Partial Passive Ownership and Environmental Innovation: The Role of Emission Taxes and Information Sharing

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Abstract

This paper examines the effects of partial passive ownership (PPO) on environmental research and development (ER&D), environmental quality, and social welfare in a polluting duopoly. We develop a model in which firms can reduce emissions through ER&D investments in clean technologies, subject to either exogenous or endogenous emission tax regimes, and with or without ER&D information sharing. The analysis shows that PPOs consistently reduce aggregate emissions, thereby improving environmental quality. Under exogenous taxation, PPOs increase total ER&D, while under endogenous taxation the effect depends on the severity of environmental damage and the degree of information disclosure. When environmental damage is low, PPOs either leave total ER&D unchanged or reduce welfare; when damage is high, PPOs may enhance both ER&D and welfare, particularly when firms share ER&D information. The results highlight the need for antitrust and environmental regulators to adopt context-specific approaches when assessing PPOs in high-polluting industries.

Keywords: partial passive ownership; environmental R&D; environment; welfare; environmental tax; duopoly

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1 Introduction

Partial passive ownership (PPO)—the acquisition of non-controlling minority stakes in rival firms—is pervasive in pollution-intensive industries. For instance, in 2009 Daimler acquired nearly 10% of Tesla, linking a traditional carmaker with an emerging clean-technology competitor. In the fossil fuel sector, BP long held a 19.75% stake in Rosneft before its withdrawal in 2022, while TotalEnergies continues to hold roughly 19.4% of Novatek. In the European energy market, Repsol maintained about 20% of Gas Natural Fenosa (now Naturgy), illustrating cross-holdings between upstream oil companies and downstream utilities. Although such non-controlling minority holdings do not typically alter formal control rights, they partially internalize rivals' profits, which may soften price and output competition, reshape the marginal incentives for abatement and clean-technology R&D, and thereby exert a "hidden hand" on environmental outcomes and social welfare. Despite their prevalence and economic significance, PPOs remain relatively underexplored in environmental economics. Meanwhile, emission taxes are among the most widely used and administratively feasible instruments for pollution control. Once PPOs are present, however, the transmission of emission taxes to firms' output and R&D decisions may be refracted through ownership structures, potentially altering the environmental and welfare effects of tax policy in systematic ways.

To reduce pollutant emissions, firms may either adopt end-of-pipe abatement technologies or invest in cleaner production technologies. Cleaner production emphasizes pollution prevention at the source—through measures such as raw material substitution or technological upgrading—whereas end-of-pipe solutions target pollutants after they are generated, for example via exhaust gas purification or wastewater treatment. While end-of-pipe technologies can mitigate environmental impacts, they often generate secondary pollution, making cleaner production comparatively more advantageous. Empirical evidence shows that in many OECD countries, investments in cleaner technologies far exceed those in end-of-pipe solutions (Frondel et al., 2007). By contrast, the theoretical literature on strategic environmental R&D under environmental policy has primarily focused on end-of-pipe approaches, with far fewer contributions examining cleaner production. This mismatch between practice and academic attention provides the central motivation for our study. A key innovation of our analysis is therefore to incorporate cleaner production technologies into the framework and to examine how PPOs affect firms' incentives for such strategic environmental R&D.

Building on this motivation, our analysis is guided by four interrelated research questions that, despite their importance, have not yet been systematically addressed in the environmental economics literature. First, we ask how PPOs affect firms' incentives to undertake environmental R&D in a duopoly. While existing work on cross-ownership has largely focused on cost-reducing or product-quality R&D, little is known about how the partial internalization of rivals' profits interacts with incentives to invest in cleaner production technologies that reduce emissions. Understanding this mechanism is crucial, as R&D choices determine the long-run trajectory of environmental performance. Second, we investigate the implications of PPOs for environmental quality itself. By shaping firms' output and abatement incentives, PPOs may either alleviate or exacerbate pollution. Yet the direction and magnitude of these effects remain unclear once firms can adopt cleaner technologies rather than relying solely on end-of-pipe solutions. This raises the question of whether PPOs can, under certain conditions, lead to improvements in aggregate environmental outcomes. Third, we consider the welfare consequences of PPOs. The conventional view in industrial organization is that cross-ownership tends to soften competition, thereby reducing consumer surplus. However, when combined with environmental externalities, the welfare calculus becomes more complex: reductions in emissions may offset losses in consumer surplus, potentially yielding net gains. Whether PPOs systematically reduce or enhance welfare in pollution-intensive industries is therefore an open question. Finally, we examine the role of policy design and information disclosure in shaping these outcomes. Emission taxes are the predominant regulatory instrument in many jurisdictions, but their interaction with ownership structures has received little attention. Moreover, whether firms disclose or share information about their environmental R&D may critically alter the effects of PPOs on both innovation incentives and welfare. Clarifying these interactions is essential for designing policies that account for the realities of modern ownership patterns.

To address these questions, we construct a duopoly model in which one firm holds a minority stake in its rival. Firms can reduce emissions by investing in clean technologies, and the effectiveness of abatement depends on their environmental R&D effort. The government levies an emission tax on firms' pollution, and we consider two distinct regimes: one in which the tax rate is taken as exogenously fixed and another in which it is endogenously chosen to maximize social welfare.[1]

Our analysis yields three main insights. First, PPOs always raise aggregate R&D when the tax rate is exogenous, but their effect under endogenous taxation depends crucially on the severity of environmental damage and on whether firms disclose R&D information. When damages are minor, total R&D remains unchanged; when damages are severe, PPOs reduce R&D in the absence of information sharing but increase it when full disclosure occurs. Second, PPOs consistently improve environmental quality across both tax regimes, as they lead to lower aggregate emissions. Third, the welfare implications of PPOs are more nuanced. With low environmental damage, PPOs invariably reduce welfare regardless of the policy setting. With high damage, however, their effect on welfare can be either positive or negative, depending on the

[1] In practice, emission taxes rarely coincide with the welfare-maximizing level. Rates in most developed countries are often set above the optimum, while in many developing countries they remain below it. Such deviations justify treating the tax as exogenous in certain contexts.

tax regime, the extent of information disclosure, and the level of ownership. Taken together, these findings highlight that under conditions of severe environmental harm, PPOs can, in conjunction with appropriate tax policies and information sharing, stimulate innovation, improve environmental quality, and enhance social welfare. They also suggest that antitrust authorities should evaluate PPOs in pollution-intensive industries with careful attention to the institutional environment in which they operate.

This study is related to two strands of PPO literature. The first examines the impact of PPOs on strategic R&D. Existing work typically considers cost-reducing or quality-improving innovation, showing the conditions under which cross-ownership may stimulate R&D investment (Shelegia and Spiegel, 2015; Bayona and López, 2018; López and Vives, 2019; Papadopoulos et al., 2019; Brito et al., 2020; Vives, 2020; Xing and Tan, 2023; Mukherjee, 2023; Xing et al., 2024; Chen et al., 2024; Cheng et al., 2024; Antón et al., 2025). Liu et al. (2020), departing from this focus, analyzes product-differentiating R&D in a mixed duopoly and finds that PPOs can increase incentives to innovate. Unlike these contributions, however, the existing literature has not addressed environmental R&D (ER&D), and therefore has overlooked how ownership ties may affect incentives to adopt clean technologies. This gap is particularly notable since the concept of PPO itself has been extensively studied in industrial organization and corporate finance (Papadopoulos et al., 2019; Leonardos et al., 2021; Xing and Tan, 2023), but remains largely unexplored in the context of environmental economics.

The second strand investigates the interaction between PPOs and environmental regulation. Prior studies show that PPOs may alter environmental damage and welfare when emission taxes are imposed, either cooperatively across countries or domestically by governments (Bárcena-Ruiz and Campo, 2012, 2017; Lian et al., 2025). Yet these analyses generally assume end-of-pipe abatement technologies, which lower total emissions without affecting output. By contrast, clean technologies reduce emissions per unit of output, thereby intertwining production and abatement decisions in ways that can fundamentally change welfare outcomes (Tsai et al., 2016; Xing et al., 2022). Moreover, earlier contributions concentrate almost exclusively on endogenous emission taxes, where governments adjust rates optimally in response to damage, while in practice taxes are often fixed exogenously by national or local authorities for extended periods (Hirose and Matsumura, 2020; Li and Shang, 2022; Xing et al., 2023). These assumptions limit the applicability of existing results to real-world policy environments.

Against this background, our paper makes the following contribution. On the academic side, we provide the first analysis that integrates PPOs with clean-technology ER&D, explicitly comparing exogenous and endogenous emission tax regimes and allowing for different degrees of information disclosure. This framework uncovers novel mechanisms through which ownership structures influence firms' innovation incentives, aggregate emissions, and welfare outcomes. On the policy side, our results highlight the need to consider the interaction between competition policy and environmental regulation. PPOs are often scrutinized by antitrust authorities because of their potential to soften product market competition, yet our findings suggest that their implications depend critically on the environmental policy environment. Recognizing the joint determination of market power, innovation, and externalities can help regulators in high-polluting industries to design more effective and context-specific interventions.

Methodologically, our analysis further contributes by employing the Cylindrical Algebraic Decomposition (CAD) method. This approach allows us to conduct comparative statics in the endogenous-tax framework even in cases where closed-form solutions for the welfare-maximizing tax are unavailable. It is worth noting that the CAD method is grounded in symbolic computation, ensuring that the results are exact and can therefore serve as a basis for proving theorems. We believe that the application of the CAD method will greatly facilitate future research on strategic environmental R&D problems under endogenous policy regimes.

The remainder of the paper is organized as follows. Section 2 introduces the basic model. Section 3 examines the effects of PPOs without ER&D information sharing under both exogenous and endogenous emission tax regimes, while Section 4 extends the analysis to the case with information sharing. Section 5 concludes.

2 Model Setup

Consider a polluting duopoly with two firms, indexed by $i = 1, 2$, producing a homogeneous good. The inverse demand function is given by

$$p = 1 - q_1 - q_2,$$

where p denotes the market price and q_i the output of firm i. Production generates environmental pollution, and for tractability we assume that each unit of output produces one unit of pollutant.

