Closed Ideals in the Algebra of Bounded Operators on Baernstein and Schreier Spaces

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A thesis submitted for the degree of

Doctor of Philosophy

February, 2026

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Abstract

This thesis is broadly centered around the study of closed ideals within the Banach algebra $\mathscr{B}(E)$ of bounded linear operators on E, where E is a Banach space from the following two families: the Baernstein spaces B_p $(1 < p < \infty)$ and p-convexified Schreier spaces S_p $(1 \leq p < \infty)$.

In Chapters 2 and 3 we study the complemented subspaces in Baernstein and p-convexified Schreier spaces, where our main goal is to demonstrate that a continuum $\mathfrak{c}$ of pairwise non-isomorphic complemented subspaces exist in the Baernstein and p-convexified Schreier spaces. This had already been established for the Schreier space S_1 by Gasparis and Leung, with the aid of a numerical index that characterizes when two subsequences of the unit vector basis are equivalent in S_1. We surprisingly find that the *same* index works for the Baernstein spaces and p-convexified Schreier spaces, producing analogous results.

In Chapter 4, we study the closed ideals of $\mathscr{B}(E)$ generated by projections. The findings represent the main results of this thesis; $\mathscr{B}(E)$ has at least $\mathfrak{c}$ maximal ideals and $2^{\mathfrak{c}}$ closed ideals.

In Chapter 5 we focus on the ideal of strictly singular operators on a Baernstein or p-convexified Schreier space. Our main finding is that the composition of two strictly singular operators is compact. We then study several applications of this

result.

In Chapter 6, we study operator algebra structure which can arise from 1-unconditional bases when equipped with the coordinate-wise multiplication. It was shown by Varopoulos that ℓ_p is an operator algebra in the coordinate-wise multiplication. We show that many more Banach spaces are operator algebras in this respect. Whether this property extends to *all* 1-unconditional bases, remains unknown.

Contents

Acknowledgements

First and foremost, I acknowledge my fortune in having Professor Niels J.Laustsen as my supervisor. Your dedicated and attentive supervision has created the perfect environment for someone such as myself to develop and improve as a mathematician. When I look back fondly on my days as a PhD student, I will not forget the importance of having you as my supervisor.

Secondly, I would like to thank Dr Yemon Choi as my HDC chair for all the helpful ideas and conversations throughout my PhD. This allowed me to broaden my research horizons and see the world of Banach algebras from different viewpoints.

I also give thanks to Dr Graham Jameson for his input during my second year. It took you just one week to dispatch of a problem that had defeated me for weeks! I was very fortunate to benefit from your experience during a challenging moment.

I would also like to thank my fellow PhD student Nathan; your company and conversations about research were an important part of a pleasant working environment.

During the PhD, I was able to travel to new places and participate in more conferences than I had expected to. Thus, I would like to thank all the event organizers and academics that I had the pleasure of meeting, for the chance to discuss mathematics and also giving me the opportunity to platform my research.

I gratefully acknowledge the support offered by Texas A&M, the Czech Academy of Sciences and the Universities of Waterloo and Kent with regards to travel and accommodation.

Finally, I wholeheartedly thank my family for their encouragement and enthusiasm over the years. To Mom and Dad, supporting my decision to pursue a Mathematics degree will always be of special importance to me.

Declaration

I declare that throughout this thesis, any results that are not standard textbook material are either clearly referenced or original. The material has not been submitted, either in whole or in part, for a degree at this, or any other university.

Chapters 2,3,4 and 5 contain results from the published paper [42], of which N.J.Laustsen is a co-author. Chapters 3 and 5 contain results from the preprint [43], which has been submitted for review and of which N.J.Laustsen is a co-author. Chapter 6 contains unpublished work.

This thesis does not exceed the maximum permitted word length of 80,000 words including appendices and footnotes, but excluding the bibliography.
James Samuel Smith

Chapter 1

Preliminaries

The purpose of this chapter is to introduce notation, terminology and fundamental results that will be repeatedly used throughout this thesis. Section 1.1 introduces basic concepts concerning Banach spaces and Schauder bases thereof. Section 1.2 introduces Banach algebras, bounded operators and their accompanying terminology. Section 1.3 introduces classical Banach spaces which feature throughout this thesis. The chapter is concluded with Section 1.4 which introduces the theory of operator ideals in the sense of Pietsch.

The symbols $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, rational, real and complex numbers respectively. We adopt the convention that $0 \notin \mathbb{N}$. Moreover we let $\aleph_0, \mathfrak{c}$ denote the cardinalities of $\mathbb{N}, \mathbb{R}$ respectively. For two nonempty sets A, B we define $A^B := \{f : B \to A : f \text{ is a function}\}$.

1.1 Fundamentals of Banach space theory

A Banach space X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ (commonly denoted by $\mathbb{K}$) equipped with a norm $\|\cdot\|_X$ such that all Cauchy sequences converge. We define the *unit ball of* X as $B_X := \{x \in X : \|x\|_X \leq 1\}$. We denote by X^* the *dual* of X, which is the vector space of continuous linear functionals $x^* : X \to \mathbb{K}$ equipped with the norm

$$\|x^*\|_{X^*} := \sup\{|x^*(x)| : \|x\|_X \leq 1\},$$

which makes X^* a Banach space. Throughout this thesis, we adopt the angle bracket notation, meaning we write $x^*(x) = \langle x, x^* \rangle$. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is said to be *semi-normalized* if $0 < \inf_{n\in\mathbb{N}}\|x_n\|_X \leq \sup_{n\in\mathbb{N}}\|x_n\| < \infty$.

Definition 1.1.1. A sequence $(b_n)_{n\in\mathbb{N}} \subset X$ is said to be a *Schauder basis* for X if, for every $x \in X$ there exists a unique sequence of scalars $(\alpha(n))_{n\in\mathbb{N}} \subset \mathbb{K}$ such that $\sum_{n=1}^{\infty} \alpha(n)b_n$ converges to x.

Since all infinite bases in this thesis will be Schauder, we will just refer to Schauder bases as bases throughout.

Remark 1.1.2. Separable Banach spaces need not have a basis; in [24] P.Enflo constructed such an example.

For a subset Y in a Banach space X, we denote by $\overline{Y}$ the norm closure:

$$\overline{Y} := \{x \in X : \|x - y_n\|_X \to 0 \text{ for some } (y_n)_{n\in\mathbb{N}} \subseteq Y\}.$$

Definition 1.1.3. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is a *basic sequence* if it forms a basis for the subspace $\overline{\text{span}}(x_n : n \in \mathbb{N})$.

The following is a practical criterion for determining if a sequence is basic.

Proposition 1.1.4. *[20, Theorem 1, Chapter V] For a sequence $(x_n)_{n\in\mathbb{N}}$ in X, the following are equivalent.*

1. *$(x_n)_{n\in\mathbb{N}}$ is a basic sequence,*
2. *$x_n \neq 0$ for every $n \in \mathbb{N}$ and there exists $C > 0$ such that for all $n, k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_{n+k} \in \mathbb{K}$ we have $\|\sum_{i=1}^{n} \alpha_i x_i\| \leq C\|\sum_{i=1}^{n+k} \alpha_i x_i\|$.*

A key result in basis theory is the following result attributed to Mazur.

Theorem 1.1.5. *[54, Proposition] Every infinite-dimensional Banach space contains a basic sequence.*

In the presence of a basis $(b_n)_{n\in\mathbb{N}}$, we denote by $(b_n^*)_{n\in\mathbb{N}}$ the *coordinate functionals associated with* $(b_n)_{n\in\mathbb{N}}$, where $\langle x, b_n^*\rangle := \alpha(n)$ for $x = \sum_{j=1}^{\infty} \alpha(j) b_j$. It is well-known that if $(b_n)_{n\in\mathbb{N}}$ is semi-normalized, $(b_n^*)_{n\in\mathbb{N}}$ is a uniformly bounded subset of X^* and that (b_n^*) is a basic sequence in X^*.

Definition 1.1.6. A basis $(b_n)_{n\in\mathbb{N}}$ for X is said to be *shrinking* if the coordinate functionals (b_n^*) form a basis for X^*.

Definition 1.1.7. A basic sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X *dominates* a basic sequence $(y_n)_{n\in\mathbb{N}}$ in a Banach space Y if there is a constant $C > 0$ such that

$$\left\|\sum_{n=1}^{m} \alpha_n y_n\right\|_Y \leqslant C \left\|\sum_{n=1}^{m} \alpha_n x_n\right\|_X \qquad (m \in \mathbb{N},\ \alpha_1, \dots, \alpha_m \in \mathbb{K}). \tag{1.1}$$

If we wish to record the value of the constant C, we say that $(x_n)_{n\in\mathbb{N}}$ *C-dominates* $(y_n)_{n\in\mathbb{N}}$.

A crucial notion in this thesis will be that of unconditionality in bases. First, we state the definition of unconditionality and then provide a series of equivalent formulations. In most circumstances, those formulations are easier to verify when considering unconditionality.

Definition 1.1.8. A basis $(b_n)_{n\in\mathbb{N}}$ for a Banach space X is said to be *unconditional* if, for any sequence of scalars $(\alpha(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, convergence of $\sum_{n=1}^{\infty} \alpha(n)b_n$ in X implies the convergence of $\sum_{n=1}^{\infty} \alpha(\pi(n))b_{\pi(n)}$ in X for any permutation $\pi : \mathbb{N} \to \mathbb{N}$.

Proposition 1.1.9. *[64, Theorem 17.1] Let X be a Banach space. If $(b_n)_{n\in\mathbb{N}}$ is a basis for X then the following are equivalent.*

(1) The basis $(b_n)_{n\in\mathbb{N}}$ is unconditional,

(2) There exists $1 \leq M < \infty$ such that for all $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and $\alpha(1), \ldots, \alpha(n) \in \mathbb{K}$:

$$\left\| \sum_{k=1}^{n} \varepsilon_k \alpha(k) b_k \right\| \leq M \left\| \sum_{k=1}^{n} \alpha(k) b_k \right\|.$$

(3) There exists $1 \leq C < \infty$ such that for all $\alpha(1), \ldots, \alpha(n), \beta(1), \ldots, \beta(n) \in \mathbb{K}$ satisfying $|\alpha(k)| \leq |\beta(k)|$ for every $1 \leq k \leq n$:

$$\left\| \sum_{k=1}^{n} \alpha(k) b_k \right\| \leq C \left\| \sum_{k=1}^{n} \beta(k) b_k \right\|.$$

In this case, the basis shall be referred to as C-unconditional.

Unconditionality of a basis passes to the associated coordinate functionals as explained in the following.

Lemma 1.1.10. *[64, Theorem 17.7] Let $(b_n)_{n\in\mathbb{N}}$ be an unconditional basis of a Banach space X. Then $(b_n^*)_{n\in\mathbb{N}}$ is an unconditional basic sequence in X^*.*

In the presence of an unconditional basis, we will use the notation $\sum_{n=1}^{\infty} \alpha(n)b_n$ and $\sum_{n\in\mathbb{N}} \alpha(n)b_n$ interchangeably.

1.2 Banach algebras and the algebra of operators on a Banach space

We shall say that A is a *Banach algebra* over $\mathbb{K}$ if A is an algebra over $\mathbb{K}$ equipped with a complete norm satisfying $\|ab\| \leq \|a\|\|b\|$ for each $a, b \in A$. If a Banach algebra A contains a norm one element 1_A satisfying $1_A a = a 1_A = a$ for all $a \in A$, we call A *unital*. We shall say a non-empty subset I of a Banach algebra A is an *ideal* if it is a subspace of A such that $ax, xa \in I$ for all $x \in I, a \in A$. That is, all of our ideals will be two-sided. An ideal I is said to be:

(a) *closed* if $I = \overline{I}$,

(b) *maximal* if there is no ideal J satisfying $I \subsetneq J \subsetneq A$,

(c) *non-trivial* if $\{0\} \subsetneq I \subsetneq A$.

For vector spaces V, W we denote by $\mathscr{L}(V, W)$ the vector space of linear maps from V to W. For Banach spaces X, Y we shall say a map $T : X \to Y$ is an *operator* if it is linear and there exists $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$. We also adopt the following standard conventions

$$\begin{aligned} \ker T &:= \{x \in X : Tx = 0\}, \\ \operatorname{im} T &:= \{y \in Y : y = Tx \text{ for some } x \in X\}. \end{aligned}$$

We denote by $\mathscr{B}(X, Y)$ the vector space of all operators from X to Y over $\mathbb{K}$. This becomes a Banach space when equipped with the following norm

$$\|T\| := \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}.$$

Moreover $\mathscr{B}(X) := \mathscr{B}(X, X)$ becomes a Banach algebra when equipped with composition as multiplication. This Banach algebra is always unital due to the identity operator $I_X \in \mathscr{B}(X), x \mapsto x$. Associated with each operator $T : X \to Y$ is the *adjoint operator*

$$T^* : Y^* \to X^*, \langle x, T^*(y^*) \rangle := \langle Tx, y^* \rangle.$$

We shall say that $T \in \mathscr{B}(X,Y)$ is *bounded below* if there exists $C > 0$ such that $\|x\|_X \leq C\|Tx\|_Y$ for all $x \in X$. Furthermore, we shall say that T is an *isomorphism* if there exists $S \in \mathscr{B}(Y,X)$ satisfying $ST = I_X$ and $TS = I_Y$. In this scenario, we will say that X and Y are *isomorphic* and write $X \approx Y$. If in addition $\|Tx\|_Y = \|x\|_X$ for each $x \in X$, then we say T is *isometric* and write $X \cong Y$. We shall say that $P \in \mathscr{B}(X)$ is a *projection* if $P^2 = P$. Then we say that a subspace $Z \subseteq X$ is *complemented in* X if there is a projection $P \in \mathscr{B}(X)$ with $\operatorname{im} P = Z$. In the presence of a basis, projections allow us to define the following.

Definition 1.2.1. Let X be a Banach space with a basis $(x_n)_{n\in\mathbb{N}}$. For $N \in \mathbb{N}$, we define the *Nth basis projection* P_N as follows

$$P_N : X \to X,\ \ P_N\left(\sum_{n=1}^{\infty} \alpha(n)x_n\right) := \sum_{n=1}^{N} \alpha(n)x_n.$$

It is a standard result that $P_N \in \mathscr{B}(X)$ for each $N \in \mathbb{N}$. Moreover, that $\sup_N\|P_N\|$ is finite. This enables the following definition.

Definition 1.2.2. Let X be a Banach space with basis $(x_n)_{n\in\mathbb{N}}$. We say that the *basis constant* of $(x_n)_{n\in\mathbb{N}}$ is the quantity $\sup_{N\in\mathbb{N}}\|P_n\|$ and denote this quantity by $K_{(x_n)}$.

Next, we introduce a property which will be important throughout this thesis.

Definition 1.2.3. A Banach space X is said to be *saturated* with copies of a Banach space Y if every closed, infinite-dimensional subspace of X contains a closed subspace Z which is isomorphic to Y.

1.3 Classical Banach spaces and direct sums

Throughout this thesis, the following Banach spaces will appear. For $(\alpha(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ we define

$$\begin{aligned}\|(\alpha(n))_{n\in\mathbb{N}}\|_\infty &:= \sup\{|\alpha(n)| : n \in \mathbb{N}\},\\ \ell_\infty &:= \{(\alpha(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|(\alpha(n))_{n\in\mathbb{N}}\|_\infty < \infty\},\\ c_0 &:= \{(\alpha(n))_{n\in\mathbb{N}} \in \ell_\infty : |\alpha(n)| \to 0\}.\end{aligned}$$

It is well known that ℓ_∞ and c_0 are Banach spaces when equipped with the $\|\cdot\|_\infty$ norm. For $1 \leq p < \infty$ we define

$$\begin{aligned}\|(\alpha(n))_{n\in\mathbb{N}}\|_{\ell_p} &:= \left(\sum_{n\in\mathbb{N}} |\alpha(n)|^p\right)^{\frac{1}{p}},\\ \ell_p &:= \{(\alpha(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|(\alpha(n))_{n\in\mathbb{N}}\|_{\ell_p} < \infty\}.\end{aligned}$$

Again, it is well known that ℓ_p is a Banach space when equipped with the $\|\cdot\|_{\ell_p}$ norm. Whilst c_0, ℓ_1 and ℓ_∞ are not reflexive, the space ℓ_p is reflexive for all $1 < p < \infty$. We shall denote the unit vector basis in these spaces by $(d_n)_{n\in\mathbb{N}}$ (given by $d_n(m) = 1$ if $m = n$ and $d_n(m) = 0$ otherwise), so as to differentiate them from the unit vector bases in the Baernstein and Schreier spaces later on. Then $(d_n)_{n\in\mathbb{N}}$ forms a 1-unconditional basis for c_0 and ℓ_p for each $1 \leq p < \infty$, while ℓ_∞ is non-separable. At certain points we will require the finite-dimensional versions of these spaces. For $N \in \mathbb{N}$ we simply denote by ℓ_p^N, ℓ_∞^N the vector space $\mathbb{K}^N$ when equipped with the $\|\cdot\|_{\ell_p}, \|\cdot\|_\infty$ norms respectively. In light of the following well-known fact, the specific norm we choose is mostly irrelevant.

Proposition 1.3.1. *Let X be a finite dimensional vector space. If $\|\cdot\|_X$ and $\|\!|\cdot|\!\|_X$ are two norms on X, then there exist constants $C_1, C_2 > 0$ such that for all $x \in X$*

$$C_1\|x\|_X \leq \|\!|x|\!\|_X \leq C_2\|x\|_X.$$

We now define infinite direct sums of Banach spaces over D, where $D = c_0$ or $D = \ell_p$

for some $1 \le p < \infty$. For a sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces over $\mathbb{K}$ we define

$$\Big(\bigoplus_{n\in\mathbb{N}} X_n\Big)_D = \begin{cases} \{(x_n)_{n\in\mathbb{N}} : x_n \in X_n \text{ for each } n \in \mathbb{N}, \|x_n\|_{X_n} \to 0\} & \text{if } D = c_0, \\ \{(x_n)_{n\in\mathbb{N}} : x_n \in X_n \text{ for each } n \in \mathbb{N}, \sum_{n=1}^{\infty} \|x_n\|_{X_n}^p < \infty\} & \text{if } D = \ell_p, \end{cases}$$

This set becomes a vector space over $\mathbb{K}$ when equipped with coordinate-wise addition and multiplication, and then a Banach space over $\mathbb{K}$ when equipped with the following norm:

$$\|(x_n)_{n\in\mathbb{N}}\| := \begin{cases} \sup_{n\in\mathbb{N}} \|x_n\|_{X_n} & \text{if } D = c_0, \\ \big(\sum_{n\in\mathbb{N}} \|x_n\|_{X_n}^p\big)^{\frac{1}{p}} & \text{if } D = \ell_p. \end{cases}$$

Lastly, we define the following subset of $\mathbb{K}^{\mathbb{N}}$ which is not equipped with Banach space structure, but will be useful in terms of notation. Letting $x = (x(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ we define

$$\begin{aligned} \operatorname{supp} x &:= \{n \in \mathbb{N} : x(n) \neq 0\}, \\ c_{00} &:= \{(\alpha(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : |\operatorname{supp} x| < \infty\}. \end{aligned}$$

1.4 Ideals of operators on Banach spaces

Following the terminology of Pietsch in [59], an *operator ideal* is an assignment $\mathscr{I}$ which designates to each pair (X, Y) of Banach spaces, a subspace $\mathscr{I}(X, Y)$ of $\mathscr{B}(X, Y)$ such that

(i) There exist Banach spaces X, Y such that $\mathscr{I}(X, Y) \neq \{0\}$,

(ii) For any Banach spaces W, X, Y, Z and operators $R \in \mathscr{B}(W, X), T \in \mathscr{I}(X, Y)$, $S \in \mathscr{B}(Y, Z)$ we have that $STR \in \mathscr{I}(W, Z)$.

We now define certain operator ideals which feature later on in this thesis. An operator $T \in \mathscr{B}(X, Y)$ between Banach spaces X and Y is:

(a) *finite rank* if the image of T is finite-dimensional,

(b) *compact* if $\overline{T(B_X)}$ is a compact subset of Y,

(c) *strictly singular* if the restriction of T to W is not an isomorphic embedding for any infinite-dimensional subspace W of X,

(d) *finitely strictly singular* if, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that every subspace of X of dimension at least n contains a unit vector x for which $\|Tx\| \leq \varepsilon$.

(e) *inessential* if $I_X + UT$ is a Fredholm operator (meaning that its kernel is finite-dimensional and its range has finite codimension in X) for every operator $U \in \mathscr{B}(Y, X)$,

(f) *weakly compact* if the image under T of the unit ball in X is relatively weakly compact.

We write $\mathscr{F}(X,Y)$, $\mathscr{K}(X,Y)$, $\mathscr{S}(X,Y)$, $\mathscr{E}(X,Y)$ and $\mathscr{W}(X,Y)$ for the sets of finite-rank, compact, strictly singular, inessential and weakly compact operators between X and Y, respectively, with the usual convention that $\mathscr{S}(X) = \mathscr{S}(X,X)$, *etc.* It is well known that each of these form operator ideals in the sense of the definition above. Generally, certain inclusions always hold among these ideals, which we now state.

Proposition 1.4.1. *Let X, Y be Banach spaces.*

1. *[59, Theorem 1.2.2] If $\mathscr{I}$ is any operator ideal then $\mathscr{F}(X,Y) \subseteq \mathscr{I}(X,Y)$.*

2. *[59, Sections 1.11 and 26.7] For any Banach spaces X, Y we have the following*

inclusions:

$$\begin{tikzcd}
\mathscr{B}(X,Y) & \\
& \mathscr{E}(X,Y) \arrow[ul] \\
\mathscr{W}(X,Y) \arrow[uu] & \\
& \mathscr{S}(X,Y) \arrow[uu] \\
\mathscr{K}(X,Y) \arrow[uu] \arrow[ur] & \\
\{0\} \arrow[u] &
\end{tikzcd}$$

Remark 1.4.2. We shall say $R \in \mathscr{F}(X)$ satisfying $\dim(\operatorname{im} R) = 1$ is a *rank one* operator. Then

(i) It is straightforward to check there exist $f \in X^*$ and $y \in X$ so that $Rx = \langle x, f\rangle y$ for each $x \in X$. We then denote this by $R = y \otimes f$.

(ii) $\mathscr{F}(X,Y) = \operatorname{span}(y \otimes f : y \in X, f \in X^*)$.

We now provide a well-known result about how two-sided ideals in $\mathscr{B}(X)$ relate to $\mathscr{F}(X)$. Note this differs from operator ideals defined above.

Proposition 1.4.3. *Let X be a Banach space and $\mathcal{I}$ be a non-zero ideal in $\mathscr{B}(X)$. Then $\mathscr{F}(X) \subseteq \mathcal{I}$.*

Proof. It suffices to show for any rank one operator $R \in \mathscr{B}(X)$ that $R \in \mathcal{I}$, because of Remark 1.4.2. There exist $f \in X^*$ and $y \in X$ such that $Rx = \langle x, f\rangle y$ for each $x \in X$. Take $T \in \mathcal{I} \setminus \{0\}$. Suppose $Tx_0 \neq 0$. Then by the Hahn-Banach Theorem there exists $\varphi \in X^*$ such that $\varphi(Tx_0) = 1$. We then calculate for $z \in X$

$$\begin{aligned}(x \otimes \varphi)T(x_0 \otimes f)(z) &= (x \otimes \varphi)T(\langle z, f\rangle x_0) \\ &= \langle z, f\rangle (x \otimes \varphi)Tx_0 \\ &= \langle z, f\rangle \langle Tx_0, \varphi\rangle x \\ &= \langle z, f\rangle x \\ &= Rx.\end{aligned}$$

Thus $R = (x \otimes \varphi)T(x_0 \otimes f) \in \mathcal{I}$ as required. □

Finally, we discuss the matrix representations of operators on direct sums of Banach spaces. We follow the exposition provided in [41, Section 2]. Let $X = X_1 \oplus \cdots \oplus X_n$ for some Banach spaces $X_1, \dots, X_n$. For $1 \leq j \leq n$ let $Q_j : X \to X_j$ and $\iota_j : X_j \to X$ denote the canonical jth projection and inclusion maps. We equip X with the following norm to make it a Banach space:

$$\|(x_1, \dots, x_n)\| := \max\{\|x_1\|_{X_1}, \dots, \|x_n\|_{X_n}\}, \quad (x_1, \dots, x_n) \in X.$$

With an operator $T : X \to X$ we can associate the $(n \times n)$-matrix $(Q_j T \iota_i)_{i,j=1}^n$. This yields a bijective correspondence between $\mathscr{B}(X)$ and $\{(m_{ij})_{i,j=1}^n : m_{i,j} \in \mathscr{B}(X_i, X_j)\}$. This correspondence is linear, and crucially, multiplicative. That is, composition of maps in $\mathscr{B}(X)$ corresponds to multiplication of the associated matrices. The matrix representation of operators combines nicely with operator ideals as summarized in the following.

Proposition 1.4.4. *[41, Remark 3.2] Let $\mathscr{I}$ be an operator ideal. For any Banach spaces $X_1, \dots, X_n$, an operator $T \in \mathscr{B}(X_1 \oplus \cdots \oplus X_n)$ belongs to $\mathscr{I}(X_1 \oplus \cdots \oplus X_n)$ if and only if $Q_j T \iota_i \in \mathscr{I}(X_i, X_j)$ for every $i, j \in \{1, \dots, n\}$.*

Chapter 2

Subspaces of the Baernstein and Schreier spaces

2.1 Introducing the Schreier spaces

Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ and write $[\mathbb{N}]$ and $[\mathbb{N}]^{<\infty}$ for the families of infinite and finite subsets of $\mathbb{N}$, respectively. By the *Schreier sets*, we mean the following collection:

$$\mathcal{S}_1 = \{F \in [\mathbb{N}]^{<\infty} : |F| \leq \min F\}$$

where $|F|$ denotes the cardinality of the set F (and by convention $\min \emptyset = 0$, so $\emptyset \in \mathcal{S}_1$). Observe that $\mathcal{S}_1$ is closed under taking subsets, and with respect to the following operation.

Definition 2.1.1. Let $\{j_1 < \cdots < j_n\} \in [\mathbb{N}]^{<\infty}$. We say that $\{k_1 < \cdots < k_n\} \in [\mathbb{N}]^{<\infty}$ is a *spread* of $\{j_1 < \cdots < j_n\}$ if $j_i \leq k_i$ for each $1 \leq i \leq n$.

A Schreier set is *maximal* if it is not contained in any strictly larger Schreier set. Clearly, this is equivalent to $F \neq \emptyset$ and $|F| = \min F$.

It will be convenient to express the p-convexified Schreier and Baernstein norms as the suprema over certain families of semi-norms. This is straightforward for the Schreier norms and was used extensively in [7]: for $1 \leq p < \infty$ and $x = (x(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, we can write

$$\|x\|_{S_p} = \sup\bigl\{\mu_p(x, F) : F \in \mathcal{S}_1\bigr\} \in [0, \infty],$$

where

$$\mu_p(x, F) = \begin{cases} 0 & \text{if } F = \emptyset \\ \Bigl(\sum_{n\in F} |x(n)|^p\Bigr)^{\frac{1}{p}} & \text{otherwise.} \end{cases}$$

As observed in [7, Lemma 3.3(iv)], $\|\cdot\|_{S_p}$ defines a complete norm on the subspace $Z_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{S_p} < \infty\}$ of $\mathbb{K}^{\mathbb{N}}$, and we define the p-convexified Schreier space S_p as the closed subspace of Z_p spanned by the "unit vector basis" $(e_n)_{n\in\mathbb{N}}$ given by $e_n(m) = 1$ if $m = n$ and $e_n(m) = 0$ otherwise. Justifying its name, $(e_n)_{n\in\mathbb{N}}$ is a normalized basis for S_p that is 1-unconditional and shrinking, as shown in [7, Propositions 3.5, 3.10 and Corollary 3.12]. It is not hard to verify that S_p is a proper subspace of Z_p; in fact, Z_p is non-separable by [7, Corollary 5.6]. It is also established in [7, p. 3.14] that Z_p is the bidual of S_p meaning that S_p fails to be reflexive.

Remark 2.1.2. (i) For each $1 \leq p < \infty$, S_p clearly contains isometric copies of ℓ_p^n for all $n \in \mathbb{N}$; select $F \in \mathcal{S}_1$ such that $|F| = n$ and consider the subspace $\operatorname{span}(e_i : i \in F)$. This suggests an isomorphic copy of ℓ_p may reside in S_p; in light of [5, Corollary 5.4] by Bird and Laustsen the space S_p is saturated with copies of c_0 rendering this impossible.

(ii) A much more commonly researched variant of the Banach space S_1 is the family of *higher-order Schreier spaces* $X[\mathcal{S}_\xi]$, defined by Alspach and Argyros [2] for every countable ordinal ξ; the correspondence is that the space we denote S_1 is equal to $X[\mathcal{S}_1]$ (and $X[\mathcal{S}_0] = c_0$). We shall not add to the theory of these spaces for $\xi \geq 2$, but instead develop counterparts of some of the main results about them for the Baernstein and p-convexified Schreier spaces.

2.2 Introducing the Baernstein spaces

The second family of spaces we investigate, is the Baernstein spaces B_p for $1 < p < \infty$. The origin of these spaces is in the special case of B_2, constructed by Baernstein in [5] to demonstrate that reflexive Banach spaces need not have the Banach-Saks property (whilst the converse is true generally). In his thesis [63], Seifert initiated a study of B_p for $1 < p < \infty$. Whilst he successfully established that each is reflexive and fails the Banach-Saks property, he made several fundamental claims about the spaces which we shall later demonstrate are false. For now, we proceed to the construction.

To analogously express the Baernstein norm as the supremum of a certain family of seminorms, we introduce the following notion: a *Schreier chain* is a non-empty, finite collection $\mathcal{C}$ of non-empty, consecutive Schreier sets; that is, $\mathcal{C} = \{F_1 < F_2 < \cdots < F_m\}$, where $m \in \mathbb{N}$, $F_1, \ldots, F_m \in \mathcal{S}_1 \setminus \{\emptyset\}$, and the notation $F_1 < F_2 < \cdots <$

F_m signifies that $\max F_j < \min F_{j+1}$ for each $1 \le j < m$. We write SC for the collection of all Schreier chains. Then, for $1 < p < \infty$ and a Schreier chain $\mathcal{C}$, we can define a seminorm $\beta_p(\,\cdot\,, \mathcal{C})$ on $\mathbb{K}^{\mathbb{N}}$ by

$$\beta_p(x, \mathcal{C}) = \Big(\sum_{F \in \mathcal{C}} \Big(\sum_{n \in F} |x(n)|\Big)^p\Big)^{\frac{1}{p}} \qquad (x = (x(n))_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}),$$

and

$$\|x\|_{B_p} = \sup\big\{\beta_p(x, \mathcal{C}) : \mathcal{C} \in \mathrm{SC}\big\} \in [0, \infty] \qquad (x \in \mathbb{K}^{\mathbb{N}}).$$

We then define $B_p = \overline{\operatorname{span}}(e_n : n \in \mathbb{N})$ with respect to the norm $\|\cdot\|_{B_p}$. In contrast to the Schreier spaces, it turns out that the p^{th} Baernstein space is precisely the collection of vectors $x \in \mathbb{K}^{\mathbb{N}}$ with finite Baernstein norm: $B_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{B_p} < \infty\}$; this follows by replacing the exponent 2 with p in Baernstein's argument given in [5, page 92, first paragraph of "Proof of (2)"]. As is the case for the Schreier spaces, the unit vector basis $(e_n)_{n \in \mathbb{N}}$ forms a 1-unconditional, normalized basis for B_p. Some elementary observations below offer us a chance to become acquainted with the behaviour of the Baernstein norms.

Remark 2.2.1. (i) The omission of $p = 1$ in our family of Baernstein spaces requires some explanation. In that case, it is clear that if $\mathcal{C} = \{\{1\} < \cdots < \{n\}\}$ then $\beta_1(x, \mathcal{C}) = \sum_{i=1}^n |x(i)|$. We immediately conclude that B_1 is simply ℓ_1 in disguise.

(ii) Since there is no limit to how large a Schreier chain can be, we can again consider $\mathcal{C} = \{\{1\} < \{2\} < \cdots < \{n\}\}$ to conclude that $\|x\|_{B_p} \ge \|x\|_{\ell_p}$ for any $x \in B_p$.

(iii) It is also clear that each B_p contains an isometric copy of ℓ_1^n for each $n \in \mathbb{N}$; consider $\operatorname{span}(e_i : i \in F)$ where $F \in \mathcal{S}_1$ has cardinality n. This suggests a copy of ℓ_1 may reside in B_p. However, it is well-known that a closed subspace of a reflexive Banach space (such as B_p) is reflexive. Thus B_p does not contain ℓ_1.

Although the Baernstein and Schreier spaces exhibit numerous differences, the majority of our operator ideal results will apply to both. When this happens, a tuple (E, D) will be used, which means the result will apply to either of the following:

(1) $E = B_p$ and $D = \ell_p$ for some $1 < p < \infty$,

(2) $E = S_p$ and $D = c_0$ for some $1 \leq p < \infty$.

In both cases, we shall denote by (d_n) the unit vector basis of D.

2.3 Preliminary norm results and an inequality due to G.Jameson

The goal of this section is to further acquaint ourselves with the norms of S_p and B_p through a series of preliminary norm estimates. Clearly, when computing $\|x\|_{S_p}$, it suffices to consider $\mu_p(x, F)$ for $F \in \mathcal{S}_1$ with $F \subseteq \operatorname{supp}(x)$. Similarly, when computing $\|x\|_{B_p}$, it suffices to consider $\beta_p(x, \mathcal{C})$ for $\mathcal{C} \in \mathrm{SC}$ with $\bigcup \mathcal{C} \subseteq \operatorname{supp}(x)$. On certain occasions we will be working with vectors that form absolutely decreasing sequences; the following observation will be useful.

Lemma 2.3.1. *Let $1 < p < \infty$, and suppose that the non-zero coordinates of $x \in c_{00}$ are absolutely decreasing. Then x attains its Baernstein norm at some Schreier chain covering* $\operatorname{supp}(x)$; *that is, $\|x\|_{B_p} = \beta_p(x, \mathcal{C})$ for some $\mathcal{C} \in \mathrm{SC}$ with $\bigcup \mathcal{C} = \operatorname{supp}(x)$.*

Proof. Take $\mathcal{C} = \{F_1 < F_2 < \cdots < F_m\} \in \mathrm{SC}$ with $\bigcup \mathcal{C} \subsetneq \operatorname{supp}(x)$. For $1 \leq j \leq m$, let F'_j be the set which contains precisely the first $|F_j|$ points of $\operatorname{supp}(x) \cap [\min(F_j), \max(F_j)]$. Then $\mathcal{C}' = \{F'_1 < F'_2 < \cdots < F'_m\} \in \mathrm{SC}$ and

$$\sum_{n \in F_j} |\langle x, e_n^* \rangle| \leq \sum_{n \in F'_j} |\langle x, e_n^* \rangle| \qquad (1 \leq j \leq m)$$

because the non-zero coordinates of x are decreasing in absolute value. Hence $\beta_p(x, \mathcal{C}) \leq \beta_p(x, \mathcal{C}')$. The set $\mathcal{C}'' = \big\{\{n\} : n \in \operatorname{supp}(x) \setminus \bigcup \mathcal{C}'\big\}$ is non-empty because $\bigcup \mathcal{C} \subsetneq \operatorname{supp}(x)$. Clearly $\mathcal{D} = \mathcal{C}' \cup \mathcal{C}''$ is a Schreier chain, and

$$\beta_p(x, \mathcal{C}) \leq \beta_p(x, \mathcal{C}') < \beta_p(x, \mathcal{D}) \leq \|x\|_{B_p},$$

so $\|x\|_{B_p}$ is not attained at $\mathcal{C}$. However, it must be attained at some Schreier chain because x is finitely supported. $\square$

For later reference, we now record an estimate for the norm of certain "flat" vectors.

Lemma 2.3.2. *Let $\{F_1 < F_2 < \cdots < F_m\}$ be a chain of maximal Schreier sets for some $m \in \mathbb{N}$. Then*

$$1 \leq \left\| \sum_{j=1}^{m} \frac{1}{|F_j|^{\frac{1}{p}}} \sum_{k \in F_j} e_k \right\|_{S_p} \leq 2^{\frac{1}{p}} \qquad (1 \leq p < \infty) \tag{2.1}$$

and

$$m^{\frac{1}{p}} \leq \left\| \sum_{j=1}^{m} \frac{1}{|F_j|} \sum_{k \in F_j} e_k \right\|_{B_p} \leq 2m^{\frac{1}{p}} \qquad (1 < p < \infty). \tag{2.2}$$

Proof. To prove (2.1), set $x = \sum_{j=1}^{m} |F_j|^{-\frac{1}{p}} \sum_{k \in F_j} e_k \in S_p$. By definition, we have $\mu_p(x, F_j) = 1$ for $1 \leq j \leq m$; the lower bound on the norm follows. For the upper bound, we claim that $\|x\|_{S_p} = \mu_p(x, G)$ for some set $G \in \mathcal{S}_1$ such that $G \subseteq F_j \cup F_{j+1}$ for some $1 \leq j < m$. Indeed, x has finite support, so it attains its norm at some set $G \in \mathcal{S}_1$ contained in $\operatorname{supp}(x) = \bigcup_{i=1}^{m} F_i$. We may suppose that G forms an interval within $\operatorname{supp}(x)$ because the non-zero coordinates of x are decreasing; that is, $G = [\min G, \max G] \cap \operatorname{supp}(x)$. Take $k \in \{1, \ldots, m\}$ such that $\min G \in F_k$. If $k = m$, the claim is trivially satisfied for $j = k-1$. Otherwise $k < m$; the assumption that $\max G \notin F_k \cup F_{k+1}$ would imply that F_{k+1} is a proper subset of G, contradicting the maximality of F_{k+1}. Therefore we must have $G \subseteq F_k \cup F_{k+1}$, so the claim is satisfied for $j = k$. Therefore we have $\mu_p(x, G)^p \leq \mu_p(x, F_j)^p + \mu_p(x, F_{j+1})^p = 2$, which proves the upper bound.

Turning our attention to (2.2), we define $x = \sum_{j=1}^{m} \frac{1}{|F_j|} \sum_{k \in F_j} e_k \in B_p$. The lower bound on the norm of x follows from the fact that $\mathcal{C} = \{F_1 < F_2 < \cdots < F_m\}$ is a Schreier chain for which $\beta_p(x, \mathcal{C}) = m^{\frac{1}{p}}$.

For the upper bound, Lemma 2.3.1 implies that we can take $\mathcal{C} = \{G_1 < \cdots < G_n\} \in \mathrm{SC}$ such that $\|x\|_{B_p} = \beta_p(x, \mathcal{C})$ and $\bigcup_{k=1}^n G_k = \bigcup_{j=1}^m F_j$. Define a map $\varphi\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ by

$$\varphi(k) = \min\{j : G_k \cap F_j \neq \emptyset\}.$$

We observe that φ is surjective because otherwise we would have $F_j \subsetneq G_k$ for some j and k, contradicting that F_j is a maximal Schreier set.

Fix $j \in \{1, \ldots, m\}$, set $h(j) = \max \varphi^{-1}(\{j\})$, and note that $\bigcup\big\{G_k : \varphi(k) = j,\, k \neq h(j)\big\} \subseteq F_j$ because the sets $G_1, \ldots, G_n$ are successive. Since the ℓ_1-norm dominates the ℓ_p-norm, it follows that

$$\sum_{k \in \varphi^{-1}(\{j\})} \Big(\sum_{i \in G_k} |\langle x, e_i^* \rangle|\Big)^p \leq \bigg(\sum_{k \in \varphi^{-1}(\{j\})} \sum_{i \in G_k} |\langle x, e_i^* \rangle|\bigg)^p \leq \bigg(1 + \frac{|G_{h(j)} \cap F_{j+1}|}{|F_{j+1}|}\bigg)^p \leq 2^p.$$

Combining this with the fact that the sets $\{\varphi^{-1}(\{j\}) : 1 \leq j \leq m\}$ partition $\{1, \ldots, n\}$, we conclude that

$$\|x\|_{B_p}^p = \beta_p(x, \mathcal{C})^p = \sum_{j=1}^m \sum_{k \in \varphi^{-1}(\{j\})} \Big(\sum_{i \in G_k} |\langle x, e_i^* \rangle|\Big)^p \leq 2^p m,$$

which gives the upper bound. $\square$

Example 2.3.3. The upper bound given by Inequality (2.1) in Lemma 2.3.2 cannot be improved generally. Indeed it suffices to put $m = 2$ and consider the following situation: let $F_1 = \{n\} \cup [n^2 - (n-1), n^2 - 1] \cap \mathbb{N}$, $F_2 = [n^2, 2n^2 - 1] \cap \mathbb{N}$ and $G := [n^2 - (n-1), 2n^2 - (2n-1)] \cap \mathbb{N}$. We claim that $F_1, F_2, G \in \mathcal{S}_1$:

$$|F_1| = 1 + (n^2 - 1) - (n^2 - n + 1) + 1 = n = \min(F_1),$$
$$|F_2| = n^2 = \min(F_2),$$
$$|G| = (2n^2 - (2n-1)) - (n^2 - (n-1)) + 1 = n^2 - n + 1 = \min(G).$$

In particular they are maximal. We then estimate as follows

$$\begin{aligned}\left\|\frac{1}{|F_1|^{\frac{1}{p}}}\sum_{j\in F_1}e_j+\frac{1}{|F_2|^{\frac{1}{p}}}\sum_{j\in F_2}e_j\right\|_{S_p}^p &\geq \mu_p\left(\frac{1}{|F_1|^{\frac{1}{p}}}\sum_{j\in F_1}e_j+\frac{1}{|F_2|^{\frac{1}{p}}}\sum_{j\in F_2}e_j, G\right)^p\\ &= \frac{n-1}{n}+\frac{n^2+1-n-(n-1)}{n^2}\\ &= \frac{2n^2-3n+2}{n^2}.\end{aligned}$$

Letting $n\to\infty$, the last fraction tends to 2. We conclude that our upper bound cannot be improved generally. Despite this, certain sets F_j can be chosen for each $j\in\mathbb{N}$ so that the upper bound is 1 (in particular, they span an isometric copy of c_0). We elaborate on this in Section 2.5.

