Time integrals under the Black-Scholes-Merton and Margrabe economies*

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Abstract

The problem of integrating the Black and Scholes (1973) and Merton (1973) (BSM) formula with respect to the time variable is paramount for an economist. Inspired by the real options literature, Shackleton and Wojakowski (2007) offer analytic formulae for valuing finite maturity (profit) caps and floors that are contingent on continuous flows following a lognormal distribution. Alternative, but equivalent, closed-form solutions have been recently proposed in Dias et al. (2024b) by solving the time integral of options using a direct approach that does not rely on the real options intuition. This paper further extends and simplifies the computation of time integrals under the BSM world, considering not only plain-vanilla but also several exotic, including path-dependent options. We also provide a new closed-form solution of the time integral under the Margrabe (1978) economy. The method proposed in this paper makes the evaluation easier, cements the “non-real options” route and opens the way for more analytical work in BSM, Margrabe and other areas.

Keywords: Caps and floors; Continuous flows; Finite maturity; Time integral of options.

JEL Classification: G12; G13; G31.

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1. Introduction

The real options approach emerged in the literature as an alternative to the so called traditional capital budgeting approach in order to better evaluate the firms' investment and divestment decisions by introducing three important characteristics into the economic analysis of a project: irreversibility, timing flexibility and uncertainty. This stream of the literature typically uses stochastic processes for modeling perpetual continuous cash flows, prices or revenues from a project. Early examples of such real options applications with a continuous uncertain profit (or revenue) flow include McDonald and Siegel (1985), McDonald and Siegel (1986) and Dixit and Pindyck (1994), just to name a few.

For example, McDonald and Siegel (1985) show that the functional form $J = \int_0^T v_0(t)\, dt$ can be interpreted as the (present) value of a project producing a continuous cash flow $v_0(t)$ that captures the positive part of a stochastic net profit at time t while avoiding losses. Even though McDonald and Siegel (1985) motivate and discuss this interesting problem, they do not provide the analytical formulae for the functional form J. This problem is simple to be formulated, though it is difficult to be solved analytically. Its solution is of fundamental importance as it involves integrating the widely known Black and Scholes (1973) and Merton (1973) (BSM) formula along its most likely useful variable for an economist: the time variable.

Shackleton and Wojakowski (2007) produce closed-form solutions for such finite-lived profit caps (and floors) that are contingent on continuous flows following a lognormal distribution using perpetual methods inspired in the real options literature. More specifically, a finite maturity profit cap expiring in T years is decomposed (or replicated) by a portfolio that includes a long position in a perpetual profit cap and a short position in a forward start perpetual profit cap that begins after T years. The use of their time decomposition technique emerged in the real options literature as the stepping-stone to many other recent applications of real options models with related caps, floors and collars—e.g., Barbosa et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Paxson et al. (2022) and Dias et al. (2024a).

More recently, alternative, but equivalent, analytical representations have been proposed by Dias et al. (2024b), who tackle the time integral of options using a direct approach that does not rely on the real options intuition. However, both approaches have considered only positive interest rates and dividend yields, which prevents their use in a wider range of applications.

This paper offers four contributions to the literature. First, it simplifies and extends the solutions of Dias et al. (2024b) by producing analytic formulae accommodating the possibility of interest rate and/or dividend yield to be zero. Second, it offers the time integral solutions of binary options and barrier options under the BSM setup. Third, it provides the analytical solution of the time integral under the Margrabe (1978) economy. Finally, it proves the mathematical equivalence between both methods and shows that the analytic solution of Shackleton and Wojakowski (2007) can be further simplified when the underlying asset price is equal to the strike price, thus avoiding the computation of unnecessary terms when the cap is at-the-money. Hence, the method proposed in this paper further simplifies and facilitates the computation of time integrals of BSM and other option types, cements the "non-real options" route and opens the way for more analytical work in BSM, Margrabe and other areas.

We recall that profit caps and floors are also used in other streams of the literature. For example, Ebrahim et al. (2011) establish an analytic framework for studying Shared Income Mortgages and Shared Equity Mortgages, that involve positions in profit caps, whose design is aimed to improve the efficiency of financial intermediation with participating mortgages. Moreover, Shiller et al. (2013) and Shiller et al. (2019) advocate the viability of using Continuous Workout Mortgages, that include positions in profit floors, for mitigating financial fragility. Hence, a complete comprehension and relation between alternative formulations for valuing profit caps and floors are important in a wide range of applications.

The remainder of the paper is organized as follows. Section 2 presents a brief summary of the time decomposition technique offered by Shackleton and Wojakowski (2007) and extends the time integral solution of Dias et al. (2024b) for any option pricing scenario under both the BSM and Margrabe economies. Section 3 reconciles both pricing methodologies under the BSM model by showing their mathematical equivalence. In addition, a simplification of the analytic formulae produced by Shackleton and Wojakowski (2007) is proposed. Section 4 provides some numerical examples of the theoretical results. Finally, Section 5 highlights the main conclusions.

2. Time integrals in the BSM world

2.1. The time integral definition

Let us consider a continuous price process $\{S_t \in \mathbb{R}^+ : t \geq 0\}$ that generates cash at an instantaneous rate of flow $S_t dt$ and that is assumed to be governed by the risk-neutral dynamics

$$dS_t = (r - q) S_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \tag{1}$$

where r, q and σ are the (constant) risk free interest rate, dividend yield (or rate of return shortfall) and volatility, respectively, while $\{W_t^{\mathbb{Q}} \in \mathbb{R} : t \geq 0\}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.

Let $V(S_0, K, T, \sigma, r, q)$ be the time-0 value of a finite-lived profit cap. Its value can be computed from a cash flow of an instantaneous maximum flow rate $\Pi(S_t)\, dt$, with $\Pi(S_t) := (S_t - K)^+$. Under the martingale (or risk-neutral) measure $\mathbb{Q}$, its time-0 value is given by

$$V(S_0, K, T, \sigma, r, q) = \int_0^T c(S_0, K, t, \sigma, r, q)\, dt, \tag{2}$$

where $c(S_0, K, t, \sigma, r, q)$ is interpreted as the time-0 price of a European-style call option (or caplet) offered in Black and Scholes (1973) and Merton (1973) on an underlying asset with spot price S_0, strike price K and with expiry date at time t (> 0), that is

$$c(S_0, K, t, \sigma, r, q) = S_0 e^{-qt} N(d_1(t)) - K e^{-rt} N(d_0(t)), \tag{3}$$

with

$$d_\beta(t) = \frac{\ln(S_0/K) + (r - q + (\beta - 1/2)\sigma^2)t}{\sigma\sqrt{t}}, \tag{4}$$

and where $N(d_\beta(t))$ represents the cumulative distribution function of the univariate standard normal distribution for $\beta \in \{0, 1\}$.

Similarly, the value of a finite-lived profit floor can be computed from a cash flow of an instantaneous maximum flow rate $\Pi(S_t)\,dt$, with $\Pi(S_t) := (K - S_t)^+$. Its time-0 value is given by

$$F(S_0, K, T, \sigma, r, q) = \int_0^T p\,(S_0, K, t, \sigma, r, q)\,dt, \tag{5}$$

where $p\,(S_0, K, t, \sigma, r, q)$ is interpreted as the time-0 price of a European-style put option (or floorlet) given in Black and Scholes (1973) and Merton (1973) on an underlying asset with spot price S_0, strike price K and with expiry date at time t (> 0), that is

$$p\,(S_0, K, t, \sigma, r, q) = Ke^{-rt}N\left(-d_0(t)\right) - S_0 e^{-qt} N\left(-d_1(t)\right). \tag{6}$$

2.2. The time integral solution using the time decomposition technique of Shackleton and Wojakowski (2007)

Shackleton and Wojakowski (2007) use perpetual methods inspired in the real options literature to evaluate the time integral (2). More specifically, the value of a finite-lived profit cap expiring in T years is decomposed (or replicated) by a portfolio that includes a long position in a perpetual profit cap, $V(S_0, K, \infty, \sigma, r, q)$, and a short position in a forward start perpetual profit cap that begins after T years, $V(S_T, K, \infty, \sigma, r, q)$, that is[1]

$$\begin{aligned}
& V(S_0, K, T, \sigma, r, q) \\
= \;& V(S_0, K, \infty, \sigma, r, q) - e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[V(S_T, K, \infty, \sigma, r, q)|\mathcal{F}_0\right] \\
= \;& \frac{S_0}{q}\left[\mathbb{1}_{\{S_0 \geq K\}} - e^{-qT}N(d_1(T))\right] - \frac{K}{r}\left[\mathbb{1}_{\{S_0 \geq K\}} - e^{-rT}N(d_0(T))\right] \\
& + B(K)S_0^{\beta_2}\left[\mathbb{1}_{\{S_0 \geq K\}} - N(d_{\beta_2}(T))\right] - A(K)S_0^{\beta_1}\left[\mathbb{1}_{\{S_0 \geq K\}} - N(d_{\beta_1}(T))\right],
\end{aligned} \tag{7}$$

with $\beta \in \{0, 1, \beta_1, \beta_2\}$,

$$A(K) = \frac{K^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{q}\right), \tag{8}$$

[1] The corresponding profit floors can be tackled similarly.

$$B(K) = \frac{K^{1-\beta_2}}{\beta_1 - \beta_2} \left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{q} \right) \tag{9}$$

and

$$\beta_{1,2} = \frac{1}{2} - \frac{r-q}{\sigma^2} \pm \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}, \tag{10}$$

where the constants β_1 and β_2 are the (real) roots of the quadratic equation $\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + (r-q)\beta - r = 0$, for $\beta \in \{\beta_1, \beta_2\}$, that characterizes the linearly independent solutions of a second-order homogeneous ordinary differential equation.

We recall that the analytic representation (7) is the solution of the time integral (2) that is composed by a continuum of European-style call options (or caplets) given by equation (3). Since the caplets contained within the integral are independent (i.e., the valuation of each caplet is not path dependent), it is possible to isolate the finite cap integral from the perpetual cap by subtracting the (discounted) risk-neutral expectation of the forward start perpetual cap.

Clearly, the time integral solution (7) cannot accommodate the cases where the risk free interest rate and/or the dividend yield are zero, which prevents its use to other applications under the BSM world. Nevertheless, it is possible to extend the time integral solution approach of Dias et al. (2024b) to obtain closed forms for these particular cases and safely tackle all option pricing scenarios.

2.3. Extending the time integral solution of Dias et al. (2024b)

The next propositions further simplify the original solution of Dias et al. (2024b, Proposition 1) and produce analytic formulae accommodating the possibility of r and/or q to be zero.

Proposition 1 *Assume the lognormal process* (1) *with positive* σ.
*(i) The time-*0 *value of a finite-lived cap,* $V(S_0, K, T, \sigma, r, q)$*, is given by*

$$V(S_0, K, T, \sigma, r, q) = S_0\, I_c(0, T, S_0, K, 1, q) - K\, I_c(0, T, S_0, K, -1, r), \tag{11}$$

where, for $\phi \in \{-1, 1\}$ *and* $v \in \{r, q\}$,

$$\begin{aligned} & I_c(0, T, S_0, K, \phi, v) \\ := & \frac{1}{v}\left[\frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right)e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right]\right. \\ & -\frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right)e^{-a\left(c_v+b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\ & \left.-e^{-vT}N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)+\mathbb{1}_{\{S_0>K\}}\right], \end{aligned} \tag{12}$$

for $v > 0$, *and*[2]

$$\begin{aligned} & I_c(0, T, S_0, K, \phi, v) \\ := & TN\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)+\frac{\sqrt{T}}{b_\phi}n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right) \\ & +\frac{\operatorname{sign}(b_\phi)}{2b_\phi^2}e^{-2ab_\phi 1_{\{b_\phi<0\}}}\left(2ab_\phi\mathbb{1}_{\{b_\phi>0\}}-1\right)\left[N\left(\frac{a}{\sqrt{T}}+|b_\phi|\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\ & -\frac{\operatorname{sign}(b_\phi)}{2b_\phi^2}e^{-2ab_\phi 1_{\{b_\phi>0\}}}\left(2ab_\phi\mathbb{1}_{\{b_\phi<0\}}-1\right)\left[N\left(\frac{a}{\sqrt{T}}-|b_\phi|\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right], \end{aligned} \tag{13}$$

for $v = 0$, *with* $N(.)$ *and* $n(.)$ *representing, respectively, the cumulative distribution function and probability density function of the standard univariate normal distribution,*

$$a := \frac{\ln(S_0/K)}{\sigma}, \tag{14}$$

$$b_\phi := \frac{r - q + \phi\sigma^2/2}{\sigma} \tag{15}$$

and

$$c_v := \sqrt{b_\phi^2 + 2v}, \tag{16}$$

with $a \in \mathbb{R}$, $b_\phi \in \mathbb{R}$ *and* $c_v \in \mathbb{R}^+$.[3]

*(ii) The time-*0 *value of a finite-lived floor,* $F(S_0, K, T, \sigma, r, q)$, *is given by*

$$\begin{aligned} F(S_0, K, T, \sigma, r, q) &= V(S_0, K, T, \sigma, r, q) + \frac{K}{r}\left(1 - e^{-rT}\right)\mathbb{1}_{\{r>0\}} + KT\mathbb{1}_{\{r=0\}} \\ & -\frac{S_0}{q}\left(1 - e^{-qT}\right)\mathbb{1}_{\{q>0\}} - S_0 T\mathbb{1}_{\{q=0\}}. \end{aligned} \tag{17}$$

[2]We notice that the first argument of the function $I_c(0, T, S_0, K, \phi, v)$ represents the initial date $t = 0$. The same rationale is applied hereafter to similar functions.

[3]Even though the parameter defined in equation (16) could be formally stated as $c_{\phi,v}$, we only define it by c_v to lighten notation and because c_q and c_r are always associated with $\phi = 1$ and $\phi = -1$, respectively.

