Angular Constraints on Planar Frameworks

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Abstract

Consider a collection of points in the plane and the sets of slopes or directions of the lines between pairs of points. It is known that the algebraic matroid on the set of direction constraints between the points is equivalent to the algebraic matroid on the set of distances between the points. This is the well-studied generic 2-dimensional rigidity matroid of a graph. This article studies a higher-level construction built on the slope data: an angle constraint system obtained by prescribing relationships between pairs of slopes. The central question we analyze is: when is an angle system rigid, in the sense that every nontrivial motion alters one of the fixed angles?

We formulate the problem in matricial terms for certain edge-colored graphs, finding precise necessary conditions for when such edge-colored graphs are rigid, and a combinatorial characterization of generic rigidity for a special case. We also prove the validity of an equivalent formulation of the angle matroid as the algebraic matroid of a field extension.

1 Introduction

Consider a collection of points $p = (p_v)_{v\in V})$ indexed by some finite set V and the set $d_{vw} = ||p_v - p_w||^2$ of pairwise squared distances between them. For generic points p_v, (e. g. with algebraically independent coordinates), the algebraic matroid on the set of distances d_{vw} was characterized by Pollaczek-Geiringer [PG27] and later rediscovered by Laman [Lam70]. This matroid, called the (generic 2-dimensional) *rigidity matroid*, is denoted $\mathcal{R}_2$ throughout this paper. Viewing the points and squared distances as vertices and edge lengths of a complete graph on vertex set V, a set E of edges is independent in $\mathcal{R}_2$ if and only if it is $(2, 3)$-sparse; that is, the subgraph induced by any non-empty $E' \subset E$ satisfies the inequality $|E'| \leq 2|V(E')| - 3$. The pair $(G = (V, E), p)$ is called a *bar-joint framework*.

A natural related question examines the sets of slopes or directions of the lines between pairs of points. In particular, the equations $m_{vw} = (y_v - y_w)/(x_v - x_w)$, where $p_v = (x_v, y_v)$ for each $v \in V$. Surprisingly, the algebraic matroid on this set of elements is precisely the rigidity matroid $\mathcal{R}_2$. This was proved by Whiteley [Whi87, Proposition AB.14] using an analysis of the rigidity matrix. It was reproved later by Martin using techniques from algebraic geometry [Mar03].

In this paper, we examine a construction on top of the slope matroid, given by an angle constraint system. Instead of fixing slopes of edges, we fix the angles between chosen pairs of edges. Consider the example in Figure 1. Noting that for any subset of m edges, fixing $m-1$ pairwise angles determines all pairwise angles, such an angle constraint system partitions the edges into sets such that the pairwise angles are fixed within each set.

The corresponding matroid governing angles appears to be a more stubborn object to characterize. In this paper, we approach this problem from algebro-geometric and combinatorial perspectives. Our main results include Theorem 2.12 which gives a matrix-based characterization of angle rigidity, Theorem 4.4 which derives a complete combinatorial characterization in a rich special case, and Corollary 5.4 and Theorem 5.6 which provide algebraic matroidal re-formulations.

There are several potential applications of analyzing rigidity for angle frameworks. The multi-agent formation literature uses angle constraint systems to guide multiple agents toward a desired

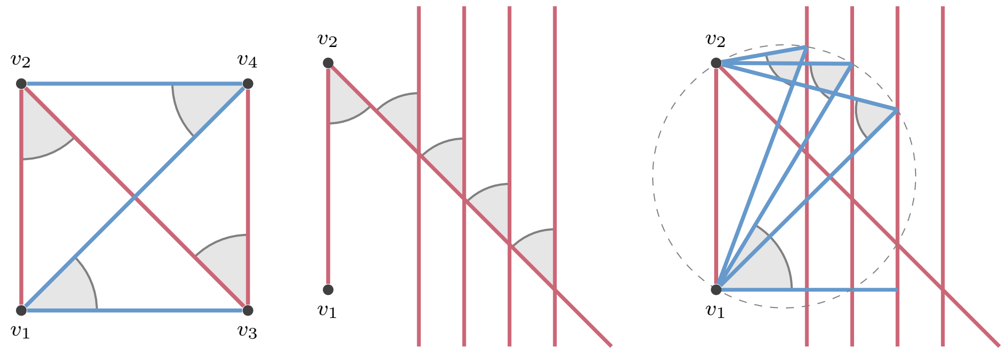


Figure 1: K_4 is dependent in $\mathcal{R}_2$, i. e. the distance/slope of one of the pairs of points (vertices) is dependent on the distances/slopes of the remaining pairs, however, the shown set of angles is independent.

formation based on angle information [ZZ15, JZLW19, CCL21]. Lacking a combinatorial analysis of angle systems they use combinatorial characterizations arising in the bar-joint setting as a weak approximation. Combinatorial characterizations as in Theorem 4.4 are important for computer-aided design software that computes a solution, i.e., a framework, to a user-specified geometric constraint system, such as those studied by Haller et al. in [HLSS+12], which include pairs of angle-constrained lines. Such constraint systems typically take doubly exponential time in the number of variables to solve directly via computational algebra.

1.1 Previous work on angles

Mathematically, the problem has been primarily discussed in two bodies of literature. In the rigidity theory literature, especially the work of Walter Whiteley, angle constraints are considered in the context of direction and length constraints. More recently, various engineering groups have studied this problem as part of multi-agent formation control.

First, we summarize the rigidity theory results. In [SW99], Servatius and Whiteley proved the necessary sparsity bound $|A| \leq 2|V(E)| - 4$, as well as the fact that this sparsity condition is not sufficient to guarantee independence. Polygon constraints are cited as one demonstration of the failure of sufficiency, but they add that even if polygons are accounted for, the sparsity bound is not sufficient. In a subsequent paper [EWM+03], Whiteley and co-authors stated further results on angle arrangements. They introduced the *first-order angle matrix* as an analogue to the traditional rigidity matrix. They conjectured that there is no polynomial-time algorithm to check independence of angle arrangements. They also described Henneberg-type moves to extend angle arrangements: the 0-extension adds one new vertex v to a graph as well as two angle constraints both centered at vertex v. The 1-extension adds a vertex v, deletes one angle constraint, and adds three new angle constraints. The theorems are stated there unproven, with reference made to an unpublished article entitled “Constraining plane geometric configurations in CAD: Angle”. They conclude that these two types of extensions were insufficient to construct all angle-rigid arrangements, since they do not produce vertices of degree five.

In the 2006 doctoral thesis of Zhou [Zho06], the author considers the problem of angle constraints. A rigorous proof of the validity of 0-extensions (which the author calls “gradual construction”) is presented, as are some necessary combinatorial conditions for rigidity. Other studies of angles have been made but usually with some fundamental twist, e. g. considering circle arrangements with constrained angles of intersection as in [SW04].

The multi-agent formation literature is differentiated from the rigidity literature both in its goals and its mathematical approach. In particular, they often feature practical results that can be used to guide multiple agents toward a desired formation based on angle information. The mathematical tools tend to come from control theory and analysis. For example, the 2015 paper [ZZ15] begins from the 2-d rigidity-theoretic work and generalizes to higher dimensions. Then they use Lyapunov methods to define a control law that can stabilize bearing-rigid formations. Other recent papers [JZLW19, CCL21] stay focused on the 2-d setting, but further explore the control law and how perturbed formations stabilize under a control law. Given the different objectives and toolkit, we leave further integration of the literature for future work.

1.2 Analogous cases in the literature

At first glance, the angle matroid appears to be a special case of the point-line incidence structures studied by Jackson and Owen in [JO16]. In that article, the authors characterize structures with two types of objects—points and lines—and three types of relations: (i) angles between pairs of lines; (ii) perpendicular distance from a point to a line; (iii) distance from point to point. Assuming these quantities are generic, they fully characterize the resulting matroid. If you set the distance between p and ℓ to 0 at p lies on ℓ and the resulting setup is exactly of the type we consider. Unfortunately, once genericity is broken, the characterization of [JO16] no longer applies.

Another seemingly similar setup is that of frameworks with coordinated edge motions, studied by Schulze, Serocold and Theran in [SST22]. In their setting, subcollections of edges are assigned to "coordinated classes." In addition to the standard rigid motions, the edges of each color are allowed to change length (additively) by the same fixed amount while still being considered equivalent. They characterize the resulting matroid (in the $d = 2$ case) as the matroid union of the rigidity matroid $\mathcal{R}_2$ with the transversal matroid whose bases take one element from each of the coordinated classes. We believe that the matroid here takes a very similar form, though our characterization is incomplete. Indeed, they note in Section 5.3 of their paper, "the rigidity analysis of such coordinated frameworks seems more complex than the one considered in this paper."

1.3 Outline

The structure of the paper is as follows: In Section 2, we formally define angular analogues of concepts from rigidity theory. In Section 3, we prove an important property of angle-rigid frameworks and conjecture that the property is also sufficient for angle-rigidity. In Section 4, we define extension moves on angle frameworks, much like Henneberg moves, that allow us to characterize an important subclass of arrangements. Finally, in Section 5, we derive an algebraic matroid formalizing angle-rigidity, much as we have in standard rigidity theory. Taken as a whole, this represents an important step in developing angular constraints as an analog to classical rigidity theory.

2 Basic definitions

In this section we develop the general theory of angle-rigid frameworks, from the perspective of the geometry of realizations.

Definition 2.1. Let $G = (V, E)$ be a graph. We call an injective map $p : V \to \mathbb{R}^2$ a *realization* of G, and we denote by $R^G \subset (\mathbb{R}^2)^V$ the (Zariski-open) *realization space* of G.

- An *angle index set* A is a set of distinct unordered pairs in E; for example, $A = \{\{(a_1b_1), (c_1d_1)\}, \ldots, \{(a_kb_k), (c_kd_k)\} \mid a_ib_i, c_id_i \in E\}$.
- The triple (G, A, p) is called an *angle framework*.

