Degree Growth in Graded Cluster Algebras

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A thesis submitted for the degree of
Doctor of Philosophy

January, 2026

To Matías.

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Abstract

This thesis focuses on graded cluster algebras, looking specifically at degree growth. We begin by considering the rank 3 skew-symmetric case, building on earlier work by Booker-Price. We establish the existence of fastest growing paths, and compare the behaviour for different initial conditions.

The central part of the thesis concerns the cluster algebra structure on the homogeneous coordinate ring of the Grassmannian. We construct a distinguished mutation path with certain nice properties. In particular, we suggest a way of using this mutation path to define a partial order on cluster variables, making use of perfect matchings on the exchange quivers. We show that, at least in the finite type case, the partial order we obtain coincides with the 'standard' partial order which appears in work of Lenagan and Rigal on quantum graded algebras with a straightening law. We hope that the connection with Lenagan and Rigal's work could be used to transfer the techniques they use in order to establish the homological properties of other classes of (quantum) cluster algebra.

In the final part of the thesis we show that, under mild assumptions, the Segre product of two graded cluster algebras has a natural cluster structure.

Acknowledgements

First of all, I would like to thank my supervisor, Jan Grabowski, for all of his advice and guidance, and for providing interesting research questions for me to work on.

Thank you also to the EPSRC for providing the funding that made this research possible.

I am extremely grateful to my former Master's supervisor, Martina Balagović, for all of the support and encouragement throughout my Masters year and during the PhD application process.

I also extend my thanks to my HDC Chair, David Pauksztello, for making the department a much less scary place.

Thanks should also go to Matthew Pressland for providing helpful comments that helped shape Chapter 4 of this thesis, and to the anonymous referee for many helpful suggestions that have greatly improved the exposition of the results in Chapter 5.

Thank you to my fellow maths PhD students, for providing great company, and for making the department such an enjoyable place to work. A special thank you to Björn for the constant support and reassurance, both mathematical and otherwise, and for reminding me how to show that a sequence converges...

My examiners, Bethany Marsh and Raquel Coelho Simões, deserve thanks for their thoughtful questions, careful reading, and for making the examination process both engaging and constructive.

Finally, but most importantly, thank you to my family. To my Mam and Dad, for supporting me in everything I've ever done.

Declaration

I declare that the work presented in this thesis is, to the best of my knowledge and belief, original and my own work, except for where indicated in Chapter 5. The material has not been submitted, either in whole or in part, for a degree at this, or any other university. This thesis does not exceed the maximum permitted word length of 80,000.

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CHAPTER 1

Introduction

The notion of a cluster algebra was introduced by Fomin and Zelevinsky in the early 2000s in the series of papers [**FZ01, FZ03, BFZ03**], the latter of which was coauthored by Berenstein. Their original goal was to provide an algebraic and combinatorial tool with which to study total positivity and dual canonical bases in algebraic Lie theory. Since their introduction, cluster algebras have found applications in a diverse range of areas of mathematics, with representation theory, algebraic and symplectic geometry, and mathematical physics being a few notable examples. An introductory survey, including many useful references for the interested reader, can be found in [**Kel12**].

Cluster algebras are a class of commutative algebras defined from some initial data via a recursive process known as *mutation*. We will see that this often results in having many more generators than one would expect, but with relations of a particularly nice form. It is the case, for example, that an algebra can be finite dimensional whilst still having infinitely many generators as a cluster algebra. Cluster algebras have a rich combinatorial structure, in particular since the process of mutation is governed by directed graphs, usually called *quivers* in this context.

In its simplest guise, the initial data required in order to define a cluster algebra comes in the form of a pair $(\underline{x}, Q)$, where $\underline{x}$ is an n-tuple of algebraically independent variables known as *cluster variables*, and Q is a quiver on n vertices. We also ask that Q has no loops or 2-cycles. The pair $(\underline{x}, Q)$ is known as a *seed*, and $\underline{x}$ is called a *cluster*. The process of mutation involves replacing one cluster variable in $\underline{x}$ with a new one, a certain rational function in the elements of $\underline{x}$, via a mutation rule determined by the quiver Q. In addition to this, the quiver itself, and hence the cluster variable mutation rule, is changed at each step via 'quiver mutation'. The resulting cluster algebra, denoted by $\mathscr{A}(\underline{x}, Q)$, is the algebra generated by all cluster variables obtained from the initial cluster via mutation in all possible directions.

Often, we will also consider cluster algebras with additional 'frozen' variables (sometimes referred to as coefficient variables in the literature). Frozen variables are simply extra cluster variables at which we do not allow mutation, and which therefore appear in every cluster. It should be pointed out that the cluster algebras described above are, specifically, *skew-symmetric cluster algebras of geometric type.* In fact, the majority of the cluster algebras considered in this thesis will be of this type. In the more general setting, we may no longer consider an exchange quiver, but rather an exchange *matrix*. In the skew-symmetric case, we can recover this matrix by simply taking the skew-symmetrisation of the adjacency matrix of the exchange quiver Q.

One of the main results in Fomin and Zelevinsky's initial series of papers is the *Laurent Phenomenon*. This states that every cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of any given cluster. This is a surprising result—it is not at all obvious, a priori, that cluster variables should have such a nice form. The expressions obtained after mutation often look very complicated, but the Laurent Phenomenon tells us that there is always a way to simplify them into the form described above. Moreover, it is conjectured that every cluster variable can be expressed as a Laurent polynomial with *positive* integer coefficients in the elements of a given cluster. This has been proven to be true in several cases, including for all skew-symmetric cluster algebras in [**LS15**], but it remains open in full generality.

It was shown in [**FZ03**] that 'finite type' cluster algebras, i.e. those with finitely many seeds, admit a classification in terms of Dynkin diagrams or, equivalently, finite type Cartan matrices:

THEOREM 1.0.1 ([**FZ03**, Theorem 1.8]). *For a cluster algebra $\mathscr{A}$, the following are equivalent:*

- (i) *$\mathscr{A}$ is of finite type;*
- (ii) *the set of all cluster variables is finite;*
- (iii) *for every seed $(\underline{x}, B)$ in $\mathscr{A}$, the entries of the matrix B satisfy the inequalities $|b_{ij}b_{ji}| \leq 3$, for all $x_i, x_j \in \underline{x}$;*
- (iv) *the (principal part of the) exchange matrix B is mutation equivalent to a matrix whose 'Cartan counterpart' is of finite type.*

A very accessible, though quite long, proof of this result can be found in Chapter 5 of [**FWZ21d**]. The definition of the Cartan counterpart of a matrix can also be found in [**FWZ21d**].

In this thesis the focus will be on graded cluster algebras, as defined in [**Gra15**]. An additional piece of initial data is required to define a grading on a cluster algebra $\mathscr{A}(\underline{x}, B)$—a grading vector $G \in \mathbb{Z}^n$ such that $B^T G = 0$. In his thesis, [**BP17**], Booker-Price studies gradings on rank 3 skew-symmetric cluster algebras. In particular, he determines which initial exchange matrices yield infinitely/finitely many cluster variables of each degree. We wish to further understand the behaviour of this class of cluster algebras by looking at the growth of cluster variable degrees along certain mutation paths.

Many already well-known algebras have been shown to admit a cluster structure. One important example of such an algebra is $\mathbb{C}[\mathrm{Gr}(k,n)]$; the homogeneous coordinate ring of the *Grassmannian*, i.e. the space of k-dimensional subspaces of an n-dimensional vector space. This was shown to have a cluster structure by Scott in [**Sco06**], though the $k = 2$ case appeared already in [**FZ01**]. The Grassmannian will be our main focus in Chapter 4.

In [**GL09**] the quantisation, $\mathbb{C}_q[\mathrm{Gr}(k,n)]$, of the coordinate ring of the Grassmannian was shown to have the structure of a *quantum cluster algebra*. Quantum cluster algebras were introduced in [**BZ05**] and, like graded cluster algebras, involve an extra piece of initial data. This comes in the form of a matrix Λ which determines a rule for quasi-commutation of cluster variables. In [**LR04**], the notion of a *quantum graded algebra with a straightening law* (QGASL) was introduced as a tool for establishing some nice homological properties of certain rings. In particular, they show that $\mathbb{C}_q[\mathrm{Gr}(k,n)]$ is a QGASL. This structure is then utilised to show that $\mathbb{C}_q[\mathrm{Gr}(k,n)]$ is both 'AS-Cohen-Macaulay' and 'AS-Gorenstein'. An important aspect of Lenagan and Rigal's work is the use of the standard partial order on the generators of $\mathbb{C}_q[\mathrm{Gr}(k,n)]$. We wish to study the connection between this partial order and the (quantum) cluster structure.

It is a very natural question to ask how, if given two cluster algebras, we can combine them to make a new cluster algebra. In [**Pre23**], the *Segre product* of two cluster algebras is shown to have a cluster structure in one particular case—coordinate rings of positroid varieties in the Grassmannian. In Chapter 5 we generalise this construction to the case of graded skew-symmetric cluster algebras.

1.1. Thesis Outline

In Chapter 2 we provide an introduction to the basic theory of (quantum, graded) cluster algebras used throughout this thesis. We begin by covering background on cluster algebras and discussing the main results obtained in Fomin and Zelevinsky's original papers—the *Laurent Phenomenon*, and the classification of finite type cluster algebras. We give a brief introduction to both graded and quantum cluster algebras, the former being the main focus of this thesis. Our attention then turns to two main examples—the cluster structures on the homogeneous coordinate rings of the Matrix algebra and the Grassmannian respectively. These examples will be of particular importance in Chapter 4. Finally, we briefly outline the work carried out in [**LR04**] on *quantum graded algebras with a straightening law*. Whilst the majority of this work will not be used directly in this thesis, we include it here as motivation and to help illustrate a possible future direction for research.

In his thesis, [**BP17**], Booker-Price studies gradings on rank 3 skew-symmetric cluster algebras. He provides a classification of such cluster algebras, detailing which initial gradings produce (in)finitely many cluster variables of each degree. In Chapter 3, we begin by summarising Booker-Price's work. Inspired by this, we then study 'growth' of cluster variable degrees along certain mutation paths in the rank 3 skew-symmetric case. Our goal is to determine a suitable growth function, akin to GK-dimension (see e.g. [**KL00**] for details), which captures the different behaviour occurring dependent upon initial grading conditions. We define the notion of a 'fastest growing path', and compare the growth rates for different initial grading conditions. Unfortunately, with this approach, we were unable to determine a suitable growth function to capture different behaviour coming from different initial grading vectors. What we see instead is that, in general, cluster variable degrees seem to grow extremely fast.