(iii) The time-0 *value of a perpetual cap,* $V(S_0, K, \infty, \sigma, r, q)$*, is given by*

$$V(S_0, K, \infty, \sigma, r, q) = S_0\, I_c(0, \infty, S_0, K, 1, q) - K\, I_c(0, \infty, S_0, K, -1, r), \tag{18}$$

where

$$\begin{aligned} & I_c(0, \infty, S_0, K, \phi, v) \\ := \ & \frac{1}{v}\left[\frac{1}{2}\left(\frac{b_\phi}{c_v} + 1\right) e^{a\left(c_v - b_\phi\right)} \mathbb{1}_{\{S_0 \le K\}} + \left(\frac{1}{2}\left(\frac{b_\phi}{c_v} - 1\right) e^{-a\left(c_v + b_\phi\right)} + 1\right) \mathbb{1}_{\{S_0 > K\}}\right], \end{aligned} \tag{19}$$

for $v > 0$*.*

(iv) The time-0 *value of a perpetual floor,* $F(S_0, K, \infty, \sigma, r, q)$*, is given by*

$$F(S_0, K, \infty, \sigma, r, q) = V(S_0, K, \infty, \sigma, r, q) + \frac{K}{r} - \frac{S_0}{q}, \tag{20}$$

for $v > 0$*.*

Proof. Please see Appendix A. ■

Remark 1 *We note that the integral solutions* (12) *and* (13) *have two important economic interpretations depending on the parameter* $v \in \{r, q\}$*:*

(i) for $v = q$ *(with* $\phi = 1$*), such integrals can be interpreted as the delta of a finite-lived profit cap, i.e.,* $\Delta_V(S_0, K, T, \sigma, r, q) := \partial V(S_0, K, T, \sigma, r, q)/\partial S_0 = I_c(0, T, S_0, K, 1, q)$*;*

(ii) for $v = r$ *(with* $\phi = -1$*),* $I_c(0, T, S_0, K, -1, r)$ *can be understood as the value of a finite-lived continuum of cash-or-nothing calls with a unit contract size (as will be discussed in more detail in Section 2.5).*

The integral solution (19) *has similar interpretations, but for the perpetual case.*

We notice that the perpetual profit cap solution (18) is restricted to $v > 0$. Nevertheless, it is still possible to cover the cases with $r = 0$ provided that $q > 0$, as shown in the next proposition.

Proposition 2 *Assume the lognormal process* (1) *with* $\sigma > 0$, $r = 0$ *and* $q > 0$. *The time*-0 *value of a perpetual cap,* $V(S_0, K, \infty, \sigma, r, q)$, *is given by*

$$\begin{aligned} & V(S_0, K, \infty, \sigma, r, q) \\ = \; & S_0 \, I_c(0, \infty, S_0, K, 1, q) - \frac{K}{2b_{-1}^2} e^{-2ab_{-1}} \mathbb{1}_{\{S_0 \leq K\}} + \frac{K}{b_{-1}} \left(a - \frac{1}{2b_{-1}} \right) \mathbb{1}_{\{S_0 > K\}}, \end{aligned} \tag{21}$$

where the integral $I_c(0, \infty, S_0, K, 1, q)$ *is calculated via equation* (19).

Proof. Please see Appendix B. ■

Remark 2 *Similarly, the perpetual profit floor solution* (20) *is restricted to* $v > 0$. *In this case, it is still possible to cover the cases with* $q = 0$ *provided that* $r > 0$. *Fortunately, it is possible to invoke the caplet-floorlet duality so that the price of a perpetual profit floor is recovered from the price of a perpetual profit cap through a suitable change in its arguments, that is*

$$F(S_0, K, \infty, \sigma, r, q) = V(K, S_0, \infty, \sigma, q, r). \tag{22}$$

Clearly, Propositions 1 and 2 and Remark 2 provide already a complete guide for computing time integrals under the BSM world. Nevertheless, it is still possible to evaluate finite-lived and perpetual profit floors directly, i.e., without the need of using the corresponding solutions for profit caps. The next two propositions present these results.

Proposition 3 *Assume the lognormal process* (1) *with positive* σ.
(i) The time-0 *value of a finite-lived floor,* $F(S_0, K, T, \sigma, r, q)$, *is given by*

$$F(S_0, K, T, \sigma, r, q) = K \, I_f(0, T, S_0, K, -1, r) - S_0 \, I_f(0, T, S_0, K, 1, q), \tag{23}$$

where

$$I_f(0, T, S_0, K, \phi, v) := \frac{1 - e^{-vT}}{v} - I_c(0, T, S_0, K, \phi, v), \tag{24}$$

for $v > 0$, *with* $I_c(0, T, S_0, K, \phi, v)$ *being given by equation* (12), *and*

$$I_f(0, T, S_0, K, \phi, v) := T - I_c(0, T, S_0, K, \phi, v), \tag{25}$$

for $v = 0$*, with* $I_c(0, T, S_0, K, \phi, v)$ *being given by equation* (13)*.*

*(ii) The time-*0 *value of a perpetual floor,* $F(S_0, K, \infty, \sigma, r, q)$*, is given by*

$$F(S_0, K, \infty, \sigma, r, q) = K\, I_f(0, \infty, S_0, K, -1, r) - S_0\, I_f(0, \infty, S_0, K, 1, q), \tag{26}$$

$$\begin{aligned} & I_f(0, \infty, S_0, K, \phi, v) \\ := \; & -\frac{1}{v}\left[\left(\frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right)e^{a\left(c_v-b_\phi\right)}-1\right)\mathbb{1}_{\{S_0\leq K\}}+\frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right)e^{-a\left(c_v+b_\phi\right)}\mathbb{1}_{\{S_0>K\}}\right] \quad (27) \\ = \; & \frac{1}{v} - I_c(0, \infty, S_0, K, \phi, v), \quad (28) \end{aligned}$$

for $v > 0$*, with* $I_c(0, \infty, S_0, K, \phi, v)$ *being given by equation* (19)*.*

Proof. Please see Appendix C. ■

We notice that the perpetual profit floor solution (26) is restricted to $v > 0$, where the required integrals can be computed either via equation (27) or through equation (28). Nevertheless, it is still possible to cover the cases with $q = 0$ provided that $r > 0$, as shown in the next proposition.

Proposition 4 *Assume the lognormal process* (1) *with* $\sigma > 0$*,* $r > 0$ *and* $q = 0$*. The time-*0 *value of a perpetual floor,* $F(S_0, K, \infty, \sigma, r, q)$*, is given by*

$$\begin{aligned} & F(S_0, K, \infty, \sigma, r, q) \\ = \; & K\, I_f(0, \infty, S_0, K, -1, r) + \frac{S_0}{b_1}\left(a - \frac{1}{2b_1}\right)\mathbb{1}_{\{S_0\leq K\}} - \frac{S_0}{2b_1^2}e^{-2ab_1}\mathbb{1}_{\{S_0>K\}}, \quad (29) \end{aligned}$$

where the integral $I_f(0, \infty, S_0, K, -1, r)$ *is calculated either via equation* (27) *or through equation* (28)*.*

Proof. Please see Appendix D. ■

In summary, Propositions 1, 2, 3 and 4 provide a complete guide for computing time integrals of profit caps and profit floors on continuous flows under the BSM world and for any combination of parameters. We recall that the contracts under analysis have instantaneous maximum flow rates $\Pi(S_t)\,dt$, with $\Pi(S_t) := (S_t - K)^+$ and $\Pi(S_t) := (K - S_t)^+$ for profit caps and profit floors, respectively. Nevertheless, our approach can be straightforwardly extended to cope with many other contingent claims with continuous flows (and still for any combination of parameters), as will be shown in the next four subsections.

2.4. Time integrals for price caps, price floors and price collars

The goal now is to show how to value the price caps, floors and collars considered in Dias et al. (2024b, Sections 3 and 4), but now for any combination of option pricing parameters under the BSM world. To accomplish this purpose, we will consider only the arbitrage-free relations involving finite-lived profit caps.[4]

A price cap offers an instantaneous flow rate $\Pi(S_t) := \min(S_t, H) = S_t - (S_t - H)^+$ and can be understood as a contingent claim containing a cap level H that provides a ceiling to the underlying asset price. The arbitrage-free relation of Dias et al. (2024b, equation 20) implies that the fair value of a price cap, $V_c(S_0, H, T, \sigma, r, q)$, can be computed as

$$V_c(S_0, H, T, \sigma, r, q) = \frac{S_0}{q}\left(1 - e^{-qT}\right)\mathbb{1}_{\{q>0\}} + S_0 T \mathbb{1}_{\{q=0\}} - V(S_0, H, T, \sigma, r, q). \tag{30}$$

A price floor provides an instantaneous flow rate $\Pi(S_t) := \max(S_t, L) = (S_t - L)^+ + L$ and can be considered as a contingent claim containing a floor level L that guarantees a minimum to the prevailing market price in the face of adverse scenarios. The arbitrage-free relation of Dias et al. (2024b, equation 23) implies that the fair value of a price floor, $V_f(S_0, L, T, \sigma, r, q)$, is determined as

$$V_f(S_0, L, T, \sigma, r, q) = V(S_0, L, T, \sigma, r, q) + \frac{L}{r}\left(1 - e^{-rT}\right)\mathbb{1}_{\{r>0\}} + LT\mathbb{1}_{\{r=0\}}. \tag{31}$$

The instantaneous flow rate of a price collar is defined as $\Pi(S_t) := \min(\max(L, S_t), H) = L + (S_t - L)^+ - (S_t - H)^+$, which implies that the underlying asset price floats freely subject to a price floor level L and a price cap level H (with $H \geq L$), so that the investor receives

[4] The arbitrage-free relations involving finite maturity profit floors and perpetual profit cap and floors considered in Dias et al. (2024b, Sections 3 and 4) can be treated similarly.

L if $S_t < L$, receives the unit price S_t if $L \leq S_t < H$ and receives H if $S_t \geq H$. The arbitrage-free relation of Dias et al. (2024b, equation 32) implies that the fair value of a price collar, $V_{col}(S_0, L, H, T, \sigma, r, q)$, is calculated as

$$\begin{aligned} & V_{col}(S_0, L, H, T, \sigma, r, q) \\ = \;& \frac{L}{r}\left(1 - e^{-rT}\right) \mathbb{1}_{\{r>0\}} + LT\mathbb{1}_{\{r=0\}} + V(S_0, L, T, \sigma, r, q) - V(S_0, H, T, \sigma, r, q). \end{aligned} \quad (32)$$

2.5. Time integrals for binary options

The analytic formulae for the profit caps and floors offered in Propositions 1, 2, 3 and 4 are obtained for contracts with instantaneous maximum flow rates in the spirit of plain-vanilla options and, therefore, they possess a continuum of smooth payoff patterns. The purpose now is to show that our approach can also be used to evaluate continuous flows of binary (or digital) call options in the sense of Rubinstein and Reiner (1991b) containing a continuum of discontinuous payoffs.[5]

A European-style cash-or-nothing call pays off nothing if the underlying asset price at maturity, S_t, ends up below or equals the strike price K, or pays out a predetermined fixed cash amount X (also known as the contract size) if the underlying asset finishes above the strike. Therefore, a cash-or-nothing call option pays out the fixed amount X at maturity t if the option finishes in-the-money, i.e., the time-t payoff of each cash-or-nothing call is equal to $\Pi\left(S_t\right) := X\mathbb{1}_{\{S_t>K\}}$.

Hence, it is straightforward to conclude that the time-0 price of a finite-lived contract on continuous flows of cash-or-nothing call options can be calculated as

$$\begin{aligned} V_{con}(S_0, K, X, T, \sigma, r, q) &= \int_0^T X e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{S_t>K\}}|\mathcal{F}_0\right] dt \\ &= X\int_0^T e^{-rt} N\left(d_0(t)\right) dt \\ &= X\, I_c(0, T, S_0, K, -1, r), \end{aligned} \quad (33)$$

[5]The corresponding binary put options can be obtained similarly.

with the ingredient $I_c(0, T, S_0, K, -1, r)$ being interpreted as the time-0 price of a finite-lived continuum of cash-or-nothing calls with a unit contract size and computed via equation (12) if $r > 0$ or equation (13) if $r = 0$.

A European-style asset-or-nothing call pays off nothing if the underlying asset price at maturity, S_t, ends up below or equals the strike price K, or pays out a cash amount that is equal to the underlying asset value at maturity (i.e, an amount equal to S_t) if the underlying asset finishes above the strike. Therefore, these option contracts are similar to cash-or-nothing call options, except that when they pay off, the cash amount is not predetermined, but rather is equal to the underlying asset price at expiration.

Since the time-t payoff of each asset-or-nothing call is equal to $\Pi(S_t) := S_t \mathbb{1}_{\{S_t > K\}}$, it follows that the time-0 price of a finite-lived contract on continuous flows of asset-or-nothing call options can be computed as

$$\begin{aligned} V_{aon}(S_0, K, T, \sigma, r, q) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[S_t \mathbb{1}_{\{S_t > K\}} | \mathcal{F}_0 \right] dt \\ &= \int_0^T S_0 e^{-qt} N\left(d_1(t)\right) dt \\ &= S_0 \, I_c(0, T, S_0, K, 1, q), \end{aligned} \tag{34}$$

with the ingredient $I_c(0, T, S_0, K, 1, q)$ being computed via equation (12) if $q > 0$ or equation (13) if $q = 0$.

Armed with the integral solutions (33) and (34), we can now price another exotic contingent claim on continuous flows under the BSM world: a finite-lived contract on continuous flows of gap call options. We recall that gap options highlight the dual role played by the strike price of a plain-vanilla option: the strike price K determines not only the exercise decision at maturity but also the resulting payoff. The time-t payoff of each gap call option is equal to $\Pi(S_t) := (S_t - X) \mathbb{1}_{\{S_t > K\}} = [S_t - (K + g)] \mathbb{1}_{\{S_t > K\}}$, with the gap value defined as $g := X - K$, where X is the strike price of the gap call option whereas K is the price level above which the gap call option finishes in-the-money and triggers the exercise of the option.

