can be expected to improve the SNR of a given signal. There has been much confusion on this point in the literature, with more than one

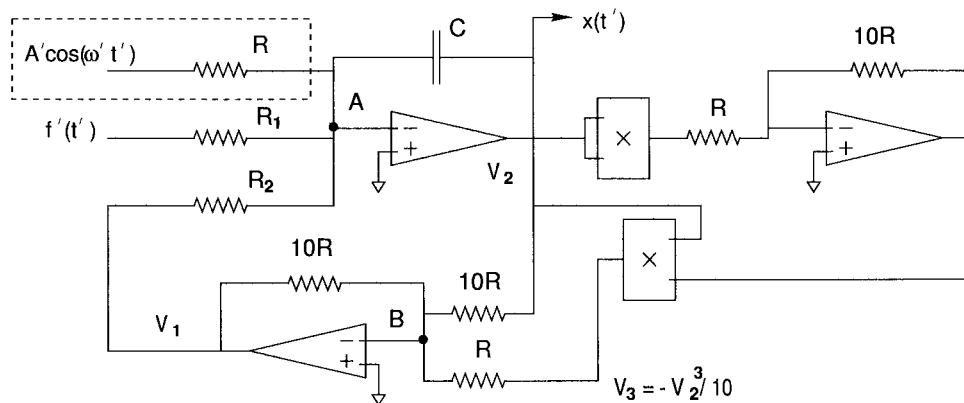


Fig. 5. Block diagram of a circuit model of the overdamped double-well Duffing oscillator.

research group claiming that absolute SNR improvements have been observed in experiments. However, Dykman *et al.* [50] and DeWeese and Bialek [84], using different LRT-based arguments, have pointed out that, for small-amplitude signals, the SNR at the output of a system driven by a stationary Gaussian noise does not exceed that at the input, even if the system displays SR. Indeed, the Fourier components of the noise are statistically independent and the total power of the noise $\Xi(\Omega) d\Omega$ in a small spectral interval $d\Omega$ about the frequency of the signal Ω is small. The SNR at the input is given by $1/4A^2/\Xi(\Omega)$, whereas that at the output is $1/4|\chi(\Omega)|^2 A^2 / [|\chi(\Omega)|^2 \Xi(\Omega) + Q^{(0)}(\Omega)]$. The quantity $Q^{(0)}(\Omega)$ gives the value of the spectral density of fluctuations in the system at frequency Ω as it would be if there was no signal and the spectral components of the noise at frequency Ω were suppressed, i.e., as if the power spectrum of the input noise had a hole at frequency Ω . By construction $Q^{(0)}(\Omega) \geq 0$, which proves the statement. (In linear systems, on the other hand, which do not mix frequencies, $Q^{(0)}(\Omega) = 0$ and the SNR at the output must be the same as at the input.) These points deserve, and have recently received [85], emphasis given the level of misinformation in some of the recent literature on SR.

The situation with large-amplitude signals (beyond the LRT range) is less clear. Generally, such signals are distorted by a nonlinear system: the response to a sinusoidal signal has overtones, and therefore may not be characterized by the SNR at the signal frequency. In some models, the SR effect decreases with signal amplitude for large signals [86], whereas for specific types of nonlinear systems and signals, and/or definitions of the SNR, an increase of the SNR has been reported by Kiss [87] and by Khovanov and Anishchenko [88].

IV. CONVENTIONAL STOCHASTIC RESONANCE IN CIRCUITS

The idea that SR might be describable in terms of LRT was tested [32], [33] through analog electronic simulation of underdamped 1-D Brownian motion

$$\ddot{x} + 2\Gamma\dot{x} + \frac{\partial U}{\partial x} = f(t) + A \cos(\Omega t) \quad (21)$$

$$\langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 4\Gamma D\delta(t - t')$$

in the simple symmetric bistable potential

$$U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4. \quad (22)$$

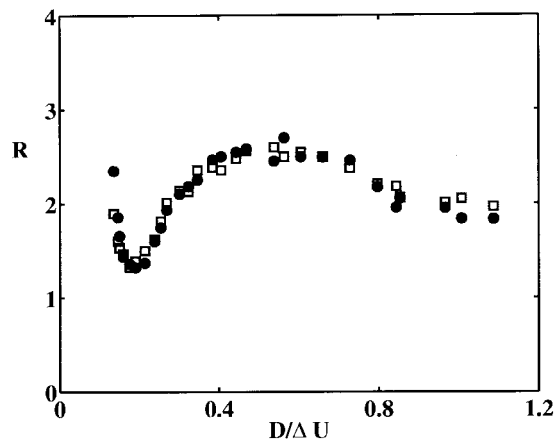


Fig. 6. Demonstration that stochastic resonance can be described in terms of LRT. The filled circles represent direct measurements of the SNR R for the system (21) and (22) as a function of scaled noise intensity $D/\Delta U$, where ΔU is the depth of each potential well below the central potential maximum; the open squares represent LRT calculations based on *measured* spectral densities and the use of (18)–(20). There are no adjustable parameters.