- The *angle map* is the map sending each realization to the resulting angles indexed by A: $\theta_A : R^G \to \mathbb{R}^A,\ \mathbf{p} = (p_v)_{v\in V} \mapsto \left(\arccos\left(\frac{(p_a-p_b)\cdot(p_c-p_d)}{\|p_a-p_b\|\|p_c-p_d\|}\right)\right)_{\{ab,cd\}\in A}$.

- The *angle graph* $\tilde{G}(A)$ is the graph with vertices indexed by E and edges by A.

- The *edge support* $E(A)$ denotes the set of edges in G appearing as half of some pair in the angle index set.

Example 2.2. Let us consider the graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{(v_1v_2), (v_1v_3), (v_1v_4), (v_2v_3), (v_2v_4)\}$. Let further $A = \{\{(v_1v_2), (v_1v_3)\}, \{(v_2v_3), (v_2v_4)\}\}$. Figure 2 (left) shows the angle framework (G, A, p), where A is indicated by colors and $p_{v_1} = (0, 0)$, $p_{v_2} = (1, 0)$, $p_{v_3} = (1, 1)$ and $p_{v_4} = (0, 1)$. The angle map θ_A is then defined by $\theta_A(\{(v_1v_2), (v_1v_3)\}) = \theta_A(\{(v_2v_3), (v_2v_4)\}) = \pi/4$. The angle graph $\tilde{G}(A)$ is depicted on the right.

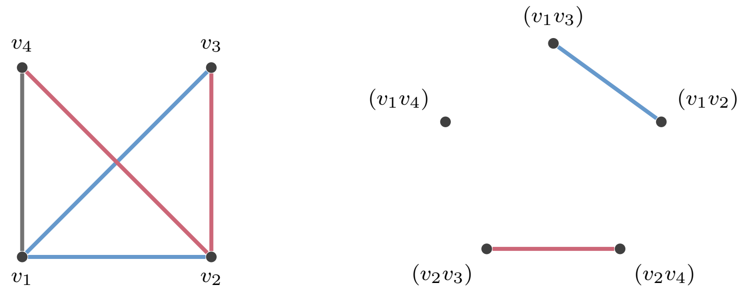


Figure 2: An angle framework (left) with A indicated in colors, and the corresponding angle graph (right).

We would like to drop some of the redundancy in the definition of angle frameworks. To do this, we require notions of equivalence up to some transformation:

Definition 2.3. Let G be a graph and A an angle index set. Let (G, A, p) and (G, A, q) be two angle frameworks.

- (G, p) and (G, q) are called *similar* if there exists a linear isometry T, a vector $z \in \mathbb{R}^2$ and a scalar $\lambda > 0$ such that $q_v = \lambda T(p_v) + z$ for each $v \in V$; the affine map $x \mapsto \lambda T(x) + z$ is called a *similarity*.

- (G, A, p) and (G, A, q) are *equivalent* if $\theta_A(p) = \theta_A(q)$.

Let A_1 and A_2 be angle index sets of G. Observe that if $\tilde{G}(A_1)$ and $\tilde{G}(A_2)$ partition E into the same set of connected components $c_1, \ldots, c_k$, then (G, A_1, p) is equivalent to (G, A_1, q) if and only if (G, A_2, p) is equivalent to (G, A_2, q). Hence, equivalence is completely determined by the connected components of $\tilde{G}(A)$, which can be viewed in terms of edge-colored graphs.

Definition 2.4. Let $G = (V, E)$ be a graph. We define the *color map* $\mathbf{c}$, associating a color to every edge, and call $(G, \mathbf{c})$ an *edge-colored graph*. By abuse of notation justified above, we also call $(G, \mathbf{c}, p)$ an angle framework. Furthermore, we let $G_c = (V, E_c)$ where E_c is the set of all edges of color c, and we denote by $C_G = \mathbf{c}(E)$ and omit G in case it is clear.

2.1 The angle-rigidity matrix

Let $G = (V, E)$ be a graph and $(G, \mathbf{c}, p)$ an angle framework. For every point $p_v = (x_v, y_v) \in \mathbb{R}^2$, $p_v^\perp$ denotes the point formed by applying a 90° counter-clockwise rotation to p_v. As a short-hand, we use the notation $p^\perp$ for $(p_v^\perp)_{v\in V}$. For any two vertices $v, w \in V$, we set $X_{vw} = x_v - x_w$, $Y_{vw} = y_v - y_w$, and the vector $P_{vw} = p_v - p_w = (X_{vw}, Y_{vw})$.

By differentiating the angle constraints $\theta_A(p_v) = constant$ and scaling the result, we observe that a map $u : V \to \mathbb{R}^2$ is an infinitesimal deformation of $(G, \mathbf{c}, p)$ that preserves angles between pairs of edges of the same color (now referred to as an *infinitesimal flex*) if and only if for every pair $ab, vw \in E$ with $\mathbf{c}(ab) = \mathbf{c}(vw)$, we have

$$\left(\frac{(P_{ab} \cdot P_{vw})P_{ab} - (P_{ab} \cdot P_{ab})P_{vw}}{P_{ab} \cdot P_{ab}}\right) \cdot (u_a - u_b) \; + \; \left(\frac{(P_{ab} \cdot P_{vw})P_{vw} - (P_{vw} \cdot P_{vw})P_{ab}}{P_{vw} \cdot P_{vw}}\right) \cdot (u_v - u_w) = 0.$$

By cancellation, we derive that

$$((P_{ab} \cdot P_{vw})P_{ab} - (P_{ab} \cdot P_{ab})P_{vw}) \cdot P_{ab} = 0 \quad \text{and} \quad ((P_{ab} \cdot P_{vw})P_{vw} - (P_{vw} \cdot P_{vw})P_{ab}) \cdot P_{vw} = 0.$$

Hence,

$$(P_{ab} \cdot P_{vw})P_{ab} - (P_{ab} \cdot P_{ab})P_{vw} = \alpha_{ab}P_{ab}^\perp \quad \text{and} \quad (P_{ab} \cdot P_{vw})P_{vw} - (P_{vw} \cdot P_{vw})P_{ab} = \alpha_{vw}P_{vw}^\perp.$$

for some scalars $\alpha_{ab}, \alpha_{vw} \in \mathbb{R}$. By measuring the two vectors, we see that $\alpha_{ab} = \alpha_{vw} = -P_{ab}^\perp \cdot P_{vw}$. This allows us to simplify the infinitesimal flex constraint condition to obtain

$$\left(\frac{P_{ab}^\perp}{P_{ab} \cdot P_{ab}}\right) \cdot (u_a - u_b) - \left(\frac{P_{vw}^\perp}{P_{vw} \cdot P_{vw}}\right) \cdot (u_v - u_w) = 0. \tag{1}$$

We say that an infinitesimal flex is *trivial* if it is a restriction of an infinitesimal similarity to the points $\{p_v : v \in V\}$. It can be easily checked that eq. (1) holds for any choice of vertices a, b, c, d with $a \neq b$ and $c \neq d$ when u is a trivial infinitesimal flex. The set of trivial infinitesimal flexes forms a linear subspace of the linear space of infinitesimal flexes. Since similarities are formed from translations, rotations and scalings, the following result is immediate.

Lemma 2.5. *Let $(G, \mathbf{c}, p)$ be an angle framework. If there exist vertices $v, w \in V$ where $p_v \neq p_w$, then the following vectors form a basis of the trivial infinitesimal flexes of $(G, \mathbf{c}, p)$:*

$$u^{(1,0)} = ((1, 0))_{v\in V}, \qquad u^{(0,1)} = ((0, 1))_{v\in V}, \qquad u^p = (p_v)_{v\in V}, \qquad u^{p^\perp} = (p_v^\perp)_{v\in V}.$$

Lemma 2.6. *Let $(G, \mathbf{c}, p)$ be an angle framework. Let $u \in (\mathbb{R}^2)^V$. Then u is an infinitesimal flex of $(G, \mathbf{c}, p)$ if and only if for each color $c \in C$, there exists $\lambda_c \in \mathbb{R}$ so that for each $vw \in E$ with $\mathbf{c}(vw) = c$,*

$$P_{vw}^\perp \cdot (u_v - u_w) = \lambda_c P_{vw} \cdot P_{vw}.$$

Proof. For each color c choose an edge $vw \in E$ with $\mathbf{c}(vw) = c$. We now set

$$\lambda_c := \frac{P_{vw}^\perp \cdot (u_v - u_w)}{P_{vw} \cdot P_{vw}}.$$

The result follows by inspection of the constraint system given in eq. (1). □

Definition 2.7. The *angle-rigidity matrix* is the $|E| \times (2|V| + |C|)$ matrix, defined blockwise as

$$R(G, \mathbf{c}, p) := [R(G, p) \quad M(G, \mathbf{c}, p)] ,$$

where $R(G, p)$ is the standard 2-d rigidity matrix, and $M(G, \mathbf{c}, p)$ is the $|E| \times |C|$ matrix with entries

$$M(G, \mathbf{c}, p)_{vw,c} := \begin{cases} -P_{vw} \cdot P_{vw} = -X_{vw}^2 - Y_{vw}^2, & \text{if } \mathbf{c}(vw) = c, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 2.6, a map $u \colon V \to \mathbb{R}^2$ is an infinitesimal flex of $(G, \mathbf{c}, p)$ if and only if there exists $\lambda : \{1, \ldots, |C|\} \to \mathbb{R}$ such that $(u, \lambda) \in \ker R(G, c, p^\perp)$. From this we can immediately deduce the following lemma.

Lemma 2.8. *Let $(G, \mathbf{c}, p)$ be an angle framework. If there exist vertices $v, w \in V$ where $p_v \neq p_w$, then the following vectors are contained in the kernel of $R(G, \mathbf{c}, p)$:*

$$(u^{(1,0)}, \mathbf{0}), \qquad (u^{(0,1)}, \mathbf{0}), \qquad (u^{p^\perp}, \mathbf{0}), \qquad (u^p, \mathbf{1}), \tag{2}$$

where each of the u vectors is defined as in Lemma 2.5, and $\mathbf{0}, \mathbf{1} : \{1, \ldots, |C|\} \to \mathbb{R}$ are the constant maps $i \mapsto 0$ and $i \mapsto 1$ respectively.