Chapter 4 contains the most significant portion of this thesis. Our focus turns to the Grassmannian cluster algebra, and we construct a distinguished sequence of mutations in $\mathbb{C}[\mathrm{Gr}(k,n)]$ satisfying certain 'nice' properties. In the finite type cases, we have been able to explicitly compute this sequence of mutations (mutation path), and we see that it has the following properties:

(P1) The path includes all Plücker coordinates.

(P2) The sequence of Plücker coordinates obtained is monotonically increasing with respect to the partial order $\leq_{\mathrm{st}}$.

(P3) The path begins (resp. ends) at some well-defined 'minimal' (resp. 'maximal') cluster.

We conjecture that a path with these properties can be found in the infinite type case, and we suggest a potential connection with perfect matchings on the exchange quivers. We see, by explicit computation, that in the finite type case the partial order arising from this mutation path coincides with the standard partial order on Plücker coordinates used in [**LR04**].

A potential future direction would be to further explore possible connections to the work of Lenagan and Rigal. The hope would be to extend the techniques used in their work to other (quantum) cluster algebras, utilising a mutation path to define a partial order on a set of generators. In Section 4.3 we describe a 'nice' mutation path for the coordinate ring of the (quantum) matrix algebra. Whilst this case is very closely related to the Grassmannian, the fact that the techniques carry over so nicely supports the idea that it may be possible to extend this further to other classes of (quantum) graded cluster algebras.

Chapter 5 is joint work with Jan E. Grabowski and can be found at [**GH24**]. Drawing inspiration from [**Pre23**] we show that, with suitable assumptions, the Segre product of two graded cluster algebras has a natural cluster algebra structure. We show that the Segre product is formed via a gluing operation on suitable frozen variables, this is illustrated in the example below. We obtain the following result, and state some basic properties of the Segre product of two cluster algebras.

THEOREM 5.2.7. *Let $\mathscr{A}_i = (\underline{\tilde{x}}_i, \underline{x}_i, B_i, G_i)$, $i = 1, 2$ be graded cluster algebras such that there exist $x \in \underline{\tilde{x}}_1 \setminus \underline{x}_1$ and $y \in \underline{\tilde{x}}_2 \setminus \underline{x}_2$ both of degree 1.*

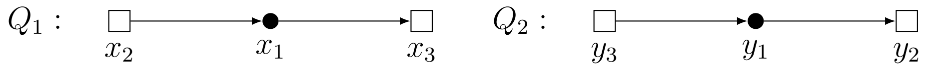
Then the map $\varphi : \mathscr{A}_1 \square \mathscr{A}_2 \to \mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$ given on initial cluster variables by

$$\varphi(x_j) = x_j \otimes y^{\deg x_j} \quad \text{for } x_j \in \underline{\tilde{x}}_1 \setminus \{x\},$$
$$\varphi(y_j) = x^{\deg y_j} \otimes y_j \quad \text{for } y_j \in \underline{\tilde{x}}_2 \setminus \{y\} \text{ and}$$
$$\varphi(z) = x \otimes y$$

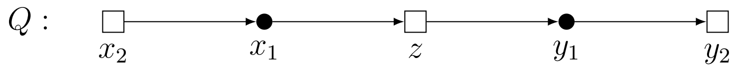
is a graded algebra isomorphism, with the property that the above formulæ hold for any cluster of $\mathscr{A}_1 \square \mathscr{A}_2$.

Thus the construction above endows $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$ *with the structure of a cluster algebra.*

EXAMPLE 5.2.2. *Let* $\mathscr{A}_1 = (\underline{\tilde{x}}_1 = \{x_1, x_2, x_3\}, \underline{x}_1 = \{x_1\}, Q_1, G_1 = \mathbb{1})$ *and* $\mathscr{A}_2 = (\underline{\tilde{x}}_2 = \{y_1, y_2, y_3\}, \underline{x}_2 = \{y_1\}, Q_2, G_1 = \mathbb{1})$ *be cluster algebras with exchange quivers as follows:*



The quiver obtained by 'gluing' at the frozen variables x_3 *and* y_3 *is shown below—we denote the new variable by* z*.*



The cluster algebra $\mathscr{A}_1 \square \mathscr{A}_2$ *is then given by the initial data*

$$(\underline{\tilde{x}} = \{x_1, x_2, y_1, y_2, z\}, \underline{x} = \{x_1, y_1\}, Q, G = \mathbb{1}).$$

Theorem 5.2.7 shows that this gives a cluster structure on the Segre product $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$*.*

The Segre product construction above should be useful for gaining a greater understanding of Segre products of those algebraic varieties known to have cluster structures.

CHAPTER 2

Preliminaries

2.1. Background on Cluster Algebras

2.1.1. Basic Definitions. We begin by setting up notation and recalling the basic definitions and concepts required in order to define a cluster algebra. We broadly follow the notation used in Section 2 of [**BZ05**]. The reader unfamiliar with cluster algebras may also wish to consult e.g. [**FWZ21c**], [**Mar13**].

Definition 2.1.1 (Ambient field). Let $m \geq n$ be positive integers. The *ambient field* $\mathscr{F}$ is the field of rational functions over $\mathbb{Q}$ in m algebraically independent variables.

Cluster algebras, as we will see in what follows, are subrings of the ambient field $\mathscr{F}$ defined above. We start by defining (extended) seeds—these will form the initial data required to define a cluster algebra.

Definition 2.1.2 (Extended seed). An *extended seed* in $\mathscr{F}$ is a pair $(\underline{\tilde{x}}, \tilde{B})$ where

(i) $\underline{\tilde{x}} = \{x_1, \dots, x_m\}$ is a transcendence basis of $\mathscr{F}$ which generates $\mathscr{F}$.
(ii) $\tilde{B}$ is an $m \times n$ integer matrix with rows labelled by $[1, m]$ and columns by the subset $\underline{ex} = [1, n]$ of $[1, m]$.
(iii) The upper $n \times n$ submatrix B of $\tilde{B}$ is skew-symmetrisable.

The set $\underline{\tilde{x}}$ (sometimes considered as a tuple) is called the *extended cluster*, and $\tilde{B}$ is the *extended exchange matrix*. The elements of $\underline{\tilde{x}}$ are referred to as *cluster variables*.

Let $(\underline{\tilde{x}}, \tilde{B})$ be an extended seed, B the principal part of $\tilde{B}$, and $\underline{x} = \{x_1, \dots, x_n\} \subseteq \underline{\tilde{x}}$. The pair $(\underline{x}, B)$ is known as a *seed*—$\underline{x}$ and B are called the *cluster* and *exchange matrix* respectively.

The definition of an extended seed allows us to consider cluster algebras in which some of the cluster variables are 'frozen'. A frozen variable is simply one at which we will not be allowed to mutate. As a result, we will see that frozen variables appear in every cluster. The set of frozen variables is precisely the set $\underline{\tilde{x}} \backslash \underline{x}$, where $\underline{\tilde{x}}$ and $\underline{x}$ are as above. Cluster variables which are not frozen are called *mutable* or, sometimes, *exchangeable*. We note that in the literature it is common to refer to frozen variables as *coefficient* variables.

The recursive procedure by which a cluster algebra is defined, known as mutation, is governed by the (extended) exchange matrix $\tilde{B}$. In order to state the definition of a cluster algebra, we must first describe how to mutate an (extended) seed. Let us begin with cluster mutation.

Given an (extended) seed $(\underline{\tilde{x}} = (x_1, \dots, x_n, \dots, x_m), \tilde{B})$, the *cluster mutation* at x_k for $k \in \{1, \dots, n\}$ is defined as follows:

$$\mu_k(x_i) = \begin{cases} \frac{1}{x_k}\left(\prod_{b_{jk}>0} x_j^{b_{jk}} + \prod_{b_{jk}<0} x_j^{-b_{jk}}\right) & \text{if } i = k, \\ x_i & \text{otherwise.} \end{cases}$$

The mutated cluster is then $\mu_k(\underline{x})$. As mentioned earlier, the exchange matrix is also changed at each step along a sequence of mutations—the *matrix mutation* at k is defined as follows:

$$\mu_k(\tilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise} \end{cases}$$

where $[n]_+ = \max\{n, 0\}$ and $[n]_- = \max\{-n, 0\}$.

We are now able to define the mutated seed $\mu_k((\underline{\tilde{x}}, \tilde{B})) := (\mu_k(\underline{\tilde{x}}), \mu_k(\tilde{B}))$. It is a fairly straightforward exercise to verify that both cluster and matrix mutation are involutions, that is, two successive mutations at the same position will return the seed you began with. Note that we often refer to k above as the 'direction' of mutation.

Informally speaking, the cluster algebra with initial (extended) seed $(\underline{\tilde{x}}, \tilde{B})$, which we will denote by $\mathscr{A}(\underline{\tilde{x}}, \tilde{B})$, is the algebra whose generating set is the set of all cluster variables obtained by repeated mutation of $(\underline{\tilde{x}}, \tilde{B})$ in all possible directions. Before we can proceed with a more formal definition, we must define *mutation paths* and the notion of *mutation equivalence*.

Definition 2.1.3 (Mutation path). A *mutation path* is a sequence of mutations; $\mu_{p_r} \circ \cdots \circ \mu_{p_1}$. For convenience, since our mutation paths will sometimes be fairly long, we borrow the following notation from [**BP17**] and write $[\underline{p}] = [p_r, \dots, p_1]$ to denote the mutation path $\mu_{p_r} \circ \cdots \circ \mu_{p_1}$. Note that in what follows we use the convention that mutations are carried out from right to left.

Definition 2.1.4 (Mutation equivalence). Two (extended) seeds (or clusters, or exchange matrices) are called *mutation equivalent* if one can be obtained from the other via a sequence of mutations, and *essentially equivalent* if they are equal up to permutation of indices.

We are now ready to give a formal definition of a cluster algebra (of geometric type).

Definition 2.1.5 (Cluster algebra). The *cluster algebra* $\mathscr{A}(\underline{\tilde{x}}, \tilde{B})$ is the subring of $\mathscr{F}$ generated by the union of all (extended) clusters mutation equivalent to $\underline{\tilde{x}}$.

Associated to a cluster algebra $\mathscr{A}$ are the following two combinatorial objects.

Definition 2.1.6 (Exchange tree/exchange graph).