The terminal payoff of each gap call option allows us to conclude that the time-0 price of a finite-lived contract on continuous flows of gap call options can be obtained by subtracting the value of a continuum of cash-or-nothing calls from the value of a continuum of asset-or-nothing calls, that is

$$\begin{aligned} V_{gap}(S_0,K,X,T,\sigma,r,q) &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(S_t-X)\,\mathbb{1}_{\{S_t>K\}}|\mathcal{F}_0\right]dt \\ &= V_{aon}(S_0,K,T,\sigma,r,q)-V_{con}(S_0,K,X,T,\sigma,r,q). \end{aligned} \tag{35}$$

Clearly, the special case with $X = K$ (i.e., with a gap value $g = 0$) yields immediately the value of a finite-lived profit cap, that is

$$\begin{aligned} V_{gap}(S_0,K,K,T,\sigma,r,q) &= V_{aon}(S_0,K,T,\sigma,r,q)-V_{con}(S_0,K,K,T,\sigma,r,q) \\ &= V(S_0,K,T,\sigma,r,q). \end{aligned} \tag{36}$$

2.6. Time integrals for path-dependent options

We note that each cash-or-nothing and asset-or-nothing call contained within the time integrals yielding the closed-form solutions (33) and (34) are independent (i.e., the valuation of each binary option in the integrals does not depend on the others). Moreover, these contingent claims belong to the class of path-independent binary options in the sense that there is no contractual clause triggering any knock-in or knock-out event. Nevertheless, our approach can also be used to evaluate finite-lived contracts on continuous flows of barrier options or binary barrier options in the spirit of Rubinstein and Reiner (1991a) and Rubinstein and Reiner (1991b), respectively.[6] For illustrative purposes (and due to space constraints), we will analyze only the case of finite-lived contracts on continuous flows of down-and-out and down-and-in calls, though similar contingent claims composed by a continuum of other types of barrier options, binary barrier options or the whole family of lookback options can be treated similarly.

[6] Additional background on hitting times in finance applications can be consulted, for instance, in Rich (1994), Jeanblanc et al. (2009, Chapter 3) and Zaevski (2020).

The time-t payoff of a unit face value and zero rebate European-style down-and-out call option on the asset price S, with strike price K, knock-out barrier level L (with $L < S_0$) and maturity at time t is equal to $\Pi(S_t) := (S_t - K)^+ \mathbb{1}_{\{\tau_L > t\}}$, where $\tau_L := \inf\{u > 0 : S_u = L\}$ is the first hitting time of the lower barrier L by the asset price S_u. In other words, the owner of a down-and-out call will receive the payoff $(S_t - K)^+$ only if the random hitting timing τ_L does not occur before the maturity date t.

Under the assumptions of the BSM world, the time-0 price of a unit face value and zero rebate European-style down-and-out call option on the asset price S, with strike K, barrier level L $(< S_0)$ and maturity at time t (> 0) is equal to

$$\begin{aligned} & DOC\left(S_0, K, L, t, \sigma, r, q\right) \\ = \; & c\left(S_0, \max(L,K), t, \sigma, r, q\right) - \left(\frac{L}{S_0}\right)^{\frac{2\mu}{\sigma^2}} c\left(\frac{L^2}{S_0}, \max(L,K), t, \sigma, r, q\right) \\ & + \left[\max(L,K) - K\right] e^{-rt} \left\{ N\left[d_0\left(S_0, L, t\right)\right] - \left(\frac{L}{S_0}\right)^{\frac{2\mu}{\sigma^2}} N\left[d_0\left(L, S_0, t\right)\right] \right\}, \end{aligned} \tag{37}$$

with

$$d_\beta(x,y,t) = \frac{\ln(x/y) + (r - q + (\beta - 1/2)\sigma^2)t}{\sigma\sqrt{t}}, \tag{38}$$

$$\mu = r - q - \frac{\sigma^2}{2} \tag{39}$$

and $c\left(x, y, t, \sigma, r, q\right)$ being the time-0 price of a European-style plain-vanilla call on the asset price x, with strike price y and expiry date t.[7]

[7] Notice that equation (38) is exactly equivalent to equation (4), but now with a functional form showing its explicit dependence on the asset price and the strike price.

Therefore, the time-0 price of a finite-lived contract on continuous flows of down-and-out call options can be computed as

$$\begin{aligned}
& V_{doc}(S_0, K, L, T, \sigma, r, q) \qquad (40) \\
= & \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[(S_t - K)^+ \mathbb{1}_{\{\tau_L > t\}} | \mathcal{F}_0 \right] dt \\
= & \int_0^T DOC\left(S_0, K, L, t, \sigma, r, q\right) dt \\
= & V\left(S_0, \max(L, K), T, \sigma, r, q\right) - \left(\frac{L}{S_0}\right)^{\frac{2\mu}{\sigma^2}} V\left(\frac{L^2}{S_0}, \max(L, K), T, \sigma, r, q\right) \\
& + [\max(L, K) - K] \left[V_{con}(S_0, L, 1, T, \sigma, r, q) - \left(\frac{L}{S_0}\right)^{\frac{2\mu}{\sigma^2}} V_{con}(L, S_0, 1, T, \sigma, r, q) \right],
\end{aligned}$$

which requires calculations of finite maturity profit caps (11) and finite-lived continuum of cash-or-nothing call options (33).

Notice that the valuation of each down-and-out call option contained within the time integral (40) does not depend on the others, which implies that there is a continuum of independent knock-out events. However, each individual down-and-out call option is path-dependent in the sense that a specific knock-out event will occur for that particular down-and-out call if the barrier level L is hit before its specific maturity date t, for $t \in]0, T]$.

Finally, we note that a finite-lived continuum of down-and-out call options limits the range of possible outcomes due to the existence of a barrier level triggering a continuum of (independent) random knock-out events and, hence, it is cheaper than the corresponding finite maturity profit cap. Moreover, it is straightforward to show that

$$\lim_{L \to 0} V_{doc}(S_0, K, L, T, \sigma, r, q) = V(S_0, K, T, \sigma, r, q). \qquad (41)$$

Let us consider now an example of a contract composed by a finite-lived continuum of (independent) knock-in clauses. We first recall that the holder of a down-and-in call will receive the payoff $(S_t - K)^+$ at the maturity date t if at any time between the inception of the contract and its expiry date t the barrier level L is touched. Hence, the time-t payoff of a unit face value and zero rebate European-style down-and-in call option on the asset price S, with strike price K, knock-in barrier level L (with $L < S_0$) and maturity at time t is equal to $\Pi(S_t) := (S_t - K)^+ \mathbb{1}_{\{\tau_L \le t\}}$. In other words, if the random hitting timing τ_L does not occur before or at the maturity date t for a specific down-and-in call, then the knock-in clause for that particular option is not triggered.

The time-t price of a unit face value and zero rebate European-style down-and-in call option on the asset price S, with strike K, barrier level L ($< S_0$) and maturity at time t (> 0) is equal to

$$\begin{aligned} & DIC\left(S_0, K, L, t, \sigma, r, q\right) \\ = \; & \left(\frac{L}{S_0}\right)^{\frac{2\mu}{\sigma^2}} \left\{ c\left(\frac{L^2}{S_0}, \max(L,K), t, \sigma, r, q\right) + [\max(L,K) - K] e^{-rt} N\left[d_0\left(L, S_0, t\right)\right] \right\} \\ & + \left\{ p\left(S_0, K, t, \sigma, r, q\right) - p\left(S_0, L, t, \sigma, r, q\right) + (L-K) e^{-rt} N\left[-d_0\left(S_0, L, t\right)\right] \right\} \mathbb{1}_{\{L>K\}}, \end{aligned} \tag{42}$$

where $c(x, y, t, \sigma, r, q)$ and $p(x, y, t, \sigma, r, q)$ are, respectively, the time-0 prices of European-style plain-vanilla calls and puts on the asset price x, with strike price y and expiry date t.

Therefore, the time-0 price of a finite-lived contract on continuous flows of down-and-in call options can be computed as

$$\begin{aligned} & V_{dic}(S_0, K, L, T, \sigma, r, q) \\ = \; & \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[(S_t - K)^+ \mathbb{1}_{\{\tau_L \le t\}} | \mathcal{F}_0 \right] dt \\ = \; & \int_0^T DIC\left(S_0, K, L, t, \sigma, r, q\right) dt \\ = \; & \left(\frac{L}{S_0}\right)^{\frac{2\mu}{\sigma^2}} \left[V\left(\frac{L^2}{S_0}, \max(L,K), T, \sigma, r, q\right) + [\max(L,K) - K]\, V_{con}(L, S_0, 1, T, \sigma, r, q) \right] \\ & + \left[F\left(S_0, K, T, \sigma, r, q\right) - F\left(S_0, L, T, \sigma, r, q\right) + (L-K)\, F_{con}(S_0, L, 1, T, \sigma, r, q) \right] \mathbb{1}_{\{L>K\}}, \end{aligned} \tag{43}$$

which requires calculations of finite maturity profit caps (11), finite maturity profit floors (23), finite-lived continuum of cash-or-nothing call options (33) and finite-lived continuum of cash-or-nothing put options defined as

$$\begin{aligned} F_{con}(S_0, K, X, T, \sigma, r, q) &= \int_0^T X e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[\mathbb{1}_{\{S_t < K\}} | \mathcal{F}_0 \right] dt \\ &= X \int_0^T e^{-rt} N\left(-d_0(t)\right) dt \\ &= X\, I_f(0, T, S_0, K, -1, r), \end{aligned} \tag{44}$$

with the ingredient $I_f(0, T, S_0, K, -1, r)$ being interpreted as the time-0 price of a finite-lived continuum of cash-or-nothing puts with a unit contract size and computed via equation (24) if $r > 0$ or equation (25) if $r = 0$.

As an alternative to equation (43), and since $\mathbb{1}_{\{\tau_L \leq t\}} = 1 - \mathbb{1}_{\{\tau_L > t\}}$, it follows that

$$\begin{aligned}
& V_{dic}(S_0, K, L, T, \sigma, r, q) \\
= & \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(S_t - K)^+ \mathbb{1}_{\{\tau_L \leq t\}} | \mathcal{F}_0\right] dt \\
= & \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(S_t - K)^+ | \mathcal{F}_0\right] dt - \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(S_t - K)^+ \mathbb{1}_{\{\tau_L > t\}} | \mathcal{F}_0\right] dt \\
= & V(S_0, K, T, \sigma, r, q) - V_{doc}(S_0, K, L, T, \sigma, r, q),
\end{aligned} \tag{45}$$

which constitutes a novel in-out parity relation between finite-lived contracts on continuous flows. Similar relationships can be obtained immediately if we use other path-dependent barrier options.

2.7. Time integrals under the Margrabe world

Dias et al. (2024b, Proposition 3) show how to compute profit caps and floors on continuous exchange flows under the two-factor model of Margrabe (1978), though restricting the dividend yield of both risky assets, denoted by q_s and q_k, to be strictly positive. Nevertheless, it is possible to adopt the rationale used in Section 2.3 to produce analytic formulae accommodating the possibility of q_s and/or q_k to be zero. To save space and avoid bloating the paper with similar expressions, we just note that the time integrals under the Margrabe economy can be straightforwardly obtained from Propositions 1, 2, 3 and 4 with the following changes: (i) K is replaced by the time-0 value of the risky asset K_0; (ii) r is replaced by the dividend yield q_k that is associated to the risky asset K; (iii) q is replaced by the dividend yield q_s that is associated to the risky asset S; and (iv) the volatility σ is replaced by $\sigma := \sqrt{\sigma_s^2 + \sigma_k^2 - 2\rho\sigma_s\sigma_k}$, where $\sigma_i > 0$, for $i \in \{s, k\}$, is the volatility for the random variable i and ρ is the correlation coefficient between the random variables S and K.[8]

[8]Full details of the proofs are available upon request.

These results under the Margrabe world can be immediately applied in the valuation of financial instruments delivering the overprice between two (random) cash flows as shown in Dias et al. (2024b, Section 7.1). However, we can now price such contingent claims considering any possible combination of parameters under the Margrabe economy. For example, the finite-lived profit cap on continuous exchange flows presented in Dias et al. (2024b, equation 101) is restricted to positive dividend yields q_s and q_k and, hence, does not accommodate the scenario studied by Margrabe (1978), where both risky assets were assumed to be non-dividend paying. Fortunately, the novel analytic representation (11) allows the calculation of a finite-lived continuum of exchange options with the possibility of q_s and/or q_k to be zero.

Furthermore, and following the insights of Rubinstein (1991) on European-style exchange options, our solutions can also be used for valuing financial instruments involving a continuum of options on the minimum (or worse performing) and on the maximum (or better performing) of two underlying assets delivering, respectively, the instantaneous flow rates $\Pi(S_t, K_t) := \min(S_t, K_t) = S_t - (S_t - K_t)^+$ and $\Pi(S_t, K_t) := \max(S_t, K_t) = K_t + (S_t - K_t)^+$. Hence, a finite-lived continuum of options on the minimum and on the maximum can be calculated as[9]

$$V_{min}(S_0, K_0, T, \sigma, q_k, q_s) = \frac{S_0}{q_s}\left(1 - e^{-q_s T}\right)\mathbb{1}_{\{q_s>0\}} + S_0 T \mathbb{1}_{\{q_s=0\}} - V(S_0, K_0, T, \sigma, q_k, q_s) \quad (46)$$

and

$$V_{max}(S_0, K_0, T, \sigma, q_k, q_s) = \frac{K_0}{q_k}\left(1 - e^{-q_k T}\right)\mathbb{1}_{\{q_k>0\}} + K_0 T \mathbb{1}_{\{q_k=0\}} + V(S_0, K_0, T, \sigma, q_k, q_s), \quad (47)$$

respectively, with the finite-lived profit cap $V(S_0, K_0, T, \sigma, q_k, q_s)$ being still computed via equation (11), but with $\sigma := \sqrt{\sigma_s^2 + \sigma_k^2 - 2\rho\sigma_s\sigma_k}$.

[9]For the sake of completeness, we note that Dias et al. (2024b, equation 122) contains a text typo. In their notation, the parameter q_p appearing in the denominator of the first term should be replaced by q_x.

3. Reconciling the two option pricing solutions

3.1. Solving the time integral using integration by parts

Dias et al. (2024b) solve analytically the time integral (2) using integration by parts and obtain an alternative, but equivalent, solution to (7). The goal now is to show how to use the integration by parts technique and obtain exactly the closed-form solution (7).

Let us first recall the method of integration known as integration by parts:

$$\int_a^b p'(t)q(t)dt = [p(t)q(t)]_{t=a}^{t=b} - \int_a^b p(t)q'(t)dt, \tag{48}$$

where $p'(t)$ is the derivative of a primitive function $p(t)$, $q(t)$ is a second function whose derivative is $q'(t)$, and a and b are the lower and upper limits of the integral, respectively.