Remark 2.9. It follows from Lemma 2.8 that for $(G, \mathbf{c}, p)$ with vertices $v, w \in V$ such that $p_v \neq p_w$, the rank of the matrix $R(G, \mathbf{c}, p)$ is the same as the rank of the matrix formed by deleting the columns corresponding to the vertices v and w.

Example 2.10. Let us consider the triangle graph where all edges have the same color and the same graph where there are edges in two colors (see Figure 3). Then $R(K_3, c, p)$ and $R(K_3, c', p)$ are given

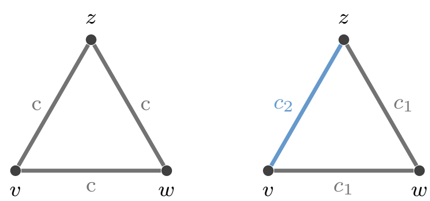


Figure 3: Colored triangles $(K_3, \mathbf{c})$ and $(K_3, \mathbf{c}')$.

by

$$\begin{pmatrix} X_{wv} & Y_{wv} & -X_{wv} & -Y_{wv} & 0 & 0 & -X_{wv}^2 - Y_{wv}^2 \\ X_{wz} & Y_{wz} & 0 & 0 & -X_{wz} & -Y_{wz} & -X_{wz}^2 - Y_{wz}^2 \\ 0 & 0 & X_{vz} & Y_{vz} & -X_{vz} & -Y_{vz} & -X_{vz}^2 - Y_{vz}^2 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} X_{wv} & Y_{wv} & -X_{wv} & -Y_{wv} & 0 & 0 & -X_{wv}^2 - Y_{wv}^2 & 0 \\ X_{wz} & Y_{wz} & 0 & 0 & -X_{wz} & -Y_{wz} & -X_{wz}^2 - Y_{wz}^2 & 0 \\ 0 & 0 & X_{vz} & Y_{vz} & -X_{vz} & -Y_{vz} & 0 & -X_{vz}^2 - Y_{vz}^2 \end{pmatrix}, \quad \text{respectively.}$$

2.2 Three types of angle-rigidity

In this section we define rigidity properties for angle frameworks and colored graphs.

Definition 2.11. Let $(G, \mathbf{c}, p)$ be an angle framework. We define three types of angle-rigidity in analogy with standard notions of rigidity, see for example, [GSS93, Whi96]:

(i) $(G, \mathbf{c}, p)$ is *locally angle-rigid* if all angle frameworks equivalent and sufficiently close to $(G, \mathbf{c}, p)$ are similar to $(G, \mathbf{c}, p)$.

(ii) $(G, \mathbf{c}, p)$ is *globally angle-rigid* if all angle frameworks equivalent to $(G, \mathbf{c}, p)$ are similar to $(G, \mathbf{c}, p)$.

(iii) $(G, \mathbf{c}, p)$ is *infinitesimally angle-rigid* if the null space of the angle-rigidity matrix $R(G, \mathbf{c}, p)$ is precisely the space of trivial infinitesimal flexes.

Furthermore, $(G, \mathbf{c}, p)$ is *minimally locally/infinitesimally angle-rigid* if, in addition to being locally/infinitesimally angle-rigid, deleting any edge (and adjusting the coloring accordingly) produces an angle framework that is not locally/infinitesimally angle-rigid.

From the previous section, we see that the following holds.

Theorem 2.12. *Let $(G, \mathbf{c}, p)$ be an angle framework with $|V| \geq 2$. Then $(G, \mathbf{c}, p)$ is infinitesimally angle-rigid if and only if* $\operatorname{rank} R(G, \mathbf{c}, p) = 2|V| + |C| - 4$.

Proof. The matrix $R(G, \mathbf{c}, p)$ is formed from $R(G, c, p^{\perp})$ by applying a column reordering and multiplying some columns by -1, hence $\operatorname{rank} R(G, \mathbf{c}, p) = \operatorname{rank} R(G, c, p^{\perp})$. The result now follows from Lemma 2.8. □

Proposition 2.13. *For a fixed angle framework $(G, \mathbf{c}, p)$, both infinitesimal and global angle-rigidity imply local angle-rigidity. No other implication among the notions of angle-rigidity holds.*

Proof. (Infinitesimal angle-rigidity $\implies$ Local angle-rigidity). We first observe that if $(G, \mathbf{c}, p)$ is not *affinely spanning* (i.e., the affine span of $\{p_v : v \in V\}$ is $\mathbb{R}^2$), then $(G, \mathbf{c}, p)$ is either a single vertex or two vertices joined by an edge (in which case the result is obvious) or else $(G, \mathbf{c}, p)$ is not infinitesimally angle-rigid.

Hence, suppose that $(G, \mathbf{c}, p)$ is affinely spanning. To "quotient out" the trivial motions of $(G, \mathbf{c}, p)$, we can "pin" two vertices a and b which are joined by an edge to the points p_a and p_b, respectively (it is easy to see that the framework requires at least one edge to be infinitesimally angle-rigid). This corresponds to restricting the domain of the map θ_A (where A is any set of angles with corresponding edge coloring c) to the set of realizations with a at p_a and b at p_b. Since $(G, \mathbf{c}, p)$ is infinitesimally angle-rigid, it follows from Lemmas 2.6 and 2.8 that the Jacobian of our domain-restriction for θ_A at p has rank $2|V| - 4$ (since we have now removed the trivial infinitesimal flexes), which is maximal. By applying the constant rank theorem to our new map, we see that any equivalent framework (G, c, q) with $q_a = p_a$ and $q_b = p_b$ that is sufficiently close to $(G, \mathbf{c}, p)$ is exactly $(G, \mathbf{c}, p)$, which gives the desired result.

(Global angle-rigidity $\implies$ Local angle-rigidity). By definition.

(Infinitesimal angle-rigidity $\not\Rightarrow$ Global angle-rigidity and Local angle-rigidity $\not\Rightarrow$ Global angle-rigidity). We present the following angle framework which is infinitesimally angle-rigid (thus locally angle-rigid) but not globally angle-rigid. The graph G is K_4, the color map assigns the edges $\{(vw), (vb), (wa), (ab)\}$ to color c_1, and $\{(va), (wb)\}$ to color c_2, and the embedding is given by

$$p_v = (0, 0), \quad p_w = (1, 0), \quad p_a = \left(1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right), \quad p_b = \left(\frac{1 + \sqrt{2}}{2}, \frac{1 + \sqrt{2}}{2\sqrt{3}}\right).$$

Computing the angle-rigidity matrix explicitly demonstrates that $(G, \mathbf{c}, p)$ is infinitesimally angle-rigid. However, there is another realization (Figure 4) with the same angles:

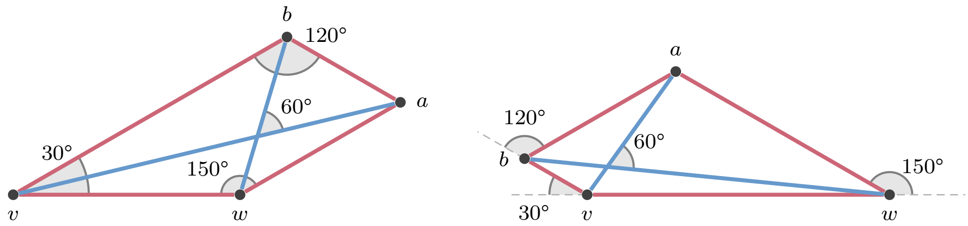


Figure 4: An angle framework with two different realizations.

$$p_v = (0,0), \quad p_w = (1,0), \quad p_a = \left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right), \quad p_b = \left(\frac{1-\sqrt{2}}{2}, \frac{\sqrt{2}-1}{2\sqrt{3}}\right).$$

Direct computation of the angle between 12 and 24 shows that the realizations are dissimilar. Note that our definition of similarity considers two angles to be the same as long as the pairs of defining lines intersect in the same set of four angles. These realizations were obtained by computing the defining system of polynomial equations, and solving explicitly for the vertex coordinates after fixing some angles.

(Global angle-rigidity $\not\Rightarrow$ Infinitesimal angle-rigidity and Local angle-rigidity $\not\Rightarrow$ Infinitesimal angle-rigidity). We present an angle framework which is globally angle-rigid (thus locally angle-rigid) but not infinitesimally angle-rigid. Again we set $G = K_4$. This time, the edges $\{(vw), (va), (vb), (wa)\}$ have color c_1 and $\{(wb), (ab)\}$ have c_2. The embedding into $\mathbb{R}^2$ (Figure 5) is given by

$$p_v = (0,0), \quad p_w = (1,0), \quad p_a = (2,1), \quad p_b = (0,1).$$

Plugging these values into the angle-rigidity matrix, we see that the null space has something extra that corresponds to the infinitesimal vertical motion of p_b. This implies that the angle framework is *not* infinitesimally angle-rigid.

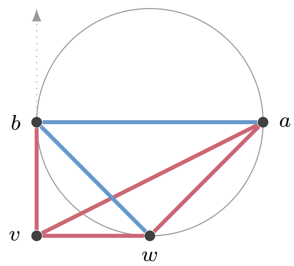


Figure 5: An infinitesimal vertical motion of a globally angle-rigid angle framework.