Firms can reduce emissions by adopting clean technologies, the effectiveness of which depends on their environmental R&D (ER&D) efforts (Xing and Tan, 2021). After ER&D investment, the emission per unit of output of firm i decreases from 1 to $1 - y_i - by_j$, where y_i denotes firm i's ER&D effort ($y_i + by_j \in [0, 1]$) and $b \in [0, 1]$ captures the degree of spillovers. The ER&D cost function is

$$I(y_i) = \frac{\gamma}{2} y_i^2,$$

with γ representing the efficiency parameter; for simplicity we normalize $\gamma = 1$. Hence, firm i's total emissions are

$$e_i = (1 - y_i - by_j)q_i.$$

Following Youssef and Dinar (2011), environmental damage is specified as

$$D(E) = dE,$$

where $E = e_1 + e_2$ denotes aggregate emissions and d is the marginal damage parameter.[2]

The government imposes an emission tax on each firm's pollutant emissions. When firm i generates emissions e_i, it must pay

$$T_i(e_i) = te_i,$$

where t denotes the emission tax rate (Li and Shang, 2022). Throughout the analysis we assume $0 \leq t < d$.[3] The profit function for firm i is thus:

$$\pi_i = pq_i - T_i(e_i) - I(y_i). \tag{1}$$

Firm 1 holds a partial passive ownership (PPO) stake in firm 2, owning a share $k \in [0, 0.5]$ of its rival's profits (Leonardos et al., 2021). Both firms maximize total profits, leading to the following objective functions:

$$\Pi_1 = \pi_1 + k\pi_2, \tag{2}$$

$$\Pi_2 = (1 - k)\pi_2. \tag{3}$$

Social welfare is defined as the sum of producer surplus, consumer surplus, and total emission tax revenues, net of environmental damage:

$$SW = (\Pi_1 + \Pi_2) + CS + T - D(E), \tag{4}$$

where $T = T_1(e_1) + T_2(e_2)$ denotes total tax payments and CS denotes consumer surplus. Given linear

[2]This linear specification is also adopted by Tsai et al. (2016), Nie et al. (2019), and Xing and Tan (2021). To ensure $y_i + by_j \in [0, 1]$ and $q_i \geq 0$ in equilibrium, we assume $0 < d \leq 0.55$. d represents the environmental externality relative to the unit value of the good. Empirically, the threshold between low-damage ($d = 0.05$) and high-damage ($d = 0.55$) regions differentiates between industries where market-power distortions are the primary welfare concern (such as light manufacturing with low emission intensity) and high-polluting sectors (such as fossil fuel power generation or heavy chemicals) where the social cost of pollution internalizes a larger portion of the total welfare calculus.

[3]This assumption, which posits that the tax rate is lower than the marginal environmental damage, is commonly accepted in the literature on imperfect competition Ohori (2006), Tsai et al. (2016), Lee and Xu (2018), and Xing and Tan (2021).

demand, consumer surplus is

$$CS = \frac{1}{2}(q_1 + q_2)^2.$$

We distinguish between two regimes depending on whether the emission tax rate t is exogenous or endogenous. In Case I, t is exogenously given. Firms first choose their ER&D efforts in stage 1 and subsequently determine output in stage 2. In Case II, t is endogenously determined. The government sets the welfare-maximizing tax rate in stage 1, after which firms choose ER&D efforts in stage 2 and output in stage 3.

Furthermore, firms may differ in the degree of ER&D information disclosure. They may either withhold their research outcomes ($b = 0$) or fully share them ($b = 1$). The subsequent sections examine these scenarios in detail.

3 Equilibrium Analysis: No Information Sharing

In this section, firms choose their ER&D investments non-cooperatively and do not share their ER&D information.

3.1 Exogenous Emission Tax

We first analyze the case in which the emission tax rate is exogenously determined. The model is solved by backward induction, beginning with the output stage. At this stage, firm i chooses its optimal output by maximizing Π_i. The corresponding first-order conditions are

$$\frac{\partial \Pi_1}{\partial q_1} = 1 - t(1 - y_1) - 2q_1 - (1 + k)q_2 = 0 \tag{5}$$

$$\frac{\partial \Pi_2}{\partial q_2} = (1 - k)[1 - t(1 - y_2) - q_1 - 2q_2] = 0 \tag{6}$$

Solving (5)–(6) yields reaction output

$$q_1(y_1, y_2) = \frac{(1 - t)(1 - k) + t[2y_1 - (1 + k)y_2]}{3 - k} \tag{7}$$

$$q_2(y_1, y_2) = \frac{1 - t - t(y_1 - 2y_2)}{3 - k} \tag{8}$$

The second-order conditions (SOCs) are satisfied since

$$\frac{\partial^2 \Pi_1}{\partial q_1^2} = -2 < 0 \quad \text{and} \quad \frac{\partial^2 \Pi_2}{\partial q_2^2} = -2(1 - k) < 0.$$

Substituting (7)–(8) into the profit functions yields

$$\pi_1(y_1, y_2) = \frac{1}{(3-k)^2}\big[(1-t) + t(2-k)y_1 - t(1-k)y_2\big]\big[(1-t)(1-k) + 2ty_1 - t(1+k)y_2\big] - \frac{1}{2}y_1^2, \tag{9}$$

$$\pi_2(y_1, y_2) = \frac{\big[(1-t) - t(y_1 - 2y_2)\big]^2}{(3-k)^2} - \frac{1}{2}y_2^2 \tag{10}$$

In the first stage, firms choose their ER&D efforts. The FOCs are

$$\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1} = \frac{\partial \pi_1(y_1, y_2)}{\partial y_1} + k\,\frac{\partial \pi_2(y_1, y_2)}{\partial y_1} = 0, \qquad \frac{\partial \Pi_2(y_1, y_2)}{\partial y_2} = (1-k)\,\frac{\partial \pi_2(y_1, y_2)}{\partial y_2} = 0,$$

which can be written explicitly as

$$\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1} = \frac{(1-t)t(4-k)(1-k) - \big[(3-k)^2 - 2t^2(4-k)\big]y_1 - t^2(4-k)(1+k)y_2}{(3-k)^2} = 0, \tag{11}$$

$$\frac{\partial \Pi_2(y_1, y_2)}{\partial y_2} = (1-k)\frac{4t(1-t) - 4t^2 y_1 - \big[(3-k)^2 - 8t^2\big]y_2}{(3-k)^2} = 0. \tag{12}$$

Solving (11)–(12) delivers equilibrium ER&D efforts

$$y_1^{ex} = \frac{t(1-t)(4-k)(3-k)\big[(1-k)(3-k) - 4t^2\big]}{\big[(3-k)^2 - 2t^2(4-k)\big]\big[(3-k)^2 - 8t^2\big] - 4t^4(4-k)(1+k)}, \tag{13}$$

$$y_2^{ex} = \frac{4t(1-t)(3-k)\,\big[(3-k) - t^2(4-k)\big]}{\big[(3-k)^2 - 2t^2(4-k)\big]\big[(3-k)^2 - 8t^2\big] - 4t^4(4-k)(1+k)}. \tag{14}$$

It can be shown that $0 \leq y_i^{ex} \leq 1$ for $i = 1, 2$. The SOCs hold since

$$\frac{\partial^2 \Pi_1}{\partial y_1^2} = \frac{(3-k)^2 - 2t^2(4-k)}{(3-k)^2} < 0 \quad \text{and} \quad \frac{\partial^2 \Pi_2}{\partial y_2^2} = \frac{\big[(3-k)^2 - 8t^2\big](1-k)}{(3-k)^2} < 0.$$

Substituting (13)–(14) into (7)–(8) yields equilibrium output

$$q_1^{ex} = \frac{(1-t)(1-k)\,w + t\,[\,2\theta - (1+k)\psi\,]}{(3-k)\,w}, \tag{15}$$

$$q_2^{ex} = \frac{(1-t)\,w - t\,(\theta - 2\psi)}{(3-k)\,w}, \tag{16}$$

where

$$\theta = t(1-t)(4-k)(3-k)\big[(1-k)(3-k) - 4t^2\big], \quad \psi = 4t(1-t)(3-k)\big[(3-k) - t^2(4-k)\big],$$

and

$$w = \left[(3-k)^2 - 2t^2(4-k)\right]\left[(3-k)^2 - 8t^2\right] - 4t^4(4-k)(1+k).$$

Differentiation shows that

$$\frac{\partial q_1^{ex}}{\partial k} < 0, \qquad \frac{\partial q_2^{ex}}{\partial k} > 0, \qquad \frac{\partial (q_1^{ex} + q_2^{ex})}{\partial k} < 0.$$

Hence, as k rises, firm 1 internalizes more of firm 2's profits and competes less aggressively, reducing q_1^{ex}; by contrast, firm 2 expands (q_2^{ex} increases), yet the contraction of q_1^{ex} dominates so that total output falls.

Using (13)–(16), one can derive equilibrium profits, environmental damage, and social welfare (reported in Appendix A). The main results are summarized as follows.[4]

Proposition 1. *When t is exogenous and firms do not share ER&D information:*

(i) $\frac{\partial y_1^{ex}}{\partial k} < 0$, $\frac{\partial y_2^{ex}}{\partial k} > 0$, *and* $\frac{\partial (y_1^{ex} + y_2^{ex})}{\partial k} > 0$;

(ii) $\frac{\partial D^{ex}}{\partial k} < 0$;

(iii) $\frac{\partial SW^{ex}}{\partial k} < 0$ *if d is low, while* $\frac{\partial SW^{ex}}{\partial k} > (<)\, 0$ *if d is high and t is low (high).*

Proposition 1(i) demonstrates that under an exogenous emission tax with no ER&D information sharing, PPO has a differential effect on the firms' ER&D efforts. Specifically, PPOs negatively impact the ER&D of the owning firm (firm 1) while positively affecting the ER&D of the partially owned firm (firm 2). The net effect is an overall increase in total ER&D. This finding challenges the conventional concern among antitrust authorities that PPOs, which can be likened to "partial" mergers, might stifle innovation (Vives, 2020). Our results suggest that PPOs can, in fact, boost total ER&D investment in a polluting industry, even without information sharing. The mechanism parallels the output effects: as k increases, firm 1 partially internalizes its rival's profit and therefore reduces its output q_1^{ex}, which dampens its ER&D incentives. By contrast, q_2^{ex} increases, strengthening firm 2's incentives. The latter effect dominates in the aggregate.

Proposition 1(ii) establishes that PPOs consistently reduce total environmental damage, thereby enhancing environmental quality. The intuition is twofold. First, a higher PPO stake (k) lowers both the output and ER&D of firm 1, with emissions primarily reduced through lower output. Second, a higher k raises both the output and ER&D of firm 2, which tends to increase its emissions. Since the reduction in firm 1's emissions dominates the increase in firm 2's, the net effect of a higher k is a decline in aggregate emissions and, consequently, total environmental damage.

[4] Proofs of all propositions and lemmas are provided in Appendix B.