Note that it is the *response* of the system in the presence of noise that is expected to be linear [82], proportional to the amplitude A of the periodic force; the system itself is, of course, highly nonlinear.

The circuit model was very similar to that of Fig. 3, except that the V_2 term was not inverted before being fed back to point A. More accurate measurements have been made on the equivalent overdamped system

$$\dot{x} + \frac{\partial U}{\partial x} = f(t) + A \cos(\Omega t)$$

$$f(t) = 0, \quad \langle f(t)f(t') \rangle = 2D\delta(t - t'). \quad (23)$$

Note the different conventional definitions of noise intensity that are used for underdamped (21) and overdamped (23) systems.

A. SNR Measurements

The circuit used to model (22) and (23) is shown in Fig. 5. Some data obtained from it are plotted in Fig. 6. They are of two entirely different kinds, obtained as follows. First, SDF's were measured for several values of D with the weak periodic force applied ($A \neq 0$), and the SNR was determined through measurements of the ratio of the height of the spike at Ω to

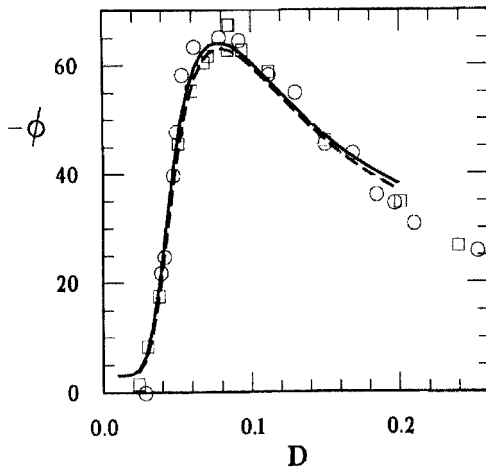


Fig. 7. The phase shift $-\phi$ (degrees) between the periodic force of amplitude A and the averaged coordinate $\langle x(t) \rangle$ measured [88] as a function of noise intensity D in the electronic model of (22) and (23) for $\Omega = 0.1$ and $A = 0.04$ (circles); $A = 0.2$ (squares). The full curve is a theoretical prediction based on LRT and the fluctuation dissipation theorem.

that of the noisy background [30]. These direct measurements of the SNR are shown as filled circle data points in Fig. 6. Secondly, SDF's were measured in the *absence* of the periodic force ($A = 0$) for several different noise intensities. These results were used to compute the SNR to be expected from the LRT relations (18)–(20), which are shown as open squares in the figure. Given that there are *no adjustable parameters*, the agreement between the theory and the direct measurement may be considered an impressive vindication of the LRT approach to SR. Of course, the SDF $Q^{(0)}(\omega)$ for insertion in (18)–(20) does not necessarily need to be measured, as here. It can also be calculated analytically over a wide range of parameters [89], [90].

Note that the onset of SR in a bistable system can be understood in terms of the evolution with D of its SDF and, in particular, the broadening of the zero-frequency peak [89], [90] associated with the interwell fluctuational transitions. As the peak broadens, $Q^{(0)}(\Omega)$ for Ω finite but small, rises exponentially fast. Thus, from (19) and (20), it is clear that both $\text{Re}\chi(\omega)$ and $\text{Im}\chi(\omega)$ for $\omega = \Omega$ will also increase exponentially fast. It is clear from (16) and (18) that a and R will also both increase with D . This perception of bistable SR is, of course, a counterpart of the picture of noise-enhanced hopping between the wells in response to the external driving, discussed by [52].

B. Phase-Shift Measurements

In close analogy with conventional resonance phenomena, it is to be expected that a phase difference will exist in stochastic resonance between the periodic driving force and the periodic response of the system. The magnitude of this phase shift, and its variation with noise intensity, was initially a subject of controversy; the matter was resolved partly through the relative simplicity and transparency of the LRT approach [see (16), (19), (20)], and partly through measurements on an analog electronic model of (22) and (23).

The phase shift ϕ between the force and the response was determined experimentally by measurement of ensemble-

averages $\langle q(t) \rangle$, comparing the resultant sinusoid directly with the weak periodic driving force $A \cos(\Omega t)$. Results are shown by the data points of Fig. 7. It can be seen that, as D is increased, $-\phi$ starts from a very small value, rises extremely fast, passes through a maximum, and then decreases more slowly again. A theory able to describe this behavior is readily constructed [91] on the basis of LRT, and leads to the full curve. LRT also provides a satisfactory description of the system behavior [92], even when Ω has become so large that the SR effect has disappeared, and R just decreases monotonically with increasing D .

V. CONCLUSION

The successful development of the LRT approach to SR is attributable in significant measure to the model experiments based on analog electronic circuits, which enabled the approximate analytic theory to be tested while it was being developed. As was made clear, LRT provides a very general perception of SR and shows that the phenomenon does not necessarily require bistability, or even a static potential, but that it should appear in diverse systems and contexts. We consider some examples of such *nonconventional SR* in the companion paper [1].

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