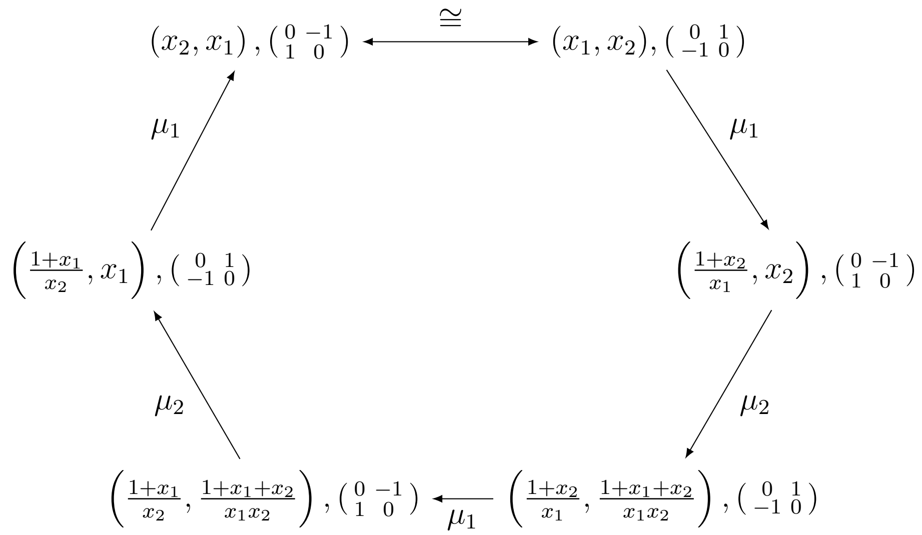
(i) The *exchange tree* of a cluster algebra is the n-regular tree whose vertices correspond to seeds, and whose edges correspond to mutations.

(ii) The *exchange graph* of a cluster algebra is the exchange tree modulo essential equivalence of seeds.

Definition 2.1.7 (Rank). A cluster algebra has rank m if it has m variables in each extended cluster.

Definition 2.1.8 (Finite type). A cluster algebra is of *finite type* if it has finitely many seeds, otherwise it is of *infinite type*.

Example 2.1.9. *The exchange graph for the cluster algebra with initial seed* $(\underline{x}, B) = ((x_1, x_2), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ *is shown below. Note that this is an example of a finite type cluster algebra with five cluster variables.*



2.1.2. Cluster Algebras from Quivers. Throughout this thesis, we will usually work with *skew-symmetric* cluster algebras.

DEFINITION 2.1.10 (Skew-symmetric). A cluster algebra $\mathscr{A}(\underline{x}, B)$ is *skew-symmetric* if its exchange matrix B is a skew-symmetric matrix, i.e. if $B^T = -B$.

When working with skew-symmetric cluster algebras, in place of the exchange matrix, we will often consider the exchange *quiver*. By quiver, we simply mean directed graph. In order for a quiver Q to be an exchange quiver, we require the following:

- No loops, i.e. no arrows $i \to i$.
- No 2-cycles, i.e. no pairs of arrows $i \rightleftarrows j$.

Definition 2.1.2 remains unchanged; we simply replace the (extended) exchange matrix with an exchange *quiver*.

Working with exchange quivers over exchange matrices where possible is often convenient, in particular since it provides a nice way to visualise mutation. It also allows us to have the data of both the cluster and the exchange matrix in one picture. In Section 2.1.1 we saw how to mutate an exchange matrix—we now describe the corresponding procedure for exchange quivers.

Let $(\underline{\tilde{x}}, \tilde{Q})$ be an extended seed. Then the cluster mutation at $x_k \in \underline{\tilde{x}}$ is defined as follows:

$$\mu_k(x_i) = \begin{cases} \frac{1}{x_k}\left(\prod_{i\to k\in Q} x_i^{|\{i\to k\in Q\}|} + \prod_{k\to i\in Q} x_i^{|\{k\to i\in Q\}|}\right) & \text{if } i = k, \\ x_i & \text{otherwise.} \end{cases}$$

The mutated cluster is then $\mu_k(\underline{\tilde{x}}) = (x_1, \ldots, \mu_k(x_k), \ldots, x_m)$. The mutation $\mu_k(Q)$ of a quiver Q in direction k is obtained in the following way:

(i) For any pair $i \to k \to j$ in Q, add an arrow $i \to j$.

(ii) Reverse arrows incident to k.

(iii) Remove any resulting 2-cycles.

The mutated seed $\mu_k(\underline{x}, Q)$ is $(\mu_k(\underline{x}), \mu_k(Q))$. An example of quiver mutation is illustrated in Figure 2.1.

REMARK 2.1.11. *We can recover the corresponding exchange matrix $B = (b_{ij})$ by taking the skew-symmetrisation of the adjacency matrix of $\tilde{Q}$, i.e. by setting $b_{ij} = |\{\text{arrows } i \to j\}| - |\{\text{arrows } j \to i\}|$.*

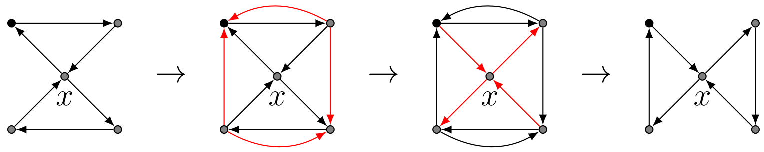
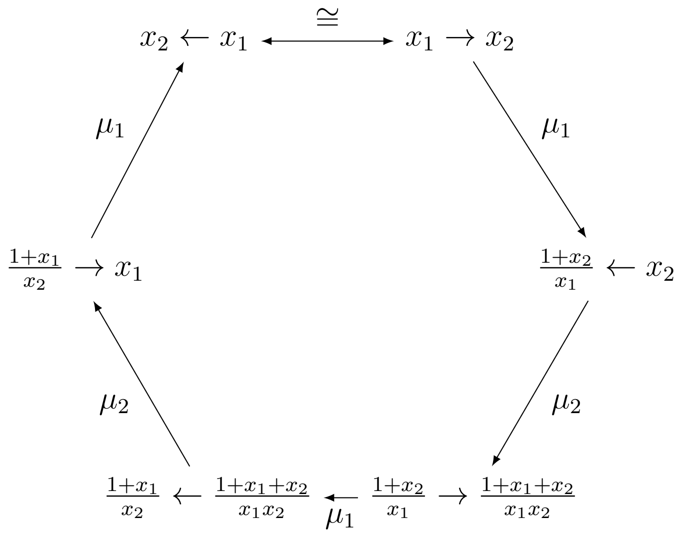


FIGURE 2.1. A step-by-step example of quiver mutation at the vertex x. New or reversed arrows are shown in red at each stage.

REMARK 2.1.12. *We typically omit any arrows between pairs of vertices corresponding to frozen variables, since these play no role in mutation. However, we will see in Chapter 4 that these arrows are useful when we wish to consider an exchange quiver as a 'quiver with faces'.*

EXAMPLE 2.1.13. *The exchange graph for the cluster algebra with initial seed* $(\underline{x}, Q) = ((x_1, x_2), 1 \to 2)$ *is shown below.*



We see that after 5 mutation steps, the seed we obtain is essentially equivalent to our initial seed. We can therefore conclude that we have found all cluster variables—in this case there are five:

$$\left\{ x_1, x_2, x_3 = \frac{1+x_2}{x_1}, x_4 = \frac{1+x_1+x_2}{x_1x_2}, x_5 = \frac{1+x_1}{x_2} \right\}.$$

The resulting cluster algebra is generated by this set of 5 cluster variables, and is hence a cluster algebra of finite type. In fact, this is precisely the cluster algebra from Example 2.1.9 framed in terms of quivers instead of matrices.

2.1.3. Main Results. In Example 2.1.13 above, we notice that all of the cluster variables have a particularly nice form—they are *Laurent polynomials* in the initial cluster variables. It is not at all obvious, a priori, that this should be the case, but it turns out that this is just one instance of a phenomenon that occurs in all cluster algebras. In fact the following result, which appeared in [**FZ01**], is one of the most important results in the study of cluster algebras.

THEOREM 2.1.14 (The Laurent Phenomenon). *Every cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of any single (extended) cluster.*

It was conjectured by Fomin and Zelevinsky that an even stronger result should hold—that every cluster variable can be expressed as a Laurent polynomial with *positive* integer coefficients in the elements of any single (extended) cluster. This result has now been proven in several cases, including for all skew-symmetric cluster algebras in [**LS15**], though it remains open in full generality.

It was shown in [**FZ03**] that cluster algebras of *finite type*, i.e. those with only finitely many cluster variables, admit a classification in terms of finite type Cartan matrices. This classification uses the following relationship between Cartan matrices and skew-symmetrisable matrices.

DEFINITION 2.1.15 (Cartan counterpart). Let $B = (b_{ij})$ be a skew-symmetrisable integer matrix, e.g. an exchange matrix. Then its *Cartan counterpart* is the symmetrisable generalised Cartan matrix

$$A = A(B) = (a_{ij})$$

of the same size, defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

THEOREM 2.1.16 (Finite type classification). *A cluster algebra is of finite type if and only if the exchange matrix B is (mutation equivalent to) a matrix whose Cartan counterpart is a Cartan matrix of finite type.*

Rephrasing in terms of exchange quivers, Theorem 2.1.16 says that a cluster algebra is of finite type if and only if the mutable part of the exchange quiver is mutation equivalent to some orientation of a finite type Dynkin diagram. A very accessible, though quite long, proof of Theorem 2.1.16 can be found in [**FWZ21c**].

We do not repeat this here, but note that a key tool used throughout is the correspondence between quivers and triangulations. This can be seen for the type A_n case in Section 2.2.1 below.

2.1.4. Graded Cluster Algebras. We now turn our attention to *graded* cluster algebras as defined in [**Gra15**], building on earlier work carried out in [**BZ05**] and [**GSV03**]. In order to define a grading on a cluster algebra we require an additional piece of initial data, which we call a *grading vector*, assigning a degree to each initial cluster variable.

DEFINITION 2.1.17 (Graded seed). A *graded seed* is a triple $(\underline{x}, B, G)$ such that:

(i) $(\underline{x}, B)$ is a seed, with $|\underline{x}| = n$,
(ii) $G = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ is such that $B^T G = 0$.

We have $\deg(x_i) := g_i$ for $i = 1, \ldots, n$.

REMARK 2.1.18. *In* [**Gra15**]*, the more general concept of a multi-graded seed is defined. In that case, G is an $n \times d$ integer matrix, and the resulting cluster algebra is a $\mathbb{Z}^d$-graded algebra. In what follows, we will only consider the case in which $d = 1$.*

DEFINITION 2.1.19 (Graded cluster algebra). The *graded cluster algebra* denoted $\mathscr{A}(\underline{x}, B, G)$ is the cluster algebra $\mathscr{A}(\underline{x}, B)$ with the grading given on initial cluster variables by G extended in the obvious way via mutation.