Any configuration of the framework has a representative with $p_v = (0,0)$, $p_w = (1,0)$ up to the group action. Take any embedding realizing $(G, \mathbf{c})$ with the same angles. Then, it has a curve of representatives keeping p_v and p_w constant. The angles between $(vw), (va)$, and (wa) would then determine the location of a to be (w, v). Then p_b must be on the y-axis based on the angle $\angle wvb$; it must also be on the black circle so that the angle $\angle wba$ is fixed. The circle and line intersect in only one point, so the graph is globally angle-rigid. □

Having defined infinitesimal rigidity of angle-frameworks, we now connect it to infinitesimal rigidity of underlying bar-joint frameworks. Fix coker M to be the cokernel (also known as the left kernel or left null space) of the matrix M. We say that $(G, \mathbf{c}, p)$ is *independent* if coker $R(G, \mathbf{c}, p) = \{0\}$. An element of coker $R(G, \mathbf{c}, p)$ is called an *equilibrium stress* of $(G, \mathbf{c}, p)$. Hence, $(G, \mathbf{c}, p)$ is independent if and only if it has no non-zero equilibrium stresses. The structure of $R(G, \mathbf{c}, p)$ implies that $\omega \in \mathbb{R}^E$ is an equilibrium stress of $(G, \mathbf{c}, p)$ if and only if it is an equilibrium stress of (G, p) and for every color c we have

$$\sum_{vw \in E_c} \omega_{vw} \|p_v - p_w\|^2 = 0.$$

Proposition 2.14. *Let $(G, \mathbf{c}, p)$ be an angle framework. For $c_i \in C$, an equilibrium stress of (G_{c_i}, p) induces an equilibrium stress of $(G, \mathbf{c}, p)$. If $(G, \mathbf{c}, p)$ is independent, then each bar-joint framework (G_{c_j}, p) is independent for each color c_j.*

Proof. Let $(G, \mathbf{c}, p)$ be an angle framework and fix a color c_i. Let ω be an equilibrium stress of the bar-joint framework (G_{c_i}, p). Define $\tilde{\omega} \in \mathbb{R}^E$ by $\tilde{\omega}_{vw} = \omega_{vw}$ for all $vw \in E_{c_i}$ and $\tilde{\omega}_{vw} = 0$ otherwise. As ω is an equilibrium stress of (G_{c_i}, p), we have that $\sum_{vw \in E_{c_i}} \omega_{vw} \|p_v - p_w\|^2 = 0$, see [Con82]. Thus, $\tilde{\omega}$ lies in the cokernel of $R(G, \mathbf{c}, p)$. Since any equilibrium stress of (G_{c_j}, p) would induce an equilibrium stress of $(G, \mathbf{c}, p)$, independence of $(G, \mathbf{c}, p)$ forces independence of (G_{c_j}, p). □

Corollary 2.15. *Let $(G, \mathbf{c}, p)$ be a monochromatic angle framework. Then $(G, \mathbf{c}, p)$ is infinitesimally angle-rigid (resp. independent) if and only if the bar-joint framework (G, p) is infinitesimally rigid (resp. independent).*

Proof. It follows from Proposition 2.14 that $\omega \in \mathbb{R}^E$ is an equilibrium stress of $(G, \mathbf{c}, p)$ if and only if it is an equilibrium stress of (G, p). Hence, $\operatorname{rank} R(G, \mathbf{c}, p) = \operatorname{rank} R(G, p)$. □

2.3 Genericity and angle-rigid graphs

Figure 5 gives an example of a locally angle-rigid (in fact, globally angle-rigid) graph which is not infinitesimally angle-rigid, however it relies on a very specific geometric coincidence. In particular, the tangent line to the unique circle containing points w, a and b contains the edge vb. This coincidence leads us to define an angle-version of the rigidity-theoretic concept of generic rigidity.

Definition 2.16. The colored graph $(G, \mathbf{c})$ is said to be *(minimally) angle-rigid* if there exists a non-empty Zariski open subset $S \subset \mathbb{R}^{2|V|}$ such that for all $p \in S$ the associated angle framework $(G, \mathbf{c}, p)$ is (minimally) locally angle-rigid.

It follows, using Proposition 2.13, that our Zariski open subset S can be chosen such that each $p \in S$ is associated to an infinitesimally angle-rigid angle framework $(G, \mathbf{c}, p)$.

Proposition 2.17. *Suppose there exists an embedding p such that $(G, \mathbf{c}, p)$ is (minimally) infinitesimally angle-rigid. Then the colored graph $(G, \mathbf{c})$ is (minimally) angle-rigid.*

Proof. If the rank of $R(G, \mathbf{c}, p)$ is $2|V| + |C| - 4$, then there is a nonzero $(2|V| + |C| - 4)$-minor determinant. As a function of the parameters p, this minor is a polynomial function not identically equal to zero. Thus, taking the ideal of all non-zero $(2|V| + |C| - 4)$-minor determinants yields a nonzero ideal defining a Zariski-closed subset of parameter space. The angle framework $(G, \mathbf{c}, p)$ is (minimally) infinitesimally angle-rigid for any p in the complement of that subset. □

3 A necessary condition for minimal angle-rigidity

In this section we prove necessary conditions for minimal angle-rigidity. First, using basic linear algebra and dimension counting techniques, one can derive the following Maxwell-type necessary condition. We omit the proof.

Lemma 3.1. *Let $(G, \mathbf{c})$ be minimally angle-rigid. Then for each subgraph $H \subseteq G$, the following inequality holds:*

$$|E(H)| \leq 2|V(H)| + \chi(H) - 4,$$

where $\chi(H)$ is the number of colors among the edges of H.

We next compare the necessary condition described in Lemma 3.1 to two related combinatorial statements. Throughout the remainder of the section we fix $\mathcal{R}_2(V)$ to be the restriction of the rigidity matroid to the complete graph with vertex set V.

Proposition 3.2. *Let $(G, \mathbf{c})$ be a colored graph with $G = (V, E)$. Consider the following three conditions:*

(i) There exists a set $F = \{e_1, \ldots, e_{|C|}\}$ where $\mathbf{c}(e_i) = c_i$ for each color c_i, and $(E \setminus F) + e_i$ is a basis of $\mathcal{R}_2(V)$ for each color c_i.

(ii) For each color c_i, there exists a set $F_i = \{e_j\}_{j \neq i} \subseteq E$ where $\mathbf{c}(e_j) = c_j$ for each j such that $E \setminus F_i$ is a basis of $\mathcal{R}_2(V)$.

(iii) For each subgraph $H \subseteq G$, the following inequality holds: $|E(H)| \leq 2|V(H)| + \chi(H) - 4$, where $\chi(H)$ is the number of colors among the edges of H.

Condition (i) implies (ii), which implies (iii). Condition (iii) does not imply (i) or (ii), and (ii) does not imply (i).

Proof. (i) $\Rightarrow$ (ii). For each i we simply set $F_i = F \setminus e_i$. The sets $F_1, \ldots, F_{|C|}$ satisfy the desired conditions by the definition of the set F.
(ii) $\Rightarrow$ (iii). Consider a subgraph $H \subseteq G$. Suppose, without loss of generality, that c_1 is among the colors on the edges of H. Let $H_1 = H \cap (E \setminus F_1)$ and $H_2 = H \cap F_1$. As $E \setminus F_1$ is a basis in $\mathcal{R}_2$, we have that $|E(H_1)| \leq 2|V(H_1)| - 3$. Since $|E(H_2)| \leq \chi(H) - 1$, this implies

$$|E(H)| = |E(H_1)| + |E(H_2)| \leq 2|V(H_1)| - 3 + (\chi(H) - 1) \leq 2|V(H)| + \chi(H) - 4.$$

(iii) $\not\Rightarrow$ (i) or (ii). Consider the left-hand graph in Figure 6. One may check directly that condition (iii) is satisfied. Condition (ii), on the other hand, is not satisfied: If (ii) held then there would be a red edge that could be omitted leaving behind a basis of $\mathcal{R}_2(V)$. However, removing either red edge leaves a K_4 in the graph. If condition (ii) fails, then condition (i) must also fail.

(ii) $\not\Rightarrow$ (i). Consider the right-hand graph in Figure 6. Condition (ii) is satisfied since $E \setminus (e_1 \cup e_2^*)$, $E \setminus (e_1 \cup e_3^*)$ and $E \setminus (e_2 \cup e_3^*)$ are bases of $\mathcal{R}_2(V)$. We claim that Condition (i) fails. Suppose such an $F = \{x, y, z\}$ exists. If F contains an edge not contained in one of the two copies of K_4, say the edge x, then $(E \setminus F) + y = E \setminus \{x, z\}$ leaves some K_4 intact, which means it cannot be a basis of $\mathcal{R}_2(V)$. We conclude that every edge of F must be contained in some K_4. If F has all three edges in the two K_4's, one of the copies of K_4 must contain two edges while the other contains one, say x. This means that $(E \setminus F) + x$ contains all the edges of one of the copies of K_4, so also fails to be a basis of $\mathcal{R}_2(V)$. □

The graph on the left of Figure 6 is not minimally angle-rigid (see Section 4 for more details). Hence, property (iii) of Proposition 3.2 (see Lemma 3.1) is a necessary but not sufficient condition for minimal angle-rigidity. The graph on the right of Figure 6 is minimally angle-rigid but does not satisfy property (i) of Proposition 3.2. Hence, property (i) of Proposition 3.2 is not a necessary condition for minimal angle-rigidity. We discuss the potential sufficiency of (ii) of Proposition 3.2 in Theorem 3.5. However, we do have positive results in some specific cases. In Section 4 we use two simple extension

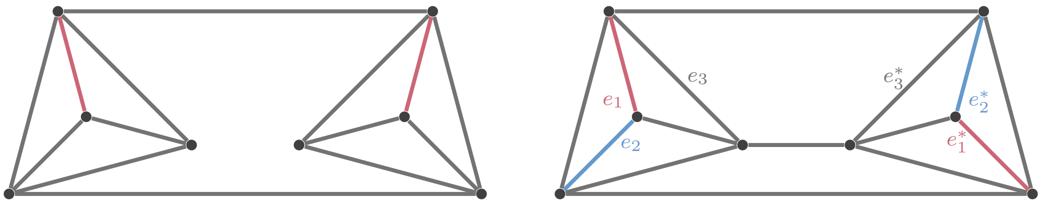


Figure 6: Counterexample graphs.

operations to generate rigid angle frameworks and show that (ii) of Proposition 3.2 is sufficient (and hence provide a combinatorial characterization) when there are only 2 color classes.

We now give a sharper result showing that property (ii) of Proposition 3.2 is necessary. To this end let $(G, \mathbf{c})$ be a colored graph, and let $c_j \in C$ be a color. Denote by $T_{\mathbf{c},j}(G)$, where the ground set is $E(G)$ and the bases are sets of $|C|-1$ elements, where for each color $c_i \neq c_j$ there exists exactly one element e_i where $\mathbf{c}(e_i) = c_i$. The next lemma, whose proof is simply unpacking the definitions, reformulates property (ii) of Proposition 3.2 in terms of transversal.