We finish this section with an inequality which relates the norms of ℓ_p, S_1, B_p and ℓ_∞. Its proof is due to Graham Jameson; we are very grateful for his permission to include it here, and also in our paper [42]. The inequality will play a key role in the proof of Theorem 4.1.1(i) in understanding stictly singular operators on the Baernstein spaces.

Theorem 2.3.4 (Jameson). *For every $1<p<\infty$, there is a constant $K_p\in\left[\frac{2^p-1}{2^{p-1}-1},\frac{3\cdot 2^{p-1}-2}{2^{p-1}-1}\right]$ such that*

$$\|x\|_{\ell_p}^p\leq K_p\|x\|_\infty^{p-1}\|x\|_{S_1}\qquad (x\in\mathbb{K}^{\mathbb{N}}). \tag{2.3}$$

Consequently, $\|x\|_{\ell_p}^p\leq K_p\|x\|_\infty^{p-1}\|x\|_{B_p}$ for every $x\in B_p$.

We begin with a lemma that will help us reduce to the case of decreasing sequences.

Lemma 2.3.5. *Let $x\colon\mathbb{N}\to[0,\infty)$ be decreasing with limit 0. Then $\|x\|_{S_p}\leq\|x\circ\sigma\|_{S_p}$ for every $1\leq p<\infty$ and every permutation $\sigma\colon\mathbb{N}\to\mathbb{N}$.*

Proof. Take $F\in\mathcal{S}_1\setminus\{\emptyset\}$, and let $k=\min F$. Since σ is surjective, the set $\sigma^{-1}([1,2k)\cap\mathbb{N})\setminus[1,k)$ contains a subset G of cardinality k. Then $G\in\mathcal{S}_1$, and therefore

$$\|x\circ\sigma\|_{S_p}^p\geq\sum_{n\in G}x(\sigma(n))^p\geq\sum_{j=k}^{2k-1}x(j)^p\geq\mu_p(x,F)^p,$$

where the second inequality follows because $\sigma(G)$ is a k-element subset of $[1, 2k) \cap \mathbb{N}$ and x is decreasing. Now the conclusion follows by taking the supremum over F. $\square$

Proof of Theorem 2.3.4. Since all three norms in (2.3) depend only on the moduli of the coordinates of x, we may suppose that $x(n) \geq 0$ for every $n \in \mathbb{N}$. There is nothing to prove if $\|x\|_{S_1} = \infty$, so we may suppose otherwise. Then $x \in c_0$. We claim that we may also suppose that x is decreasing. The reason is that if the support of x is either finite or equal to $\mathbb{N}$, then we can find a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $x \circ \sigma$ is decreasing, and Lemma 2.3.5 implies that $\|x \circ \sigma\|_{S_p} \leq \|x\|_{S_p}$, while $\|x \circ \sigma\|_{\ell_p} = \|x\|_{\ell_p}$. Hence, in these cases we can replace x with the decreasing sequence $x \circ \sigma$. However, this argument does not work if $\operatorname{supp} x$ is an infinite, proper subset of $\mathbb{N}$. In that case, let $\theta\colon \mathbb{N} \to \operatorname{supp} x$ be the order isomorphism, and set $y = x \circ \theta$. Then we have $\operatorname{supp} y = \mathbb{N}$, so arguing as in the previous case, we can find a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $y \circ \sigma$ is decreasing, and

$$\begin{aligned} \|y \circ \sigma\|_{S_p} &\leq \|y\|_{S_p} && \text{by Lemma 2.3.5} \\ &\leq \|x\|_{S_p} && \text{because } \mathcal{S}_1 \text{ is spreading.} \end{aligned}$$

On the other hand, $\|x\|_{\ell_p} = \|y\|_{\ell_p} = \|y \circ \sigma\|_{\ell_p}$, so we can replace x with the decreasing sequence $y \circ \sigma$. This completes the proof of the claim. Finally, by homogeneity, we may suppose that $\|x\|_{S_1} = 1$.

Take $n \in \mathbb{N}_0$, and let $F_n = [2^n, 2^{n+1}) \cap \mathbb{N} \in \mathcal{S}_1$. Since x is non-negative and decreasing, we have $\|x\|_\infty = x(1)$ and $x(2^{n+1}) \leq x(j) \leq x(2^n)$ for $j \in F_n$. Combining this with the fact that $\mu_1(x, F_n) \leq \|x\|_{S_1} = 1$, we obtain

$$x(2^{n+1}) \leq \frac{1}{2^n} \qquad \text{and} \qquad \mu_p(x, F_n)^p \leq x(2^n)^{p-1}\mu_1(x, F_n) \leq x(2^n)^{p-1}. \tag{2.4}$$

Write $x(1) = \frac{\theta}{2^k}$, where $k \in \mathbb{N}$ and $1 \leq \theta \leq 2$. Then we have

$$\sum_{n=0}^{k-1} \mu_p(x, F_n)^p = \sum_{j=1}^{2^k-1} x(j)^p \leq (2^k - 1)x(1)^p \leq \theta x(1)^{p-1}.$$

Furthermore, (2.4) implies that $\mu_p(x, F_k)^p \leq x(1)^{p-1}$ and

$$\sum_{n=k+1}^{\infty} \mu_p(x, F_n)^p \leq \sum_{n=k+1}^{\infty} \frac{1}{2^{(n-1)(p-1)}} = \left(\frac{1}{2^k}\right)^{p-1} \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{x(1)^{p-1}2^{p-1}}{\theta^{p-1}(2^{p-1} - 1)}.$$

Since $(F_n)_{n=0}^{\infty}$ is a partition of $\mathbb{N}$, we conclude that

$$\|x\|_{\ell_p}^p = \sum_{n=0}^{\infty} \mu_p(x, F_n)^p \leq f(\theta)x(1)^{p-1} = f(\theta)\|x\|_{\infty}^{p-1},$$
$$\text{where} \quad f(\theta) = \theta + 1 + \frac{2^{p-1}}{(2^{p-1}-1)\theta^{p-1}}.$$

This defines a smooth function $f\colon (0,\infty) \to (1,\infty)$ whose second derivative $f''(\theta) = \frac{2^{p-1}p(p-1)}{(2^{p-1}-1)\theta^{p+1}}$ is positive. Hence f is convex, so $\max\{f(\theta) : 1 \leq \theta \leq 2\} = \max\{f(1), f(2)\}$. We find that

$$f(1) = f(2) = \frac{3 \cdot 2^{p-1} - 2}{2^{p-1} - 1},$$

and therefore the inequality (2.3) is satisfied for some constant $K_p \leq \frac{3 \cdot 2^{p-1}-2}{2^{p-1}-1}$.

To verify that this constant is at least $\frac{2^p-1}{2^{p-1}-1}$, take $k \in \mathbb{N}$ and define $x\colon \mathbb{N} \to (0,\infty)$ by

$$x(j) = \begin{cases} \dfrac{1}{2^k} & \text{for } 1 \leq j < 2^{k+1}, \\[2ex] \dfrac{1}{2^n} & \text{for } 2^n \leq j < 2^{n+1}, \text{ where } n \in (k,\infty) \cap \mathbb{N}. \end{cases}$$

Then $\|x\|_{\infty} = \frac{1}{2^k}$ and

$$\|x\|_{\ell_p}^p = \frac{2^{k+1}-1}{2^{kp}} + \sum_{n=k+1}^{\infty} \frac{2^n}{2^{np}} = \frac{2}{2^{k(p-1)}} - \frac{1}{2^{kp}} + \frac{1}{2^{k(p-1)}(2^{p-1}-1)}.$$

We claim that $\|x\|_{S_1} = 1$. Since x is decreasing, it suffices to consider Schreier sets of the form $[j, 2j) \cap \mathbb{N}$ for $j \in \mathbb{N}$ when computing $\|x\|_{S_1}$. Clearly $\mu_1(x, [j, 2j) \cap \mathbb{N}) = j/2^k \leq 1$ for $1 \leq j \leq 2^k$. Otherwise $j = 2^n + m$ for some $n \geq k$ and $1 \leq m \leq 2^n$, and we have

$$\mu_1(x, [j, 2j) \cap \mathbb{N}) = \frac{2^n - m}{2^n} + \frac{2m}{2^{n+1}} = 1.$$

This proves the claim. Hence

$$\begin{aligned} K_p &\geq \frac{\|x\|_{\ell_p}^p}{\|x\|_{\infty}^{p-1}\|x\|_{S_1}} = \Big(\frac{2}{2^{k(p-1)}} - \frac{1}{2^{kp}} + \frac{1}{2^{k(p-1)}(2^{p-1}-1)}\Big)2^{k(p-1)} \\ &= 2 - \frac{1}{2^k} + \frac{1}{2^{p-1}-1} \to 2 + \frac{1}{2^{p-1}-1} = \frac{2^p-1}{2^{p-1}-1} \qquad \text{as} \qquad k \to \infty. \end{aligned}$$

The inequality stated in the last line of the theorem follows immediately from (2.3) because the p^{th} Baernstein norm 1-dominates the first Schreier norm due to the fact that

$$\mu_1(x, F) = \beta_p(x, \{F\}) \qquad (x \in \mathbb{K}^{\mathbb{N}},\ F \in \mathcal{S}_1). \qquad \square$$

Remark 2.3.6. The conclusion of Lemma 2.3.5 fails if S_p is replaced by B_p. Let $t > 0$ and define

$$x = e_2 + te_3 + e_4, \qquad y = e_2 + e_3 + te_4.$$

We claim that $\|y\|_{B_p} > \|x\|_{B_p}$ for sufficiently small $t > 0$. Evidently

$$\|y\|_{B_p}^p = \beta_p(y, \{\{2,3\},\{4\}\})^p = 2^p + t^p.$$

To evaluate $\|x\|_{B_p}$, it suffices to consider the following cases:

$$\begin{aligned}
\mathcal{C}_1 &:= \{\{2\},\{3,4\}\}: \qquad \beta_p(x, \mathcal{C}_1)^p = 1 + (t+1)^p \\
\mathcal{C}_2 &:= \{\{2,3\},\{4\}\}: \qquad \beta_p(x, \mathcal{C}_2)^p = (1+t)^p + 1 \\
\mathcal{C}_3 &:= \{\{2\},\{4\}\}: \qquad \beta_p(x, \mathcal{C}_3)^p = 2^p.
\end{aligned}$$

Since $\lim_{t\to 0}(1 + (t+1)^p) = 2 < 2^p$, the claim is proven.

Finally, we deduce a domination result for the S_p spaces.

Proposition 2.3.7. *Let $1 \le p < q < \infty$. Then there exists $C > 0$ such that whenever $x \in S_p$, we have $\|x\|_{\ell_q} \le C\|x\|_{S_p}$.*

Proof. Since both norms depend only on the moduli of the coordinates of x, we may suppose that $x(n) \ge 0$ for every $n \in \mathbb{N}$. There is nothing to prove if $\|x\|_{S_p} = \infty$, so we may suppose otherwise. Then $x \in c_0$. We claim that we may also suppose that x is decreasing. The reason is that if the support of x is either finite or equal to $\mathbb{N}$, then we can find a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $x \circ \sigma$ is decreasing, and Lemma 2.3.5 implies that $\|x \circ \sigma\|_{S_p} \le \|x\|_{S_p}$, while $\|x \circ \sigma\|_{\ell_q} = \|x\|_{\ell_q}$. Hence, in these cases we can replace x with the decreasing sequence $x \circ \sigma$. However, this argument does not work if $\operatorname{supp} x$ is an infinite, proper subset of $\mathbb{N}$. In that case, let $\theta\colon \mathbb{N} \to \operatorname{supp} x$ be the order isomorphism, and set $y = x \circ \theta$. Then we have $\operatorname{supp} y = \mathbb{N}$, so arguing as in the previous case, we can find a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $y \circ \sigma$ is decreasing, and

$$\begin{aligned} \|y \circ \sigma\|_{S_p} &\leq \|y\|_{S_p} && \text{by Lemma 2.3.5} \\ &\leq \|x\|_{S_p} && \text{because } \mathcal{S}_1 \text{ is spreading.} \end{aligned}$$

On the other hand, $\|x\|_{\ell_q} = \|y\|_{\ell_q} = \|y \circ \sigma\|_{\ell_q}$, so we can replace x with the decreasing sequence $y \circ \sigma$. This completes the proof of the claim. Finally, by homogeneity, we may suppose that $\|x\|_{S_p} = 1$. Let $n \geq 0$ and $F_n := [2^n, 2^{n+1} - 1] \cap \mathbb{N}$. First we estimate

$$\mu_q(x, F_n)^q = \sum_{j \in F_n} x(j)^{q-p} x(j)^p \leq x(2^n)^{q-p} \mu_p(x, F_n)^p \leq x(2^n)^{q-p}.$$

Then since $\mu_p(x, F_n) \leq 1$ we obtain $x(2^{n+1}) \leq 2^{-\frac{n}{p}}$. Combining the two estimates

$$\mu_q(x, F_n)^q \leq (2^{-\frac{n-1}{p}})^{q-p} = (2^{\frac{p-q}{p}})^{n-1}$$

Finally, we estimate $\|x\|_{\ell_q}$

$$\|x\|_{\ell_q}^q = x(1)^q + \sum_{n=1}^{\infty} \mu_q(x, F_n)^q \leq 1 + \sum_{n=1}^{\infty} (2^{\frac{p-q}{p}})^{n-1} = 1 + \frac{1}{1 - 2^{\frac{p-q}{p}}}.$$

□

2.4 Complemented copies of c_0 and ℓ_p in Schreier and Baernstein spaces

The goal of this section is to study isomorphic and complemented copies of ℓ_p in B_p and copies of c_0 in S_p. Across the literature, several authors have obtained results in this direction. We begin this section by accounting for what is known. Due to the following isomorphic variant of Sobczyk's Theorem, copies of c_0 will automatically be complemented in Schreier spaces.

Theorem 2.4.1. *[31, Theorem] Let A be an isomorphism mapping c_0 into a separable Banach space X. Then there exists a projection from X onto* $\operatorname{im} A$ *of norm at most* $2\|A\| \cdot \|A^{-1}\|$.

Below are the known results on subspaces of S_p and B_p in this direction.

Theorem 2.4.2. (i) *[10, Proposition 0.7] The following sequence in S_1 is isometrically equivalent to the unit vector basis of c_0:*

$$u_n := \frac{1}{2^n} \sum_{i=2^n}^{2^{n+1}-1} e_i.$$

(ii) *[7, Corollary 5.4] Every closed, infinite-dimensional subspace of S_p contains an isomorphic copy of c_0.*

(iii) *[63, Theorem 3] Every closed, infinite-dimensional subspace of B_p contains an isomorphic copy of ℓ_p.*

We establish a quantitative version of the aforementioned Theorem 2.4.2(ii). Our statement strengthens these results by providing explicit norm bounds on the projections and isomorphisms, using the following terminology.

Definition 2.4.3. Let X and Y be Banach spaces. We say that X is *C-uniformly saturated with complemented copies of* Y for some constant $C \geq 1$ if every closed, infinite-dimensional subspace of X contains a closed subspace Z for which

(i) there exists an isomorphism U of Y onto Z with $\|U\| \cdot \|U^{-1}\| \leq C$, and

(ii) there exists a projection P of X onto Z with $\|P\| \leq C$.

Theorem 2.4.4. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$. Then E is C-uniformly saturated with complemented copies of D for every $C > 1$.*

We provide a full proof of this theorem, even though the norm bounds it provides may seem only a modest improvement of existing knowledge. However, we shall

explicitly make use of the quantitative aspect of Theorem 2.4.4 later, as well as the complemented aspect when discussing operator ideals in Chapter 4.
Having explained *why* Theorem 2.4.4 requires a detailed proof, we shall now present one, proceeding through a series of lemmas.

Definition 2.4.5. Let X be a Banach space with a basis $(x_n)_{n\in\mathbb{N}}$.

(1) A basic sequence $(u_n)_{n\in\mathbb{N}} \subset X$ is said to be *block basic* if there exists a sequence $1 = k_1 < k_2 < \cdots$ in $\mathbb{N}$ such that $u_n \in \operatorname{span}(x_i : k_n \leq i < k_{n+1})$ for each $n \in \mathbb{N}$.

(2) A closed subspace $W \subseteq X$ is said to be a *block subspace of* X if $W = \overline{\operatorname{span}}(u_n : n \in \mathbb{N})$ for some block basic sequence $(u_n)_{n\in\mathbb{N}}$ of X.

Lemma 2.4.6. *Let X and Y be Banach spaces, where X has a basis $(x_n)_{n\in\mathbb{N}}$. Suppose that there is a constant $C_1 \geq 1$ for which every block subspace W of X admits operators $U \in \mathscr{B}(Y, W)$ and $V \in \mathscr{B}(X, Y)$ such that $V|_W U = I_Y$ and $\|U\| \cdot \|V\| \leq C_1$. Then X is C_2-uniformly saturated with complemented copies of Y for every constant $C_2 > C_1$.*

Proof. Let K be the basis constant of $(x_n)_{n\in\mathbb{N}}$, and let $(P_n)_{n\in\mathbb{N}}$ be the corresponding basis projections. Given $C_2 > C_1$, choose $\varepsilon \in (0, 1)$ such that

$$\frac{7(C_2 - C_1)}{4(C_1 + C_2)} \geq \varepsilon. \tag{2.5}$$

Set $m_0 = 0$ and $P_0 = 0$, and let Z be a closed, infinite-dimensional subspace of X. By recursion, we can choose natural numbers $m_1 < m_2 < \cdots$ and unit vectors $z_n \in Z \cap \ker P_{m_{n-1}}$ such that $\|z_n - P_{m_n} z_n\| \leq \varepsilon/(2^{n+2}K)$ for every $n \in \mathbb{N}$. Set $w_n = P_{m_n} z_n \in X$, and note that

$$\|w_n\| = \|z_n - (z_n - w_n)\| \geq 1 - \frac{\varepsilon}{2^{n+2}K} \geq \frac{7}{8} \qquad (n \in \mathbb{N}). \tag{2.6}$$

In particular, $w_n \neq 0$, and since $z_n \in \ker P_{m_{n-1}}$, it follows that $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence of $(x_n)_{n\in\mathbb{N}}$. Set $W = \overline{\operatorname{span}}\,(w_n : n \in \mathbb{N})$. By hypothesis, we can find operators $U_1 \in \mathscr{B}(Y, W)$ and $V_1 \in \mathscr{B}(X, Y)$ such that $V_1|_W U_1 = I_Y$ and $\|U_1\| \cdot \|V_1\| \leq C_1$.

For each $n \in \mathbb{N}$, choose a functional $f_n \in X^*$ of norm 1 such that $\langle w_n, f_n \rangle = \|w_n\|$. Then, using (2.6), we have

$$\sum_{n=1}^{\infty} \frac{\|(P_{m_n} - P_{m_{n-1}})^* f_n\|}{\|w_n\|} \cdot \|w_n - z_n\| \leq \sum_{n=1}^{\infty} \frac{2K}{7/8} \cdot \frac{\varepsilon}{2^{n+2}K} = \frac{4\varepsilon}{7},$$

so we can define an operator $R \in \mathscr{B}(X)$ by

$$R = \sum_{n=1}^{\infty} \frac{(P_{m_n} - P_{m_{n-1}})^* f_n}{\|w_n\|} \otimes (w_n - z_n),$$

where we recall from Remark 1.4.2 that for $x \in X$ and $f \in X^*$, $f \otimes x$ denotes the rank-one operator $y \mapsto \langle y, f \rangle x$. Since $\|R\| \leq 4\varepsilon/7 < 1$, the Neumann series implies that the operator $S = I_X - R \in \mathscr{B}(X)$ is invertible, and $\|S^{-1}\| \leq (1 - 4\varepsilon/7)^{-1} = 7/(7 - 4\varepsilon)$. The definition of R shows that

$$Rw_j = \sum_{n=1}^{\infty} \frac{\langle (P_{m_n} - P_{m_{n-1}}) w_j, f_n \rangle}{\|w_n\|} (w_n - z_n) = \frac{\langle w_j, f_j \rangle}{\|w_j\|} (w_j - z_j) = w_j - z_j,$$

so $Sw_j = z_j$ for each $j \in \mathbb{N}$, and therefore $S[W] \subseteq Z$. It follows that $Z_0 = (S|_W U_1)[Y]$ is a subspace of Z, and the operators $U = S|_W U_1 \in \mathscr{B}(Y, Z_0)$ and $V = V_1 S^{-1} \in \mathscr{B}(X, Y)$ satisfy $V|_{Z_0} U = V_1|_W U_1 = I_Y$. Since U is surjective by definition, this implies that U is invertible with inverse $V|_{Z_0}$ and $P = UV$ is a projection of X onto Z_0. In particular, Z_0 is a closed subspace of Z, and the norm bounds on $\|U\| \cdot \|U^{-1}\|$ and $\|P\|$ specified in Definition 2.4.3(i)–(ii) follow from the fact that

$$\|U\| \cdot \|V\| \leq \|S\| \cdot \|U_1\| \cdot \|V_1\| \cdot \|S^{-1}\| \leq \Big(1 + \frac{4\varepsilon}{7}\Big) C_1 \cdot \frac{7}{7 - 4\varepsilon} \leq C_2,$$

where the final inequality is a direct consequence of (2.5). □

It turns out that the supremum norm $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x(n)|$ for $x = (x(n))_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ plays an important auxiliary role in a number of results about the Baernstein and Schreier spaces. We shall sometimes use the coordinate functionals to express it in the alternative form $\|x\|_\infty = \sup_{n \in \mathbb{N}} |\langle x, e_n^* \rangle|$ for $x \in B_p$ or $x \in S_p$.

Lemma 2.4.7. *Every block basic sequence of the unit vector basis for B_p (for $1 < p < \infty$) or S_p (for $1 \leq p < \infty$) admits a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ for which $\|u_n\|_\infty \to 0$ as $n \to \infty$.*

Proof. As usual, let $E = B_p$ or $E = S_p$, and let $(w_n)_{n\in\mathbb{N}}$ be a block basic sequence of the unit vector basis $(e_n)_{n\in\mathbb{N}}$ for E. Replacing $(w_n)_{n\in\mathbb{N}}$ with the block basic sequence $(w_n/\|w_n\|_E)_{n\in\mathbb{N}}$, we may suppose that $(w_n)_{n\in\mathbb{N}}$ is normalized in the E-norm. If $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w_{n_j})_{j\in\mathbb{N}}$ such that $\|w_{n_j}\|_\infty \to 0$ as $j \to \infty$, there is nothing to prove.

Otherwise $\delta := \inf_{n\in\mathbb{N}}\|w_n\|_\infty > 0$, so for each $n \in \mathbb{N}$, we can choose $m_n \in \mathbb{N}$ such that $|\langle w_n, e^*_{m_n}\rangle| \geq \delta$. Since $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence, we have $m_1 < m_2 < \cdots$ and $F_n = \{m_j : 2^{n-1} \leq j < 2^n\}$ is a Schreier set, being a spread of the interval $[2^{n-1}, 2^n) \cap \mathbb{N} \in \mathcal{S}_1$. This implies that the block basic sequence $(v_n)_{n\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ defined by

$$v_n = \sum_{j=2^{n-1}}^{2^n-1} w_j \qquad (n \in \mathbb{N})$$

is unbounded because

$$\|v_n\|_E \geq \begin{cases} \mu_p(v_n, F_n) \geq 2^{(n-1)/p}\delta \to \infty & \text{as} \quad n \to \infty \quad \text{for} \quad E = S_p, \\ \beta_p(v_n, \{F_n\}) \geq 2^{n-1}\delta \to \infty & \text{as} \quad n \to \infty \quad \text{for} \quad E = B_p. \end{cases}$$

On the other hand, $\|v_n\|_\infty = \max\big\{\|w_j\|_\infty : 2^{n-1} \leq j < 2^n\big\} \leq 1$ because $(w_j)_{j\in\mathbb{N}}$ is a normalized block basic sequence of $(e_n)_{n\in\mathbb{N}}$, so $u_n := v_n/\|v_n\|_E$ is a normalized block basic sequence of $(w_n)_{n\in\mathbb{N}}$ such that

$$\|u_n\|_\infty = \frac{\|v_n\|_\infty}{\|v_n\|_E} \leq \frac{1}{\|v_n\|_E} \to 0 \quad \text{as} \quad n \to \infty,$$

as required. □

Remark 2.4.8. In what follows, we adopt the convention that $\beta_p(x, \emptyset) := 0$. This is because we will be decomposing $\mathcal{S}_1$ chains in a general manner that will sometimes involve $\emptyset$.

Lemma 2.4.9. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and suppose that $(u_n)_{n\in\mathbb{N}}$ is a normalized block basic sequence of the unit vector basis for E with $\inf_{n\in\mathbb{N}} \|u_n\|_\infty = 0$. Then, for every constant $C > 1$, $(u_n)_{n\in\mathbb{N}}$ admits a subsequence which is C-dominated by the unit vector basis for D.*

Proof. Set $\varepsilon = C^p - 1 > 0$. We begin with the easier case, which is the Schreier space; that is, $(E, D) = (S_p, c_0)$. Recursively, we can choose integers $1 = j_1 < j_2 < \cdots$ such that

$$\|u_{j_{k+1}}\|_\infty^p \leq \frac{\varepsilon}{\max(\operatorname{supp}(u_{j_k}))} \qquad (k \in \mathbb{N}). \tag{2.7}$$

In order to verify that the unit vector basis for c_0 C-dominates the basic sequence $(u_{j_k})_{k\in\mathbb{N}}$ in S_p, we must show that $\mu_p(x, F) \leq C$ whenever $x = \sum_{k=1}^n \alpha_k u_{j_k}$ for some $n \in \mathbb{N}$ and some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ with $\max_{1\leq k\leq n}|\alpha_k| \leq 1$, and $F \in \mathcal{S}_1 \setminus \{\emptyset\}$ with $F \subseteq \operatorname{supp}(x)$. Set

$$m = \min\{k \in \mathbb{N} : F \cap \operatorname{supp}(u_{j_k}) \neq \emptyset\}.$$

Then we have $|F| \leq \min F \leq \max(\operatorname{supp}(u_{j_m}))$, so $|F| \cdot \|u_{j_k}\|_\infty^p \leq \varepsilon$ for $k > m$ by (2.7), and therefore

$$\begin{aligned}\mu_p(x, F)^p &= \mu_p(\alpha_m u_{j_m}, F)^p + \mu_p\Big(\sum_{k=m+1}^n \alpha_k u_{j_k}, F\Big)^p \\ &\leq |\alpha_m|^p \,\|u_{j_m}\|_{S_p}^p + |F|\Big(\max_{m<k\leq n} |\alpha_k|\, \|u_{j_k}\|_\infty\Big)^p \leq 1 + \varepsilon = C^p,\end{aligned}$$

as required.

Proceeding to the case $(E, D) = (B_p, \ell_p)$, we use the fact that the function $t \mapsto t^p$ is uniformly continuous on $[0, 2]$ to choose numbers $\delta_k \in (0, 1)$ such that

$$(s + t)^p \leq s^p + \frac{\varepsilon}{2^k} \qquad (k \in \mathbb{N},\ s \in [0, 1],\ t \in [0, \delta_k]). \tag{2.8}$$

After replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, we may suppose that $\|u_n\|_\infty \to 0$ as $n \to \infty$. We can then recursively choose integers $1 = j_1 < j_2 < \cdots$ such that

$$\|u_i\|_\infty \leq \frac{\delta_k}{\max(\operatorname{supp}(u_{j_k}))} \qquad (k \in \mathbb{N},\ i \geq j_{k+1}). \tag{2.9}$$

We seek to verify that the unit vector basis for ℓ_p C-dominates the basic sequence $(u_{j_k})_{k\in\mathbb{N}}$ in B_p. This amounts to showing that $\beta_p(x, \mathcal{C}) \leq C$ whenever $x =$

$\sum_{k=1}^{n} \alpha_k u_{j_k}$ for some $n \in \mathbb{N}$ and some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ with $\sum_{k=1}^{n} |\alpha_k|^p \leq 1$, and $\mathcal{C}$ is a Schreier chain contained in $\operatorname{supp}(x)$. Set

$$\mathcal{C}_k = \{F \in \mathcal{C} : \min F \in \operatorname{supp}(u_{j_k})\} \qquad (1 \leq k \leq n).$$

Then we can write

$$\beta_p(x, \mathcal{C})^p = \sum_{k=1}^{n} \sum_{F \in \mathcal{C}_k} \Big(\sum_{i \in F} |\langle x, e_i^* \rangle|\Big)^p = \sum_{k=1}^{n} \beta_p(x, \mathcal{C}_k)^p. \tag{2.10}$$

We claim that

$$\beta_p(x, \mathcal{C}_k)^p \leq |\alpha_k|^p + \frac{\varepsilon}{2^k} \qquad (1 \leq k \leq n), \tag{2.11}$$

from which the conclusion will follow because substituting (2.11) into (2.10), we obtain

$$\beta_p(x, \mathcal{C})^p \leq \sum_{k=1}^{n} \Big(|\alpha_k|^p + \frac{\varepsilon}{2^k}\Big) \leq 1 + \varepsilon = C^p.$$

It remains to prove (2.11). Take $k \in \{1, \ldots, n\}$ with $\mathcal{C}_k \neq \emptyset$, let G_k be the final set in $\mathcal{C}_k$ (in the sense that G_k is the set in $\mathcal{C}_k$ with the largest minimum), and define

$$G_k' = G_k \cap \operatorname{supp}(u_{j_k}) \qquad \text{and} \qquad G_k'' = G_k \setminus \operatorname{supp}(u_{j_k}).$$

Then $G_k' \neq \emptyset$ because $\min G_k \in \operatorname{supp}(u_{j_k})$, and we have

$$\begin{aligned} \beta_p(x, \mathcal{C}_k)^p &= \sum_{F \in \mathcal{C}_k \setminus \{G_k\}} \Big(\sum_{i \in F} |\langle x, e_i^* \rangle|\Big)^p + \Big(\sum_{i \in G_k} |\langle x, e_i^* \rangle|\Big)^p \\ &= |\alpha_k|^p \sum_{F \in \mathcal{C}_k \setminus \{G_k\}} \Big(\sum_{i \in F} |\langle u_{j_k}, e_i^* \rangle|\Big)^p + (s_k + t_k)^p, \end{aligned} \tag{2.12}$$

where we have introduced the quantities

$$s_k = \sum_{i \in G_k'} |\langle x, e_i^* \rangle| = |\alpha_k| \sum_{i \in G_k'} |\langle u_{j_k}, e_i^* \rangle| \qquad \text{and} \qquad t_k = \sum_{i \in G_k''} |\langle x, e_i^* \rangle|.$$

Now we observe that $0 \leq s_k \leq |\alpha_k| \, \|u_{j_k}\|_{B_p} \leq 1$ because $G_k' \in \mathcal{S}_1$, and

$$0 \leq t_k \leq |G_k''| \max_{k < i \leq n} |\alpha_i| \, \|u_{j_i}\|_\infty \leq \delta_k,$$

where we have used (2.9) together with the fact that $|G_k''| < |G_k| \leq \min(G_k) \leq \max(\operatorname{supp}(u_{j_k}))$. Hence (2.8) implies that $(s_k + t_k)^p \leq s_k^p + \varepsilon/2^k$. Substituting this into (2.12) and defining the Schreier chain $\mathcal{C}_k' = (\mathcal{C}_k \setminus \{G_k\}) \cup \{G_k'\}$, we obtain

$$\begin{aligned}\beta_p(x, \mathcal{C}_k)^p &\leq |\alpha_k|^p \sum_{F \in \mathcal{C}_k \setminus \{G_k\}} \Big(\sum_{i \in F} |\langle u_{j_k}, e_i^* \rangle|\Big)^p + |\alpha_k|^p \Big(\sum_{i \in G_k'} |\langle u_{j_k}, e_i^* \rangle|\Big)^p + \frac{\varepsilon}{2^k} \\ &= |\alpha_k|^p \beta_p(u_{j_k}, \mathcal{C}_k')^p + \frac{\varepsilon}{2^k} \leq |\alpha_k|^p + \frac{\varepsilon}{2^k},\end{aligned}$$

which establishes Equation (2.11) and thus completes the proof. □

Lemma 2.4.10. *Let $\mathcal{C} = \{F_1 < F_2 < \cdots\}$ be an infinite chain of successive Schreier sets, and take $1 < p < \infty$. Then, for each $x \in B_p$,*

$$\Sigma_{\mathcal{C}} x = \Big(\sum_{j \in F_n} \langle x, e_j^* \rangle\Big)_{n \in \mathbb{N}} \tag{2.13}$$

defines an element of ℓ_p with $\|\Sigma_{\mathcal{C}} x\|_{\ell_p} \leq \|x\|_{B_p}$. Hence (2.13) defines a map $\Sigma_{\mathcal{C}} \colon B_p \to \ell_p$, which is bounded and linear with norm 1.

Proof. For $x \in B_p$ and $m \in \mathbb{N}$, we have

$$\sum_{n=1}^{m} \Big|\sum_{j \in F_n} \langle x, e_j^* \rangle\Big|^p \leq \sum_{n=1}^{m} \Big(\sum_{j \in F_n} |\langle x, e_j^* \rangle|\Big)^p = \beta_p\big(x, \{F_1 < F_2 < \cdots < F_m\}\big)^p \leq \|x\|_{B_p}^p.$$

This shows that $\Sigma_{\mathcal{C}} x \in \ell_p$ with $\|\Sigma_{\mathcal{C}} x\|_{\ell_p} \leq \|x\|_{B_p}$ because the upper bound $\|x\|_{B_p}^p$ is independent of m, hence $\|\Sigma_{\mathcal{C}}\| \leq 1$. Letting $n \in F_1$ we observe $\|\Sigma_{\mathcal{C}} e_n\|_{B_p} = 1$ hence $\|\Sigma_{\mathcal{C}}\| = 1$. □

Lemma 2.4.11. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and suppose that $(u_n)_{n \in \mathbb{N}}$ is a normalized block basic sequence of the unit vector basis for E. Then there exists an operator $V \in \mathscr{B}(E, D)$ of norm 1 such that $Vu_n = d_n$ for every $n \in \mathbb{N}$. In particular, the basic sequence $(u_n)_{n \in \mathbb{N}}$ 1-dominates $(d_n)_{n \in \mathbb{N}}$.*

Proof. We begin with the case $(E, D) = (S_p, c_0)$. For $n \in \mathbb{N}$, set $m_n = \max(\operatorname{supp}(u_n))$, and use the Hahn–Banach Theorem to find a functional $f_n \in S_p^*$ such that $\langle u_n, f_n \rangle = 1 = \|f_n\|$. Then, for each $x \in S_p$, we can define

$$Vx = \big(\langle (P_{m_n} - P_{m_{n-1}})x, f_n \rangle\big)_{n \in \mathbb{N}} \in \ell_\infty, \tag{2.14}$$

where we have introduced $m_0 = 0$ and $P_0 = 0$ for notational convenience. We see that $Vx \in c_0$ because

$$\big|\langle (P_{m_n} - P_{m_{n-1}})x, f_n\rangle\big| \leq \|(P_{m_n} - P_{m_{n-1}})x\|_{S_p} \leq \|(I - P_{m_{n-1}})x\|_{S_p} \to 0 \quad \text{as} \quad n \to \infty,$$

so (2.14) defines a map $V\colon S_p \to c_0$, which is clearly linear. Furthermore, V is bounded with $\|V\| \leq 1$ because $\|P_{m_n} - P_{m_{n-1}}\| = 1 = \|f_n\|$ for $n \in \mathbb{N}$, and we have $Vu_n = d_n$ because $\operatorname{supp}(u_n) \subseteq (m_{n-1}, m_n]$ and $\langle u_n, f_n\rangle = 1$.

As before, the case $(E, D) = (B_p, \ell_p)$ is somewhat more involved. We begin by choosing a sequence of scalars $(\sigma_j)_{j\in\mathbb{N}}$ as follows. If $j \in \operatorname{supp}(u_n)$ for a (necessarily unique) $n \in \mathbb{N}$, we take $\sigma_j \in \mathbb{K}$ of modulus 1 such that $\sigma_j \cdot \langle u_n, e_j^*\rangle > 0$. Otherwise (that is, for $j \in \mathbb{N} \setminus \bigcup_{n=1}^\infty \operatorname{supp}(u_n)$), set $\sigma_j = 1$. The 1-unconditionality of the unit vector basis $(e_j)_{j\in\mathbb{N}}$ for B_p means that we can define an isometric isomorphism $\Delta \in \mathscr{B}(B_p)$ by $\Delta x = \sum_{j=1}^\infty \sigma_j \langle x, e_j^*\rangle e_j$. Our choice of the sequence $(\sigma_j)_{j\in\mathbb{N}}$ implies that

$$\Delta(u_n) = |u_n| \qquad (n \in \mathbb{N}), \tag{2.15}$$

where we have used the standard notion of *modulus* for an element of a Banach space with a 1-unconditional basis (justified by the fact that such a Banach space is a Banach lattice), that is, $\big|\sum_{j=1}^\infty \alpha_j e_j\big| = \sum_{j=1}^\infty |\alpha_j| e_j$.

For each $n \in \mathbb{N}$, take a Schreier chain $\mathcal{C}_n$ contained in $\operatorname{supp}(u_n)$ with $\beta_p(u_n, \mathcal{C}_n) = 1$, and set $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. Defining $m_0 = 0$ and $m_n = \sum_{k=1}^n |\mathcal{C}_k|$ for $n \in \mathbb{N}$, we can enumerate $\mathcal{C}_n$ as

$$\mathcal{C}_n = \{F_{m_{n-1}+1} < F_{m_{n-1}+2} < \cdots < F_{m_n}\}.$$

Since $(u_n)_{n\in\mathbb{N}}$ is a block basic sequence, we have $F_{m_n} < F_{m_n+1}$ for $n \in \mathbb{N}$. Consequently $\mathcal{C} = \{F_1 < F_2 < \cdots\}$ is an infinite chain of successive Schreier sets, so it induces an operator $\Sigma_{\mathcal{C}} \in \mathscr{B}(B_p, \ell_p)$ of norm 1 by Lemma 2.4.10.

Set $D_n = \operatorname{span}(d_j : m_{n-1} < j \leq m_n) \subset \ell_p$, and let $Q_n \in \mathscr{B}(\ell_p, D_n)$ be the basis projection onto D_n; that is, $Q_n d_j = d_j$ for $m_{n-1} < j \leq m_n$ and $Q_n d_j = 0$

otherwise. Then we can define an isometric isomorphism $\Theta \in \mathscr{B}\big(\ell_p, \big(\bigoplus_{n\in\mathbb{N}} D_n\big)_{\ell_p}\big)$ by $\Theta x = (Q_n x)_{n\in\mathbb{N}}$.

In view of (2.15), the vector $y_n = \Sigma_{\mathcal{C}}\Delta(u_n) \in \ell_p$ satisfies

$$y_n = \Sigma_{\mathcal{C}}|u_n| = \sum_{j=m_{n-1}+1}^{m_n} \Big(\sum_{k\in F_j} |\langle u_n, e_k^*\rangle|\Big) d_j \in D_n.$$

In particular, we have

$$\|y_n\|_{\ell_p}^p = \sum_{j=m_{n-1}+1}^{m_n} \Big(\sum_{k\in F_j} |\langle u_n, e_k^*\rangle|\Big)^p = \beta_p(u_n, \mathcal{C}_n)^p = 1,$$

so by the Hahn–Banach Theorem, we can take $f_n \in D_n^*$ such that $\langle y_n, f_n\rangle = 1 = \|f_n\|_{\ell_p^*}$. This enables us to define an operator $\Gamma \in \mathscr{B}\big(\big(\bigoplus_{n\in\mathbb{N}} D_n\big)_{\ell_p}, \ell_p\big)$ of norm 1 by $\Gamma(x_n)_{n\in\mathbb{N}} = \big(\langle x_n, f_n\rangle\big)_{n\in\mathbb{N}}$.