Proposition 1(iii) shows that PPOs may enhance social welfare, contrary to the conventional view that they generally reduce welfare (Bárcena-Ruiz and Campo, 2017). Combined with Proposition 1(ii), this suggests that PPOs can generate a "win–win" outcome, improving both environmental quality and welfare, when marginal damage is high and the emission tax rate is low. The mechanism operates as follows. PPOs reduce consumer surplus because total output falls, but they increase producer surplus by softening market competition. Moreover, since total emissions decline with k, both environmental damage and tax revenues decrease. When d is low, environmental harm is minor, and when d is high but t is high, most of the pollution externality is already internalized through taxation. In both cases, the negative effects of PPOs (lower consumer surplus and lower tax revenue) outweigh the positive effects, leading to reduced welfare. By contrast, when d is high and t is low, the externality is insufficiently internalized, so the positive effects of PPOs (higher producer surplus and lower environmental damage) dominate, and welfare increases with k.

To clarify the structural origins of these welfare changes, we decompose the total change in social welfare (ΔSW) into its constituent economic channels: $\Delta SW = \Delta CS + \Delta PS + \Delta TR - \Delta D$. Within this framework, the impact of PPOs is determined by the shifting dominance between product-market distortions and environmental externalities. The product-market channel is characterized by a tradeoff where the increase in k softens competition, leading to a loss in consumer surplus ($\Delta CS < 0$) that is partially offset by an increase in producer surplus ($\Delta PS > 0$). Simultaneously, the fiscal and environmental channels reflect the reduction in aggregate emissions; while this generates a significant welfare gain through lower environmental damage ($-\Delta D > 0$), it also results in a contraction of the tax base and a subsequent decline in fiscal revenue ($\Delta TR < 0$). In the low-damage region (d is small), the environmental benefit is negligible, allowing the standard consumer-surplus loss from softened competition to dominate the welfare calculus, resulting in $\Delta SW < 0$. Conversely, in the high-damage region (d is large), the environmental improvement $-\Delta D$ becomes the primary driver of welfare. In this context, PPOs serve as an informal corrective mechanism for under-internalized externalities, and the reduction in environmental harm—often coupled with the gain in producer surplus—becomes sufficiently large to outweigh the combined losses in consumer surplus and tax revenue, thereby yielding a net welfare improvement.

3.2 Endogenous Emission Tax

We now consider the case in which the emission tax is endogenously determined by the government, which acts as a welfare-maximizing policymaker. The interaction unfolds as a three-stage game: (i) in stage 1, the government sets the emission tax rate to maximize social welfare; (ii) in stage 2, firms independently choose their ER&D levels; and (iii) in stage 3, firms determine their production quantities. This structure allows us to assess how PPOs affect ER&D incentives, environmental outcomes, and social welfare when taxation is policy-driven.

The equilibrium output and ER&D efforts in stages 2 and 3 are identical to those derived in the exogenous-tax case, given by (7), (8), (13) and (14). The social welfare function as a function of t is reported in Appendix A. In stage 1, the government chooses the welfare-maximizing tax rate, denoted t^{en}. We distinguish two cases depending on the marginal damage parameter: (i) a low-damage case ($d = 0.05$) and (ii) a high-damage case ($d = 0.55$).

In the low-damage case, the optimal tax rate is $t^{en} = 0$. Substituting this value into (13) and (14) yields

$$y_1^{en} = y_2^{en} = 0 \tag{17}$$

The corresponding equilibrium output is

$$q_1^{en} = \frac{1-k}{3-k} \tag{18}$$

and

$$q_2^{en} = \frac{1}{3-k} \tag{19}$$

Hence, $\partial q_1^{en}/\partial k < 0$, $\partial q_2^{en}/\partial k > 0$, and $\partial(q_1^{en} + q_2^{en})/\partial k < 0$. That is, as k increases, firm 1 and aggregate output decline, while firm 2's output expands. The associated equilibrium profits, environmental damage, and social welfare are reported in Appendix A.

In the high-damage case, a closed-form expression for t^{en} cannot be obtained. However, symbolic computation establishes that $0.25 < t^{en} < 0.32$ when $d = 0.55$, implying that the optimal tax rate is positive but strictly below the marginal damage. In this case, the comparative statics with respect to k remain qualitatively unchanged: $\partial q_1^{en}/\partial k < 0$, $\partial q_2^{en}/\partial k > 0$, and $\partial(q_1^{en} + q_2^{en})/\partial k < 0$.

Lemma 1. *When t is endogenous and firms do not share ER&D information:*

(i) if d is low, then $t^{en} = 0$ and $\partial t^{en}/\partial k = 0$;

(ii) if d is high, then $t^{en} > 0$ and $\partial t^{en}/\partial k < 0$.

The intuition is straightforward. When d is low, environmental damage is relatively minor. Introducing an emission tax would reduce production, thereby lowering both consumer surplus and firms' profits, without generating substantial environmental benefits. Consequently, the optimal tax rate is zero and unaffected by PPOs. By contrast, when d is high, emissions impose significant damage. Taxation mitigates this effect at the cost of lower output and profits, thereby improving welfare. As k rises, industry output falls, which reduces environmental harm even in the absence of taxation. Hence, the government optimally sets a lower tax rate.

Proposition 2. *When t is endogenous and firms do not share ER&D information:*

(a) If d is low: (i) $\frac{\partial y_1^{en}}{\partial k} = 0$, $\frac{\partial y_2^{en}}{\partial k} = 0$ and $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} = 0$; (ii) $\frac{\partial D^{en}}{\partial k} < 0$; (iii) $\frac{\partial SW^{en}}{\partial k} < 0$.

(b) If d is high: (i) $\frac{\partial y_1^{en}}{\partial k} < 0$, $\frac{\partial y_2^{en}}{\partial k} > 0$ and $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} < 0$; (ii) $\frac{\partial D^{en}}{\partial k} < 0$; (iii) $\frac{\partial SW^{en}}{\partial k} \lesseqgtr 0$ depending on whether k is small (negative effect) or large (positive effect).

In contrast to the exogenous-tax case, PPOs do not influence firms' ER&D investment when t is endogenous and d is low (Proposition 2(a)(i)). The reason is straightforward: with negligible environmental damage, the government sets $t^{en} = 0$, eliminating any incentive for abatement. Consequently, both firms refrain from ER&D. Proposition 2(a)(ii) follows the same intuition as Proposition 1(ii): PPOs reduce environmental damage because the decline in firm 1's output outweighs the increase in firm 2's. Proposition 2(a)(iii) implies that an increase in k reduces consumer surplus through lower total output, raises producer surplus by softening competition, and mitigates environmental damage through reduced emissions. When d is low, however, the environmental gains are minor. In this case, the negative effects—lower consumer surplus and reduced tax revenue—dominate, leading to a decline in social welfare as k rises. [5]

The situation differs when d is high. In this case, PPOs affect firms' ER&D decisions and may have a non-monotonic effect on social welfare. Comparing the exogenous and endogenous regimes reveals that while the effects of PPOs on individual ER&D and environmental damage are qualitatively similar, their implications for total ER&D and welfare diverge. Under exogenous taxation, PPOs increase total ER&D and may raise or lower welfare. Under endogenous taxation, by contrast, PPOs reduce total ER&D and may generate a U-shaped relationship between k and social welfare. The key mechanism lies in the tax-setting rule. When t is exogenous, it can be arbitrarily low or high, amplifying either the positive or negative effects of PPOs. When t is endogenous, however, the government selects a moderate tax rate that partially internalizes environmental damage. This moderating effect alters the welfare implications: when k is small, the negative consequences of PPOs dominate, while for sufficiently large k, the positive effects—higher producer surplus and lower environmental damage—prevail, resulting in an improvement in welfare. This

[5] The optimality of $t^{en} = 0$ in the low-damage region serves as a theoretical baseline where the output-distortion cost of taxation in a duopoly exceeds the marginal environmental benefits. To ensure the robustness of this result, we consider several extensions in Appendix C, including the use of a convex damage function $D(E) = dE^2$ and the introduction of a fiscal revenue motive $\lambda > 1$. We find that while a sufficiently large fiscal motive or institutional constraint can induce a positive tax rate, our qualitative conclusion regarding the welfare-reducing effect of PPOs under low environmental damage remains robust, as the negative impact on consumer surplus and fiscal revenue continues to dominate modest gains in producer surplus and environmental quality.

non-monotonic, U-shaped relationship between k and social welfare illustrates the shifting dominance of the structural components identified in the preceding decomposition; specifically, for low levels of ownership, the welfare losses from the consumer channel and reduced tax revenue prevail, whereas the environmental channel ($-\Delta D$) only begins to dominate and drive a net welfare improvement once k reaches a critical threshold.

In institutional settings where a regulatory floor $t \geq \bar{t} > 0$ is imposed—due to international treaties, policy inertia, or minimum standards—the endogenous tax would simply bind at $\bar{t}$ whenever the unconstrained optimum is zero. Under such conditions, the endogenous regime effectively converges to the exogenous analysis presented in Section 3.1, where the tax rate is treated as a fixed institutional parameter. This constraint truncates the corner solution without overturning the fundamental economic trade-off between output distortion and environmental benefit.

4 Equilibrium Analysis: Full Information Sharing

In practice, firms may establish technology alliances through PPOs and share the outcomes of their R&D activities. We therefore consider the case in which firms make their ER&D investments non-cooperatively but fully disclose their research outcomes (i.e., $b = 1$). This setting corresponds to competition under an environmental research joint venture (ERJV), where firms independently choose ER&D levels while benefiting from complete information sharing.

4.1 Exogenous Emission Tax

The analytical procedure mirrors that in Section 3.1; for brevity, we present only the equilibrium results. The equilibrium ER&D levels are given by

$$y_1^{ex} = \frac{2t(1-t)}{(3-k)^2 - 4t^2}, \tag{20}$$

$$y_2^{ex} = \frac{2t(1-t)}{(3-k)^2 - 4t^2}. \tag{21}$$

It can be verified that $0 \leq y_1^{ex} + y_2^{ex} \leq 1$. The second-order conditions are satisfied since

$$\frac{\partial^2 \Pi_1}{\partial y_1^2} = \frac{(3-k)^2 - 2t^2}{(3-k)^2} < 0, \quad \frac{\partial^2 \Pi_2}{\partial y_2^2} = \frac{(1-k)[(3-k)^2 - 2t^2]}{(3-k)^2} < 0.$$

Hence, the corresponding equilibrium output is

$$q_1^{ex} = \frac{(1-t)(3-k)(1-k)}{(3-k)^2 - 4t^2}, \tag{22}$$

$$q_2^{ex} = \frac{(1-t)(3-k)}{(3-k)^2 - 4t^2}. \tag{23}$$

Comparative statics yield $\partial q_1^{ex}/\partial k < 0$, $\partial q_2^{ex}/\partial k > 0$, and $\partial (q_1^{ex} + q_2^{ex})/\partial k < 0$. Thus, as in the no-sharing case, firm 1's and total output fall with k, while firm 2's output rises.