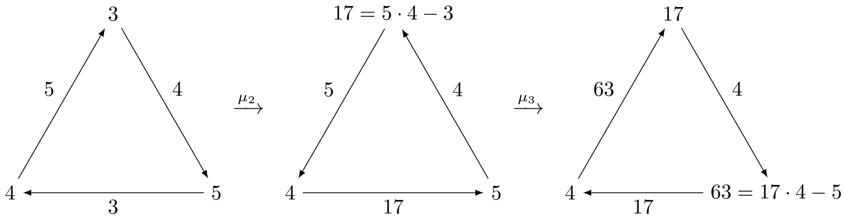
PROPOSITION 2.1.20 ([**Gra15**, Prop. 3.2]). *The cluster algebra $\mathscr{A}(\underline{x}, B, G)$ associated to an initial graded seed $(\underline{x}, B, G)$, with $G \in \mathbb{Z}^n$, is a $\mathbb{Z}$-graded algebra. Every cluster variable of $\mathscr{A}(\underline{x}, B, G)$ is homogeneous with respect to this grading.*

In Chapter 3 we will be interested solely in the degrees of cluster variables, rather than their precise forms. It will therefore be useful to work with *degree seeds.*

DEFINITION 2.1.21 (Degree seed). A *degree seed* is a pair (G, B) of a grading vector G and a corresponding exchange matrix B.

We may then directly mutate the degree seed, conveniently forgetting the cluster variables when we do not need to know them explicitly. This is illustrated in the example below.

EXAMPLE 2.1.22. *The diagram below shows two mutation steps starting from the initial degree seed* $G = (4, 3, 5)$, $B = \begin{pmatrix} 0 & 5 & -3 \\ -5 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$.



2.1.5. Quantum Cluster Algebras. Quantum analogues of cluster algebras were introduced in [**BZ05**] by Berenstein and Zelevinsky. A quantum cluster algebra is a $\mathbb{Q}(q)$-algebra in which every cluster is a quasi-commuting family, and defining these will again involve an additional piece of initial data, this time in the form of a matrix Λ which determines the rule for quasi-commutation. Whilst we will not directly refer to the quantum cluster algebra structure, the results in Chapter 4 will hold in the quantum setting, and so we provide a brief overview here. We do not include proofs in this section—these, as well as more details, can be found in [**BZ05**].

DEFINITION 2.1.23 (Compatible pair). Let $\tilde{B}$ be an $m \times n$ integer matrix with rows labelled by $[1, m]$ and columns by $[1, n]$. Let Λ be a skew-symmetric $m \times m$ integer matrix with both rows and columns labelled by $[1, m]$. We call the pair $(\Lambda, \tilde{B})$ *compatible* if, for every $j \in [1, n]$ and every $i \in [1, m]$, we have

$$\sum_{k=1}^{m} b_{kj}\lambda_{ki} = \delta_{ij} d_j$$

for some $d_j \in \mathbb{Z}_{>0}$.

We now wish to understand how to mutate a compatible pair. Fix $k \in [1, n]$ (the direction in which we will mutate) and $\varepsilon \in \{\pm 1\}$. Then the matrix $\mu_k(\tilde{B})$ can be expressed as $\mu_k(\tilde{B}) = E_\varepsilon \tilde{B} F_\varepsilon$, where

- E_ε is the $m \times m$ matrix with entries

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ \max(0, -\varepsilon b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

- F_ε is the $n \times n$ matrix with entries

$$f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k; \\ -1 & \text{if } i = j = k; \\ \max(0, \varepsilon b_{kj}) & \text{if } i = k \neq j. \end{cases}$$

For a compatible pair $(\Lambda, \tilde{B})$, we set $\mu_k(\Lambda) = E_\varepsilon^T \Lambda E_\varepsilon$, and we call this the *mutation* of Λ in direction k.

PROPOSITION 2.1.24. *Let $(\Lambda, \tilde{B})$ be a compatible pair. Then:*

(i) The pair $(\mu_k(\Lambda), \mu_k(\tilde{B}))$ is again compatible.

(ii) The mutated matrix $\mu_k(\Lambda)$ is independent of the choice of sign ε.

(iii) Mutation of compatible pairs is involutive.

A quantum cluster algebra will be defined as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathscr{F}$, where $\mathscr{F}$ is the skew-field of fractions of the 'based quantum torus' T, and q is a formal variable.

DEFINITION 2.1.25 (Based quantum torus). Let L be a lattice of rank m, with skew-symmetric bilinear form $\Lambda : L \times L \to \mathbb{Z}$, and let q be a formal variable. Denote by $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ the ring of Laurent polynomials with integer coefficients in the variable $q^{\frac{1}{2}}$. Then the *based quantum torus* associated with L is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra $\mathscr{T} = \mathscr{T}(\Lambda)$ with $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis $\{X^e : e \in L\}$ and multiplication

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f} \text{ for } e, f \in L.$$

DEFINITION 2.1.26 (Toric frame). A *toric frame* in $\mathscr{F}$ is a mapping $M : \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ of the form

$$M(c) = \varphi(X^{\eta(c)}),$$

where φ is an automorphism of $\mathscr{F}$, and $\eta : \mathbb{Z}^m \to L$ is a lattice isomorphism.

The elements $M(c)$ form a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis of the image of $\mathscr{T}$ under φ, with multiplication and commutation relations

$$M(c)M(d) = q^{\Lambda_M(c,d)/2} M(c+d)$$

$$M(c)M(d) = q^{\Lambda_M(c,d)} M(d)M(c).$$

Here, Λ_M is the bilinear form on $\mathbb{Z}^m$ obtained from Λ via the isomorphism η. We have

$$M(0) = 1 \text{ and } M(c)^{-1} = M(-c) \text{ for } c \in \mathbb{Z}^m.$$

We denote also by Λ_M the $m \times m$ integer matrix whose entries are $\lambda_{ij} = \Lambda_M(e_i, e_j)$, where $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{Z}^m$.

Given a toric frame M, we set $X_i = M(e_i)$ for $i \in [1, m]$. The elements X_i quasi-commute, i.e. we have

$$X_i X_j = q^{\lambda_{ij}} X_j X_i.$$

PROPOSITION 2.1.27. *A toric frame M is uniquely determined by the elements $X_i = M(e_i)$ for $i \in [1, m]$.*

DEFINITION 2.1.28 (t-binomial coefficient). The t-binomial coefficient $\binom{r}{p}_t$ is defined as follows:

$$\binom{r}{p}_t = \frac{(t^r - t^{-r}) \cdots (t^{r-p+1} - t^{r+p-1})}{(t^p - t^{-p}) \cdots (t - t^{-1})}.$$

DEFINITION 2.1.29 (Quantum seed, [**BZ05**, Def 4.5]). A *quantum seed* is a pair $(M, \tilde{B})$ where

- M is a toric frame in $\mathscr{F}$.
- $\tilde{B}$ is an $m \times n$ integer matrix with rows labelled by $[1, m]$ and columns labelled by an n-element subset $\underline{\text{ex}} \subset [1, m]$.
- The pair $(\Lambda_M, \tilde{B})$ is compatible in the sense of Definition 2.1.23.

Let $(M, \tilde{B})$ be a quantum seed. Fix an index $k \in \underline{\text{ex}}$ and a sign $\varepsilon \in \{\pm 1\}$. Define the mapping $M' : \mathbb{Z} \to \mathscr{F} \backslash \{0\}$ by

$$M'(c) = \sum_{p=0}^{c_k} \binom{c_k}{p}_{q^{d_k/2}} M(E_\varepsilon c + \varepsilon p b^k) \text{ and } M'(-c) = M'(c)^{-1}$$

for $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Here, $\binom{c_k}{p}_{q^{d_k/2}}$ is the t-binomial coefficient as in Definition 2.1.28, and b_k is the k^{th} column of $\tilde{B}$.

PROPOSITION 2.1.30. (i) *The mapping M' is a toric frame.*

(ii) *We have $\mu_k(\Lambda_M) = \Lambda_{M'}$.*

(iii) *The pair $(\mu_k(M), \mu_k(\tilde{B}))$ is a quantum seed as in Definition 2.1.29. We call this the mutation of $(M, \tilde{B})$ in direction k, and we write $\mu_k(M, \tilde{B})$.*

(iv) *Mutation of quantum seeds is involutive, i.e. we have* $\mu_k(\mu_k(M, \tilde{B})) = (M, \tilde{B})$.

As in the classical setting, two (quantum) seeds are called *mutation equivalent* if one can be obtained from the other via a sequence of quantum seed mutations. Given a quantum seed $(M, \tilde{B})$, we denote by $\underline{\tilde{X}} = \{X_1, \ldots, X_m\}$ the *extended (quantum) cluster* given by taking $X_i = M(e_i)$. The subset $\underline{X} = \{X_j : j \in \underline{\mathrm{ex}}\} \subset \underline{\tilde{X}}$ is called the *(quantum) cluster*.

We can now give the quantum analogue of Definition 2.1.5.

DEFINITION 2.1.31 (Quantum cluster algebra). Let $(M, \tilde{B})$ be a quantum seed. The associated *quantum cluster algebra* $\mathscr{A}(M, \tilde{B})$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathscr{F}$, generated by the union of clusters of all seeds mutation equivalent to $(M, \tilde{B})$, together with the set of frozen variables $\underline{\tilde{X}} \backslash \underline{X}$ and their inverses.

REMARK 2.1.32. *In* [**GL13**] *the notion of a* $\mathbb{Z}$*-grading for a quantum cluster algebra was introduced—we do not use this directly, and so refer the reader to* [**GL13**] *for details. Note, however, that the grading data is independent of the quasi-commutation data. The theory of graded quantum cluster algebras therefore mirrors that of graded cluster algebras. The main theorem of* [**BZ05**] *states that quantum cluster variables are in bijection with those in the* $q = 1$ *case. Hence, without loss of generality, it is possible to study graded quantum cluster algebras by looking at their* $q = 1$ *specialisations.*

2.2. Examples of Cluster Algebras

We now present two explicit examples of algebras which admit cluster structures; both of which will play a significant role in Chapter 4. The first of these examples is the Grassmannian, $\mathrm{Gr}(k, n)$. Roughly following [**Bau21**], we will describe the cluster algebra structure on its homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(k, n)]$. We will then discuss a second closely related example—the homogeneous coordinate ring of matrices. For completeness, we briefly recall two definitions from algebraic geometry. For further background on projective varieties see, for example, [**Vak**] or [**Har77**].