Finally, we compose these operators to obtain an operator $V = \Gamma\Theta\Sigma_{\mathcal{C}}\Delta \in \mathscr{B}(B_p, \ell_p)$; that is,

$$B_p \xrightarrow{\quad\Delta\quad} B_p \xrightarrow{\quad\Sigma_{\mathcal{C}}\quad} \ell_p \xrightarrow{\quad\Theta\quad} \Big(\bigoplus_{n\in\mathbb{N}} D_n\Big)_{\ell_p} \xrightarrow{\quad\Gamma\quad} \ell_p.$$

Recalling that $y_n = \Sigma_{\mathcal{C}}\Delta(u_n) \in D_n$ and then using the definitions of the operators Θ and Γ, we conclude that

$$Vu_n = \Gamma\Theta y_n = \big(\langle Q_j y_n, f_j\rangle\big)_{j\in\mathbb{N}} = \langle y_n, f_n\rangle d_n = d_n \qquad (n \in \mathbb{N}).$$

In particular, since u_n and d_n are unit vectors, we have

$$1 \le \|V\| \le \|\Gamma\|\,\|\Theta\|\,\|\Sigma_{\mathcal{C}}\|\,\|\Delta\| = 1,$$

so V has norm 1. $\square$

Proof of Theorem 2.4.4. By Lemma 2.4.6 (applied with $X = E$ and $Y = D$), it suffices to show that for every constant $C > 1$ and every block subspace $W = \overline{\operatorname{span}}\,(w_n : n \in \mathbb{N})$ of E, where $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence of the unit vector basis for E, there are operators $U \in \mathscr{B}(D, W)$ and $V \in \mathscr{B}(E, D)$ such that $V|_W U = I_D$ and $\|U\|\,\|V\| \le C$.

Lemma 2.4.7 implies that we can find a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$. By Lemma 2.4.9, $(u_n)_{n\in\mathbb{N}}$ admits a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ which is C-dominated by the unit vector basis $(d_j)_{j\in\mathbb{N}}$ for D. Therefore we can define an operator $U \in \mathscr{B}(D, W)$ by $Ud_j = u_{n_j}$ for every $j \in \mathbb{N}$, and $\|U\| \leq C$. Lemma 2.4.11 shows that there exists an operator $V \in \mathscr{B}(E, D)$ of norm 1 such that $Vu_{n_j} = d_j$ for every $j \in \mathbb{N}$. It follows that $VUd_j = d_j$ for every $j \in \mathbb{N}$, so $V|_W U = I_D$, and $\|U\| \cdot \|V\| \leq C \cdot 1 = C$, as required. $\square$

Definition 2.4.12. A Banach space X is *subprojective* if every closed, infinite-dimensional subspace of X contains a closed, infinite-dimensional subspace which is complemented in X.

Theorem 2.4.4 implies that the Baernstein and Schreier spaces have this property. We record this observation formally for later reference.

Corollary 2.4.13. *The Baernstein spaces B_p for $1 < p < \infty$ and the Schreier spaces S_p for $1 \leq p < \infty$ are subprojective.*

Remark 2.4.14. Originally, Baernstein [5] proved that the Banach space B_2 is reflexive by verifying that the unit vector basis is a shrinking and boundedly complete basis for it and then appealing to a well-known theorem of James [45][Theorem 1.b.5]. We can now give an alternative proof of this result using Theorem 2.4.4, valid for any $1 < p < \infty$: the fact that B_p is ℓ_p-saturated implies that it does not contain any subspace isomorphic to either c_0 or ℓ_1. Since B_p has an unconditional basis, it follows from another well-known theorem of James [46][1.c.12(a)] that B_p is reflexive.

With a small amount of extra effort, we can characterize the normalized block basic sequences of the unit vector basis for the Baernstein and Schreier spaces that admit a subsequence which is equivalent to the unit vector basis for ℓ_p or c_0, respectively.

Proposition 2.4.15. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$. The following conditions are equivalent for a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the unit vector basis for E:*

(a) $\inf_{n\in\mathbb{N}}\|u_n\|_\infty = 0$;

(b) *for every $C > 1$ the sequence $(u_n)_{n\in\mathbb{N}}$ admits a subsequence which is C-equivalent to the unit vector basis for D,*

(c) *$(u_n)_{n\in\mathbb{N}}$ admits a subsequence which is dominated by the unit vector basis for D.*

Proof. To see that (a) implies (b), suppose that $\inf_{n\in\mathbb{N}}\|u_n\|_\infty = 0$, and take $C > 1$. By Lemma 2.4.9, $(u_n)_{n\in\mathbb{N}}$ admits a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ that is C-dominated by $(d_j)_{j\in\mathbb{N}}$. On the other hand, $(u_{n_j})_{j\in\mathbb{N}}$ 1-dominates $(d_j)_{j\in\mathbb{N}}$ because Lemma 2.4.11 shows that there is an operator $V \in \mathscr{B}(E, D)$ with $\|V\| = 1$ such that $Vu_{n_j} = d_j$ for every $j \in \mathbb{N}$. Hence $(u_{n_j})_{j\in\mathbb{N}}$ and $(d_j)_{j\in\mathbb{N}}$ are C-equivalent.

The implication (b)⇒(c) is trivial.

We complete the proof by proving that (c) implies (a), arguing contrapositively. Suppose that $\delta := \inf_{n\in\mathbb{N}}\|u_n\|_\infty > 0$, and take a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$. To verify that $(d_j)_{j\in\mathbb{N}}$ does not dominate $(u_{n_j})_{j\in\mathbb{N}}$, it suffices to show that for every $C \geq 1$, there exists $k \in \mathbb{N}$ such that

$$\Big\|\sum_{j=k}^{2k-1} u_{n_j}\Big\|_E > C\Big\|\sum_{j=k}^{2k-1} d_j\Big\|_D = \begin{cases} C & \text{for } D = c_0, \\ Ck^{\frac{1}{p}} & \text{for } D = \ell_p. \end{cases} \tag{2.16}$$

Choose $k \in \mathbb{N}$ such that $k > (C/\delta)^p$ if $E = S_p$ and $k > (C/\delta)^{\frac{p}{p-1}}$ if $E = B_p$, and set $x = \sum_{j=k}^{2k-1} u_{n_j} \in E$. By hypothesis, we can find $m_j \in \operatorname{supp}(u_{n_j})$ such that $|\langle u_{n_j}, e^*_{m_j}\rangle| \geq \delta$ for each $j \in \{k, \dots, 2k-1\}$. Then $F = \{m_j : k \leq j < 2k\}$ is a Schreier set because $|F| = k \leq m_k = \min F$. Hence we have

$$\|x\|_{S_p} \geq \mu_p(x, F) = \Big(\sum_{j=k}^{2k-1}|\langle x, e^*_{m_j}\rangle|^p\Big)^{\frac{1}{p}} = \Big(\sum_{j=k}^{2k-1}|\langle u_{n_j}, e^*_{m_j}\rangle|^p\Big)^{\frac{1}{p}} \geq k^{\frac{1}{p}}\delta > C$$

and

$$\|x\|_{B_p} \geq \beta_p(x, \{F\}) = \sum_{j=k}^{2k-1}|\langle x, e^*_{m_j}\rangle| = \sum_{j=k}^{2k-1}|\langle u_{n_j}, e^*_{m_j}\rangle| \geq k\delta > Ck^{\frac{1}{p}},$$

where the final inequalities follow from the choice of k in both cases. This establishes (2.16). $\square$

2.5 Isometric copies of ℓ_p in Baernstein spaces

In this section, we identify isometric copies of ℓ_p in B_p for each $1 < p < \infty$. Along the way, we will show that for a bounded block basic sequence $(u_n)_{n\in\mathbb{N}}$ to be isomorphically equivalent to the unit vector basis of ℓ_p in B_p, it is sufficient that $(u_n)_{n\in\mathbb{N}}$ be isomorphically equivalent to the unit vector basis of c_0 when viewed as a sequence in S_1. This consolidates our earlier findings, that the norms of the two spaces share a close relationship despite their cosmetic differences.

Proposition 2.5.1. *Let (u_n) be a bounded block basic sequence in B_p. Let $\theta = \sup_n \|u_n\|_{B_p}$. Suppose (u_n) is C-dominated by the unit vector basis of c_0 when viewed as a sequence in S_1, for some $C \geq 1$. Then*

(i) *in B_p, $(u_n)_{n\in\mathbb{N}}$ is $(\theta^p + 2C^p)^{\frac{1}{p}}$-dominated by the unit vector basis of ℓ_p,*

(ii) *if $\sup_n \|u_n\|_{\ell_1} \leq 1$ and $C = 1$, then as a sequence in B_p, $(u_n)_{n\in\mathbb{N}}$ is 1-dominated by the unit vector basis of ℓ_p.*

Proof. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$. Let $\mathcal{C} \in \mathrm{SC}$ such that $\cup_{E\in\mathcal{C}} E \subseteq \mathrm{supp}(x)$. We claim that

$$\beta_p\left(\sum_{k=1}^m \lambda_k u_k, \mathcal{C}\right)^p \leq (\theta^p + 2C^p) \sum_{k=1}^m |\lambda_k|^p.$$

Write $x = \sum_{k=1}^m \lambda_k u_k$. For $1 \leq k \leq m$ we define

$$\mathcal{C}_k := \{F \in \mathcal{C} : F \subseteq \mathrm{supp}(u_k)\},$$
$$J := \{1 \leq k \leq m : \mathcal{C}_k \neq \emptyset\},$$
$$\mathcal{C}' := \mathcal{C} \setminus \bigcup_{k\in J} \mathcal{C}_k.$$

We can then write $\beta_p(x,\mathcal{C})^p$ as follows

$$\begin{aligned}\beta_p(x,\mathcal{C})^p &= \sum_{k\in J}\beta_p(x,\mathcal{C}_k)^p+\beta_p(x,\mathcal{C}')^p\\ &= \sum_{k\in J}|\lambda_k|^p\beta_p(u_k,\mathcal{C}_k)^p+\beta_p(x,\mathcal{C}')^p \qquad (2.17)\\ &\le \theta^p\sum_{j\in K}|\lambda_k|^p+\beta_p(x,\mathcal{C}')^p.\end{aligned}$$

It remains to show $\beta_p(x,\mathcal{C}')^p \le 2C^p\sum_{k=1}^m|\lambda_k|^p$. Writing $\mathcal{C}'=\{F_1<\dots<F_n\}$ we define

$$K_r := \{1\le k\le m : F_r\cap \operatorname{supp}(u_k)\neq\emptyset\}.$$

The following two properties of these sets will be needed:

(1) For each $1\le r<n$ the intersection $|K_r\cap K_{r+1}|\le 1$ because $F_1<\dots<F_n$ and $\operatorname{supp}(u_1)<\dots<\operatorname{supp}(u_m)$,

(2) For $s\ge r+2$ in $\{1,\dots,n\}$ the intersection $K_r\cap K_s=\emptyset$. Otherwise if $k\in K_r\cap K_s$ then $F_{r+1}\subseteq\operatorname{supp}(u_k)$ which contradicts the definition of $\mathcal{C}'$.

We then proceed to estimate

$$\begin{aligned}\beta_p(x,\mathcal{C}')^p=\sum_{r=1}^n\mu_1(x,F_r)^p &= \sum_{r=1}^n\left(\sum_{k\in K_r}\mu_1(x,F_r\cap\operatorname{supp}(u_k))\right)^p\\ &= \sum_{r=1}^n\left(\sum_{k\in K_r}|\lambda_k|\mu_1(u_k,F_r)\right)^p. \qquad (2.18)\end{aligned}$$

Jensen's inequality for the convex function $t\mapsto t^p$ on $[0,\infty)$ yields

$$\frac{\left(\sum_{k\in K_r}\mu_1(u_k,F_r)|\lambda_k|\right)^p}{\left(\sum_{k\in K_r}\mu_1(u_k,F_r)\right)^p}\le\frac{\sum_{k\in K_r}\mu_1(u_k,F_r)|\lambda_k|^p}{\sum_{k\in K_r}\mu_1(u_k,F_r)}.$$

Upon rearranging we obtain

$$\left(\sum_{k\in K_r}\mu_1(u_k,F_r)|\lambda_k|\right)^p\le\left(\sum_{k\in K_r}\mu_1(u_k,F_r)\right)^{p-1}\sum_{k\in K_r}\mu_1(u_k,F_r)|\lambda_k|^p$$

Now by the assumption that (u_k) in S_1 is dominated by the unit vector basis of c_0

$$\sum_{k\in K_r} \mu_1(u_k, F_r) = \mu_1\left(\sum_{k\in K_r} u_k, F_r\right) \leq \left\|\sum_{k=1}^{m} u_k\right\|_{S_1} \leq C.$$

from which we deduce

$$\left(\sum_{k\in K_r} \mu_1(u_k, F_r)|\lambda_k|\right)^p \leq C^{p-1} \sum_{k\in K_r} \mu_1(u_k, F_r)|\lambda_k|^p.$$

Substituting this into 2.18 we obtain

$$\beta_p(x, \mathcal{C}')^p \leq C^p \sum_{r=1}^{n} \sum_{k\in K_r} |\lambda_k|^p.$$

By the properties (1) and (2) above we make the final estimate

$$C^p \sum_{r=1}^{n} \sum_{k\in K_r} |\lambda_k|^p \leq 2C^p \sum_{k=1}^{m} |\lambda_k|^p.$$

For part (ii), the further assumptions will allow us to sharpen the estimate of part (i). Using the same notation as in part (i) we recall the estimate

$$\begin{aligned}
\beta_p(x, \mathcal{C})^p &= \sum_{k\in J} |\lambda_k|^p \beta_p(u_k, \mathcal{C}_k)^p + \beta_p(x, \mathcal{C}')^p \\
&= \sum_{k\in J} |\lambda_k|^p \sum_{F\in\mathcal{C}_k} \mu_1(u_k, F)^p + \beta_p(x, \mathcal{C}')^p \\
&\leq \sum_{k\in J} |\lambda_k|^p \sum_{F\in\mathcal{C}_k} \mu_1(u_k, F)^p + C^{p-1} \sum_{r=1}^{n} \sum_{k\in K_r} \mu_1(u_k, F_r)|\lambda_k|^p.
\end{aligned}$$

By assumption $C = 1$ so it remains to estimate the quantity

$$\sum_{k\in J} |\lambda_k|^p \sum_{F\in\mathcal{C}_k} \mu_1(u_k, F)^p + \sum_{r=1}^{n} \sum_{k\in K_r} \mu_1(u_k, F_r)|\lambda_k|^p.$$

For each $1 \leq k \leq m$, we claim the coefficient of $|\lambda|_k^p$ does not exceed 1. In light of properties (1) and (2) proven in part (i) the following three cases are exhaustive:

Case 1: $k \notin K_r$ for any $1 \leq r \leq n$. Then the coefficient of $|\lambda_k|^p$ is

$$\sum_{F\in\mathcal{C}_k} \mu_1(u_k, F)^p \leq \|u_k\|_{\ell_1} \leq 1.$$

Case 2: $k \in K_r$ for precisely one $1 \leq r \leq n$. Then the coefficient of $|\lambda_k|^p$ is

$$\sum_{F \in \mathcal{C}_k} \mu_1(u_k, F)^p + \mu_1(u_k, F_r) \leq \|u_k\|_{\ell_1} \leq 1.$$

Case 3: $k \in K_r \cap K_{r+1}$ for some $1 \leq r \leq n-1$. Then the coefficient of $|\lambda_k|^p$ is

$$\sum_{F \in \mathcal{C}_k} \mu_1(u_k, F)^p + \mu_1(u_k, F_r) + \mu_1(u_k, F_{r+1}) \leq \|u_k\|_{\ell_1} \leq 1.$$

□

We now produce the analogue of Theorem 2.4.2(i) in B_p.

Corollary 2.5.2. *Let $1 < p < \infty$. For $n \in \mathbb{N}$ we define*

$$u_n := \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}-1} e_k \in B_p.$$

Then $(u_n)_{n\in\mathbb{N}}$ is isometrically equivalent to the unit vector basis of ℓ_p.

Proof. By Theorem 2.4.2(i) the sequence satisfies the hypotheses of Proposition 2.5.1(ii) and we conclude $(u_n)_{n\in\mathbb{N}}$ is 1-dominated by $(d_n)_{n\in\mathbb{N}}$ in ℓ_p. Finally we invoke Lemma 2.4.11 to also conclude that the two sequences are 1-equivalent. □

Remark 2.5.3.

- Whilst Proposition 2.5.1 can be used to provide similar estimates to the earlier Lemma 2.3.2, it does not render it superfluous. More specifically, not all flat vectors presented in Lemma 2.3.2 satisfy the hypotheses of Proposition 2.5.1(ii); see Example 2.3.3 for such an example.
- We are unsure if a converse to Proposition 2.5.1(i) exists; that ℓ_p-domination of a block basic sequence in B_p implies c_0 domination of that sequence in S_1.

2.6 Corrections to the thesis of C.Seifert

We are now in a position to correct several mistakes made by C.Seifert in his 1977 thesis [63]. It is likely that these mistakes have remained unaddressed because Seifert never published his thesis, and subsequent invesigations into the Baernstein spaces have been few in number. Despite this, one error has actually been reproduced in the published literature leading to a mistake. As such, the motivation for offering these corrections is self-evident. The first error appears as [63, Chapter II, Section 3, Lemma 2]:

Every semi-normalized block basic sequence of the unit vector basis in B_p admits a subsequence equivalent to the unit vector basis of ℓ_p.

That is not true; for instance, no subsequence of the unit vector basis for B_p admits a subsequence equivalent to the unit vector basis of ℓ_p. As we have already seen in Lemma 2.4.9, the correct replacement for such a claim involves allowing for *block* subsequences, as opposed to subsequences alone. This error has been reproduced in the following sources:

(1) [10, Theorem 0.15(c)–(d)], the lecture notes on Tsirelson's space by Casazza and Shura,

(2) [26, page 233], as a special case in a book on the James Tree space,

In [27, page 334] by Flores *et al.* the authors state that a consequence of [10, Theorem 0.15(c)–(d)] is that Seifert's result implies that B_p is 'disjointly homogeneous'. But this is impossible because [27, Theorem 2.13] would then imply that every strictly singular operator on B_p is compact. In Chapter 4, we shall see that this is far from true.
The second error appears as [63, Chapter II, Section 3, Corollary 4]:

For all $1 < q < p < \infty$ any bounded linear operator $T : B_p \to B_q$ is compact.

This result was reproduced in [10, Theorem 0.15(f)]. However, it has been shown to be incorrect by N.J.Laustsen and H.Wirzenius in [44, Corollary 3.9].
The third error appears as [63, Chapter II, Section 3, Lemma 1] and requires more work to correct:

Every semi-normalized block basic sequence of the unit vector basis in B_p spans a complemented subspace of B_p.

Whilst every normalized block basic sequence in ℓ_p spans a complemented subspace of ℓ_p (an argument is provided in [45, Proposition 2.a.1]), I.Gasparis and D.Leung established in [29, Proposition 4.7] that certain semi-normalized block basic sequences in $X[\mathcal{S}_n]$ fail to span complemented subspaces for each $n \geq 1$. We follow their approach to produce analogous results for Baernstein and p-convexified Schreier spaces. It is based on a lemma of Lindenstrauss and Tzafriri and involves the following standard notion.

Definition 2.6.1. Let X be a Banach space with a basis $(x_n)_{n\in\mathbb{N}}$. A block basic sequence $(u_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ is *skipped* if there are integers $0 = m_0 < m_1 < m_2 < \cdots$ such that

$$u_n \in \operatorname{span}\{x_j : m_{n-1} < j < m_n\} \qquad (n \in \mathbb{N}).$$

Proposition 2.6.2. *The Baernstein spaces B_p, for $1 < p < \infty$, and the Schreier spaces S_p, for $1 \leq p < \infty$, contain block basic sequences whose closed span is not complemented. More precisely, let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and let $(u_n)_{n\in\mathbb{N}}$ be a skipped, normalized block basic sequence of the unit vector basis for E such that the unit vector basis for D dominates $(u_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, take $m_n \in (\max\operatorname{supp} u_n, \min\operatorname{supp} u_{n+1}) \cap \mathbb{N}$, and set*

$$t_n = \frac{1}{k} \qquad (n \in [2^{k-1}, 2^k) \cap \mathbb{N},\ k \in \mathbb{N}).$$

Then $(u_n+t_ne_{m_n})_{n\in\mathbb{N}}$ is a block basic sequence whose closed span is not complemented in E.

Proof. We can choose m_n as specified because the block basic sequence $(u_n)_{n\in\mathbb{N}}$ is skipped, and it ensures that $(u_n + t_ne_{m_n})_{n\in\mathbb{N}}$ is a block basic sequence. Assume towards a contradiction that its closed span is complemented in E. Since $t_n \to 0$ as $n \to \infty$, [45, Lemma 2.a.11] implies that the basic sequence $(u_n)_{n\in\mathbb{N}}$ dominates

$(t_n e_{m_n})_{n\in\mathbb{N}}$. By hypothesis, the unit vector basis $(d_n)_{n\in\mathbb{N}}$ for D dominates $(u_n)_{n\in\mathbb{N}}$, and $(t_n e_{m_n})_{n\in\mathbb{N}}$ dominates $(t_n e_n)_{n\in\mathbb{N}}$ because of spreading; in particular for $F \in \mathcal{S}_1$ the set $\{m_n : n \in F\} \in \mathcal{S}_1$. Hence we can find a constant $C > 0$ such that $(d_n)_{n\in\mathbb{N}}$ C-dominates $(t_n e_n)_{n\in\mathbb{N}}$, so in particular, we have

$$C\Big\|\sum_{n=2^{k-1}}^{2^k-1} d_n\Big\|_D \geq \Big\|\sum_{n=2^{k-1}}^{2^k-1} t_n e_n\Big\|_E = \frac{1}{k}\Big\|\sum_{n=2^{k-1}}^{2^k-1} e_n\Big\|_E \tag{2.19}$$

for every $k \in \mathbb{N}$. However,

$$\Big\|\sum_{n=2^{k-1}}^{2^k-1} d_n\Big\|_D = \begin{cases} 1 & \text{for } D = c_0 \\ 2^{\frac{k-1}{p}} & \text{for } D = \ell_p \end{cases} \quad \text{and} \quad \Big\|\sum_{n=2^{k-1}}^{2^k-1} e_n\Big\|_E = \begin{cases} 2^{\frac{k-1}{p}} & \text{for } E = S_p \\ 2^{k-1} & \text{for } E = B_p \end{cases}$$

because $[2^{k-1}, 2^k) \cap \mathbb{N} \in \mathcal{S}_1$. Substituting these values into (2.19), we conclude that

$$C \geq \begin{cases} \dfrac{2^{\frac{k-1}{p}}}{k} & \text{for } (E, D) = (S_p, c_0) \\ \dfrac{\left(2^{1-\frac{1}{p}}\right)^{k-1}}{k} & \text{for } (E, D) = (B_p, \ell_p), \end{cases}$$

which is absurd because the right-hand sides are unbounded as $k \to \infty$ in both cases. □

In order to apply Proposition 2.6.2, we must find a skipped, normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the unit vector basis for E that is dominated by the unit vector basis for D. Proposition 2.4.15 ensures that many such sequences exist.

Chapter 3

The Gasparis–Leung index and complemented subspace structure

3.1 Introducing the Gasparis–Leung Index

The aim of this section is to study the machinery developed by Gasparis and Leung. They introduced a numerical index for each $n \in \mathbb{N}$ and every pair M, N of infinite subsets of $\mathbb{N}$ which characterizes when the subspaces spanned by the infinite subsequences of the unit vector basis for $X[\mathcal{S}_n]$ corresponding to M and N are isomorphic.

In order to define it, we must first introduce the *Schreier covering number* of a set $A \in [\mathbb{N}]^{<\infty}$:

$$\tau_1(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \min\Big\{|\mathcal{C}| : \mathcal{C} \in \mathrm{SC},\ A \subseteq \bigcup \mathcal{C}\Big\} & \text{otherwise.} \end{cases} \tag{3.1}$$

Unpacking the somewhat condensed notation for $A \neq \emptyset$, we can restate this definition as

$$\tau_1(A) = \min\Big\{m \in \mathbb{N} : A \subseteq \bigcup_{j=1}^{m} F_j,\ \text{where } F_1, \dots, F_m \in \mathcal{S}_1 \text{ and } F_1 < F_2 < \cdots < F_m\Big\}.$$

Furthermore, as observed in [6, Remark 4.2], we can refine it as follows. Let $m \in \mathbb{N}$. Then $\tau_1(A) = m$ if and only if there is a Schreier chain $\{F_1 < \cdots < F_m\}$ such that $A = \bigcup_{j=1}^{m} F_j$ and $F_1, \dots, F_{m-1}$ are maximal Schreier sets; it is important to note that F_m need not be maximal.

Remark 3.1.1. If $K \in [\mathbb{N}]^{<\infty}$ is a spread of $J \in [\mathbb{N}]^{<\infty}$ then $\tau_1(K) \leq \tau_1(J)$.

As in [6], for a set $M = \{m_1 < m_2 < \cdots\} \in [\mathbb{N}]$ and $J \subseteq \mathbb{N}$, we define

$$M(J) = \{m_j : j \in J\}.$$

This piece of notation enables us to state [29, Definition 3.3] in the following compact form. For $M, N \in [\mathbb{N}]$, the *Gasparis–Leung index* of M with respect to N is

$$\Gamma L_1(M, N) = \sup\{\tau_1(M(J)) : J \in [\mathbb{N}]^{<\infty},\ N(J) \in \mathcal{S}_1\}. \tag{3.2}$$

Gasparis and Leung denoted this quantity $d_1(M, N)$. We have chosen the more distinctive symbol $\Gamma L_1(M, N)$ in their honour, noting that the Greek spelling of "Gasparis" begins with the letter Γ.

To provide motivation for investigating this index further, we state the characterization discovered by Gasparis and Leung (putting $\xi = 1$ in their statement to restrict attention to S_1 alone).

Lemma 3.1.2. *[29, Lemma 3.4] Let $M, N \in [\mathbb{N}]$. Then $(e_{m_i})_{i\in\mathbb{N}}$ dominates $(e_{n_i})_{i\in\mathbb{N}}$ in S_1 if and only if $\Gamma L_1(M, N)$ is finite. In that case, the domination constant does not exceed $\Gamma L_1(M, N)$.*

Later in this chapter, we shall see that this relationship deepens, characterizing when such subsequences span isomorphic subspaces. For now, we acquaint ourselves with this index via some fundamental properties.

Remark 3.1.3. For $M \in [\mathbb{N}]$, $J \in [\mathbb{N}]^{<\infty} \setminus \{\emptyset\}$ and $n \in \mathbb{N}$, we have $\tau_1(M(J)) \leq n$ if and only if we can find a natural number $m \leq n$ and Schreier sets $F_1 < \cdots < F_m$ such that $M(J) = \bigcup_{j=1}^{m} F_j$. Writing $F_j = M(J_j)$, we see that $\tau_1(M(J)) \leq n$ if and only if we can decompose J as $J = \bigcup_{j=1}^{m} J_j$ for some natural number $m \leq n$, where the sets $J_1 < \cdots < J_m$ satisfy $M(J_j) \in \mathcal{S}_1$ for each $1 \leq j \leq m$.

Lemma 3.1.4. *Let L, $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$ be infinite subsets of $\mathbb{N}$. Then:*

(i) $\Gamma L_1(L, N) \leq \Gamma L_1(L, M) \cdot \Gamma L_1(M, N)$.

(ii) *Suppose that N is a spread of M. Then*

$$\Gamma L_1(N, M) = 1 \qquad \text{and} \qquad \Gamma L_1(L, M) \leq \Gamma L_1(L, N).$$

(iii) $\Gamma L_1(N, N \setminus F) \leq \tau_1(\{n_j : 1 \leq j \leq |F|\}) + 1$ *for every $F \in [N]^{<\infty}$.*

(iv) *Suppose that $m_j \leq n_{j+1}$ for each $j \in \mathbb{N}$. Then $\Gamma L_1(N, M) \leq 2$.*

Proof. (i). Take $J \in [\mathbb{N}]^{<\infty} \setminus \{\emptyset\}$ such that $N(J) \in \mathcal{S}_1$. Then $\tau_1(M(J)) \leq \Gamma L_1(M,N)$, so by Remark 3.1.3, we can write $J = \bigcup_{j=1}^m J_j$ for some natural number $m \leq \Gamma L_1(M,N)$, where the sets $J_1 < \cdots < J_m$ satisfy $M(J_j) \in \mathcal{S}_1$ for each $1 \leq j \leq m$. Applying Remark 3.1.3 again, for each $1 \leq j \leq m$, we can find a natural number $r_j \leq \Gamma L_1(L,M)$ and sets $J_{j,1} < J_{j,2} < \cdots < J_{j,r_j}$ such that $J_j = \bigcup_{k=1}^{r_j} J_{j,k}$ and $L(J_{j,k}) \in \mathcal{S}_1$ for each $1 \leq k \leq r_j$. It follows that $J = \bigcup_{j=1}^m \bigcup_{k=1}^{r_j} J_{j,k}$, where

$$J_{1,1} < J_{1,2} < \cdots < J_{1,r_1} < J_{2,1} < J_{2,2} < \cdots < J_{2,r_2} < \cdots < J_{m,1} < \cdots < J_{m,r_m},$$

so appealing to Remark 3.1.3 once more, we conclude that

$$\tau_1(L(J)) \leq \sum_{j=1}^m r_j \leq m \cdot \max_{1\leq j\leq m} r_j \leq \Gamma L_1(M,N) \cdot \Gamma L_1(L,M).$$

(ii). Suppose that N is a spread of M. Then $N(J)$ is a spread of $M(J)$ for every $J \in [\mathbb{N}]^{<\infty}$, so in particular $N(J) \in \mathcal{S}_1$ whenever $M(J) \in \mathcal{S}_1$. This proves that $\Gamma L_1(N,M) = 1$, and also that $\tau_1(L(J)) \leq \Gamma L_1(L,N)$ for every $J \in [\mathbb{N}]^{<\infty}$ such that $M(J) \in \mathcal{S}_1$. The second inequality follows.

(iii). Set $k = |F|$, $G = \{n_j : 1 \leq j \leq k\}$ and $N' = N \setminus G = \{n_j : j > k\}$. Then N' is a spread of $N \setminus F$, so by (ii), it suffices to show that $\Gamma L_1(N,N') \leq \tau_1(G)+1$; that is, $\tau_1(N(J)) \leq \tau_1(G)+1$ for every $J \in [\mathbb{N}]^{<\infty}$ such that $N'(J) \in \mathcal{S}_1$. Clearly, we may suppose that J is non-empty, say $J = \{j_1 < \cdots < j_m\}$. Set $K = \{j_i : 1 \leq i \leq \min\{k,m\}\}$, which is a spread of the interval $\{1,2,\ldots,\min\{k,m\}\}$. Consequently $N(K)$ is a spread of $\{n_j : 1 \leq i \leq \min\{k,m\}\}$, which is a subset of G, and hence $\tau_1(N(K)) \leq \tau_1(G)$. This completes the proof if $J = K$. Otherwise $k < m$; then $\max K = j_k$, so $\min(J \setminus K) = j_{k+1}$, and therefore

$$\begin{aligned}\min N(J\setminus K) = n_{j_{k+1}} \geq n_{j_1+k} = \min N'(J) \geq |N'(J)| \qquad &\text{because } N'(J) \in \mathcal{S}_1\\ = |J| > |J \setminus K| = |N(J\setminus K)|.&\end{aligned}$$

This proves that $N(J\setminus K) \in \mathcal{S}_1$, so $\tau_1(N(J\setminus K)) = 1$. Since $K < J\setminus K$, we conclude that

$$\tau_1(N(J)) \leq \tau_1(N(K)) + \tau_1(N(J\setminus K)) \leq \tau_1(G)+1,$$

as desired.

(iv). We have

$$\Gamma L_1(N,M) \leq \Gamma L_1(N, N\setminus\{n_1\}) \cdot \Gamma L_1(N\setminus\{n_1\}, M) \leq \big(\tau_1(\{n_1\})+1\big)\cdot 1 = 2,$$

where the first inequality follows from (i) and the second from (iii) and (ii) because the hypothesis implies that $N\setminus\{n_1\}$ is a spread of M. $\square$

We now present a concrete example, with the aim of demonstrating how the Gasparis–Leung index relates to simpler measures of growth rate between infinite sequences. Recall for $M=\{m_n : n\in\mathbb{N}\}\in[\mathbb{N}]$ and $k\in\mathbb{N}$ we define $kM=\{km_n : n\in\mathbb{N}\}$.

Example 3.1.5. *Let $k>2$ be an integer and $M\in[\mathbb{N}]$. Then $\Gamma L_1(M,kM)\leq \lceil\log_2(k+1)\rceil$. Moreover if $N\in[\mathbb{N}]$ satisfies $\sup_i n_i/m_i<\infty$, then $\Gamma L_1(M,N)<\infty$.*

Proof. Suppose $kM(J)\in\mathcal{S}_1$. Then $|J|\leq k\min(M(J))$. It is therefore required to estimate $\tau_1(M(J))$. First, we observe that $M(J)$ is a spread of the set $[\min(M(J)),\min(M(J))+|J|-1]\cap\mathbb{N}$, which is a subset of $L=[\min(M(J)),\min(M(J))+k\min(M(J))]\cap\mathbb{N}$. As $M(J)$ is a subset of a spread of L, we observe that $\tau_1(M(J))\leq\tau_1(L)$ and seek an estimate for $\tau_1(L)$. Putting $r=\min(M(J))$, there exists an integer $t\geq 1$ such that

$$L\subseteq[r,2r-1]\cup[2r,4r-1]\cup\cdots\cup[2^t r,2^{t+1}r-1].$$

it is clear $\tau_1(L)$ will be the smallest integer t satisfying this relationship. For such t, we deduce

$$2^t r\geq \max(L)=r+kr$$

from which we deduce t is the smallest integer which exceeds $\log_2(t+1)$. This means $t=\lceil\log_2(k+1)\rceil$.

For the second claim, suppose $n_i\leq km_i$ for all $i\in\mathbb{N}$. This simply means that kM is a spread of N, so by applying Lemma 3.1.4(ii) we deduce that $\Gamma L_1(M,N)\leq\Gamma L_1(M,kM)<\infty$. $\square$

Corollary 3.1.6. *In general for* $M, N \in [\mathbb{N}]$*, the condition that* $\Gamma L_1(M, N) < \infty$ *is a strictly weaker condition than* $\sup_i\{n_i/m_i\} < \infty$*.*

Proof. Select $N = \{n_1, n_2, \ldots\} \in [\mathbb{N}]$ such that $n_{i+1}/n_i \to \infty$ and put $m_1 = n_1$, $m_i = n_{i-1} + 1$ for $i \geq 2$. By Lemma 3.1.4(iv) $\Gamma L_1(M, N) \leq 2$ but clearly the ratio n_i/m_i will diverge. □

Finally, we observe that this index is *not symmetric*; $\Gamma L_1(\{n^2\}_{n\in\mathbb{N}}, \mathbb{N}) = 1$ by Lemma 3.1.4(i) but interchanging the two leads to the following.

Example 3.1.7. $\Gamma L_1(\mathbb{N}, \{n^2\}_{n\in\mathbb{N}})$ is not finite.

Proof. Let $J = [j, j + 2^j - 1] \cap \mathbb{N}$. It is clear $J \in \mathcal{S}_1$ hence $\{n^2\}(J) \in \mathcal{S}_1$. It therefore suffices to show that $\tau_1(J) \to \infty$ as $j \to \infty$. Arguing as in Example 3.1.5 $\tau_1(J)$ will be the smallest integer t satisfying $2^t j \geq j + j^2$ from which we deduce $t \geq \log_2(1 + j)$ which diverges as $j \to \infty$ as required. □

3.2 The Gasparis–Leung index and equivalence of basic sequences

Now that we have introduced the Gasparis–Leung index and some of its fundamental behaviours, we demonstrate its relevance to the Baernstein spaces by generalizing the findings of Lemma 3.1.2.

Proposition 3.2.1. *Let* $E = B_p$ *for some* $1 < p < \infty$ *or* $E = S_p$ *for some* $1 \leq p < \infty$*, take* $M, N \in [\mathbb{N}]$ *for which* $\Gamma L_1(M, N) < \infty$*, and define*

$$C = \begin{cases} \Gamma L_1(M, N) & \text{for } E = B_p, \\ \Gamma L_1(M, N)^{\frac{1}{p}} & \text{for } E = S_p. \end{cases} \tag{3.3}$$

Then the basic sequence $(e_m)_{m\in M}$ *C-dominates* $(e_n)_{n\in N}$*.*

Proof. Enumerate M and N as $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$, respectively. Our aim is to show that $\|y\|_E \leq C\|x\|_E$ whenever $x = \sum_{j=1}^{k} \alpha_j e_{m_j}$

and $y = \sum_{j=1}^k \alpha_j e_{n_j}$ for some $k \in \mathbb{N}$ and some $\alpha_1, \ldots, \alpha_k \in \mathbb{K}$. We may of course suppose that $\alpha_1, \ldots, \alpha_k$ are not all 0.

We consider the Baernstein and Schreier spaces separately, but emphasize that the proofs follow similar strategies, originating in the proof of [29, Lemma 3.4]. For readability, we begin with the easier case, which is $E = S_p$. Since y is finitely supported, we can choose a Schreier set F such that $\|y\|_{S_p} = \mu_p(y, F)$ and $F \subseteq \operatorname{supp} y \subseteq \{n_i : 1 \leq i \leq k\}$. Take $J \subseteq \{1, \ldots, k\}$ such that $N(J) = F$. Then $\tau_1(M(J)) \leq \Gamma L_1(M, N) = C^p$ by (3.2) and (3.3), so there is a Schreier chain $\{G_1 < \cdots < G_{C^p}\}$ such that $M(J) \subseteq \bigcup_{i=1}^{C^p} G_i$ by (3.1). Set $K_i = \{j \in J : m_j \in G_i\}$ for $i \in \{1, \ldots, C^p\}$. Then we have $J = \bigcup_{i=1}^{C^p} K_i$ and $K_h \cap K_i = \emptyset$ for $h \neq i$, from which we deduce that

$$\|y\|_{S_p}^p = \mu_p(y, F)^p = \sum_{j \in J} |\alpha_j|^p = \sum_{i=1}^{C^p} \sum_{j \in K_i} |\alpha_j|^p = \sum_{i=1}^{C^p} \mu_p(x, G_i)^p \leq C^p \|x\|_{S_p}^p,$$

where the final inequality follows from the fact that $G_1, \ldots, G_{C^p} \in \mathcal{S}_1$.

Having completed the proof for $E = S_p$, we turn our attention to $E = B_p$. We begin in the same way as above: using that y is finitely supported, we can find a Schreier chain $\mathcal{C} = \{F_1 < \cdots < F_t\}$ such that $\|y\|_{B_p} = \beta_p(y, \mathcal{C})$ and $\bigcup_{r=1}^t F_r \subseteq \operatorname{supp} y \subseteq \{n_i : 1 \leq i \leq k\}$.

Fix $r \in \{1, \ldots, t\}$, and choose $J_r \subseteq \{1, \ldots, k\}$ such that $N(J_r) = F_r \in \mathcal{S}_1$. Then we have $\tau_1(M(J_r)) \leq \Gamma L_1(M, N) = C$ by (3.2) and (3.3), so we can find a Schreier chain $\{G_1^r < \cdots < G_C^r\}$ such that $M(J_r) \subseteq \bigcup_{i=1}^C G_i^r$ by (3.1). Set

$$K_i^r = \{j \in J_r : m_j \in G_i^r\} \qquad \text{and} \qquad \gamma_i^r = \sum_{j \in K_i^r} |\alpha_j| \qquad (i \in \{1, \ldots, C\}),$$

and choose $\iota(r) \in \{1, \ldots, C\}$ for which $\gamma_{\iota(r)}^r = \max\{\gamma_i^r : 1 \leq i \leq C\}$. Since $J_r = \bigcup_{i=1}^C K_i^r$ and $K_h^r \cap K_i^r = \emptyset$ whenever $h \neq i$, we have

$$\sum_{j \in J_r} |\alpha_j| = \sum_{i=1}^C \gamma_i^r \leq C \gamma_{\iota(r)}^r = C \sum_{j \in K_{\iota(r)}^r} |\alpha_j|.$$

Furthermore, $\mathcal{D} = \{M(K_{\iota(r)}^r) : 1 \leq r \leq t\}$ is a Schreier chain, as the following two facts show:

- $M(K^r_{\iota(r)}) \in \mathcal{S}_1$ for each $1 \le r \le t$ because $M(K^r_{\iota(r)}) \subseteq G^r_{\iota(r)} \in \mathcal{S}_1$, and
- the sets $M(K^1_{\iota(1)}), M(K^2_{\iota(2)}), \ldots, M(K^t_{\iota(t)})$ are successive because $M(K^r_{\iota(r)}) \subseteq M(J_r)$ for each $1 \le r \le t$ and $M(J_1) < M(J_2) < \cdots < M(J_t)$.

In conclusion, we have

$$\begin{aligned}\|y\|^p_{B_p} = \beta_p(y, \mathcal{C})^p = \sum_{r=1}^{t}\Big(\sum_{j \in J_r}|\alpha_j|\Big)^p &\le C^p \sum_{r=1}^{t}\Big(\sum_{j \in K^r_{\iota(r)}}|\alpha_j|\Big)^p \\ &= C^p \beta_p(x, \mathcal{D})^p \le C^p \|x\|^p_{B_p}. \qquad \square\end{aligned}$$

3.3 Isomorphism classes of complemented subspaces

For the Baernstein and Schreier spaces, we have now seen that finiteness of the Gasparis–Leung index offers a purely set–theoretic characterization for when subsequences of $(e_n)_{n\in\mathbb{N}}$ are equivalent. In this section our core endeavor will be to show that for Baernstein and Schreier spaces, the equivalence of basic subsequences of $(e_n)_{n\in\mathbb{N}}$ occurs *if and only if* they span isomorphic subspaces.