Lemma 3.3. *Let $G = (V, E)$ and let $(G, \mathbf{c})$ be a colored graph. Then for each color c_i, the following are equivalent:*

(i) there exists a set $F_i = \{e_j\}_{j \neq i} \subseteq E$ where $\mathbf{c}(e_j) = c_j$ for each j such that $E \setminus F_i$ is a basis of $\mathcal{R}_2(V)$;

(ii) there exists a transversal $X \in T_{\mathbf{c},i}(G)$ and a minimally rigid graph H such that $G = H + X$ and $X \cap E(H) = \emptyset$.

Lemma 3.4. *Suppose $(G, \mathbf{c})$ is minimally angle-rigid. Then the submatrix of $R(G, \mathbf{c}, p)$ excluding the column for any color c_j has rank $2|V| + |C| - 4$.*

Proof. Take p generic. Since $(G, \mathbf{c})$ is minimally angle-rigid, $R(G, \mathbf{c}, p)$ has $2|V| + |C| - 4$ rows and $2|V| + |C|$ columns and has rank $2|V| + |C| - 4$. In particular, this implies that it has a 4-dimensional kernel as in Lemma 2.8. Only the vector $(u^p, \mathbf{1})$ is supported on the color columns, and it has full support.

Now consider $R_{\hat{i}}(G, \mathbf{c}, p)$, the $(2|V| + |C| - 4) \times (2|V| + |C| - 1)$ matrix obtained by dropping one color column. Obviously it retains the three first kernel vectors of Lemma 2.8, namely $(u^{(1,0)}, \mathbf{0})$, $(u^{(0,1)}, \mathbf{0})$, and $(u^{p^\perp}, \mathbf{0})$. Let $\mathbf{v}$ be in $\ker R_{\hat{i}}(G, \mathbf{c}, p)$. The vector $(\mathbf{v}, 0)$ must then be in $\ker R(G, \mathbf{c}, p)$, which implies it is in the span of $(u^{(1,0)}, \mathbf{0})$, $(u^{(0,1)}, \mathbf{0})$, $(u^{p^\perp}, \mathbf{0})$, and $(u^p, \mathbf{1})$. The zero in the final coordinate implies that the final vector has coefficient zero in the expansion of $(\mathbf{v}, 0)$, but this in turn implies that $\mathbf{v}$ is in the span of $(u^{(1,0)}, \mathbf{0})$, $(u^{(0,1)}, \mathbf{0})$, and $(u^{p^\perp}, \mathbf{0})$. Thus the kernel of $R_{\hat{i}}(G, \mathbf{c}, p)$ had dimension precisely 3. Thus its rank is also $2|V| + |C| - 4$. □

Theorem 3.5. *If a colored graph $(G, \mathbf{c})$ is minimally angle-rigid, then for each color c_j, there exists a transversal $X \in T_{\mathbf{c},j}(G)$ and a minimally rigid graph H such that $G = H + X$ and $X \cap E(H) = \emptyset$.*

Proof. For a generic angle framework $(G, \mathbf{c}, p)$, fix the linear map

$$\Phi : \ker R(G, p)^T \to \mathbb{R}^c, \ \omega \mapsto \left(\sum_{vw \in E_{c_i}} \omega_{vw} \|p_v - p_w\|^2 \right)_{c_i \in C}.$$

We now calculate the rank of Φ. The graph G has $2|V| + |C| - 4$ edges and a rigid subgraph, so $R(G, p)$ has $2|V| + |C| - 4$ rows and rank $2|V| - 3$. Rank-nullity thus implies that $\ker R(G, p)^T$ has dimension $|C|-1$. Note that $\omega \in \ker \Phi$ if and only if it is an equilibrium stress of $(G, \mathbf{c}, p)$; minimal angle-rigidity of $(G, \mathbf{c}, p)$ thus implies Φ is injective. Therefore, $\operatorname{rank} \Phi = |C|-1$.

We can actually specify $\operatorname{im} \Phi$ explicitly. Since $\sum_{vw \in E} \omega_{vw} \|p_v - p_w\|^2 = 0$ (see, for example, [Con82]), every point in the image of Φ is orthogonal to the all-ones vector. Since the rank is $|C|-1$, we conclude that Φ surjects onto the orthogonal complement of the all-ones vector. In particular, there exists $\omega_i \in \ker R(G,p)^T$ such that $\Phi(\omega_i) = \mathbf{u}_i - \mathbf{u}_j$, where $\mathbf{u}_i$ is the vector with entry i equal to 1 and all others equal to 0. Moreover, for each $i \neq j$ we can pick $e_i \in E_{c_i}$ so that $\omega_i(e_i) \neq 0$. With this, set $X = \{e_i : i \neq j\}$.

Order the rows of $R_{\hat{j}}(G, \mathbf{c}, p)$ so that rows corresponding to X are at the top with row e_s above row e_t if $s < t$. We now apply the following matrix operations to $R_{\hat{j}}(G, \mathbf{c}, p)$ for each $i \neq j$ in turn: (i) multiply row e_i by $\omega_i(e_i)$, (ii) add $\omega_i(e)$ times row e to row e_i for each $e \in E$. These two steps cancel out the entries in $R(G,p)$ in rows corresponding to X, since $\omega \in \ker R(G,p)^T$. As for the color columns in the rows of X, our choice of ω_i mapping to $\mathbf{u}_i - \mathbf{u}_j$ forces this to output a 1 in the column corresponding to color c_i and zero elsewhere (since color c_j has been dropped). We end up with the following matrix:

$$M = \begin{bmatrix} \mathbf{0}_{|F| \times 2|V|} & I_{c-1} \\ R(G-X, p) & A \end{bmatrix}$$

for some $(|E \setminus F|) \times (c-1)$ matrix A. As $\operatorname{rank} R_{\hat{j}}(G, \mathbf{c}, p) = \operatorname{rank} R(G, \mathbf{c}, p)$ (Lemma 3.4) and $\operatorname{rank} M = \operatorname{rank} R_{\hat{j}}(G, \mathbf{c}, p)$, we have $\operatorname{rank} R(G-X, p) = \operatorname{rank} M - \operatorname{rank} I_{c-1} = 2|V|-3$. Thus $(G-X, p)$ is infinitesimally rigid. □

We conclude by setting forth the conjecture that the matroid takes the form specified by the second property in Proposition 3.2, i.e., that the converse of Theorem 3.5 holds.

Conjecture 3.6. *A colored graph $(G, \mathbf{c})$ is minimally angle-rigid if and only if for each color c_j, there exists a transversal edge set $X \in T_{\mathbf{c},j}(G)$ and a minimally rigid graph H such that $G = H \cup X$ and $X \cap E(H) = \emptyset$.*

4 Extension moves for angle-rigid graphs

Given a graph $G = (V, E)$, a *0-extension* creates a new graph G' which is obtained from G by adding one new vertex w and 2 new edges both incident to w. A *1-extension* creates a new graph G' by deleting an edge xy from E and adding a new vertex w and 3 new edges all incident to w including the edges wx, wy.

We generalize these operations to the colored case to form a new colored graph $(G', \mathbf{c}')$ from $(G, \mathbf{c})$. In the 0-extension case the only constraint is that the two new edges use colors from the color set of the original graph. For the 1-extension the three new edges use colors from the color set of the original graph but we need an extra constraint. Specifically, we say that a 1-extension is *color-preserving* if either $\mathbf{c}'(wx) = \mathbf{c}(xy)$ or $\mathbf{c}'(wy) = \mathbf{c}(xy)$.

Lemma 4.1. *Let $(G', \mathbf{c}')$ be formed from $(G, \mathbf{c})$ by a 0-extension. Then $(G, \mathbf{c})$ is independent if and only if $(G', \mathbf{c}')$ is independent.*

Proof. Let w be the new vertex in G' that is adjacent to $x, y \in V$. Choose a generic realization p' of G' and define p to be the realization of G with $p_v = p'_v$ for all $v \in V$. Let A be the 2×2 non-singular matrix with rows $(p_w - p_x)^T$ and $(p_w - p_y)^T$ and O be the $|E| \times 2$ all zeroes matrix. Then there exists a $2 \times (2|V| + |C|)$ matrix B so that

$$R(G',c',p') = \begin{pmatrix} O & R(G,\mathbf{c},p) \\ A & B \end{pmatrix}.$$

Hence, rank $R(G',c',p') = \text{rank}\, R(G,\mathbf{c},p) + 2$ as required. □

The same holds for color-preserving 1-extensions.

Lemma 4.2. *Let $(G',\mathbf{c}')$ be formed from $(G,\mathbf{c})$ by a color-preserving 1-extension. If $(G,\mathbf{c})$ is independent then (G',c') is independent.*

Proof. Suppose $(G',\mathbf{c}')$ is formed from $(G,\mathbf{c})$ by a 1-extension that removes an edge xy, adds a new vertex w and adds the edges wx, wy, wz for some other vertex z. Further suppose that $\mathbf{c}'(wx) = \mathbf{c}(xy)$. Choose a generic realization p of $(G,\mathbf{c})$ and define for each $t \in \mathbb{R} \setminus \{0,1\}$ the realization p^t of (G',c') where $p^t_v = p_v$ for all $v \in V$ and $p^t_w = tp_x + (1-t)p_y$.

Define M'_t to be the matrix formed from $R(G',c',p^t)$ by adding $(1-t)/t$ times row wy to row wx. The only rows with non-zero entries in the w columns are row wy (with $p^t_w - p^t_y$) and row wz (with $p^t_w - p^t_z$); this is because in row wx and column w we have

$$P^t_{wx} + ((1-t)/t)P^t_{wy} = (1-t)P_{yx} + (1-t)P_{xy} = 0$$

Now define M_t to be the matrix formed from M'_t by deleting rows wy and wz and the columns corresponding to w, and then multiplying row wx by $(1-t)^{-1}$. As P^t_{wy} and P^t_{wz} are linearly independent, we have rank $M_t = \text{rank}(G',\mathbf{c}',p^t) - 2$.