Replacing the plain-vanilla call (or caplet) solution (3) in the time integral (2) and applying the integration by parts technique yields

$$\begin{aligned}
V(S_0,K,T,\sigma,r,q) &= \int_0^T \left[S_0 e^{-qt} N(d_1(t)) - K e^{-rt} N(d_0(t))\right] dt \\
&= -\frac{S_0}{q}\int_0^T (-q)e^{-qt}N(d_1(t))dt + \frac{K}{r}\int_0^T (-r)e^{-rt}N(d_0(t))dt \\
&= -\frac{S_0}{q}\left(\left[e^{-qt}N(d_1(t))\right]_{t=0}^{t=T} - \int_0^T e^{-qt}\frac{\partial N(d_1(t))}{\partial t}dt\right) \\
&\quad +\frac{K}{r}\left(\left[e^{-rt}N(d_0(t))\right]_{t=0}^{t=T} - \int_0^T e^{-rt}\frac{\partial N(d_0(t))}{\partial t}dt\right) \\
&= V_1 + V_2 + V_3 + V_4,
\end{aligned} \tag{49}$$

with

$$V_1 = -\frac{S_0}{q}\left[e^{-qt}N(d_1(t))\right]_{t=0}^{t=T}, \tag{50}$$

$$V_2 = \frac{K}{r}\left[e^{-rt}N(d_0(t))\right]_{t=0}^{t=T}, \tag{51}$$

$$V_3 = \frac{S_0}{q}\int_0^T e^{-qt}\frac{\partial N(d_1(t))}{\partial t}dt \tag{52}$$

and

$$V_4 = -\frac{K}{r}\int_0^T e^{-rt}\frac{\partial N(d_0(t))}{\partial t}dt. \tag{53}$$

Clearly, the ingredients V_1 and V_2 are easily determined.[10] The components V_3 and V_4 are trickier to be obtained since both require solving integrals with an exponential function of time multiplied by the partial derivative of the cumulative distribution function of a univariate standard normal distribution with respect to time. Fortunately, we can apply the insights of Dias et al. (2024b, Proposition 1 and Appendix A) to solve analytically such integrals and, hence, show the links to equation (7).

3.2. The mathematical equivalence between both methods

Using the notation of Dias et al. (2024b, Proposition 1 and Appendix A), it is straightforward to show that

$$\begin{aligned} V_1 &= -\frac{S_0}{q} e^{-qT} N\left(\frac{a}{\sqrt{T}} + b_1\sqrt{T}\right) + \frac{S_0}{q}\left(1 \times \mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}}\right) \\ &= \frac{S_0}{q}\left(\mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}} - e^{-qT} N\left(d_1(T)\right)\right), \end{aligned} \tag{54}$$

with a and b_ϕ, for $\phi \in \{-1, 1\}$, being defined by equations (14) and (15), respectively, and, therefore,

$$\begin{aligned} \frac{a}{\sqrt{T}} + b_1\sqrt{T} &= \frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{r - q + \sigma^2/2}{\sigma}\sqrt{T} \\ &= d_1(T). \end{aligned} \tag{55}$$

Similarly, and following again Dias et al. (2024b, Proposition 1 and Appendix A), it is possible to conclude that

$$\begin{aligned} V_2 &= \frac{K}{r} e^{-rT} N\left(\frac{a}{\sqrt{T}} + b_{-1}\sqrt{T}\right) - \frac{K}{r}\left(1 \times \mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}}\right) \\ &= -\frac{K}{r}\left(\mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}} - e^{-rT} N\left(d_0(T)\right)\right), \end{aligned} \tag{56}$$

because

$$\begin{aligned} \frac{a}{\sqrt{T}} + b_{-1}\sqrt{T} &= \frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{r - q - \sigma^2/2}{\sigma}\sqrt{T} \\ &= d_0(T). \end{aligned} \tag{57}$$

[10] Nevertheless, care must be taken when computing their limits as $t \to 0$, as shown in Dias et al. (2024b, equation A7). For example, Barbosa et al. (2018, equation B.7) gives incorrect limits when the spot price is equal to the strike price.

Using the insights and notation of Dias et al. (2024b, Proposition 1 and Appendix A), it follows that

$$\begin{aligned}
& V_3 \\
= & \frac{S_0}{2q}\left(\frac{b_1}{c_q}+1\right)e^{a(c_q-b_1)}\left[N\left(\frac{a}{\sqrt{T}}+c_q\sqrt{T}\right)-1\times \mathbb{1}_{\{S_0>K\}}-\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}\right] \\
& +\frac{S_0}{2q}\left(\frac{b_1}{c_q}-1\right)e^{-a(c_q+b_1)}\left[N\left(-\frac{a}{\sqrt{T}}+c_q\sqrt{T}\right)-1+1\times \mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}\right] \\
= & \frac{S_0}{2q}\frac{b_1+c_q}{c_q}e^{a(c_q-b_1)}\left[N\left(\frac{a}{\sqrt{T}}+c_q\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}-\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}\right] \\
& +\frac{S_0}{2q}\frac{b_1-c_q}{c_q}e^{-a(c_q+b_1)}\left[\mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}-N\left(\frac{a}{\sqrt{T}}-c_q\sqrt{T}\right)\right] \qquad (58)
\end{aligned}$$

and

$$\begin{aligned}
& V_4 \\
= & -\frac{K}{2r}\left(\frac{b_{-1}}{c_r}+1\right)e^{a(c_r-b_{-1})}\left[N\left(\frac{a}{\sqrt{T}}+c_r\sqrt{T}\right)-1\times \mathbb{1}_{\{S_0>K\}}-\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}\right] \\
& -\frac{K}{2r}\left(\frac{b_{-1}}{c_r}-1\right)e^{-a(c_r+b_{-1})}\left[N\left(-\frac{a}{\sqrt{T}}+c_r\sqrt{T}\right)-1+1\times \mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}\right] \\
= & -\frac{K}{2r}\frac{b_{-1}+c_r}{c_r}e^{a(c_r-b_{-1})}\left[N\left(\frac{a}{\sqrt{T}}+c_r\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}-\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}\right] \\
& -\frac{K}{2r}\frac{b_{-1}-c_r}{c_r}e^{-a(c_r+b_{-1})}\left[\mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}-N\left(\frac{a}{\sqrt{T}}-c_r\sqrt{T}\right)\right], \qquad (59)
\end{aligned}$$

with $c_v \in \mathbb{R}^+$, for $v \in \{r, q\}$, being defined by equation (16).

Now it is necessary to perform some auxiliary calculations that are collected in Appendix E. Using such auxiliary computations and summing the option components (54), (56) and (E.30), allows equation (49) to be rewritten as

$$\begin{aligned}
V(S_0,K,T,\sigma,r,q) = & \frac{S_0}{q}\left(\mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}-e^{-qT}N\left(d_1(T)\right)\right) \\
& -\frac{K}{r}\left(\mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}-e^{-rT}N\left(d_0(T)\right)\right) \\
& +B(K)S_0^{\beta_2}\left[\mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}-N\left(d_{\beta_2}(T)\right)\right] \\
& -A(K)S_0^{\beta_1}\left[\mathbb{1}_{\{S_0>K\}}+\frac{1}{2}\times \mathbb{1}_{\{S_0=K\}}-N\left(d_{\beta_1}(T)\right)\right]. \qquad (60)
\end{aligned}$$

At first glance, it seems that the finite cap solutions (7) and (60) are different due to the distinct indicator functions appearing in each analytical formula. However, they are mathematically equivalent. This is so because the sum of the terms involving the indicator function $\mathbb{1}_{\{S_0=K\}}$ is zero, that is

$$\frac{1}{2}\frac{S_0}{q}\mathbb{1}_{\{S_0=K\}} - \frac{1}{2}\frac{K}{r}\mathbb{1}_{\{S_0=K\}} + \frac{1}{2}B(K)S_0^{\beta_2}\mathbb{1}_{\{S_0=K\}} - \frac{1}{2}A(K)S_0^{\beta_1}\mathbb{1}_{\{S_0=K\}} = 0. \tag{61}$$

To show this, we note that replacing S_0 by K and definitions (8) and (9) in equation (61) yields

$$\begin{aligned}
& \frac{1}{2}\frac{K}{q} - \frac{1}{2}\frac{K}{r} + \frac{1}{2}B(K)K^{\beta_2} - \frac{1}{2}A(K)K^{\beta_1} \\
= \; & \frac{1}{2}\frac{K}{q} - \frac{1}{2}\frac{K}{r} + \frac{1}{2}\frac{K^{1-\beta_2}}{\beta_1-\beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1-1}{q}\right)K^{\beta_2} - \frac{1}{2}\frac{K^{1-\beta_1}}{\beta_1-\beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2-1}{q}\right)K^{\beta_1} \\
= \; & \frac{K}{2}\left(\frac{1}{q} - \frac{1}{r} + \frac{1}{\beta_1-\beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_2}{r} - \frac{\beta_1-1}{q} + \frac{\beta_2-1}{q}\right)\right) \\
= \; & \frac{K}{2}\left(\frac{1}{q} - \frac{1}{r} + \frac{1}{\beta_1-\beta_2}\left(\frac{\beta_1-\beta_2}{r} - \frac{\beta_1-\beta_2}{q}\right)\right) \\
= \; & 0.
\end{aligned} \tag{62}$$

In summary, if we sum equations (60) and (61) we obtain exactly equation (7). This is so because $\mathbb{1}_{\{S_0>K\}} + \frac{1}{2}\times\mathbb{1}_{\{S_0=K\}} + \frac{1}{2}\times\mathbb{1}_{\{S_0=K\}} = \mathbb{1}_{\{S_0\geq K\}}$. Nevertheless, since for any $\alpha \in \mathbb{R}$ we know that

$$\alpha\left(\frac{S_0}{q}\mathbb{1}_{\{S_0=K\}} - \frac{K}{r}\mathbb{1}_{\{S_0=K\}} + B(K)S_0^{\beta_2}\mathbb{1}_{\{S_0=K\}} - A(K)S_0^{\beta_1}\mathbb{1}_{\{S_0=K\}}\right) = 0, \tag{63}$$

equation (7), i.e., Shackleton and Wojakowski (2007, equation 21), can be further simplified to

$$\begin{aligned}
& V(S_0, K, T, \sigma, r, q) \\
= \; & \frac{S_0}{q}\left[\mathbb{1}_{\{S_0>K\}} - e^{-qT}N(d_1(T))\right] - \frac{K}{r}\left[\mathbb{1}_{\{S_0>K\}} - e^{-rT}N(d_0(T))\right] \\
& + B(K)S_0^{\beta_2}\left[\mathbb{1}_{\{S_0>K\}} - N(d_{\beta_2}(T))\right] - A(K)S_0^{\beta_1}\left[\mathbb{1}_{\{S_0>K\}} - N(d_{\beta_1}(T))\right].
\end{aligned} \tag{64}$$

Therefore, the indicator functions $\mathbb{1}_{\{S_0\geq K\}}$ appearing in the delta and gamma sensitivity measures shown in Shackleton and Wojakowski (2007, equations 28 and 29), respectively, can be also safely replaced by $\mathbb{1}_{\{S_0>K\}}$.

4. Numerical examples

In this section we present some numerical experiments that should be helpful for providing a clear understanding of the theoretical results offered in the proposed novel pricing formulae. Table 1 shows the prices of finite-lived profit caps computed via equation (11), with $S_0 = 100$, $T = 1$, $\sigma = 0.25$ and considering different moneyness levels with strike prices $K \in \{95, 100, 105\}$ and different drift specifications (i.e., positive, zero and negative drifts $r - q$).

Although the results documented in Panel A can be calculated using Shackleton and Wojakowski (2007, equation 21) and Dias et al. (2024b, equation 4), these pricing methodologies cannot be used for valuing the contracts shown in Panel B that consider option pricing scenarios with r and/or q equal to zero. Fortunately, such parameter combinations can now be safely tackled in closed-form using the analytical solution (13) when $v = 0$ for $v \in \{r, q\}$.[11]

Table 1: Finite-lived profit caps for different moneyness levels and drift specifications.

			Strike price K		
#	r	q	95	100	105
Panel A: r and q are both positive					
1	0.05	0.03	9.728	6.981	4.955
2	0.03	0.03	9.191	6.520	4.575
3	0.03	0.05	8.574	6.007	4.162
Panel B: r and/or q are zero					
1	0	0.03	8.416	5.865	4.041
2	0.03	0	10.175	7.350	5.252
3	0	0	9.347	6.639	4.662

Note: This table values finite-lived profit caps for different strike prices K, risk free interest rates r and dividend yields q computed via equation (11). Other parameters used in the calculations: $S_0 = 100$, $T = 1$ and $\sigma = 0.25$.

Figure 1 illustrates an attempt to compute the value of an out-of-the-money profit cap using the Shackleton and Wojakowski (2007, equation 21) when both the interest rate r and dividend rate q are equal to zero. Direct substitution of $r = q = 0$ leads to numerical

[11] Notice that all these results can be straightforwardly checked against the ones obtained via equation (2) computed through a numerical integration scheme. Similar robustness tests have been performed to all the other contracts considered in the remaining numerical examples of this section.

issues such as division by zero and other indeterminate expressions. To circumvent this, increasingly small positive values are substituted, starting from $r = q = \varepsilon = 10^{-13}$ (marks in red) and then $r = q = \varepsilon = 10^{-14}$ (marks in blue). Paradoxically, reducing ε leads to larger numerical errors, and for sufficiently small values the formula collapses. By contrast, the new formula (11) remains numerically stable and produces the correct values across the domain (black line).