Proposition 3. *When t is exogenously given and firms fully share their ER&D information:*

(i) $\frac{\partial y_1^{ex}}{\partial k} > 0$, $\frac{\partial y_2^{ex}}{\partial k} > 0$ *and* $\frac{\partial (y_1^{ex}+y_2^{ex})}{\partial k} > 0$;

(ii) $\frac{\partial D^{ex}}{\partial k} < 0$;

(iii) $\frac{\partial SW^{ex}}{\partial k} < 0$ *if d is low, while* $\frac{\partial SW^{ex}}{\partial k} > 0$ *if d is high.*

The results in Proposition 3 largely parallel those in Proposition 1, particularly with respect to the effects of k on firm 2's ER&D, total ER&D, and environmental damage. This indicates that, under an exogenous tax, the qualitative insights derived without information sharing remain robust when firms fully disclose their ER&D outcomes. A notable difference arises, however, in the behavior of the holding firm. With full information sharing, PPOs stimulate rather than discourage firm 1's ER&D. The reason is that firm 1's investment now benefits not only itself but also firm 2, by further reducing firm 2's emissions and associated tax payments. Since firm 1 appropriates part of firm 2's profits through its ownership stake, a larger k increases the return from such spillovers, thereby strengthening firm 1's incentives to invest in ER&D.

The welfare implications of PPOs also differ in the high-damage case. When d is low, the mechanism is similar to that described in Proposition 1(iii): total output falls, consumer surplus decreases, and welfare declines with k. By contrast, when d is high, PPOs always improve social welfare. In this case, the pollution externality is largely internalized by the emission tax, and the main welfare effects operate through changes in producer and consumer surplus. PPOs enhance producer surplus by encouraging greater ER&D and raising industry profits, while reducing consumer surplus through higher prices and lower output. Under full information sharing, the positive effect on producer surplus is amplified because ER&D investments generate joint benefits for both firms. As a result, the increase in producer surplus is sufficient to offset the loss in consumer surplus, implying that PPOs have an unambiguously positive effect on social welfare when environmental damage is severe.

4.2 Endogenous Emission Tax

The solution procedure in this case parallels that of Section 3.2. For brevity, we present only the equilibrium results below.

When d is low, the optimal emission tax is $t^{en} = 0$. Substituting this into the equilibrium conditions yields

$$y_1^{en} = y_2^{en} = 0, \qquad q_1^{en} = \frac{1-k}{3-k}, \qquad q_2^{en} = \frac{1}{3-k}.$$

The remaining equilibrium expressions are reported in Appendix A. When d is high, a closed-form solution for t^{en} is not available. Symbolic computation shows, however, that the qualitative properties of equilibrium are preserved:

$$\frac{\partial q_1^{en}}{\partial k} < 0, \qquad \frac{\partial q_2^{en}}{\partial k} > 0, \qquad \frac{\partial (q_1^{en} + q_2^{en})}{\partial k} < 0.$$

Thus, as in the low-damage case, firm 1's output and total output decline with k, while firm 2's output increases.

Lemma 2. *When t is endogenous and firms share ER&D information:*

(i) if d is low, then $t^{en} = 0$ and $\frac{\partial t^{en}}{\partial k} = 0$;

(ii) if d is high, then $t^{en} > 0$ and $\frac{\partial t^{en}}{\partial k} < 0$.

The intuition is straightforward. When environmental damage is minor (d low), taxation reduces output without generating substantial environmental benefits, so the government optimally sets $t^{en} = 0$, and PPOs do not affect the tax rate. When damage is severe (d high), taxation is welfare-improving. However, as k rises, total industry output declines, which reduces emissions even absent taxation. The government therefore sets a lower optimal tax rate, implying $\partial t^{en}/\partial k < 0$.

Proposition 4. *When t is endogenous and firms share ER&D information:*

(a) If d is low: (i) $\frac{\partial y_1^{en}}{\partial k} = 0$, $\frac{\partial y_2^{en}}{\partial k} = 0$, and $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} = 0$; (ii) $\frac{\partial D^{en}}{\partial k} < 0$; (iii) $\frac{\partial SW^{en}}{\partial k} < 0$.

(b) If d is high: (i) $\frac{\partial y_1^{en}}{\partial k} > 0$, $\frac{\partial y_2^{en}}{\partial k} > 0$, and $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} > 0$; (ii) $\frac{\partial D^{en}}{\partial k} < 0$; (iii) $\frac{\partial SW^{en}}{\partial k} > 0$.

Proposition 4 highlights two important contrasts. First, when d is low, PPOs reduce environmental damage but leave ER&D incentives unchanged, while welfare declines due to lower consumer surplus and reduced tax revenues. Second, when d is high, PPOs encourage ER&D investment by both firms, thereby enhancing total ER&D. In this case, PPOs generate a joint improvement in environmental quality and welfare.

Relative to Proposition 2, the effects of PPOs under low damage are unchanged regardless of whether firms share ER&D information. In contrast, under high damage, information sharing alters the results: PPOs increase firm 1's ER&D as well as overall welfare. This suggests that the welfare implications of PPOs in highly polluting industries depend critically on the extent of ER&D information disclosure.

The mechanisms underlying Proposition 4(a) and Proposition 4(b)(ii) are analogous to those discussed in Proposition 2. The intuition for Proposition 4(b)(i) follows Proposition 3(i). The novel element concerns Proposition 4(b)(iii): when d is high and firms share ER&D information, the positive effects of PPOs—higher producer surplus and lower environmental damage—dominate the negative effects—reduced tax revenues and consumer surplus. As a result, social welfare unambiguously rises with k.

The structural origins of the welfare gains observed under high environmental damage are further illuminated by the role of information disclosure. Under full information sharing ($b = 1$), the impact of PPOs on the producer channel (ΔPS) is significantly amplified because ER&D investments generate joint benefits through complete spillovers. Unlike the no-sharing scenario, where PPOs may discourage the owning firm's innovation, the presence of information disclosure ensures that PPOs stimulate ER&D investment for both the owning and the partially owned firm. This enhanced innovation efficiency strengthens the producer channel and accelerates the reduction in environmental harm ($-\Delta D$), ensuring that these positive welfare components unambiguously dominate the losses in consumer surplus and fiscal revenue when environmental damage is severe. Consequently, the "win-win" outcome for environmental quality and social welfare is more robustly achieved under an ERJV structure, as the synergy between ownership ties and information sharing mitigates the traditional market-power distortions.

To assess whether our results are sensitive to the linear damage assumption $D(E) = dE$, we test our model against a convex specification $D(E) = dE^2$(see Appendix C). Under this form, marginal damages increase with aggregate emissions, capturing tipping-point dynamics or non-linear health risks. Our symbolic and numerical verifications confirm that the fundamental effects of PPOs on aggregate emissions reduction and individual ER&D efforts (Propositions 1-4) remain qualitatively unchanged. However, the welfare trajectory under the endogenous tax regime (Proposition 2) exhibits higher sensitivity; when d is sufficiently large (e.g., $d = 1$), the welfare-enhancing effect of PPOs becomes monotonic rather than U-shaped. This reinforces our core finding that in high-polluting contexts, the environmental benefit of ownership ties acts as a robust corrective force against externalities, even when those externalities follow non-linear growth patterns.

5 Policy Implications and Conclusions

This paper has examined the effects of partial passive ownership (PPOs) on environmental R&D (ER&D), environmental quality, and social welfare in a polluting duopoly. We considered both exogenous and endogenous emission tax regimes and allowed for the presence or absence of ER&D information sharing. Several key findings emerge.

First, PPOs consistently improve environmental quality by reducing aggregate emissions. Second, the impact of PPOs on ER&D investment depends critically on the policy environment and the severity of environmental damage. When marginal damage is low, PPOs increase total ER&D under exogenous taxation but leave it unchanged under endogenous taxation. In both cases, however, welfare declines due to reduced consumer surplus and lower tax revenues. By contrast, when marginal damage is high, PPOs may stimulate total ER&D, reduce environmental damage, and raise welfare. The specific outcome depends on the interaction between the tax regime, the degree of ER&D information sharing, and the level of PPOs.

These results carry important policy implications. They suggest that antitrust authorities should not adopt a uniform stance toward PPOs in polluting industries. Instead, the welfare consequences of PPOs depend on institutional and environmental factors, including emission tax design, the extent of ER&D information disclosure, and the severity of environmental externalities. Under certain conditions, PPOs can generate a "win–win" outcome by simultaneously improving environmental quality and enhancing welfare.

Three directions for future research are worth highlighting. First, this study has focused on symmetric private firms. Extending the analysis to mixed oligopolies, where ownership structures differ across firms, would yield further insights into the strategic role of PPOs. Second, while our analysis has centered on emission taxes, other environmental policy instruments, such as R&D subsidies or tradable permits, may interact with PPOs in distinct ways. Finally, while our model assumes a purely welfare-maximizing regulator, the robustness of our findings under low-damage scenarios suggests that the underlying mechanisms are not strictly dependent on the linear damage specification or the absence of fiscal motives. Incorporating richer institutional constraints, such as minimum regulatory standards or multi-faceted government objectives involving revenue needs, represents a natural and valuable direction for future work. Exploring these extensions would enrich our understanding of how ownership structures and environmental policy jointly shape firms' innovation incentives and welfare outcomes.

Conflict of interest

No potential conflict of interest was reported by the authors.