Suppose $\mathbf{c}'(wx) = \mathbf{c}'(wy) = \mathbf{c}(xy)$. The row wx of M_t is of the form

$$\big(\overbrace{P_{xy}}^{x} \quad \dots \quad \overbrace{-P_{xy}}^{y} \quad \dots \quad \overbrace{-P_{xy}\cdot P_{xy}}^{\mathbf{c}(xy)} \quad \big).$$

Hence, $M_t = R(G,\mathbf{c},p)$ for all values of t, and so rank $R(G',\mathbf{c}',p^{1/2}) = \text{rank}\, R(G,\mathbf{c},p)+2$ as required. Now suppose $\mathbf{c}'(wx) \neq \mathbf{c}'(wy)$. The row wx of M_t is of the form

$$\big(\overbrace{(1-t)P_{xy}}^{x} \quad \dots \quad \overbrace{-(1-t)P_{xy}}^{y} \quad \dots \quad \overbrace{-(1-t)P_{xy}\cdot P_{xy}}^{\mathbf{c}(xy)} \quad \overbrace{-tP_{xy}\cdot P_{xy}}^{\mathbf{c}'(wy)} \big).$$

We now note that $\lim_{t\to 0} M_t = R(G,\mathbf{c},p)$, hence for sufficiently small t we get rank $R(G',c',p^t) = \text{rank}\, R(G,\mathbf{c},p) + 2$ as required. □

We next show that in the special case when $|C| = 2$ they are enough to derive a complete characterization. To this end we need the following basic lemma.

Lemma 4.3. *Let (K_4,c) be a colored graph with $|C| \geq 2$. Then, there exists an independent angle framework (K_4,c,p).*

Proof. There are 5 non-isomorphic bichromatic graphs (K_4,c), see Figure 7, which we can easily check are independent by choosing random realizations.

Suppose the result holds for any coloring of K_4 with at least 2 and at most $k \geq 2$ colors, and suppose (K_4,c) is a colored graph with $|C| = k+1$. Let c' be the k-coloring of K_4 formed by setting every edge of color c_{k+1} to be of color c_k. The result now follows as (K_4,c') is independent and rank $R(K_4,c',p) \leq \text{rank}\, R(K_4,c,p)$ for any realization p of K_4. □

For the next result, if $G = (V,E)$ is a graph and E is a circuit in $\mathcal{R}_2$ then we say that G is an *$\mathcal{R}_2$-circuit.*

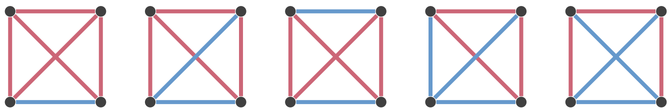


Figure 7: The 5 non-isomorphic bichromatic colorings of K_4.

Theorem 4.4. *Let $(G, \mathbf{c})$ be a colored graph with $|C| = 2$. Then the following are equivalent:*

(i) $(G, \mathbf{c})$ *is minimally angle-rigid.*

(ii) $|E| = 2|V| - 2$, *and G contains a unique $\mathcal{R}_2$-circuit that contains edges of both colors.*

(iii) $(G, \mathbf{c})$ *can be constructed from a bichromatic copy of K_4 by a sequence of 0-extensions and color-preserving 1-extensions.*

In the proof, we need the following: Let $G = (V, E)$ be a graph and, for $X \subset V$, let $i_G(X)$ denote the number of edges in the subgraph of G induced by X. We say that G is *Laman* if $i_G(X) \leq 2|X| - 3$ for all X with $|X| \geq 2$ and $i_G(V) = 2|V| - 3$. Also $X \subset V$ is *critical* if $i_G(X) = 2|X| - 3$. We also use $d(X, Y)$ to denote the number of edges in G of the form xy with $x \in X, y \in Y$ for disjoint sets $X, Y \subset V$.

Recall that [PG27, Lam70] showed that G is minimally rigid (as a generic bar-joint framework) in the plane if and only if G is Laman. Hence, all $\mathcal{R}_2$-circuits are rigid. It is now immediate that condition (ii) implies that G is rigid as a bar-joint framework. An elementary property of such rigid graphs is that they are 2-edge-connected and any 2-edge-separation has one component of size 1 (i. e. the separation simply separates a degree 2 vertex from the rest of the graph). The condition is also checkable efficiently by the pebble game algorithm [JH97].

Lemma 4.5 ([PG27, Lam70])**.** *Let $G = (V, E)$ be a Laman graph and let $v \in V$ be a vertex of degree 3 in G. Then there exists a pair $\{x, y\} \subset N(v)$ such that $G - v + xy$ is a Laman graph.*

It is convenient to introduce the terms *0-reduction* and *color-preserving 1-reduction* for the colored graph operations that 'undo' a 0-extension and a color preserving 1-extension respectively.

Proof of Theorem 4.4. (i) $\Rightarrow$ (ii): This follows from Theorem 3.5.

(ii) $\Rightarrow$ (iii): Let $\mathcal{D}$ be the set of all colored graphs that satisfy (ii). It suffices to show that any colored graph with at least 5 vertices in $\mathcal{D}$ can be reduced to another graph in $\mathcal{D}$ by a 0-reduction or color-preserving 1-reduction. Choose any colored graph $(G, \mathbf{c}) \in \mathcal{D}$ with $|V| \geq 5$. Since $|E| = 2|V| - 2$, G has a vertex of degree less than 4 and since G contains a unique $\mathcal{R}_2$-circuit it has no vertex of degree less than 2. If $(G, \mathbf{c})$ has a vertex v of degree 2 then we can apply a 0-reduction to that vertex. (Note that if $G - v$ was monochromatic then it would contain an $\mathcal{R}_2$-circuit which violates condition (ii) and this circuit would also be contained in G.) Hence, we may assume that the minimum degree in G is 3.

Since $|E| = 2|V| - 2$ and the minimum degree is 3, G contains at least four vertices of degree 3. Let H be the unique $\mathcal{R}_2$-circuit and suppose that there is a vertex v with degree 3 that is not in H. If v has three neighbors in H, then for any $e \in H$, $(H - e) \cup v$ fails the Laman count; this would imply it contains a circuit H' distinct from H, contrary to assumption. Thus, v has at most two neighbors in H. Delete from G an edge e of H (not incident to both of the two neighbors) and we have a minimally rigid graph. Hence, by Lemma 4.5 there is a 1-reduction on v to a smaller minimally rigid graph with any color on the new edge. Moreover the vertex set of $H - e$ is a critical set so the new edge has at most one end-vertex in H. We can now re-add e (with its color) to get the same unique $\mathcal{R}_2$-circuit.

If all degree 3 vertices are in H, we may assume $H \neq K_4$ (otherwise, G would be disconnected). Since $|V(H)| \geq 5$, we may delete xy from $H - v$ such that $|\{x, y\} \cap N(v)| \leq 1$. Then $G - xy$ is a minimally rigid graph and Lemma 4.5 implies there exists a 1-reduction deleting v in $G - xy$, and adding ab (with color prescribed by the color-preserving reduction) for some $a, b \in N(v)$, that results in a smaller minimally rigid graph. Now add xy back with its original color to obtain G'. The graph G' contains a unique $\mathcal{R}_2$-circuit H'.

It remains to show that H' contains edges of both colors. If the 3 edges incident to v have the same color this is trivial. Suppose v is incident to two blue edges av, bv and one red edge cv. The conclusion is also trivial if either ac or bc is added in the 1-reduction. Hence, we may suppose that ab is added, this forces us to color ab blue. The conclusion fails if and only if ac was the unique red edge of $(G, \mathbf{c})$. However, there are at least 4 vertices of degree 3 in H. So we may choose $u \neq v$ in H of degree 3 and repeat the argument. Clearly, at the final step, there is more than one red edge in $(G, \mathbf{c})$.

(iii) ⇒ (i): This follows from Lemmas 4.1 to 4.3. □

We expect that it may be possible to use similar techniques (though at least one additional operation is certainly needed) to resolve the case when $|C| = 3$. However, in general, different techniques seem to be needed. The 2-color case has a nice application for bar-and-joint frameworks.

Corollary 4.6. *Let (G, p) be a generic framework in $\mathbb{R}^2$ such that G is an $\mathcal{R}_2$-circuit. Let ω be an equilibrium stress of (G, p). Then, for a non-empty $F \subseteq E$, we have*

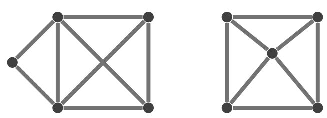
$$\sum_{vw \in F} \omega_{vw} \|p_v - p_w\|^2 = 0 \qquad \textit{if and only if} \qquad F = E.$$

Proof. Choose any proper subset $F \subset E$. Define $\mathbf{c}$ to be the coloring of G, where $\mathbf{c}(e) = c_1$ if $e \in F$ and $\mathbf{c}(e) = c_2$ if $e \in E \setminus F$. By Theorem 4.4, $(G, \mathbf{c}, p)$ is minimally infinitesimally angle-rigid. Hence, $\sum_{vw \in F} \omega_{vw} \|p_v - p_w\|^2 \neq 0$ as required. □

4.1 Computational results

Theorem 4.4 gives a tool to construct all 2-colored graphs $(G, \mathbf{c})$ that are minimally angle-rigid. Our other tool to test a given $(G, \mathbf{c})$ for angle-rigidity is to apply Proposition 2.17 to a random realization p. In the case when $(G, \mathbf{c}, p)$ is infinitesimally angle-rigid we know $(G, \mathbf{c})$ is also angle-rigid. On the other hand, if $(G, \mathbf{c}, p)$ is not infinitesimally angle-rigid, $(G, \mathbf{c})$ can only be assumed to be non-angle-rigid, if we assume our random realization was sufficiently generic. Using these two tools we can analyze the angle-rigid colored graphs computationally.

