

STRICTLY SINGULAR OPERATORS ON THE BAERNSTEIN AND SCHREIER SPACES

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ABSTRACT. Every composition of two strictly singular operators is compact on the Baernstein space B_p for $1 < p < \infty$ and on the p -convexified Schreier space S_p for $1 \leq p < \infty$. Furthermore, every subsymmetric basic sequence in B_p (respectively, S_p) is equivalent to the unit vector basis for ℓ_p (respectively, c_0), and the Banach spaces B_p and S_p contain block basic sequences whose closed span is not complemented.

1. INTRODUCTION

This note is a continuation of the line of research we initiated in [12], investigating the Baernstein and p -convexified Schreier spaces, particularly their lattices of closed ideals of bounded operators. Two of the main conclusions of [12] are that on each of these spaces, there are 2^c closed ideals lying between the ideals of compact and strictly singular operators, as well as 2^c closed ideals that contain infinite-rank projections.

These results are counterparts of, and were motivated by, the seminal work [10] of Johnson and Schechtman concerning closed ideals of bounded operators on the Lebesgue spaces $L_p[0, 1]$ for $p \in (1, 2) \cup (2, \infty)$. Johnson and Schechtman [11] went on to show that among all these ideals, only the ideal of compact operators possesses any kind of approximate identity. (Monotonicity of the basis and reflexivity ensure that the sequence of basis projections is a contractive, two-sided approximate identity for the ideal of compact operators on $L_p[0, 1]$.)

This raises the question: Which closed ideals of bounded operators on the Baernstein and Schreier spaces possess approximate identities? We shall answer it for the “small” ideals (those that are contained in the ideal of strictly singular operators), obtaining the same conclusion as Johnson and Schechtman; that is, only the ideal of compact operators does. The reason is that the composition of two strictly singular operators on any Baernstein or p -convexified Schreier space is compact.

Spaces with this property are well known in the literature. Probably the oldest and most famous instances are the continuous functions $C(K)$ on a compact Hausdorff space K and the Lebesgue spaces $L_1(\mu)$ for a σ -finite measure μ . This follows by combining two classical results:

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- $C(K)$ and $L_1(\mu)$ have the Dunford–Pettis property (see for example [1, Theorem 5.4.5] or [8, Theorems VI.7.4 and VI.8.12]); this implies that the composition of two weakly compact operators is compact.
- A bounded operator on $C(K)$ or $L_1(\mu)$ is strictly singular if and only if it is weakly compact (this is due to Pełczyński [15, 16]).

Milman [14] complemented the L_1 -case by showing that the composition of two strictly singular operators on $L_p[0, 1]$ is also compact when $1 < p < \infty$.

Much closer to our investigation, Causey and Pelczar-Barwacz [6, Theorem 7.6, (3)–(4)] have recently studied compositions of strictly singular operators on the higher-order Schreier spaces $X[\mathcal{S}_\xi]$ for every countable ordinal $\xi > 0$, proving that there is a (necessarily unique) natural number k , dependent only on the ordinal ξ , such that:

- every composition of $k + 1$ strictly singular operators on $X[\mathcal{S}_\xi]$ is compact;
- there are k strictly singular operators on $X[\mathcal{S}_\xi]$ whose composition is not compact.

We note that $k = \xi$ when ξ is finite, so in particular $k = 1$ for $X[\mathcal{S}_1]$, which is the space we denote S_1 . As indicated above, our main result extends this conclusion to the p -convexified variants of S_1 , as well as the Baernstein spaces B_p for $1 < p < \infty$. We now state it formally.

Theorem 1.1. *Let T and U be strictly singular operators on either the Baernstein space B_p for some $1 < p < \infty$ or the p -convexified Schreier space S_p for some $1 \leq p < \infty$. Then the composite operator TU is compact.*

Organization and overview of content. In Section 2, we introduce the set-up and main objects that we study, notably giving the precise definitions of the Banach spaces S_p and B_p , as well as the Gasparis–Leung index, which we then use to prove some results about domination and equivalence of subsequences of normalized block basic sequences of the unit vector bases for S_p and B_p .

These results lay the foundations for our proof of Theorem 1.1, which we present in Section 3. In fact, Section 3 contains *two* proofs of it: The first is self-contained, while the other relies on the theory of Schreier spreading basic sequences and equivalence thereof, which is due to Androulakis *et al* [2]. However, it also depends on some of the lemmas that we established in the course of our first proof, so the two proofs are not truly independent.

We conclude with two short sections. In Section 4, we apply one of our results about basic sequences to deduce that the only subsymmetric basic sequences in B_p (respectively, S_p) are those equivalent to the unit vector basis for ℓ_p (respectively, c_0). Finally, in Section 5, we show that B_p and S_p contain block basic sequences whose closed span is not complemented.

2. BLOCK BASIC SEQUENCES IN THE BAERNSTEIN AND SCHREIER SPACES

Before proving our first results, we state the main definitions and conventions regarding notation and terminology, all of which are similar to [12]. In particular, \mathbb{K} denotes the scalar field, either \mathbb{R} or \mathbb{C} , and we use function notation for sequences, so that $x(n)$ is the n^{th} coordinate of a sequence $x \in \mathbb{K}^{\mathbb{N}}$. As usual, $\text{supp } x = \{n \in \mathbb{N} : x(n) \neq 0\}$ for $x \in \mathbb{K}^{\mathbb{N}}$, and c_{00} denotes the subspace of finitely supported elements of $\mathbb{K}^{\mathbb{N}}$. It is spanned by the “unit vector basis” $(e_n)_{n \in \mathbb{N}}$ given by $e_n(m) = 1$ if $m = n$ and $e_n(m) = 0$ otherwise. We

shall sometimes consider two Banach spaces D and E simultaneously, both of which have the unit vector basis as a (Schauder) basis. In such instances, we write $(d_n)_{n \in \mathbb{N}}$ for the unit vector basis of D and $(e_n)_{n \in \mathbb{N}}$ for the unit vector basis of E to keep track of the ambient spaces.

By an *operator*, we understand a bounded linear map between Banach spaces. We write $\mathcal{B}(X, Y)$ for the Banach space of operators from a Banach space X into a Banach space Y , abbreviated $\mathcal{B}(X)$ when $X = Y$; I_X denotes the identity operator on X . An operator $T \in \mathcal{B}(X, Y)$ is *strictly singular* if no restriction of T to an infinite-dimensional subspace of X is an isomorphic embedding.

Our focus is on two classes of Banach spaces: the p -convexified Schreier spaces and the Baernstein spaces. The cornerstone of their definitions is the notion of a *Schreier set*, that is, a finite subset F of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ such that either $F = \emptyset$ or $|F| \leq \min F$, where $|F|$ denotes the cardinality of F . In line with standard practice, \mathcal{S}_1 denotes the family of Schreier sets. A frequently used property of \mathcal{S}_1 is that it is *spreading* in the sense that if $F = \{f_1 < \dots < f_n\}$ belongs to \mathcal{S}_1 and $G = \{g_1 < \dots < g_n\}$ is another subset of \mathbb{N} which satisfies $f_j \leq g_j$ for each $1 \leq j \leq n$, then also $G \in \mathcal{S}_1$.

For $1 \leq p < \infty$ and $F \in \mathcal{S}_1$, we can define a seminorm $\mu_p(\cdot, F)$ on $\mathbb{K}^{\mathbb{N}}$ by

$$\mu_p(x, F) = \begin{cases} 0 & \text{if } F = \emptyset, \\ \left(\sum_{n \in F} |x(n)|^p \right)^{1/p} & \text{otherwise} \end{cases} \quad (x \in \mathbb{K}^{\mathbb{N}}).$$

Set $\|x\|_{S_p} = \sup\{\mu_p(x, F) : F \in \mathcal{S}_1\} \in [0, \infty]$ for $x \in \mathbb{K}^{\mathbb{N}}$ and $Z_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{S_p} < \infty\}$. Standard arguments show that Z_p is a subspace of $\mathbb{K}^{\mathbb{N}}$ and $\|\cdot\|_{S_p}$ a complete norm on it. Furthermore, by [4, Propositions 3.5 and 3.10 and Corollary 3.12], the unit vector basis $(e_n)_{n \in \mathbb{N}}$ is a 1-unconditional, shrinking, normalized basic sequence in Z_p and hence a basis for its closed linear span, which we denote S_p and call the *p -convexified Schreier space*. We remark that S_p is a proper subspace of Z_p ; in fact, the latter is non-separable, as observed in [4, Corollary 5.6].

To analogously define the Baernstein spaces, we require the notion of a *Schreier chain*, which is a non-empty, finite collection \mathcal{C} of non-empty, consecutive Schreier sets; that is, $\mathcal{C} = \{F_1, \dots, F_m\}$ for some $m \in \mathbb{N}$ and Schreier sets $F_1, \dots, F_m \in \mathcal{S}_1 \setminus \{\emptyset\}$ satisfying $F_1 < F_2 < \dots < F_m$, where we use the standard notational convention that $F_j < F_{j+1}$ means $\max F_j < \min F_{j+1}$ for (finite, non-empty) subsets F_j and F_{j+1} of \mathbb{N} .