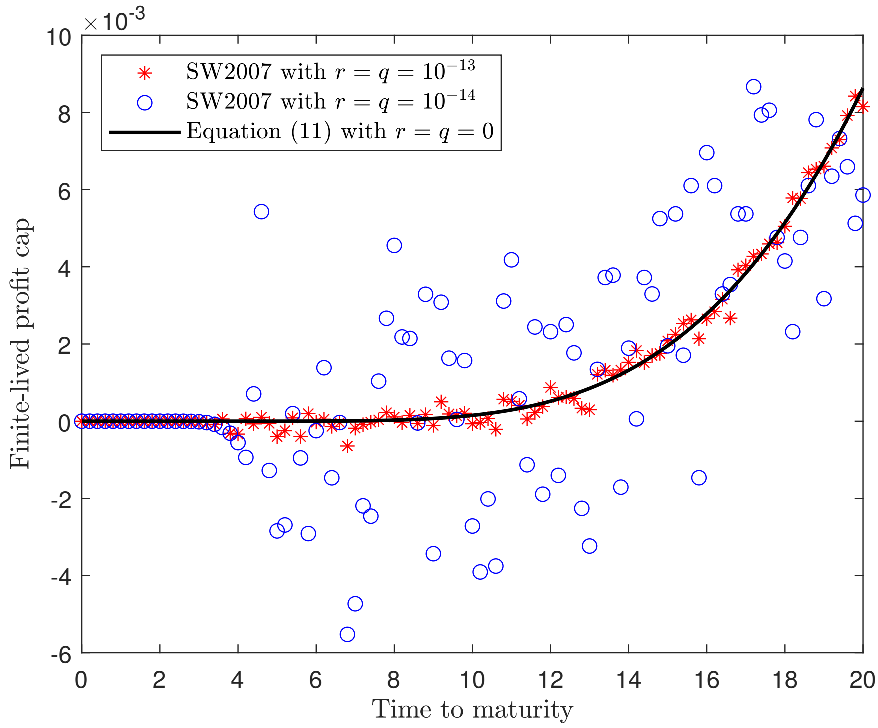

Figure 1: Numerical instability of the Shackleton and Wojakowski (2007, equation 21) when the interest rate and dividend rate approach zero.

Note: This figure plots finite-lived profit caps as a function of the time to maturity T, with $T \in [0, 20]$ (in years) divided into 100 evenly-spaced time points. The plots with marks in red and in blue implement Shackleton and Wojakowski (2007, equation 21) with $r = q = 10^{-13}$ and $r = q = 10^{-14}$, respectively, whereas the black line uses equation (11) with $r = q = 0$. Other parameters used in the calculations: $S_0 = 0.4$, $K = 1$ and $\sigma = 0.10$.

Figure 2 plots finite-lived profit caps as a function of the risk free interest rate r, with $r \in [-0.001, 0.001]$ divided into 200,000 evenly-spaced interest rate points, for the constellation of parameters $S_0 = 100$, $K = 100$, $T = 1$, $\sigma = 0.25$ and $q = 0.03$. The top-left plot computes Shackleton and Wojakowski (2007, equation 21), the top-right plot implements Dias et al. (2024b, equation 4) and the bottom plot uses equation (11). The two plots on the top show that the calculations become numerically unstable in the limit as $r \to 0$, resulting in messy or unreliable graphs. This pattern is encountered when computing and plotting profit cap (and profit floor) values as r and/or q approach zero. Unlike the approaches of Shackleton and Wojakowski (2007) and Dias et al. (2024b) that rely on taking limits on the neighborhood of 0, the new formula (11) resolves this issue by yielding the correct value directly, as shown in the plot on the bottom.

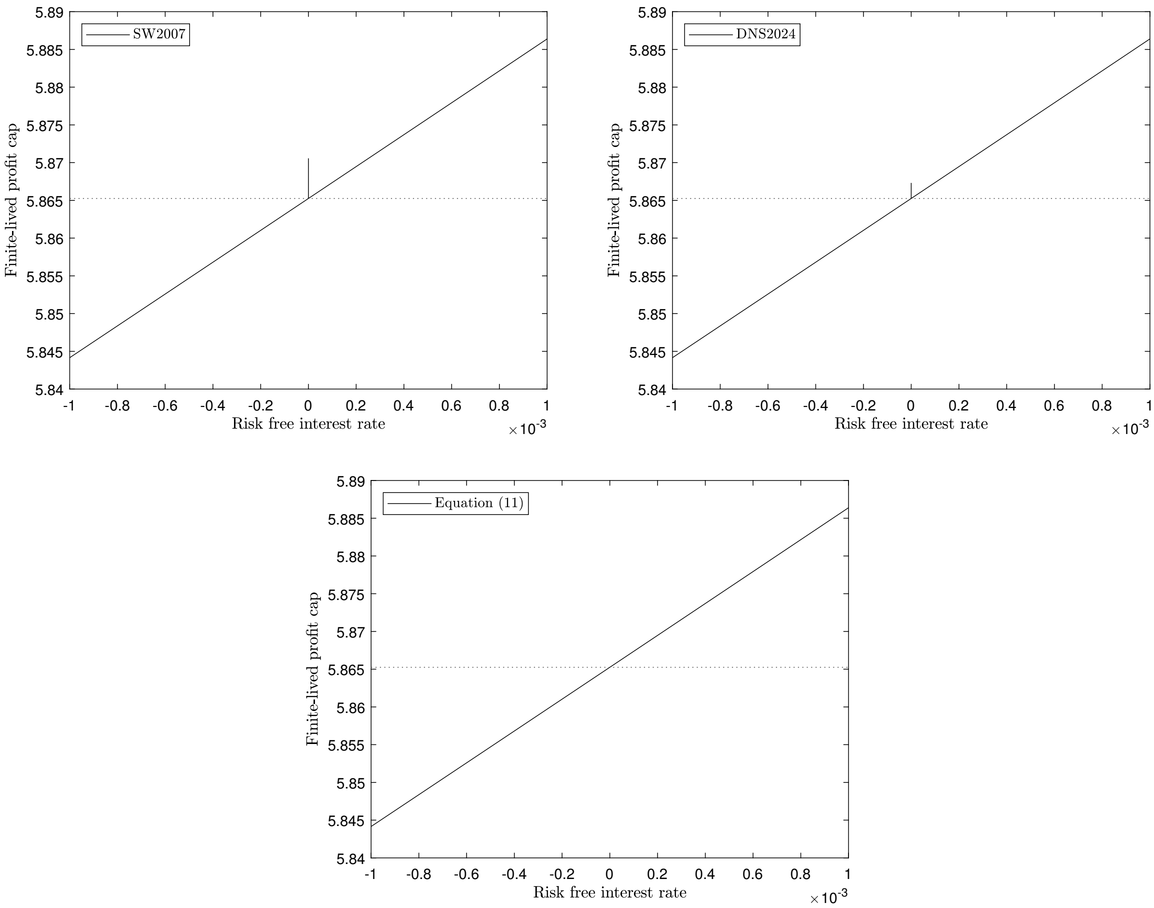

Figure 2: Finite-lived profit caps as a function of the risk free interest rate.

Note: This figure plots finite-lived profit caps as a function of the risk free interest rate r, with $r \in [-0.001, 0.001]$ divided into 200,000 evenly-spaced interest rate points. The top-left plot computes Shackleton and Wojakowski (2007, equation 21), the top-right plot implements Dias et al. (2024b, equation 4) and the bottom plot uses equation (11). Other parameters used in the calculations: $S_0 = 100$, $K = 100$, $T = 1$, $\sigma = 0.25$ and $q = 0.03$. The dotted line in each plot shows the value of the contract for $r = 0$ (i.e., contract #1 of Panel B of Table 1 with $K = 100$), that is equal to 5.8652510796.

Additionally, the original formulas of Shackleton and Wojakowski (2007, equation 21) and Dias et al. (2024b, equation 4) remain numerically well behaved for negative values of r and/or q.[12] This is so because both approaches are (alternative) analytical solutions of the same integral representation (2) and the BSM framework is mathematically and economically consistent with negative r and/or q, as long as the usual BSM assumptions are kept. Nevertheless, $q < 0$ is best understood as holding the underlying has a net cost proportional to its value (rather than a benefit like dividends in equity instruments). Moreover, it is well known that q plays the role of the foreign risk free interest rate in currency options. Hence, the case with $q < 0$ might make sense in foreign exchange options applications with negative foreign interest rates.

As a result, an alternative way for tackling such cases was to approximate the profit cap value by averaging results for small negative and positive values of r and/or q. The numerical experiments highlighted in the two plots on the top of Figure 2 suggest that Shackleton and Wojakowski (2007, equation 21) and Dias et al. (2024b, equation 4) are continuous everywhere except for a single "hole" at exactly $r = 0$. The new formula (11) bridges this gap.

Panels A, B and C of Table 2 show the prices of finite-lived continuums of cash-or-nothing (CON), asset-or-nothing (AON) and gap calls calculated through equations (33), (34), and (35), respectively, with $S_0 = 100$, $T = 1$, $\sigma = 0.25$ and considering different moneyness levels with strike prices $K \in \{95, 100, 105\}$, different drift specifications (i.e., positive, zero and negative drifts $r - q$), different predetermined fixed cash amounts $X \in \{90, 100\}$ for finite-lived continuums of cash-or-nothing calls and different gap values $g := X - K$ (thus yielding positive, zero and negative gaps) for the case of finite-lived continuums of gap options. As expected, and following the insights enunciated in equation (36), the contracts #4, #5 and #6 of Panel C with $X = K = 100$ (i.e., with a gap value $g = 0$) should be understood as a

[12]For example, equation (12) is still valid for less probable option pricing scenarios with $v < 0$, provided that $b_\phi^2 + 2v \geq 0 \Leftrightarrow \left|r - q + \phi\sigma^2/2\right| \geq \sigma\sqrt{-2v}$. This is a necessary and sufficient condition to avoid the square root of a negative number in equation (16) whenever $v < 0$. The same rationale should be applied to compute equation (10) or, alternatively, to equations (E.10) and (E.11).

finite-lived profit cap and, hence, their prices are equal to the values reported in Panel A of Table 1 when $K = 100$.

Table 2: Finite-lived continuums of cash-or-nothing, asset-or-nothing and gap calls for different moneyness levels, predetermined fixed cash amounts and drift specifications.

				Strike price K		
#	X	r	q	95	100	105
Panel A: Finite-lived continuums of CON calls						
1	90	0.05	0.03	55.605	42.848	30.741
2	90	0.03	0.03	54.317	41.397	29.321
3	90	0.03	0.05	52.491	39.532	27.603
4	100	0.05	0.03	61.783	47.609	34.156
5	100	0.03	0.03	60.352	45.997	32.579
6	100	0.03	0.05	58.323	43.925	30.670
Panel B: Finite-lived continuums of AON calls						
1	n/a	0.05	0.03	68.422	54.590	40.819
2	n/a	0.03	0.03	66.526	52.518	38.783
3	n/a	0.03	0.05	63.981	49.932	36.365
4	n/a	0.05	0.03	68.422	54.590	40.819
5	n/a	0.03	0.03	66.526	52.518	38.783
6	n/a	0.03	0.05	63.981	49.932	36.365
Panel C: Finite-lived continuums of gap calls						
1	90	0.05	0.03	12.817	11.742	10.078
2	90	0.03	0.03	12.208	11.120	9.461
3	90	0.03	0.05	11.490	10.400	8.762
4	100	0.05	0.03	6.639	6.981	6.662
5	100	0.03	0.03	6.173	6.520	6.204
6	100	0.03	0.05	5.658	6.007	5.695

Note: Panels A, B and C of this table value finite-lived continuums of cash-or-nothing (CON), asset-or-nothing (AON) and gap calls for different strike prices K, predetermined fixed cash amounts X, risk free interest rates r and dividend yields q using equations (33), (34) and (35), respectively. Other parameters used in the calculations: $S_0 = 100$, $T = 1$ and $\sigma = 0.25$. n/a stands for not applicable.

Finally, Panels A and B of Table 3 report the prices of finite-lived continuums of down-and-out calls calculated through the analytic representation (40), with $S_0 = 100$, $T = 1$, $\sigma = 0.25$ and considering different moneyness levels with strike prices $K \in \{95, 100, 105\}$ and different drift specifications (i.e., positive, zero and negative drifts $r - q$). Panel A uses a knock-out barrier level $L = 90$, whereas Panel B adopts the special case of $L \to 0$ (using the eps value of Matlab that is equal to 2.2204E−16). As expected, and following the rationale of equation (41), the prices of the finite-lived continuums of down-and-out calls shown in

Panel B of Table 3 are indistinguishable from the ones of finite-lived profit caps reported in Table 1. Moreover, Panel C of Table 3 shows the prices of finite-lived continuums of down-and-in calls with a knock-in barrier level $L = 90$, computed via the closed-form solution (43). Notice that the sum of the prices contained in Panels A and C of Table 3 results in the profit cap values presented in Table 1, thus validating the in-out parity relation (45).

Table 3: Finite-lived continuums of down-and-out (resp., down-and-in) calls for different moneyness levels, knock-out (resp., knock-in) barrier levels and drift specifications.

			Strike price K		
#	r	q	95	100	105
Panel A: DOC with $L = 90$					
1	0.05	0.03	7.958	5.837	4.218
2	0.03	0.03	7.489	5.435	3.884
3	0.03	0.05	6.966	4.995	3.526
4	0	0.03	6.821	4.868	3.419
5	0.03	0	8.333	6.151	4.474
6	0	0	7.609	5.529	3.956
Panel B: DOC with $L \to 0$					
1	0.05	0.03	9.728	6.981	4.955
2	0.03	0.03	9.191	6.520	4.575
3	0.03	0.05	8.574	6.007	4.162
4	0	0.03	8.416	5.865	4.041
5	0.03	0	10.175	7.350	5.252
6	0	0	9.347	6.639	4.662
Panel C: DIC with $L = 90$					
1	0.05	0.03	1.770	1.144	0.737
2	0.03	0.03	1.702	1.086	0.691
3	0.03	0.05	1.608	1.012	0.635
4	0	0.03	1.595	0.997	0.622
5	0.03	0	1.842	1.199	0.778
6	0	0	1.738	1.110	0.706

Note: Panels A and B of this table value finite-lived continuums of down-and-out calls (DOC) for different strike prices K, knock-out barrier levels L, risk free interest rates r and dividend yields q computed via equation (40). Panel C shows the values of finite-lived continuums of down-and-in calls (DIC) with a knock-in barrier level $L = 90$ obtained through equation (43). Other parameters used in the calculations: $S_0 = 100$, $T = 1$ and $\sigma = 0.25$.

5. Conclusions

Two different analytical representations have been proposed in the real options literature for valuing finite-lived profit caps and floors on continuous flows following a lognormal distribution, namely: (i) the one proposed by Shackleton and Wojakowski (2007), who use a time decomposition technique inspired in perpetual real options models; and (ii) the one offered by Dias et al. (2024b), who solve analytically the required time integral using a direct approach that employs the integration by parts method. This paper extends the time integral solution of Dias et al. (2024b) in order to accommodate the possibility of r and/or q to be zero, to price continuums of exotic options under the BSM economy and to adapt the proposed analytic formulae to the two-factor model of Margrabe (1978). In addition, the paper shows the mathematical equivalence between the two alternative pricing methodologies and highlights that it is still possible to simplify the analytic solution produced by Shackleton and Wojakowski (2007). Furthermore, this paper also contributes to the options literature by providing closed form solutions for price caps, price floors, price collars as well as continuums of "exotic" (cash-or-nothing, asset-or-nothing, gap) and path-dependent (down-and-out, down-and-in) options. Therefore, the method proposed in this paper further simplifies and facilitates the evaluation of time integrals of BSM and other option types, cements the "non-real options" route and opens the way for more analytical work in the BSM and other areas.