Appendix A: Equilibrium Results

(i) The situation of $b = 0$ (i.e., firms do not share ER&D information)

When t is exogenous, there are:

$$\pi_1^{ex} = \frac{[(1-t)w + t(2-k)\theta - t(1-k)\psi][(1-t)(1-k)w + 2t\theta - t(1+k)\psi]}{(3-k)^2 w^2} - \frac{1}{2}\left(\frac{\theta}{w}\right)^2,$$

$$\pi_2^{ex} = \frac{[(1-t)w - t(\theta - 2\psi)]^2}{(3-k)^2 w^2} - \frac{1}{2}\left(\frac{\psi}{w}\right)^2, \quad CS^{ex} = \frac{\{(1-t)(2-k)w + t[\theta + (1-k)\psi]\}^2}{2(3-k)^2 w^2},$$

$$D^{ex} = \frac{d((w-\theta)\{((1-t)(1-k)w + t[2\theta - (1+k)\psi]\} + (w-\psi)[(1-t)w - t(\theta - 2\psi)])}{(3-k)w^2},$$

$$T^{ex} = \frac{t((w-\theta)\{(1-t)(1-k)w + t[2\theta - (1+k)\psi]\} + (w-\psi)[(1-t)w - t(\theta - 2\psi)])}{(3-k)w^2},$$

$$\begin{aligned} SW^{ex} = \frac{1}{2(3-k)^2 w^2} \Bigg\{ & 2\big[(1-t)w + t(2-k)\theta - t(1-k)\psi\big]\big[(1-t)(1-k)w + 2t\theta - t(1+k)\psi\big] \\ & + 2\big[(1-t)w - t(\theta - 2\psi)\big]^2 + \big\{(1-t)(2-k)w + t[\theta + (1-k)\psi]\big\}^2 \\ & + 2(t-d)(3-k)\Big\{(1-t)(1-k)(w-\theta)w + t(w-\theta)\big[2\theta - (1+k)\psi\big] \\ & + (w-\psi)\big[(1-t)w - t(\theta - 2\psi)\big]\Big\} - (3-k)^2\big(\theta^2 + \psi^2\big)\Bigg\} \end{aligned}$$

In addition, when t is endogenous and d is low, there are: $\pi_1^{en} = \frac{(1-k)}{(3-k)^2}$, $\pi_2^{en} = \frac{1}{(3-k)^2}$,

$$CS^{en} = \frac{(2-k)^2}{2(3-k)^2}, \quad D^{en} = \frac{d(2-k)}{(3-k)}, \quad T^{en} = 0, \quad SW^{en} = \frac{(2-k)}{2(3-k)^2}[(4-k) - 2d(3-k)].$$

(ii) The situation of $b = 1$ (i.e., firms share ER&D information)

When t is exogenous, there are:

$$\pi_1^{ex} = \frac{(1-t)^2[(1-k)(3-k)^2 - 2t^2]}{[(3-k)^2 - 4t^2]^2}, \quad \pi_2^{ex} = \frac{(1-t)^2[(3-k)^2 - 2t^2]}{[(3-k)^2 - 4t^2]^2},$$

$$CS^{ex} = \frac{(2-k)^2(3-k)^2(1-t)^2}{2[(3-k)^2-4t^2]^2}, \quad D^{ex} = \frac{(2-k)(3-k)(1-t)[(3-k)^2-4t]d}{[(3-k)^2-4t^2]^2},$$

$$T^{ex} = \frac{(2-k)(3-k)(1-t)t[(3-k)^2-4t]}{[(3-k)^2-4t^2]^2},$$

$$SW^{ex} = \frac{1}{2\big[(3-k)^2-4t^2\big]^2}\Bigg\{(1-t)\Big(-(4-2k)(3-k)[(3-k)^2-4t]d+8t^3 - (8k^2-40k+56)t^2+\big(k^4-10k^3+37k^2-60k+36\big)t + k^4-12k^3+53k^2-102k+72\Big)\Bigg\}$$

In addition, when t is endogenous and d is low, there are: $\pi_1^{en} = \frac{(1-k)}{(3-k)^2}$, $\pi_2^{en} = \frac{1}{(3-k)^2}$,

$$CS^{en} = \frac{(2-k)^2}{2(3-k)^2}, \quad D^{en} = \frac{d(2-k)}{(3-k)}, \quad T^{en} = 0, \quad SW^{en} = \frac{(2-k)}{2(3-k)^2}[(4-k)-2d(3-k)].$$

Appendix B: Proofs

In our proofs, the Cylindrical Algebraic Decomposition (CAD) method plays a central role. Specifically, we employ the CAD method to provide computational evidence showing that the system under consideration admits no real solution. For this purpose, the **SamplePoints** function from the **RegularChains** package in **Maple 2022** can be directly utilized.

The CAD method, first introduced by Collins (1975) as the earliest practical approach to quantifier elimination over the reals, is often referred to as Collins' algorithm. The procedure partitions any semi-algebraic subset of $\mathbb{R}^n$ into a finite collection of mutually disjoint semi-algebraic cells, each of which is sign-invariant with respect to a prescribed set of polynomials. In other words, the sign of every polynomial in the input set remains constant within each cell. Importantly, the CAD method not only produces the decomposition but also provides a representative sample point for each cell, thereby enabling an effective examination of the associated sign conditions. Although Collins' original formulation was computationally demanding, significant refinements were later proposed by Collins and Hong (1991) and Brown (2001), considerably improving its practical applicability.

Proof of Proposition 1

Note that $0 \le t \le 0.55$ and $0 \le k \le 0.5$.

(i) We have

$$\frac{\partial y_1^{ex}}{\partial k} = -\frac{t(t-1)A}{\left(k^3+\left(2t^2-9\right)k^2+\left(4t^4-22t^2+27\right)k-16t^4+48t^2-27\right)^2},$$

where

$$A = \left(2t^2-1\right)k^4+\left(8t^4-36t^2+16\right)k^3+\left(-72t^4+198t^2-90\right)k^2+\left(192t^4-432t^2+216\right)k-96t^4+324t^2-189.$$

Since $t-1<0$, proving $\frac{\partial y_1^{ex}}{\partial k}<0$ is equivalent to proving $A<0$. We just need to prove that the following system has no real solution:

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ A \ge 0. \end{cases}$$

By using the CAD method, the above system has no real solution, which completes the proof.

Also, we have

$$\frac{\partial y_2^{ex}}{\partial k} = -\frac{t(t-1)B}{\left(k^3+\left(2t^2-9\right)k^2+\left(4t^4-22t^2+27\right)k-16t^4+48t^2-27\right)^2},$$

where

$$B = \left(8t^2-8\right)k^3+\left(8t^4-92t^2+72\right)k^2+\left(-64t^4+336t^2-216\right)k+144t^4-396t^2+216.$$

Since $t-1<0$, proving $\frac{\partial y_2^{ex}}{\partial k}>0$ is equivalent to proving $B>0$. We just need to prove that the following system has no real solution:

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ B \le 0. \end{cases}$$

By using the CAD method, the above system has no real solution, which completes the proof.

In addition,

$$\frac{\partial (y_1^{ex}+y_2^{ex})}{\partial k} = -\frac{t(t-1)C}{\left(k^3+\left(2t^2-9\right)k^2+\left(4t^4-22t^2+27\right)k-16t^4+48t^2-27\right)^2},$$

where

$$C = \left(2t^2 - 1\right) k^4 + \left(8t^4 - 28t^2 + 8\right) k^3 + \left(-64t^4 + 106t^2 - 18\right) k^2 + \left(128t^4 - 96t^2\right) k + 48 \left(t^2 - \frac{3}{4}\right)^2 .$$

Since $t - 1 < 0$, proving $\frac{\partial(y_1^{ex} + y_2^{ex})}{\partial k} > 0$ is equivalent to proving $C > 0$. We just need to prove that the following system has no real solution:

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ C \le 0. \end{cases}$$

By using the CAD method, the above system has no real solution, which completes the proof.

(ii) It can be obtained that

$$\frac{\partial D^{ex}}{\partial k} = -\frac{d\,(t-1)\,D}{\left((4k-16)\,t^4 + (2k^2 - 22k + 48)\,t^2 + (k-3)^3\right)^3},$$

where

$$\begin{aligned} D = {} & 16\,k^6t^6 - 16\,k^4t^8 + 16\,k^3t^9 + 4\,k^7t^4 - 16\,k^6t^5 - 240\,k^5t^6 + 56\,k^4t^7 + 144\,k^3t^8 - 192\,k^2t^9 - 4\,k^7t^3 \\ & - 92\,k^6t^4 + 264\,k^5t^5 + 1296\,k^4t^6 - 744\,k^3t^7 - 192\,k^2t^8 + 832\,kt^9 - 2\,k^7t^2 + 102\,k^6t^3 + 768\,k^5t^4 \\ & - 1804\,k^4t^5 - 2472\,k^3t^6 + 3552\,k^2t^7 - 1440\,kt^8 - 1280\,t^9 + 3\,k^7t + 30\,k^6t^2 - 990\,k^5t^3 - 2820\,k^4t^4 \\ & + 6492\,k^3t^5 - 2160\,k^2t^6 - 7008\,kt^7 + 3456\,t^8 - k^7 - 63\,k^6t - 138\,k^5t^2 + 4876\,k^4t^3 + 3240\,k^3t^4 \\ & - 12948\,k^2t^5 + 13608\,kt^6 + 4608\,t^7 + 21\,k^6 + 543\,k^5t - 90\,k^4t^2 - 13232\,k^3t^3 + 7128\,k^2t^4 + 13644\,kt^5 \\ & - 12960\,t^6 - 189\,k^5 - 2475\,k^4t + 2970\,k^3t^2 + 19566\,k^2t^3 - 23004\,kt^4 - 6048\,t^5 + 945\,k^4 + 6345\,k^3t \\ & - 10854\,k^2t^2 - 14094\,kt^3 + 17496\,t^4 - 2835\,k^3 - 8829\,k^2t + 17010\,kt^2 + 3456\,t^3 + 5103\,k^2 + 5589\,kt \\ & - 10206\,t^2 - 5103\,k - 729\,t + 2187. \end{aligned}$$

By using the CAD method, we derive that the following two systems have no real solution:

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ D \le 0, \end{cases}$$

and

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ (4k-16)\,t^4 + \left(2k^2 - 22k + 48\right)t^2 + (k-3)^3 \ge 0. \end{cases}$$

Therefore, $D > 0$ and $(4k-16)\,t^4 + \left(2k^2 - 22k + 48\right)t^2 + (k-3)^3 < 0$, implying that $\frac{\partial D^{ex}}{\partial k} < 0$.