Given $1 < p < \infty$ and a Schreier chain \mathcal{C} , we can define a seminorm $\beta_p(\cdot, \mathcal{C})$ on $\mathbb{K}^{\mathbb{N}}$ by

$$\beta_p(x, \mathcal{C}) = \left(\sum_{F \in \mathcal{C}} \left(\sum_{n \in F} |x(n)| \right)^p \right)^{1/p} = \left(\sum_{F \in \mathcal{C}} \mu_1(x, F)^p \right)^{1/p} \quad (x \in \mathbb{K}^{\mathbb{N}}).$$

Set $\|x\|_{B_p} = \sup\{\beta_p(x, \mathcal{C}) : \mathcal{C} \in \text{SC}\} \in [0, \infty]$ for $x \in \mathbb{K}^{\mathbb{N}}$, where SC denotes the collection of all Schreier chains. Then $B_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{B_p} < \infty\}$ is a subspace of $\mathbb{K}^{\mathbb{N}}$ and $\|\cdot\|_{B_p}$ a complete norm on it. In contrast to the Schreier spaces, c_{00} is dense in B_p , which is the p^{th} *Baernstein space*. It is reflexive, and the unit vector basis $(e_n)_{n \in \mathbb{N}}$ is a 1-unconditional,

normalized basis for it. Baernstein [3] originally defined the space B_2 , while Seifert [18] observed that Baernstein's definition generalizes to arbitrary $p > 1$.

As in [12], we shall often consider the Baernstein and Schreier spaces simultaneously, using the letter E to denote either B_p for some $1 < p < \infty$ or S_p for some $1 \leq p < \infty$. Then, for a subset N of \mathbb{N} , we write E_N for the closed subspace of E spanned by $\{e_n : n \in N\}$. This subspace is 1-complemented in E by 1-unconditionality of the unit vector basis. We write $(e_n^*)_{n \in \mathbb{N}}$ for the sequence of coordinate functionals associated with the unit vector basis. They form a 1-unconditional basis for the dual space E^* because the unit vector basis for E is shrinking.

Gasparis and Leung [9] introduced a numerical index $\Gamma L_1(M, N)$ for every pair (M, N) of infinite subsets of \mathbb{N} to help analyze the closed subspaces spanned by subsequences of the unit vector basis for the Schreier space S_1 and its higher-order counterparts. We showed in [12, Section 4] that this index has similar properties for the Baernstein and Schreier spaces B_p and S_p for $p > 1$. In the context of the present investigation, its most important property is that for infinite subsets M and N of \mathbb{N} , the basic sequence $(e_m)_{m \in M}$ dominates the basic sequence $(e_n)_{n \in N}$ in either B_p or S_p if and only if $\Gamma L_1(M, N) < \infty$.

In order to state concisely [9, Definition 3.3], in which the index $\Gamma L_1(M, N)$ is defined, we require two pieces of notation. First, for an infinite subset $M = \{m_1 < m_2 < \dots\}$ of \mathbb{N} , set

$$M(J) = \{m_j : j \in J\} \quad (J \subseteq \mathbb{N}).$$

Second, the *Schreier covering number* of a finite subset A of \mathbb{N} is given by $\tau_1(\emptyset) = 0$ and

$$\tau_1(A) = \min \left\{ m \in \mathbb{N} : A \subseteq \bigcup_{j=1}^m F_j, \text{ where } F_1, \dots, F_m \in \mathcal{S}_1 \text{ and } F_1 < F_2 < \dots < F_m \right\}$$

for $A \neq \emptyset$. Now the *Gasparis–Leung index* of an infinite subset M of \mathbb{N} with respect to another infinite subset N of \mathbb{N} is

$$\Gamma L_1(M, N) = \sup \{ \tau_1(M(J)) : J \subset \mathbb{N} \text{ finite, } N(J) \in \mathcal{S}_1 \} \in \mathbb{N} \cup \{\infty\}. \quad (2.1)$$

(We remark that Gasparis and Leung denoted this index $d_1(M, N)$. We have chosen the more distinctive symbol $\Gamma L_1(M, N)$ in their honour, noting that Γ is the first letter of the Greek spelling of ‘‘Gasparis’’.)

Remark 2.1. For $n \in \mathbb{N}$ and subsets M and J of \mathbb{N} , where M is infinite and J is finite and non-empty, we have $\tau_1(M(J)) \leq n$ if and only if we can find a natural number $m \leq n$ and Schreier sets $F_1 < \dots < F_m$ such that $M(J) = \bigcup_{j=1}^m F_j$. Writing $F_j = M(J_j)$, we see that $\tau_1(M(J)) \leq n$ if and only if we can decompose J as $J = \bigcup_{j=1}^m J_j$ for some natural number $m \leq n$, where the sets $J_1 < \dots < J_m$ satisfy $M(J_j) \in \mathcal{S}_1$ for each $1 \leq j \leq m$.

We shall now establish some basic properties of the Gasparis–Leung index. We state them in greater generality than is strictly necessary for the applications we have in mind, for two reasons. First, we expect that they will be useful in future studies of the Baernstein and Schreier spaces, and second, we hope that they may inspire others to continue this line

of research, investigating the Gasparis–Leung index from a purely combinatorial/number-theoretic point of view.

Lemma 2.2. *Let $L, M = \{m_1 < m_2 < \dots\}$ and $N = \{n_1 < n_2 < \dots\}$ be infinite subsets of \mathbb{N} . Then:*

- (i) $\Gamma_{L_1}(L, N) \leq \Gamma_{L_1}(L, M) \cdot \Gamma_{L_1}(M, N)$.
- (ii) *Suppose that N is a spread of M in the sense that $n_j \geq m_j$ for every $j \in \mathbb{N}$. Then*

$$\Gamma_{L_1}(N, M) = 1 \quad \text{and} \quad \Gamma_{L_1}(L, M) \leq \Gamma_{L_1}(L, N).$$
- (iii) $\Gamma_{L_1}(N, N \setminus F) \leq \tau_1(\{n_j : 1 \leq j \leq |F|\}) + 1$ for every finite subset F of N .
- (iv) *Suppose that $m_j \leq n_{j+1}$ for each $j \in \mathbb{N}$. Then $\Gamma_{L_1}(N, M) \leq 2$.*

Proof. (i). Take a non-empty, finite subset J of \mathbb{N} for which $N(J) \in \mathcal{S}_1$. Then we have $\tau_1(M(J)) \leq \Gamma_{L_1}(M, N)$, so Remark 2.1 implies that we can write $J = \bigcup_{j=1}^m J_j$ for some natural number $m \leq \Gamma_{L_1}(M, N)$, where the sets $J_1 < \dots < J_m$ satisfy $M(J_j) \in \mathcal{S}_1$ for each $1 \leq j \leq m$. Applying Remark 2.1 again, for each $1 \leq j \leq m$, we can find a natural number $r_j \leq \Gamma_{L_1}(L, M)$ and sets $J_{j,1} < J_{j,2} < \dots < J_{j,r_j}$ such that $J_j = \bigcup_{k=1}^{r_j} J_{j,k}$ and $L(J_{j,k}) \in \mathcal{S}_1$ for each $1 \leq k \leq r_j$. It follows that $J = \bigcup_{j=1}^m \bigcup_{k=1}^{r_j} J_{j,k}$, where

$$J_{1,1} < J_{1,2} < \dots < J_{1,r_1} < J_{2,1} < J_{2,2} < \dots < J_{2,r_2} < \dots < J_{m,1} < \dots < J_{m,r_m},$$

so appealing to Remark 2.1 once more, we conclude that

$$\tau_1(L(J)) \leq \sum_{j=1}^m r_j \leq m \cdot \max_{1 \leq j \leq m} r_j \leq \Gamma_{L_1}(M, N) \cdot \Gamma_{L_1}(L, M).$$

(ii). Suppose that N is a spread of M . Then $N(J)$ is a spread of $M(J)$ for every $J \subseteq \mathbb{N}$, so in particular $N(J) \in \mathcal{S}_1$ whenever $M(J) \in \mathcal{S}_1$. This proves that $\Gamma_{L_1}(N, M) = 1$, and also that $\tau_1(L(J)) \leq \Gamma_{L_1}(L, N)$ for every $J \subset \mathbb{N}$ such that $M(J) \in \mathcal{S}_1$. The second inequality follows.

(iii). Set $k = |F|$, $G = \{n_j : 1 \leq j \leq k\}$ and $N' = N \setminus G = \{n_{k+j} : j \in \mathbb{N}\}$. Then N' is a spread of $N \setminus F$, so by (ii), it suffices to show that $\Gamma_{L_1}(N, N') \leq \tau_1(G) + 1$; that is, $\tau_1(N(J)) \leq \tau_1(G) + 1$ for every non-empty, finite subset $J = \{j_1 < \dots < j_m\}$ of \mathbb{N} such that $N'(J) \in \mathcal{S}_1$. Set $K = \{j_i : 1 \leq i \leq \min\{k, m\}\}$, which is a spread of the interval $[1, \min\{k, m\}] \cap \mathbb{N}$. Consequently $N(K)$ is a spread of $\{n_i : 1 \leq i \leq \min\{k, m\}\}$, which is a subset of G , and hence $\tau_1(N(K)) \leq \tau_1(G)$. This completes the proof if $J = K$. Otherwise $k < m$; then $\max K = j_k$, so $\min(J \setminus K) = j_{k+1}$, and therefore

$$\begin{aligned} \min N(J \setminus K) &= n_{j_{k+1}} \geq n_{k+j_1} = \min N'(J) \geq |N'(J)| \quad \text{because } N'(J) \in \mathcal{S}_1 \\ &= |J| > |J \setminus K| = |N(J \setminus K)|. \end{aligned}$$

This proves that $N(J \setminus K) \in \mathcal{S}_1$. Since $K < J \setminus K$, we conclude that

$$\tau_1(N(J)) \leq \tau_1(N(K)) + 1 \leq \tau_1(G) + 1,$$

as desired.