First, let us introduce a piece of notation that we shall use frequently. Given a Banach space X with a basis $(x_n)_{n\in\mathbb{N}}$, we set

$$X_N = \overline{\operatorname{span}}\,(x_n : n \in N) \qquad (N \subseteq \mathbb{N}). \tag{3.4}$$

Further, we write $\|x\|_\infty = \sup_{n\in\mathbb{N}}|\langle x, e^*_n\rangle|$ for $x \in B_p$ or $x \in S_p$ in line with standard usage. We now state the main result, which is clearly a counterpart of [29, Theorem 1.1] for the Baernstein and p-convexified Schreier spaces.

Theorem 3.3.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \le p < \infty$, equipped with the unit vector basis $(e_n)_{n\in\mathbb{N}}$. The following conditions are equivalent for $M, N \in [\mathbb{N}]$:*

(a) *The Gasparis–Leung index $\Gamma L_1(M, N)$ is finite.*

(b) *The basic sequence $(e_m)_{m\in M}$ dominates $(e_n)_{n\in N}$.*

(c) *There exists an operator* $T \in \mathscr{B}(E_M, E_N)$ *for which* $\inf_{m\in M}\|Te_m\|_\infty > 0$.

Proof of Theorem 3.3.1, (a)⇒(b)⇒(c). Proposition 3.2.1 shows that (a) implies (b).

To see that (b) implies (c), suppose that $(e_m)_{m\in M}$ dominates $(e_n)_{n\in N}$. By definition, this means that the linear map $T\colon \operatorname{span}(e_m : m \in M) \to E_N$ determined by $Te_{m_j} = e_{n_j}$ for every $j \in \mathbb{N}$ is bounded, where $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$ are the increasing enumerations. Therefore T extends uniquely to an operator in $\mathscr{B}(E_M, E_N)$, also denoted T, which satisfies $\langle Te_{m_j}, e^*_{n_k}\rangle = \delta_{j,k}$ for every $j, k \in \mathbb{N}$. Hence $\inf_{m\in M}\|Te_m\|_\infty = 1 > 0$. □

It remains to prove the implication (c)⇒(a) in Theorem 3.3.1. For this, we follow the approach of [29] closely, although we can shorten certain steps because we consider only the first Schreier family $\mathcal{S}_1$.

We begin by generalizing [29, Proposition 3.13], which Gasparis and Leung established for the higher-order Schreier spaces $X[\mathcal{S}_\xi]$ for $\xi < \omega_1$. However, as we shall show, it applies to a much larger class of Banach spaces, including the Baernstein and Schreier spaces that we are investigating. We provide a detailed proof for the reader's convenience.

Lemma 3.3.2. *Let X and Y be Banach spaces with unconditional, normalized bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively, and suppose that the basis $(x_n)_{n\in\mathbb{N}}$ for X is shrinking. The following conditions are equivalent:*

(a) *There is an operator* $T \in \mathscr{B}(X, Y)$ *for which*

$$\inf_{k\in\mathbb{N}} \sup_{j\in\mathbb{N}} |\langle Tx_k, y^*_j\rangle| > 0. \tag{3.5}$$

(b) *There is an operator* $U \in \mathscr{B}(X, Y)$ *for which* $Ux_k \in \{y_j : j \in \mathbb{N}\}$ *for every* $k \in \mathbb{N}$.

It is easy to see that (b) implies (a). The proof of the converse relies on a careful analysis of the matrix $(T_{j,k})_{j,k\in\mathbb{N}}$ associated with the operator T. We recall the standard definition of this matrix: for $j, k \in \mathbb{N}$, the $(j,k)^{\text{th}}$ coefficient of the matrix associated with an operator $T \in \mathscr{B}(X, Y)$ between Banach spaces X and Y with bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively, is

$$T_{j,k} = \langle Tx_k, y_j^* \rangle; \tag{3.6}$$

that is, the k^{th} column of the matrix associated with T contains the coordinates with respect to the basis $(y_j)_{j\in\mathbb{N}}$ of the image under T of the k^{th} basis vector x_k. Dualizing, we have $T_{j,k} = \langle x_k, T^* y_j^* \rangle$, so if the basis $(x_n)_{n\in\mathbb{N}}$ for X is shrinking, then the j^{th} row of the matrix associated with T contains the coordinates with respect to the basis $(x_k^*)_{k\in\mathbb{N}}$ for X^* of the image under T^* of the j^{th} coordinate functional y_j^*.

In the proof of Lemma 3.3.2, we require a variant of a result due to Tong [67]. It involves the following notion: a matrix $\Gamma = (\gamma_{j,k})_{j,k\in\mathbb{N}}$ is a *block diagonal* of a matrix $A = (\alpha_{j,k})_{j,k\in\mathbb{N}}$ if there are increasing sequences $0 \leq r_1 < r_2 < \cdots$ and $0 \leq s_1 < s_2 < \cdots$ of integers for which

$$\gamma_{j,k} = \begin{cases} \alpha_{j,k} & \text{if } (j,k) \in \bigcup_{i=1}^{\infty} (r_i, r_{i+1}] \times (s_i, s_{i+1}] \\ 0 & \text{otherwise} \end{cases} \qquad (j, k \in \mathbb{N}).$$

Lemma 3.3.3. *Let $T \in \mathscr{B}(X, Y)$ be an operator between Banach spaces X and Y with unconditional bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively, and suppose that $\Gamma = (\gamma_{j,k})_{j,k\in\mathbb{N}}$ is a block diagonal of the matrix associated with T. Then there is an operator $R \in \mathscr{B}(X, Y)$ whose matrix is Γ; that is,*

$$\langle Rx_k, y_j^* \rangle = \gamma_{j,k} \qquad (j, k \in \mathbb{N}).$$

Proof. As already mentioned, Tong proved a similar result in [67], using very different terminology. A simple proof of the above statement is outlined in the first remark after [45, Proposition 1.c.8]. Note, however, that the definition stated in the text above [45, Proposition 1.c.8] of the matrix associated with an operator $T \in \mathscr{B}(X, Y)$ produces the transpose of the matrix given by (3.6). Fortunately this difference does not matter, as the transpose of a block diagonal is again a block diagonal of the transposed matrix. $\square$

Proof of Lemma 3.3.2. To see that (b) implies (a), suppose that $U \in \mathscr{B}(X,Y)$ is an operator for which $Ux_k \in \{y_j : j \in \mathbb{N}\}$ for every $k \in \mathbb{N}$. Then $\sup_{j\in\mathbb{N}}|\langle Ux_k, y_j^*\rangle| = 1$ for every $k \in \mathbb{N}$, so $T = U$ satisfies (3.5).

We prove that (a) implies (b) by expanding on the approach Gasparis and Leung took in their proof of [29, Proposition 3.13]. In view of (3.5) and (3.6), we can choose $\delta > 0$ such that, for every $k \in \mathbb{N}$, $|T_{j,k}| \geq \delta$ for some $j \in \mathbb{N}$. This allows us to define a map $\psi\colon \mathbb{N} \to \mathbb{N}$ by

$$\psi(k) = \min\{j \in \mathbb{N} : |T_{j,k}| \geq \delta\}.$$

Take $j \in \mathbb{N}$. Since the basis $(x_n)_{n\in\mathbb{N}}$ for X is shrinking, the series $\sum_{k=1}^{\infty} T_{j,k}x_k^*$ is convergent with sum $T^*y_j^*$, as explained in the text below (3.6). It follows that $|T_{j,k}| \to 0$ as $k \to \infty$, so every natural number has finite (possibly empty) pre-image under ψ. Therefore ψ has infinite image; let $\psi(\mathbb{N}) = \{n_1 < n_2 < \cdots\}$ be its increasing enumeration. Then $\{\psi^{-1}(n_j) : j \in \mathbb{N}\}$ partitions $\mathbb{N}$ into non-empty, finite, disjoint sets, so there is a unique permutation $\rho\colon \mathbb{N} \to \mathbb{N}$ such that

$$\rho(k) < \rho(m) \iff \begin{cases} \psi(k) < \psi(m), \text{ or} \\ \psi(k) = \psi(m) \text{ and } k < m \end{cases} \qquad (k, m \in \mathbb{N}).$$

In more concrete terms, we can define ρ as follows. Set $s_0 = 0$ and $s_j = \sum_{i=1}^{j}|\psi^{-1}(n_i)|$ for $j \in \mathbb{N}$. Then each $k \in \mathbb{N}$ belongs to the interval $(s_{j-1}, s_j]$ for a unique $j \in \mathbb{N}$, and $\rho(k)$ is the $(k - s_{j-1})^{\text{th}}$ smallest element of the set $\psi^{-1}(n_j)$. In particular, we have $\psi(\rho(k)) = n_j$, so the definition of ψ implies that $|T_{n_j,\rho(k)}| \geq \delta$, and therefore we can define a diagonal operator $\Delta \in \mathscr{B}(X)$ of norm at most K/δ by $\Delta x_m = T_{n_j,m}^{-1}x_m$ for each $m \in \mathbb{N}$, where $j \in \mathbb{N}$ is chosen such that $s_{j-1} < \rho^{-1}(m) \leq s_j$ and K denotes the unconditional basis constant of $(x_n)_{n\in\mathbb{N}}$.

The unconditionality of the basis $(x_n)_{n\in\mathbb{N}}$ means that any reordering of it is also a basis for X. Hence, viewing the composite operator $P_{\psi(\mathbb{N})}T\Delta$ as a map from X to $Y_{\psi(\mathbb{N})} = \overline{\operatorname{span}}\,(y_{n_j} : j \in \mathbb{N})$, we may consider its matrix with respect to the bases $(x_{\rho(k)})_{k\in\mathbb{N}}$ for X and $(y_{n_j})_{j\in\mathbb{N}}$ for $Y_{\psi(\mathbb{N})}$. Suppose that $j, k \in \mathbb{N}$ satisfy

$s_{j-1} < k \leq s_j$. Then we have

$$(P_{\psi(\mathbb{N})}T\Delta)_{j,k} = \langle P_{\psi(\mathbb{N})}T\Delta x_{\rho(k)}, y^*_{n_j}\rangle = \frac{\langle Tx_{\rho(k)}, P^*_{\psi(\mathbb{N})}y^*_{n_j}\rangle}{T_{n_j,\rho(k)}} = \frac{\langle Tx_{\rho(k)}, y^*_{n_j}\rangle}{T_{n_j,\rho(k)}} = 1,$$

so the matrix $\Gamma = (\gamma_{j,k})_{j,k\in\mathbb{N}}$ defined by

$$\gamma_{j,k} = \begin{cases} 1 & \text{if } s_{j-1} < k \leq s_j \\ 0 & \text{otherwise} \end{cases} \qquad (j,k \in \mathbb{N}) \tag{3.7}$$

is a block diagonal of the matrix associated with the operator $P_{\psi(\mathbb{N})}T\Delta$. Lemma 3.3.3 implies that there is an operator $R \in \mathscr{B}(X, Y_{\psi(\mathbb{N})})$ whose matrix is Γ; that is, $\langle Rx_{\rho(k)}, y^*_{n_j}\rangle = \gamma_{j,k}$ for every $j,k \in \mathbb{N}$. In view of (3.7), this means that $Rx_{\rho(k)} = y_{n_j}$ for every $k \in \mathbb{N}$, where $j \in \mathbb{N}$ is the unique number such that $s_{j-1} < k \leq s_j$. Hence, writing $J\colon Y_{\psi(\mathbb{N})} \to Y$ for the inclusion map, we obtain an operator $U = JR \in \mathscr{B}(X,Y)$ which satisfies $Ux_m = Rx_m \in \{y_n : n \in \mathbb{N}\}$ for every $m \in \mathbb{N}$, as required. □

Lemma 3.3.4. *Let $x = \sum_{j=1}^k \alpha_j e_{m_j}$, where $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in [0,\infty)$, and $m_1, \ldots, m_k \in \mathbb{N}$ are (not necessarily distinct) numbers for which $\{m_1, \ldots, m_k\} \in \mathcal{S}_1$. Then $\|x\|_E = \sum_{j=1}^k \alpha_j$ for $E = S_1$ and $E = B_p$, while $\|x\|_{S_p} \geq \big(\sum_{j=1}^k \alpha_j^p\big)^{1/p}$ for $1 < p < \infty$.*

Proof. Take $J \subseteq \{1, \ldots, k\}$ such that $\{m_j : j \in J\} = \{m_1, \ldots, m_k\}$ and $m_i \neq m_j$ for distinct $i, j \in J$, and set $K_j = \{i \in \{1, \ldots, k\} : m_i = m_j\}$ for each $j \in J$. Then $\{K_j : j \in J\}$ partitions $\{1, \ldots, k\}$, and we have $x = \sum_{j\in J}\big(\sum_{i\in K_j}\alpha_i\big)e_{m_j}$. Since $\{m_j : j \in J\}$ is a Schreier set, we conclude that $\|x\|_E = \sum_{j\in J}\big(\sum_{i\in K_j}\alpha_i\big) = \sum_{j=1}^k \alpha_j$ for $E = S_1$ and $E = B_p$, while

$$\|x\|_{S_p}^p = \sum_{j\in J}\Big(\sum_{i\in K_j}\alpha_i\Big)^p \geq \sum_{j\in J}\sum_{i\in K_j}\alpha_i^p = \sum_{j=1}^k \alpha_j^p$$

for $1 < p < \infty$, where the inequality follows from the fact that the ℓ_1-norm dominates the ℓ_p-norm. □

Lemma 3.3.5. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$, let $M \in [\mathbb{N}]$, and suppose that $\theta\colon M \to \mathbb{N}$ is a map for which the linear map $U\colon \operatorname{span}(e_m : m \in M) \to E$ determined by $Ue_m = e_{\theta(m)}$ for $m \in M$ is bounded. Then*

$$\sup\{|\theta^{-1}(n)| : n \in \mathbb{N}\} < \infty \qquad \textit{and} \qquad \sup\{\tau_1(\theta^{-1}(F)) : F \in \mathcal{S}_1\} < \infty. \tag{3.8}$$

Proof. The hypothesis means that U extends uniquely to an operator in $\mathscr{B}(E_M, E)$, also denoted U. We begin by showing that the second supremum in (3.8) is finite. Note that this will include showing that the pre-image under θ of every Schreier set F is finite, as otherwise $\tau_1(\theta^{-1}(F))$ is not defined. Our strategy is as follows: given $F \in \mathcal{S}_1$, we take $m \in \mathbb{N}$ for which $\theta^{-1}(F)$ contains a chain $\{G_1 < G_2 < \cdots < G_m\}$ of maximal Schreier sets. As we shall verify below, m is then dominated by a constant times a power of the norm of the operator U; that is, $m \leq C\|U\|^t$ for some constants $C, t \in (0, \infty)$ that will depend only on p. This will give the desired conclusion because (i) if $\theta^{-1}(F)$ were infinite, it would contain arbitrarily long chains of maximal Schreier sets, contradicting the uniform bound on m; (ii) we can therefore use the characterization of τ_1 stated in the paragraph below its definition (3.1) to deduce that $\tau_1(\theta^{-1}(F)) \leq C\|U\|^t + 1$. Since the right-hand side of this inequality is independent of F, it provides an upper bound on the second supremum in (3.8).

It remains to establish the inequality $m \leq C\|U\|^t$. We consider the two types of spaces separately. For $E = B_p$, set $x = \sum_{k=1}^m |G_k|^{-1} \sum_{j \in G_k} e_j$, and recall from (2.2) that $\|x\|_{B_p}^p \leq 2^p m$. Consequently, we have

$$2^p m \|U\|^p \geq \|Ux\|_{B_p}^p = \left\| \sum_{k=1}^m \frac{1}{|G_k|} \sum_{j \in G_k} e_{\theta(j)} \right\|_{B_p}^p = \left(\sum_{k=1}^m \frac{|G_k|}{|G_k|} \right)^p = m^p, \tag{3.9}$$

where the penultimate step follows from Lemma 3.3.4 and the fact that $\bigcup_{k=1}^m \theta(G_k)$ is a Schreier set because it is contained in $F \in \mathcal{S}_1$. Rearranging (3.9), we obtain $m \leq (2\|U\|)^{\frac{p}{p-1}}$, which provides an upper bound on m of the desired form for $C = 2^{\frac{p}{p-1}}$ and $t = p/(p-1)$.

The argument for $E = S_p$ is very similar, except that we use the vector

$$x = \sum_{k=1}^{m} \frac{1}{|G_k|^{\frac{1}{p}}} \sum_{j \in G_k} e_j.$$

It has S_p-norm at most $2^{\frac{1}{p}}$ by (2.1), so

$$2\|U\|^p \geq \|Ux\|_{S_p}^p = \left\| \sum_{k=1}^{m} \frac{1}{|G_k|^{\frac{1}{p}}} \sum_{j \in G_k} e_{\theta(j)} \right\|_{S_p}^p \geq \sum_{k=1}^{m} \frac{|G_k|}{|G_k|} = m$$

by another application of Lemma 3.3.4. This establishes the desired upper bound on m for $C = 2$ and $t = p$, thereby completing our proof that the second supremum in (3.8) is finite.

We now turn our attention to the first supremum in (3.8). Assume towards a contradiction that the set $\{|\theta^{-1}(n)| : n \in \mathbb{N}\}$ is unbounded. Given a non-empty set $G \subseteq \mathbb{N}$, it will be convenient to introduce the notation $G^\dagger = G \setminus \{\min G\}$. Arguing as in the proof of [29, Proposition 3.11], we can recursively construct an increasing sequence of maximal Schreier sets $G_1 < G_2 < \cdots$, each contained in $M \cap [2, \infty)$, and a sequence $(n_j)_{j \in \mathbb{N}}$ of natural numbers such that

$$\theta(i) = n_j \qquad (j \in \mathbb{N},\, i \in G_j^\dagger). \tag{3.10}$$

We include the details of this recursion for the reader's convenience. Set $m_1 = \min M \cap [2, \infty)$. By hypothesis, we can choose a number $n_1 \in \mathbb{N}$ such that $|\theta^{-1}(n_1)| > m_1$, so we can find a subset $F_1 \subseteq \theta^{-1}(n_1) \setminus \{1, m_1\}$ of cardinality $m_1 - 1$. Then $G_1 = \{m_1\} \cup F_1 \subseteq M \cap [2, \infty)$ is a maximal Schreier set, and (3.10) is satisfied for $j = 1$ because $G_1^\dagger = F_1$.

Now assume recursively that $G_1 < \cdots < G_{j-1}$ have been chosen for some $j \geq 2$. Set $m_j = \min M \cap (\max G_{j-1}, \infty)$, choose $n_j \in \mathbb{N}$ such that $|\theta^{-1}(n_j)| \geq m_j + |M \cap [1, m_j)|$, and take a subset $F_j \subseteq \theta^{-1}(n_j) \cap (m_j, \infty)$ of cardinality $m_j - 1$. Then $G_j = \{m_j\} \cup F_j \subseteq M \cap [m_j, \infty)$ is a maximal Schreier set such that (3.10) is satisfied for the given value of j, and $G_j > G_{j-1}$ because $\min G_j = m_j > \max G_{j-1}$. Hence the recursion continues.

Choose an integer m such that $m > (4\|U\|)^{\frac{p}{p-1}}$ if $E = B_p$ and $m > 4\|U\|^p$ if $E = S_p$. We observe that the set $\{n_j : j \in \mathbb{N}\}$ is unbounded, or else we could take $n \in \mathbb{N}$ and $J \in [\mathbb{N}]$ such that $n_j = n$ for every $j \in J$, which would imply that $\bigcup_{j\in J} G_j^\dagger \subseteq \theta^{-1}(n)$, contradicting that $\theta^{-1}(n)$ is finite, as shown in the first part of the proof. Consequently, we can find a set $K \in [\mathbb{N}]^{<\infty}$ such that $|K| = m \leq \min\{n_k : k \in K\}$, and therefore $\{n_k : k \in K\} \in \mathcal{S}_1$.

For $E = B_p$, consider the vector $y = \sum_{k\in K}|G_k|^{-1} \sum_{j\in G_k^\dagger} e_j \in B_p$, which has norm at most $2m^{\frac{1}{p}}$ by (2.2) and the 1-unconditionality of the basis $(e_j)_{j\in\mathbb{N}}$. We can now argue as in (3.9) to obtain

$$2^p m\|U\|^p \geq \|Uy\|_{B_p}^p = \Big\|\sum_{k\in K} \frac{|G_k|-1}{|G_k|} e_{n_k}\Big\|_{B_p}^p = \Big(\sum_{k\in K} \frac{|G_k|-1}{|G_k|}\Big)^p \geq \Big(\frac{m}{2}\Big)^p,$$

where we have used (3.10), Lemma 3.3.4 and the fact that $|G_k| - 1 \geq |G_k|/2$ for every $k \in \mathbb{N}$. Rearranging this inequality, we find $4^p\|U\|^p \geq m^{p-1}$, which contradicts our choice of m.

Again, the argument for $E = S_p$ is very similar, just using the vector

$$y = \sum_{k\in K} \frac{1}{|G_k|^{\frac{1}{p}}} \sum_{j\in G_k^\dagger} e_j \in S_p,$$

whose norm is at most $2^{\frac{1}{p}}$. Following the same steps as above, we obtain

$$2\|U\|^p \geq \|Uy\|_{S_p}^p = \Big\|\sum_{k\in K} \frac{|G_k|-1}{|G_k|^{\frac{1}{p}}} e_{n_k}\Big\|_{S_p}^p \geq \sum_{k\in K} \frac{(|G_k|-1)^p}{|G_k|} \geq \frac{m}{2},$$

once again contradicting the choice of m. ☐

Proposition 3.3.6. *Let $M, N \in [\mathbb{N}]$, and suppose that there exists a map $\theta\colon M \to N$ for which*

$$\sup\{|\theta^{-1}(n)| : n \in N\} < \infty \qquad \textit{and} \qquad \sup\{\tau_1(\theta^{-1}(F)) : F \in \mathcal{S}_1 \cap [N]^{<\infty}\} < \infty. \tag{3.11}$$

Then $\Gamma L_1(M, N) < \infty$.

Proof. This is a restatement of [29, Proposition 3.12] for $\xi = 1$, bearing in mind that

$$\sup\{|\theta^{-1}(n)| : n \in N\} = \sup\{\tau_0(\theta^{-1}(F)) : F \in \mathcal{S}_0 \cap [N]^{<\infty}\}$$

because $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\tau_0(A) = |A|$ for every $A \in [\mathbb{N}]^{<\infty}$. $\square$

Proof of Theorem 3.3.1*,* (c)$\Rightarrow$(a)*.* Suppose that $T \in \mathscr{B}(E_M, E_N)$ is an operator for which

$$\inf_{m\in M} \|Te_m\|_\infty > 0.$$

Then T satisfies condition (3.5) with respect to the bases $(e_m)_{m\in M}$ and $(e_n)_{n\in N}$ for E_M and E_N, respectively, so Lemma 3.3.2 shows that there is an operator $U \in \mathscr{B}(E_M, E_N)$ for which $Ue_m = e_{\theta(m)}$ for every $m \in M$, where $\theta(m) \in N$ is a suitably chosen index. Regarding θ as a map of M into $\mathbb{N}$, we can apply Lemma 3.3.5 to deduce that both suprema in (3.8) are finite. However, they are equal to the suprema in (3.11) because $\theta(M) \subseteq N$, so Proposition 3.3.6 implies that $\Gamma L_1(M, N) < \infty$, as required. $\square$

With the proof of Theorem 3.3.1 complete, we state an important consequence of it that is the second main outcome of this section.

Theorem 3.3.7. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \le p < \infty$. The following conditions are equivalent for $M, N \in [\mathbb{N}]$:*

(a) *The Gasparis–Leung indices $\Gamma L_1(M, N)$ and $\Gamma L_1(N, M)$ are both finite.*

(b) *The basic sequences $(e_m)_{m\in M}$ and $(e_n)_{n\in N}$ are equivalent.*

(c) *The subspaces E_M and E_N are isomorphic.*

As already indicated, we shall deduce this result from Theorem 3.3.1. However, the implication (c)$\Rightarrow$(a) requires one additional ingredient: every isomorphic embedding of E_M into E satisfies the technical condition (c) of Theorem 3.3.1.

Lemma 3.3.8. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \le p < \infty$, and suppose that $T \in \mathscr{B}(E_M, E)$ is an isomorphic embedding for some set $M \in [\mathbb{N}]$. Then $\inf_{m\in M}\|Te_m\|_\infty > 0$.*

Proof. Assume towards a contradiction that $\inf_{m\in M}\|Te_m\|_\infty = 0$ for some $M \in [\mathbb{N}]$ and some isomorphic embedding $T \in \mathscr{B}(E_M, E)$. Take $\eta > 0$ such that $\|Tx\| \geq \eta\|x\|$ for every $x \in E_M$, and set $k_0 = 0$, $P_0 = 0$ and $\varepsilon_j = \eta/(3\cdot 2^j + 1)$ for $j \in \mathbb{N}$. We can then recursively choose increasing sequences $(m_j)_{j\in\mathbb{N}}$ in M and $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\|Te_{m_j}\|_\infty \leq \frac{\varepsilon_j}{2(k_{j-1}+1)} \qquad \text{and} \qquad \|(I_E - P_{k_j})Te_{m_j}\|_E \leq \frac{\varepsilon_j}{2} \qquad (j \in \mathbb{N}).$$

This implies that for each $j \in \mathbb{N}$, the vector $v_j = (P_{k_j} - P_{k_{j-1}})Te_{m_j} \in E$ satisfies

$$\begin{aligned}\|Te_{m_j} - v_j\|_E &\leq \|(I_E - P_{k_j})Te_{m_j}\|_E + \|P_{k_{j-1}}Te_{m_j}\|_E \\ &\leq \frac{\varepsilon_j}{2} + k_{j-1}\cdot \max_{1\leq n\leq k_{j-1}} |\langle Te_{m_j}, e_n^*\rangle| \leq \frac{\varepsilon_j}{2} + \frac{\varepsilon_j}{2} = \varepsilon_j.\end{aligned}$$

In particular we have

$$\|T\| \geq \|v_j\|_E \geq \|Te_{m_j}\|_E - \|Te_{m_j} - v_j\|_E \geq \eta - \varepsilon_j \geq \frac{6\eta}{7},$$

so $(v_j)_{j\in\mathbb{N}}$ is a semi-normalized block basic sequence of $(e_n)_{n\in\mathbb{N}}$. Furthermore, since

$$\sum_{j=1}^\infty \frac{\|Te_{m_j} - v_j\|_E}{\|v_j\|_E} \leq \sum_{j=1}^\infty \frac{\varepsilon_j}{\eta - \varepsilon_j} = \sum_{j=1}^\infty \frac{1}{3\cdot 2^j} = \frac{1}{3} < \frac{1}{2},$$

the Principle of Small Perturbations (see for instance [1, Theorem 1.3.9]) implies that $(Te_{m_j})_{j\in\mathbb{N}}$ is a basic sequence equivalent to $(v_j)_{j\in\mathbb{N}}$ (here we have used the fact that the basis constant of $(v_j)_{j\in\mathbb{N}}$ is no greater than the basis constant of $(e_n)_{n\in\mathbb{N}}$, which is 1).

Set $u_j = v_j/\|v_j\|_E$ for $j \in \mathbb{N}$. Being semi-normalized and unconditional, $(v_j)_{j\in\mathbb{N}}$ is equivalent to $(u_j)_{j\in\mathbb{N}}$, and

$$\|u_j\|_\infty = \frac{\|(P_{k_j} - P_{k_{j-1}})Te_{m_j}\|_\infty}{\|v_j\|_E} \leq \frac{\|Te_{m_j}\|_\infty}{6\eta/7} \leq \frac{7}{12(3\cdot 2^j + 1)(k_{j-1}+1)},$$

which converges to 0 as $j \to \infty$, so $\inf_{j\in\mathbb{N}}\|u_j\|_\infty = 0$. Hence Proposition 2.4.15 implies that $(u_j)_{j\in\mathbb{N}}$ admits a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ which is equivalent to the unit vector basis $(d_n)_{n\in\mathbb{N}}$ for D, where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$, as usual.

In conclusion, we have shown that $(d_n)_{n\in\mathbb{N}}$ is equivalent to $(u_{j_n})_{n\in\mathbb{N}}$, which is equivalent to $(v_{j_n})_{n\in\mathbb{N}}$, which is equivalent to $(Te_{m_{j_n}})_{n\in\mathbb{N}}$, and therefore $(e_{m_{j_n}})_{n\in\mathbb{N}}$

is equivalent to $(d_n)_{n\in\mathbb{N}}$ because T is an isomorphic embedding. However, this is absurd: no subsequence of $(e_n)_{n\in\mathbb{N}}$ is dominated by $(d_n)_{n\in\mathbb{N}}$, as is easy to see (or alternatively this is a very special case of Proposition 2.4.15). $\square$

Corollary 3.3.9. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \le p < \infty$, and suppose that E_M embeds isomorphically into E_N for some sets $M, N \in [\mathbb{N}]$. Then $\Gamma L_1(M, N) < \infty$.*

Proof. Take an isomorphic embedding $T \in \mathscr{B}(E_M, E_N)$, and let $J\colon E_N \to E$ denote the inclusion map. Then we have $0 < \inf_{m\in M}\|JTe_m\|_\infty = \inf_{m\in M}\|Te_m\|_\infty$ by Lemma 3.3.8, so the implication (c)$\Rightarrow$(a) in Theorem 3.3.1 shows that $\Gamma L_1(M, N) < \infty$. $\square$

Proof of Theorem 3.3.7. The equivalence of conditions (a) and (b) in Theorem 3.3.1 implies that conditions (a) and (b) are also equivalent in Theorem 3.3.7. The implication (b)$\Rightarrow$(c) is clear, and finally Corollary 3.3.9 shows that (c) implies (a). $\square$

The equivalence of conditions (b) and (c) in Theorem 3.3.7 for the unit vector basis can be interpreted as a rather strong property in the following sense; even within the Baernstein and Schreier spaces, we claim there exist other unconditional bases which fail to satisfy this property. In part, this is due to the following deep result of Lindenstrauss and Zippin.

Theorem 3.3.10. *[45, Theorem 2.b.10] A Banach space has, up to equivalence, a unique unconditional basis if and only if it is isomorphic to either c_0, ℓ_1 or ℓ_2.*

Letting E be as in Theorem 3.3.7, there must therefore exist an unconditional basis (x_n) in E which is non-equivalent to (e_n). Then for the space $E \oplus E$, the basis given by $y_{2n} := x_n$ and $y_{2n-1} := e_n$ is unconditional, $(E \oplus E)_{2\mathbb{N}}$ and $(E \oplus E)_{2\mathbb{N}-1}$ are isomorphic, yet the corresponding subsequences of (y_n) are non-equivalent. In Lemma 4.2.3 we shall see that $E \oplus E$ is actually isomorphic to E, so that our claim is established.

We conclude this section with two applications of Corollary 3.3.9, both establishing counterparts for the Baernstein and Schreier spaces of results of Gasparis and Leung concerning the structure of the complemented subspaces of the higher-order Schreier spaces.

Definition 3.3.11. (i) A Banach space X is *primary* if the kernel or the range of P is isomorphic to X for every idempotent operator $P \in \mathscr{B}(X)$.

(ii) Two Banach spaces X and Y are *incomparable* if no subspace of X is isomorphic to Y and no subspace of Y is isomorphic to X.

Proposition 3.3.12. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \le p < \infty$. Then:*

(i) *The subspace E_N fails to be primary for every $N \in [\mathbb{N}]$.*

(ii) *There is a subset $\mathcal{A}$ of $[\mathbb{N}]$ of cardinality $\mathfrak{c}$ such that E_L and E_M are incomparable whenever $L, M \in \mathcal{A}$ are distinct.*

Proof. We follow the approach Gasparis and Leung took in their proofs of [29, Corollary 3.15 and Theorem 1.3], respectively. The common starting point is that for any set $N \in [\mathbb{N}]$, we can equip $[N]$ with the topology of pointwise convergence obtained by identifying the elements of $[N]$ with their indicator functions; this turns $[N]$ into a Polish space.

(i). The set

$$\mathcal{F} = \{(L, M) \in [N] \times [N] : L \cup M = N,\ L \cap M = \emptyset\}$$

is closed with respect to the product topology on $[N] \times [N]$. Define

$$\mathcal{G} = \{(L, M) \in \mathcal{F} : \Gamma L_1(N, L) = \Gamma L_1(N, M) = \infty\}$$

By an argument similar to [29, Lemma 3.5], the set $\mathcal{G}$ is a G_δ–dense set. In particular $\mathcal{G}$ is non-empty, so we can take $(L, M) \in \mathcal{G}$. The fact that $N = L \cup M$ and $L \cap M = \emptyset$ implies that $E_N = E_L \oplus E_M$, but E_N is neither isomorphic to E_L

nor E_M by Corollary 3.3.9 because $\Gamma L_1(N, L) = \Gamma L_1(N, M) = \infty$. (In fact, E_L and E_M do not even contain subspaces which are isomorphic to E_N.) This proves that E_N is not primary.

(ii). By [29, Lemma 3.5],

$$\mathcal{D} = \{(L, M) \in [\mathbb{N}] \times [\mathbb{N}] : \Gamma L_1(L, M) = \Gamma L_1(M, L) = \infty\}$$

is a dense G_δ-subset of $[\mathbb{N}] \times [\mathbb{N}]$. Therefore, applying [29, Proposition 3.6], we can find a subset $\mathcal{A}$ of $[\mathbb{N}]$ which is homeomorphic to the Cantor set and satisfies $(L, M) \in \mathcal{D}$ whenever $L, M \in \mathcal{A}$ are distinct. In particular $\mathcal{A}$ has cardinality $\mathfrak{c}$, and Corollary 3.3.9 shows that E_L and E_M are incomparable for distinct $L, M \in \mathcal{A}$ because $\Gamma L_1(L, M) = \Gamma L_1(M, L) = \infty$. $\square$

Chapter 4

Closed ideals in the algebra of operators on Baernstein and Schreier spaces

4.1 Statement of the main result and background

We are now in a position to apply the results of previous chapters, to study closed ideals in the algebra $\mathscr{B}(E)$ where $E = S_p$ for $1 \leq p < \infty$ or $E = B_p$ for $1 < p < \infty$. In particular, our main objective in this chapter will be to prove the following theorem.

Theorem 4.1.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$. Then:*

(i) *$\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ many closed ideals between the ideals of compact and strictly singular operators.*

(ii) *$\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ many closed ideals which are larger than the ideal*

$$\{UV : U \in \mathscr{B}(D, E),\, V \in \mathscr{B}(E, D)\}$$

of operators factoring through D, where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$.

(iii) *$\mathscr{B}(E)$ contains at least continuum many maximal ideals.*

Beanland, Kania and Laustsen [6] used the results from [29] to demonstrate that the family of closed ideals of $\mathscr{B}(X[\mathcal{S}_n])$ that are singly generated by basis projections has a very rich structure for every $n \in \mathbb{N}$. We show that by referring to Section 3.3 instead of [29], we can transfer the arguments from [6] to the Baernstein and Schreier spaces; Theorem 4.2.4 states our main conclusions, which include Theorem 4.1.1(iii).

Answering a question raised in [6], Manoussakis and Pelczar-Barwacz [48] combined the results from [29] with the seminal idea of Johnson and Schechtman [36] to prove that $\mathscr{B}(X[\mathcal{S}_n])$ contains $2^{\mathfrak{c}}$ many closed ideals that lie between the ideals of compact and strictly singular operators for every $n \in \mathbb{N}$. Theorem 4.1.1(i) is the analogue of this result for the Baernstein and Schreier spaces. We prove it in Section 4.3, together with Theorem 4.1.1(ii), whose proof turns out to be the easier of the two. The reason is that Theorem 4.1.1(i) requires the non-trivial fact that the formal inclusion map from B_p into ℓ_p is strictly singular. In contrast to Manoussakis and Pelczar-Barwacz, we express our arguments in terms of the numerical index of Gasparis and Leung, thereby elucidating their combinatorial nature and providing a blueprint for other Banach spaces admitting a suitable index.

To provide additional context and background for our results, we conclude this introduction with a survey of separable Banach spaces X for which the Banach algebra $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals. As far as we know, Gowers' hyperplane space X_{G} originally introduced in [32] is the first example of this kind; more precisely, Laustsen in [41, Theorem 8.4] classified the maximal ideals of $\mathscr{B}(X_{\mathrm{G}})$ and noted that there are $2^{\mathfrak{c}}$ of them.

A major breakthrough occurred when Johnson and Schechtman [36] showed that $\mathscr{B}(L_p[0,1])$ contains $2^{\mathfrak{c}}$ many closed ideals for every $p \in (1,2)\cup(2,\infty)$. Their key technique has proved very versatile and spawned many new results. Theorem 4.3.1 states a variant of it, formulated by Freeman, Schlumprecht and Zsák [28], who used it to verify that $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals for a number of direct sums of Banach spaces, notably $X = \ell_p \oplus \ell_q$, $X = \ell_q \oplus c_0$ and $X = \ell_q \oplus \ell_\infty$ for $1 \leq p < q < \infty$, as well as the Hardy space H_1 and its predual VMO.

Also building on the Johnson–Schechtman technique, Manoussakis and Pelczar-Barwacz [48] showed that $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals for Schlumprecht's arbitrarily distortable Banach space [62] and the higher-order Schreier spaces $X[\mathcal{S}_n]$ for $n \in \mathbb{N}$, as already mentioned. In collaboration with Causey, Pelczar-Barwacz [12] has subsequently extended the latter result to the Schreier spaces $X[\mathcal{S}_\xi]$ of any countable order ξ, as well as their duals and biduals.

Finally, Chu and Schlumprecht [16] have shown that $\mathscr{B}(T[\mathcal{S}_\xi,\theta])$ contains $2^{\mathfrak{c}}$ many closed ideals for every countable ordinal ξ and $0 < \theta < 1$, where $T[\mathcal{S}_\xi,\theta]$ denotes the Tsirelson space of order ξ, as defined by Alspach and Argyros [2].

4.2 Spatial ideals of operators on the Baernstein and Schreier spaces

Let X be a Banach space. We write $\langle T\rangle$ for the (algebraic, two-sided) ideal of $\mathscr{B}(X)$ generated by an operator $T \in \mathscr{B}(X)$, that is,

$$\langle T\rangle = \left\{\sum_{j=1}^{k} U_j T V_j : k \in \mathbb{N},\, U_1,\ldots,U_k, V_1,\ldots,V_k \in \mathscr{B}(X)\right\}. \tag{4.1}$$

Since $\mathscr{B}(X)$ is a unital Banach algebra, the ideal $\langle T\rangle$ is proper if and only if its norm-closure $\overline{\langle T\rangle}$ is. Suppose that X has an unconditional basis. Following [6], we call the closed ideals of the form $\overline{\langle P_M\rangle}$ for some non-empty subset M of $\mathbb{N}$ *spatial*, where P_M denotes the basis projection, as usual.

The main aim of this section is to prove the following proposition, which is an extended counterpart of [6, Proposition 4.12] for the Baernstein and Schreier spaces. The key difference is the addition of a new quantitative condition, (d), that will play an essential role in the proofs of parts (i) and (ii) of Theorem 4.1.1 in the next section. We remind the reader that for $E = B_p$ or S_p and $M \in [\mathbb{N}]$ we define

$$E_M := \overline{\operatorname{span}}(e_m : m \in M).$$

Proposition 4.2.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$. The following conditions are equivalent for every pair of sets $M \subseteq \mathbb{N}$ and $N \in [\mathbb{N}]$:*

(a) $P_M \in \overline{\langle P_N\rangle}$,

(b) $\langle P_M\rangle \subseteq \langle P_N\rangle$,

(c) $\langle P_N\rangle = \langle P_{M\cup N}\rangle$,

(d) $\operatorname{dist}(P_M, \langle P_N\rangle) < 1$,

(e) $\Gamma L_1(M \cup N, N) < \infty$,

(f) *E_N contains a subspace which is isomorphic to E_M,*

(g) *E_N contains a complemented subspace which is isomorphic to E_M,*

(h) *E_N is isomorphic to $E_{M\cup N}$,*

(i) *The basic sequences $(e_n)_{n\in M\cup N}$ and $(e_n)_{n\in N}$ are equivalent.*

We require two lemmas in the proof of this proposition. The first is a variant of the Neumann series, showing that every idempotent element which is close to an ideal of a Banach algebra must in fact belong to the ideal.

Lemma 4.2.2. *Let $\mathscr{I}$ be an ideal of a Banach algebra $\mathscr{A}$, and take a non-zero idempotent $p \in \mathscr{A}$. Then $p \in \mathscr{I}$ if (and only if)* $\operatorname{dist}(p, \mathscr{I}) < \|p\|^{-2}$.

Proof. The implication $\Rightarrow$ is obvious. Conversely, suppose that $\|p - a\| < \|p\|^{-2}$ for some $a \in \mathscr{I}$. Then $\|p - pap\| < 1$, so the series $\sum_{n=1}^{\infty}(p - pap)^n$ converges absolutely. Set $b = p + \sum_{n=1}^{\infty}(p - pap)^n \in \mathscr{A}$ and observe that

$$\begin{aligned}\mathscr{I} \ni bpap &= \left(p + \sum_{n=1}^{\infty}(p - pap)^n\right)\big(p - (p - pap)\big)\\ &= p - p(p - pap) + \sum_{n=1}^{\infty}(p - pap)^n p - \sum_{n=2}^{\infty}(p - pap)^n = p.\end{aligned}$$

□

The second lemma is the counterpart of [6, Proposition 4.6]. It will enable us to connect the first four conditions of Proposition 4.2.1 concerning ideals with the last four (or five) concerning subspaces.