(iii) One can see that

$$\frac{\partial SW^{ex}}{\partial k} = \frac{(t-1)\,(Gd+F)}{\left((4k-16)\,t^4 + (2k^2 - 22k + 48)\,t^2 + (k-3)^3\right)^3}$$

where

$$\begin{aligned} G = {} & 16\,k^6t^6 - 16\,k^4t^8 + 16\,k^3t^9 + 4\,k^7t^4 - 16\,k^6t^5 - 240\,k^5t^6 + 56\,k^4t^7 + 144\,k^3t^8 - 192\,k^2t^9 - 4\,k^7t^3 \\ & - 92\,k^6t^4 + 264\,k^5t^5 + 1296\,k^4t^6 - 744\,k^3t^7 - 192\,k^2t^8 + 832\,kt^9 - 2\,k^7t^2 + 102\,k^6t^3 + 768\,k^5t^4 \\ & - 1804\,k^4t^5 - 2472\,k^3t^6 + 3552\,k^2t^7 - 1440\,kt^8 - 1280\,t^9 + 3\,k^7t + 30\,k^6t^2 - 990\,k^5t^3 - 2820\,k^4t^4 \\ & + 6492\,k^3t^5 - 2160\,k^2t^6 - 7008\,kt^7 + 3456\,t^8 - k^7 - 63\,k^6t - 138\,k^5t^2 + 4876\,k^4t^3 + 3240\,k^3t^4 \\ & - 12948\,k^2t^5 + 13608\,kt^6 + 4608\,t^7 + 21\,k^6 + 543\,k^5t - 90\,k^4t^2 - 13232\,k^3t^3 + 7128\,k^2t^4 + 13644\,kt^5 \\ & - 12960\,t^6 - 189\,k^5 - 2475\,k^4t + 2970\,k^3t^2 + 19566\,k^2t^3 - 23004\,kt^4 - 6048\,t^5 + 945\,k^4 + 6345\,k^3t \\ & - 10854\,k^2t^2 - 14094\,kt^3 + 17496\,t^4 - 2835\,k^3 - 8829\,k^2t + 17010\,kt^2 + 3456\,t^3 + 5103\,k^2 + 5589\,kt \\ & - 10206\,t^2 - 5103\,k - 729\,t + 2187 \end{aligned}$$

and

$$\begin{aligned} F = {} & -8\,k^6t^7 + 28\,k^4t^9 - 16\,k^3t^{10} - 2\,k^7t^5 + 8\,k^6t^6 + 132\,k^5t^7 - 68\,k^4t^8 - 308\,k^3t^9 + 192\,k^2t^{10} + 2\,k^7t^4 \\ & + 43\,k^6t^5 - 156\,k^5t^6 - 832\,k^4t^7 + 908\,k^3t^8 + 1008\,k^2t^9 - 832\,kt^{10} - 53\,k^6t^4 - 351\,k^5t^5 + 1340\,k^4t^6 \\ & + 2188\,k^3t^7 - 4368\,k^2t^8 - 208\,kt^9 + 1280\,t^{10} - k^7t^2 + 7\,k^6t^3 + 573\,k^5t^4 + 1276\,k^4t^5 - 6208\,k^3t^6 \\ & - 624\,k^2t^7 + 8656\,kt^8 - 2688\,t^9 + k^7t + 26\,k^6t^2 - 117\,k^5t^3 - 3332\,k^4t^4 - 1254\,k^3t^5 + 15732\,k^2t^6 \\ & - 7272\,kt^7 - 5376\,t^8 - 20\,k^6t - 288\,k^5t^2 + 860\,k^4t^3 + 11246\,k^3t^4 - 4887\,k^2t^5 - 19980\,kt^6 + 9504\,t^7 - k^6 \\ & + 171\,k^5t + 1705\,k^4t^2 - 3570\,k^3t^3 - 21807\,k^2t^4 + 15039\,kt^5 + 9504\,t^6 + 18\,k^5 - 810\,k^4t - 5745\,k^3t^2 \\ & + 8775\,k^2t^3 + 22059\,kt^4 - 12312\,t^5 - 135\,k^4 + 2295\,k^3t + 10908\,k^2t^2 - 11961\,kt^3 - 8640\,t^4 + 540\,k^3 \\ & - 3888\,k^2t - 10638\,kt^2 + 6966\,t^3 - 1215\,k^2 + 3645\,kt + 3969\,t^2 + 1458\,k - 1458\,t - 729. \end{aligned}$$

Similarly, the CAD method reveals that $G > 0$, $F < 0$ and $(4k-16)\,t^4 + \left(2k^2 - 22k + 48\right)t^2 + (k-3)^3 <$

0. Therefore, we derive

$$\lim_{d\to 0^+} \frac{\partial SW^{ex}}{\partial k} = \frac{(t-1)\,F}{\left((4k-16)\,t^4 + (2k^2-22k+48)\,t^2 + (k-3)^3\right)^3} < 0,$$

which implies that $\frac{\partial SW^{ex}}{\partial k} < 0$ if d is low. In contrast,

$$\lim_{d\to (55/100)^-} \frac{\partial SW^{ex}}{\partial k} = \frac{(t-1)\,S}{\left((4k-16)\,t^4 + (2k^2-22k+48)\,t^2 + (k-3)^3\right)^3},$$

where

$$\begin{aligned} S = {} & 197127/10\,t^3k - 60192/5\,t^7 + 1496/5\,t^9k^3 + 54238/5\,t^3k^3 + 55632/5\,t^7k - 6648/5\,t^7k^2 + 17376/5\,t^8 \\ & + 16\,t^{10}k^3 - 192\,t^{10}k^2 + 832\,t^{10}k - 13028\,t^4k^3 - 195363/10\,t^3k^2 - 17709/5\,t^3k^4 + 8685/4\,tk^4 \\ & + 1323/2\,t^3k^5 - 631/10\,t^3k^6 - 28\,t^9k^4 + 8\,t^7k^6 - 132\,t^7k^5 + 78192/5\,t^5 + 3639/10\,t^2k^5 - 4512/5\,t^9k^2 \\ & - 9393/20\,tk^5 - 112716/5\,t^5k + 11/5\,t^3k^7 - 21/5\,t^4k^7 + 518/5\,t^4k^6 - 4977/5\,t^4k^5 + 384/5\,t^8k^4 \\ & - 4936/5\,t^8k^3 + 1093/20\,tk^6 - 53/20\,tk^7 - 1248/5\,t^9k + 60042/5\,t^5k^2 - 11583/5\,t^5k^3 + 4006/5\,t^7k^4 \\ & + 22368/5\,t^8k^2 + 1029/5\,t^5k^5 - 171/5\,t^5k^6 + 62478/5\,t^6k - 1419/5\,t^5k^4 + 37838/5\,t^6k^3 - 8894/5\,t^7k^3 \\ & - 134379/20\,tk - 23139/4\,tk^3 + 174879/20\,tk^2 - 10264/5\,t^6k^4 - 84/5\,t^6k^6 - 211/20\,k^6 + 11/20\,k^7 \\ & - 9477/20 - 4914/5\,t^4 - 1539/4\,k^4 + 4077/4\,k^3 + 16443/10\,t^2 + 26973/20\,k - 31833/20\,k^2 \\ & + 37179/20\,t - 44334/5\,t^3 + 1719/20\,k^5 + 21/10\,t^2k^7 - 85/2\,t^2k^6 + 2565/2\,t^2k - 47034/5\,t^4k \\ & - 49383/10\,t^2k^2 + 8223/2\,t^2k^3 - 3311/2\,t^2k^4 + 89433/5\,t^4k^2 + 3392\,t^9 - 1280\,t^{10} - 2376\,t^6 + 288\,t^6k^5 \\ & - 14544\,t^6k^2 - 7864\,t^8k + 4883\,t^4k^4 + 2\,t^5k^7. \end{aligned}$$

Furthermore, it can be derived that

$$\lim_{t\to 0^+} S = \frac{(11k-13)\,(k-3)^6}{20} > 0,$$

and

$$\begin{aligned} \lim_{t\to (55/100)^-} S = {} & -\frac{7170330171087}{32000000000}k + \frac{120631385187}{2000000000} + \frac{10381504205979}{32000000000}k^2 - \frac{8635522023}{320000000}k^5 \\ & - \frac{31622580058689}{128000000000}k^3 + \frac{13800018123387}{128000000000}k^4 + \frac{570969999}{160000000}k^6 - \frac{303831}{1600000}k^7 < 0. \end{aligned}$$

Therefore, provided that d is high, $\frac{\partial SW^{ex}}{\partial k} > 0$ if t is low; otherwise, $\frac{\partial SW^{ex}}{\partial k} < 0$ if t is high.

Proof of Lemma 1

(i) It suffices to prove that $\frac{\partial SW}{\partial t} < 0$ for $d \in [0, 1/5]$ according to the assumption $0 \le t \le d$. From (4), we have

$$SW = \frac{(t-1)\,H}{\left((4k-16)\,t^4 + (2k^2-22k+48)\,t^2 + (k-3)^3\right)^2},$$

where

$$\begin{aligned}
H = & -324 + 675\,k + 486\,d - 162\,t + 2531/2\,k^2t^2 + 1224\,t^2 + 288\,t^3 + 640\,t^6 + 96\,t^5 - 1536\,t^4 - 256\,t^7 + 9\,k^5 \\
& - 67\,k^4 + 264\,k^3 - 384\,t^5d + 864\,t^4d + 576\,t^3d - 1296\,t^2d - 216\,td - 1161/2\,k^2 - 450\,k^3d + 945\,k^2d \\
& - 1053\,kd - 17\,dk^5 + 120\,dk^4 + dk^6 - 1371/2\,k^2t^4 - 19/2\,k^4t^4 + 1/2\,k^6t^2 + 7/2\,k^4t^5 + 313/2\,k^2t^3 \\
& - 1/2\,k^6t^3 - 1/2\,k^6t + 4\,t^4k^3d + 8\,t^3dk^4 + 32\,t^2dk^4 - tdk^6 + 4\,t^5k^3d + 15\,tdk^5 + 1890\,t^2kd + 126\,t^4k^2d \\
& + 138\,t^2k^3d - 816\,t^3kd + 424\,t^3k^2d - 945\,t^2k^2d - 648\,t^4kd + 459\,tkd - 12\,t^2dk^5 - 96\,t^3k^3d + 272\,t^5kd \\
& - 2\,t^4dk^4 + t^2dk^6 - 90\,tdk^4 - 60\,t^5k^2d + 278\,tk^3d - 477\,tk^2d - 729/2\,k^2t + 271/2\,k^2t^5 - 1/2\,k^6 \\
& - 4\,k^3t^6 - 16\,k^2t^7 + 7\,k^5t^3 - 37\,k^3t^5 + 76\,k^2t^6 + 128\,kt^7 - 10\,k^5t^2 - 32\,k^4t^3 + 129\,k^3t^4 - 400\,kt^6 \\
& + 8\,k^5t + 90\,k^4t^2 + 31\,k^3t^3 - 200\,kt^5 - 53\,k^4t - 447\,k^3t^2 + 1664\,kt^4 + 186\,k^3t - 426\,kt^3 - 1923\,kt^2 \\
& + 378\,kt.
\end{aligned}$$

One can see that the expression of $\frac{\partial SW}{\partial t}$ is complex, which is not given here due to space limitations. Then proving $\frac{\partial SW}{\partial t} < 0$ for $d \in [0, 1/5]$ is equivalent to proving that the following system has no real solution:

$$\begin{cases} \mathrm{Num}\left(\dfrac{\partial SW}{\partial t}\right) \cdot \mathrm{Den}\left(\dfrac{\partial SW}{\partial t}\right) \ge 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad 0 \le d \le 1/5, \end{cases}$$

where Num(·) and Den(·) represent the numerator and denominator of a rational function, respectively. By using the CAD method, we can conclude that the above system has no real solution.