(iv). We have

$$\Gamma_{L_1}(N, M) \leq \Gamma_{L_1}(N, N \setminus \{n_1\}) \cdot \Gamma_{L_1}(N \setminus \{n_1\}, M) \leq (\tau_1(\{n_1\}) + 1) \cdot 1 = 2,$$

where the first inequality follows from (i) and the second from (iii) and (ii) because the hypothesis implies that $N \setminus \{n_1\}$ is a spread of M . \square

Corollary 2.3. *Let $(m_j)_{j \in \mathbb{N}}$ and $(n_j)_{j \in \mathbb{N}}$ be increasing sequences of natural numbers, and let $(e_j)_{j \in \mathbb{N}}$ denote the unit vector basis for E , where $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$.*

- (i) *Suppose that $m_j \leq n_j$ for each $j \in \mathbb{N}$. Then the basic sequence $(e_{n_j})_{j \in \mathbb{N}}$ 1-dominates $(e_{m_j})_{j \in \mathbb{N}}$.*
- (ii) *Suppose that $m_j \leq n_{j+1}$ for each $j \in \mathbb{N}$. Then the basic sequence $(e_{n_j})_{j \in \mathbb{N}}$ C_0 -dominates $(e_{m_j})_{j \in \mathbb{N}}$, where $C_0 = 2$ for $E = B_p$ and $C_0 = 2^{1/p}$ for $E = S_p$.*
- (iii) *There exists an infinite subset J of \mathbb{N} such that the basic sequences $(e_{m_j})_{j \in J}$ and $(e_{n_j})_{j \in J}$ are equivalent.*

Proof. (i)–(ii). According to Lemma 2.2(ii) and (iv), the sets $M = \{m_1 < m_2 < \dots\}$ and $N = \{n_1 < n_2 < \dots\}$ satisfy $\Gamma L_1(N, M) = 1$ in case (i) and $\Gamma L_1(N, M) \leq 2$ in case (ii). Now the conclusions follow from [12, Lemma 4.2].

(iii). By recursion, we can choose integers $1 = j_1 < j_2 < j_3 < \dots$ such that $m_{j_k} \leq n_{j_{k+1}}$ and $n_{j_k} \leq m_{j_{k+1}}$ for each $k \in \mathbb{N}$. Then (ii) implies that the subsequences $(e_{m_{j_k}})_{k \in \mathbb{N}}$ and $(e_{n_{j_k}})_{k \in \mathbb{N}}$ C_0 -dominate each other, so the subset $J = \{j_1 < j_2 < \dots\}$ of \mathbb{N} has the required property. \square

The distinction between “flat” and “spiky” vectors plays a key role in the study of the Baernstein and Schreier spaces. The uniform norm given by $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x(n)| \in [0, \infty]$ for $x \in \mathbb{K}^{\mathbb{N}}$ is the tool that allows us to quantify this distinction. (Note that B_p and S_p are contained in c_0 as vector subspaces because the unit vector basis is a normalized basis for them, so in particular $\|x\|_\infty < \infty$ for x belonging to any of these spaces.)

The importance of certain vectors being “spiky” is already evident in the main theorem [9, Theorem 1.1] of Gasparis and Leung (who denoted the uniform norm $\|\cdot\|_0$, not $\|\cdot\|_\infty$) and in our counterpart [12, Theorem 4.1] of it for the Baernstein and p -convexified Schreier spaces.

On the other hand, “flat” block basic sequences appear in [9, Lemma 3.14] and [12, Proposition 2.14]; we include the details of the latter result for later reference and to provide context for Proposition 2.5, whose final part is a counterpart of it for “spiky” block basic sequences.

Proposition 2.4 ([12, Proposition 2.14]). *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$. The following conditions are equivalent for a normalized block basic sequence $(u_n)_{n \in \mathbb{N}}$ of the unit vector basis for E :*

- (a) $\inf_{n \in \mathbb{N}} \|u_n\|_\infty = 0$;
- (b) $(u_n)_{n \in \mathbb{N}}$ admits a subsequence which is C -equivalent to the unit vector basis for D , for every constant $C > 1$;
- (c) $(u_n)_{n \in \mathbb{N}}$ admits a subsequence which is dominated by the unit vector basis for D .

Proposition 2.5. *Let $(u_j)_{j \in \mathbb{N}}$ be a normalized block basic sequence of the unit vector basis $(e_j)_{j \in \mathbb{N}}$ for E , where $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$.*

- (i) Set $m_j = \max \text{supp } u_j$ for $j \in \mathbb{N}$. Then $(e_{m_j})_{j \in \mathbb{N}}$ C_1 -dominates $(u_j)_{j \in \mathbb{N}}$, where $C_1 = 3^{1/p}$ if $E = B_p$ and $C_1 = 1$ if $E = S_p$.
- (ii) Suppose that $\delta := \inf_{j \in \mathbb{N}} \|u_j\|_\infty > 0$, and choose $n_j \in \text{supp } u_j$ such that $|\langle u_j, e_{n_j}^* \rangle| \geq \delta$ for each $j \in \mathbb{N}$. Then $(u_j)_{j \in \mathbb{N}}$ δ^{-1} -dominates $(e_{n_j})_{j \in \mathbb{N}}$, and $\overline{\text{span}} \{u_j : j \in \mathbb{N}\}$ is C_2 -complemented in E , where $C_2 = 2 \cdot 3^{1/p} / \delta$ if $E = B_p$ and $C_2 = 2^{1/p} / \delta$ if $E = S_p$.
- (iii) The following conditions are equivalent:
 - (a) $\inf_{j \in \mathbb{N}} \|u_j\|_\infty > 0$;
 - (b) $(u_j)_{j \in \mathbb{N}}$ is equivalent to a subsequence of $(e_j)_{j \in \mathbb{N}}$;
 - (c) $(u_j)_{j \in \mathbb{N}}$ dominates a subsequence of $(e_j)_{j \in \mathbb{N}}$.

Proof. Before we embark on the proof, let us record that the sequences $(m_j)_{j \in \mathbb{N}}$ and $(n_j)_{j \in \mathbb{N}}$ in parts (i) and (ii) are increasing because $(u_j)_{j \in \mathbb{N}}$ is a block basic sequence.

(i). Set $M = \{m_j : j \in \mathbb{N}\}$. Our aim is to prove that for $k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{K}$, the elements $x = \sum_{j=1}^k \alpha_j e_{m_j}$ and $y = \sum_{j=1}^k \alpha_j u_j$ satisfy the inequality $C_1 \|x\|_E \geq \|y\|_E$. This is trivial if $\alpha_1 = \dots = \alpha_k = 0$, so we may suppose otherwise; thus $x, y \neq 0$.

If $E = S_p$, take $F \in \mathcal{S}_1$ such that $\|y\|_{S_p} = \mu_p(y, F)$. After replacing F with $F \cap \text{supp } y$, we may suppose that $F \subseteq \text{supp } y$. Set $F_j = F \cap \text{supp } u_j \in \mathcal{S}_1$ for $1 \leq j \leq k$. Then, defining $J = \{j \in \{1, \dots, k\} : F_j \neq \emptyset\}$ and $j_1 = \min J$, we have

$$\begin{aligned}
\min M(J) = m_{j_1} &\geq \min F && \text{because } \min F \in \text{supp } u_{j_1} \\
&\geq |F| && \text{because } F \in \mathcal{S}_1 \\
&\geq |J| && \text{because } (F_j)_{j \in J} \text{ are disjoint, non-empty subsets of } F \\
&= |M(J)|,
\end{aligned}$$

so $M(J) \in \mathcal{S}_1$. Consequently,

$$\begin{aligned}
\|x\|_{S_p}^p &\geq \mu_p(x, M(J))^p = \sum_{j \in J} |\alpha_j|^p = \sum_{j \in J} |\alpha_j|^p \|u_j\|_{S_p}^p \\
&\geq \sum_{j \in J} |\alpha_j|^p \mu_p(u_j, F_j)^p = \sum_{j \in J} \mu_p(y, F_j)^p = \mu_p(y, F)^p = \|y\|_{S_p}^p,
\end{aligned}$$

which proves that $(e_{m_j})_{j \in \mathbb{N}}$ 1-dominates $(u_j)_{j \in \mathbb{N}}$ in S_p .

Before we tackle the case $E = B_p$, let us mention that Causey [5, Lemma 3.2] has proved a closely related result, which states that $(e_{k_j})_{j \in \mathbb{N}}$ 4-dominates $(u_j)_{j \in \mathbb{N}}$ in B_p , where $k_j = \min \text{supp } u_j$ for $j \in \mathbb{N}$. Our result, with a larger constant, follows easily from Causey's. Nevertheless, we include a (fairly simple) proof of it to keep our work as self-contained as possible.