Appendix A: Proof of Proposition 1

This appendix is organized in four parts:

(i) For $v > 0$, the proof follows straightforwardly by simply rearranging Dias et al. (2024b, equations 4 and 5) and noting that the indicator function included in Dias et al. (2024b, equation 9) associated to the at the money case, i.e., $\frac{1}{2} \times \mathbb{1}_{\{S_0=K\}}$ can be excluded. Notice that when $S_0 = K$, $a = 0$ and, therefore, the sum of the terms appearing in Dias et al. (2024b, equation 5) involving the indicator function $1\!\!1_{\{S_0=K\}}$ is zero, that is

$$\begin{aligned}
& \frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right)\times\left(-\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}\right)+\frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right)\times\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}} \\
= & -\frac{1}{4}\frac{b_\phi}{c_v}\times 1\!\!1_{\{S_0=K\}}-\frac{1}{4}\times 1\!\!1_{\{S_0=K\}}+\frac{1}{4}\frac{b_\phi}{c_v}\times 1\!\!1_{\{S_0=K\}}-\frac{1}{4}\times 1\!\!1_{\{S_0=K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}} \\
= & \ 0.
\end{aligned}$$

The case with $v = 0$ of the finite-lived profit cap requires calculation of the limit of the corresponding functions as $v \to 0$. Let us first define the expression inside the square brackets of equation (12) as $f(0, T, S_0, K, \phi, v)$, so that $I_c(0, T, S_0, K, \phi, v) = f(0, T, S_0, K, \phi, v)/v$. It is easy to show analytically that $f(0, T, S_0, K, \phi, 0) = 0$ and, hence, $\lim_{v\to 0} I_c(0, T, S_0, K, \phi, v)$ gives an indetermination of the type $0/0$. This implies that we can apply the l'Hôpital rule. Let us first compute a few derivatives that will be needed:

$$\frac{\mathrm{d}}{\mathrm{d}v}a = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}v}b_\phi = \frac{g(v)}{\sigma},$$

with

$$g(v) := 1\!\!1_{\{v=r\}} - 1\!\!1_{\{v=q\}},$$

$$\frac{\mathrm{d}}{\mathrm{d}v}c_v = \frac{\mathrm{d}}{\mathrm{d}v}\left(b_\phi^2+2v\right)^{1/2} = \frac{1}{2}\left(b_\phi^2+2v\right)^{-1/2}\left(2b_\phi\frac{g(v)}{\sigma}+2\right) = \frac{g(v)b_\phi+\sigma}{\sigma c_v},$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{b_\phi}{c_v}+1\right) = \frac{\frac{g(v)}{\sigma}c_v - b_\phi\frac{\sigma+g(v)b_\phi}{\sigma c_v}}{c_v^2} = \frac{2vg(v)-\sigma b_\phi}{\sigma c_v^3},$$

$$\frac{\mathrm{d}}{\mathrm{d}v}e^{a\left(c_v-b_\phi\right)} = \frac{\mathrm{d}}{\mathrm{d}v}\left(a\left(c_v-b_\phi\right)\right)e^{a\left(c_v-b_\phi\right)} = a\frac{g(v)\left(b_\phi-c_v\right)+\sigma}{\sigma c_v}e^{a\left(c_v-b_\phi\right)},$$

$$\frac{\mathrm{d}}{\mathrm{d}v}e^{-a\left(c_v+b_\phi\right)} = \frac{\mathrm{d}}{\mathrm{d}v}\left(-a\left(c_v+b_\phi\right)\right)e^{-a\left(c_v+b_\phi\right)} = -a\frac{g(v)\left(b_\phi+c_v\right)+\sigma}{\sigma c_v}e^{-a\left(c_v+b_\phi\right)},$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}v}N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right) &= n\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right) \\
&= \frac{g(v)b_\phi+\sigma}{\sigma c_v}\sqrt{T}\,n\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}v}N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right) &= n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right) \\
&= \frac{g(v)}{\sigma}\sqrt{T}\, n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right).
\end{aligned}$$

Using the above derivatives, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial v}f(0,T,S_0,K,\phi,v) \\
=\ & \frac{1}{2}\frac{2vg(v)-\sigma b_\phi}{\sigma c_v^3}e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
&+\frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right)a\frac{g(v)\left(b_\phi-c_v\right)+\sigma}{\sigma c_v}e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
&+\frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right)e^{a\left(c_v-b_\phi\right)}\frac{g(v)b_\phi+\sigma}{\sigma c_v}\sqrt{T}\, n\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right) \\
&-\frac{1}{2}\frac{2vg(v)-\sigma b_\phi}{\sigma c_v^3}e^{-a\left(c_v+b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
&-\frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right)(-a)\frac{g(v)\left(b_\phi+c_v\right)+\sigma}{\sigma c_v}e^{-a\left(c_v+b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
&-\frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right)e^{-a\left(c_v+b_\phi\right)}\frac{-g(v)b_\phi-\sigma}{\sigma c_v}\sqrt{T}\, n\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right) \\
&-(-T)e^{-vT}N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)-e^{-vT}\frac{g(v)}{\sigma}\sqrt{T}\, n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right) \\
=\ & \frac{1}{2}\frac{\left(2vg(v)-\sigma b_\phi\right)\left(1-ac_v\right)+a\sigma c_v^2}{\sigma c_v^3}e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
&-\frac{1}{2}\frac{\left(2vg(v)-\sigma b_\phi\right)\left(1+ac_v\right)+a\sigma c_v^2}{\sigma c_v^3}e^{-a\left(c_v+b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
&+\frac{\sqrt{T}}{2}\frac{g(v)b_\phi+\sigma}{\sigma c_v}\left[\left(\frac{b_\phi}{c_v}+1\right)e^{a\left(c_v-b_\phi\right)}n\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)\right. \\
&\left.+\left(\frac{b_\phi}{c_v}-1\right)e^{-a\left(c_v+b_\phi\right)}n\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right)\right] \\
&+e^{-vT}\left[T\,N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)-\frac{g(v)}{\sigma}\sqrt{T}\, n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)\right], \qquad \text{(A.1)}
\end{aligned}$$

where the second equality of this expression is obtained after performing several simplifying calculations.

Now it is necessary to calculate the $\lim_{v\to 0} \partial I_c(0,T,S_0,K,\phi,v)/\partial v$, which is equal to the $\lim_{v\to 0} \partial f(0,T,S_0,K,\phi,v)/\partial v$. Using expression (A.1) and noting that $\lim_{v\to 0} c_v = |b_\phi|$, it can be shown that

$$
\begin{aligned}
& \lim_{v\to 0} \frac{\partial}{\partial v} I_c(0,T,S_0,K,\phi,v) \\
= \; & T\, N\left(\frac{a}{\sqrt{T}} + b_\phi\sqrt{T}\right) - \frac{g(v)}{\sigma}\sqrt{T}\, n\left(\frac{a}{\sqrt{T}} + b_\phi\sqrt{T}\right) \\
& + e^{a\left(|b_\phi| - b_\phi\right)} \left[\frac{b_\phi\left(a\left(|b_\phi| + b_\phi\right) - 1\right)}{2\left|b_\phi\right|^3} \left(N\left(\frac{a}{\sqrt{T}} + |b_\phi|\sqrt{T}\right) - \mathbb{1}_{\{S_0>K\}} \right) \right. \\
& \left. + \frac{\sqrt{T}}{2}\frac{g(v)b_\phi + \sigma}{\sigma\,|b_\phi|}\left(\frac{b_\phi}{|b_\phi|} + 1\right) n\left(\frac{a}{\sqrt{T}} + |b_\phi|\sqrt{T}\right) \right] \\
& + e^{-a\left(|b_\phi| + b_\phi\right)} \left[\frac{b_\phi\left(a\left(|b_\phi| - b_\phi\right) + 1\right)}{2\left|b_\phi\right|^3} \left(N\left(\frac{a}{\sqrt{T}} - |b_\phi|\sqrt{T}\right) - \mathbb{1}_{\{S_0>K\}} \right) \right. \\
& \left. + \frac{\sqrt{T}}{2}\frac{g(v)b_\phi + \sigma}{\sigma\,|b_\phi|}\left(\frac{b_\phi}{|b_\phi|} - 1\right) n\left(\frac{a}{\sqrt{T}} - |b_\phi|\sqrt{T}\right) \right],
\end{aligned} \tag{A.2}
$$

after some simplifying calculus.

To further simplify expression (A.2), we note that

$$
|b_\phi| = \begin{cases} b_\phi & \Leftarrow \; b_\phi > 0 \\ -b_\phi & \Leftarrow \; b_\phi < 0 \end{cases},
$$

$$
\begin{aligned}
\frac{b_\phi}{|b_\phi|} &= \operatorname{sign}(b_\phi), \\
\frac{b_\phi}{|b_\phi|^3} &= \frac{\operatorname{sign}(b_\phi)}{b_\phi^2}, \\
\frac{b_\phi}{|b_\phi|} + 1 &= 2\mathbb{1}_{\{b_\phi>0\}}, \\
\frac{b_\phi}{|b_\phi|} - 1 &= -2\mathbb{1}_{\{b_\phi<0\}}, \\
|b_\phi| + b_\phi &= 2b_\phi\mathbb{1}_{\{b_\phi>0\}}
\end{aligned}
$$

and

$$
|b_\phi| - b_\phi = -2b_\phi\mathbb{1}_{\{b_\phi<0\}}.
$$

Replacing these simplifications in expression (A.2) and rearranging yields

$$\begin{aligned}
&\lim_{v\to 0}\frac{\partial}{\partial v}I_c(0,T,S_0,K,\phi,v)\\
=\;& T\,N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)-\frac{g(v)}{\sigma}\sqrt{T}\,n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)\\
&+e^{-2ab_\phi 1_{\{b_\phi<0\}}}\left(\frac{\operatorname{sign}(b_\phi)}{2b_\phi^2}\left(2ab_\phi\mathbb{1}_{\{b_\phi>0\}}-1\right)\left[N\left(\frac{a}{\sqrt{T}}+|b_\phi|\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right]\right.\\
&\left.+\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma\,|b_\phi|}\,n\left(\frac{a}{\sqrt{T}}+|b_\phi|\sqrt{T}\right)\mathbb{1}_{\{b_\phi>0\}}\right)\\
&-e^{-2ab_\phi 1_{\{b_\phi>0\}}}\left(\frac{\operatorname{sign}(b_\phi)}{2b_\phi^2}\left(2ab_\phi\mathbb{1}_{\{b_\phi<0\}}-1\right)\left[N\left(\frac{a}{\sqrt{T}}-|b_\phi|\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right]\right.\\
&\left.+\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma\,|b_\phi|}\,n\left(\frac{a}{\sqrt{T}}-|b_\phi|\sqrt{T}\right)\mathbb{1}_{\{b_\phi<0\}}\right). \qquad \text{(A.3)}
\end{aligned}$$

Another simplification arises by noting that

$$\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma\,|b_\phi|}\,n\left(\frac{a}{\sqrt{T}}+|b_\phi|\sqrt{T}\right)\mathbb{1}_{\{b_\phi>0\}}=\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma b_\phi}\,n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)\mathbb{1}_{\{b_\phi>0\}}$$

and

$$\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma\,|b_\phi|}\,n\left(\frac{a}{\sqrt{T}}-|b_\phi|\sqrt{T}\right)\mathbb{1}_{\{b_\phi<0\}}=-\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma b_\phi}\,n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)\mathbb{1}_{\{b_\phi<0\}},$$

which implies that these two expressions are symmetric in the corresponding regions. Let us now define

$$W:=\sqrt{T}\,\frac{g(v)b_\phi+\sigma}{\sigma b_\phi}\,n\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right).$$

Then,

$$\begin{aligned}
& e^{-2ab_\phi 1_{\{b_\phi<0\}}}W\mathbb{1}_{\{b_\phi>0\}}-e^{-2ab_\phi 1_{\{b_\phi>0\}}}(-W)\mathbb{1}_{\{b_\phi<0\}}\\
=\;& W\left(e^{-2ab_\phi 1_{\{b_\phi<0\}}}\mathbb{1}_{\{b_\phi>0\}}+e^{-2ab_\phi 1_{\{b_\phi>0\}}}\mathbb{1}_{\{b_\phi<0\}}\right)\\
=\;& W. \qquad \text{(A.4)}
\end{aligned}$$

Finally, combining expressions (A.3) and (A.4) and rearranging the obtained terms yields equation (13).

(ii) Equation (17) corresponds to Dias et al. (2024b, equation 10), but now augmented for the possibility of having option pricing scenarios with $v = 0$.

(iii) The proof follows straightforwardly by simply rearranging Dias et al. (2024b, equations 11 and 12) and noting that the indicator function included in Dias et al. (2024b, equation 9) associated to the at the money case, i.e., $\frac{1}{2} \times \mathbb{1}_{\{S_0=K\}}$ can be excluded. Finally, the integral (19) is obtained by using the relation $1 - \mathbb{1}_{\{S_0>K\}} = \mathbb{1}_{\{S_0\leq K\}}$.

(iv) Equation (20) corresponds to Dias et al. (2024b, equation 13).

Appendix B: Proof of Proposition 2

The case with $v = 0$ for the perpetual profit cap must be treated carefully. Let us first define the expression inside the square brackets of equation (19) as $h(0, \infty, S_0, K, \phi, v)$, so that $I_c(0, \infty, S_0, K, \phi, v) = h(0, \infty, S_0, K, \phi, v)/v$. It is easy to show analytically that $h(0, \infty, S_0, K, 1, 0) = 1$ and $h(0, \infty, S_0, K, -1, 0) = 0$. Hence, the $\lim_{q\to 0} I_c(0, \infty, S_0, K, 1, 0)$ explodes to infinity and the $\lim_{r\to 0} I_c(0, \infty, S_0, K, -1, 0)$ gives an indetermination of the type $0/0$. This implies that we can apply the l'Hôpital rule only to the second limit and the problem must be restricted to $q > 0$, otherwise it is not well-behaved due to its perpetual nature. This is a well-known feature in the option pricing literature.