DEFINITION 2.2.1 (Projective variety). Let $n \in \mathbb{N}$, and let $S \subset \mathbb{C}[x_0, \ldots, x_n]$ be a set of homogeneous polynomials. The *zero locus* of S is the set

$$V(S) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S\}.$$

Subsets of the projective n-space $\mathbb{P}^n$ of this form are called *projective varieties.*

Definition 2.2.2 (Homogeneous coordinate ring). Let $Y \subset \mathbb{P}^n$ be a projective variety. The *homogeneous coordinate ring* of Y is

$$S(Y) := \mathbb{K}[x_0, \ldots, x_n]/I(Y).$$

Here, $I(Y)$ denotes the *ideal* of Y, defined as follows:

$$I(Y) := \langle f \in \mathbb{K}[x_0, \ldots, x_n] \text{ homogeneous } : f(x) = 0 \text{ for all } x \in Y \rangle .$$

2.2.1. The Grassmannian. The cluster algebra structure on the homogeneous coordinate ring of the Grassmannian was first established by Scott in [**Sco06**], though the $\mathrm{Gr}(2, n)$ case appeared already in [**FZ03**]. In this section we recall the definition of the Grassmannian—for more details see [**MS04**]. We will focus on describing the cluster structure on its homogeneous coordinate ring.

Definition 2.2.3 (The Grassmannian). Let $1 < k < n$. The Grassmannian $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$.

There is a map from $\mathrm{Gr}(k, n)$ into projective N-space, for $N = \binom{n}{k} - 1$, defined as follows.

Definition 2.2.4 (The Plücker embedding). Let $U \in \mathrm{Gr}(k, n)$ with basis $\{v_1, \ldots, v_k\}$, and consider $w := v_1 \wedge \ldots \wedge v_k \in \Lambda^k(\mathbb{C}^n)$. Define a map $\varphi : \mathrm{Gr}(k, n) \to \mathbb{P}^N$ by

$$\varphi(U) = (\Delta^{i_1,\ldots,i_k}(w))_{i_1<\cdots<i_k}$$

where $\Delta^{i_1,\ldots,i_k}$ denotes the minor of a $k \times n$ matrix with column set $\{i_1, \ldots, i_k\}$ which we call a *Plücker coordinate.* The map φ is known as the *Plücker embedding.*

In the above definition, $\Lambda^k(\mathbb{C}^n)$ is the subspace of the exterior algebra $\Lambda(\mathbb{C})$ spanned by the elements $v_1 \wedge v_2 \wedge \cdots \wedge v_k$, where $v_i \in \mathbb{C}$ for all i. Here, $\wedge$ denotes the product in $\Lambda(\mathbb{C})$.

To define the Plücker coordinate $\Delta^{i_1,\ldots,i_k}$ corresponding to an arbitrary multi-set $\{i_1, \ldots, i_k\}$ such that $i_j \in \{1, \ldots, n\}$, we set

$$\Delta^{i_1,\ldots,i_k} \begin{cases} 0 & \text{if } i_r = i_s \text{ for some } r \neq s, \\ \mathrm{sgn}(\pi)\Delta^{j_1,\ldots,j_k} & \text{if } \{i_1, \ldots, i_k\} = \{j_1 < \cdots < j_k\}. \end{cases}$$

In the above, π is the permutation such that $\pi(i_r) = j_r$ for all r. We are now able to write down the relations satisfied by the image $\varphi(\mathrm{Gr}(k,n))$.

DEFINITION 2.2.5 (Plücker relations). The *Plücker relations* for $\mathrm{Gr}(k,n)$ are:

$$\sum_{r=0}^{k}(-1)^r \Delta^{i_1,\ldots,i_{k-1},j_r}\Delta^{j_0,\ldots,\widehat{j_r},\ldots j_k} = 0.$$

The sum is taken over all tuples $(i_1,\ldots,i_{k-1})$, $(j_0,\ldots,j_k)$ such that $1 \leq i_1 < \cdots < i_{k-1} \leq n$ and $1 \leq j_0 < \cdots < j_k \leq n$. We denote by $\widehat{a}$ an omitted index a.

As suggested by Definition 2.2.4, we have the following proposition.

PROPOSITION 2.2.6. *The map φ is injective.*

PROOF. See [**MS04**]. □

The homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ is the quotient of the polynomial ring in the Plücker coordinates Δ^I, where $|I| = k$, subject to the Plücker relations as in Definition 2.2.5.

2.2.1.1. *The Cluster Structure on the Grassmannian for $k = 2$.* When $k = 2$ the Plücker relations are simply

$$\Delta^{ab}\Delta^{cd} - \Delta^{ac}\Delta^{bd} + \Delta^{ad}\Delta^{bc} = 0 \text{ for } 1 \leq a < b < c < d \leq n. \tag{1}$$

The homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(2,n)]$ is therefore the quotient of the polynomial ring in Δ^{ab} for $1 \leq a < b \leq n$ subject to the relations (1).

Let T be a triangulation of the regular polygon P^n. Then we can define a quiver Q_T such that:

(i) The vertices of Q_T correspond to the diagonals/edges in T.
(ii) We have an arrow $i \to j$ whenever i and j are edges of the same triangle in T, and j is clockwise from i. Note that we do not include arrows between vertices corresponding to boundary edges.

As mentioned earlier, a key tool used to prove Theorem 2.1.16 was the relationship between quivers and triangulations. The $\mathrm{Gr}(2,n)$ case corresponds to Dynkin diagrams of type A. We illustrate this correspondence here.

First, notice that Plücker coordinates can be parameterised by the diagonals (and edges) of a regular polygon with n vertices, labelled clockwise. The Plücker

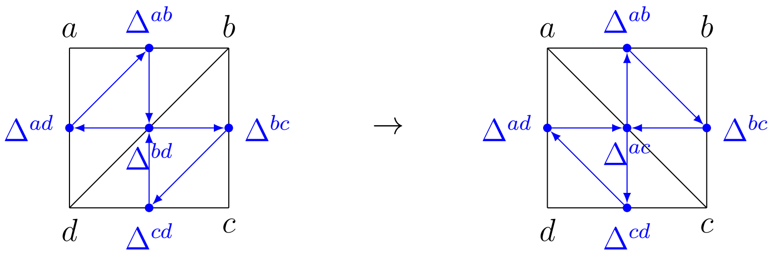


FIGURE 2.2. Cluster mutation corresponds to the quadrilateral flip as shown above. The diagram represents a quadrilateral within a triangulation—we replace the diagonal bd with the diagonal ac. The corresponding quiver is shown in blue.

coordinate Δ^{ij} with $i < j$ corresponds to the diagonal connecting vertices i and j—this is illustrated in Figure 2.2.

THEOREM 2.2.7 ([**FZ03**]). *Let P_n be a convex n-gon, with $n \geq 5$. The homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(2, n)]$ is a cluster algebra of type A_{n-3} where:*

(i) The (mutable) cluster variables are the Plücker coordinates Δ^{ij} where (i, j) are the diagonals in P^n, and the frozen variables are the Plücker coordinates corresponding to the boundary edges of P^n.

(ii) The seeds are in bijection with the triangulations T of P^n, with exchange quiver Q_T.

(iii) Cluster mutation corresponds to the quadrilateral flip (see Figure 2.2) in the triangulation T.

2.2.1.2. *The Cluster Structure on the Grassmannian for $k > 2$.* In the general case, for $\mathrm{Gr}(k, n)$, things get somewhat more complicated. In this setting the Plücker coordinates are only a subset of the set of all cluster variables, and in place of triangulations we must consider 'Postnikov diagrams'—see Figure 2.3 for an example. Since we do not explicitly use this in what follows, we refer the reader to [**Sco06**] for the details.

The cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(k, n)]$ has a well-known natural grading given by setting the degree of all Plücker coordinates to be one—see [**BP17**] for details. This grading endows $\mathbb{C}[\mathrm{Gr}(k, n)]$ with the structure of a *graded* cluster algebra as in Definition 2.1.19.

REMARK 2.2.8. *The quantum cluster structure on the quantum analogue of the homogeneous coordinate ring of the Grassmannian was given in* [**GL13**]*, by lifting the quantum cluster algebra on quantum matrices given by* [**GLS11a**]*. Since we do*

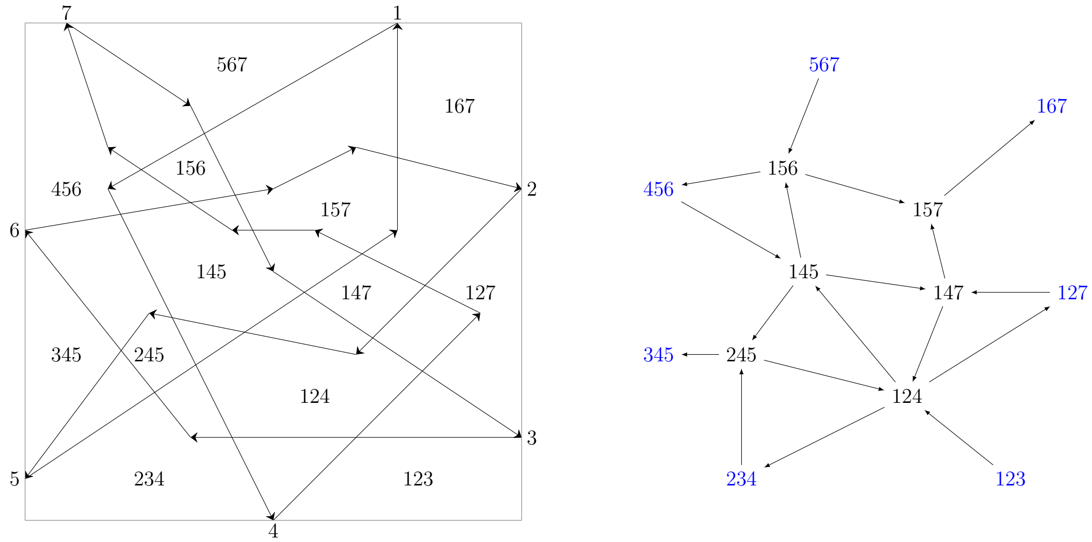


FIGURE 2.3. An example of the Postnikov diagram for a cluster in $\mathbb{C}[\mathrm{Gr}(3,7)]$ and its corresponding quiver. The boundary segments correspond to frozen variables, indicated in blue on the quiver.

not directly utilise the quantum cluster structure we will not repeat this here but note that, as detailed in Remark 2.1.32, the underlying combinatorics is identical to the commutative case—as such, all of our results in Chapter 4 will continue to hold in the quantum setting.