Take $\mathcal{C} \in \text{SC}$ such that $\|y\|_{B_p} = \beta_p(y, \mathcal{C})$. We may suppose that $F \subseteq \text{supp } y$ for every $F \in \mathcal{C}$. Define $\mathcal{C}_j = \{F \in \mathcal{C} : F \subseteq \text{supp } u_j\}$ for $1 \leq j \leq k$, $J = \{j \in \{1, \dots, k\} : \mathcal{C}_j \neq \emptyset\}$ and $\mathcal{C}' = \mathcal{C} \setminus \bigcup_{j \in J} \mathcal{C}_j$. Then \mathcal{C} is the disjoint union of $\{\mathcal{C}_j : j \in J\} \cup \{\mathcal{C}'\}$, so

$$\|y\|_{B_p}^p = \sum_{j \in J} \beta_p(y, \mathcal{C}_j)^p + \beta_p(y, \mathcal{C}')^p. \tag{2.2}$$

It is easy to estimate the first term on the right-hand side of this equation because the fact that $F \subseteq \text{supp } u_j$ for each $F \in \mathcal{C}_j$ implies that

$$\beta_p(y, \mathcal{C}_j) = |\alpha_j| \beta_p(u_j, \mathcal{C}_j) \leq |\alpha_j| \|u_j\|_{B_p} = |\alpha_j| \quad (j \in J).$$

Hence we have

$$\sum_{j \in J} \beta_p(y, \mathcal{C}_j)^p \leq \sum_{j \in J} |\alpha_j|^p \leq \|x\|_{B_p}^p, \quad (2.3)$$

using the easy observation that the B_p -norm 1-dominates the ℓ_p -norm.

This completes the proof if $\mathcal{C}' = \emptyset$. Otherwise we can write $\mathcal{C}' = \{F_1 < \dots < F_n\}$ for some $n \in \mathbb{N}$. Set $K_r = \{j \in \{1, \dots, k\} : F_r \cap \text{supp } u_j \neq \emptyset\}$ and $i_r = \min K_r$ for $1 \leq r \leq n$. Then $M(K_r) \in \mathcal{S}_1$ because

$$|M(K_r)| = |K_r| \leq |F_r| \leq \min F_r \leq m_{i_r} = \min M(K_r).$$

Furthermore, using that $\{F_r \cap \text{supp } u_j : j \in K_r\}$ partitions F_r , we obtain

$$\begin{aligned} \mu_1(y, F_r) &= \sum_{j \in K_r} \mu_1(y, F_r \cap \text{supp } u_j) = \sum_{j \in K_r} |\alpha_j| \mu_1(u_j, F_r) \\ &\leq \sum_{j \in K_r} |\alpha_j| \|u_j\|_{B_p} = \sum_{j \in K_r} |\alpha_j| = \mu_1(x, M(K_r)). \end{aligned} \quad (2.4)$$

The definition of \mathcal{C}' implies that $|K_r| \geq 2$ and $K_r \setminus \{i_{r+1}\} < K_{r+1}$. In particular, we have $K_r < K_{r+2}$, so $M(K_r) < M(K_{r+2})$, and therefore we can define two Schreier chains by

$\mathcal{D} = \{M(K_1) < M(K_3) < \dots < M(K_{n'})\}$ and $\mathcal{E} = \{M(K_2) < M(K_4) < \dots < M(K_{n''})\}$, where $(n', n'') = (n-1, n)$ if n is even and $(n', n'') = (n, n-1)$ if n is odd. Hence, applying (2.4) and using that $\mathcal{D} \cup \mathcal{E}$ partitions $\{M(K_r) : 1 \leq r \leq n\}$, we deduce that

$$\beta_p(y, \mathcal{C}')^p = \sum_{r=1}^n \mu_1(y, F_r)^p \leq \sum_{r=1}^n \mu_1(x, M(K_r))^p = \beta_p(x, \mathcal{D})^p + \beta_p(x, \mathcal{E})^p \leq 2\|x\|_{B_p}^p.$$

Finally, we substitute this upper bound together with that from (2.3) into (2.2) to conclude that $\|y\|_{B_p}^p \leq 3\|x\|_{B_p}^p$, which proves that $(e_{m_j})_{j \in \mathbb{N}}$ $3^{1/p}$ -dominates $(u_j)_{j \in \mathbb{N}}$ in B_p .

(ii). Set $N = \{n_j : j \in \mathbb{N}\}$, and recall that $E_N = \overline{\text{span}} \{e_{n_j} : j \in \mathbb{N}\}$. The 1-unconditionality of the unit vector basis for E allows us to define a bounded operator $T : E \rightarrow E_N$ of norm at most δ^{-1} by

$$Tx = \sum_{k=1}^{\infty} \frac{\langle x, e_{n_k}^* \rangle}{\langle u_k, e_{n_k}^* \rangle} e_{n_k}.$$

It satisfies $Tu_j = e_{n_j}$ for each $j \in \mathbb{N}$ because $\langle u_j, e_{n_k}^* \rangle = 0$ whenever $j \neq k$. Hence $(u_j)_{j \in \mathbb{N}}$ δ^{-1} -dominates $(e_{n_j})_{j \in \mathbb{N}}$ in E .

To prove that the subspace $W = \overline{\text{span}} \{u_j : j \in \mathbb{N}\}$ is C_2 -complemented in E , we note that $m_j = \max \text{supp } u_j < n_{j+1}$ for each $j \in \mathbb{N}$ because $(u_j)_{j \in \mathbb{N}}$ is a block basic sequence. Therefore Corollary 2.3(ii) implies that the map $U : e_{n_j} \mapsto e_{m_j}$ for $j \in \mathbb{N}$ extends uniquely to an operator $U \in \mathcal{B}(E_N, E_M)$ with $\|U\| \leq C_0$, where we recall that $C_0 = 2$ for $E = B_p$ and $C_0 = 2^{1/p}$ for $E = S_p$. By (i), we can define an operator $V \in \mathcal{B}(E_M, W)$ with $\|V\| \leq C_1$

by $Ve_{m_j} = u_j$ for each $j \in \mathbb{N}$. It follows that the composite operator $Q = VUT \in \mathcal{B}(E, W)$ satisfies $Qu_j = u_j$ for each $j \in \mathbb{N}$, so by linearity and continuity, Q acts as the identity on W ; that is, Q is a projection of E onto W . Now the conclusion follows from the fact that

$$\|Q\| \leq \frac{C_0 C_1}{\delta} \leq \begin{cases} \frac{2 \cdot 3^{1/p}}{\delta} & \text{for } E = B_p \\ \frac{2^{1/p}}{\delta} & \text{for } E = S_p. \end{cases}$$

(iii), (a) \Rightarrow (b). Suppose that $\inf_{j \in \mathbb{N}} \|u_j\|_\infty > 0$, and define the sequences $(m_j)_{j \in \mathbb{N}}$ and $(n_j)_{j \in \mathbb{N}}$ as above. Then, as we already saw in the proof of (ii), the basic sequences $(u_j)_{j \in \mathbb{N}}$, $(e_{m_j})_{j \in \mathbb{N}}$ and $(e_{n_j})_{j \in \mathbb{N}}$ are all equivalent because $(u_j)_{j \in \mathbb{N}}$ dominates $(e_{n_j})_{j \in \mathbb{N}}$ by (ii), $(e_{n_j})_{j \in \mathbb{N}}$ dominates $(e_{m_j})_{j \in \mathbb{N}}$ by Corollary 2.3(ii), and $(e_{m_j})_{j \in \mathbb{N}}$ dominates $(u_j)_{j \in \mathbb{N}}$ by (i).

The implication (b) \Rightarrow (c) is trivial.

Finally, we prove that (c) implies (a) by contradiction. Assume that $(u_j)_{j \in \mathbb{N}}$ dominates a subsequence of $(e_j)_{j \in \mathbb{N}}$ and that $\inf_{j \in \mathbb{N}} \|u_j\|_\infty = 0$. Proposition 2.4 shows that the latter assumption implies that $(u_j)_{j \in \mathbb{N}}$ admits a subsequence which is dominated by the unit vector basis $(d_j)_{j \in \mathbb{N}}$ for D , where $D = c_0$ if $E = S_p$ and $D = \ell_p$ if $E = B_p$. Combining this with the former assumption, we conclude that $(d_j)_{j \in \mathbb{N}}$ dominates a subsequence of $(e_j)_{j \in \mathbb{N}}$, which is clearly impossible. \square

3. THE COMPOSITION OF TWO STRICTLY SINGULAR OPERATORS IS COMPACT

The aim of this section is to prove Theorem 1.1. Our first step is to combine Propositions 2.4–2.5 with the Principle of Small Perturbations to obtain the following result.

Lemma 3.1. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and let $(w_n)_{n \in \mathbb{N}}$ be a weakly null sequence in E with $\inf_{n \in \mathbb{N}} \|w_n\|_E > 0$.*

- (i) *Suppose that $\liminf_{n \rightarrow \infty} \|w_n\|_\infty = 0$. Then $(w_n)_{n \in \mathbb{N}}$ admits a subsequence $(w'_n)_{n \in \mathbb{N}}$ which is a basic sequence equivalent to the unit vector basis for D , and $\|w'_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.*
- (ii) *Suppose that $\limsup_{n \rightarrow \infty} \|w_n\|_\infty > 0$. Then $(w_n)_{n \in \mathbb{N}}$ admits a subsequence $(w'_n)_{n \in \mathbb{N}}$ which is a basic sequence equivalent to a subsequence of the unit vector basis for E , and $\inf_{n \in \mathbb{N}} \|w'_n\|_\infty > 0$.*

Proof. We begin by noting that the weak convergence of $(w_n)_{n \in \mathbb{N}}$ implies that it is norm-bounded and hence seminormalized. For most of the proof, we shall consider the two cases simultaneously, but first we need to separate them:

- In case (i), we replace $(w_n)_{n \in \mathbb{N}}$ with a subsequence such that $\|w_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, and we set $\xi = \inf_{n \in \mathbb{N}} \|w_n\|_E > 0$.
- In case (ii), we replace $(w_n)_{n \in \mathbb{N}}$ with a subsequence such that $\xi := \inf_{n \in \mathbb{N}} \|w_n\|_\infty > 0$ and observe that this implies that $\inf_{n \in \mathbb{N}} \|w_n\|_E \geq \xi$.