Therefore, for $r = 0$ and $q > 0$, the perpetual profit cap can be calculated as

$$V(S_0, K, \infty, \sigma, r, q) = S_0\, I_c(0, \infty, S_0, K, 1, q) - K \lim_{r\to 0} I_c(0, \infty, S_0, K, -1, r). \tag{B.1}$$

Using the derivatives considered in Appendix A and rearranging the obtained terms yields

$$\begin{aligned} & \frac{\partial}{\partial r} h(0, \infty, S_0, K, -1, r) \\ = \; & \frac{1}{2}\left(\frac{2r - \sigma b_{-1}}{\sigma c_r^3} + \left(\frac{b_{-1}}{c_r} + 1\right) a \frac{b_{-1} - c_r + \sigma}{\sigma c_r}\right) e^{a(c_r - b_{-1})} \mathbb{1}_{\{S_0\leq K\}} \\ & + \frac{1}{2}\left(\frac{2r - \sigma b_{-1}}{\sigma c_r^3} - \left(\frac{b_{-1}}{c_r} - 1\right) a \frac{b_{-1} + c_r + \sigma}{\sigma c_r}\right) e^{-a(c_r + b_{-1})} \mathbb{1}_{\{S_0>K\}}. \end{aligned} \tag{B.2}$$

Now it is necessary to calculate the $\lim_{r\to 0} \partial I_c(0,\infty,S_0,K,-1,r)/\partial r$, which is equal to the $\lim_{r\to 0} \partial h(0,\infty,S_0,K,-1,r)/\partial r$. Using expression (B.2) and noting that $\lim_{r\to 0} c_r = |b_{-1}|$, it can be shown that

$$\begin{aligned}
&\lim_{r\to 0}\frac{\partial}{\partial r} I_c(0,\infty,S_0,K,-1,r)\\
=\;& \frac{1}{2}\left(-\frac{b_{-1}}{|b_{-1}|^3}+\left(\frac{b_{-1}}{|b_{-1}|}+1\right)a\frac{b_{-1}-|b_{-1}|+\sigma}{\sigma\,|b_{-1}|}\right)e^{a(|b_{-1}|-b_{-1})}\mathbb{1}_{\{S_0\le K\}}\\
&+\frac{1}{2}\left(-\frac{b_{-1}}{|b_{-1}|^3}-\left(\frac{b_{-1}}{|b_{-1}|}-1\right)a\frac{b_{-1}+|b_{-1}|+\sigma}{\sigma\,|b_{-1}|}\right)e^{-a(|b_{-1}|+b_{-1})}\mathbb{1}_{\{S_0> K\}}\\
=\;& \frac{1}{2}\left(-\frac{\operatorname{sign}(b_{-1})}{b_{-1}^2}+2\mathbb{1}_{\{b_{-1}>0\}}a\frac{2b_{-1}\mathbb{1}_{\{b_{-1}<0\}}+\sigma}{\sigma\,|b_{-1}|}\right)e^{-2ab_{-1}1_{\{b_{-1}<0\}}}\mathbb{1}_{\{S_0\le K\}}\\
&+\frac{1}{2}\left(-\frac{\operatorname{sign}(b_{-1})}{b_{-1}^2}+2\mathbb{1}_{\{b_{-1}<0\}}a\frac{2b_{-1}\mathbb{1}_{\{b_{-1}>0\}}+\sigma}{\sigma\,|b_{-1}|}\right)e^{-2ab_{-1}1_{\{b_{-1}>0\}}}\mathbb{1}_{\{S_0> K\}}\\
=\;& \frac{1}{2b_{-1}^2}e^{-2ab_{-1}}\mathbb{1}_{\{S_0\le K\}}-\frac{1}{b_{-1}}\left(a-\frac{1}{2b_{-1}}\right)\mathbb{1}_{\{S_0>K\}},
\end{aligned}\tag{B.3}$$

because $r=0$ and $\phi=-1$ and, hence, $b_{-1}<0$.

Finally, combining expressions (B.1) and (B.3) yields equation (21).

Appendix C: Proof of Proposition 3

This appendix is organized in two parts:

(i) Combining equations (5) and (6), the finite-lived floor can be rewritten as

$$F(S_0,K,T,\sigma,r,q)=\int_0^T\left[Ke^{-rt}N(-d_0(t))-S_0e^{-qt}N(-d_1(t))\right]dt, \tag{C.1}$$

which requires the computation of two integrals. Following a similar procedure to the one used by Dias et al. (2024b, Proposition 1) for the case of finite-lived caps, it can be shown that the integrals for the finite-lived floor case are as follows:

$$
\begin{aligned}
& I_f(0,T,S_0,K,\phi,v) \\
= \; & \frac{1}{v}\left[-\frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right)e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right]\right. \\
& +\frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right)e^{-a\left(c_v+b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}-c_v\sqrt{T}\right)-\mathbb{1}_{\{S_0>K\}}\right] \\
& \left.-e^{-vT}N\left(-\frac{a}{\sqrt{T}}-b_\phi\sqrt{T}\right)+\mathbb{1}_{\{S_0\leq K\}}\right], \qquad \text{(C.2)}
\end{aligned}
$$

for $v > 0$. This integral can then be used for valuing the finite-lived profit floor (23) when $v > 0$.

Alternatively, summing the integrals (C.2) and (12) yields

$$
\begin{aligned}
& I_f(0,T,S_0,K,\phi,v)+I_c(0,T,S_0,K,\phi,v) \\
= \; & \frac{1}{v}\left[-e^{-vT}N\left(-\frac{a}{\sqrt{T}}-b_\phi\sqrt{T}\right)+\mathbb{1}_{\{S_0\leq K\}}-e^{-vT}N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)+\mathbb{1}_{\{S_0>K\}}\right], \\
= \; & \frac{1}{v}\left[-e^{-vT}+e^{-vT}N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)-e^{-vT}N\left(\frac{a}{\sqrt{T}}+b_\phi\sqrt{T}\right)+1\right], \\
= \; & \frac{1-e^{-vT}}{v}, \qquad \text{(C.3)}
\end{aligned}
$$

which implies that the integral (C.2) can be reexpressed as

$$
I_f(0,T,S_0,K,\phi,v)=\frac{1-e^{-vT}}{v}-I_c(0,T,S_0,K,\phi,v), \qquad \text{(C.4)}
$$

as indicated in equation (24).

For $v = 0$, it suffices to note that

$$
\begin{aligned}
\lim_{v\to 0} I_f(0,T,S_0,K,\phi,v) &= \lim_{v\to 0}\frac{1-e^{-vT}}{v}-\lim_{v\to 0} I_c(0,T,S_0,K,\phi,v) \\
&= T-I_c(0,T,S_0,K,\phi,0), \qquad \text{(C.5)}
\end{aligned}
$$

as shown in equation (25).

(ii) The perpetual floor can be obtained as

$$\begin{aligned} F(S_0, K, \infty, \sigma, r, q) &= \lim_{T\to\infty} F(S_0, K, T, \sigma, r, q) \\ &= K \lim_{T\to\infty} I_f(0, T, S_0, K, -1, r) - S_0 \lim_{T\to\infty} I_f(0, T, S_0, K, 1, q) \\ &= K\, I_f(0, \infty, S_0, K, -1, r) - S_0\, I_f(0, \infty, S_0, K, 1, q), \end{aligned} \tag{C.6}$$

as given in equation (26).

Using the integral solution (C.2), it follows that

$$\begin{aligned} & I_f(0, \infty, S_0, K, \phi, v) \\ = \; & \lim_{T\to\infty} I_f(0, T, S_0, K, \phi, v) \\ = \; & \frac{1}{v}\left[-\frac{1}{2}\left(\frac{b_\phi}{c_v}+1\right) e^{a\left(c_v - b_\phi\right)} \left(1 - \mathbb{1}_{\{S_0 > K\}}\right) \right. \\ & \left. + \frac{1}{2}\left(\frac{b_\phi}{c_v}-1\right) e^{-a\left(c_v + b_\phi\right)} \left(0 - \mathbb{1}_{\{S_0 > K\}}\right) - 0 + \mathbb{1}_{\{S_0 \le K\}}\right], \end{aligned} \tag{C.7}$$

after applying the limits shown in Dias et al. (2024b, equations A14, A15 and A16). Finally, the analytical representation (27) is obtained after performing some calculus.

Alternatively, we can use the integral solution (24) to show that

$$\begin{aligned} & I_f(0, \infty, S_0, K, \phi, v) \\ = \; & \lim_{T\to\infty} \frac{1 - e^{-vT}}{v} - \lim_{T\to\infty} I_c(0, T, S_0, K, \phi, v) \\ = \; & \frac{1}{v} - I_c(0, \infty, S_0, K, \phi, v), \end{aligned} \tag{C.8}$$

as documented in equation (28).

Appendix D: Proof of Proposition 4

The case with $v = 0$ for the perpetual profit floor must be treated carefully. Let us first define the expression inside the square brackets of equation (27) as $j(0, \infty, S_0, K, \phi, v)$, so that $I_f(0, \infty, S_0, K, \phi, v) = -j(0, \infty, S_0, K, \phi, v)/v$. It is easy to show analytically that $j(0, \infty, S_0, K, 1, 0) = 0$ and $j(0, \infty, S_0, K, -1, 0) = -1$. Hence, the $\lim_{q\to 0} I_f(0, \infty, S_0, K, 1, 0)$

gives an indetermination of the type 0/0 and the $\lim_{r\to 0} I_f(0,\infty,S_0,K,-1,0)$ explodes to infinity. This implies that we can apply the l'Hôpital rule only to the first limit and the problem must be restricted to $r > 0$, otherwise it is not well-behaved due to its perpetual nature. This is, again, a well-known feature in the option pricing literature.

Therefore, for $r > 0$ and $q = 0$, the perpetual profit floor can be calculated as

$$F(S_0,K,\infty,\sigma,r,q) = K\, I_f(0,\infty,S_0,K,-1,r) - S_0 \lim_{q\to 0} I_f(0,\infty,S_0,K,1,q). \tag{D.1}$$

Using the derivatives computed in Appendix A and rearranging the obtained terms yields

$$\begin{aligned}
&\frac{\partial}{\partial q} j(0,\infty,S_0,K,1,q) \\
= \;& \frac{1}{2}\left(\frac{-2q-\sigma b_1}{\sigma c_q^3} + \left(\frac{b_1}{c_q}+1\right) a \frac{c_q - b_1 + \sigma}{\sigma c_q}\right) e^{a(c_q-b_1)} \mathbb{1}_{\{S_0\le K\}} \\
& + \frac{1}{2}\left(\frac{-2q-\sigma b_1}{\sigma c_q^3} + \left(\frac{b_1}{c_q}-1\right) a \frac{c_q + b_1 - \sigma}{\sigma c_q}\right) e^{-a(c_q+b_1)} \mathbb{1}_{\{S_0> K\}}.
\end{aligned} \tag{D.2}$$

Now it is necessary to calculate the $\lim_{q\to 0} \partial I_f(0,\infty,S_0,K,1,q)/\partial q$, which is equal to $-\lim_{q\to 0} \partial j(0,\infty,S_0,K,1,q)/\partial q$. Using expression (D.2) and noting that $\lim_{q\to 0} c_q = |b_1|$, it can be shown that

$$\begin{aligned}
&\lim_{q\to 0} \frac{\partial}{\partial q} I_f(0,\infty,S_0,K,1,q) \\
= \;& -\frac{1}{2}\left(-\frac{b_1}{|b_1|^3} + \left(\frac{b_1}{|b_1|}+1\right) a \frac{|b_1| - b_1 + \sigma}{\sigma\,|b_1|}\right) e^{a(|b_1|-b_1)} \mathbb{1}_{\{S_0\le K\}} \\
& -\frac{1}{2}\left(-\frac{b_1}{|b_1|^3} + \left(\frac{b_1}{|b_1|}-1\right) a \frac{|b_1| + b_1 - \sigma}{\sigma\,|b_1|}\right) e^{-a(|b_1|+b_1)} \mathbb{1}_{\{S_0> K\}} \\
= \;& -\frac{1}{b_1}\left(a - \frac{1}{2b_1}\right) \mathbb{1}_{\{S_0\le K\}} + \frac{1}{2b_1^2} e^{-2ab_1} \mathbb{1}_{\{S_0> K\}},
\end{aligned} \tag{D.3}$$

because $q = 0$ and $\phi = 1$ and, hence, $b_1 > 0$.

Finally, combining expressions (D.1) and (D.3) yields equation (29).

Appendix E: Some auxiliary computations

Taking $\beta \in \{\beta_1, \beta_2\}$ and using equation (10), we notice that

$$\begin{aligned}
\left(\beta_{1,2} - \frac{1}{2}\right)\sigma^2 &= \left(\frac{1}{2} - \frac{r-q}{\sigma^2} \pm \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} - \frac{1}{2}\right)\sigma^2 \\
&= -(r-q) \pm \sigma^2 \sqrt{\left(\left(\frac{r-q-\sigma^2/2}{\sigma}\right)^2 + 2r\right)\frac{1}{\sigma^2}} \\
&= -(r-q) \pm \sigma \sqrt{\left(\frac{r-q-\sigma^2/2}{\sigma}\right)^2 + 2r} \\
&= -(r-q) \pm \sigma \sqrt{b_{-1}^2 + 2r} \\
&= -(r-q) \pm \sigma c_r. \qquad \text{(E.1)}
\end{aligned}$$

Therefore, combining equations (14) and (E.1), it follows that

$$\begin{aligned}
\frac{a}{\sqrt{T}} \pm c_r\sqrt{T} &= \frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{r-q+(\beta_{1,2}-1/2)\sigma^2}{\sigma}\sqrt{T} \\
&= d_{\beta_{1,2}}(T). \qquad \text{(E.2)}
\end{aligned}$$

Moreover,

$$\frac{a}{\sqrt{T}} \pm c_q\sqrt{T} = d_{\beta_{1,2}}(T), \qquad \text{(E.3)}$$

because $c_q = c_r$. To demonstrate this equality, we note that

$$\begin{aligned}
c_r^2 - c_q^2 &= b_{-1}^2 + 2r - b_1^2 - 2q \\
&= (b_{-1} + b_1)(b_{-1} - b_1) + 2r - 2q \\
&= \left(\frac{r-q-\sigma^2/2}{\sigma} + \frac{r-q+\sigma^2/2}{\sigma}\right)\left(\frac{r-q-\sigma^2/2}{\sigma} - \frac{r-q+\sigma^2/2}{\sigma}\right) + 2r - 2q \\
&= \frac{2(r-q)}{\sigma}(-\sigma) + 2r - 2q \\
&= -2r + 2q + 2r - 2q \\
&= 0, \qquad \text{(E.4)}
\end{aligned}$$

which is valid for the assumed parameters $r \in \mathbb{R}^+$ and $q \in \mathbb{R}^+$.