(ii) We focus on the case when $d = 0.55$, i.e., $d = 55/100$. We know t^{en} is the optimal emission tax rate that maximizes the social welfare SW. Thus, we have $\frac{\partial SW}{\partial t}(t^{en}) = 0$ and $\frac{\partial^2 SW}{\partial t^2}(t^{en}) < 0$. To obtain $\frac{\partial t^{en}}{\partial k}$, one can see $\frac{\partial SW}{\partial t}(t^{en}) = 0$ implies that

$$\frac{\partial \frac{\partial SW}{\partial t}(t^{en})}{\partial k} = J(d,k,t)\frac{\partial t^{en}}{\partial k} - K(d,k,t) = 0,$$

where $J(d,k,t)$ and $K(d,k,t)$ are rational functions in d, k, and t. Thus, $\frac{\partial t^{en}}{\partial k} = K(d,k,t)/J(d,k,t)$, which is also a rational function in d, k, and t. First, it is needed to derive that there exists $t^{en} \in [0, d]$. This is equivalent to there existing at least one real solution for

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \end{cases}$$

which can be computationally proved by using the CAD method. To derive $\frac{\partial t^{en}}{\partial k} < 0$, we just need to prove that the following system has no real solution:

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \\ \operatorname{Num}\left(K(d,k,t)/J(d,k,t)\right) \cdot \operatorname{Den}\left(K(d,k,t)/J(d,k,t)\right) \ge 0. \end{cases}$$

By using the CAD method, the above system has no real solution, which completes the proof.

Proof of Proposition 2

(a) According to Lemma 1 (i), $t^{en} = 0$ if d is low. Thus, we have (i) $\frac{\partial y_1^{en}}{\partial k} = 0$, $\frac{\partial y_2^{en}}{\partial k} = 0$, $\frac{\partial (y_1^{en} + y_2^{en})}{\partial k} = 0$. Furthermore, we derive (ii)

$$\frac{\partial D^{en}}{\partial k} = -\frac{d}{(k-3)^2} < 0,$$

and (iii)

$$\frac{\partial SW^{en}}{\partial k} = \frac{1 + (k-3)\, d}{(k-3)^3} < 0 \;\text{ if }\; d < \frac{1}{3-k}.$$

(b) We consider the case when $d = 0.55$, i.e., $d = 55/100$. (i) According to the proof the Lemma 1, we know $\frac{\partial t^{en}}{\partial k} = K(d,k,t)/J(d,k,t)$. Thus,

$$\frac{\partial y_1^{en}}{\partial k} = \frac{\partial y_1^{en}}{\partial t^{en}} \cdot \frac{\partial t^{en}}{\partial k} = \frac{\partial y_1^{en}}{\partial t^{en}} \cdot \frac{K}{J}, \quad \frac{\partial y_2^{en}}{\partial k} = \frac{\partial y_2^{en}}{\partial t^{en}} \cdot \frac{\partial t^{en}}{\partial k} = \frac{\partial y_2^{en}}{\partial t^{en}} \cdot \frac{K}{J}, \quad \frac{\partial (y_1^{en} + y_2^{en})}{\partial k} = \frac{\partial y_1^{en}}{\partial k} + \frac{\partial y_2^{en}}{\partial k}.$$

Then $\frac{\partial y_1^{en}}{\partial k} < 0$ is equivalent to the following system having no real solution:

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \\ \operatorname{Num}\left(\dfrac{\partial y_1^{en}}{\partial t^{en}} \cdot \dfrac{K}{J}\right) \cdot \operatorname{Den}\left(\dfrac{\partial y_1^{en}}{\partial t^{en}} \cdot \dfrac{K}{J}\right) \ge 0, \end{cases}$$

which can be proved by using the CAD method. In addition, $\frac{\partial y_2^{en}}{\partial k} > 0$ and $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} < 0$ can by similarly derived.

(ii) We know $\frac{\partial D^{en}}{\partial k} = \frac{\partial D^{en}}{\partial t^{en}} \cdot \frac{K}{J}$. Hence, proving $\frac{\partial D^{en}}{\partial k} < 0$ can be transformed into proving no real solution for

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \\ \text{Num}\left(\dfrac{\partial D^{en}}{\partial t^{en}} \cdot \dfrac{K}{J}\right) \cdot \text{Den}\left(\dfrac{\partial D^{en}}{\partial t^{en}} \cdot \dfrac{K}{J}\right) \ge 0. \end{cases}$$

We can prove this by employing the CAD method.

(iii) Similarly, $\frac{\partial SW^{en}}{\partial k} = \frac{\partial SW^{en}}{\partial t^{en}} \cdot \frac{K}{J}$. Therefore, $\frac{\partial SW^{en}}{\partial k} < 0$ is equivalent to the following system having no real solution:

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \\ \text{Num}\left(\dfrac{\partial SW^{en}}{\partial t^{en}} \cdot \dfrac{K}{J}\right) \cdot \text{Den}\left(\dfrac{\partial SW^{en}}{\partial t^{en}} \cdot \dfrac{K}{J}\right) \ge 0. \end{cases}$$

By using the method of real root classification, we conclude that the above system has no real solution if $k < 0.1429526859$ and has one real solution if $k > 0.1429526859$. Put differently, $\frac{\partial SW^{en}}{\partial k} < 0$ if k is small and $\frac{\partial SW^{en}}{\partial k} > 0$ if k is large.

Proof of Proposition 3

Note that $0 \le t \le 0.55$ and $0 \le k \le 0.5$.

(i) From (20) and (21), it can be readily derived that

$$\frac{\partial y_1^{ex}}{\partial k} = \frac{\partial y_2^{ex}}{\partial k} = \frac{4t\,(t-1)\,(k-3)}{\left(k^2 - 4t^2 - 6k + 9\right)^2} > 0.$$

It follows that $\frac{\partial (y_1^{ex}+y_2^{ex})}{\partial k} > 0$.

(ii) We have

$$D^{ex} = \frac{(t-1)\,(2-k)\,(k-3)\left(k^2 - 6k - 4t + 9\right)d}{\left(k^2 - 4t^2 - 6k + 9\right)^2}.$$

Therefore,

$$\frac{\partial D^{ex}}{\partial k} = \frac{d\,(t-1)\,L}{\left(k^2 - 4t^2 - 6k + 9\right)^3},$$

where

$$L = k^4+\left(16t^2-8t-12\right)k^3+\left(-132t^2+60t+54\right)k^2+\left(-32t^3+360t^2-144t-108\right)k+80t^3-324t^2+108t+81.$$

The CAD method shows that the following two systems have no real solution:

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ L \le 0, \end{cases}$$

and

$$\begin{cases} 0 \le k \le 1/2, \quad 0 \le t \le 55/100, \\ k^2-4t^2-6k+9 \le 0, \end{cases}$$

implying that $L > 0$ and $k^2 - 4t^2 - 6k + 9 > 0$. Since $t - 1 < 0$, $\frac{\partial D^{ex}}{\partial k} < 0$.

(iii) We have

$$\frac{\partial SW^{ex}}{\partial k} = \frac{(1-t)(Pd+Q)}{(k^2-4t^2-6k+9)^3},$$

where

$$P = k^4+\left(16t^2-8t-12\right)k^3+\left(-132t^2+60t+54\right)k^2+\left(-32t^3+360t^2-144t-108\right)k+80t^3-324t^2+108t+81,$$

and

$$\begin{aligned} Q = {} & tk^4+\left(-8t^3+11t+1\right)k^3+\left(60t^3+12t^2-45t-9\right)k^2+\left(32t^4-164t^3-52t^2+81t+27\right)k \\ & -80t^4+168t^3+48t^2-54t-27. \end{aligned}$$

Similarly, the CAD method shows that $P > 0$, $Q < 0$, and $k^2 - 4t^2 - 6k + 9 > 0$.

Therefore, we derive

$$\lim_{d\to 0^+} \frac{\partial SW^{ex}}{\partial k} = \frac{(1-t)\,Q}{(k^2-4t^2-6k+9)^3} < 0,$$

meaning that $\frac{\partial SW^{ex}}{\partial k} < 0$ if d is low. In contrast,

$$\lim_{d\to (55/100)^-} \frac{\partial SW^{ex}}{\partial k} = \frac{(1-t)\,R}{(k^2-4t^2-6k+9)^3},$$

where

$$R = \left(\frac{11}{20} - t\right)k^4 + \left(-8t^3 + \frac{44}{5}t^2 + \frac{33}{5}t - \frac{28}{5}\right)k^3 + \left(60t^3 - \frac{303}{5}t^2 - 12t + \frac{207}{10}\right)k^2$$
$$+ \left(32t^4 - \frac{908}{5}t^3 + 146t^2 + \frac{9}{5}t - \frac{162}{5}\right)k - 80t^4 + 212t^3 - \frac{651t^2}{5} + \frac{27t}{5} + \frac{351}{20}.$$

From $0 \le t \le 0.55$ and $0 \le k \le 0.5$, the CAD method shows $R > 0$, implying that $\frac{\partial SW^{ex}}{\partial k} > 0$ if d is high.

Proof of Lemma 2

(i) It suffices to prove that $\frac{\partial SW}{\partial t} < 0$ for $d \in [0, 1/5]$ according to the assumption $0 \le t \le d$. From (4), we have

$$SW = \frac{(t-1)\,M}{(k^2 - 4t^2 - 6k + 9)^2},$$

where

$$M = \left(d - \frac{t}{2} - \frac{1}{2}\right)k^4 + (-11d + 5t + 6)\,k^3 + \left(4t^2 + \left(-4d - \frac{37}{2}\right)t + 45d - \frac{53}{2}\right)k^2$$
$$+ \left(-20t^2 + (20d + 30)\,t - 81d + 51\right)k - 4t^3 + 28t^2 + (-24d - 18)\,t + 54d - 36.$$

The expression of $\frac{\partial SW}{\partial t}$ is complex, which is not given here due to space limitations. Then proving $\frac{\partial SW}{\partial t} < 0$ for $d \in [0, 1/5]$ is equivalent to proving that the following system has no real solution:

$$\begin{cases} \text{Num}\left(\dfrac{\partial SW}{\partial t}\right) \cdot \text{Den}\left(\dfrac{\partial SW}{\partial t}\right) \ge 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad 0 \le d \le 1/5. \end{cases}$$

By using the CAD method, we can conclude that the above system has no real solution.