REMARK 2.2.9. *It was established in* [**Sco06**] *that the cluster algebra structures on the following Grassmannians are* finite-type*:*

$$\mathbb{C}[\mathrm{Gr}(2,n)], \mathbb{C}[\mathrm{Gr}(3,6)], \mathbb{C}[\mathrm{Gr}(3,7)], \mathbb{C}[\mathrm{Gr}(3,8)].$$

The cluster algebra structure for any other Grassmannian is of infinite type*.*

2.2.2. The Matrix Algebra. Another example of a (quantum) graded cluster algebra closely related to the Grassmannian is the (quantum) matrix algebra. This was shown to have a cluster algebra structure in [**GLS11b**], and quantum cluster structure in [**GLS11a**]. We briefly describe the cluster structure here.

DEFINITION 2.2.10 (Coordinate ring of $M(k,j)$). Denote by $M(k,j)$ the set of $k \times j$ matrices. The coordinate ring is defined to be $\mathbb{C}[M(k,j)] = \mathbb{C}[x_{r,s}]$, where $1 \leq r \leq k$, $1 \leq s \leq j$, and $x_{r,s} : M(k,j) \to \mathbb{C}$ is given by $x_{r,s}(A) = a_{rs}$.

Let $\left[\begin{smallmatrix} J \\ I \end{smallmatrix}\right]$ denote the minor of the matrix $\begin{pmatrix} x_{11} & \dots & x_{1j} \\ \vdots & \ddots & \vdots \\ x_{k1} & \dots & x_{kj} \end{pmatrix}$ corresponding to the row set I and column set J.

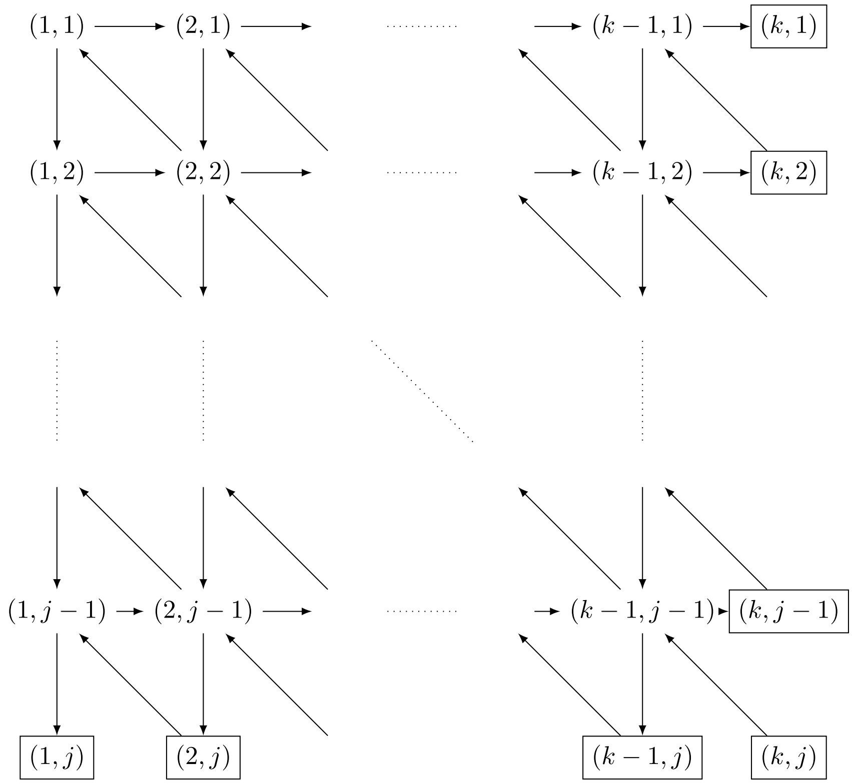


FIGURE 2.4. The initial quiver for the cluster structure on $\mathbb{C}[M(k,j)]$.

Now, for $1 \leq r \leq k$ and $1 \leq s \leq j$, we define the following sets:

$$R(r,s) = \{k-r+1, k-r+2, \ldots k-r+s\} \cap \{1, \ldots, k\},$$

$$C(r,s) = \{j-s+1, j-s+2, \ldots j-s+r\} \cap \{1, \ldots, j\}.$$

The quiver corresponding to the initial cluster is shown in Figure 2.4; the cluster variable in position (r,s) is the minor $\begin{bmatrix} C(r,s) \\ R(r,s) \end{bmatrix}$.

REMARK 2.2.11. *The cluster structure on* $\mathbb{C}[M(k,j)]$ *has a well-known natural grading given by declaring all matrix entries* x_{rs} *to have degree one, and extending this in the obvious way using mutation. See* [**BP17**] *for details. We may also endow the quantisation* $\mathbb{C}_q[M(k,j)]$ *with the structure of a quantum cluster algebra. We do not directly use the quantum structure but, as with the Grassmannian case, our results in Chapter 4 will continue to hold in the quantum setting since the underlying combinatorics is unchanged.*

2.3. Quantum Graded Algebras with a Straightening Law

In [**LR04**], Lenagan and Rigal introduce the idea of a *quantum graded algebra with a straightening law* (QGASL). They show that $\mathbb{C}_q[\mathrm{Gr}(k,n)]$ is a QGASL, utilising this structure to prove that it has some nice homological properties—it is *AS-Cohen-Macaulay* and *AS-Gorenstein*. The definition of a QGASL involves a partial order on a generating set—we will make use of this partial order on the Grassmannian in Chapter 4.

DEFINITION 2.3.1 ([**LR04**, Def 1.1.1]). Let A be an $\mathbb{N}$-graded $\mathbb{C}$-algebra, and Π a finite subset of A equipped with a partial order $\leq_{\mathrm{st}}$. We say that A is a *quantum graded algebra with a straightening law* (QGASL) on the poset $(\Pi, \leq_{\mathrm{st}})$ if the following conditions are satisfied.

(i) The elements of Π are homogeneous with positive degree.
(ii) The elements of Π generate A as a $\Bbbk$-algebra.
(iii) The set of standard monomials on Π is a linearly independent set.
(iv) If $\alpha, \beta \in \Pi$ are not comparable for $\leq_{\mathrm{st}}$, then $\alpha\beta$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\mathrm{st}} \mu$ and $\lambda \leq_{\mathrm{st}} \alpha, \beta$.
(v) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha\beta} \in \mathbb{C}^*$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\mathrm{st}} \mu$ and $\lambda \leq_{\mathrm{st}} \alpha, \beta$.

As mentioned above, it has been shown that the quantised coordinate ring of the Grassmannian $\mathrm{Gr}(k, n)$ is a QGASL. In this case, the poset Π consists of all Plücker coordinates, with partial order defined as follows.

Denote by $J = \{j_1 < \cdots < j_k\}$ the index set $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$ with $1 \leq j_1 < \cdots < j_k \leq n$. The set of all index sets of cardinality k is denoted by $\Pi_{k,n}$, or simply Π when k and n are clear (note that, as the two can be identified, we will also denote by Π the set of all Plücker coordinates). The standard partial order, $\leq_{\mathrm{st}}$, on $\Pi = \Pi_{k,n}$ is defined as follows:

$$\{i_1 < \cdots < i_k\} \leq_{\mathrm{st}} \{j_1 < \cdots < j_k\} \text{ if and only if } i_s \leq j_s \text{ for all } 1 \leq s \leq k. \qquad (2)$$

We may then consider the induced partial order on the set of all Plücker coordinates, which we again denote by $\leq_{\mathrm{st}}$.

Lenagan and Rigal use the QGASL structure on $\mathbb{C}_q[\mathrm{Gr}(k, n)]$ in order to prove that it is both *AS-Cohen-Macaulay* and *AS-Gorenstein*. They also show that the coordinate ring $\mathbb{C}[M(k, j)]$ has both of these properties. An interested reader may find more details about these homological properties in [**JZ00**]—we will not make use of them, and so will not define them here.

The goal in Chapter 4 of this thesis will be to identify a connection between the QGASL structure and the cluster algebra structure. The hope is that doing this will make it possible in future to extend the techniques used by Lenagan and Rigal to prove results regarding the homological properties of other classes of (quantum) cluster algebras.

CHAPTER 3

Growth in Graded Cluster Algebras of Rank 3

In this chapter, we study gradings on rank 3 cluster algebras. Building on work done in [**BP17**], we wish to further understand the growth of cluster variable degrees along mutation paths, with a focus on the differences in behaviour in the 'mutation-cyclic' and 'mutation-acyclic' cases.

DEFINITION 3.0.1 (Mutation-(a)cyclic). A quiver Q is called *mutation-cyclic* if every quiver mutation equivalent to Q is cyclic, otherwise it is called *mutation-acyclic*. We will also use this terminology to refer to the corresponding exchange matrices.

In [**BP17**], the classification shown in Figure 3.1 is given for rank 3 graded cluster algebras with initial degree seed

$$\left((b, c, a), \left(\begin{smallmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{smallmatrix}\right)\right) \tag{3}$$

where $a, b, c \in \mathbb{N}_0$ and $a \geq b \geq c$. For the mutation-infinite, mutation-acyclic case, it was shown in [**BP17**] that we have the following cases:

$$a, b, c \geq 2$$

$$a, b \geq 2, c = 1$$

$$a \geq 2, b = c = 1$$

$$a, b \geq 2, c = 0$$

$$a \geq 2, b = 1, c = 0.$$

The intention was to define a suitable 'growth function', similar to GK-dimension, which should capture the differences shown in Figure 3.1 between the mutation-cyclic and mutation-acyclic cases, since having infinitely *(resp. finitely)* many variables in each degree corresponds to slow *(resp. fast)* growth. See, for example, [**KL00**] for details regarding GK-dimension. Unfortunately, we have not been able to define

For mutation-finite matrices (which give rise to finitely many degrees)		
Finite type	**Mixed**	**Infinitely many variables per degree**
$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$
For mutation-infinite matrices (which give rise to infinitely many degrees)		
Finitely many variables per degree	**Mixed**	**Infinitely many variables per degree**
$\begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix}$, mutation-cyclic with $c > 2$	$\emptyset$	$\begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix}$, mutation-acyclic

FIGURE 3.1. The classification of rank 3 graded cluster algebras given in [**BP17**].

such a function—it appears that degree growth is simply very fast, and we do not obtain significantly different results for different initial conditions.