Returning to the unified approach, we define $m_0 = 0$, $P_0 = 0$ and $\varepsilon_j = \xi / (3 \cdot 2^j + 1)$ for each $j \in \mathbb{N}$. Using the fact that $(w_n)_{n \in \mathbb{N}}$ is weakly null, we can recursively construct increasing

sequences $(k_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ of natural numbers such that the vectors

$$v_j := (P_{m_j} - P_{m_{j-1}})w_{k_j} \in E$$

satisfy $\|w_{k_j} - v_j\|_E \leq \varepsilon_j$ for each $j \in \mathbb{N}$, where P_m denotes the m^{th} basis projection.

These choices imply that $(v_j)_{j \in \mathbb{N}}$ is a seminormalized block basic sequence because

$$\sup_{n \in \mathbb{N}} \|w_n\|_E + \varepsilon_j \geq \|w_{k_j}\|_E + \|w_{k_j} - v_j\|_E \geq \|v_j\|_E \geq \|w_{k_j}\|_E - \|w_{k_j} - v_j\|_E \geq \xi - \varepsilon_j$$

and $\varepsilon_j \leq \xi/7$ for each $j \in \mathbb{N}$. Hence $(v_j)_{j \in \mathbb{N}}$ is equivalent to its normalization $(v_j/\|v_j\|_E)_{j \in \mathbb{N}}$. Another application of the lower bound $\xi - \varepsilon_j$ on $\|v_j\|_E$ gives

$$\sum_{j=1}^{\infty} \frac{\|w_{k_j} - v_j\|_E}{\|v_j\|_E} \leq \sum_{j=1}^{\infty} \frac{\varepsilon_j}{\xi - \varepsilon_j} = \sum_{j=1}^{\infty} \frac{1}{3 \cdot 2^j} = \frac{1}{3} < \frac{1}{2},$$

so the Principle of Small Perturbations implies that $(w_{k_j})_{j \in \mathbb{N}}$ is a basic sequence equivalent to $(v_j)_{j \in \mathbb{N}}$ and therefore to $(v_j/\|v_j\|_E)_{j \in \mathbb{N}}$. For later reference, we note that $\|w_{k_j} - v_j\|_{\infty} \leq \varepsilon_j$ for each $j \in \mathbb{N}$.

Consequently, if we are in case (i), it follows that

$$\frac{\|v_j\|_{\infty}}{\|v_j\|_E} \leq \frac{\|w_{k_j}\|_{\infty} + \varepsilon_j}{6\xi/7} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Combining this with Proposition 2.4, we deduce that $(v_j/\|v_j\|_E)_{j \in \mathbb{N}}$ admits a subsequence $(v_{j_i}/\|v_{j_i}\|_E)_{i \in \mathbb{N}}$ which is equivalent to the unit vector basis for D , and therefore the same is true for the subsequence $(w_{k_{j_i}})_{i \in \mathbb{N}}$ of $(w_n)_{n \in \mathbb{N}}$.

Otherwise we are in case (ii), and we see that $\|v_j\|_{\infty} \geq \|w_{k_j}\|_{\infty} - \varepsilon_j \geq 6\xi/7$, so

$$\inf_{j \in \mathbb{N}} \frac{\|v_j\|_{\infty}}{\|v_j\|_E} \geq \frac{6\xi}{7 \sup_{j \in \mathbb{N}} \|v_j\|_E} > 0.$$

Now Proposition 2.5(iii) implies that $(v_j/\|v_j\|_E)_{j \in \mathbb{N}}$ is equivalent to a subsequence of the unit vector basis for E , and therefore the same is true for the subsequence $(w_{k_j})_{j \in \mathbb{N}}$ of $(w_n)_{n \in \mathbb{N}}$. \square

Remark 3.2. The two conditions in Lemma 3.1 are clearly not mutually exclusive, but at least one of them is always satisfied because the limit superior of a sequence dominates its limit inferior.

Lemma 3.3. *Let T be a strictly singular operator on E , where $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$, and let $(w_n)_{n \in \mathbb{N}}$ be a weakly null sequence in E for which $\inf_{n \in \mathbb{N}} \|Tw_n\|_E > 0$. Then $(w_n)_{n \in \mathbb{N}}$ admits a subsequence $(w'_n)_{n \in \mathbb{N}}$ such that*

- (i) $\inf_{n \in \mathbb{N}} \|w'_n\|_{\infty} > 0$, and $(w'_n)_{n \in \mathbb{N}}$ is equivalent to a subsequence of the unit vector basis for E ; and
- (ii) $\|Tw'_n\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$, and $(Tw'_n)_{n \in \mathbb{N}}$ is equivalent to the unit vector basis for D , where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$.

Proof. Let $(e_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$ denote the unit vector bases for E and D , respectively. Since $\inf_{n \in \mathbb{N}} \|w_n\|_E \geq \|T\|^{-1} \inf_{n \in \mathbb{N}} \|Tw_n\|_E > 0$, Lemma 3.1 implies that $(w_n)_{n \in \mathbb{N}}$ admits a subsequence $(w'_n)_{n \in \mathbb{N}}$ which is a basic sequence and satisfies one of the following two conditions:

- (I) $(w'_n)_{n \in \mathbb{N}}$ is equivalent to $(d_n)_{n \in \mathbb{N}}$, and $\|w'_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$; or
- (II) $(w'_n)_{n \in \mathbb{N}}$ is equivalent to a subsequence of $(e_n)_{n \in \mathbb{N}}$, and $\inf_{n \in \mathbb{N}} \|w'_n\|_\infty > 0$.

Being bounded, T is weakly continuous, so the sequence $(Tw'_n)_{n \in \mathbb{N}}$ converges weakly to 0. Therefore we may apply Lemma 3.1 to it, which allows us to replace $(w'_n)_{n \in \mathbb{N}}$ with a subsequence (also denoted $(w'_n)_{n \in \mathbb{N}}$) such that $(Tw'_n)_{n \in \mathbb{N}}$ is a basic sequence satisfying one of the following two conditions:

- (III) $(Tw'_n)_{n \in \mathbb{N}}$ is equivalent to $(d_n)_{n \in \mathbb{N}}$, and $\|Tw'_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$; or
- (IV) $(Tw'_n)_{n \in \mathbb{N}}$ is equivalent to a subsequence of $(e_n)_{n \in \mathbb{N}}$, and $\inf_{n \in \mathbb{N}} \|Tw'_n\|_\infty > 0$.

Importantly, we note that conditions (I) and (II) remain true when we replace $(w'_n)_{n \in \mathbb{N}}$ with a subsequence.

We see that the desired conclusions (i)–(ii) will follow if (and only if) conditions (II) and (III) are satisfied; we verify this by ruling out the other three possible combinations.

To present these arguments concisely, we require some notation. Let W and X denote the closed subspaces of E spanned by the basic sequences $(w'_n)_{n \in \mathbb{N}}$ and $(Tw'_n)_{n \in \mathbb{N}}$, respectively, and write \tilde{T} for the restriction of T to W , viewed as an operator into X . We note that \tilde{T} is strictly singular because T is. We introduce the following symbols for the isomorphisms that implement the equivalences in cases (I)–(IV):

- In cases (I) and (III), the linear maps U_1 and V_1 given by $U_1 d_j = w'_j$ and $V_1 Tw'_j = d_j$ for $j \in \mathbb{N}$ define isomorphisms $U_1 \in \mathcal{B}(D, W)$ and $V_1 \in \mathcal{B}(X, D)$, respectively.
- In case (II), we can take an infinite subset $M = \{m_1 < m_2 < \dots\}$ of \mathbb{N} and an isomorphism $U_2 \in \mathcal{B}(E_M, W)$ such that $U_2 e_{m_j} = w'_j$ for each $j \in \mathbb{N}$, where we recall that $E_M = \overline{\text{span}} \{e_{m_j} : j \in \mathbb{N}\}$.
- Similarly, in case (IV), we can take an infinite subset $N = \{n_1 < n_2 < \dots\}$ of \mathbb{N} and an isomorphism $V_2 \in \mathcal{B}(X, E_N)$ for which $V_2 Tw'_j = e_{n_j}$ for each $j \in \mathbb{N}$.