Lemma 4.2.3. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \le p < \infty$. Then*

$$E_M \approx E_M \oplus E_M \qquad (M \in [\mathbb{N}]).$$

Proof. We take the same approach as in the proof of [6, Proposition 4.6]. Set $M' = 2M - 1$ and $M'' = 2M$. Since these sets are disjoint, we have

$$E_{M'\cup M''} = E_{M'} \oplus E_{M''}$$

by unconditionality, so it will suffice to show that each of these spaces is isomorphic to E_M, which in turn will follow from Theorem 3.3.7 provided that the appropriate

Gasparis–Leung indices are finite. First, we have $\Gamma L_1(M, M' \cup M'') \leq 3$ and $\Gamma L_1(M' \cup M'', M) \leq 2$ by [6, Lemma 4.11], so $E_{M'\cup M''} \approx E_M$. Second, M'' is a spread of M and of M', so $\Gamma L_1(M'', M) = 1 = \Gamma L_1(M'', M')$. Third, we claim that

$$\Gamma L_1(M, M'') \leq 2 \qquad \text{and} \qquad \Gamma L_1(M', M'') \leq 2. \tag{4.2}$$

Indeed, suppose that $M''(J) \in \mathcal{S}_1$ for some non-empty $J \in [\mathbb{N}]^{<\infty}$. Then $2m_{j_1} \geq k$, where we have written $J = \{j_1 < \cdots < j_k\}$ and $M = \{m_1 < m_2 < \cdots\}$. This implies that we can partition J into two successive subsets, J_1 and J_2, each having at most m_{j_1} elements, and therefore $M(J_1), M(J_2), M'(J_1), M'(J_2) \in \mathcal{S}_1$. Hence we have $\tau_1(M(J)) \leq 2$ and $\tau_1(M'(J)) \leq 2$ because $M(J) = M(J_1) \cup M(J_2)$ and $M'(J) = M'(J_1) \cup M'(J_2)$. This proves (4.2), and consequently $E_M \approx E_{M''} \approx E_{M'}$. □

Proof of Proposition 4.2.1. Lemma 4.2.3 implies that $E_N \approx E_N \oplus E_N$ and $E_{M\cup N} \approx E_{M\cup N} \oplus E_{M\cup N}$ because N is infinite. Hence conditions (a), (b), (c), (g) and (h) are equivalent by [6, Lemma 2.3 and Corollary 2.5].

We have $\Gamma L_1(N, M \cup N) = 1$ because N is a spread of $M \cup N$, so conditions (e), (h) and (i) are equivalent by Theorem 3.3.7.

The implications (a)⇒(d) and (g)⇒(f) are trivial, while Lemma 4.2.2 shows that (d) implies (b). We complete the proof by showing that (f) implies (e). Suppose that E_M embeds isomorphically into E_N. Then $E_{M\cup N} = E_M \oplus E_{N\setminus M}$ embeds isomorphically into $E_N \oplus E_N \approx E_N$, so $\Gamma L_1(M \cup N, N) < \infty$ by Corollary 3.3.9. □

Proposition 4.2.1 enables us to establish a counterpart for the Baernstein and Schreier spaces of the main result of [6]. This requires one additional piece of terminology. Let X be a Banach space with an unconditional basis. The ideal $\mathscr{K}(X)$ of compact operators is always spatial because $\overline{\langle P_M \rangle} = \mathscr{K}(X)$ if (and only if) $M \in [\mathbb{N}]^{<\infty} \setminus \{\emptyset\}$. Following [6], we call a spatial ideal $\mathscr{I}$ *non-trivial* if $\mathscr{K}(X) \subsetneq \mathscr{I} \subsetneq \mathscr{B}(X)$.

Theorem 4.2.4. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$. Then:*

(i) *The family of non-trivial spatial ideals of* $\mathscr{B}(E)$ *is non-empty and has no minimal or maximal elements.*

(ii) *Let* $\mathscr{I} \subsetneq \mathscr{J}$ *be spatial ideals of* $\mathscr{B}(E)$*. Then there is a family* $\{\Gamma_L : L \in \Delta\}$ *such that:*

(1) *the index set* Δ *has the cardinality of the continuum;*

(2) *for each* $L \in \Delta, \Gamma_L$ *is an uncountable chain of spatial ideals of* $\mathscr{B}(E)$ *such that*

$$\mathscr{I} \subsetneq \mathscr{L} \subsetneq \mathscr{J} \qquad (\mathscr{L} \in \Gamma_L),$$

and $\bigcup \Gamma_L$ *is a closed ideal that is not spatial;*

(3) $\overline{\mathscr{L} + \mathscr{M}} = \mathscr{J}$ *whenever* $\mathscr{L} \in \Gamma_L$ *and* $\mathscr{M} \in \Gamma_M$ *for distinct* $L, M \in \Delta$.

(iii) *The Banach algebra* $\mathscr{B}(E)$ *contains at least continuum many maximal ideals.*

(iv) *The ideal*

$$\bigcap \{\mathscr{I} : \mathscr{I} \text{ is a non-trivial spatial ideal of } \mathscr{B}(E)\}$$

is not contained in the ideal of strictly singular operators on E.

Proof. Clauses (i)–(iii) are the counterparts for E of [6, Theorem 1.1] and can be proved in exactly the same way; see [6, pages 10–11]. This requires that we establish the counterpart of [6, Lemma 2.8] for E, which we can do by copying the proof given in [6, pages 21–24] for $n = 1$, just referring to Proposition 4.2.1 instead of [6, Proposition 4.12] throughout.

(iv) is the counterpart of [6, Theorem 1.2(ii), Equation (1.1)], and the proof is similar. Indeed, let $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$, and take a projection $Q \in \mathscr{B}(E)$ whose range is isomorphic to D. Then Q is not strictly singular, but Theorem 2.4.4 implies that $Q \in \langle P_N \rangle$ for every $N \in [\mathbb{N}]$, and therefore Q belongs to every non-trivial spatial ideal of $\mathscr{B}(E)$. $\square$

4.3 Finding $2^{\mathfrak{c}}$ many closed ideals of operators

In this section we combine our previous results about the Gasparis–Leung index and ideals generated by basis projections to prove the remaining two parts of Theorem 4.1.1; that is, taking $E = B_p$ for $1 < p < \infty$ or $E = S_p$ for $1 \leq p < \infty$, as usual, we shall show that $\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ many closed ideals that lie between the ideals of compact and strictly singular operators, as well as $2^{\mathfrak{c}}$ many closed ideals that are "large" in the sense that they contain projections of infinite rank. Note that $\mathscr{B}(E)$ cannot contain more than $2^{\mathfrak{c}}$ many closed ideals because E is separable.

Both results rely on a general theorem of Freeman, Schlumprecht and Zsák [28, Proposition 1] that extracts the key idea of the argument that Johnson and Schechtman [36] used to show that $\mathscr{B}(L_p[0,1])$ contains $2^{\mathfrak{c}}$ many closed ideals for every $p \in (1,2) \cup (2,\infty)$. Before we can state the said theorem of Freeman–Schlumprecht–Zsák precisely, we require two additional pieces of terminology. The first generalizes the classical notion of a (closed) ideal of a (Banach) algebra to the space of operators between two distinct Banach spaces X and Y: a *(closed) ideal* of $\mathscr{B}(X,Y)$ is a (norm-closed) subspace $\mathscr{J}$ of $\mathscr{B}(X,Y)$ such that $UTV \in \mathscr{J}$ whenever $V \in \mathscr{B}(X)$, $T \in \mathscr{J}$ and $U \in \mathscr{B}(Y)$. Extending (4.1), for a subset $\mathscr{T}$ of $\mathscr{B}(X,Y)$, we write $\langle \mathscr{T} \rangle$ for the ideal of $\mathscr{B}(X,Y)$ it generates.

The reason this generalization is useful for our purposes is that in the case where X contains a complemented subspace isomorphic to Y, the map

$$\mathscr{J} \mapsto \overline{\left\{ \sum_{j=1}^{n} U_j T_j : n \in \mathbb{N},\, U_1, \ldots, U_n \in \mathscr{B}(Y,X),\, T_1, \ldots, T_n \in \mathscr{J} \right\}} \tag{4.3}$$

is an injection from the lattice of closed ideals of $\mathscr{B}(X,Y)$ into the lattice of closed ideals of $\mathscr{B}(X)$. (This is a special case of an observation stated above [28, Proposition 2], and is also easy to verify directly.) Hence, to show that $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals, it suffices to find a complemented subspace Y of X for which $\mathscr{B}(X,Y)$ contains $2^{\mathfrak{c}}$ many closed ideals.

The second notion that we require is that of a 1-*unconditional finite-dimensional decomposition,* or 1-*UFDD* for short, of a Banach space X; that is, a sequence

$(X_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X such that every $x \in X$ has a unique decomposition of the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for every $n \in \mathbb{N}$, and the series $\sum_{n=1}^{\infty} \sigma_n x_n$ converges with $\|\sum_{n=1}^{\infty} \sigma_n x_n\| \leq \|x\|$ for every sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$. It follows that for every (non-empty) subset N of $\mathbb{N}$, we can define a projection $Q_N \in \mathscr{B}(X)$ of norm 1 by $Q_N x = \sum_{n\in N} x_n$.

In fact, we shall only consider 1-UFDDs of a very simple kind. Let X be a Banach space with a 1-unconditional basis $(x_n)_{n\in\mathbb{N}}$, and take a partition $J_1 < J_2 < \cdots$ of $\mathbb{N}$ into finite, successive intervals. Then the sequence of finite-dimensional subspaces given by

$$X_n = \operatorname{span}\{x_j : j \in J_n\} \qquad (n \in \mathbb{N}) \tag{4.4}$$

is a 1-UFDD for X. For later reference, we observe that in this case the projection Q_N, for $N \subseteq \mathbb{N}$, defined above is equal to the basis projection P_{L_N} induced by the set $L_N = \bigcup_{n\in N} J_n$.

Theorem 4.3.1 (Freeman, Schlumprecht and Zsák). *Let $\mathcal{A} \subset [\mathbb{N}]$ be an almost disjoint family of cardinality $\mathfrak{c}$, let X and Y be Banach spaces with 1-UFDDs $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$, respectively, and suppose that $T \in \mathscr{B}(X, Y)$ is an operator which satisfies*

(i) $T[X_n] \subseteq Y_n$ *for every* $n \in \mathbb{N}$;

(ii) $\inf\bigl\{\operatorname{dist}(TQ_M, \langle TQ_N\rangle) : M, N \in [\mathbb{N}], |M \setminus N| = \infty\bigr\} > 0.$

Then the map

$$\mathcal{N} \mapsto \overline{\langle TQ_N : N \in \mathcal{N}\rangle} \tag{4.5}$$

defines an order-preserving injection from the power set of $\mathcal{A}$ into the lattice of closed ideals of $\mathscr{B}(X, Y)$.

To enable us to apply this theorem to the Baernstein and Schreier spaces, we present a variant not involving dyadic trees of the key construction that Manoussakis and Pelczar-Barwacz used in their proof of [48, Lemma 4.3].

Construction 4.3.2. *Set $F_1 = \emptyset$. By recursion, we can partition $\mathbb{N}$ into finite, successive intervals $G_1 < F_2 < G_2 < F_3 < G_3 < \cdots$ with the following properties:*

(i) *G_n is the union of n successive maximal Schreier sets for each $n \in \mathbb{N}$ (so in particular $\tau_1(G_n) = n$),*

(ii) *$|F_n| = \sum_{m=1}^{n-1}(|F_m| + |G_m|)$ for each $n \geq 2$.*

For brevity, we introduce the notation $J_n = F_n \cup G_n$ for $n \in \mathbb{N}$ and set

$$L_N = \bigcup_{n\in N} J_n \qquad (N \subseteq \mathbb{N}). \tag{4.6}$$

We observe for later reference that we can rewrite property (ii) *as $|F_n| = \sum_{m=1}^{n-1}|J_m|$.*

Lemma 4.3.3. *For $M, N \in [\mathbb{N}]$, the set $M \setminus N$ is bounded above by $\Gamma L_1(L_M, L_N) \in \mathbb{N} \cup \{\infty\}$, where $L_M, L_N \in [\mathbb{N}]$ are defined by* (4.6)*.*

Proof. Take $m \in M \setminus N$. We seek to prove that $m \leq \Gamma L_1(L_M, L_N)$, which by the definition (3.2) of the Gasparis–Leung index means that we must find a set $K \in [\mathbb{N}]^{<\infty}$ such that

$$\tau_1(L_M(K)) \geq m \qquad \text{and} \qquad L_N(K) \in \mathcal{S}_1. \tag{4.7}$$

The case $m = 1$ is trivial, so we may suppose that $m \geq 2$. We have $J_m \subseteq L_M$ because $m \in M$, so we can find $H \in [\mathbb{N}]^{<\infty}$ such that $J_m = L_M(H)$. Note that H is an interval because J_m is, and the definition of J_m implies that we can split H in two subintervals $H_1 < H_2$ such that $L_M(H_1) = F_m$ and $L_M(H_2) = G_m$. We claim that $K = H_2$ satisfies (4.7).

The first part is immediate because $\tau_1(L_M(H_2)) = \tau_1(G_m) = m$, so it only remains to show that $L_N(H_2) \in \mathcal{S}_1$; that is, $|L_N(H_2)| \leq \min(L_N(H_2))$. Set $k = \min H_2$ and observe that $\min(L_N(H_2))$ is the k^{th} element of the set

$$L_N = \bigcup_{n\in N} J_n = \Big(\bigcup_{n\in N\cap[1,m)} J_n\Big) \cup \Big(\bigcup_{n\in N\cap(m,\infty)} J_n\Big)$$

because $m \notin N$. We have

$$\Big|\bigcup_{n\in N\cap[1,m)} J_n\Big| = \sum_{n\in N\cap[1,m)} |J_n| \le \sum_{j=1}^{m-1} |J_j| = |F_m| = |L_M(H_1)| = |H_1| \le \max H_1 < k,$$

so the k^{th} element of L_N must belong to the set $\bigcup_{n\in N\cap(m,\infty)} J_n$. Hence

$$\begin{aligned}\min(L_N(H_2)) &\ge \min\Big(\bigcup_{n\in N\cap(m,\infty)} J_n\Big) \ge \min J_{m+1} > \max J_m \\ &= \max G_m \ge |G_m| = |L_M(H_2)| = |H_2| = |L_N(H_2)|,\end{aligned}$$

and the conclusion follows. □

Corollary 4.3.4. *The following conditions are equivalent for $M, N \in [\mathbb{N}]$:*

(a) $\Gamma L_1(L_M, L_N) < \infty$,

(b) $\Gamma L_1(L_{M\cup N}, L_N) < \infty$,

(c) $|M \setminus N| < \infty$.

Proof. Lemma 4.3.3 shows that the set $M \setminus N$ is bounded above by $\Gamma L_1(L_M, L_N)$, so (a) implies (c).

(c)⇒(b). Suppose that $M \setminus N$ is finite. Then $L_{M\cup N} \setminus L_N = L_{M\setminus N}$ is finite, too, so $\Gamma L_1(L_{M\cup N}, L_N) < \infty$.

(b)⇒(a). This is a consequence of the fact that $\mathcal{S}_1$ is closed under spreading. □

Proof of Theorem 4.1.1(ii). We shall apply Theorem 4.3.1 with $X = Y = E$, $T = I_E$ and the 1-UFDDs given by $X_n = Y_n = \operatorname{span}\{e_j : j \in J_n\}$ for every $n \in \mathbb{N}$, where $J_1 < J_2 < \cdots$ are the intervals defined in Construction 4.3.2. These choices ensure that condition (i) of Theorem 4.3.1 is trivially satisfied. To verify condition (ii), we recall that for $N \subseteq \mathbb{N}$, the projection Q_N associated with the chosen 1-UFDDs is the basis projection P_{L_N}. Taking $M, N \in [\mathbb{N}]$ with $|M \setminus N| = \infty$, we have $\Gamma L_1(L_M \cup L_N, L_N) = \Gamma L_1(L_{M\cup N}, L_N) = \infty$ by Corollary 4.3.4, so Proposition 4.2.1 implies that $\operatorname{dist}(Q_M, \langle Q_N\rangle) = \operatorname{dist}(P_{L_M}, \langle P_{L_N}\rangle) = 1$. Hence Theorem 4.3.1 shows that

$$\big\{\overline{\langle P_{L_N} : N \in \mathcal{N}\rangle} : \mathcal{N} \subseteq \mathcal{A}\big\}$$

is a collection of $2^{\mathfrak{c}}$ many distinct closed ideals of $\mathscr{B}(E)$ for any almost disjoint family $\mathcal{A} \subset [\mathbb{N}]$ of cardinality $\mathfrak{c}$.

The set $\{UV : U \in \mathscr{B}(D, E), V \in \mathscr{B}(E, D)\}$ is closed under addition and therefore an ideal of $\mathscr{B}(E)$ because $D \approx D \oplus D$, where we recall that $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$, as usual. An easy standard argument shows that this ideal is equal to $\langle Q \rangle$ for any projection $Q \in \mathscr{B}(E)$ whose range is isomorphic to D. As we saw in the proof of Theorem 4.2.4(iv), $Q \in \langle P_N \rangle$ for every $N \in [\mathbb{N}]$, so $\langle Q \rangle \subseteq \overline{\langle P_{L_N} : N \in \mathcal{N} \rangle}$ for every non-empty subset $\mathcal{N}$ of $\mathcal{A}$. □

Whilst the spatial ideals identified in Theorem 4.2.4 guaranteed the existence of $\mathfrak{c}$ maximal ideals in $\mathscr{B}(E)$, the $2^{\mathfrak{c}}$ large ideals constructed in Theorem 4.1.1 offer no additional insight regardless of the almost disjoint family. In particular, we now explain why all of the ideals constructed in Theorem 4.3.1 reside within a *single* maximal ideal.

Proposition 4.3.5. *Let $\mathcal{A} \subset [\mathbb{N}]$ be an almost disjoint family of cardinality $\mathfrak{c}$. Let X, Y be Banach spaces with 1-UFDDs $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ respectively. Suppose $T \in \mathscr{B}(X, Y)$ satisfies the conditions of Theorem 4.3.1. Then the ideal $\overline{\langle TQ_N : N \in \mathcal{A} \rangle}$ is proper in $\mathscr{B}(X, Y)$.*

Proof. We claim that $T \notin \overline{\langle TQ_N : N \in \mathcal{A} \rangle}$. By Theorem 4.3.1 there exists $\eta > 0$ such that whenever $|M \setminus N| = \infty$ we have

$$\operatorname{dist}(TQ_M, \langle TQ_N \rangle) \geq \eta.$$

Towards a contradiction we suppose $T \in \overline{\langle TQ_N : N \in \mathcal{A} \rangle}$. Then we can identify $S \in \langle TQ_N : N \in \mathcal{A} \rangle$ such that $\|T - S\| < \eta$. Write

$$S = \sum_{j=1}^{k} U_j TQ_{N_j} V_j \qquad \text{where } N_1, \ldots, N_k \in \mathcal{A}.$$

Let $N = N_1 \cup \cdots \cup N_k$. Then we can re-write S as

$$S = \sum_{j=1}^{k} U_j TQ_N Q_{N_j} V_j \in \langle TQ_N \rangle$$

But this implies

$$\eta > \|T - S\| \geq \operatorname{dist}(TQ_{\mathbb{N}}, \langle TQ_N \rangle).$$

Subsequently, this would imply that $|\mathbb{N} \setminus N| < \infty$ which is impossible for $\mathcal{A}$. $\square$

The proof of Theorem 4.1.1(i) follows a similar path, but some aspects require additional work. We begin with a standard characterization of strictly singular operators defined on a Banach space with a basis, and include a short proof for completeness.

Lemma 4.3.6. *Let $T \in \mathscr{B}(X, Y)$ be an operator between Banach spaces X and Y, and suppose that X has a basis. Then T is strictly singular if (and only if) the restriction of T to any block subspace of X fails to be an isomorphic embedding.*

Proof. The implication $\Rightarrow$ is trivial because block subspaces are infinite-dimensional.

Conversely, suppose that T fails to be strictly singular, so that its restriction to some closed, infinite-dimensional subspace Z of X is bounded below by some number $\eta > 0$. We use the same notation and approach as in the first part of the proof of Lemma 2.4.6; that is, $(x_n)_{n\in\mathbb{N}}$ denotes the basis of X, K is the basis constant, P_n is the n^{th} basis projection for $n \in \mathbb{N}$, and we set $m_0 = 0$ and $P_0 = 0$. By recursion, we choose natural numbers $m_1 < m_2 < \cdots$ and unit vectors $z_n \in Z \cap \ker P_{m_{n-1}}$ such that for $n \in \mathbb{N}$

$$\|z_n - w_n\| \leq \varepsilon_n, \quad \text{where} \quad w_n = P_{m_n} z_n \quad \text{and} \quad \varepsilon_n = \frac{\eta}{2^{n+2} K(\eta + \|T\|) + \eta}.$$

Then $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence of $(x_n)_{n\in\mathbb{N}}$ because $\|w_n\| \geq 1 - \varepsilon_n > 0$ for every $n \in \mathbb{N}$.

We shall now complete the proof by showing that the restriction of T to the block subspace spanned by $(w_n)_{n\in\mathbb{N}}$ is bounded below by $\eta/2$. Take a unit vector $w = \sum_{n=1}^N \alpha_n w_n$ for some $N \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{K}$, and set $z = \sum_{n=1}^N \alpha_n z_n \in Z$. We have

$$\begin{aligned} \|Tw\| &\geq \|Tz\| - \|T(z - w)\| \geq \eta\|z\| - \|T\|\|z - w\| \qquad (4.8) \\ &\geq \eta(\|w\| - \|z - w\|) - \|T\|\|z - w\| = \eta - (\eta + \|T\|)\|z - w\|. \end{aligned}$$

To find an upper bound on $\|z - w\|$, we observe that $\alpha_n w_n = (P_{m_n} - P_{m_{n-1}})w$, so

$$|\alpha_n| \leq \frac{2K}{\|w_n\|} \leq \frac{2K}{1 - \varepsilon_n} \qquad (1 \leq n \leq N),$$

and therefore

$$\|z - w\| \leq \sum_{n=1}^{N} |\alpha_n| \|z_n - w_n\| \leq \sum_{n=1}^{N} \frac{2K\varepsilon_n}{1 - \varepsilon_n} = \sum_{n=1}^{N} \frac{\eta}{(\eta + \|T\|)2^{n+1}} \leq \frac{\eta}{2(\eta + \|T\|)},$$

where the equality in the middle follows from the choice of ε_n. Substituting this estimate into (4.8), we conclude that $\|Tw\| \geq \eta/2$, which establishes the result. $\square$

Proposition 4.3.7. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and let $(e_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ denote the unit vector bases for E and D, respectively. Then the formal inclusion map given by $\iota\colon e_n \mapsto d_n$ for $n \in \mathbb{N}$ extends to a bounded linear injection $\iota\colon E \to D$ of norm 1. Furthermore, ι is strictly singular, but not compact.*

Proof. It is obvious that the formal inclusion map $\iota\colon S_p \to c_0$ is a bounded linear injection of norm 1, while the same conclusion for $\iota\colon B_p \to \ell_p$ is an easy consequence of the definition of the norm on B_p, or alternatively it follows by applying Lemma 2.4.10 to the chain $\mathcal{C} = \big\{\{n\} : n \in \mathbb{N}\big\}$. The non-compactness of ι is witnessed by its action on the unit vector basis in both cases, so it only remains to verify that ι is strictly singular.

Lemma 4.3.6 implies that it suffices to show that the restriction of ι to the closed subspace spanned by a block basic sequence $(w_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$ is not an isomorphic embedding. By Lemma 2.4.7, $(w_n)_{n\in\mathbb{N}}$ admits a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ for which $\|u_n\|_\infty \to 0$ as $n \to \infty$. This completes the proof for the Schreier space S_p because $\|\iota(u_n)\| = \|u_n\|_\infty$ in this case. The argument for the Baernstein space B_p is more subtle, relying on Jameson's inequality stated in the last line of Theorem 2.3.4. Applying it here, we obtain

$$\|\iota(u_n)\|^p = \|u_n\|_{\ell_p}^p \leq K_p \|u_n\|_\infty^{p-1} \|u_n\|_{B_p} = K_p \|u_n\|_\infty^{p-1} \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

Proof of Theorem 4.1.1(i). We shall apply Theorem 4.3.1 with $(X,Y) = (E,D)$, that is, either $(X,Y) = (B_p,\ell_p)$ for some $1 < p < \infty$ or $(X,Y) = (S_p,c_0)$ for some $1 \le p < \infty$, endowed with the 1-UFDDs obtained by blocking the unit vector bases as follows:

$$X_n = \operatorname{span}\{e_j : j \in J_n\} \qquad \text{and} \qquad Y_n = \operatorname{span}\{d_j : j \in J_n\} \qquad (n \in \mathbb{N}), \tag{4.9}$$

where $J_1 < J_2 < \cdots$ are the intervals defined in Construction 4.3.2, and $T = \iota \in \mathscr{B}(E,D)$ is the formal inclusion map.

Condition (i) of Theorem 4.3.1 is trivially satisfied because $\iota(e_j) = d_j$ for $j \in \mathbb{N}$. We claim that the infimum in condition (ii) equals 1. To prove that, we begin by recalling that $Q_N = P_{L_N}$ for every $N \subseteq \mathbb{N}$, where $Q_N \in \mathscr{B}(E)$ denotes the projection associated with the 1-UFDD $(X_n)_{n\in\mathbb{N}}$ of E, the set L_N is given by (4.6), and $P_{L_N} \in \mathscr{B}(E)$ is the corresponding basis projection, as usual. Hence the claim will follow provided that we show that

$$\operatorname{dist}(\iota P_{L_M}, \langle \iota P_{L_N}\rangle) = 1 \qquad (M,N \in [\mathbb{N}],\ |M \setminus N| = \infty).$$

The inequality $\le$ is trivial because $\|\iota P_{L_M}\| = 1$. We shall verify the opposite inequality by showing that if $\operatorname{dist}(\iota P_{L_M}, \langle \iota P_{L_N}\rangle) < 1$ for some $M,N \in [\mathbb{N}]$, then $|M \setminus N| < \infty$. Hence, suppose that $\|\iota P_{L_M} - R\| < 1$ for some operator $R \in \langle \iota P_{L_N}\rangle$, say $R = \sum_{j=1}^k U_j \iota P_{L_N} V_j$, where $k \in \mathbb{N}$, $U_1,\ldots,U_k \in \mathscr{B}(D)$ and $V_1,\ldots,V_k \in \mathscr{B}(E)$. By replacing U_j with $\|V_j\|U_j$ and V_j with $\frac{V_j}{\|V_j\|}$ if $\|V_j\| > 0$, we may suppose that $\|V_j\| \le 1$ for each $j \in \{1,\ldots,k\}$.

Take $m \in L_M$. Since e_m and $d_m = \iota P_{L_M} e_m$ are unit vectors, we have

$$\begin{aligned} \|\iota P_{L_M} - R\| \ge \|(\iota P_{L_M} - R)e_m\|_D &\ge \|\iota P_{L_M} e_m\|_D - \|Re_m\|_D \\ &= 1 - \|Re_m\|_D \end{aligned}$$

so

$$1 - \|\iota P_{L_M} - R\| \le \|Re_m\|_D \le \sum_{j=1}^k \|U_j\|\, \|\iota P_{L_N} V_j e_m\|_D \tag{4.10}$$

$$\le k \cdot \max_{1\le j\le k} \|U_j\| \cdot \|\iota P_{L_N} V_{\varphi(m)} e_m\|_D, \tag{4.11}$$

where we have chosen $\varphi(m) \in \{1, \ldots, k\}$ such that

$$\max_{1 \le j \le k} \|\iota P_{L_N} V_j e_m\|_D = \|\iota P_{L_N} V_{\varphi(m)} e_m\|_D.$$

This defines a map $\varphi\colon L_M \to \{1, \ldots, k\}$ which in view of (4.10) satisfies

$$\|\iota P_{L_N} V_{\varphi(m)} e_m\|_D \ge \eta \qquad (m \in L_M), \qquad \text{where} \qquad \eta = \frac{1 - \|\iota P_{L_M} - R\|}{k \cdot \max\limits_{1 \le j \le k} \|U_j\|} > 0. \tag{4.12}$$

We use this map to introduce a new operator

$$W = P_{L_N} \sum_{j \in \varphi(L_M)} V_j P_{\varphi^{-1}(\{j\})}|_{E_{L_M}} \in \mathscr{B}(E_{L_M}, E_{L_N}).$$

Our aim is to show that it satisfies

$$\inf_{m \in L_M} \|W e_m\|_\infty \ge \begin{cases} \eta & \text{for } E = S_p, \\ \left(\dfrac{\eta^p}{K_p}\right)^{\frac{1}{p-1}} & \text{for } E = B_p, \end{cases} \tag{4.13}$$

where $K_p > 0$ denotes the constant from Theorem 2.3.4. Take $m \in L_M$, and observe that $\|W e_m\|_\infty = \|P_{L_N} V_{\varphi(m)} e_m\|_\infty$ because

$$P_{\varphi^{-1}(\{j\})} e_m = \begin{cases} e_m & \text{if } j = \varphi(m), \\ 0 & \text{otherwise.} \end{cases}$$

If $E = S_p$, then $D = c_0$, so $\|P_{L_N} V_{\varphi(m)} e_m\|_\infty = \|\iota P_{L_N} V_{\varphi(m)} e_m\|_D \ge \eta$ by (4.12), which establishes (4.13) in the first case. Otherwise $E = B_p$ and $D = \ell_p$; combining (4.12) with Jameson's inequality stated in the last line of Theorem 2.3.4, we obtain

$$\eta^p \le \|\iota P_{L_N} V_{\varphi(m)} e_m\|_{\ell_p}^p \le K_p \|P_{L_N} V_{\varphi(m)} e_m\|_\infty^{p-1} \|P_{L_N} V_{\varphi(m)} e_m\|_{B_p} \le K_p \|W e_m\|_\infty^{p-1}, \tag{4.14}$$

where the simple estimate $\|P_{L_N} V_{\varphi(m)} e_m\|_{B_p} \le \|P_{L_N}\| \, \|V_{\varphi(m)}\| \, \|e_m\|_{B_p} \le 1$ justifies the final inequality. The second case of (4.13) follows by rearranging (4.14).

Hence the operator W satisfies condition (c) of Theorem 3.3.1, so $\Gamma L_1(L_M, L_N) < \infty$, and therefore $|M \setminus N| < \infty$ by Corollary 4.3.4, as required.

We have thus verified both conditions of Theorem 4.3.1. It follows that the map (4.5) is injective. Composing it with the injection (4.3), we obtain $2^{\mathfrak{c}}$ many closed ideals of $\mathscr{B}(E)$. They are contained in the ideal of strictly singular operators because the operator ι is strictly singular, as we showed in Proposition 4.3.7. $\square$

Finally, the exact number of maximal operator ideals for Baernstein and Schreier spaces still remains unknown.

Question 4.3.8. Are there $2^{\mathfrak{c}}$ maximal operator ideals of operators on Baernstein or Schreier spaces?

Chapter 5

Strictly singular operators on Baernstein and Schreier spaces

5.1 Introduction

In this chapter, we conduct a more focused investigation into the ideal of strictly singular operators $\mathscr{S}(E)$, where E is a Baernstein or p-convexified Schreier space. To begin, we present a variety of characterizations for strictly singular operators on these spaces, including coincidence results with several other operator ideals. We then present the main finding of this section, that the composition of two strictly singular operators on E is compact. We finally present several applications of these ideas, exposing some unexplored aspects of ideals in $\mathscr{B}(X)$.

5.2 Coincidence of strictly singular operators with other operator ideals

In this section, we apply Theorem 2.4.4 to demonstrate how the ideal of strictly singular operators coincides with other well-known operator ideals in the sense of Pietsch; we refer the reader back to Section 1.4 for the relevant definitions and notation. In [58], Pfaffenberger has shown that $\mathscr{S}(X) = \mathscr{E}(X)$ for every subprojective Banach space X. Hence, in view of Corollary 2.4.13, we can immediately make the following observation.

Proposition 5.2.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$. Then $\mathscr{S}(E) = \mathscr{E}(E)$.*

As the Baernstein spaces are reflexive, $\mathscr{B}(B_p) = \mathscr{W}(B_p)$ for each $1 < p < \infty$ by the Banach-Alaoglu Theorem.

In the following result we establish that $\mathscr{W}(S_p) = \mathscr{S}(S_p)$ for each $1 \leq p < \infty$ by relating several different properties of operators under saturation conditions. We remark that the equivalence of conditions (a)-(c) is well-known to specialists. Another definition will be needed; an operator $T \in \mathscr{B}(X, Y)$ between Banach spaces X and Y is said to be *unconditionally converging* if the series $\sum_{n=1}^{\infty} Tx_n$ converges unconditionally in norm for every series $\sum_{n=1}^{\infty} x_n$ in X which is weakly unconditionally Cauchy in the sense that the series $\sum_{n=1}^{\infty}\langle x_n, f\rangle$ converges absolutely for every functional $f \in X^*$. Finally, we refer the reader back to Definition 1.2.3 for the notion of saturation by copies of a Banach space.

Proposition 5.2.2. *Let $T \in \mathscr{B}(X)$ be an operator, where X is a Banach space saturated with complemented copies of some infinite-dimensional Banach space D. The following are equivalent:*

(a) *T is strictly singular;*

(b) *T is inessential;*

(c) *the identity operator on D does not factor through T in the sense that there are no operators $U \in \mathscr{B}(X, D)$ and $V \in \mathscr{B}(D, X)$ such that $UTV = I_D$;*

Suppose that $D = c_0$ and that X embeds into a Banach space with an unconditional basis. Then conditions (a) to (c) are also equivalent to the following:

(d) *T is unconditionally converging;*

(e) *T is weakly compact.*

The proof relies on a classical result of Pełczyński [55, Proposition 9, 1°].

Theorem 5.2.3 (Pełczyński)**.** *Let X be a Banach space which embeds into a Banach space with an unconditional basis, and suppose that X does not contain any subspace which is isomorphic to ℓ_1. Then every unconditionally converging operator from X into a Banach space is weakly compact.*

Proof of Proposition 5.2.2. The proof has two parts. In part (I), we show that conditions (a)–(c) are equivalent, while part (II) contains the proof that (c) - (e) are equivalent under the further assumptions proposed.

Part (I). The implication (a) $\implies$ (b) is always true, and so is (b) $\implies$ (c) because the identity operator on an infinite-dimensional Banach space is never inessential. Next, we prove (c) $\implies$ (a) by contraposition. Suppose that T is bounded below on an infinite-dimensional subspace $Z \subseteq X$, so there exists a constant $\delta > 0$ such that $\|z\|_X \leq \delta \|Tz\|_X$ for each $z \in Z$. By assumption the subspace $T[Z]$ contains a complemented copy of D, which takes the form $T[Z_0]$ for some closed subspace $Z_0 \subseteq Z$. Thus, we can write $X = T[Z_0] \oplus Y$ for some subspace Y and let $P \in \mathscr{B}(X)$ denote the projection onto $T[Z_0]$. Since $D \approx T[Z_0] \approx Z_0$ there exists an isomorphism $R : Z_0 \to D$, where for some $\eta > 0$ we have $\|z\|_{Z_0} \leq \eta \|Rz\|_D$. Define a map

$$U : X \to D, (Tz + y) \to Rz \qquad (y \in Y, z \in Z_0).$$

Clearly U is linear. To see that U is bounded let $y \in Y$ and $z \in Z_0$. We compute

$$\|U(Tz + y)\|_D = \|Rz\|_D \leq \|R\| \cdot \|z\| \leq \|R\|\delta\|Tz\|_X \leq \|R\|\delta\|P\| \cdot \|Tz + y\|_X$$

Letting $V : D \to X, d \mapsto R^{-1}d$ we can factor $I_D = UTV$ as required.
Part (II). Under these assumptions, (a) is equivalent to an operator not being bounded below on any subspace isomorphic to c_0. By [60, Theorem 8.4', Part C.II], we have (a) $\implies$ (d) in general (that is, without any restrictions on the Banach space X). The hypothesis that X is now c_0-saturated ensures that ℓ_1 does not embed into X, so Theorem 5.2.3 shows that (d) implies (e). Finally, the implication (e) $\implies$ (c) must be true otherwise I_{c_0} would be weakly compact, but c_0 is not reflexive. $\square$

5.3 The composition of strictly singular operators

This section contains the main finding of this chapter, that the composition of two strictly singular operators on a Baernstein or p-convexified Schreier space is compact. Results of this kind have a long history. Probably the first spaces known to satisfy this property are $C(K)$ for any compact Hausdorff space K and $L_1(\mu)$ for any σ-finite measure μ; this is a consequence of two famous results: (i) $C(K)$ and $L_1(\mu)$ have the Dunford–Pettis property (see for instance [1, Theorem 5.4.5] or

[22, Theorems VI.7.4 and VI.8.12]); (ii) an operator on $C(K)$ or $L_1(\mu)$ is strictly singular if and only if it is weakly compact (this is due to Pełczyński [56, 57]).

Much more recently, and much closer to the spaces we study, Causey and Pelczar-Barwacz [12, Theorem 7.6, (3)–(4)] have obtained results of a slightly more general form for the higher-order Schreier spaces $X[\mathcal{S}_\xi]$ for every countable ordinal $\xi > 0$: there is a (necessarily unique) natural number k, dependent only on ξ, such that:

- every composition of $k+1$ strictly singular operators on $X[\mathcal{S}_\xi]$ is compact;
- there are k strictly singular operators on $X[\mathcal{S}_\xi]$ whose composition is not compact.

That a composition of two strictly singular operators need not be compact in $X[\mathcal{S}_k]$ for $k > 1$ is pertinent, given the proximity of the higher order Schreier spaces to those we have studied throughout this thesis. It is also important to acknowledge that for certain Banach spaces X, we have a stronger property that $\mathscr{S}(X) = \mathscr{K}(X)$. For instance:

- c_0, ℓ_p for each $1 \leq p < \infty$ due to [45, Proposition 2.c.3]
- The quasireflexive spaces J_p for $p = 2$ due to Saksman and Tylli in [61, Remark 3.9] and later $1 < p < \infty$ due to Laustsen in [41, Proposition 4.9].
- Tsirelson's space T due to Argyros, Beanland and Motakis in [4].

Lemma 5.3.1. *Let $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ be increasing sequences of natural numbers, and let $E = S_p$ for some $1 \leq p < \infty$ or $E = B_p$ for some $1 < p < \infty$.*

(i) *Suppose that $m_j \leq n_j$ for each $j \in \mathbb{N}$. Then the basic sequence $(e_{n_j})_{j\in\mathbb{N}}$ 1-dominates $(e_{m_j})_{j\in\mathbb{N}}$ in E.*

(ii) *Suppose that $m_j \leq n_{j+1}$ for each $j \in \mathbb{N}$. Then the basic sequence $(e_{n_j})_{j\in\mathbb{N}}$ C-dominates $(e_{m_j})_{j\in\mathbb{N}}$, where $C = 2^{1/p}$ for $E = S_p$ and $C = 2$ for $E = B_p$.*

(iii) *There exists an infinite subset J of $\mathbb{N}$ such that the basic sequences $(e_{m_j})_{j\in J}$ and $(e_{n_j})_{j\in J}$ are equivalent.*

Proof. According to Lemma 3.1.4(ii) and (iv), the sets $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$ satisfy $\Gamma L_1(N, M) = 1$ in case (i) and $\Gamma L_1(N, M) \le 2$ in case (ii). Now the conclusions follow from Proposition 3.2.1.

(iii). Set $j_1 = 1$. By recursion, we can choose integers $1 < j_2 < j_3 < \cdots$ such that $m_{j_k} \le n_{j_{k+1}}$ and $n_{j_k} \le m_{j_{k+1}}$ for each $k \in \mathbb{N}$. Then (ii) implies that the subsequences $(e_{m_{j_k}})_{k\in\mathbb{N}}$ and $(e_{n_{j_k}})_{k\in\mathbb{N}}$ C-dominate each other, so the set $J = \{j_1 < j_2 < \cdots\} \in [\mathbb{N}]$ has the required property. $\square$

Lemma 5.3.2. *Let $(u_j)_{j\in\mathbb{N}}$ be a normalized block basic sequence of the unit vector basis $(e_j)_{j\in\mathbb{N}}$ for E, where $E = S_p$ for some $1 \le p < \infty$ or $E = B_p$ for some $1 < p < \infty$.*

(i) *Set $m_j = \max \operatorname{supp} u_j$ for $j \in \mathbb{N}$. Then $(e_{m_j})_{j\in\mathbb{N}}$ C_1-dominates $(u_j)_{j\in\mathbb{N}}$, where $C_1 = 1$ if $E = S_p$ and $C_1 = 3^{1/p}$ if $E = B_p$.*

(ii) *Suppose that $\delta := \inf_{j\in\mathbb{N}}\|u_j\|_\infty > 0$, and choose $n_j \in \operatorname{supp} u_j$ such that $|\langle u_j, e^*_{n_j}\rangle| \ge \delta$ for each $j \in \mathbb{N}$. Then $(u_j)_{j\in\mathbb{N}}$ δ^{-1}-dominates $(e_{n_j})_{j\in\mathbb{N}}$, and $\overline{\operatorname{span}}\{u_j : j \in \mathbb{N}\}$ is C_2-complemented in E, where $C_2 = 2^{1/p}/\delta$ if $E = S_p$ and $C_2 = 2 \cdot 3^{1/p}/\delta$ if $E = B_p$.*

(iii) *The following conditions are equivalent:*

  (a) $\inf_{j\in\mathbb{N}}\|u_j\|_\infty > 0$;

  (b) *$(u_j)_{j\in\mathbb{N}}$ is equivalent to a subsequence of $(e_j)_{j\in\mathbb{N}}$;*

  (c) *$(u_j)_{j\in\mathbb{N}}$ dominates a subsequence of $(e_j)_{j\in\mathbb{N}}$.*

Proof. Before we embark on the proof, let us point out that the sequences $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ in parts (i) and (ii) are strictly increasing because $(u_j)_{j\in\mathbb{N}}$ is a block basic sequence.