(ii) We consider the case when $d = 0.55$, i.e., $d = 55/100$. We know $\frac{\partial SW}{\partial t}(t^{en}) = 0$ and $\frac{\partial^2 SW}{\partial t^2}(t^{en}) < 0$. To obtain $\frac{\partial t^{en}}{\partial k}$, one can see $\frac{\partial SW}{\partial t}(t^{en}) = 0$ implies that

$$\frac{\partial \frac{\partial SW}{\partial t}(t^{en})}{\partial k} = U(d,k,t)\frac{\partial t^{en}}{\partial k} - V(d,k,t) = 0,$$

where $U(d,k,t)$ and $V(d,k,t)$ are rational functions in d, k, and t. Thus, $\frac{\partial t^{en}}{\partial k} = U(d,k,t)/V(d,k,t)$, which is also a rational function in d, k, and t. First, it is needed to derive that there exists $t^{en} \in [0, d]$. This is equivalent to there existing at least one real solution for

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \end{cases}$$

which can be proved by using the method of real root classification. To derive $\frac{\partial t^{en}}{\partial k} < 0$, we just need to prove that the following system has no real solution:

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \\ \mathrm{Num}\left(U(d,k,t)/V(d,k,t)\right) \cdot \mathrm{Den}\left(U(d,k,t)/V(d,k,t)\right) \ge 0. \end{cases}$$

The CAD method reveals that the above system has no real solution, which completes the proof.

Proof of Proposition 4

(a) From Lemma 2 (i), $t^{en} = 0$ if d is low. Thus, we have (i) $\frac{\partial y_1^{en}}{\partial k} = 0$, $\frac{\partial y_2^{en}}{\partial k} = 0$, $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} = 0$. Furthermore, we derive (ii)

$$\frac{\partial D^{en}}{\partial k} = -\frac{d}{(k-3)^2} < 0,$$

and (iii)

$$\frac{\partial SW^{en}}{\partial k} = \frac{1 + (k-3)\,d}{(k-3)^3} < 0 \ \text{ if } \ d < \frac{1}{3-k}.$$

(b) We consider the case when $d = 55/100$. (i) According to the proof the Lemma 2, we know $\frac{\partial t^{en}}{\partial k} = U(d,k,t)/V(d,k,t)$. Thus,

$$\frac{\partial y_1^{en}}{\partial k} = \frac{\partial y_1^{en}}{\partial t^{en}} \cdot \frac{\partial t^{en}}{\partial k} = \frac{\partial y_1^{en}}{\partial t^{en}} \cdot \frac{U}{V}, \quad \frac{\partial y_2^{en}}{\partial k} = \frac{\partial y_2^{en}}{\partial t^{en}} \cdot \frac{\partial t^{en}}{\partial k} = \frac{\partial y_2^{en}}{\partial t^{en}} \cdot \frac{U}{V}, \quad \frac{\partial (y_1^{en} + y_2^{en})}{\partial k} = \frac{\partial y_1^{en}}{\partial k} + \frac{\partial y_2^{en}}{\partial k}.$$

Then $\frac{\partial y_1^{en}}{\partial k} > 0$ is equivalent to the following system having no real solution:

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \le k \le 1/2, \quad 0 \le t \le d, \quad d = 55/100, \\ \mathrm{Num}\left(\dfrac{\partial y_1^{en}}{\partial t^{en}} \cdot \dfrac{U}{V}\right) \cdot \mathrm{Den}\left(\dfrac{\partial y_1^{en}}{\partial t^{en}} \cdot \dfrac{U}{V}\right) \le 0, \end{cases}$$

which can be proved by using the CAD method. In addition, $\frac{\partial y_2^{en}}{\partial k} > 0$ and $\frac{\partial (y_1^{en}+y_2^{en})}{\partial k} > 0$ can by similarly derived.

(ii) We know $\frac{\partial D^{en}}{\partial k} = \frac{\partial D^{en}}{\partial t^{en}} \cdot \frac{U}{V}$. Hence, proving $\frac{\partial D^{en}}{\partial k} < 0$ can be transformed into proving no real solution for

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \leq k \leq 1/2, \quad 0 \leq t \leq d, \quad d = 55/100, \\ \text{Num}\left(\dfrac{\partial D^{en}}{\partial t^{en}} \cdot \dfrac{U}{V}\right) \cdot \text{Den}\left(\dfrac{\partial D^{en}}{\partial t^{en}} \cdot \dfrac{U}{V}\right) \geq 0. \end{cases}$$

We can prove this by employing the CAD method.

(iii) Similarly, $\frac{\partial SW^{en}}{\partial k} = \frac{\partial SW^{en}}{\partial t^{en}} \cdot \frac{U}{V}$. Therefore, $\frac{\partial SW^{en}}{\partial k} > 0$ is equivalent to the following system having no real solution:

$$\begin{cases} \dfrac{\partial SW}{\partial t} = 0, \quad \dfrac{\partial^2 SW}{\partial t^2} < 0, \\ 0 \leq k \leq 1/2, \quad 0 \leq t \leq d, \quad d = 55/100, \\ \text{Num}\left(\dfrac{\partial SW^{en}}{\partial t^{en}} \cdot \dfrac{U}{V}\right) \cdot \text{Den}\left(\dfrac{\partial SW^{en}}{\partial t^{en}} \cdot \dfrac{U}{V}\right) \leq 0. \end{cases}$$

The CAD method shows that the above system has no real solution.

References

Antón, M., Ederer, F., Giné, M., and Schmalz, M. (2025). Innovation: the bright side of common ownership? *Management Science*, 71(5):3713–3733.

Bárcena-Ruiz, J. C. and Campo, M. L. (2012). Partial cross-ownership and strategic environmental policy. *Resource and Energy Economics*, 34(2):198–210.

Bárcena-Ruiz, J. C. and Campo, M. L. (2017). Taxes versus standards under cross-ownership. *Resource and Energy Economics*, 50:36–50.

Bayona, A. and López, Á. L. (2018). Silent financial interests and product innovation. *Economics letters*, 170:109–112.

Brito, D., Ribeiro, R., and Vasconcelos, H. (2020). Overlapping ownership, endogenous quality, and welfare. *Economics Letters*, 190:109074.

Brown, C. W. (2001). Simple CAD construction and its applications. *Journal of Symbolic Computation*, 31(5):521–547.

Chen, J., Li, Z., and Lee, S.-H. (2024). Research spillovers and partial passive ownership by a public enterprise. *Technology Analysis & Strategic Management*, 36(12):4810–4828.

Cheng, H., Ding, X., and Zeng, C. (2024). Cross-ownership on r&d and welfare in a mixed oligopoly: revisiting with convex cost. *Economics of Innovation and New Technology*, 33(6):819–829.

Collins, G. E. (1975). Quantifier elimination for real closed fields by cylindrical algebraic decompostion. In Brakhage, H., editor, *Automata Theory and Formal Languages*, pages 134–183, Berlin, Heidelberg. Springer.

Collins, G. E. and Hong, H. (1991). Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12(3):299–328.

Frondel, M., Horbach, J., and Rennings, K. (2007). End-of-pipe or cleaner production? an empirical comparison of environmental innovation decisions across oecd countries. *Business strategy and the environment*, 16(8):571–584.

Hirose, K. and Matsumura, T. (2020). A comparison between emission intensity and emission cap regulations. *Energy Policy*, 137:111115.

Lee, S.-H. and Xu, L. (2018). Endogenous timing in private and mixed duopolies with emission taxes. *Journal of Economics*, 124(2):175–201.

Leonardos, S., Petrakis, E., Skartados, P., and Stamatopoulos, G. (2021). Partial passive ownership holdings and licensing. *Economics Letters*, 204:109910.

Li, D. and Shang, C. (2022). When does environmental innovation crowd out process innovation? a dynamic analysis. *Managerial and Decision Economics*, 43(6):2275–2283.

Lian, X., Wang, L. F. S., and Zhang, K. (2025). Cross-ownership, green managerial delegation, and emission taxation. *Managerial and Decision Economics*, 46(2):1191–1205.

Liu, L., Wang, X. H., and Zeng, C. (2020). Endogenous horizontal product differentiation in a mixed duopoly. *Review of Industrial Organization*, 56(3):435–462.

López, Á. L. and Vives, X. (2019). Overlapping ownership, r&d spillovers, and antitrust policy. *Journal of Political Economy*, 127(5):2394–2437.

Mukherjee, A. (2023). Merger and product innovation under cross ownership and cooperative r&d. *Economics Letters*, 233:111418.

Nie, P., Wang, C., and Meng, Y. (2019). An analysis of environmental corporate social responsibility. *Managerial and Decision Economics*, 40(4):384–393.

Ohori, S. (2006). Optimal environmental tax and level of privatization in an international duopoly. *Journal of Regulatory Economics*, 29(2):225–233.

Papadopoulos, K. G., Petrakis, E., and Skartados, P. (2019). Product innovation transfer under passive partial ownership holdings. *Economics Letters*, 177:22–25.

Shelegia, S. and Spiegel, Y. (2015). Cost-reducing investments under partial cross-ownership. Working paper.

Tsai, T.-H., Wang, C.-C., and Chiou, J.-R. (2016). Can privatization be a catalyst for environmental r&d and result in a cleaner environment? *Resource and Energy Economics*, 43:1–13.

Vives, X. (2020). Common ownership, market power, and innovation. *International Journal of Industrial Organization*, 70:102528.

Xing, M. and Tan, T. (2021). Environmental attitudes and impacts of privatization on r&d, environment and welfare in a mixed duopoly. *Economic research-Ekonomska istraživanja*, 34(1):807–827.

Xing, M. and Tan, T. (2023). Partial passive ownership holdings and r&d risk choices in a differentiated duopoly. *Economic research-Ekonomska istraživanja*, 36(2):2135554.

Xing, M., Tan, T., Ma, X., and Wang, X. (2023). Environmental tax policy and optimal privatization in a polluting mixed duopoly. *Managerial and Decision Economics*, 44(6):3016–3025.

Xing, M., Wang, L. F. S., and Zhou, C. (2024). Cross-ownership on r&d and social welfare in mixed oligopoly. *Economics of Innovation and New Technology*, 33(2):206–217.

Xing, M., Yang, L., Tan, T., Wang, X., Li, Q., and Wang, B. (2022). Environmental taxes and the effects of partial privatization on environmental r&d, environment and welfare. *Economic research-Ekonomska istraživanja*, 35(1):5164–5183.

Youssef, S. B. and Dinar, Z. (2011). Regulation of a duopoly and environmental r&d. *Environmental Economics*, 2(3):47–51.