We are now ready to complete the argument by ruling out the three combinations of [(I) or (II)] and [(III) or (IV)] that are not (II) and (III):

- (I)+(III): In this case we have $V_1 \tilde{T} U_1 d_j = d_j$ for each $j \in \mathbb{N}$, so $V_1 \tilde{T} U_1 = I_D$, which contradicts that \tilde{T} is strictly singular.
- (I)+(IV): In this case we see that $V_2 \tilde{T} U_1 d_j = e_{n_j}$ for each $j \in \mathbb{N}$, which is impossible because $(d_j)_{j \in \mathbb{N}}$ does not dominate any subsequence of $(e_j)_{j \in \mathbb{N}}$.
- (II)+(IV): In this case Corollary 2.3(iii) shows that the map $e_{m_{j_k}} \mapsto e_{n_{j_k}}$ for $k \in \mathbb{N}$ extends to an isomorphism of $E_{M(J)}$ onto $E_{N(J)}$ for some infinite subset $J = \{j_1 < j_2 < \dots\}$ of \mathbb{N} . However, this would imply that the restriction of $V_2 \tilde{T} U_2$ to $E_{M(J)}$ is an isomorphism onto $E_{N(J)}$, which again contradicts that \tilde{T} is strictly singular. \square

Lemma 3.4. *Let T be a non-compact operator from a Banach space X with a basis $(x_n)_{n \in \mathbb{N}}$ into a Banach space Y . Then $(x_n)_{n \in \mathbb{N}}$ admits a normalized block basic sequence $(u_n)_{n \in \mathbb{N}}$ for which $\inf_{n \in \mathbb{N}} \|Tu_n\|_Y > 0$.*

Proof. Take $\eta \in (0, \|T + \mathcal{K}(X, Y)\|)$, where $\mathcal{K}(X, Y)$ denotes the closed subspace of $\mathcal{B}(X, Y)$ consisting of compact operators, and set $m_0 = 0$. We shall recursively choose natural numbers $m_1 < m_2 < \dots$ and unit vectors $u_n \in \text{span}\{x_j : m_{n-1} < j \leq m_n\}$ such that $\|Tu_n\|_Y \geq \eta/(K+1)$ for each $n \in \mathbb{N}$, where K denotes the basis constant of $(x_n)_{n \in \mathbb{N}}$.

We begin the construction by observing that since $\text{span}\{x_j : j \in \mathbb{N}\}$ is dense in X and $\|T\| > \eta$, we can find $m_1 \in \mathbb{N}$ and a unit vector $u_1 \in \text{span}\{x_j : 1 \leq j \leq m_1\}$ such that $\|Tu_1\|_Y > \eta > \eta/(K+1)$.

Assume recursively that we have chosen natural numbers $m_1 < m_2 < \dots < m_n$ and unit vectors u_1, \dots, u_n for some $n \in \mathbb{N}$. Since the basis projection P_{m_n} has finite rank, we have $\|T(I_X - P_{m_n})\| \geq \|T + \mathcal{K}(X, Y)\| > \eta$, so we can find $m_{n+1} > m_n$ and a unit vector $v_{n+1} \in \text{span}\{x_j : 1 \leq j \leq m_{n+1}\}$ such that $\|T(I_X - P_{m_n})v_{n+1}\|_Y > \eta$. Then

$$u_{n+1} = \frac{(I_X - P_{m_n})v_{n+1}}{\|(I_X - P_{m_n})v_{n+1}\|_X} \in \text{span}\{x_j : m_n < j \leq m_{n+1}\}$$

is a unit vector for which $\|Tu_{n+1}\|_Y \geq \eta/(K+1)$ because $\|(I_X - P_{m_n})v_{n+1}\|_X \leq K+1$. Hence the recursion continues, and the result follows. \square

Proof of Theorem 1.1. Let T and U be strictly singular operators on E , where $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$, and assume towards a contradiction that their composition TU is not compact. By Lemma 3.4, we can find a normalized block basic sequence $(u_n)_{n \in \mathbb{N}}$ of the unit vector basis $(e_n)_{n \in \mathbb{N}}$ for E such that $\inf_{n \in \mathbb{N}} \|TUu_n\|_E > 0$. We note that $(u_n)_{n \in \mathbb{N}}$ is weakly null because $(e_n)_{n \in \mathbb{N}}$ is shrinking, so $(Uu_n)_{n \in \mathbb{N}}$ is also weakly null. Hence, applying Lemma 3.3 with $w_n = Uu_n$ for $n \in \mathbb{N}$, we can extract a subsequence $(u'_n)_{n \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ for which

$$\inf_{n \in \mathbb{N}} \|Uu'_n\|_\infty > 0. \quad (3.1)$$

Since $(u'_n)_{n \in \mathbb{N}}$ is weakly null and $\inf_{n \in \mathbb{N}} \|Uu'_n\|_E \geq \inf_{n \in \mathbb{N}} \|Uu'_n\|_\infty > 0$, another application of Lemma 3.3 shows that $(u'_n)_{n \in \mathbb{N}}$ admits a subsequence $(u''_n)_{n \in \mathbb{N}}$ such that $\|Uu''_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. However, this contradicts (3.1). \square

Having presented a self-contained proof of Theorem 1.1, we shall next outline how it may alternatively be deduced from the work of Androulakis, Dodos, Sirotkin and Troitsky [2] concerning compositions of strictly singular operators. This argument still relies on Lemmas 3.1 and 3.3, so it is not independent of the above proof. It is based on the observation that every strictly singular operator on the Baernstein and Schreier spaces is “finitely strictly singular” in the following sense, a result that may be of interest in its own right.

Definition 3.5. An operator $T \in \mathcal{B}(X, Y)$ between Banach spaces X and Y is *finitely strictly singular* if, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that every subspace of X of dimension at least n contains a unit vector x for which $\|Tx\| \leq \varepsilon$.

Proposition 3.6. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$. The following conditions are equivalent for an operator $T \in \mathcal{B}(E)$:*

- (a) T is finitely strictly singular;
- (b) T is strictly singular;
- (c) T does not fix a copy of D ;
- (d) the identity operator on D does not factor through T .

The implications (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) are clear, and (d) implies (b) because E is saturated with complemented copies of D by [12, Theorem 2.4], so it only remains to prove that (b) implies (a). This relies on Milman’s “flat-vector lemma”, originally published in [14]; we refer to [7, Lemma 13] for an easily accessible presentation in English.

Lemma 3.7 (Milman). *Let W be a non-zero subspace of c_0 , and take a natural number $n \leq \dim W$. Then W contains a unit vector w which attains its norm in at least n coordinates; that is, $\|w\|_\infty = 1$ and the set $\{j \in \mathbb{N} : |w(j)| = 1\}$ has cardinality at least n .*

Proof of Proposition 3.6, (b) \Rightarrow (a). Assume towards a contradiction that $T \in \mathcal{B}(E)$ is a strictly singular operator which fails to be finitely strictly singular. Then, for some $\varepsilon > 0$, there is a sequence $(W_n)_{n \in \mathbb{N}}$ of subspaces of E such that

$$\dim W_n \geq 2n - 1 \quad \text{and} \quad \|Tw\|_E \geq \varepsilon \|w\|_E \quad (n \in \mathbb{N}, w \in W_n). \quad (3.2)$$

As previously observed, E is a vector subspace of c_0 because the unit vector basis is a normalized basis for E . Hence Lemma 3.7 applies, showing that for each $n \in \mathbb{N}$, we can choose $v_n \in W_n$ such that $\|v_n\|_\infty = 1$ and the set $J_n = \{j \in \mathbb{N} : |v_n(j)| = 1\}$ has cardinality at least $2n - 1$. Then $J_n \cap [n, \infty)$ has cardinality at least n , so J_n contains a Schreier set F_n of cardinality n , and therefore $\|v_n\|_{S_p} \geq \mu_p(v_n, F_n) = n^{1/p}$ for every $p \in [1, \infty)$. Since $\|\cdot\|_{B_p} \geq \|\cdot\|_{S_1}$, we conclude that $\|v_n\|_E \rightarrow \infty$ as $n \rightarrow \infty$. Consequently, defining $w_n = v_n/\|v_n\|_E \in W_n$ for each $n \in \mathbb{N}$, we have

$$\|w_n\|_\infty = 1/\|v_n\|_E \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (3.3)$$

In particular, $(w_n)_{n \in \mathbb{N}}$ is a norm-bounded sequence with $\langle w_n, e_j^* \rangle \rightarrow 0$ as $n \rightarrow \infty$ for each $j \in \mathbb{N}$, so $(w_n)_{n \in \mathbb{N}}$ is weakly null because the basis $(e_j)_{j \in \mathbb{N}}$ for E is shrinking. Since $\inf_{n \in \mathbb{N}} \|Tw_n\|_E \geq \varepsilon$ by (3.2), Lemma 3.3 implies that $(w_n)_{n \in \mathbb{N}}$ admits a subsequence $(w'_n)_{n \in \mathbb{N}}$ for which $\inf_{n \in \mathbb{N}} \|w'_n\|_\infty > 0$. However, this contradicts (3.3). \square

Remark 3.8. In the case $E = S_p$, $1 \leq p < \infty$, conditions (a)–(c) in Proposition 3.6 remain equivalent for operators $T \in \mathcal{B}(S_p, Y)$ into any Banach space Y . The implications (a) \Rightarrow (b) \Rightarrow (c) are true in general, and (c) implies (b) because S_p is saturated with copies of c_0 , as before. To see that (b) implies (a), we proceed as in the above proof, noting that we do not use the facts that T is strictly singular or that the codomain of T is E until we invoke Lemma 3.3 in the penultimate line. For an operator $T \in \mathcal{B}(E, Y)$, still assumed not to be finitely strictly singular, but now with arbitrary codomain Y , we can instead apply Lemma 3.1(i): since $(w_n)_{n \in \mathbb{N}}$ is weakly null with $\|w_n\|_E = 1$ and $\|w_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, it admits a subsequence $(w'_n)_{n \in \mathbb{N}}$ which is equivalent to the unit vector basis $(d_n)_{n \in \mathbb{N}}$ for D .