(i). Set $M = \{m_j : j \in \mathbb{N}\}$. Our aim is to prove that for $k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{K}$, the elements $x = \sum_{j=1}^k \alpha_j e_{m_j}$ and $y = \sum_{j=1}^k \alpha_j u_j$ satisfy the inequality $C_1\|x\|_E \ge \|y\|_E$. This is trivial if $\alpha_1 = \cdots = \alpha_k = 0$, so we may suppose otherwise; thus $x, y \ne 0$.

If $E = S_p$, take $F \in \mathcal{S}_1$ such that $F \subseteq \operatorname{supp}(y)$ and $\|y\|_{S_p} = \mu_p(y, F)$, and set $F_j = F \cap \operatorname{supp} u_j \in \mathcal{S}_1$ for $1 \le j \le k$. Then, defining $J = \{j \in \{1, \dots, k\} : F_j \neq \emptyset\}$ and $j_1 = \min J$, we have

$$\begin{aligned}
\min M(J) = m_{j_1} &\ge \min F && \text{because } \min F \in \operatorname{supp} u_{j_1} \\
&\ge |F| && \text{because } F \in \mathcal{S}_1 \\
&\ge |J| && \text{because } (F_j)_{j \in J} \text{ are disjoint, non-empty subsets of } F \\
&= |M(J)|,
\end{aligned}$$

so $M(J) \in \mathcal{S}_1$. Consequently,

$$\begin{aligned}
\|x\|_{S_p}^p \ge \mu_p(x, M(J))^p &= \sum_{j \in J} |\alpha_j|^p = \sum_{j \in J} |\alpha_j|^p \|u_j\|_{S_p}^p \\
&\ge \sum_{j \in J} |\alpha_j|^p \mu_p(u_j, F_j)^p = \sum_{j \in J} \mu_p(y, F_j)^p = \mu_p(y, F)^p = \|y\|_{S_p}^p,
\end{aligned}$$

which proves that $(e_{m_j})_{j \in \mathbb{N}}$ 1-dominates $(u_j)_{j \in \mathbb{N}}$ in S_p.

Before we tackle the case $E = B_p$, let us mention that Causey [11, Lemma 3.2] has proved a closely related result, which states that $(e_{k_j})_{j \in \mathbb{N}}$ 4-dominates $(u_j)_{j \in \mathbb{N}}$ in B_p, where $k_j = \min \operatorname{supp} u_j$ for $j \in \mathbb{N}$. Our result (with a poorer constant) follows easily from Causey's. Nevertheless, we include a (fairly simple) proof of it to keep our work as self-contained as possible.

Take $\mathcal{C} \in \mathrm{SC}$ such that $\|y\|_{B_p} = \beta_p(y, \mathcal{C})$. We may suppose that $F \subseteq \operatorname{supp} y$ for every $F \in \mathcal{C}$. Define $\mathcal{C}_j = \{F \in \mathcal{C} : F \subseteq \operatorname{supp} u_j\}$ for $1 \le j \le k$, $J = \{j \in \{1, \dots, k\} : \mathcal{C}_j \neq \emptyset\}$ and $\mathcal{C}' = \mathcal{C} \setminus \bigcup_{j \in J} \mathcal{C}_j$. Then $\mathcal{C}$ is the disjoint union of $\{\mathcal{C}_j : j \in J\} \cup \{\mathcal{C}'\}$, so

$$\|y\|_{B_p}^p = \sum_{j \in J} \beta_p(y, \mathcal{C}_j)^p + \beta_p(y, \mathcal{C}')^p. \tag{5.1}$$

It is is easy to estimate the first term on the right-hand side of this equation because the fact that $F \subseteq \operatorname{supp} u_j$ for each $F \in \mathcal{C}_j$ implies that

$$\beta_p(y, \mathcal{C}_j) = |\alpha_j| \beta_p(u_j, \mathcal{C}_j) \le |\alpha_j| \, \|u_j\|_{B_p} = |\alpha_j| \qquad (j \in J).$$

Hence we have

$$\sum_{j\in J}\beta_p(y,\mathcal{C}_j)^p \le \sum_{j\in J}|\alpha_j|^p \le \|x\|_{B_p}^p, \tag{5.2}$$

using the easy observation that the B_p-norm 1-dominates the ℓ_p-norm.

This completes the proof if $\mathcal{C}' = \emptyset$. Otherwise we can write $\mathcal{C}' = \{F_1 < \cdots < F_n\}$ for some $n \in \mathbb{N}$. Set $K_r = \{j \in \{1,\ldots,k\} : F_r \cap \operatorname{supp} u_j \neq \emptyset\}$ and $i_r = \min K_r$ for $1 \le r \le n$. Then $M(K_r) \in \mathcal{S}_1$ because

$$|M(K_r)| = |K_r| \le |F_r| \le \min F_r \le m_{i_r} = \min M(K_r).$$

Furthermore, using that $\{F_r \cap \operatorname{supp} u_j : j \in K_r\}$ is a partition of F_r, we obtain

$$\begin{aligned}\mu_1(y,F_r) &= \sum_{j\in K_r}\mu_1(y, F_r\cap \operatorname{supp} u_j) = \sum_{j\in K_r}|\alpha_j|\mu_1(u_j,F_r)\\ &\le \sum_{j\in K_r}|\alpha_j|\,\|u_j\|_{B_p} = \sum_{j\in K_r}|\alpha_j| = \mu_1(x, M(K_r)).\end{aligned} \tag{5.3}$$

The definition of $\mathcal{C}'$ implies that $|K_r| \ge 2$ and $K_r \setminus \{i_{r+1}\} < K_{r+1}$. In particular, we have $K_r < K_{r+2}$, so $M(K_r) < M(K_{r+2})$, and therefore we can define Schreier chains by

$$\begin{aligned}\mathcal{D} &:= \{M(K_1) < M(K_3) < \cdots < M(K_{n'})\}\\ \mathcal{E} &:= \{M(K_2) < M(K_4) < \cdots < M(K_{n''})\}\end{aligned}$$

where $(n', n'') = (n-1, n)$ if n is even and $(n', n'') = (n, n-1)$ if n is odd. Hence, applying (5.3), we deduce that

$$\beta_p(y,\mathcal{C}')^p = \sum_{r=1}^{n}\mu_1(y,F_r)^p \le \sum_{r=1}^{n}\mu_1(x,M(K_r))^p = \beta_p(x,\mathcal{D})^p + \beta_p(x,\mathcal{E})^p \le 2\|x\|_{B_p}^p.$$

Finally, we substitute this upper bound together with that from (5.2) into (5.1) to conclude that $\|y\|_{B_p}^p \le 3\|x\|_{B_p}^p$, which proves that $(e_{m_j})_{j\in\mathbb{N}}$ $3^{1/p}$-dominates $(u_j)_{j\in\mathbb{N}}$ in B_p.

(ii). Set $N = \{n_j : j \in \mathbb{N}\}$. The 1-unconditionality of the unit vector basis for E allows us to define a bounded operator $T\colon E \to E_N$ of norm at most δ^{-1} by

$$Tx = \sum_{k=1}^{\infty}\frac{\langle x, e_{n_k}^*\rangle}{\langle u_k, e_{n_k}^*\rangle}e_{n_k}.$$

It satisfies $Tu_j = e_{n_j}$ for each $j \in \mathbb{N}$ because $\langle u_j, e_{n_k}^* \rangle = 0$ whenever $j \neq k$. Hence $(u_j)_{j\in\mathbb{N}}$ δ^{-1}-dominates $(e_{n_j})_{j\in\mathbb{N}}$ in E.

Next, we prove that the subspace $W = \overline{\operatorname{span}}\{u_j : j \in \mathbb{N}\}$ is C_2-complemented in E. We have $m_j = \max\operatorname{supp} u_j < n_{j+1}$ for each $j \in \mathbb{N}$ because $(u_j)_{j\in\mathbb{N}}$ is a block basic sequence, so Lemma 5.3.1(ii) shows that the map $U\colon e_{n_j} \mapsto e_{m_j}$ for $j \in \mathbb{N}$ extends uniquely to an operator $U \in \mathscr{B}(E_N, E_M)$ with $\|U\| \leq C_0$, where $C_0 = 2^{1/p}$ for $E = S_p$ and $C_0 = 2$ for $E = B_p$. By (i), we can define an operator $V \in \mathscr{B}(E_M, W)$ with $\|V\| \leq C_1$ by $Ve_{m_j} = u_j$ for each $j \in \mathbb{N}$. It follows that the composite operator $Q = VUT \in \mathscr{B}(E, W)$ satisfies $Qu_j = u_j$ for each $j \in \mathbb{N}$, so by linearity and continuity, Q acts as the identity on $\overline{\operatorname{span}}\{u_j : j \in \mathbb{N}\} = W$; that is, Q is a projection of E onto W. Now the conclusion follows from the fact that

$$\|Q\| \leq \frac{C_0 C_1}{\delta} \leq \begin{cases} \dfrac{2^{1/p}}{\delta} & \text{for } E = S_p \\ \dfrac{2 \cdot 3^{1/p}}{\delta} & \text{for } E = B_p. \end{cases}$$

(iii), (a)$\Rightarrow$(b). Suppose that $\inf_{j\in\mathbb{N}}\|u_j\|_\infty > 0$, and define the sequences (m_j) and (n_j) as above. Then (ii) shows that $(u_j)_{j\in\mathbb{N}}$ dominates $(e_{n_j})_{j\in\mathbb{N}}$, which in turn dominates $(e_{m_j})_{j\in\mathbb{N}}$ by Lemma 5.3.1(ii), as we already observed in the previous paragraph. Furthermore, $(e_{m_j})_{j\in\mathbb{N}}$ dominates $(u_j)_{j\in\mathbb{N}}$ by (i), so $(u_j)_{j\in\mathbb{N}}$, $(e_{m_j})_{j\in\mathbb{N}}$ and $(e_{n_j})_{j\in\mathbb{N}}$ are all equivalent.

The implication (b)$\Rightarrow$(c) is trivial.

Finally, we prove that (c) implies (a) by contradiction. Assume that $(u_j)_{j\in\mathbb{N}}$ dominates a subsequence of $(e_j)_{j\in\mathbb{N}}$ and that $\inf_{j\in\mathbb{N}}\|u_j\|_\infty = 0$. By Lemma 2.4.9, the latter assumption implies that $(u_j)_{j\in\mathbb{N}}$ admits a subsequence which is dominated by the unit vector basis $(d_j)_{j\in\mathbb{N}}$ for D, where $D = c_0$ if $E = S_p$ and $D = \ell_p$ if $E = B_p$. Combining this with the former assumption, we conclude that $(d_j)_{j\in\mathbb{N}}$ dominates a subsequence of $(e_j)_{j\in\mathbb{N}}$, which is impossible. $\square$

Remark 5.3.3. Note that Lemma 5.3.2(ii) tells us that semi-normalization with respect to $\|\cdot\|_\infty$ in the Baernstein spaces will ensure that a block basic sequence spans a complemented subspace, in contrast to the false claim of C.Seifert that semi-normalization with respect to $\|\cdot\|_{B_p}$ offered a characterization (see Section 2.6).

Applying the Principle of Small Perturbations, we can establish the following lemma.

Lemma 5.3.4. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$, and let $(w_n)_{n\in\mathbb{N}}$ be a weakly null sequence in E for which $\inf_{n\in\mathbb{N}}\|w_n\|_E > 0$.*

(i) *Suppose that $\liminf_{n\to\infty}\|w_n\|_\infty = 0$. Then $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w'_n)_{n\in\mathbb{N}}$ which is a basic sequence equivalent to the unit vector basis for D, and $\|w'_n\|_\infty \to 0$ as $n \to \infty$.*

(ii) *Suppose that $\limsup_{n\to\infty}\|w_n\|_\infty > 0$. Then $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w'_n)_{n\in\mathbb{N}}$ which is a basic sequence equivalent to a subsequence of the unit vector basis for E, and $\inf_{n\in\mathbb{N}}\|w'_n\|_\infty > 0$.*

Proof. We begin by noting that the weak convergence of $(w_n)_{n\in\mathbb{N}}$ implies that it is norm-bounded and hence semi-normalized. For most of the proof, we shall consider the two cases simultaneously, but first we need to separate them:

- In case (i), we replace $(w_n)_{n\in\mathbb{N}}$ with a subsequence such that $\|w_n\|_\infty \to 0$ as $n \to \infty$, and we set $\xi = \inf_{n\in\mathbb{N}}\|w_n\|_E > 0$.

- In case (ii), we replace $(w_n)_{n\in\mathbb{N}}$ with a subsequence such that $\xi := \inf_{n\in\mathbb{N}}\|w_n\|_\infty > 0$, and observe that this implies that $\inf_{n\in\mathbb{N}}\|w_n\|_E \geq \xi$.

Returning to the unified approach, we define $m_0 = 0$, $P_0 = 0$ and $\varepsilon_j = \xi/(3 \cdot 2^j + 1)$ for each $j \in \mathbb{N}$. Using the fact that $(w_n)_{n\in\mathbb{N}}$ is weakly null, we can recursively choose increasing sequences $(k_j)_{j\in\mathbb{N}}$ and $(m_j)_{j\in\mathbb{N}}$ of natural numbers such that the vectors $v_j := (P_{m_j} - P_{m_{j-1}})w_{k_j} \in E$ satisfy $\|w_{k_j} - v_j\|_E \leq \varepsilon_j$ for each $j \in \mathbb{N}$, where we recall that P_m denotes the m^{th} basis projection.

We have

$$\|v_j\|_E \le \|w_{k_j}\|_E + \|w_{k_j} - v_j\|_E \le \sup_{n\in\mathbb{N}} \|w_n\|_E + \varepsilon_j,$$

and similarly

$$\|v_j\|_E \ge \|w_{k_j}\|_E - \|w_{k_j} - v_j\|_E \ge \xi - \varepsilon_j \qquad (j \in \mathbb{N}).$$

Since $\varepsilon_j \le \xi/7$, it follows that $(v_j)_{j\in\mathbb{N}}$ is a semi-normalized block basic sequence and therefore equivalent to its normalization $(v_j/\|v_j\|_E)_{j\in\mathbb{N}}$ by unconditionality. Furthermore,

$$\sum_{j=1}^{\infty} \frac{\|w_{k_j} - v_j\|_E}{\|v_j\|_E} \le \sum_{j=1}^{\infty} \frac{\varepsilon_j}{\xi - \varepsilon_j} = \sum_{j=1}^{\infty} \frac{1}{3\cdot 2^j} = \frac{1}{3} < \frac{1}{2},$$

so the Principle of Small Perturbations implies that $(w_{k_j})_{j\in\mathbb{N}}$ is a basic sequence equivalent to $(v_j)_{j\in\mathbb{N}}$ and hence to $(v_j/\|v_j\|_E)_{j\in\mathbb{N}}$. We note that $\|w_{k_j} - v_j\|_\infty \le \varepsilon_j$ for each $j \in \mathbb{N}$.

Consequently, if we are in case (i), then

$$\frac{\|v_j\|_\infty}{\|v_j\|_E} \le \frac{\|w_{k_j}\|_\infty + \varepsilon_j}{\inf_{n\in\mathbb{N}}\|v_n\|_E} \to 0 \quad \text{as} \quad j \to \infty,$$

so $(v_j/\|v_j\|_E)_{j\in\mathbb{N}}$ admits a subsequence $(v_{j_i}/\|v_{j_i}\|_E)_{i\in\mathbb{N}}$ which is equivalent to the unit vector basis for D by Proposition 2.4.15, and therefore the same is true for the subsequence $(w_{k_{j_i}})_{i\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$.

Otherwise we are in case (ii), and we see that $\|v_j\|_\infty \ge \|w_{k_j}\|_\infty - \varepsilon_j \ge 6\xi/7$, so

$$\inf_{j\in\mathbb{N}} \frac{\|v_j\|_\infty}{\|v_j\|_E} \ge \frac{6\xi}{7\sup_{j\in\mathbb{N}}\|v_j\|_E} > 0.$$

Hence Lemma 5.3.2(iii) implies that $(v_j/\|v_j\|_E)_{j\in\mathbb{N}}$ is equivalent to a subsequence of the unit vector basis for E, and therefore the same is true for the subsequence $(w_{k_j})_{j\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$. □

Remark 5.3.5. The two conditions in Lemma 5.3.4 are clearly not mutually exclusive, but at least one of them is always satisfied because the limit superior of a sequence dominates its limit inferior.

Lemma 5.3.6. *Let T be a strictly singular operator on E, where $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$, and let $(w_n)_{n\in\mathbb{N}}$ be a weakly null sequence in E for which $\inf_{n\in\mathbb{N}}\|Tw_n\|_E > 0$. Then $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w'_n)_{n\in\mathbb{N}}$ such that*

$$\inf_{n\in\mathbb{N}}\|w'_n\|_\infty > 0 \qquad \textit{and} \qquad \|Tw'_n\|_\infty \to 0 \quad \textit{as} \quad n \to \infty. \tag{5.4}$$

Proof. Following our usual convention, let $(d_n)_{n\in\mathbb{N}}$ denote the unit vector basis for D, where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$. Since $\inf_{n\in\mathbb{N}}\|w_n\|_E \geq \|T\|^{-1}\inf_{n\in\mathbb{N}}\|Tw_n\|_E > 0$, Lemma 5.3.4 implies that $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w'_n)_{n\in\mathbb{N}}$ which is a basic sequence and satisfies one of the following two conditions:

(i) $(w'_n)_{n\in\mathbb{N}}$ is equivalent to $(d_n)_{n\in\mathbb{N}}$, and $\|w'_n\|_\infty \to 0$ as $n \to \infty$; or

(ii) $(w'_n)_{n\in\mathbb{N}}$ is equivalent to a subsequence of $(e_n)_{n\in\mathbb{N}}$, and $\inf_{n\in\mathbb{N}}\|w'_n\|_\infty > 0$.

Being bounded, T is weakly continuous, so the sequence $(Tw'_n)_{n\in\mathbb{N}}$ converges weakly to 0, and therefore we may apply Lemma 5.3.4 to it, concluding that after replacing $(w'_n)_{n\in\mathbb{N}}$ with a subsequence, we may suppose that $(Tw'_n)_{n\in\mathbb{N}}$ is a basic sequence satisfying:

(iii) $(Tw'_n)_{n\in\mathbb{N}}$ is equivalent to $(d_n)_{n\in\mathbb{N}}$, and $\|Tw'_n\|_\infty \to 0$ as $n \to \infty$; or

(iv) $(Tw'_n)_{n\in\mathbb{N}}$ is equivalent to a subsequence of $(e_n)_{n\in\mathbb{N}}$, and $\inf_{n\in\mathbb{N}}\|Tw'_n\|_\infty > 0$.

Importantly, we note that conditions (i) and (ii) remain true when we replace $(w'_n)_{n\in\mathbb{N}}$ with a subsequence.

We see that (5.4) will follow if (and only if) conditions (ii) and (iii) are satisfied and shall verify this by ruling out the other three possible combinations. To present these arguments concisely, we define $W = \overline{\operatorname{span}}\,\{w'_n : n \in \mathbb{N}\}$, $X = \overline{\operatorname{span}}\,\{Tw'_n : n \in \mathbb{N}\}$ and write $\widetilde{T}$ for the restriction of T to W, viewed as an operator into X; we note that $\widetilde{T}$ is strictly singular because T is.

(i)+(iii) In this case we can take isomorphisms $U \in \mathscr{B}(D, W)$ and $V \in \mathscr{B}(X, D)$ such that $Ud_n = w'_n$ and $VTw'_n = d_n$ for each $n \in \mathbb{N}$. It follows that $V\widetilde{T}U = I_D$, which contradicts that $\widetilde{T}$ is strictly singular.

(i)+(iv) In this case we can find a set $M = \{m_1 < m_2 < \cdots\} \in [\mathbb{N}]$ and isomorphisms $U \in \mathscr{B}(D, W)$ and $V \in \mathscr{B}(X, E_M)$ such that $Ud_n = w'_n$ and $VTw'_n = e_{m_n}$ for each $n \in \mathbb{N}$. Then $V\widetilde{T}Ud_n = e_{m_n}$, which is impossible: $(d_n)_{n\in\mathbb{N}}$ does not dominate any subsequence of $(e_n)_{n\in\mathbb{N}}$.

(ii)+(iv) In this case there are sets $M = \{m_1 < m_2 < \cdots\}, N = \{n_1 < n_2 < \cdots\} \in [\mathbb{N}]$ and isomorphisms $U \in \mathscr{B}(E_M, W)$ and $V \in \mathscr{B}(X, E_N)$ such that $Ue_{m_j} = w'_j$ and $VTw'_j = e_{n_j}$ for each $j \in \mathbb{N}$. Lemma 5.3.1(iii) implies that we can find a set $J = \{j_1 < j_2 < \cdots\} \in [\mathbb{N}]$ such that the map $e_{m_{j_k}} \mapsto e_{n_{j_k}}$ for $k \in \mathbb{N}$ extends to an isomorphism of $E_{M(J)}$ onto $E_{N(J)}$. However, this would imply that the restriction of $V\widetilde{T}W$ to $E_{M(J)}$ is an isomorphism onto $E_{N(J)}$, which contradicts that $\widetilde{T}$ is strictly singular. $\square$

Lemma 5.3.7. *Let T be a non-compact operator from a Banach space X with a basis $(x_n)_{n\in\mathbb{N}}$ into a Banach space Y. Then $(x_n)_{n\in\mathbb{N}}$ admits a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ for which* $\inf_{n\in\mathbb{N}}\|Tu_n\|_Y > 0$.

Proof. Take $\eta \in (0, \|T + \mathscr{K}(X, Y)\|)$, and set $m_0 = 0$. We shall recursively choose natural numbers $m_1 < m_2 < \cdots$ and unit vectors $u_n \in \operatorname{span}\{x_j : m_{n-1} < j \le m_n\}$ such that $\|Tu_n\|_Y \ge \eta/(K+1)$ for each $n \in \mathbb{N}$, where K denotes the basis constant of $(x_n)_{n\in\mathbb{N}}$.

We begin the construction by observing that since $\operatorname{span}\{x_j : j \in \mathbb{N}\}$ is dense in X and $\|T\| > \eta$, we can find $m_1 \in \mathbb{N}$ and a unit vector $u_1 \in \operatorname{span}\{x_j : 1 \le j \le m_1\}$ such that $\|Tu_1\|_Y > \eta > \eta/(K+1)$.

Assume recursively that we have chosen natural numbers $m_1 < \cdots < m_n$ and unit vectors $u_1, \ldots, u_n$ for some $n \in \mathbb{N}$. Since the basis projection P_{m_n} has finite rank, we have $\|T(I_X - P_{m_n})\| \ge \|T + \mathscr{K}(X, Y)\| > \eta$, so we can find $m_{n+1} > m_n$ and a unit vector $v_{n+1} \in \operatorname{span}\{x_j : 1 \le j \le m_{n+1}\}$ such that $\|T(I_X - P_{m_n})v_{n+1}\|_Y > \eta$. Then

$$u_{n+1} = \frac{(I_X - P_{m_n})v_{n+1}}{\|(I_X - P_{m_n})v_{n+1}\|_X} \in \operatorname{span}\{x_j : m_n < j \le m_{n+1}\}$$

is a unit vector for which $\|Tu_{n+1}\|_Y \geq \eta/(K+1)$ because $\|(I_X - P_{m_n})v_{n+1}\|_X \leq K+1$. Hence the recursion continues, and the result follows. $\square$

Before proving the main result of the chapter, we present a further application of these ideas.

Proposition 5.3.8. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$. Then for any operator $T \in \mathscr{B}(E)$, conditions (a)-(e) in Proposition 5.2.2 are equivalent to T being finitely strictly singular.*

The proof relies on Milman's "flat-vector lemma", originally published in [50]; we refer to [13, Lemma 13] for an easily accessible presentation in English.

Lemma 5.3.9 (Milman). *Let W be a non-zero subspace of c_0, and take a natural number $n \leq \dim W$. Then W contains a unit vector w which attains its norm in at least n coordinates; that is, $\|w\|_\infty = 1$ and the set $\{j \in \mathbb{N} : |w(j)| = 1\}$ has cardinality at least n.*

Proof of Proposition 5.3.8. Assume towards a contradiction that $T \in \mathscr{B}(E)$ is a strictly singular operator which fails to be finitely strictly singular. Then, for some $\varepsilon > 0$, there is a sequence $(W_n)_{n\in\mathbb{N}}$ of subspaces of E such that

$$\dim W_n \geq 2n - 1 \qquad \text{and} \qquad \|Tw\|_E \geq \varepsilon \|w\|_E \qquad (n \in \mathbb{N},\ w \in W_n). \tag{5.5}$$

Lemma 5.3.9 implies that for every $n \in \mathbb{N}$, we can choose $v_n \in W_n$ such that $\|v_n\|_\infty = 1$ and the set $J_n = \{j \in \mathbb{N} : |v_n(j)| = 1\}$ has cardinality at least $2n - 1$. Then $J_n \cap [n, \infty)$ has cardinality at least n, so J_n contains a Schreier set A_n of cardinality n, and therefore $\|v_n\|_{S_p} \geq \mu_p(v_n, A_n) = n^{1/p}$ for any $p \in [1, \infty)$. Since $\|\cdot\|_{B_p} \geq \|\cdot\|_{S_1}$, we conclude that $\|v_n\|_E \to \infty$ as $n \to \infty$. Consequently, defining $w_n = v_n/\|v_n\|_E \in W_n$ for $n \in \mathbb{N}$, we have

$$\|w_n\|_\infty = 1/\|v_n\|_E \to 0 \quad \text{as} \quad n \to \infty. \tag{5.6}$$

In particular, $(w_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence with $\langle w_n, e_j^* \rangle \to 0$ as $n \to \infty$ for each $j \in \mathbb{N}$, so $(w_n)_{n\in\mathbb{N}}$ is weakly null because the basis $(e_j)_{j\in\mathbb{N}}$ for E is shrinking.

Since $\inf_{n\in\mathbb{N}}\|Tw_n\|_E \geq \varepsilon$ by (5.5), Lemma 5.3.6 implies that $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w'_n)_{n\in\mathbb{N}}$ such that $\inf_{n\in\mathbb{N}}\|w'_n\|_\infty > 0$. However, this contradicts (5.6). □

We are now ready to prove the main theorem of the chapter.

Theorem 5.3.10. *Let T_1 and T_2 be strictly singular operators on either the Baernstein space B_p for some $1 < p < \infty$ or the Schreier space S_p for some $1 \leq p < \infty$. Then the composite operator T_2T_1 is compact.*

Proof of Theorem 5.3.10. Let T_1 and T_2 be strictly singular operators on E, where $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leq p < \infty$, and assume towards a contradiction that $T_2T_1 \notin \mathscr{K}(E)$. By Lemma 5.3.7, we can find a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the unit vector basis $(e_n)_{n\in\mathbb{N}}$ for E such that $\inf_{n\in\mathbb{N}}\|T_2T_1u_n\|_E > 0$. We note that $(u_n)_{n\in\mathbb{N}}$ is weakly null because $(e_n)_{n\in\mathbb{N}}$ is shrinking, and therefore $(T_1u_n)_{n\in\mathbb{N}}$ is weakly null as well. Hence, applying Lemma 5.3.6 with $T = T_2$ and $w_n = T_1u_n$ for $n \in \mathbb{N}$, we can extract a subsequence $(u'_n)_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ for which

$$\inf_{n\in\mathbb{N}}\|T_1u'_n\|_\infty > 0. \tag{5.7}$$

Since $(u'_n)_{n\in\mathbb{N}}$ is weakly null and $\inf_{n\in\mathbb{N}}\|T_1u'_n\|_E > 0$, another application of Lemma 5.3.6 shows that $(u'_n)_{n\in\mathbb{N}}$ admits a subsequence $(u''_n)_{n\in\mathbb{N}}$ such that $\|T_1u''_n\|_\infty \to 0$ as $n \to \infty$. This, however, contradicts (5.7). □

5.4 Application 1: approximate identities

We are now in a position to profit from Theorem 5.3.10, with two applications that broaden our understanding of the closed ideals. The first is to approximate identities. Let B be a non-unital Banach algebra. Recall that a *left (resp. right) approximate identity* for B is a net $(e_\lambda)_{\lambda\in\Lambda} \subset B$ such that for all $x \in B$ we have $\lim_{\lambda\in\Lambda}\|x - e_\lambda x\| = 0$ (resp. $\lim_{\lambda\in\Lambda}\|x - xe_\lambda\| = 0$). If $\sup_{\lambda\in\Lambda}\|e_\lambda\| < \infty$ then we say the approximate identity is *bounded*. In the case that B is a closed ideal of $\mathscr{B}(X)$, there has been success in determining whether closed ideals posses a left (or right) approximate identity:

(1) If X has a basis, the sequence of projections $(P_n)_{n\in\mathbb{N}}$ forms a left bounded approximate identity for $\mathscr{K}(X)$.

(2) When $X = L_p[0,1]$ for $1 \leq p < \infty$, the *only* non-trivial closed ideal in $\mathscr{B}(L_p[0,1])$ with a left approximate identity is $\mathscr{K}(L_p[0,1])$ due to Johnson and Schechtman in [36].

(3) When $X = J_2$ (the quasireflexive space of James) $\mathscr{W}(J_2)$ has a left bounded approximate identity due to Odell and Tylli in [52], and a right bounded approximate identity due to Loy and Willis in [47].

(4) When $X = C_0[0,\omega_1)$, the unique maximal ideal $\mathcal{M} \subseteq \mathscr{B}(X)$ has a left bounded approximate identity due to Kania, Koszmider and Laustsen in [38, Corollary 1.9] as well as a right bounded approximate identity due to Loy and Willis in [47].

(5) When $X = Y \oplus X_{\mathrm{AH}}$, where X_{AH} is the space of Argyros and Haydon from [3] and Y is the closed subspace of X_{AH} identified by Kania and Laustsen in [39] for which they obtain the following closed ideal classification:

$$\begin{tikzcd}
& \mathscr{B}(X) & \\
\mathcal{M}_1 \arrow[ur] & & \mathcal{M}_2 \arrow[ul] \\
& \mathcal{M}_1 \cap \mathcal{M}_2 \arrow[ul] \arrow[ur] & \\
& \mathscr{K}(X) \arrow[u] & \\
& \{0\} \arrow[u] &
\end{tikzcd}$$

Furthermore, the authors show in [39, Theorem 1.8] that

(5.i) $\mathcal{M}_1$ has a left bounded approximate identity but not a right approximate one,

(5.ii) $\mathcal{M}_2$ has a right bounded approximate identity but not a left approximate one,

(5.iii) $\mathcal{M}_1 \cap \mathcal{M}_2$ has no left/right approximate identity,

(5.iv) $\mathscr{K}(X)$ has a two sided bounded approximate identity.

In light of our findings and those of Causey and Pelczar-Barwacz, the following can be said.

Corollary 5.4.1. *1. Let $E = B_p$ for $1 < p < \infty$ or S_p for $1 \leq p < \infty$. Let $\mathcal{I} \subseteq \mathscr{S}(E)$ be a closed ideal. If $\mathcal{I}$ possesses a left or right approximate identity then $\mathcal{I} = \{0\}$ or $\mathscr{K}(E)$.*

2. Let $E = X[\mathcal{S}_n]$ for $n \geq 1$. Then $\mathscr{S}(E)$ does not posess a left or right approximate identity.

Proof. For 1 the claim follows from Theorem 5.3.10. For 2 recall from [12, Theorem 7.6 (3)-(4)] that there exist $T_1, \dots, T_n \in \mathscr{S}(X[\mathcal{S}_n])$ whose composition is not compact. Then $\mathscr{S}(X[\mathcal{S}_n])$ cannot posess an approximate identity on either side as the compositions $RT_1 \cdots T_n$, $T_1 \cdots T_n R \in \mathscr{K}(X[\mathcal{S}_n])$ for all $R \in \mathscr{S}(X[\mathcal{S}_n])$. □

5.5 Application 2: closed prime ideals

The second application is to closed prime ideals. Although the application itself in Proposition 5.5.14 is fairly straightforward, we conduct a more detailed investigation around primality of operator ideals given the scarcity of results in the literature. Let I, J be ideals of a ring R. Let IJ denote the following ideal of R

$$IJ := \left\{ \sum_{i=1}^{n} x_i y_i : x_1, \dots, x_n \in I, y_1, \dots, y_n \in J \right\}.$$

We shall say that a proper ideal I is

(a) *prime* if, for all ideals $J, J' \subset R$ the condition $JJ' \subseteq I$ implies at least one of J, J' lies in I,

(b) *completely prime* if $xy \in I$ implies that $x \in I$ or $y \in I$ for any $x, y \in R$

In the setting of Banach algebras, non-maximal prime ideals in uniform algebras share a close relationship with the existence of discontinuous algebra homomorphisms. First, we acknowledge the seminal works of H.G.Dales in [17, Theorem 1.1] and J.Esterle in [25, Corollaire 4.2], who independently proved the existence of discontinuous algebra homomorphisms from Banach algebras.

Theorem 5.5.1. *(H.G.Dales) Let X be an infinite compact Hausdorff space. Then, assuming the continuum hypothesis, there exists a discontinuous algebra monomorphism from $C(X, \mathbb{C})$ into a Banach algebra.*

Theorem 5.5.2. *(J.Esterle) Let K be an infinite compact space. Let A be a complex commutative radical Banach algebra with an approximate identity. Let B denote the unitisation of A. Assuming the continuum hypothesis, there exists a discontinuous algebra homomorphism from $C(K)$ to B whose kernel is a non-maximal prime ideal.*

For a comprehensive overview of this theory see [18, Section 4.8 and Chapter 5] by H.G.Dales. It is fruitless to seek analogues of these results for $\mathscr{B}(X)$ and arbitrary infinite-dimensional Banach spaces X due to the following corollary of [35, Theorem 3.3].

Corollary 5.5.3. *Let X be a Banach space satisfying $X \approx X \oplus X$. Then every algebra homomorphism from $\mathscr{B}(X)$ is continuous.*

Our brief study into primality will be focused on *closed* prime ideals in $\mathscr{B}(X)$. The following remarks motivate this.

- Because $\mathcal{J}\mathcal{J}' \subseteq \mathcal{J} \cap \mathcal{J}'$ for any ideals $\mathcal{J}, \mathcal{J}'$, a prime ideal cannot be written as an intersection of two distinct ideals properly containing it. The existence of prime ideals in $\mathscr{B}(X)$ therefore implies a richer ideal lattice structure in $\mathscr{B}(X)$.

- Any maximal ideal in $\mathscr{B}(X)$ is prime, so an investigation into prime ideals offers a chance to apply existing knowledge of maximal ideals in $\mathscr{B}(X)$.

Remark 5.5.4. Primality of a closed ideal only requires us to check the definition for closed ideals. Suppose $\mathcal{I} \subseteq \mathscr{B}(X)$ is a closed ideal and $\mathcal{J}, \mathcal{J}' \subseteq \mathscr{B}(X)$ are ideals satisfying $\mathcal{J}\mathcal{J}' \subseteq \mathcal{I}$. Closure of $\mathcal{I}$ means that $(\overline{\mathcal{J}})(\overline{\mathcal{J}'}) \subseteq \mathcal{I}$. Assuming that one of $\overline{\mathcal{J}}, \overline{\mathcal{J}'}$ lies in $\mathcal{I}$, one of $\mathcal{J}, \mathcal{J}'$ does too.

Since a maximal ideal is always prime, the task lies in classifying the non-maximal closed prime ideals. Using well-known properties of operator ideals, there will always be at least one.

Proposition 5.5.5. *For any infinite-dimensional Banach space X, $\{0\}$ is a non-maximal closed prime ideal in $\mathscr{B}(X)$.*

Proof. Suppose $\mathcal{J}\mathcal{J}' \subseteq \{0\}$ for two ideals $\{0\} \subsetneq \mathcal{J}, \mathcal{J} \subseteq \mathscr{B}(X)$. Then by Proposition 1.4.3 $\mathscr{F}(X)^2 \subseteq \{0\}$. But this is impossible because $\mathscr{F}(X)$ always contains non-zero idempotents. We conclude that either $\mathcal{J}$ or $\mathcal{J}'$ is equal to $\{0\}$. Finally $\{0\} \subsetneq \mathscr{K}(X) \subsetneq \mathscr{B}(X)$ as X is infinite-dimensional. □

Simple deductions reveal that completely prime ideals will only exist in $\mathscr{B}(X)$ if the space X satisfies certain conditions relating to its cartesian square.

Lemma 5.5.6. *Suppose X is a Banach space for which contains a complemented copy of $X \oplus X$. Then no ideal of $\mathscr{B}(X)$ can be completely prime.*

Proof. Suppose $\mathcal{I} \subset \mathscr{B}(Y \oplus X \oplus X)$ is a completely prime ideal. If P denotes a projection onto one of the X summands, then I_X will factor through P. However $(I_X - P)$ is also bounded below on a copy of X; namely the other X summand. This means I_X also factors through $(I_X - P)$. Since $P(I_X - P) = 0$ we conclude $P(I_X - P) \in \mathcal{I}$, which by complete primality means either $P \in \mathcal{I}$ or $(I_X - P) \in \mathcal{I}$. Either way this forces $\mathcal{I} = \mathscr{B}(Y \oplus X \oplus X)$ which is a contradiction. □

The following well-known observation will generate many positive examples of completely prime ideals.

Lemma 5.5.7. *Let A be a unital Banach algebra. Suppose $\mathcal{M}$ is a maximal ideal of A of codimension 1. Then $\mathcal{M}$ is completely prime.*

Proof. Suppose $a, b \in A$ satisfy $ab \in \mathcal{M}$. There exist $\lambda, \mu \in \mathbb{K}$ such that $a = \lambda 1 + x$ and $b = \mu 1 + y$ for some $x, y \in \mathcal{M}$. Then

$$ab = \lambda\mu 1 + \lambda y + \mu x + xy$$

which implies $\lambda\mu 1 \in \mathcal{M}$, forcing $\lambda\mu 1 = 0$. Without loss of generality this means $\lambda = 0$ hence $a \in \mathcal{M}$ as required. $\square$

Example 5.5.8. *Let X denote one of the following spaces, for which $\mathscr{B}(X)$ has been shown to have an ideal of codimension 1. Then by Lemma 5.5.7 that ideal is completely prime:*

(i) $X = X_{\mathrm{AH}}$, the Banach space constructed by Argyros and Haydon in [3], who showed that $\mathscr{K}(X_{\mathrm{AH}})$ is the unique non-trivial closed ideal of codimension 1 in $\mathscr{B}(X_{\mathrm{AH}})$.

(ii) $X = J_p$ for $1 < p < \infty$. The canonical image $\kappa(J_p)$ in the bidual J_p^{**} has codimension 1, which appears to have first been observed for arbitrary $1 < p < \infty$ by Edelstein and Mityagin in [23]. Consequently $\mathscr{W}(J_p)$ is codimension 1 in $\mathscr{B}(J_p)$.

(iii) $X = C[0, \omega_1)$ for which Loy and Willis observed in [47] that the unique maximal ideal of $\mathscr{B}(C[0, \omega_1))$ has codimension 1 in $\mathscr{B}(C[0, \omega_1))$.

(iv) X is a complex *hereditarily indecomposable space*, so that no closed subspace $Y \subseteq X$ can be written as $Y = Z_1 \oplus Z_2$ for any two infinite-dimensional subspaces Z_1, Z_2. By [33, Theorem 18] $\mathscr{S}(X)$ is a codimension 1 ideal.

Example 5.5.9. *Let $X = \mathscr{X}_\infty$ be the space constructed by Tarbard in his thesis [66] with the property that $\mathscr{B}(X)/\mathscr{K}(X) \approx \ell_1(\mathbb{N}_0)$ as a Banach algebra in the convolution product. Then $\mathscr{K}(X)$ is a non-maximal closed completely prime ideal.* The quotient

$\mathscr{B}(X)/\mathscr{K}(X)$ is therefore an integral domain which forces $\mathscr{K}(X)$ to be completely prime. Since $\ell_1(\mathbb{N}_0)$ has non-trivial ideals, $\mathscr{K}(X)$ cannot be maximal.

Example 5.5.10. *Let $X = \ell_p$ for $1 \leq p < \infty$. There is precisely one non-trivial closed prime ideal in $\mathscr{B}(X)$, namely $\mathscr{K}(X)$.* By the Gohberg-Markus-Feldman Theorem [30], $\mathscr{K}(\ell_p)$ is the unique non-trivial closed ideal, thus is prime. Since $\ell_p \approx \ell_p \oplus \ell_p$, it is not completely prime.