Let G be a graph with $|E| = 2|V| - 2$. We say that G is *2-color-rigid* if there is a color map $\mathbf{c}$ with $|C| = 2$ such that $(G, \mathbf{c})$ is minimally angle-rigid. For $|V| = 5$ we have two 2-color-rigid graphs (G_1 and G_2 from Figure 8). G_1 admits 45 different 2-color maps (up to isomorphism) such that $(G_1, \mathbf{c})$ is angle-rigid. G_2 has 26 such maps. Hence, in total there are 71 non-isomorphic colored graphs[1] $(G, \mathbf{c})$ with $|C| = 2$ that are minimally angle-rigid. See Table 1 for further data.



G_1 G_2

Figure 8: Two graphs with 5 vertices that are 2-color-rigid.

[1] Here we say two colored graphs $(G, \mathbf{c})$ and (G', c') are isomorphic if there exists a graph isomorphism $\phi : G \to G'$ so that $c = c' \circ \phi$.

$\|V\|$	graphs	2-color-rigid	2-colored angle-rigid
4	1	1	5
5	2	2	71
6	12	12	2227
7	97	91	99148
8	1113	1003	
9	17117	14870	

Table 1: Graphs with $|E| = 2|V| - 2$ and minimum degree 2 in the second column and the number of 2-color-rigid graphs in column three. Last column: number of non-isomorphic colored graphs $(G, \mathbf{c})$ with $|C| = 2$ which are minimally angle-rigid.

We have seen in Figure 7 that K_4 has 5 possible color maps in two colors which give a rigid structure. We have computed the number of such color maps for all graphs up to 7 vertices. There is for instance a graph with only 7 vertices that has more than 2000 possible color maps. In Table 2 we show the minimum and maximum number of color maps obtained by graphs with less than 8 vertices.

Similarly to 2-color-rigid graphs we can take more general *k-color-rigid* graphs which have $|E| = 2|V| + k - 4$ and a k-color map $\mathbf{c}$ that gives a minimally angle-rigid structure. Note that, computations are done using Proposition 2.17, hence, they use random realizations and might therefore yield false negatives. Table 3 shows how many k-color-rigid graphs there are with few vertices.

$\|V\|$	minimum 2-color maps	maximum 2-color maps
4	5	5
5	26	45
6	67	304
7	46	2047

Table 2: For all 2-color-rigid graphs we count the number of possible 2-color maps. The minimum and maximum of these numbers for a given $|V|$ is shown in the table.

$\|V\|$	2-color-rigid	3-color-rigid	4-color-rigid
4	1	-	-
5	2	1	1
6	12	8	5
7	91	80	59
8	1003	1168	
9	14870		

Table 3: Number of k-color-rigid graphs with $k \leq 6$ and less than 10 vertices.

5 The Algebraic Angle-Rigidity Matroid

One important toolbox frequently employed in rigidity theory is the theory of matroids. The "independent sets" in the classical rigidity setting can be defined using linear independence among rows of the rigidity matrix, yielding a *linear matroid*. Independence can also be taken as algebraic independence of the distances $\{d_{vw} = X_{vw}^2 + Y_{vw}^2\}$ in the field extension $\mathbb{C}(x_v, y_v)$; this defines an *algebraic matroid* (see [RST20] for an elementary introduction). Since our angle rigidity matrix indexes its rows by edges as opposed to angles, the linear matroid is not immediately adaptable. Instead, we construct an algebraic matroid.

Let $G = (V, E)$ be a graph with an angle index set A and let $e, f \in E$. Our previously defined angle map $\theta_A : R^G \to \mathbb{R}^A$ has coordinates $\left(\arccos\left(\frac{(p_a - p_b)\cdot(p_c - p_d)}{\|p_a - p_b\|\|p_c - p_d\|}\right)\right)_{\{ab,cd\}\in A}$ which are related only via transcendental functions. In this section, we construct the map $\theta_{ef} \to \exp(2\mathrm{i}\,\theta_{ef})$, obtaining a set of complex numbers which are related algebraically precisely when the original angles satisfy a geometric constraint. This allows us to define the *algebraic angle-rigidity matroid* in terms of algebraic relations.

Definition 5.1. Let $G = (V, E)$ be a graph, with $e, f \in E$. The *algebraic angle-rigidity matroid* $\mathcal{A}(G)$ (over $\mathbb{K}$) is the algebraic matroid on the elements $\alpha_{ef} = X_e Y_f X_f^{-1} Y_e^{-1}$ in the function field $\mathbb{K}(x_1, \ldots, x_n, y_1, \ldots, y_n)$. For A an angle index set of G, let $\alpha_A = \{\alpha_{ef} : \{e, f\} \in A\}$.

The justification of this formula for angles is as follows: Suppose $p_1 = (\tilde{x}_1, \tilde{y}_1), \ldots, p_4 = (\tilde{x}_4, \tilde{y}_4)$ and we want to measure the angle between line segment L_{12} joining p_1 and p_2 and L_{34} joining p_3 and p_4. We perform the following geometric operations:

Translate Segments to the Origin Then, both L_{12} and L_{34} have an endpoint at the origin. After this $p_1' = (\tilde{X}_{12}, \tilde{Y}_{12})$ and $p_3' = (\tilde{X}_{34}, \tilde{Y}_{34})$, while $p_2' = p_4' = 0$.

Rescale to the Unit Circle We apply a complex change of coordinates inspired by [CGG+18]. Send $(\tilde{x}_k, \tilde{y}_k) \mapsto (\tilde{x}_k + \mathrm{i}\tilde{y}_k, \tilde{x} - \mathrm{i}\tilde{y}_k)$ a conjugate pair of complex numbers. Rename these coordinates $x_k = \tilde{x}_k + \mathrm{i}\tilde{y}$ and $y_k = \tilde{x}_k - \mathrm{i}\tilde{y}_k$. This can be extended to an invertible linear map on $\mathbb{C}^2$, so preserves the underlying geometry. Applying this transformation, p_1' and p_3' become

$$p_1'' = (\tilde{X}_{12} + \mathrm{i}\tilde{Y}_{12}, \quad \tilde{X}_{12} - \mathrm{i}\tilde{Y}_{12}) \qquad\qquad p_3'' = (\tilde{X}_{34} + \mathrm{i}\tilde{Y}_{34}, \tilde{X}_{34} - \mathrm{i}\tilde{Y}_{34})$$

or in the new variable names: $p_1'' = (X_{12}, Y_{12})$ and $p_3'' = (X_{34}, Y_{34})$. Convert these coordinates to polar form, keeping in mind that the pairs are complex conjugates:

$$p_1'' = (X_{12}, Y_{12}) = (r_{12}e^{\mathrm{i}\theta_{12}}, r_{12}e^{-\mathrm{i}\theta_{12}}) \qquad\qquad p_3'' = (X_{34}, Y_{34}) = (r_{34}e^{\mathrm{i}\theta_{34}}, r_{34}e^{-\mathrm{i}\theta_{34}})$$

To drop the magnitude r_{kl}, we can divide the two coordinates. In particular:

$$p_1''' = X_{12}/Y_{12} = e^{2\mathrm{i}\theta_{12}}, \qquad\qquad p_3''' = X_{34}/Y_{34} = e^{2\mathrm{i}\theta_{34}}.$$

The unfortunate side effect is that θ_{kl} gets doubled, but this still specifies θ_{kl} up to π. Now both line segments (with angle doubled) terminate at the unit circle.

Rotate the Unit Circle. We divide everything by $e^{2\mathrm{i}\theta_{34}} = X_{34}/Y_{34}$ so that p_3''' rotates to 1 and p_1''' rotates to

$$e^{2\mathrm{i}(\theta_{12}-\theta_{34})} = X_{12}Y_{34}X_{34}^{-1}Y_{12}^{-1}$$

Thus, we define the angle variable $\alpha_{(12)(34)}$ as the rational function on the right. Note that the traditional angle value in $(0, 2\pi)$ can be computed via $\frac{1}{2\mathrm{i}} \ln a_{(12)(34)}$, using the principal branch of the complex logarithm as claimed above. The field elements themselves are compactly summarized as a Laurent monomial function of linear forms.

Remark 5.2. The angles can also be defined in terms of the slope matroid defined by [Mar03]. Here Martin defines, for each $e = (uv)$, the equation $m_e = Y_e X_e^{-1} = Y_{uv} X_{uv}^{-1}$. In Martin's work, no complex coordinate change is used; instead, he defines slopes in the traditional geometric sense. Still, using that formula, we have: $a_{ef} = m_f/m_e$, where m_e and m_f are elements in the function field $\mathbb{K}(m_e)/\mathcal{S}(G)$, where $\mathcal{S}(G)$ is the ideal of the *slope variety* defined in [Mar03].

In this definition, we do not use the fact that there is a difference between α_{ef} and α_{fe}. This ambiguity is justified by (i) in the next proposition, which observes that the field extension generated by the set of angles is invariant under shuffling the edges in an angle index.

Proposition 5.3. *Let A be an angle index set. The following properties are satisfied by α_A:*

(i) $\alpha_{ef} = (\alpha_{fe})^{-1}$.

(ii) If $\tilde{G}(A)$ has a cycle, then α_A satisfies a nontrivial polynomial relation.

(iii) If $\tilde{G}(A)$ is a tree and α_A satisfies a nontrivial polynomial relation, then so does every A' such that $\tilde{G}(A')$ is connected on the same support.

(iv) Suppose $\tilde{G}(A)$ is a forest with connected components $T_1, T_2, \ldots, T_k$ and α_A satisfies a nontrivial polynomial relation. If A' is another angle index set for which $\tilde{G}(A')$ is a forest, and its connected components $T'_1, \ldots, T'_k$ satsify $E(T_i) = E(T'_i)$, then $\alpha_{A'}$ also satisfies a nontrivial polynomial relation.

Proof. (i) falls directly out of the formula.

For (ii), take the edge sequence of the cycle $e_1, e_2, \ldots e_k$. Observe that multiplying all the $\alpha_{e_1 e_2}, \ldots \alpha_{e_k e_1}$ results in every factor X_{e_i} and Y_{e_i} appearing exactly once each in numerator and denominator; hence, $\alpha_{e_1 e_2}\alpha_{e_2 e_3} \cdots \alpha_{e_k e_1} = 1$.