Taking $U \in \mathcal{B}(D, E)$ with $Ud_n = w'_n$ for each $n \in \mathbb{N}$, we have

$$\inf_{n \in \mathbb{N}} \|TUd_n\|_Y = \inf_{n \in \mathbb{N}} \|Tw'_n\|_Y \geq \varepsilon.$$

If $D = c_0$, this condition implies that the restriction of TU to the closed subspace of c_0 spanned by $\{d_n : n \in N\}$ is an isomorphic embedding for some infinite subset N of \mathbb{N} by a famous result of Rosenthal, originally stated as the first remark following [17, Theorem 3.4]; that is, T fixes a copy of c_0 .

To present our alternative proof of Theorem 1.1 based on the results of [2], we require the following definitions from that paper; throughout, X denotes a Banach space.

- A seminormalized basic sequence $(x_n)_{n \in \mathbb{N}}$ in X is *Schreier spreading* if there is a constant $C \geq 1$ such that, for every pair of non-empty Schreier sets $F, G \in \mathcal{S}_1$ of the same cardinality n , the finite basic sequences $(x_j)_{j \in F}$ and $(x_j)_{j \in G}$ are C -equivalent; that is, writing $F = \{f_1 < f_2 < \dots < f_n\}$ and $G = \{g_1 < g_2 < \dots < g_n\}$, we have

$$\frac{1}{C} \left\| \sum_{j=1}^n \alpha_j x_{f_j} \right\|_X \leq \left\| \sum_{j=1}^n \alpha_j x_{g_j} \right\|_X \leq C \left\| \sum_{j=1}^n \alpha_j x_{f_j} \right\|_X \quad (\alpha_1, \dots, \alpha_n \in \mathbb{K}). \quad (3.4)$$

The set of seminormalized basic sequences in X that are Schreier spreading and weakly null is denoted $\text{SP}_{1,w}(X)$.

- Given two seminormalized Schreier spreading basic sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in X , define $(x_n)_{n \in \mathbb{N}} \approx_1 (y_n)_{n \in \mathbb{N}}$ if there is a constant $C \geq 1$ such that the finite basic sequences $(x_n)_{n \in F}$ and $(y_n)_{n \in F}$ are C -equivalent for every $F \in \mathcal{S}_1$. This defines an equivalence relation \approx_1 on the set of seminormalized Schreier spreading basic sequences in X .

We shall repeatedly use the following simple facts about these notions.

- Equivalence of basic sequences implies \approx_1 -equivalence; that is, suppose that $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are seminormalized Schreier spreading basic sequences in X which are equivalent. Then $(x_n)_{n \in \mathbb{N}} \approx_1 (y_n)_{n \in \mathbb{N}}$.
- Suppose that $(x'_n)_{n \in \mathbb{N}}$ is a subsequence of a seminormalized Schreier spreading basic sequence $(x_n)_{n \in \mathbb{N}}$ in X . Then $(x'_n)_{n \in \mathbb{N}}$ is Schreier spreading, and $(x'_n)_{n \in \mathbb{N}} \approx_1 (x_n)_{n \in \mathbb{N}}$; see [2, Proposition 3.5(i)].

Lemma 3.9. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$.*

- The unit vector basis for E belongs to $\text{SP}_{1,w}(E)$.*
- Suppose that $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \text{SP}_{1,w}(E)$ satisfy either*

$$\liminf_{n \rightarrow \infty} \|x_n\|_\infty = \liminf_{n \rightarrow \infty} \|y_n\|_\infty = 0 \quad (3.5)$$

or

$$\min \left\{ \limsup_{n \rightarrow \infty} \|x_n\|_\infty, \limsup_{n \rightarrow \infty} \|y_n\|_\infty \right\} > 0. \quad (3.6)$$

Then $(x_n)_{n \in \mathbb{N}} \approx_1 (y_n)_{n \in \mathbb{N}}$.

Proof. (i). We already know that the unit vector basis $(e_n)_{n \in \mathbb{N}}$ for E is normalized and shrinking, and thus weakly null, so it remains only to verify (3.4). However, for $F \in \mathcal{S}_1$ and scalars $\alpha_j \in \mathbb{K}$, $j \in F$, we have

$$\left\| \sum_{j \in F} \alpha_j e_j \right\|_E = \begin{cases} \sum_{j \in F} |\alpha_j| & \text{if } E = B_p \\ \left(\sum_{j \in F} |\alpha_j|^p \right)^{1/p} & \text{if } E = S_p, \end{cases}$$

which shows that (3.4) is satisfied for $C = 1$.

(ii). Lemma 3.1 applies because the sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are seminormalized and weakly null. Hence, if (3.5) is satisfied, $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ admit subsequences $(x'_n)_{n \in \mathbb{N}}$ and $(y'_n)_{n \in \mathbb{N}}$ which are equivalent to the unit vector basis for D , where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$. It follows that $(x'_n)_{n \in \mathbb{N}}$ and $(y'_n)_{n \in \mathbb{N}}$ are equivalent to each other, and using the two bullet points above, we obtain

$$(x_n)_{n \in \mathbb{N}} \approx_1 (x'_n)_{n \in \mathbb{N}} \approx_1 (y'_n)_{n \in \mathbb{N}} \approx_1 (y_n)_{n \in \mathbb{N}}.$$

On the other hand, if (3.6) is satisfied, $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ admit subsequences $(x'_n)_{n \in \mathbb{N}}$ and $(y'_n)_{n \in \mathbb{N}}$ which are equivalent to subsequences $(e_{j_n})_{n \in \mathbb{N}}$ and $(e_{k_n})_{n \in \mathbb{N}}$ of the unit vector basis $(e_n)_{n \in \mathbb{N}}$ for E . Since $(e_n)_{n \in \mathbb{N}} \in \text{SP}_{1,w}(E)$ by (i), the two bullet points above imply that

$$(x_n)_{n \in \mathbb{N}} \approx_1 (x'_n)_{n \in \mathbb{N}} \approx_1 (e_{j_n})_{n \in \mathbb{N}} \approx_1 (e_n)_{n \in \mathbb{N}} \approx_1 (e_{k_n})_{n \in \mathbb{N}} \approx_1 (y'_n)_{n \in \mathbb{N}} \approx_1 (y_n)_{n \in \mathbb{N}}.$$

In both cases, we conclude that $(x_n)_{n \in \mathbb{N}} \approx_1 (y_n)_{n \in \mathbb{N}}$ because \approx_1 is transitive. \square

Alternative proof of Theorem 1.1, based on Androulakis et al [2]. We check that the conditions for applying the ‘‘Moreover’’ statement in the first part of [2, Theorem 4.1] are satisfied for $\xi = 1$ and $n = 2$:

- Lemma 3.9 implies that $\text{SP}_{1,w}(E)$ contains at most two distinct \approx_1 -equivalence classes because every sequence $(x_j)_{j \in \mathbb{N}}$ in E trivially satisfies

$$\limsup_{j \rightarrow \infty} \|x_j\|_\infty \geq \liminf_{j \rightarrow \infty} \|x_j\|_\infty \geq 0.$$

- Combining Proposition 3.6 with [2, Proposition 2.4(i)], we see that every strictly singular operator on E is ‘‘ \mathcal{S}_1 -strictly singular’’ in the terminology of [2, Definition 2.1].
- No subspace of E is isomorphic to ℓ_1 because E has a shrinking basis.

Hence [2, Theorem 4.1] shows that the composition of two strictly singular operators on E is compact. \square

4. SUBSYMMETRIC BASIC SEQUENCES IN THE BAERNSTEIN AND SCHREIER SPACES

A basis for a Banach space is *subsymmetric* if it is unconditional and equivalent to all its subsequences. The unit vector bases for c_0 and ℓ_p , for $1 \leq p < \infty$, are standard examples

of subsymmetric bases. Using our previous results, we can show that every subsymmetric basic sequence in the Baernstein and Schreier spaces is equivalent to one of those.

This relies on the following lemma, which is certainly known to specialists; we include a short, elementary proof for ease of reference.

Lemma 4.1. *Let $(x_n)_{n \in \mathbb{N}}$ be a norm-bounded unconditional basis for a Banach space X . Then either $(x_n)_{n \in \mathbb{N}}$ is weakly null or it admits a subsequence which is equivalent to the unit vector basis for ℓ_1 .*

Proof. Let K denote the unconditional constant of the basis $(x_n)_{n \in \mathbb{N}}$, and suppose that $(x_n)_{n \in \mathbb{N}}$ is not weakly null. Then there is a functional $x^* \in X^*$ of norm 1 such that the sequence $(\langle x_n, x^* \rangle)_{n \in \mathbb{N}}$ does not tend to 0, so we can pass to a subsequence $(x_{n_j})_{j \in \mathbb{N}}$ for which $\eta := \inf_{j \in \mathbb{N}} |\langle x_{n_j}, x^* \rangle| > 0$.