By taking finite direct sums of ℓ_p spaces, we can arrange that precisely n closed prime ideals exist within $\mathscr{B}(X)$, and they are all maximal. The ideal structure of such spaces is sufficiently well understood to demonstrate this. The contents of the following are due to several authors, who we credit in turn. Recall for $X = X_1 \oplus \cdots \oplus X_n$ that $Q_i : X \to X_i$ and $\iota_i : X_i \to X$ denote the projection and inclusion maps respectively.

Theorem 5.5.11. *Let $X = \ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$ for $1 \leq p_1 < \cdots < p_n$. For $1 \leq j \leq n$ we define $\mathbb{E}_j = \{\mathscr{K}(\ell_{p_j}), \mathscr{B}(\ell_{p_j})\}$.*

(a) *Equip $\mathbb{E}_1 \times \cdots \times \mathbb{E}_n$ with the product order. There is a lattice isomorphism*

$$\begin{aligned} \Gamma : \mathbb{E}_1 \times \cdots \times \mathbb{E}_n &\to \{\mathcal{I} \text{ is a closed ideal} : \mathscr{E}(X) \subseteq \mathcal{I} \subseteq \mathscr{B}(X)\} \\ \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n) &= \{T \in \mathscr{B}(X) : \mathcal{I}_j = \mathscr{K}(\ell_{p_j}) \implies Q_j T \iota_j \in \mathscr{K}(\ell_{p_j})\}. \end{aligned}$$

(b) *The composition of n strictly singular operators on X is compact.*

Proof. Part (a) follows as a special case of [41, Theorem 6.12] by excluding the James spaces as instructed by [41, Remark 6.14(ii)].
For part (b), we understand the result is originally due to [69, Theorem 4.8]; a paper in preparation at the time of writing this thesis. However, the result is also a special case of [44, Theorem 1.1]. □

In the context of ring theory, the following observation is elementary and well-known. It will be useful when considering ideals that do not contain $\mathscr{S}(X)$.

Lemma 5.5.12. *Let R be a ring. Suppose P is a prime ideal in R. Then for all ideals $I \subseteq R$ satisfying $I^n \subseteq P$ for some $n \in \mathbb{N}$, we have $I \subseteq P$.*

Proof. By induction on n: if $n = 1$ then we have nothing to prove. Suppose now that the statement holds for $n \geq 1$. If $I^{n+1} \subseteq P$ then $I \cdot I^n \subseteq P$. By primality of P either $I \subseteq P$ or $I^n \subseteq P$. By the inductive hypothesis we conclude that $I \subseteq P$ as required. $\square$

Example 5.5.13. *Let $X = \ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$ for $1 < p_1 < \cdots < p_n$. Then $\mathscr{B}(X)$ has precisely n non-trivial closed prime ideals, all of which are maximal.* First, we observe that $\mathscr{E}(X) = \mathscr{S}(X)$; indeed each ℓ_{p_j} is subprojective and finite direct sums of subprojective spaces are subprojective by [53, Proposition 2.2(a)]. We now consider two cases for a closed ideal $\{0\} \subsetneq \mathcal{I} \subseteq \mathscr{B}(X)$.

Case 1: $\mathscr{S}(X) \subseteq \mathcal{I}$. If $\mathcal{I}$ is maximal, we have nothing to prove. Assuming $\mathcal{I}$ is non-maximal, let $(\mathcal{I}_1, \ldots, \mathcal{I}_n) = \Gamma^{-1}(\mathcal{I})$ where Γ is the isomorphism in Theorem 5.5.11(a). Define

$$N = \{1 \leq j \leq n : \mathcal{I}_j = \mathscr{K}(\ell_{p_j})\}.$$

By non-maximality of $\mathcal{I}$ we deduce $|N| \geq 2$. Let $N', N'' \subsetneq N$ be distinct such that $N' \cup N'' = N$. Consider the following associated closed ideals in $\mathscr{B}(X)$:

$$\mathcal{I}' = \{T \in \mathscr{B}(X) : j \in N' \implies Q_j T \iota_j \in \mathscr{K}(\ell_{p_j})\},$$
$$\mathcal{I}'' = \{T \in \mathscr{B}(X) : j \in N'' \implies Q_j T \iota_j \in \mathscr{K}(\ell_{p_j})\}$$

Then $\mathcal{I}'$ and $\mathcal{I}''$ are not contained in $\mathcal{I}$ by Theorem 5.5.11 but we claim $\mathcal{I}'\mathcal{I}'' \subseteq \mathcal{I}$. Let $m \in N$. We must show that if $S \in \mathcal{I}'$ and $R \in \mathcal{I}''$ then $Q_m S R \iota_m \in \mathscr{K}(\ell_{p_m})$. By the exposition in Section 1.4 this corresponds to the following sum:

$$\sum_{r=1}^{m-1} (Q_r S \iota_m)(Q_m R \iota_r) + (Q_m S \iota_m)(Q_m R \iota_m) + \sum_{r=m+1}^{n} (Q_r S \iota_m)(Q_m R \iota_r).$$

The compactness of $Q_r S \iota_m$ for $1 \leq r \leq m-1$ and $Q_m R \iota_r$ for $m+1 \leq r \leq n$ ensures that the two sums above lie in $\mathscr{K}(\ell_{p_m})$. Because $N' \cup N'' = N$, the middle term lies

in $\mathscr{K}(\ell_{p_m})$ also. This proves the claim, so non-primality of $\mathcal{I}$ is witnessed.
Case 2: $\mathscr{S}(X) \nsubseteq \mathcal{I}$. Then because of $\mathscr{S}(X)^n \subseteq \mathcal{I}$ and Lemma 5.5.12, the ideal $\mathcal{I}$ cannot be prime. Thus, every non-trivial closed prime ideal in $\mathscr{B}(X)$ is maximal.

With the exception of $\mathscr{X}_\infty$, all the examples presented so far have been maximal ideals or $\{0\}$. The following sufficiency condition will generate lots of examples where $\mathscr{S}(X)$ is prime, and includes the aforementioned application of Theorem 5.3.10 mentioned in the section title. First, we define

$$\mathscr{G}_D(E) := \overline{\{UV : U \in \mathscr{B}(D,E), V \in \mathscr{B}(E,D)\}}$$

Proposition 5.5.14. *Let X be a Banach space which is saturated with complemented copies of some infinite-dimensional Banach space D. Let $\mathcal{I} \subseteq \mathscr{B}(X)$ be a closed ideal.*

(a) *If $\mathcal{I} \nsubseteq \mathscr{S}(X)$ then $\mathscr{G}_D(X) \subseteq \mathcal{I}$.*

(b) *The ideal $\mathscr{S}(X)$ is prime.*

(c) *If $(X,D) = (B_p, \ell_p)$ for $1 < p < \infty$ or $(X,D) = (S_p, c_0)$ for $1 \leq p < \infty$ then no closed ideal $\{0\} \subsetneq I \subsetneq \mathscr{S}(E)$ is prime.*

Proof. For (a) it suffices to show for $S \in \mathscr{B}(X,D)$ and $R \in \mathscr{B}(D,X)$ that $RS \in \mathcal{I}$. Suppose $T \in \mathcal{I} \setminus \mathscr{S}(X)$. Then by Proposition 5.2.2 there exist $U \in \mathscr{B}(X,D)$ and $V \in \mathscr{B}$ such that $I_D = UTV$. But then $RS = R(UTV)S = (RU)T(VS) \in \mathcal{I}$. Item (b) now readily follows; for any ideals $\mathcal{J}, \mathcal{J}' \nsubseteq \mathscr{S}(X)$ we have $\mathscr{G}_D(X) \subseteq \mathcal{J}, \mathcal{J}'$. Since $\mathscr{G}_D(X) = \overline{\langle P \rangle}$ where $P : X \to X$ is a projection onto a copy of D, we have $P = P^2 \in \mathcal{J}\mathcal{J}'$ and thus $\mathcal{J}\mathcal{J}' \nsubseteq \mathscr{S}(X)$ as required. For (c) suppose that $\{0\} \neq \mathcal{I} \subsetneq \mathscr{S}(E)$. Then $\mathscr{K}(E) \subseteq \mathcal{I}$ and non-primality is witnessed by $\mathcal{J} = \mathcal{J}' = \mathscr{S}(E)$ and Theorem 5.3.10. □

Remark 5.5.15. Whilst complemented saturation of X implies $\mathscr{S}(X)$ is prime the converse fails spectacularly; recall from [33, Theorem 18] that $\mathscr{S}(X)$ is completely prime for any *hereditarily indecomposable* space X. By definition though, there are no infinite-dimensional subspaces $Y, Z \subseteq X$ satisfying $X = Y \oplus Z$.

Example 5.5.16. *For the following Banach spaces X and any $1 \leq p < \infty$, the ideal $\mathscr{S}(X)$ is prime and non-maximal:*

(1) $X = d(w,p)$, a Lorentz sequence space (see Section 6.5 for details),

(2) $X = J_p$, a quasireflexive James space.

Indeed for (1) The space $d(w,p)$ is complementably ℓ_p-saturated by [45] and $\mathscr{S}(d(w,p))$ is non-maximal by [37, Theorem 5.4]. For (2) the space J_p is complementably ℓ_p saturated by [9] and $\mathscr{S}(J_p) \subsetneq \mathscr{W}(J_p)$. In both cases Proposition 5.5.14(b) applies and $\mathscr{S}(X)$ is prime.

Our final example demonstrates that for certain separable Banach spaces X, infinitely many non-maximal closed prime ideals reside in $\mathscr{B}(X)$. We will require some machinery which generalises the matrix representation ideas discussed in Section 1.4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Let $X := (\oplus_{n\in\mathbb{N}} X_n)_{c_0}$. For $i \in \mathbb{N}$ denote by $\iota_i : X_i \to X$ and $Q_i : X \to X_i$ the ith canonical inclusion and projection maps respectively.

Corollary 5.5.17. *[40, Corollary 5.11(i)] Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Suppose X_n fails to contain a subspace isomorphic to c_0 for each $n \in \mathbb{N}$. Then matrix multiplication works in X; that is, for each $S,T \in \mathscr{B}(X)$ and each $j,l \in \mathbb{N}$ the following is true with respect to the norm in $\mathscr{B}(X_l, X_j)$*

$$Q_j(ST)\iota_l = \sum_{r=1}^{\infty}(Q_j S\iota_r)(Q_r T\iota_l).$$

Proposition 5.5.18. *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of infinite-dimensional Banach spaces such that for all $1 \leq k \neq l < \infty$*

(1) no subspace of X_k is isomorphic to c_0,

(2) $\mathscr{S}(X_k)$ is not a maximal ideal in $\mathscr{B}(X_k)$,

(3) No infinite-dimensional subspace of X_k is isomorphic to a subspace of X_l (they are totally incomparable).

Let $X = (\bigoplus_{k=1}^{\infty} X_k)_{c_0}$. *Then* $\mathscr{B}(X)$ *has infinitely many closed prime ideals which are non-maximal.*

Proof. Fix $1 \leq k < \infty$ and define

$$\mathcal{I}_k = \{T \in \mathscr{B}(X) : Q_k T \iota_k \in \mathscr{S}(X_k)\}.$$

It is clear that $\mathcal{I}_k$ is closed under addition and norm-closed. Let $R \in \mathscr{B}(X)$ and $T \in \mathcal{I}_k$. To show that $RT \in \mathcal{I}_k$, we first acknowledge that for each $i \in \mathbb{N}$, the space X_i fails to contain a copy of c_0. Consequently, Corollary 5.5.17 applies and the operator $Q_k RT \iota_k$ is given by the following series

$$\sum_{r=1}^{\infty} (Q_k R \iota_r)(Q_r T \iota_k).$$

By assumption (3) the summands indexed by $r \neq k$ are strictly singular. Because $Q_k T \iota_k \in \mathscr{S}(X_k)$ the $r = k$ summand is also strictly singular, so closure of the ideal $\mathscr{S}(X_k)$ implies that $Q_k RT \iota_k \in \mathscr{S}(X_k)$. Thus $RT \in \mathcal{I}_k$, and by a similar argument $TR \in \mathcal{I}_k$. Thus $\mathcal{I}_k$ is a closed ideal. To see that $\mathcal{I}_k$ is non-maximal, fix a maximal ideal $\mathscr{S}(X_k) \subsetneq \mathcal{M}_k \subsetneq \mathscr{B}(X_k)$. Define

$$\mathcal{M} = \{T \in \mathscr{B}(X) : Q_k T \iota_k \in \mathcal{M}_k\}.$$

By an argument similar to $\mathcal{I}_k$, $\mathcal{M}$ is also a closed ideal such that $\mathcal{I}_k \subsetneq \mathcal{M} \subsetneq \mathscr{B}(X)$ for each $k \in \mathbb{N}$. We claim that $\mathcal{I}_k$ is prime; suppose $\mathcal{J}\mathcal{J}' \subseteq \mathcal{I}_k$ for some ideals $\mathcal{J}, \mathcal{J}' \subseteq \mathscr{B}(X)$. We define associated subsets of $\mathscr{B}(X_k)$ as follows

$$\mathcal{J}_\odot := \{R \in \mathscr{B}(X_k) : \iota_k R Q_k \in \mathcal{J}\},$$
$$\mathcal{J}'_\odot := \{R \in \mathscr{B}(X_k) : \iota_k R Q_k \in \mathcal{J}'\}.$$

By [41, Lemma 6.1(ii)], both are ideals of $\mathscr{B}(X_k)$. We claim that

$$(\mathcal{J}_\odot)(\mathcal{J}'_\odot) \subseteq \mathscr{S}(X_k).$$

Let $R, R' \in \mathscr{B}(X_k)$ such that $\iota_k R Q_k \in \mathcal{J}, \iota_k R' Q_k \in \mathcal{J}'$. Then because $Q_k \iota_k = I_{X_k}$ we have

$$RR' = (Q_k \iota_k) R (Q_k \iota_k) R' (Q_k \iota_k) = Q_k (\iota_k R Q_k)(\iota_k R' Q_k) \iota_k.$$

Since $\mathcal{J}\mathcal{J}' \subseteq \mathcal{I}_k$ the rightmost expression belongs to $\mathscr{S}(X_k)$. This establishes the inclusion, whence by primality of $\mathscr{S}(X_k)$ we conclude without loss of generality that $\mathcal{J}_\odot \subseteq \mathscr{S}(X_k)$. Finally, let $T \in \mathcal{J}$. Then $Q_k T \iota_k \in \mathcal{J}_\odot \subseteq \mathscr{S}(X_k)$ so $\mathcal{J} \subseteq \mathcal{I}_k$ as required. Thus we have infinitely many closed non-maximal prime ideals in $\mathscr{B}(X)$. $\square$

This produces many examples. Note that c_0 saturation of each S_p for $1 \leq p < \infty$ excludes the p-convexified Schreier spaces from the following construction.

Example 5.5.19. *Let $1 < p_1 < p_2 < \cdots < \infty$. Then for the following Banach spaces X, the algebra $\mathscr{B}(X)$ has infinitely many non-maximal closed prime ideals:*

(1) $X = (\bigoplus_{k\in\mathbb{N}} B_{p_k})_{c_0}$ where each B_{p_k} is a Baernstein space,

(2) $X = (\bigoplus_{k\in\mathbb{N}} d(w,p_k))_{c_0}$ where each $d(w,p_k)$ is a Lorentz sequence space (see Section 6.5 for details).

(3) $X = (\bigoplus_{k\in\mathbb{N}} J_{p_k})_{c_0}$ where each J_{p_k} is a quasireflexive James space.

Indeed the spaces $B_p, d(w,p)$ and J_p are ℓ_p saturated which means they satisfy conditions (1) and (3) of Proposition 5.5.18. Condition (2) of Proposition 5.5.18 has also already been established for each.

In contrast, non-separable examples are immediate from the following work of Daws.

Example 5.5.20. *$X = \ell_p(I), c_0(I)$ for some infinite set I. Then $\mathscr{B}(X)$ has precisely $|I|$ closed non-maximal prime ideals.* Due to M.Daws in [20, Theorem 7.4], $\mathscr{B}(X)$ contains precisely the following closed ideals:

$$\{0\} \subsetneq \mathscr{K}(X) \subsetneq \mathscr{K}_{\aleph_1}(X) \subsetneq \ldots \subsetneq \mathscr{K}_{|I|}(X) \subsetneq \mathscr{B}(X).$$

As explained in [19, Proposition 3.7] if $\sigma < \kappa$ there exists an idempotent $P \in \mathscr{K}_\kappa(X) \setminus \mathscr{K}_\sigma(X)$ which means every non-trivial closed ideal in $\mathscr{B}(X)$ will be prime.

Given that recent progress has involved showing that $2^{\mathfrak{c}}$ closed ideals can reside in $\mathscr{B}(X)$ for X separable, the prospect of finding large families of closed ideals with certain properties (such as primality) offers another avenue for research.

Question 5.5.21. Let $\aleph_0 < \lambda \leq 2^{\mathfrak{c}}$. Does there exist a separable Banach space X with λ closed non-maximal prime ideals in $\mathscr{B}(X)$?

5.6 Subsymmetric basic sequences in the Baernstein and Schreier spaces

A basis for a Banach space is *subsymmetric* if it is unconditional and equivalent to all its subsequences. The unit vector bases for c_0 and ℓ_p, for $1 \leq p < \infty$, are standard examples of subsymmetric bases. Using our previous results, we can show that every subsymmetric basic sequence in the Baernstein and Schreier spaces is equivalent to one of those.

This relies on the following lemma, which is certainly known to specialists; we include a short proof for ease of reference.

Lemma 5.6.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a norm-bounded unconditional basis for a Banach space X. Then either $(x_n)_{n\in\mathbb{N}}$ is weakly null or it admits a subsequence which is equivalent to the unit vector basis for ℓ_1.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ is not weakly null. Then there is a functional $x^* \in X^*$ of norm 1 such that the sequence $(\langle x_n, x^*\rangle)_{n\in\mathbb{N}}$ does not tend to 0, so we can pass to a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ such that $\eta := \inf_{j\in\mathbb{N}}|\langle x_{n_j}, x^*\rangle| > 0$.

Given $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$, for each $1 \leq j \leq m$, we can choose $\sigma_j, \tau_j \in \mathbb{K}$ with $|\sigma_j| = |\tau_j| = 1$ such that $\sigma_j\langle x_{n_j}, x^*\rangle \geq \eta$ and $\tau_j\lambda_j \geq 0$. Let K denote the unconditional constant of the basis $(x_n)_{n\in\mathbb{N}}$. Then we have

$$\begin{aligned} K\Big\|\sum_{j=1}^{m}\lambda_j x_{n_j}\Big\| &\geq \Big\|\sum_{j=1}^{m}\sigma_j\tau_j\lambda_j x_{n_j}\Big\| \geq \Big|\Big\langle\sum_{j=1}^{m}\sigma_j\tau_j\lambda_j x_{n_j}, x^*\Big\rangle\Big| \\ &= \Big|\sum_{j=1}^{m}\tau_j\lambda_j\sigma_j\langle x_{n_j}, x^*\rangle\Big| = \sum_{j=1}^{m}|\lambda_j|\,|\langle x_{n_j}, x^*\rangle| \geq \eta\sum_{j=1}^{m}|\lambda_j|. \end{aligned}$$

This shows that the subsequence $(x_{n_j})_{j\in\mathbb{N}}$ η/K-dominates the unit vector basis for ℓ_1, which in turn trivially dominates every bounded sequence by the triangle inequality. The result follows. □

Remark 5.6.2. A subsymmetric basis $(x_n)_{n\in\mathbb{N}}$ must be semi-normalized. Indeed, if it had no lower norm bound, we could recursively choose an increasing sequence $1 \leq j_1 < j_2 < \cdots$ of integers such that $\|x_{j_n}\| \leq \|x_n\|/n$ for each $n \in \mathbb{N}$. However, this would contradict that $(x_{n_j})_{j\in\mathbb{N}}$ dominates $(x_n)_{n\in\mathbb{N}}$. A similar argument shows that $\sup_{n\in\mathbb{N}}\|x_n\| < \infty$.

Proposition 5.6.3. *Let $(E, D) = (S_p, c_0)$ for some $1 \leq p < \infty$ or $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$. A basic sequence in E is subsymmetric if and only if it is equivalent to the unit vector basis for D.*

Proof. The backward implication is clear. For the forward implication, suppose that $(u_n)_{n\in\mathbb{N}}$ is a subsymmetric basic sequence in E. Remark 5.6.2 shows that it is semi-normalized, so combining Lemma 5.6.1 with the fact ℓ_1 does not embed into E (for instance because E is D-saturated, as shown in [42, Theorem 2.4]), we conclude that (u_n) is weakly null. Hence, by Lemma 5.3.4, it admits a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ which is equivalent to either the unit vector basis for D or a subsequence $(e_{m_n})_{n\in\mathbb{N}}$ of the unit vector basis for E. In the former case, the result follows because (u_n) is equivalent to (u_{k_n}). The latter case is impossible because it would imply that $(e_{m_n})_{n\in\mathbb{N}}$ is subsymmetric, contradicting [42, Theorem 4.1]. □

Chapter 6

Operator algebra structure arising from sequence algebras

6.1 Introduction and statement of main results

Let $1 \leq p < \infty$. As defined by Tonge in [68], a commutative Banach algebra X is said to be a *p-summing algebra* if, for every $\varphi \in X^*$ the following induced operator is p-summing (see Definition 6.3.1):

$$L_\varphi : X \to X^*, L_\varphi x(y) := \varphi(xy).$$

Determining whether X is a p-summing algebra is a worthwhile task, as justified by the results of Charpentier, Tonge and Cole in [14], [68] and [71] respectively. Following their convention, by a *uniform algebra* we mean a Banach algebra isomorphic to a closed subalgebra of $C(K)$ for some compact Hausdorff space K. By a *Q-algebra* we mean a Banach algebra isomorphic to a quotient of a uniform algebra by a closed ideal. By an *operator algebra* we mean a Banach algebra isomorphic to a closed subalgebra of $\mathscr{B}(H)$ for some Hilbert space H.

Theorem 6.1.1. *Let X be a commutative Banach algebra.*

(a) *(Charpentier) X is a Q-algebra if and only if X is a quotient of a commutative 1-summing algebra.*

(b) *(Tonge) If X is a 2-summing algebra, then X is an operator algebra.*

(c) *(Cole) If X is a Q-algebra, then X is an operator algebra.*

An example motivating our work is ℓ_p, in the coordinate-wise multiplication. One sees that coordinate-wise multiplication in ℓ_p forms a Banach algebra due to *unconditionality* of the unit vector basis. Varopoulos established in [70] that ℓ_p is a Q-algebra (1972). Separately, the list of Banach spaces with unconditional bases grew in this period to include those of Schreier, Baernstein, and Tsirelson; constructed to exhibit Banach *space* pathologies. Our goal is to demonstrate how the findings of Varopoulos extend to these spaces and several others. We adopt the convention that $(d_n)_{n\in\mathbb{N}}$ denotes the unit vector basis of ℓ_p and refer the reader forward to Section 2 for the notion of p-convexity.

Theorem 6.1.2. *Let X denote a Banach space over $\mathbb{K}$ with a normalized 1-unconditional basis $(b_n)_{n\in\mathbb{N}}$ and equip X with the coordinatewise multiplication.*

(a) *If the linear map determined by $b_n \to d_n$ extends to an operator $X \to \ell_2$ then X is a 1-summing algebra.*

(b) *Suppose $\mathbb{K} = \mathbb{R}$. If X is p-convex for some $1 < p < \infty$, then X is a p^*-summing algebra. In particular, if $p \geq 2$ then X is an operator algebra.*

We remark here that the condition in Theorem 6.1.2(a) is equivalent to (b_n) being a *Besselian basis* which is explained in [64, Theorem 11.1]. The dual notion of a *Hilbertian basis* can also be found therein. Since Theorem 6.1.2(b) only applies to real Banach algebras, we address the complex case separately. If X is a real Banach space spanned by a normalized 1-unconditional basis, its complex analogue over $\mathbb{C}$ can be recovered by the *complexification* $X_{\mathbb{C}}$ of X. By equipping $X_{\mathbb{C}}$ with Banach algebra structure, we prove the following sufficiency result that, although straightforward, to our knowledge remains absent in the literature.

Theorem 6.1.3. *Let X be a real operator algebra. Then $X_{\mathbb{C}}$ is an operator algebra.*

These investigations originate from discussion with Dr Yemon Choi, who asked the question of whether or not the James algebra J_2 is an operator algebra in the coordinate-wise multiplication (in J_2, the unit vector basis is conditional, placing it outside the scope of our results).
Despite the rather technical assumptions made in Theorem 6.1.2, we have not managed to find a Banach space with a normalized 1-unconditional basis that is *not* isomorphic to an operator algebra in the coordinate-wise multiplication. As such, we make the following ambitious conjecture.

Conjecture 6.1.4. *Let X be a Banach space with a normalized 1-unconditional basis (b_n). Then X is an operator algebra when regarded as a sequence algebra in the coordinate-wise multiplication.*

6.2 Banach algebras and lattices

A 1-unconditional basis induces a rather natural Banach algebra structure as set out in [49, Theorem 4.2.20], which provides our main object of study in this chapter.

Lemma 6.2.1. *Let (b_n) be a normalized 1-unconditional basis for a Banach space X. Whenever $\sum_{n\in\mathbb{N}} \alpha(n)b_n, \sum_{n\in\mathbb{N}} \beta(n)b_n \in X$ we define*

$$\left(\sum_{n\in\mathbb{N}} \alpha(n)b_n\right) \cdot \left(\sum_{n\in\mathbb{N}} \beta(n)b_n\right) := \sum_{n\in\mathbb{N}} \alpha(n)\beta(n)b_n.$$

Then X is a Banach algebra in this multiplication.

Certain results we shall use for summing operators apply to *Banach lattices.* Throughout this text, all our Banach lattices shall be real, that is, over the scalar field $\mathbb{R}$. Whenever a Banach space X over $\mathbb{R}$ has a 1-unconditional basis (b_n), the following operations satisfy the axioms set out in [46, Definition 1.a.1]. For $\sum_{n\in\mathbb{N}} \alpha(n)b_n, \sum_{n\in\mathbb{N}} \alpha(n)'b_n \in X$ we say

$$\sum_{n\in\mathbb{N}} \alpha(n)b_n \vee \sum_{n\in\mathbb{N}} \alpha(n)'b_n := \sum_{n\in\mathbb{N}} \max\{\alpha(n), \alpha(n)'\}b_n,$$

which induces the following positivity and absolute value notions:

$$\sum_n \alpha(n)b_n \geq 0 \text{ if and only if } \alpha(n) \geq 0 \text{ for each } n \in \mathbb{N},$$

$$|x| := x \vee (-x) \text{ so that } |\sum_{n\in\mathbb{N}} \alpha(n)b_n| = \sum_{n\in\mathbb{N}} |\alpha(n)|b_n.$$

As remarked in Section 1.a of [46], the dual of a Banach lattice can also be furnished with a Banach lattice structure. First, we define for a Banach lattice X the *positive cone*

$$X_+ := \{x \in X : x \geq 0\}.$$

Let $x^*, y^* \in X^*$. For $x \in X$ we define

$$(x^* \vee y^*)(x) := \begin{cases} \sup\{x^*(u) + y^*(x-u) : 0 \leq u \leq x\} & \text{if } x \in X_+ \\ \sup\{x^*(u) + y^*(x-u) : 0 \leq u \leq -x\} & \text{if } x \notin X_+ \end{cases}$$

Then $x^* \geq 0$ if and only if $x^*(x) \geq 0$ whenever $x \geq 0$ in X and $|x^*| := x^* \vee (-x^*)$.

Definition 6.2.2. Let X, Y be two Banach lattices. An operator $T : X \to Y$ is *positive* if $Tx \geq 0$ whenever $x \geq 0$ for any $x \in X$.

The notions of *p-convex* and *p-concave* Banach lattices that we shall use are the result of a functional calculus developed by Yudin and Krivine, which can be found in [46, Definition 1.d.3]. For $x_1, \ldots, x_N \in X$, this result gives meaning to the positively homogeneous expression $\left(\sum_{k=1}^{N} |x_k|^p\right)^{\frac{1}{p}}$ so that the following definitions make sense.

Definition 6.2.3. Let X be a Banach lattice and $1 \leq p < \infty$. Then X is said to be *p-convex* if there exists $C > 0$ such that for all $N \in \mathbb{N}$ and vectors $x_1, \ldots, x_N \in X$:

$$\left\| \left(\sum_{k=1}^{N} |x_k|^p \right)^{\frac{1}{p}} \right\| \leq C \left(\sum_{k=1}^{N} \|x_k\|^p \right)^{\frac{1}{p}}. \tag{6.1}$$

Definition 6.2.4. Let X be a Banach lattice and $1 \leq q < \infty$. Then X is said to be *q-concave* if there exists $C > 0$ such that for all $N \in \mathbb{N}$ and vectors $x_1, \ldots, x_N \in X$:

$$\left(\sum_{k=1}^{N} \|x_k\|^q \right)^{\frac{1}{q}} \leq C \left\| \left(\sum_{k=1}^{N} |x_k|^q \right)^{\frac{1}{q}} \right\| \tag{6.2}$$

The following result from [46, Proposition 1.d.4(iii)] establishes the duality between these notions.

Proposition 6.2.5. *Let X be a p-convex Banach lattice. Then X^* is p^*-concave.*

The connection between Definition 6.2.3 and another notion of convexity will be critical for applying Theorem 6.1.2. Earlier in this thesis, we studied the p-convexified Schreier spaces for $1 \leq p < \infty$. Elsewhere in the literature, a similar procedure has been used to p-convexify other Banach spaces. Despite this, we could not find a result which explains why an *arbitrary* Banach space with 1-unconditional basis can be convexified in this manner. In particular, there does not seem to be a simple reason for why the triangle inequality would be satisfied. For completeness, we include an argument for why Banach spaces with 1-unconditional bases can always be p-convexified in this way.

Let X be a Banach space over $\mathbb{K}$ with 1-unconditional basis (b_n). Let $1 < p < \infty$. For a finitely supported vector $\sum_{n\in\mathbb{N}} \alpha(n)b_n \in \operatorname{span}(b_n : n \in \mathbb{N})$ we define the following:

$$\left\|\sum_{n\in\mathbb{N}} \alpha(n)b_n\right\|_{X_p} := \left\|\sum_{n\in\mathbb{N}} |\alpha(n)|^p b_n\right\|_X^{\frac{1}{p}}. \tag{6.3}$$

To verify that $\|\cdot\|_{X_p}$ is a norm on $\operatorname{span}(b_n : n \in \mathbb{N})$, we appeal to the following result of Graham Jameson, which can be found in the thesis of Richard Skillicorn as referenced in the following.

Lemma 6.2.6. *[65, Lemma 5.2.4] Let X be a vector space over $\mathbb{K}$. Let $\nu : X \to [0, \infty)$ have the following properties for all $x, y \in X$ and $\lambda, \mu \in \mathbb{K}$:*

(a) $\nu(\lambda x) = |\lambda|\nu(x)$,

(b) $\nu(x), \nu(y) \leq 1 \implies \nu(x + y) \leq 2$,

(c) *If* $\lambda_n \to \lambda, \mu_n \to \mu$ *as* $n \to \infty$ *then* $\nu(\lambda_n x + \mu_n y) \to \nu(\lambda x + \mu y)$ *as* $n \to \infty$,

(d) $\nu(x) = 0 \implies x = 0$.

Then $\nu(x + y) \leq \nu(x) + \nu(y)$ *so that* ν *is a norm on* X.

Proposition 6.2.7. *Let X be a Banach space over $\mathbb{K}$ with 1-unconditional basis (b_n). Let $1 < p < \infty$. Then*

(i) The map $\|\cdot\|_{X_p} : \operatorname{span}(b_n : n \in \mathbb{N}) \to [0, \infty)$ defined by (6.3) is a norm,

(ii) if X_p denotes the completion of $\operatorname{span}(b_n : n \in \mathbb{N})$ with respect to $\|\cdot\|_{X_p}$, then X_p is a Banach space with 1-unconditional basis (b_n).

Proof. Beginning with part (i), we shall verify the conditions of Lemma 6.2.6 with $\nu(x) = \|x\|_{X_p}$. First, we acknowledge that Lemma 6.2.6(a) and (d) are trivially satisfied. Towards the verification of Lemma 6.2.6(b) suppose for vectors $\sum_n \alpha(n)b_n, \sum_{n\in\mathbb{N}} \beta(n)b_n \in \operatorname{span}(b_n : n \in \mathbb{N})$ that

$$\left\|\sum_{n\in\mathbb{N}} |\alpha(n)|^p b_n\right\|_X^{\frac{1}{p}}, \left\|\sum_{n\in\mathbb{N}} |\beta(n)|^p b_n\right\|_X^{\frac{1}{p}} \leq 1.$$

We claim that

$$\left\|\sum_{n\in\mathbb{N}} |\alpha(n) + \beta(n)|^p b_n\right\|_X^{\frac{1}{p}} \leq 2.$$

First, note that the Hölder inequality implies that for all $\alpha, \beta \in \mathbb{K}$

$$\begin{aligned} |\alpha+\beta|^p &\leq (|\alpha|^p + |\beta|^p)(1^{p*} + 1^{p*})^{\frac{p}{p*}} \\ &\leq 2^{p-1}(|\alpha|^p + |\beta|^p). \end{aligned}$$

Together with 1-unconditionality of (b_n) in X, this implies

$$\begin{aligned} \left\|\sum_{n\in\mathbb{N}} |\alpha(n) + \beta(n)|^p b_n\right\|^{\frac{1}{p}} &\leq \left\|\sum_{n\in\mathbb{N}} 2^{p-1}(|\alpha(n)|^p + |\beta(n)|^p) b_n\right\|^{\frac{1}{p}} \\ &= 2^{\frac{p-1}{p}} \left\|\sum_{n\in\mathbb{N}} (|\alpha(n)|^p + |\beta(n)|^p) b_n\right\|^{\frac{1}{p}} \\ &\leq 2^{\frac{p-1}{p}} \left(\left\|\sum_{n\in\mathbb{N}} |\alpha(n)|^p b_n\right\| + \left\|\sum_{n\in\mathbb{N}} |\beta(n)|^p b_n\right\|\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{p-1}{p}} 2^{\frac{1}{p}} \\ &= 2, \end{aligned}$$

which proves the claim. To verify Lemma 6.2.6(c) let $\sum_{n=1}^N \alpha(n)b_n, \sum_{n=1}^N \beta(n)b_n \in \operatorname{span}(b_n : n \in \mathbb{N})$ and $\lambda_m \to \lambda, \mu_m \to \mu$ as $m \to \infty$. For each $1 \leq n \leq N$

$$\lambda_m \alpha_n \to \lambda \alpha_n \quad \text{and} \quad \mu_m \beta_n \to \mu \beta_n \quad \text{as} \quad m \to \infty.$$

It then follows that

$$\sum_{n=1}^N |\lambda_m \alpha_n + \mu_m \beta_n|^p b_n \to \sum_{n=1}^N |\lambda \alpha_n + \mu \beta_n|^p b_n$$

and hence by continuity of the norm in X

$$\left\|\sum_{n=1}^{N}|\lambda_m\alpha_n+\mu_m\beta_n|^p b_n\right\|_X^{\frac{1}{p}} \to \left\|\sum_{n=1}^{N}|\lambda\alpha_n+\mu\beta_n|^p b_n\right\|^{\frac{1}{p}}.$$

This verifies condition (c). By Lemma 6.2.6 we conclude that $\|\cdot\|_{X_p}$ is a norm on $\mathrm{span}(b_n : n\in\mathbb{N})$.

For part (ii) we define X_p as the completion of $\mathrm{span}(b_n : n\in\mathbb{N})$ with respect to $\|\cdot\|_{X_p}$ and the result is a Banach space. Finally, we observe that 1-unconditionality of (b_n) in X_p is an clear consequence of condition (2) in Proposition 1.1.9 being satisfied by the basis in X. □

It is now clear that X_p can be equipped with the Banach lattice structure for a 1-unconditional basis described at the start of this section. Crucially, we now explain why this construction is actually a convexification in the sense of Definition 6.2.3. While this will not enlighten specialists, it does keep our work self-contained.

Lemma 6.2.8. *Let X be a Banach space with 1-unconditional basis $(b_n)_{n\in\mathbb{N}}$. Then as a Banach lattice, X_p is p-convex.*

Proof. Let $x_1,\ldots,x_N\in X_p$. We claim that the following is true:

$$\left(\sum_{k=1}^{N}|x_k|^p\right)^{\frac{1}{p}} \leq \sum_{j\in\mathbb{N}}\left(\sum_{k=1}^{N}|\langle x_k,b_j^*\rangle|^p\right)^{\frac{1}{p}} b_j \tag{6.4}$$

Assuming this, checking Equation (6.1) is straightforward:

$$\begin{aligned}\left\|\left(\sum_{k=1}^{N}|x_k|^p\right)^{\frac{1}{p}}\right\|_{X_p} &\leq \left\|\sum_{j\in\mathbb{N}}\left(\sum_{k=1}^{N}|\langle x_k,b_j^*\rangle|^p\right)^{\frac{1}{p}} b_j\right\|_{X_p} = \left\|\sum_{j\in\mathbb{N}}\sum_{k=1}^{N}|\langle x_k,b_j^*\rangle|^p b_j\right\|_X^{\frac{1}{p}}\\ &= \left\|\sum_{k=1}^{N}\sum_{j\in\mathbb{N}}|\langle x_k,b_j^*\rangle|^p b_j\right\|_X^{\frac{1}{p}} \leq \left(\sum_{k=1}^{N}\|x_k\|_{X_p}^p\right)^{\frac{1}{p}}.\end{aligned}$$

We begin by justifying that the series on the right hand side of (6.4) converges in X_p, by verifying it forms a Cauchy sequence. Let $\varepsilon>0$. Since $x_1,\ldots,x_N\in X_p$, the series expansion of each forms a Cauchy sequence. Hence, there exists $m \in \mathbb{N}$ such that for all $1 \leq i \leq N$ and integers $m \leq r \leq s$ we have

$$\left\| \sum_{j=r}^{s} |\langle x_k, b_j^* \rangle|^p b_j \right\|_X \leq \frac{\varepsilon}{N}.$$

We then deduce

$$\begin{aligned}
&\left\| \sum_{j=1}^{s} \left(\sum_{k=1}^{N} |\langle x_k, b_j^* \rangle|^p \right)^{\frac{1}{p}} b_j - \sum_{j=1}^{r} \left(\sum_{k=1}^{N} |\langle x_k, b_j^* \rangle|^p \right)^{\frac{1}{p}} b_j \right\|_{X_p} \\
&= \left\| \sum_{j=r+1}^{s} \left(\sum_{k=1}^{N} |\langle x_k, b_j^* \rangle|^p \right) b_j \right\|_X^{\frac{1}{p}} \\
&= \left\| \sum_{k=1}^{N} \left(\sum_{j=r+1}^{s} |\langle x_k, b_j^* \rangle|^p b_j \right) \right\|_X^{\frac{1}{p}} \\
&\leq \left(\sum_{k=1}^{N} \left\| \sum_{j=r+1}^{s} |\langle x_k, b_j^* \rangle|^p b_j \right\|_X \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{N} \frac{\varepsilon}{N} \right)^{\frac{1}{p}} \leq \varepsilon^{\frac{1}{p}}.
\end{aligned}$$

Turning our attention back towards the inequality, we recall for instance from [21, Proposition 16.2(a)]:

$$\left(\sum_{k=1}^{N} |x_k|^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{k=1}^{N} \alpha(k) x_k : (\alpha(k)) \in B_{\ell_{p^*}^N} \right\}.$$

For $(\alpha(k)) \in B_{\ell_{p^*}^N}$ we have

$$\sum_{k=1}^{N} \alpha(k) x_k = \sum_{k=1}^{N} \alpha(k) \sum_{j \in \mathbb{N}} \langle x_k, b_j^* \rangle b_j = \sum_{j \in \mathbb{N}} \left(\sum_{k=1}^{N} \alpha(k) \langle x_k, b_j^* \rangle \right) b_j.$$

Applying the Hölder Inequality for each $j \in \mathbb{N}$ we get

$$\left| \sum_{k=1}^{N} \alpha(k) \langle x_k, b_j^* \rangle \right| \leq \left(\sum_{k=1}^{N} |\langle x_k, b_j^* \rangle|^p \right)^{\frac{1}{p}}.$$

By definition of the lattice ordering this proves Equation (6.4). □

6.3 Summing operators and summing algebras

Definition 6.3.1. Let $1 \leq p < \infty$. An operator $T : X \to Y$ is said to be *p-summing* if there exists $C > 0$ such that for all $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in X$:

$$\left(\sum_{k=1}^{N} \|x_k\|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left\| \sum_{k=1}^{N} \alpha(k) x_k \right\| : (\alpha(k)) \in B_{\ell_{p^*}^N} \right\}.$$

The smallest C for which this inequality holds is denoted by $\pi_p(T)$.

The classes of p-summing operators satisfy an inclusion theorem, which we cite from [21, Theorem 2.8].

Theorem 6.3.2. *Let $1 \leq p < q < \infty$. If $T : X \to Y$ is p-summing, then T is q-summing. Moreover $\pi_q(T) \leq \pi_p(T)$.*

It is straightforward to check that the p-summing operators from X to Y form a subspace of $\mathscr{B}(X, Y)$. In fact, the class of p-summing operators form an operator ideal in the following sense; see [21, Proposition 2.4] for details.