A further simple and very intuitive way of checking this equality is by expanding the expressions of c_r and c_q, that is

$$\begin{aligned} c_r &= \sqrt{b_{-1}^2 + 2r} \\ &= \sqrt{\left(\frac{r - q - \frac{\sigma^2}{2}}{\sigma}\right)^2 + 2r} \\ &= \sqrt{r + q + \frac{r^2}{\sigma^2} + \frac{q^2}{\sigma^2} + \frac{\sigma^2}{4} - \frac{2rq}{\sigma^2}} \end{aligned} \tag{E.5}$$

and

$$\begin{aligned} c_q &= \sqrt{b_1^2 + 2q} \\ &= \sqrt{\left(\frac{r - q + \frac{\sigma^2}{2}}{\sigma}\right)^2 + 2q} \\ &= \sqrt{r + q + \frac{r^2}{\sigma^2} + \frac{q^2}{\sigma^2} + \frac{\sigma^2}{4} - \frac{2rq}{\sigma^2}}. \end{aligned} \tag{E.6}$$

Hence, noting that $c_q = c_r$ and substituting expression (E.3) in equation (58) yields

$$\begin{aligned} V_3 &= \frac{S_0}{2q} \frac{b_1 + c_r}{c_r} e^{a(c_r - b_1)} \left[N\left(d_{\beta_1}(T)\right) - \mathbb{1}_{\{S_0 > K\}} - \frac{1}{2} \times \mathbb{1}_{\{S_0 = K\}} \right] \\ &+ \frac{S_0}{2q} \frac{b_1 - c_r}{c_r} e^{-a(c_r + b_1)} \left[\mathbb{1}_{\{S_0 > K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0 = K\}} - N\left(d_{\beta_2}(T)\right) \right]. \end{aligned} \tag{E.7}$$

Similarly, replacing expression (E.2) in equation (59) yields

$$\begin{aligned} V_4 &= -\frac{K}{2r} \frac{b_{-1} + c_r}{c_r} e^{a(c_r - b_{-1})} \left[N\left(d_{\beta_1}(T)\right) - \mathbb{1}_{\{S_0 > K\}} - \frac{1}{2} \times \mathbb{1}_{\{S_0 = K\}} \right] \\ &- \frac{K}{2r} \frac{b_{-1} - c_r}{c_r} e^{-a(c_r + b_{-1})} \left[\mathbb{1}_{\{S_0 > K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0 = K\}} - N\left(d_{\beta_2}(T)\right) \right]. \end{aligned} \tag{E.8}$$

Some additional auxiliary calculations are required again. We first note that the ingredients b_1 and b_{-1} are related since

$$\begin{aligned} b_1 &= \frac{r - q + \sigma^2/2}{\sigma} \\ &= \frac{r - q + \sigma^2/2}{\sigma} - \sigma + \sigma \\ &= \frac{r - q - \sigma^2/2}{\sigma} + \sigma \\ &= b_{-1} + \sigma. \end{aligned} \tag{E.9}$$

Moreover,

$$\begin{aligned}\beta_1 &= \frac{1}{2} - \frac{r-q}{\sigma^2} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ &= -\frac{r-q-\sigma^2/2}{\sigma^2} + \frac{1}{\sigma}\sqrt{\left(\frac{r-q-\sigma^2/2}{\sigma}\right)^2 + 2r} \\ &= \frac{c_r - b_{-1}}{\sigma} \end{aligned} \quad \text{(E.10)}$$

and

$$\begin{aligned}\beta_2 &= \frac{1}{2} - \frac{r-q}{\sigma^2} - \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ &= -\frac{r-q-\sigma^2/2}{\sigma^2} - \frac{1}{\sigma}\sqrt{\left(\frac{r-q-\sigma^2/2}{\sigma}\right)^2 + 2r} \\ &= -\frac{c_r + b_{-1}}{\sigma}. \end{aligned} \quad \text{(E.11)}$$

Combining equations (14), (E.9), (E.10) and (E.11) gives:

$$\begin{aligned} a(c_r - b_1) &= a(c_r - b_{-1} - \sigma) \\ &= \frac{\ln(S_0/K)}{\sigma}\beta_1\sigma - \frac{\ln(S_0/K)}{\sigma}\sigma \\ &= \ln\left(\frac{S_0}{K}\right)(\beta_1 - 1), \end{aligned} \quad \text{(E.12)}$$

$$\begin{aligned} -a(c_r + b_1) &= -a(c_r + b_{-1} + \sigma) \\ &= -\frac{\ln(S_0/K)}{\sigma}(-\beta_2\sigma) - \frac{\ln(S_0/K)}{\sigma}\sigma \\ &= \ln\left(\frac{S_0}{K}\right)(\beta_2 - 1), \end{aligned} \quad \text{(E.13)}$$

$$\begin{aligned} a(c_r - b_{-1}) &= \frac{\ln(S_0/K)}{\sigma}\beta_1\sigma \\ &= \ln\left(\frac{S_0}{K}\right)\beta_1 \end{aligned} \quad \text{(E.14)}$$

and

$$\begin{aligned} -a(c_r + b_{-1}) &= -\frac{\ln(S_0/K)}{\sigma}(-\beta_2\sigma) \\ &= \ln\left(\frac{S_0}{K}\right)\beta_2. \end{aligned} \quad \text{(E.15)}$$

Computing the exponential functions of equations (E.12), (E.13), (E.14) and (E.15) yields

$$\begin{aligned} e^{a(c_r-b_1)} &= e^{\ln(S_0/K)(\beta_1-1)} \\ &= \left(\frac{S_0}{K}\right)^{\beta_1-1}, \end{aligned} \tag{E.16}$$

$$\begin{aligned} e^{-a(c_r+b_1)} &= e^{\ln(S_0/K)(\beta_2-1)} \\ &= \left(\frac{S_0}{K}\right)^{\beta_2-1}, \end{aligned} \tag{E.17}$$

$$\begin{aligned} e^{a(c_r-b_{-1})} &= e^{\ln(S_0/K)\beta_1} \\ &= \left(\frac{S_0}{K}\right)^{\beta_1} \end{aligned} \tag{E.18}$$

and

$$\begin{aligned} e^{-a(c_r+b_{-1})} &= e^{\ln(S_0/K)\beta_2} \\ &= \left(\frac{S_0}{K}\right)^{\beta_2}. \end{aligned} \tag{E.19}$$

Using equations (E.9), (E.10) and (E.11), it follows that

$$\begin{aligned} \frac{b_1+c_r}{c_r} &= \frac{b_{-1}+\sigma+c_r}{c_r} \\ &= \frac{-\sigma\left(-\frac{c_r+b_{-1}}{\sigma}-1\right)}{c_r} \\ &= \frac{-\sigma\left(\beta_2-1\right)}{c_r}, \end{aligned} \tag{E.20}$$

$$\begin{aligned} \frac{b_1-c_r}{c_r} &= \frac{b_{-1}+\sigma-c_r}{c_r} \\ &= \frac{-\sigma\left(\frac{c_r-b_{-1}}{\sigma}-1\right)}{c_r} \\ &= \frac{-\sigma\left(\beta_1-1\right)}{c_r}, \end{aligned} \tag{E.21}$$

$$\begin{aligned} \frac{b_{-1}+c_r}{c_r} &= \frac{-\sigma\left(-\frac{c_r+b_{-1}}{\sigma}\right)}{c_r} \\ &= \frac{-\sigma\beta_2}{c_r} \end{aligned} \tag{E.22}$$

and

$$\frac{b_{-1}-c_r}{c_r} = \frac{-\sigma\frac{c_r-b_{-1}}{\sigma}}{c_r} = \frac{-\sigma\beta_1}{c_r}. \tag{E.23}$$

Substituting expressions (E.16), (E.17), (E.20) and (E.21) in equation (E.7) and rearranging yields

$$\begin{aligned} V_3 &= \frac{S_0}{2q}\times\frac{-\sigma(\beta_2-1)}{c_r}\left(\frac{S_0}{K}\right)^{\beta_1-1}\left[N\left(d_{\beta_1}(T)\right)-1\!\!1_{\{S_0>K\}}-\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}\right] \\ &\quad+\frac{S_0}{2q}\times\frac{-\sigma(\beta_1-1)}{c_r}\left(\frac{S_0}{K}\right)^{\beta_2-1}\left[1\!\!1_{\{S_0>K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}-N\left(d_{\beta_2}(T)\right)\right] \\ &= \frac{K^{1-\beta_1}}{2q}\times\frac{-\sigma(\beta_2-1)}{c_r}S_0^{\beta_1}\left[N\left(d_{\beta_1}(T)\right)-1\!\!1_{\{S_0>K\}}-\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}\right] \\ &\quad+\frac{K^{1-\beta_2}}{2q}\times\frac{-\sigma(\beta_1-1)}{c_r}S_0^{\beta_2}\left[1\!\!1_{\{S_0>K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}-N\left(d_{\beta_2}(T)\right)\right]. \end{aligned} \tag{E.24}$$

Similarly, replacing expressions (E.18), (E.19), (E.22) and (E.23) in equation (E.8) and rearranging gives

$$\begin{aligned} V_4 &= -\frac{K}{2r}\times\frac{-\sigma\beta_2}{c_r}\left(\frac{S_0}{K}\right)^{\beta_1}\left[N\left(d_{\beta_1}(T)\right)-1\!\!1_{\{S_0>K\}}-\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}\right] \\ &\quad-\frac{K}{2r}\times\frac{-\sigma\beta_1}{c_r}\left(\frac{S_0}{K}\right)^{\beta_2}\left[1\!\!1_{\{S_0>K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}-N\left(d_{\beta_2}(T)\right)\right] \\ &= -\frac{K^{1-\beta_1}}{2r}\times\frac{-\sigma\beta_2}{c_r}S_0^{\beta_1}\left[N\left(d_{\beta_1}(T)\right)-1\!\!1_{\{S_0>K\}}-\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}\right] \\ &\quad-\frac{K^{1-\beta_2}}{2r}\times\frac{-\sigma\beta_1}{c_r}S_0^{\beta_2}\left[1\!\!1_{\{S_0>K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}-N\left(d_{\beta_2}(T)\right)\right]. \end{aligned} \tag{E.25}$$

Next, it is necessary to sum the option components (E.24) and (E.25) so that the common terms can be grouped as follows:

$$\begin{aligned} & V_3+V_4 \\ &= \left(-\frac{\sigma(\beta_2-1)}{2qc_r}+\frac{\sigma\beta_2}{2rc_r}\right)K^{1-\beta_1}S_0^{\beta_1}\left[N\left(d_{\beta_1}(T)\right)-1\!\!1_{\{S_0>K\}}-\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}\right] \\ &\quad+\left(-\frac{\sigma(\beta_1-1)}{2qc_r}+\frac{\sigma\beta_1}{2rc_r}\right)K^{1-\beta_2}S_0^{\beta_2}\left[1\!\!1_{\{S_0>K\}}+\frac{1}{2}\times 1\!\!1_{\{S_0=K\}}-N\left(d_{\beta_2}(T)\right)\right]. \end{aligned} \tag{E.26}$$

Definitions (15) and (16) imply:

$$\begin{aligned}
\frac{2c_r}{\sigma} &= \frac{2}{\sigma}\sqrt{b_{-1}^2 + 2r} \\
&= \frac{2}{\sigma}\sqrt{\left(\frac{r-q-\sigma^2/2}{\sigma}\right)^2 + 2r} \\
&= 2\sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\
&= \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\
&= \frac{1}{2} - \frac{r-q}{\sigma^2} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} - \left(\frac{1}{2} - \frac{r-q}{\sigma^2} - \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\right) \\
&= \beta_1 - \beta_2. \qquad \text{(E.27)}
\end{aligned}$$

Moreover, using equation (E.27), it is straightforward to show that

$$\begin{aligned}
-\frac{\sigma(\beta_2 - 1)}{2qc_r} + \frac{\sigma\beta_2}{2rc_r} &= -\frac{\sigma}{2c_r}\left(\frac{\beta_2 - 1}{q} - \frac{\beta_2}{r}\right) \\
&= \frac{1}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{q}\right) \qquad \text{(E.28)}
\end{aligned}$$

and

$$\begin{aligned}
-\frac{\sigma(\beta_1 - 1)}{2qc_r} + \frac{\sigma\beta_1}{2rc_r} &= -\frac{\sigma}{2c_r}\left(\frac{\beta_1 - 1}{q} - \frac{\beta_1}{r}\right) \\
&= \frac{1}{\beta_1 - \beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{q}\right). \qquad \text{(E.29)}
\end{aligned}$$

Replacing expressions (E.28) and (E.29) in equation (E.26) and rearranging yields

$$\begin{aligned}
& V_3 + V_4 \\
&= \frac{1}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{q}\right) K^{1-\beta_1} S_0^{\beta_1}\left[N\left(d_{\beta_1}(T)\right) - \mathbb{1}_{\{S_0>K\}} - \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}}\right] \\
&\quad + \frac{1}{\beta_1 - \beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{q}\right) K^{1-\beta_2} S_0^{\beta_2}\left[\mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}} - N\left(d_{\beta_2}(T)\right)\right] \\
&= B(K) S_0^{\beta_2}\left[\mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}} - N\left(d_{\beta_2}(T)\right)\right] \\
&\quad - A(K) S_0^{\beta_1}\left[\mathbb{1}_{\{S_0>K\}} + \frac{1}{2} \times \mathbb{1}_{\{S_0=K\}} - N\left(d_{\beta_1}(T)\right)\right], \qquad \text{(E.30)}
\end{aligned}$$

with $A(K)$ and $B(K)$ given by equations (8) and (9), respectively.

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