Take $k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{K}$. For each $1 \leq j \leq k$, we can choose $\sigma_j \in \mathbb{K}$ with $|\sigma_j| = 1$ such that $\sigma_j \alpha_j \langle x_{n_j}, x^* \rangle \geq 0$. Then we have

$$\begin{aligned} K \left\| \sum_{j=1}^k \alpha_j x_{n_j} \right\| &\geq \left\| \sum_{j=1}^k \sigma_j \alpha_j x_{n_j} \right\| \geq \left| \left\langle \sum_{j=1}^k \sigma_j \alpha_j x_{n_j}, x^* \right\rangle \right| \\ &= \left| \sum_{j=1}^k \sigma_j \alpha_j \langle x_{n_j}, x^* \rangle \right| = \sum_{j=1}^k |\alpha_j| |\langle x_{n_j}, x^* \rangle| \geq \eta \sum_{j=1}^k |\alpha_j|. \end{aligned}$$

This shows that the subsequence $(x_{n_j})_{j \in \mathbb{N}}$ K/η -dominates the unit vector basis for ℓ_1 , which on the other hand trivially dominates every bounded sequence by the triangle inequality. The result follows. \square

Remark 4.2. A subsymmetric basis $(x_n)_{n \in \mathbb{N}}$ must be seminormalized. Indeed, if it had no lower norm bound, we could recursively choose an increasing sequence $1 \leq m_1 < m_2 < \dots$ of integers such that $\|x_{m_n}\| \leq \|x_n\|/n$ for each $n \in \mathbb{N}$. However, this would contradict that $(x_{m_n})_{n \in \mathbb{N}}$ dominates $(x_n)_{n \in \mathbb{N}}$. A similar argument shows that $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$.

Proposition 4.3. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$. A basic sequence in E is subsymmetric if and only if it is equivalent to the unit vector basis for D .*

Proof. The backward implication is clear. For the forward implication, suppose that $(u_n)_{n \in \mathbb{N}}$ is a subsymmetric basic sequence in E . Remark 4.2 shows that it is seminormalized, so combining Lemma 4.1 with the fact that no subspace of E is isomorphic to ℓ_1 because E has a shrinking basis, we conclude that $(u_n)_{n \in \mathbb{N}}$ is weakly null. Hence, by Lemma 3.1, it admits a subsequence $(u_{k_n})_{n \in \mathbb{N}}$ which is equivalent to either the unit vector basis for D or a subsequence $(e_{m_n})_{n \in \mathbb{N}}$ of the unit vector basis for E . In the former case, the result follows because $(u_n)_{n \in \mathbb{N}}$ is equivalent to $(u_{k_n})_{n \in \mathbb{N}}$. The latter case is impossible because it would imply that $(e_{m_n})_{n \in \mathbb{N}}$ is subsymmetric, contradicting [12, Theorem 4.1]. \square

5. UNCOMPLEMENTED BLOCK SUBSPACES OF THE BAERNSTEIN AND SCHREIER SPACES

The aim of this section is to show that the Baernstein and Schreier spaces contain block basic sequences whose closed span is not complemented. We follow the approach that Gasparis and Leung took when proving [9, Proposition 4.7], which is the counterpart for the higher-order Schreier spaces $X[\mathcal{S}_n]$ for $n \in \mathbb{N}$. It is based on a lemma of Lindenstrauss and Tzafriri and involves the following standard notion.

Definition 5.1. Let X be a Banach space with a basis $(x_n)_{n \in \mathbb{N}}$. A block basic sequence $(u_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ is *skipped* if there are integers $0 = m_0 < m_1 < m_2 < \dots$ such that

$$u_n \in \text{span}\{x_j : m_{n-1} < j < m_n\} \quad (n \in \mathbb{N}).$$

Proposition 5.2. *The Baernstein spaces B_p , for $1 < p < \infty$, and the Schreier spaces S_p , for $1 \leq p < \infty$, contain block basic sequences whose closed span is not complemented.*

More precisely, let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and let $(u_n)_{n \in \mathbb{N}}$ be a skipped, normalized block basic sequence of the unit vector basis for E such that the unit vector basis for D dominates $(u_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, take $m_n \in (\max \text{supp } u_n, \min \text{supp } u_{n+1}) \cap \mathbb{N}$, and set

$$t_n = \frac{1}{k} \quad (n \in [2^{k-1}, 2^k) \cap \mathbb{N}, k \in \mathbb{N}).$$

Then $(u_n + t_n e_{m_n})_{n \in \mathbb{N}}$ is a block basic sequence whose closed span is not complemented in E .

Proof. We can choose m_n as specified because the block basic sequence $(u_n)_{n \in \mathbb{N}}$ is skipped, and it ensures that $(u_n + t_n e_{m_n})_{n \in \mathbb{N}}$ is a block basic sequence.

Assume towards a contradiction that its closed span is complemented in E . Since $t_n \rightarrow 0$ as $n \rightarrow \infty$, [13, Lemma 2.a.11] implies that the basic sequence $(u_n)_{n \in \mathbb{N}}$ dominates $(t_n e_{m_n})_{n \in \mathbb{N}}$. By hypothesis, the unit vector basis $(d_n)_{n \in \mathbb{N}}$ for D dominates $(u_n)_{n \in \mathbb{N}}$, and $(t_n e_{m_n})_{n \in \mathbb{N}}$ dominates $(t_n e_n)_{n \in \mathbb{N}}$ by Corollary 2.3(i). Hence we can find a constant $C > 0$ such that $(d_n)_{n \in \mathbb{N}}$ C -dominates $(t_n e_n)_{n \in \mathbb{N}}$, so in particular we have

$$C \left\| \sum_{n=2^{k-1}}^{2^k-1} d_n \right\|_D \geq \left\| \sum_{n=2^{k-1}}^{2^k-1} t_n e_n \right\|_E = \frac{1}{k} \left\| \sum_{n=2^{k-1}}^{2^k-1} e_n \right\|_E \quad (5.1)$$

for every $k \in \mathbb{N}$. However,

$$\left\| \sum_{n=2^{k-1}}^{2^k-1} d_n \right\|_D = \begin{cases} 1 & \text{for } D = c_0 \\ 2^{\frac{k-1}{p}} & \text{for } D = \ell_p \end{cases} \quad \text{and} \quad \left\| \sum_{n=2^{k-1}}^{2^k-1} e_n \right\|_E = \begin{cases} 2^{\frac{k-1}{p}} & \text{for } E = S_p \\ 2^{k-1} & \text{for } E = B_p \end{cases}$$

because $[2^{k-1}, 2^k) \cap \mathbb{N} \in \mathcal{S}_1$. Substituting these values into (5.1), we conclude that

$$C \geq \begin{cases} 2^{\frac{k-1}{p}}/k & \text{for } (E, D) = (S_p, c_0) \\ 2^{(k-1)(1-\frac{1}{p})}/k & \text{for } (E, D) = (B_p, \ell_p), \end{cases}$$

which is absurd because the right-hand sides are unbounded as $k \rightarrow \infty$ in both cases. \square

Remark 5.3. (i) In order to apply Proposition 5.2, we must find a skipped, normalized block basic sequence $(u_n)_{n \in \mathbb{N}}$ of the unit vector basis for E that is dominated by the unit vector basis for D . Proposition 2.4 ensures that many such sequences exist.

(ii) One reason that Proposition 5.2 is significant is that in the case of the Baernstein spaces, it disproves [18, Lemma II.3.1], in which Seifert claimed that for every $1 < p < \infty$, every closed subspace of B_p spanned by a seminormalized block basic sequence is complemented in B_p . (We note that “seminormalized” is irrelevant here because a block basic sequence $(u_n)_{n \in \mathbb{N}}$ and its normalization $(u_n/\|u_n\|)_{n \in \mathbb{N}}$ span the same subspace.)

We conclude with a dichotomy which recombines some of our previous results to show that although the closed span of a block basic sequence need not be complemented in the Baernstein or Schreier spaces, their block basic sequences are nevertheless intimately connected to complemented subspaces.

Proposition 5.4. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and let $(u_n)_{n \in \mathbb{N}}$ be a normalized block basic sequence of the unit vector basis for E .*

- (i) *If $\inf_{n \in \mathbb{N}} \|u_n\|_\infty > 0$, then $(u_n)_{n \in \mathbb{N}}$ is equivalent to a subsequence of the unit vector basis for E , and $\overline{\text{span}}\{u_n : n \in \mathbb{N}\}$ is complemented in E .*
- (ii) *Otherwise $(u_n)_{n \in \mathbb{N}}$ admits a subsequence $(u_{n_j})_{j \in \mathbb{N}}$ that is equivalent to the unit vector basis for D and $\overline{\text{span}}\{u_{n_j} : j \in \mathbb{N}\}$ is complemented in E .*

Proof. Part (i) follows from Proposition 2.5(ii)–(iii).

To prove (ii), suppose that $\inf_{n \in \mathbb{N}} \|u_n\|_\infty = 0$. As in the proof of [12, Proposition 2.14], we may apply [12, Lemma 2.8] to see that $(u_n)_{n \in \mathbb{N}}$ admits a subsequence $(u_{n_j})_{j \in \mathbb{N}}$ which is dominated by the unit vector basis $(d_j)_{j \in \mathbb{N}}$ for D , so we have an operator $U \in \mathcal{B}(D, E)$ which satisfies $Ud_j = u_{n_j}$ for $j \in \mathbb{N}$. Furthermore, [12, Lemma 2.10] shows that there is an operator $V \in \mathcal{B}(E, D)$ for which $Vu_{n_j} = d_j$ for $j \in \mathbb{N}$. Hence $(u_{n_j})_{j \in \mathbb{N}}$ is equivalent to $(d_j)_{j \in \mathbb{N}}$, and UV is a projection of E onto $\overline{\text{span}}\{u_{n_j} : j \in \mathbb{N}\}$. \square

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