Throughout this section, we will be working with degree seeds, as illustrated in Example 2.1.22. This will simplify computations as we do not need to keep track of the cluster variables themselves. We first provide an overview of the work carried out by Booker-Price.

3.1. Gradings in the Rank 3 Case

Here, we summarise the results obtained by Booker-Price in Chapter 4 of his thesis—full details, including proofs, can be found in [**BP17**].

Rank 3 graded cluster algebras can be classified in terms of:

(i) the cardinality of the set of degrees occurring,

(ii) how the cluster variables are distributed with respect to the degrees.

The first important realisation is as follows.

PROPOSITION 3.1.1 ([**BP17**, Prop. 4.1.3]). *Every 3×3 skew-symmetric matrix is either essentially equivalent to the matrix* $A = \begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix}$, *for some $a, b, c \in \mathbb{N}_0$ and $a \geq b \geq c$, or mutation equivalent to a matrix which is essentially equivalent to A. Here,* essentially equivalent *is as in Definition 2.1.4.*

This result simplifies the classification problem, since now we only have to consider matrices of the form described above. We have the following.

THEOREM 3.1.2 ([**BP17**][Thm. 4.1.10]). *A partial classification of graded cluster algebras of the form* $\mathscr{A}\left((x_1, x_2, x_3), \left(\begin{smallmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{smallmatrix}\right), (b, c, a)\right)$*, with* $a, b, c \in \mathbb{N}_0$ *and* $a \geq b \geq c$*, is given in Figure 3.1.*

Only one case is not covered in Figure 3.1—the so-called *singular cyclic* case, i.e. matrices of the form $\left(\begin{smallmatrix} 0 & a & -2 \\ -a & 0 & a \\ 2 & -a & 0 \end{smallmatrix}\right)$ where $a \geq 3$. Booker-Price conjectures that this case should have infinitely many variables in each occurring degree, but this remains unproven.

An important tool for the proof of Theorem 3.1.2 is an algorithm which determines whether or not a given 3×3 matrix is mutation-cyclic. In the acyclic case, an important result is as follows.

THEOREM 3.1.3 ([**BP17**][Thm. 4.4.1]). *Let* $\mathscr{A}$ *be a cluster algebra arising from a mutation-acyclic matrix. For any cluster variable* x *in* $\mathscr{A}$*, the seeds whose clusters contain* x *form a connected subgraph of the exchange graph.*

This result was first conjectured in [**GSV03**], before being proven in the acyclic case in [**CK06**].

3.2. Fastest Growing Mutation Paths

We begin by considering the notion of a *fastest growing path*, i.e. the path producing the largest possible degree at each mutation step.

DEFINITION 3.2.1 (Fastest growing path). Given an initial degree seed $(\underline{d}, B)$, a *fastest growing path* is a mutation path in which, at each step, we mutate in a direction yielding the largest possible degree (without two consecutive mutations at the same vertex).

LEMMA 3.2.2. *Mutation of a degree seed of the form (3) along a fastest growing path produces another degree seed of the form (3). In the mutation-infinite case, we moreover have* $ab - c \geq b$*. In other words,* b *is the new smallest degree.*

PROOF. Consider the initial degree seed $((b, c, a), A)$, where the exchange matrix A is as in Proposition 3.1.1. We have

$$\mu_c(c) = ab - c \quad \text{and} \quad \mu_c(A) = \left(\begin{smallmatrix} 0 & -a & ab-c \\ a & 0 & -b \\ c-ab & b & 0 \end{smallmatrix}\right).$$

Up to relabelling, this is again a seed of the form (3).

In the mutation-infinite case, we have

$$\begin{aligned} ab - c &\geq ab - b \\ &\geq (a-1)b \\ &\geq b \end{aligned}$$

The first inequality is a consequence of the fact that $b \geq c$, and the final inequality is due to A being mutation-infinite, since this rules out $a < 2$. □

PROPOSITION 3.2.3. *Suppose we begin with a degree seed of the form* (3)*. The fastest growing path is precisely the path in which we mutate at the cluster variable of smallest possible degree at each step, without mutating at the same variable twice in a row.*

PROOF. First, note that mutation at x, y, and z yields new cluster variables of the following degrees, respectively:

$$ac - b, \quad ab - c, \quad bc - a.$$

It is then straightforward to see that the largest possible degree is obtained via mutation at y. Noting that, by Lemma 3.2.2, the degree seed we obtain after mutation is of the same form, the claim follows iteratively. □

EXAMPLE 3.2.4. *Consider the cluster algebra with initial degree seed*

$$\left(\underline{d} = (1,1,3), B = \left(\begin{smallmatrix} 0 & 3 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right)\right).$$

This is an example of a mutation-acyclic cluster algebra. Figure 3.2 shows a portion of the exchange tree, with the degree of the mutated variable at each vertex. The fastest growing path is highlighted in blue, and it produces the following sequence of degrees:

$$2, 5, 13, 62, 801, 49649, 39768787, \ldots$$

EXAMPLE 3.2.5. *Figure 3.3 shows a portion of the exchange tree for the initial degree seed* $\left(\underline{d} = (5,3,6), Q = \left(\begin{smallmatrix} 0 & 6 & -3 \\ -6 & 0 & 5 \\ 3 & -5 & 0 \end{smallmatrix}\right)\right)$*. This is a mutation-cyclic cluster algebra. The fastest growing path is again highlighted in blue, and we obtain the following sequence of degrees:*

$$27, 157, 4233, 664554, \ldots$$

In both of the examples above, we see that the degrees produced along the fastest growing path get very large very fast. We have the following results.

Let (b, c, a) be a grading vector of the form (3) where b, c, a are strictly greater than two. In what follows, denote by $(d_i)_{i \in \mathbb{N}_0}$ the sequence defined by

$$d_0 = c, d_1 = ab - c, d_2 = d_1 a - b, \text{ and } d_{i+1} = d_i d_{i-1} - d_{i-2} \text{ for } i > 2. \tag{4}$$

REMARK 3.2.6. *Note that one of the cases we have excluded above,* $(2, 2, 2)$*, is precisely the* Markov quiver*. In this case the cluster variable degrees do not grow at all, and every path is essentially 'fastest growing' with zero growth.*

LEMMA 3.2.7. *The sequence* $(d_i)_{i \in \mathbb{N}_0}$*, where* d_i *is as in* (4) *above, is strictly increasing.*

PROOF. First recall that $c \leq b \leq a$ since our grading vector is of the form (3). We additionally require that $c > 2$. Then $d_0 < d_1$ since $a, b \geq c$ implies that $d_1 = ab - c > c = d_0$.

For $d_1 < d_2$ we will show that the difference $\Delta = d_2 - d_1$ is strictly positive. We have

$$\begin{aligned}
\Delta = d_2 - d_1 &= (ab - c)a - b - (ab - c) \\
&= a^2 b - ac - b - ab + c \\
&= b(a^2 - a - 1) - c(a - 1) \\
&\geq b(a^2 - a - 1) - b(a - 1) = b(a^2 - 2a) = ab(a - 2).
\end{aligned}$$

Since $c > 2$ we have $a \geq 3$ and $b \geq 3$, hence $ab(a - 2) \geq a > 0$ and the claim holds.

We proceed by induction. Assume that the claim holds for $i < N$. We wish to show that

$$\Delta_N = d_{N+1} - d_N > 0.$$

We have

$$\begin{aligned}\Delta_N &= d_N d_{N-1} - d_{N-2} - d_N \\ &= d_N(d_{N-1} - 1) - d_{N-2} \\ &\geq (d_{N-1} + 1)(d_{N-1} - 1) \\ &= d_{N-1}^2 - 1 \\ &\geq d_{N-1} - 1 \\ &\geq d_{N-2} \geq 0.\end{aligned}$$

Hence $\Delta_N > 0$ whenever $d_{N-2} > 0$. □

CONJECTURE 3.2.8. *The sequence* $\left(\frac{\ln d_i}{\ln d_{i-1}}\right)_{i\in\mathbb{N}_0}$ *converges.*

PROPOSITION 3.2.9. *Suppose that Conjecture 3.2.8 holds. Let* $(d_i)_{i\in\mathbb{N}_0}$ *be the sequence of degrees along a fastest growing path, starting with the initial degree seed* $\left(\underline{d} = (b, c, a), B = \left(\begin{smallmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{smallmatrix}\right)\right)$, *where* $a, b, c \in \mathbb{N}_0$ *are such that* $a \geq b \geq c > 2$, *and* B *is mutation-infinite. Then* $\lim_{i\to\infty} \frac{\ln(d_{i+1})}{\ln(d_i)} = \varphi$ *or* $1 - \varphi$, *where* φ *is the golden ratio* $\frac{1+\sqrt{5}}{2}$.

PROOF. First, recall that by Lemma 3.2.2 we know that mutation of a degree seed of the form (3) along a fastest growing path produces another degree seed of the same form. Proposition 3.2.2 also tells us that in the mutation-infinite case, we have $ab - c \geq b$. Now, notice that we have $d_0 = c$, $d_1 = ab - c$, and $d_2 = d_1 a - b$. For $i > 2$ we have $d_{i+1} = d_i d_{i-1} - d_{i-2}$. Hence, we have a sequence of the form (4). We will show that, assuming Conjecture 3.2.8, this sequence tends to φ.

$$\begin{aligned}
\lim_{i\to\infty}\frac{\ln(d_{i+1})}{\ln(d_i)} &= \lim_{i\to\infty}\frac{\ln(d_i d_{i-1}-d_{i-2})}{\ln(d_i)}\\
&= \lim_{i\to\infty}\frac{\ln(d_i d_{i-1}-d_{i-2})-\ln(d_i d_{i-1})+\ln(d_i d_{i-1})}{\ln(d_i)}\\
&= \lim_{i\to\infty}\frac{\ln\left(1-\frac{d_{i-2}}{d_i d_{i-1}}\right)+\ln(d_i d_{i-1})}{\ln(d_i)}\\
&= \lim_{i\to\infty}\frac{\ln(d_i d_{i-1})}{\ln(d_i)}\\
&= \lim_{i\to\infty}\frac{\ln(d_i)+\ln(d_{i-1})}{\ln(d_i)}\\
&= 1+\lim_{i\to\infty}\frac{\ln(d_{i-1})}{\ln(d_i)}\\
&= 1+\frac{1}{\lim_{i\to\infty}\frac{\ln(d_{i+1})}{\ln(d_i)}}
\end{aligned}$$

The fourth equality is a result of Lemma 3.2.7 since the fact that $d_{i-2} < d_{i-1} < d_i$, for all i, implies that $\frac{d_{i-2}}{d_i d_{i-1}} \to 0$. Hence we have

$$\left(\lim_{i\to\infty}\frac{\ln(d_{i+1})}{\ln(d_i)}\right)^2 - \lim_{i\to\infty}\frac{\ln(d_{i+1})}{\ln(d_i)} - 1 = 0 \text{ and } \lim_{i\to\infty}\frac{\ln(d_{i+1})}{\ln(d_i)} = \frac{1\pm\sqrt{5}}{2}.$$

□

Remark 3.2.10. *It would seem that Conjecture 3.2.8 is closely related to the fact that the ratio of terms in a generalised Fibonacci sequence (i.e. a sequence starting with any two real numbers and continuing with the usual Fibonacci rule) tends to φ. See* [**Kos01**] *for details. It can be seen in the fifth equality of the proof above that what we have closely resembles the ratio of terms in such a generalised Fibonacci sequence. Moreover, computation of a large number of examples has failed to find a situation in which the sequence does not tend to φ.*

Further examples, together with additional details, can be found in Appendix A.