For (iii), take any angle index $ef \in A \setminus A'$. As the graph $\tilde{G}(A') + ef$ contains a cycle with ef, we can apply (ii) to this cycle to write α_{ef} in terms of angles of A', here using the fact that the product of a cycle equals 1. We now note that if P is a non-trivial polynomial with $P(\alpha_A) = 0$, then replacing each of those in the nontrivial relation $P(\alpha_A) = 0$ yields a nontrivial rational function $q(\alpha_{A'}) = 0$, and clearing denominators yields the result. This method also directly implies (iv). □

Part (ii) of Proposition 5.3 implies that algebraically independent sets α_A must be *acyclic*, (i.e. $\tilde{G}(A)$ has no cycles). In addition, part (iii) and (iv) of Proposition 5.3 leads to the same observation made in Section 2, reframing the problem in terms of graphs with edge colorings.

Corollary 5.4. *The algebraic (in)dependence of an acyclic set of angles A is determined by the graph G and the partition into colors c based on the connected components of $\tilde{G}(A)$.*

It follows from Corollary 5.4 that every set of angles A generates a corresponding colored graph $(G, \mathbf{c})$ by labeling the components of $\tilde{G}(A)$ as $C_1, \ldots, C_k$ and fixing $\mathbf{c}$ to be the map where $\mathbf{c}(e) = c_i$ if and only if $e \in C_i$. This observation allows us to freely change our choice of angles A to another set A' so long as the connected components of $\tilde{G}(A)$ and $\tilde{G}(A')$ share the same vertices. This can simplify computations, as is the case in the next lemma.

Lemma 5.5. *Suppose $\tilde{G}(A)$ a tree, and $f \in E(A)$. Take $m_e = X_e/Y_e$ to be the variable defined in Remark 5.2. Then, $\mathbb{K}(\alpha_A \cup \{m_f\}) = \mathbb{K}(m_e : e \in E(A))$.*

Proof. By Proposition 5.3, A can be taken so that $\tilde{G}(A)$ is a star with central vertex f. Every $\alpha_{ef} = m_e/m_f$, so for every $e \in E(A)$, $\alpha_{ef} m_f = m_e$. □

With these basic properties established, we now demonstrate that algebraic independence among α_A corresponds precisely to minimally angle-rigid graphs.

Theorem 5.6. *An angle index set A with $\tilde{G}(A)$ acyclic defines a basis in the algebraic matroid $\mathcal{A}(G)$ if and only if the corresponding colored graph $(G, \mathbf{c})$ is minimally angle-rigid.*

Proof. For convenience, we denote squared distance $S_e = X_e Y_e$. A set of elements in a field extension $\mathbb{F}/\mathbb{K}$ of characteristic 0 is algebraically independent over $\mathbb{F}$ if and only if the corresponding set of differentials is linearly independent in $\Omega_{\mathbb{F}/\mathbb{K}}$ [Eis13, Theorem 16.14]. We set $\mathbb{F} = \mathbb{K}(x_v, y_v : v \in V)$

with all x, y transcendental over $\mathbb{K}$; note that $\mathbb{F}$ contains $X_{uv} = x_u - x_v$, Y_{uv}, S_{uv}, etc. Let $e = st$ and $f = uv$. Differentiate the defining equations of α_{ef} and m_e to obtain:

$$\mathrm{d}\alpha_{ef} = \mathrm{d}\left(m_e m_f^{-1}\right) = m_f^{-1}\mathrm{d}m_e - m_e m_f^{-2}\mathrm{d}m_f = a_{ef}\left(m_e^{-1}\mathrm{d}m_e - m_{kl}^{-1}\mathrm{d}m_{kl}\right)$$
$$\mathrm{d}m_e = \mathrm{d}\left(X_e Y_e^{-1}\right) = \frac{\mathrm{d}x_s - \mathrm{d}x_t}{Y_e} - \frac{X_e(\mathrm{d}y_s - \mathrm{d}y_t)}{Y_e^2} = \frac{1}{Y_e^2}\Big[Y_e(\mathrm{d}x_s - \mathrm{d}x_s) - X_e(\mathrm{d}y_s - \mathrm{d}y_t)\Big].$$

Without affecting linear independence, we may rescale each $\mathrm{d}\alpha_{ef}$ and instead consider the vectors:

$$v_{ef} := \frac{S_e S_f}{\alpha_{ef}}\mathrm{d}\alpha_{ef} = S_e S_f\left(m_e^{-1}\mathrm{d}m_e - m_f^{-1}\mathrm{d}m_f\right) = S_f Y_e^2 \mathrm{d}m_e - S_e Y_f^2 \mathrm{d}m_f$$
$$= S_f\big[Y_e(\mathrm{d}x_s - \mathrm{d}x_t) - X_e(\mathrm{d}y_s - \mathrm{d}y_t)\big] - S_e\big[Y_f(\mathrm{d}x_u - \mathrm{d}x_v) - X_f(\mathrm{d}y_u - \mathrm{d}y_v)\big].$$

The x, y variables are taken to be algebraically independent, so their differentials $\mathrm{d}x_v, \mathrm{d}y_v$ are linearly independent. Thus we may take these as a basis of a field extension $\mathbb{F}/\mathbb{K}$, and encode the vectors $v_{ef} \in \Omega_{\mathbb{F}/\mathbb{K}}$ as rows of a $(2|V| - 4) \times 2|V|$ matrix $M(A)$. With this in mind, $M(A)$ is of the form below, where all unspecified entries are zero:

$$\begin{array}{c} \\ \vdots \\ v_{ef} \\ \vdots \end{array}\begin{array}{c} \begin{array}{ccccccccccc} \mathrm{d}x_s & \mathrm{d}y_s & \cdots & \mathrm{d}x_t & \mathrm{d}y_t & \cdots & \mathrm{d}x_u & \mathrm{d}y_u & \cdots & \mathrm{d}x_v & \mathrm{d}y_v \end{array} \\ \begin{pmatrix} & & & & & & & & & & \\ S_f Y_e & -S_f X_e & & -S_f Y_e & S_f X_e & & -S_e Y_f & S_e X_f & & S_e Y_f & -S_e X_f \\ & & & & & & & & & & \end{pmatrix} \end{array}.$$

Noting that the coefficients of the entries in $M(A)$ are all rational, we conclude that the $\mathrm{d}\alpha_{ef}$ are linearly independent (and hence α_A is algebraically independent) if and only if $M(A)$ has full rank for any choice of x_v, y_v algebraically independent over $\mathbb{Q}$.

Now fix a set $\{x_v, y_v\}_{v\in V} \subset \mathbb{R}$, algebraically independent over $\mathbb{Q}$, and take $p : V \to \mathbb{R}^2$ to be the realization where $p_i = (x_i, y_i)$. The matrix $M(A)$ can be factored as $M(A) = T_{\widetilde{G}(A)} \cdot R(G, p^\perp)$, where $R(G, p^\perp)$ is the rigidity matrix of the rotated framework $(G, p^\perp)$ and $T_{\widetilde{G}(A)}$ is a $(2|V| - 4) \times (2|V| + |C| - 4)$ block matrix described as follows. Each block corresponds to a star in $\widetilde{G}(A)$. If the star has leaves $\ell_1, \ldots, \ell_r$ and center f, then the corresponding block is:

$$\begin{array}{c} \\ \alpha_{\ell_1,f} \\ \vdots \\ \alpha_{\ell_r,f} \end{array}\begin{array}{c} \begin{array}{cccc} m_{\ell_1} & \cdots & m_{\ell_r} & m_f \end{array} \\ \begin{pmatrix} -S_f & & & S_{\ell_1} \\ & \ddots & & \vdots \\ & & -S_f & S_{\ell_r} \end{pmatrix} \end{array}.$$

Since $(G, \mathbf{c})$ was formed from A, every edge of G is in the subscript of some angle in A, and so the matrix $T_{\widetilde{G}(A)}$ has rank $2|V| - 4$.

We now form one more matrix. Fix J to be the $(2|V| + |C| - 4) \times (2|V| - 4)$ matrix formed from $R(G, c, p^\perp)$ by replacing each entry $-(X_e^2 + Y_e^2)$ in the color column for the edge e with S_e. We observe that

$$T_{\widetilde{G}(A)} J = \left(T_{\widetilde{G}(A)} R(G, p^\perp) \quad \mathbf{0}\right) = \left(M(A) \quad \mathbf{0}\right).$$

Hence, the left kernels of $M(A)$ and $T_{\widetilde{G}(A)} J$ are the same.

We claim that the left nullity of $M(A)$ and J are equal. As $T_{\widetilde{G}(A)}$ has rank $2|V| - 4$, the left nullity of $M(A)$ is at most the left nullity of J. Now choose ω in the left kernel of J. For each color i

we have $\sum_{\mathbf{c}(e)=i} \omega_e S_e = 0$. This is equivalent to the observation that for every star in $\widetilde{G}(A)$ with star center f and leaves $\ell_1, \dots, \ell_r$, we have

$$\omega_f = -\sum_{i=1}^{r} \frac{\omega_{\ell_i}}{S_f} S_{\ell_i}. \tag{3}$$

Define the element $\nu \in \mathbb{K}^A$ where $\nu(a_{\ell_i,f}) = \omega_{\ell_i}/S_f$ for every angle $a_{\ell_i,f}$ where f is a star center in $\widetilde{G}(A)$. It now follows from Equation (3) that $\nu^T T_{\widetilde{G}(A)} = \omega$, hence

$$\nu^T T_{\widetilde{G}(A)} J = \omega^T J = 0.$$

Since no element of the left kernel can be supported only on the star centers of $\widetilde{G}(A)$, the map $\omega \mapsto \nu$ is injective and linear. Therefore, it follows that the left nullity of $M(A)$ is at most the left nullity of J. Hence, the left nullities of $M(A)$ and J are equal.

By manipulating rows and using the linear transform $x = \tilde{x} + \mathrm{i}\tilde{y}$, $y = \tilde{x} - \mathrm{i}\tilde{y}$, it is simple to show that J has the same rank as $R(G, \mathbf{c}, p)$. Hence, $M(A)$ has full rank if and only if $R(G, \mathbf{c}, p)$ has full rank and the result follows from Theorem 2.12. □

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