Proposition 6.3.3. *Let $R : W \to X$, $S : X \to Y$ and $T : Y \to Z$ be operators. If S is p-summing, then so is the composition TSR. Moreover $\pi_p(TSR) \leq \|T\| \pi_p(S) \|R\|$.*

The following result of Maurey can be found in [46, Theorem 1.d.10]; it will be crucial in the proof of Theorem 6.1.2.

Theorem 6.3.4. *Let $1 \leq q < \infty$. Let X be a real Banach lattice. Then X is q-concave if and only if for every compact Hausdorff space K, every positive operator from $C(K)$ to X is q-summing.*

We now recall the notion of a *p-summing algebra* as defined in the introduction. In [21, Proposition 18.1] two equivalent notions are presented as follows.

Proposition 6.3.5. *Let X be a commutative Banach algebra. Let $1 \leq p < \infty$. Then the following are equivalent:*

(a) *X is a p-summing algebra, so that for each $\varphi \in X^*$ the following operator is p-summing:*

$$L_\varphi : X \to X^*, L_\varphi x(y) := \varphi(xy)$$

(b) *there exists $C > 0$ so that for all $N \in \mathbb{N}$ and $x_1, \ldots, x_N, y_1, \ldots, y_N \in X$ we have*

$$\left\| \sum_{k=1}^{N} x_k y_k \right\| \leq C \left(\sum_{k=1}^{N} \|x_k\|_X^{p^*} \right)^{\frac{1}{p^*}} \sup \left\{ \left\| \sum_{k=1}^{N} \alpha(k) y_k \right\| : (\alpha(k)) \in B_{\ell_{p^*}^N} \right\}.$$

Remark 6.3.6. If X is a p-summing algebra, then X is a q-summing algebra for all $q > p$. This is immediate from the definition and Theorem 6.3.2

6.4 Proof of Theorem 6.1.2

Lemma 6.4.1. *Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$ with 1-unconditional basis (b_n). Then for any $\varphi \in X^*$, $\sum_{n=1}^\infty \alpha(n)\varphi(b_n)d_n \in \ell_1$ and the map $U_\varphi : X \to \ell_1, \sum_{n=1}^\infty \alpha(n)b_n \mapsto \sum_{n=1}^\infty \alpha(n)\varphi(b_n)d_n$ is bounded.*

Proof. For $n \in \mathbb{N}$ we let $\varepsilon_n \in \mathbb{C}$ satisfy $\varepsilon_n \varphi(b_n)\alpha(n) = |\varphi(b_n)\alpha(n)|$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}} |\varphi(b_n)\alpha(n)| = \sum_{n\in\mathbb{N}} \varepsilon_n \varphi(b_n)\alpha(n) &= \left| \varphi\left(\sum_{n\in\mathbb{N}} \varepsilon_n \alpha(n) b_n \right) \right| \\ &\leq ||\varphi|| \cdot \left\| \sum_{n\in\mathbb{N}} \varepsilon_n \alpha(n) b_n \right\| = \|\varphi\| \cdot \left\| \sum_{n\in\mathbb{N}} \alpha(n) b_n \right\|.\end{aligned}$$

We conclude $||U_\varphi|| \leq ||\varphi||$. $\square$

Proof of Theorem 6.1.2(a). By assumption there exists a well–defined operator $\iota : X \to \ell_2$ that satisfies $\iota(b_n) = d_n$ for each $n \in \mathbb{N}$. Suppose $\|\iota\| = C$. Then $\iota^* : \ell_2 \to X^*$ is also an operator. Next , we claim L_φ can be factored as follows:

$$\begin{array}{ccc} X & \xrightarrow{L_\varphi} & X^* \\ {\scriptstyle U_\varphi}\downarrow & & \uparrow{\scriptstyle \iota^*} \\ \ell_1 & \xrightarrow{J} & \ell_2 \end{array}$$

where U_φ is defined as in Lemma 6.4.1 and J is the formal inclusion operator. To see that this diagram is commutative, let $x = \sum_n \alpha(n)b_n, y = \sum_n \beta(n)b_n$ in X:

$$\begin{aligned}\langle \iota^* JU_\varphi x, y\rangle = \langle JU_\varphi x, \iota y\rangle &= \langle (\alpha(n)\varphi(b_n))_{n\in\mathbb{N}}, (\beta(n))_{n\in\mathbb{N}}\rangle \\ &= \sum_{n\in\mathbb{N}} \alpha(n)\varphi(b_n)\beta(n) = \varphi\left(\sum_{n\in\mathbb{N}} \alpha(n)\beta(n)b_n\right) = L_\varphi x(y).\end{aligned}$$

By [21, Theorem 1.13] J is known to be 1-summing, with $\pi_1(J) = \kappa_G$ the Grothendieck constant. We conclude by Proposition 6.3.3 that L_φ is 1-summing with $\pi_1(L_\varphi) \leq ||\varphi|| \cdot \kappa_G \cdot C$. □

Proof of Theorem 6.1.2(b). We will factor L_φ through c_0 and then appeal to Theorem 6.3.4. For $n \in \mathbb{N}$ let $\varepsilon_n \in \{-1, 1\}$ be such that $\varphi(b_n) = |\varphi(b_n)|$. First, we observe that the following is an operator of norm 1:

$$S : X \to c_0, \;\; S\left(\sum_{n\in\mathbb{N}} \alpha(n)b_n\right) := (\varepsilon_n\alpha(n))_{n\in\mathbb{N}}$$

Next, we let $w = (\gamma_n)_{n\in\mathbb{N}} \in c_0$. By 1-unconditionality of $(b_n)_{n\in\mathbb{N}}$ we can define a diagonal operator $R_w : X \to X$ as follows:

$$R_w\left(\sum_{n\in\mathbb{N}} \alpha(n)b_n\right) := \sum_{n\in\mathbb{N}} \varepsilon_n\gamma_n\alpha(n)b_n$$

for which we observe $\|R_w\| = \|w\|_\infty$. Hence $R_w^*\varphi \in X^*$. We claim that the following is then an operator:

$$T : c_0 \to X^*, T(w) := R_w^*\varphi.$$

To see that T is linear let $w = (\gamma_n) \in c_0$, $v = (\mu_n)_{n\in\mathbb{N}}, \lambda \in \mathbb{R}$ and $x = \sum_{n\in\mathbb{N}} \alpha(n)b_n \in X$ and then

$$\begin{aligned}\langle x, T(\lambda v + w)\rangle = \langle R_{\lambda v+w}x, \varphi\rangle &= \langle \sum_{n\in\mathbb{N}} \varepsilon_n(\lambda\mu_n + \gamma_n)\alpha(n)b_n, \varphi\rangle \\ &= \lambda\langle \sum_{n\in\mathbb{N}} \varepsilon_n\mu_n\alpha(n)b_n, \varphi\rangle + \langle \sum_{n\in\mathbb{N}} \varepsilon_n\gamma_n\alpha(n)b_n, \varphi\rangle \\ &= \lambda\langle R_v x, \varphi\rangle + \langle R_w x, \varphi\rangle \\ &= \lambda\langle x, R_v^*\varphi\rangle + \langle x, R_w^*\varphi\rangle \\ &= (\lambda Tv + Tw)(x).\end{aligned}$$

To see that $T : c_0 \to X^*$ is bounded:

$$\|Tw\| = \|R_w^* \varphi\| \leq \|R_w^*\|\|\varphi\| = \|w\|_\infty \|\varphi\|,$$

so that $\|T\| \leq \|\varphi\|$. The operator T is positive; suppose $w = (\gamma_n)_{n\in\mathbb{N}} \in c_0$ satisfies $\gamma_n \geq 0$ for each $n \in \mathbb{N}$. We must show that $Tw \geq 0$ with respect to the lattice ordering in X^*, that is $\langle x, Tw\rangle \geq 0$ whenever $x \geq 0$ in X. Writing $x = \sum_{n\in\mathbb{N}} \alpha(n) b_n$ this is equivalent to each $\alpha(n) \geq 0$. Then applying Tw to x:

$$\langle x, Tw\rangle = \sum_{n\in\mathbb{N}} \varepsilon_n \gamma_n \alpha(n) \langle b_n, \varphi\rangle = \sum_{n\in\mathbb{N}} \alpha(n)\gamma_n |\langle b_n, \varphi\rangle|,$$

which is clearly non-negative. Finally we check that $TS = L_\varphi$. Letting $\sum_{n\in\mathbb{N}} \beta(n) b_n \in X$ we calculate:

$$\begin{aligned} TS\left(\sum_{n\in\mathbb{N}} \alpha(n) b_n\right)\left(\sum_{n\in\mathbb{N}} \beta(n) b_n\right) &= R^*_{(\varepsilon_n \alpha(n))}\varphi\left(\sum_{n\in\mathbb{N}} \beta(n) b_n\right) \\ &= \sum_{n\in\mathbb{N}} \varepsilon_n^2 \alpha(n)\beta(n)\varphi(b_n) \\ &= L_\varphi\left(\sum_{n\in\mathbb{N}} \alpha(n) b_n\right)\left(\sum_{n\in\mathbb{N}} \beta(n) b_n\right). \end{aligned}$$

Since X^* is p^*-concave by Proposition 6.2.5 we may invoke Theorem 6.3.4 to conclude that T is p^*-summing. By Proposition 6.3.3, we conclude that L_φ is p^*-summing. Finally, when $p \geq 2$ Remark 6.3.6 implies that X is a 2-summing algebra; now Theorem 6.1.1(b) applies. □

6.5 Example 1: Lorentz sequence spaces

The purpose of this section is to provide another classical example where Theorem 6.1.2 applies besides ℓ_p. Let $1 \leq p < \infty$ and $w = (w_n) \in c_0 \setminus \ell_1$ be a positive decreasing sequence. For a sequence $(\alpha(n))_{n\in\mathbb{N}} \in c_0$, we can choose a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\alpha_{\pi(1)}| \geq |\alpha_{\pi(2)}| \geq \cdots$. The *Lorentz sequence space* $d(w,p)$ associated with w, p is defined as follows

$$d(w,p) := \left\{ (\alpha(n))_{n\in\mathbb{N}} \in c_0 : \left(\sum_{n\in\mathbb{N}} |a_{\pi(n)}|^p w_n\right)^{\frac{1}{p}} < \infty \right\}.$$

It is straightforward to check that the unit vector basis (d_n) is a 1-unconditional basis for any Lorentz sequence space. Although cosmetically similar to ℓ_p in the norm, there are stark differences. For instance, the dual of $d(w,p)$ is *never* isomorphic to another Lorentz space (see [45, page 178]). It is clear that $d(w,p)$ is the p-convexification of $d(w,1)$, so the following is true.

Proposition 6.5.1. *Let $w \in c_0 \setminus \ell_1$ be a decreasing positive sequence and $p \geq 2$. Then $d(w,p)$ is an operator algebra in the coordinate-wise multiplication.*

Proof. If $\mathbb{K} = \mathbb{R}$ then we invoke Theorem 6.1.2(b). If $\mathbb{K} = \mathbb{C}$ then we use Theorem 6.1.3 whose proof is given in Section 6.8. $\square$

When $p < 2$ the picture remains unclear. For instance, we cannot even say if $d(\frac{1}{n}, 1)$ is an operator algebra.

6.6 Example 2: Tsirelson's space

The purpose of this section is to provide a non-classical example where Theorem 6.1.2 applies. Tsirelson's space is one of the first 'exotic' Banach spaces, and once realised as an operator algebra, offers some optimism for a positive solution to Conjecture 6.1.4. Following [10], we actually work with the dual of Tsirelson's original space. As we shall see, our results do not rely heavily on its sophisticated construction.

Recall the Schreier chains used for the Baernstein norm. Tsirelson's norm will require chains of a slightly different nature. Let

$$\mathrm{TC} := \{\{E_k\}_{k=1}^{l} : E_1 < E_2 < \cdots < E_l \text{ and } l \leq \min(E_1)\}.$$

Note that the sets E_k do not themselves have to be in $\mathcal{S}_1$. The norm for T is actually a limit of recursively defined norms. Let $x \in c_{00}$. Then

$$\|x\|_{(0)} := \|x\|_\infty,$$
$$\|x\|_{(m+1)} := \max\left\{\|x\|_{(m)}, \frac{1}{2}\max\left\{\sum_{k=1}^{l}\|P_{E_k}x\|_{(m)} : \{E_1,\dots,E_l\} \in \mathrm{TC}\right\}\right\}.$$

Clearly $\|x\|_{(m)} \leq \|x\|_{(m+1)} \leq \|x\|_{\ell_1}$ for all $m \geq 0$, so we can define $\|x\|_T := \lim_{m\to\infty}\|x\|_{(m)}$. Finally, we define T to be the completion of c_{00} with respect to $\|\cdot\|_T$. Until the arrival of Tsirelson's construction, Banach space theorists had speculated on whether or not *every* infinite-dimensional Banach space contained an isomorphic copy of c_0 or some ℓ_p. The construction above is due to Figel and Johnson, which in fact is the dual of the original Tsirelson space. Moreover, it exhibits the same pathology as the original; see [10, Corollary I.6] for details of the following.

Theorem 6.6.1. *For all $1 \leq p < \infty$, there is no isomorphic copy of ℓ_p or c_0 in T.*

By [10, Proposition I.2 (1)], the unit vector basis forms a 1-unconditional basis for T. As shown in [10, Proposition V.10], T also admits the relevant operator required to invoke Theorem 6.1.2(a)

Lemma 6.6.2. *The unit vector basis of T dominates the unit vector basis of ℓ_p for all $p > 1$. In particular, the linear map $\iota : T \to \ell_2, \iota(e_n) = d_n$ is an operator.*

As this result applies over $\mathbb{R}$ or $\mathbb{C}$, either scalar field can be taken for the following conclusion.

Corollary 6.6.3. *In the coordinate-wise multiplications, T is a 1-summing algebra, hence an operator algebra.*

6.7 Example 3: The Schreier and Baernstein spaces

As in previous chapters, let $E = S_p$ for some $1 \leq p < \infty$ or $E = B_p$ for some $1 < p < \infty$. We have already seen that the unit vector basis forms a 1-unconditional basis therein, and we explain here why E is another instance where Theorem 6.1.2(a) applies. Recall that in Remark 2.2.1 and Proposition 2.3.7 it was established that the Baernstein and convexified Schreier norms dominate certain ℓ_p norms. These can be used to apply Theorem 6.1.2.

Corollary 6.7.1. *Let $E = S_p$ for some $1 \leq p < \infty$ or B_p for some $1 < p < \infty$. Then in the coordinate-wise product of the unit vectors, E is an operator algebra.*

Proof. Since the norm of B_p dominates the norm of S_1, which in turn dominates the norm of ℓ_2 by Proposition 2.3.7, the Baernstein spaces are all 1-summing algebras by Theorem 6.1.2(a). If $1 \leq p < 2$ then the norm of S_p dominates the norm of ℓ_2 by Proposition 2.3.7 hence is a 1-summing algebra. For S_p when $p \geq 2$, the case of $\mathbb{K} = \mathbb{R}$ follows from p-convexity and Theorem 6.1.2(b). For the case of $\mathbb{K} = \mathbb{C}$, we refer the reader forward to Theorem 6.1.3 which will be proved in Section 6.8. $\square$

6.8 Elaboration on the case $\mathbb{K} = \mathbb{C}$

It is clear that Theorem 6.1.2(a) applies with both real and complex scalars. However, the penalty of appealing to p-convexity in Theorem 6.1.2(b) is that we ultimately rely on a Banach lattice structure. To circumvent the challenge of developing analogous convexity conditions over $\mathbb{C}$, we instead explain why the complexification of a real operator algebra is itself an operator algebra. This has the advantage of greater generality beyond unconditional bases. Given a real vector space V, we start by forming the *complexified* vector space $V_{\mathbb{C}}$ as instructed in [8, Chapter 13, Definition 1]. Let $\alpha, \beta \in \mathbb{R}$, $v_1, v_2, w_1, w_2 \in V$:

(i) $V_{\mathbb{C}} := V \oplus iV$ so that $(v_1 + iv_2) + (w_1 + iw_2) = (v_1 + w_1) + i(v_2 + w_2)$,

(ii) $(\alpha + i\beta)(v_1 + iv_2) := \alpha v_1 - \beta v_2 + i(\alpha v_2 + \beta v_1)$.

In the special case of a real Hilbert space H, we can equip $H_{\mathbb{C}}$ with the following inner product to produce a complex Hilbert space:

$$\langle h + ik, h' + ik' \rangle := \langle h, h' \rangle + \langle k, k' \rangle + i(\langle k, h' \rangle - \langle h, k' \rangle) \text{ for } h, h', k, k' \in H.$$

For $h, k \in H$ the resulting norm satisfies the following:

$$\|h + ik\|_{H_\mathbb{C}}^2 = \|h\|_H^2 + \|k\|_H^2.$$

In the case of a Banach algebra X, its complexification $X_\mathbb{C}$ can be furnished with a complex Banach algebra structure as follows.

Proposition 6.8.1. *[8, Chapter 13, Proposition 3] Let X be a real Banach algebra. For $x, x', y, y' \in X$, define a product on $X_\mathbb{C}$ as follows:*

$$(x + iy)(x' + iy') := (xx' - yy') + i(xy' + yx').$$

Then there exists a norm $\|\cdot\|_{X_\mathbb{C}}$ which makes $X_\mathbb{C}$ a Banach algebra over $\mathbb{C}$. Moreover it satisfies

$$\max\{\|x\|_X, \|y\|_X\} \leq \|x + iy\|_{X_\mathbb{C}} \leq 2 \max\{\|x\|_X, \|y\|_X\}.$$

Given a homomorphism of real Banach algebras $\theta : A \to B$, there is always an induced homomorphism between the complexifications:

$$\theta_\mathbb{C} : A_\mathbb{C} \to B_\mathbb{C} \qquad \theta_\mathbb{C}(x + iy) = \theta(x) + i\theta(y).$$

Checking linearity of $\theta_\mathbb{C}$ is straightforward. To see that $\theta_\mathbb{C}$ is multiplicative let $x, x', y, y' \in A$. Then

$$\begin{aligned} \theta_\mathbb{C}((x + iy)(x' + iy')) &= \theta_\mathbb{C}(xx' - yy' + i(xy' + yx')) \\ &= \theta(xx' - yy') + i\theta(xy' + yx') \\ &= (\theta(x) + i\theta(y))(\theta(x') + i\theta(y')). \end{aligned}$$

Finally, Proposition 6.8.1 implies boundedness of $\theta_\mathbb{C}$:

$$\|\theta(x) + i\theta(y)\|_{B_\mathbb{C}} \leq 2 \max\{\|\theta(x)\|_B, \|\theta(y)\|_B\} \leq 2\|\theta\| \|x + iy\|_{A_\mathbb{C}}.$$

Proposition 6.8.2. *Suppose $\theta : A \to B$ is an algebra embedding of the real Banach algebra A into the real Banach algebra B. Then the complexification is an embedding for the following algebras:*

$$\theta_\mathbb{C} : A_\mathbb{C} \to B_\mathbb{C} \qquad \theta_\mathbb{C}(x + iy) = \theta(x) + i\theta(y).$$

Proof. As θ is an embedding, there exists $C > 0$ such that $C\|x\|_A \leq \|\theta x\|_B$. We then establish that $\theta_{\mathbb{C}}$ is bounded below

$$\begin{aligned}\|x + iy\|_{A_{\mathbb{C}}} \leq 2\max\{\|x\|_A, \|y\|_A\} &\leq 2C^{-1}\max\{\|\theta(x)\|_B, \|\theta(y)\|_B\} \\ &\leq 2C^{-1}\|\theta_{\mathbb{C}}(x + iy)\|_{B_{\mathbb{C}}}.\end{aligned}$$

Since $\theta_{\mathbb{C}}$ has been shown to be continuous we are done. □

In particular, if $\theta : X \to \mathscr{B}(H)$ is an algebra embedding that witnesses the real Banach algebra X as an operator algebra for some real Hilbert space H, we have an associated algebra embedding $\theta_{\mathbb{C}} : X_{\mathbb{C}} \to \mathscr{B}(H)_{\mathbb{C}}$. It remains to be shown that $\mathscr{B}(H)_{\mathbb{C}}$ is an operator algebra. We explain why the Banach *space* embedding suggested in [51, Section 4.6] is also homomorphic for the particular complexification we are using.

Proposition 6.8.3. *There exists an algebra embedding $\psi : \mathscr{B}(H)_{\mathbb{C}} \to \mathscr{B}(H_{\mathbb{C}})$.*

Proof. For $T + iS \in \mathscr{B}(H)_{\mathbb{C}}$ we define $\psi(T + iS) : H_{\mathbb{C}} \to H_{\mathbb{C}}$ by

$$\psi(T + iS)(h + ik) := (Th - Sk) + i(Tk + Sh) \qquad (h, k \in H).$$

Step 1: We check that $\psi(T + iS)$ is a linear map. Let $h, h', k, k' \in H$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}&\psi(T + iS)(h + ik + \lambda(h' + ik')) \\ &= (T(h + \lambda h') - S(k + \lambda k')) + i(T(k + \lambda k') + S(h + \lambda h')) \\ &= (Th + \lambda Th' - Sk - \lambda Sk') + i(Tk + \lambda Tk' + Sh + \lambda Sh') \\ &= (Th - Sk) + i(Tk + Sh) + \lambda(Th' - Sk') + \lambda i(Tk' + Sh') \\ &= \psi(T + iS)(h + ik) + \lambda\psi(T + iS)(h' + ik').\end{aligned}$$

Step 2: We check that $\psi(T + iS)$ is bounded with norm at most $4\|T + iS\|_{\mathscr{B}(H)_{\mathbb{C}}}$:

$$\begin{aligned}
&\sup\{\|\psi(T+iS)(h+ik)\|_{H_{\mathbb{C}}} : \|h+ik\|_{H_{\mathbb{C}}} \leq 1\} \\
&= \sup\{\|(Th - Sk) + i(Tk + Sh)\|_{H_{\mathbb{C}}} : \|h+ik\|_{H_{\mathbb{C}}} \leq 1\} \\
&\leq \sup\{\|(Th - Sk) + i(Tk + Sh)\|_{H_{\mathbb{C}}} : \|h\|_H, \|k\|_H \leq 1\} \\
&\leq \sup\{\|(Th - Sk)\|_H + \|(Tk + Sh)\|_H : \|h\|_H, \|k\|_H \leq 1\}
\end{aligned}$$

and since $\|Th - Sk\|, \|Tk + Sh\| \leq \|T\| + \|S\|$ for all such $h, k \in H$ we conclude with the aid of Proposition 6.8.1 that the norm is at most $2(\|T\| + \|S\|) \leq 4\max\{\|T\|, \|S\|\} \leq 4\|T + iS\|_{\mathscr{B}(H)_{\mathbb{C}}}$.

Step 3: We check that ψ itself is linear. Let $T, T', S, S' \in \mathscr{B}(H)$, $h, k \in H$ and $\lambda \in \mathbb{C}$:

$$\begin{aligned}
&\psi((T+iS) + \lambda(T' + iS')) \\
&= \psi(T + \lambda T' + i(S + \lambda S'))(h + ik) \\
&= (Th + \lambda T'h - Sk - \lambda S'k) + i(Tk + \lambda T'k + Sh + \lambda S'h) \\
&= \psi(T + iS)(h + ik) + \lambda\psi(T' + iS')(h + ik).
\end{aligned}$$

Step 4: We check that ψ is an algebra homomorphism:

$$\begin{aligned}
&\psi((T+iS)(T'+iS')) \\
&= \psi(TT' - SS' + i(TS' + ST'))(h + ik) \\
&= (TT' - SS')h - (TS' + ST')k + i[(TT' - SS')k + (TS' + ST')h] \\
&= T(T'h - S'k) - S(S'h + T'k) + i[T(T'k + S'h) + S(T'h - S'k)] \\
&= \psi(T + iS)[(T'h - S'k) + i(T'k + S'h)] \\
&= \psi(T + iS)\psi(T' + iS')(h + ik).
\end{aligned}$$

Step 5: We check that ψ is bounded below

$$\begin{aligned}
&\sup\{\|\psi(T+iS)(h+ik)\|_{H_{\mathbb{C}}} : \|h+ik\|_{H_{\mathbb{C}}} \leq 1\} \\
&= \sup\{\|(Th-Sk)+i(Tk+Sh)\|_{H_{\mathbb{C}}} : \|h+ik\|_{H_{\mathbb{C}}} \leq 1\} \\
&\geq \sup\{\max\{\|Th-Sk\|_H, \|Tk+Sh\|_H\} : \|h+ik\|_{H_{\mathbb{C}}} \leq 1\} \\
&\geq \sup\{\max\{\|Th\|_H, \|Sh\|_H\} : \|h\|_H \leq 1\} \\
&= \max\{\|T\|, \|S\|\} \\
&\geq \frac{1}{2}\|T+iS\|_{\mathscr{B}(H)_{\mathbb{C}}}.
\end{aligned}$$ □

Proof of Theorem 6.1.3. If $\theta : X \to \mathscr{B}(H)$ witnesses that X is a real operator algebra, then $\psi \circ \theta_{\mathbb{C}} : X_{\mathbb{C}} \to \mathscr{B}(H)_{\mathbb{C}} \to \mathscr{B}(H_{\mathbb{C}})$ witnesses that $X_{\mathbb{C}}$ is a complex operator algebra. □

6.9 Embedding sequence algebras into $\mathscr{K}(H)$

Next, we discuss the additional prospects of embedding Banach algebras into $\mathscr{K}(H)$, the ideal of *compact* operators on a Hilbert space H. It is easy to see that c_0 embeds isometrically into $\mathscr{K}(\ell_2)$ as a Banach algebra; for $n \in \mathbb{N}$, we embed d_n into $\mathscr{K}(\ell_2)$ as follows:

$$Q_n(d_m) := \begin{cases} d_n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

A less immediate example is ℓ_2, which does occur as a subalgebra of $\mathscr{K}(\ell_2)$; see [15, Section 3.1] for an explicit realization due to Y.Choi and E.Samei. Therein, the embedding $\theta : \ell_2 \to \mathscr{K}(\ell_2)$ satisfies the following bounds:

$$\|x\|_{\ell_2} \leq \|\theta x\| \leq \sqrt{2}\|x\|_{\ell_2} \qquad (x \in \ell_2).$$

Critically, the following result due to Holub implies that ℓ_2 is the *only* reflexive Banach space with an unconditional basis that embeds into $\mathscr{K}(\ell_2)$ as a Banach space, let alone as a Banach algebra.

Theorem 6.9.1. *[34, Theorem 1] Let E be a closed subspace of $\mathscr{K}(\ell_2)$. Then either $E \approx \ell_2$ as a Banach space or E contains a copy of c_0. If it is isomorphic to ℓ_2 then it is complemented in $\mathscr{K}(\ell_2)$.*

In particular this rules out ℓ_p for each $1 \leq p < 2$. Therefore, being a 1-summing algebra is not strong enough to ensure that *compact* operator algebra structure arises. With this in mind, the following question offers an avenue for further investigation.

Question 6.9.2. Which Banach spaces with 1-unconditional bases occur as subalgebras of $\mathscr{K}(H)$?

We can localize the approach used by Choi and Samei in [15, Section 3.1] to add the following Banach space to the list.

Example 6.9.3. Let $X = (\bigoplus_{n\in\mathbb{N}} \ell_2^n)_{c_0}$. We denote the unit vector basis of X as follows:

$$e_j^{(n)} := (0, \dots, 0, \underbrace{e_j}_{n\text{th coordinate}}, 0, \dots)$$

Then $(e_j^{(n)} : n \in \mathbb{N}, 1 \leq j \leq n)$ is a 1-unconditional basis for X. Equipping X with the coordinatewise multiplication, we claim that X algebra embeds into $\mathscr{K}(\ell_2)$ as follows. Fix $n \in \mathbb{N}$. We can define a map $\theta_n : \ell_2^n \to \mathscr{B}(\ell_2^{n+1})$ in the same manner as in [15, Section 3.1]. For $e_i \in \ell_2^{n+1}$:

$$\theta_n e_j^{(n)}(e_i^{(n+1)}) = \begin{cases} e_j^{(n+1)} & \text{if } i = j \text{ or } i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Having used the same embedding as Choi and Samei, the estimate mentioned above will be satisfied for each $n \in \mathbb{N}$:

$$\|x\| \leq \|\theta_n x\| \leq \sqrt{2}\|x\| \qquad (x \in \ell_2^n).$$

Then the associated diagonal operator:

$$\theta : \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0} \to \left(\bigoplus_{n\in\mathbb{N}} \mathscr{B}(\ell_2^{n+1})\right)_{c_0} \qquad \theta((x_n)_{n\in\mathbb{N}}) := (\theta_n x_n)_{n\in\mathbb{N}}$$

is clearly a homomorphism which is bounded above and below. Finally, there exists an algebra embedding τ into the compact operators:

$$\tau : \left(\bigoplus_{n\in\mathbb{N}} \mathscr{B}(\ell_2^{n+1})\right)_{c_0} \to \mathscr{K}\left(\left(\bigoplus_{n\in\mathbb{N}} \ell_2^{n+1}\right)_{\ell_2}\right)$$

$$\tau((T_n)_{n\in\mathbb{N}})((y_n)_{n\in\mathbb{N}}) := (T_n y_n)_{n\in\mathbb{N}}.$$

Upon inspection of the norms of $\|(T_n)_{n\in\mathbb{N}}\|$ and $\|\tau((T_n)_{n\in\mathbb{N}})\|$ it becomes clear that τ is actually isometric:

$$\sup\{\|T_n\| : n \in \mathbb{N}\} \leq \sup\left\{\left(\sum_{n\in\mathbb{N}} \|T_n y_n\|^2\right)^{\frac{1}{2}} : \sum_{n\in\mathbb{N}} \|y_n\|^2 \leq 1\right\} \leq \sup\{\|T_n\| : n \in \mathbb{N}\}.$$

Indeed we can write $\|T_n\| = \|T_n z_n\|$ for some norm 1 vector $z_n \in \ell_2^{n+1}$ to deduce the first inequality and the second is obvious. To see that $\tau((T_n)_{n\in\mathbb{N}})$ is a limit of finite-rank operators, we simply acknowledge the fact that $\|\tau((T_n)_{n>N})\| \leq \sup_{n>N}\|T_n\| \to 0$ as $N \to \infty$.

Bibliography

[1] F. Albiac and N.J. Kalton. *Topics in Banach space theory*. Graduate Texts in Mathematics 233, 2006.

[2] D.E. Alspach and S. Argyros. “Complexity of weakly null sequences”. In: *Diss. Math. (Rozprawy Mat.)* 1992.

[3] S. Argyros and R. Haydon. “A hereditarily indecomposable $\mathcal{L}_\infty$ space that solves the scalar-plus-compact problem”. In: *Acta Math.* 2011, pp. 1–54.

[4] S.A. Argyros, K. Beanland, and P. Motakis. “Strictly singular operators in Tsirelson like spaces”. In: *Illinois J. Math.* 2013, pp. 1173–1217.

[5] A. Baernstein II. “Reflexivity and summability”. In: *Studia Mathematica*. 1972, pp. 91–94.

[6] K. Beanland, T Kania, and N.J Laustsen. “Closed ideals of operators on the Tsirelson and Schreier spaces”. In: *J. Funct. Anal.* 2020, article 108668.

[7] A. Bird and N.J. Laustsen. “An amalgamation of the Banach spaces associated with James and Schreier, Part I: Banach-space structure”. In: *Proceedings of the 19th International Conference on Banach Algebras, Banach Center Publications*. 2010, pp. 45–76.

[8] F. Bonsall and J. Duncan. *Complete Normed Algebras*. Springer–Verlag, Berlin, 1973.

[9] P.G. Casazza, B.L. Lin, and R.H. Lohman. “On James’ Quasi-Reflexive Banach space”. In: *Proc. Amer. Math. Soc.* 1977, pp. 265–271.

[10] P.G. Casazza and T.J. Shura. *Tsirelson's space*. Springer Lecture Notes in Mathematics 1363, Springer-Verlag, Berlin, 1989.

[11] R.M. Causey. "Estimation of the Szlenk index of reflexive Banach spaces using generalized Baernstein spaces". In: *Fund. Math.* 2015, pp. 153–171.

[12] R.M. Causey and A. Pelczar-Barwacz. "Equivalence of block sequences in Schreier spaces and their duals". In: *J. Funct. Anal.* 2025, article 110674.

[13] I. Chalendar and J.R. Partington. "An overview of some recent developments on the invariant subspace problem". In: *Concr. Oper.* 2013, pp. 1–10.

[14] P. Charpentier. "Sur les quotients d'algebres uniformes". In: *C. R. Acad. Sci. Paris*. 1974, pp. 929–932.

[15] Y. Choi and E. Samei. "Quotients of Fourier algebras, and representations which are not completely bounded". In: *Proc. Amer. Math. Soc.* 2013, pp. 2379–2388.

[16] H.V. Chu and T. Schlumprecht. "Higher order Tsirelson spaces and their modified versions are isomorphic". In: *Banach Journal of Mathematical Analysis*. 2024, article 49.

[17] H.G. Dales. "A discontinuous homomorphism from $C(X)$". In: *American Journal of Mathematics*. 1979, pp. 647–734.

[18] H.G. Dales. *Banach algebras and automatic continuity*. London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, volume 24, 2000.

[19] M. Daws. "Closed ideals in the Banach algebra of operators on classical non–separable spaces". In: *Math. Proc. Camb. Phil. Soc.* 2006.

[20] J. Diestel. *Sequences and series in Banach spaces*. Springer-Verlag, New York, 1984.

[21] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators.* Cambridge Studies in Advanced Mathematics, 43, Cambridge University Press, 1995.

[22] N. Dunford and J.T. Schwartz. *Linear operators. Part I: General theory.* Interscience Publishers, Inc., New York, 1958.

[23] I.S. Edelstein and B.S. Mityagin. "Homotopy type of linear groups of two classes of Banach spaces". In: *Functional Anal. Appl.* 1970, pp. 221–231.

[24] P. Enflo. "A counterexample to the approximation problem in Banach spaces". In: *Acta Math.* 1973, pp. 309–317.

[25] J. Esterle. "Injection de semi-groupes divisible dans des algebres de convolution et construction d'homomorphismes discontinus de $C(K)$". In: *Proceedings of the London Mathematical Society.* 1978, pp. 59–85.

[26] H. Fetter and B. Gamboa de Buen. *The James forest.* LMS Lecture Notes 236, Cambridge University Press, 1997.

[27] J. Flores, F. Hernandez, E. Semenov, and P Tradacete. "Strictly singular and power compact operators on Banach lattices". In: *Israel J. Math.* 2012, pp. 323–352.

[28] D. Freeman, T. Schlumprecht, and A. Zsák. "Banach spaces for which the space of operators has $2^{\mathfrak{c}}$ closed ideals". In: *Forum Math., Sigma.* 2021, pp. 1–20.

[29] I. Gasparis and D.H. Leung. "On the complemented subspaces of the Schreier spaces". In: *Studia Math.* 2000, pp. 273–300.

[30] I.C. Gohberg, A.S. Markus, and I.A. Feldman. "Normally solvable operators and ideals associated with them". In: *American Math. Soc. Translat.* 1967, pp. 63–81.

[31] S. Goldberg. "On Sobczyk's Projection Theorem". In: *The American Mathematical Monthly.* 1969, pp. 523–526.

[32] W.T. Gowers. “A solution to Banach’s hyperplane problem”. In: *Bull. London Math. Soc.* 1994, pp. 523–530.

[33] W.T. Gowers and B. Maurey. “The unconditional basic sequence problem”. In: *Journal of the American Mathematical Society.* 1993, pp. 851–874.

[34] J.R. Holub. “On subspaces of separable norm ideals”. In: *Bulletin of the American Mathematical Society.* 1973.

[35] B.E. Johnson. “Continuity of homomorphisms of Banach algebras”. In: *Journal of the London Mathematical Society.* 1967, pp. 537–541.

[36] W.B. Johnson and G. Schechtman. “The number of closed ideals in $L(L_p)$”. In: *Acta Math.* 2021, pp. 103–113.

[37] A. Kamińska, A.I. Popov, A. Tcaciuc, V. Troitsky, and E. Spinu. “Norm closed operator ideals in Lorentz sequence spaces”. In: *Journal of Mathematical Analysis and Applications.* 2012, pp. 247–260.

[38] T. Kania, P. Koszmider, and N.J. Laustsen. “A Weak*-topological dichotomy with applications in operator theory”. In: *Trans. London Math. Soc.* 2014, pp. 1–28.

[39] T. Kania and N.J. Laustsen. “Ideal structure on the algebra of bounded operators acting on a Banach space”. In: *Indiana University Mathematics Journal.* 2017, pp. 1019–1043.

[40] N.J. Laustsen. “Matrix multiplication and composition of operators on the direct sum of an infinite sequence of Banach spaces”. In: *Math. Proc. Camb. Phil. Soc.* 2001, pp. 165–183.

[41] N.J. Laustsen. “Maximal ideals in the algebra of operators on certain Banach spaces”. In: *Proc. Edinb. Math. Soc.* 2002, pp. 103–113.

[42] N.J. Laustsen and J. Smith. “Closed ideals of operators on the Baernstein and Schreier spaces”. In: *Journal of Mathematical Analysis and Applications.* 2025.

[43] N.J. Laustsen and J. Smith. "Strictly singular operators on Baernstein and Schreier spaces". In: *preprint, submitted for publication.* 2025.

[44] N.J. Laustsen and H. Wirzenius. "Compactness of compositions of strictly singular operators on direct sums of Baernstein, Schreier and ℓ_p-spaces". In: *Proceedings of the American Mathematical Society, to appear.* 2025.

[45] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces Part I.* Springer-Verlag, Berlin, 1977.

[46] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces Part II.* Ergeb Math Grenzgeb 97, Springer Verlag.

[47] R.J. Loy and G.A. Willis. "Continuity of derivations on $\mathscr{B}(E)$ for certain Banach spaces E". In: *J. London Math. Soc.* 1989, pp. 327–346.

[48] A. Manoussakis and A. Pelczar-Barwacz. "Small operator ideals on the Schlumprecht and Schreier spaces". In: *J. Funct. Anal.* 2021, article 109156.

[49] R. Megginson. *An introduction to Banach space theory.* Graduate Texts in Mathematics, Springer Verlag, New York, 1998.

[50] V.D. Milman. "Operators of class C_0 and C_0^*". In: *Teor. Funktsii Funkcional. Anal. i Priložen.* 1970, pp. 15–26.

[51] M.S. Moslehian, G.A. Muñoz-Fernández, A.M. Peralta, and J.B. Seoane-Sepúlveda. "Similarities and differences between real and complex Banach spaces: an overview and recent developments". In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas.* 2022.

[52] E. Odell and H.O. Tylli. "Weakly compact approximation in Banach spaces". In: *Transactions of the American Mathematical Society.* 2005, pp. 1125–1159.

[53] T. Oikhberg and E. Spinu. "Subprojective Banach spaces". In: *Journal of Mathematical Analysis and Applications.* 2015, pp. 613–635.

[54] A. Pełczyński. "A note on the paper of I.Singer: Basic sequences and reflexivity of Banach spaces". In: *Studia Mathematica.* 1962, pp. 370–374.

[55] A. Pełczyński. "Banach spaces on which every unconditionally converging operator is weakly compact". In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 1962, pp. 641–648.

[56] A. Pełczyński. "On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in $C(S)$-spaces". In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 1965, pp. 31–36.

[57] A. Pełczyński. "On strictly singular and strictly cosingular operators. II. Strictly singular and strictly cosingular operators in $L(\nu)$-spaces". In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 1965, pp. 37–41.

[58] W. Pfaffenberger. "On the ideals of strictly singular and inessential operators". In: *Proc. Amer. Math. Soc.* 1970, pp. 603–607.

[59] A. Pietsch. *Operator Ideals.* North Holland, 1980.

[60] D. Przeworska–Rolewicz and S. Rolewicz. "Linear operators in Banach spaces". In: *Instytut Matematyczny Polskiej Akademi Nauk (Warszawa).* 1968, 1019–1043, URL https://eudml.org/doc/219294.

[61] E. Saksman and H.-O. Tylli. "Weak compactness of multiplication operators on spaces of bounded linear operators". In: *Math. Scand.* 1992, pp. 91–111.

[62] T. Schlumprecht. "An arbitrarily distortable Banach space". In: *Israel J. Math.* 1991, pp. 81–95.

[63] C.J. Seifert. "Averaging in Banach spaces". In: *Kent State University Graduate School Dissertations.* 1977.

[64] I. Singer. *Bases in Banach spaces Vol.I.* Springer Verlag, Berlin–New York, 1970.

[65] R.J. Skillicorn. *Discontinuous homomorphisms from Banach algebras of operators.* Lancaster University Thesis, 2016.

[66] M. Tarbard. *Operators on Banach spaces of Bourgain-Delbaen type.* Oxford University DPhil Thesis, 2013.

[67] A.E. Tong. “Diagonal submatrices of matrix maps”. In: *Pac. J. Math.* 1970, pp. 551–559.

[68] A. Tonge. “Banach algebras and absolutely summing operators”. In: *Mathematical Proceedings of the Cambridge Philosophical Society.* 1976, pp. 465–473.

[69] H.-O. Tylli and H. Wirzenius. *Structure of closed subideals of* $\mathcal{L}(X)$. Preprint, submitted for publication, 2025.

[70] N. Varopoulos. “Some remarks on Q-algebras”. In: *Annales de l’Institut Fourier.* 1972, pp. 1–11.

[71] J. Wermer. “Quotient algebras of uniform algebras”. In: *Symposium on Function Algebras and Rational Approximation, Univ. of Michigan.* 1969.