Remark 3.2.11. *Another reasonable suggestion would be to consider the average degree of all cluster variables at a given radius of mutation from the initial cluster, but examining examples reveals that the fastest growing path dominates, and the resulting sequence of degrees does not produce significantly different results to those detailed above. See Appendix A for examples of this.*

Unfortunately, looking at areas of fast growth in the exchange tree does not appear to provide any way of distinguishing between different initial seeds, but rather illustrates the fact that cluster variable degrees can grow extremely fast. We note that it is not so surprising that this approach fails to distinguish between the mutation-cyclic and mutation-acyclic cases since, by the proof of Proposition 3.2.9, the fastest growing path does not encounter any acyclic exchange quivers.

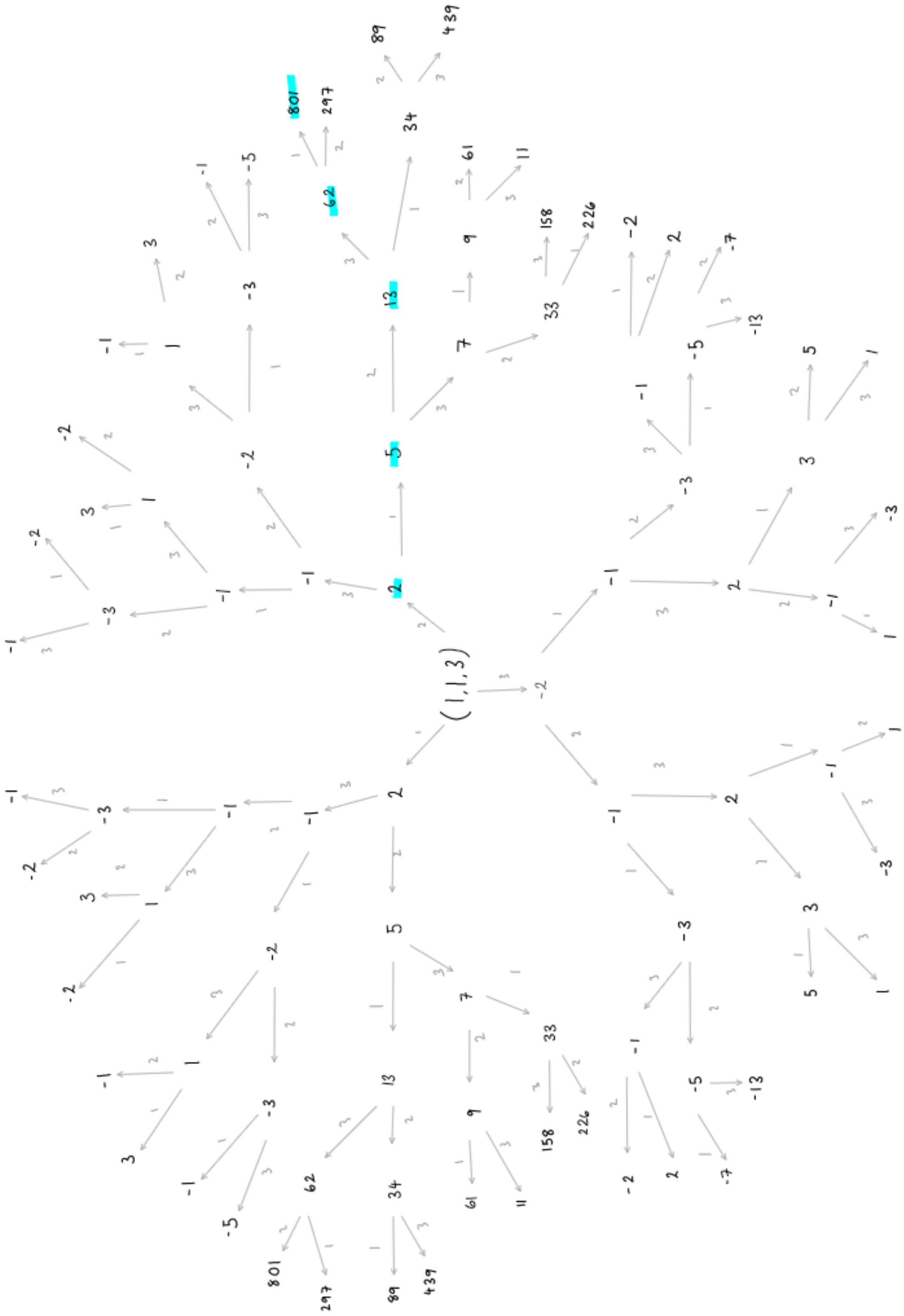


FIGURE 3.2. Part of the exchange tree for initial degree seed $(1, 1, 3)$. The fastest growing path is highlighted in blue.

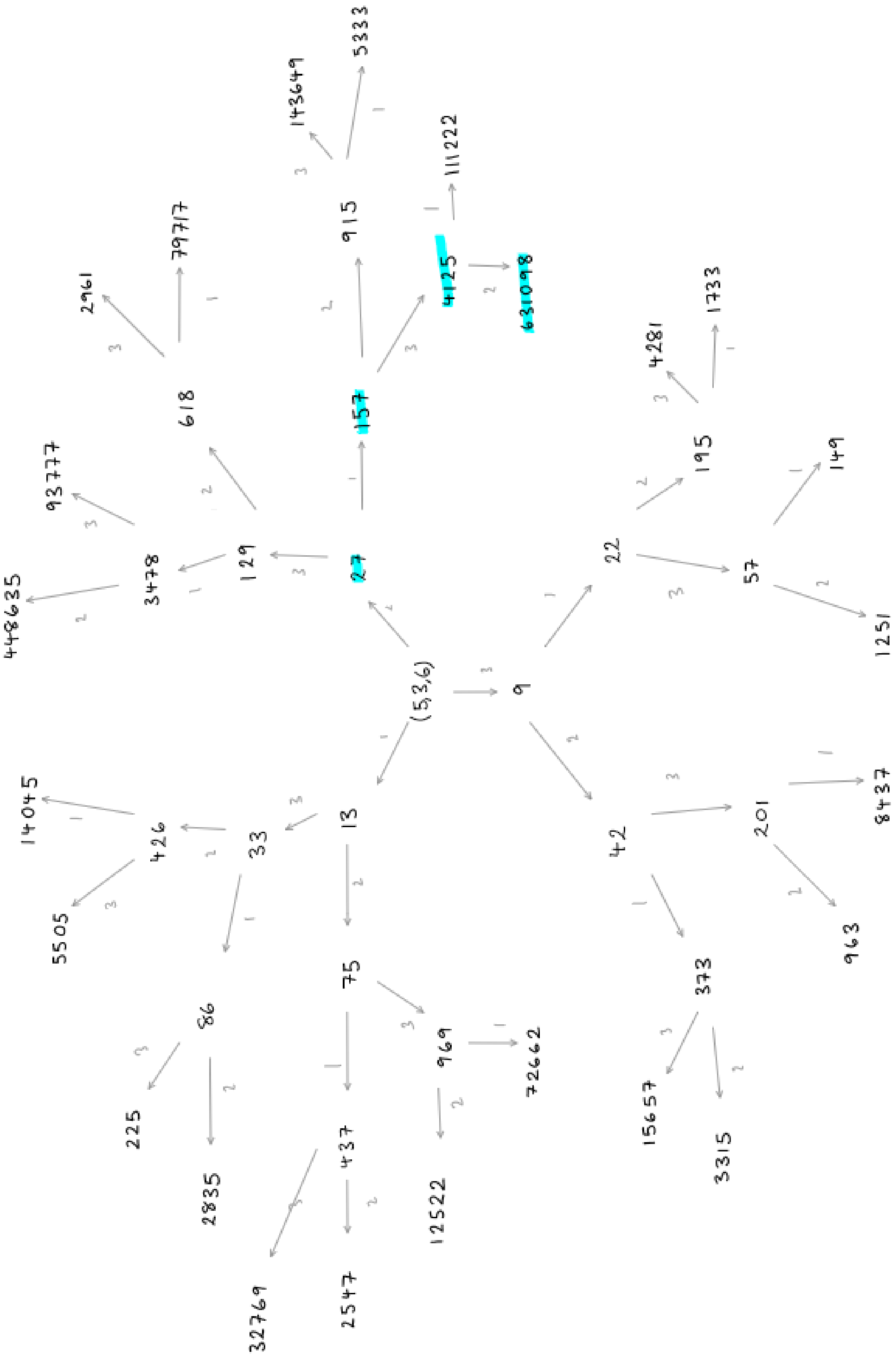


FIGURE 3.3. Part of the exchange degree for initial degree seed $(5, 3, 6)$. The fastest growing path is highlighted in blue.

3.3. Slow Growth in Rank 3 Cluster Algebras

An alternative approach could be to instead look at areas of the exchange tree exhibiting slower growth. One way to do this is by 'pruning' the exchange tree as follows:

(i) Begin with an exchange tree T_n, with initial degree cluster $\underline{d}$.
(ii) At each radius of mutation (starting from $\underline{d}$), remove the branch starting with the vertex of highest degree.

The 'pruned' version of the exchange tree with initial degree seed

$$\left((5,3,6), \begin{pmatrix} 0 & 6 & -3 \\ -6 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix}\right)$$

is shown in Figure 3.4. In theory, this process should remove all 'fastest growing' paths, starting from each possible initial seed. It seems, however, that the remaining portion of the exchange tree still exhibits very fast growth, and we remain unable to differentiate between the mutation-cyclic and mutation-acyclic cases with this method. See Appendix A for more examples and further details.

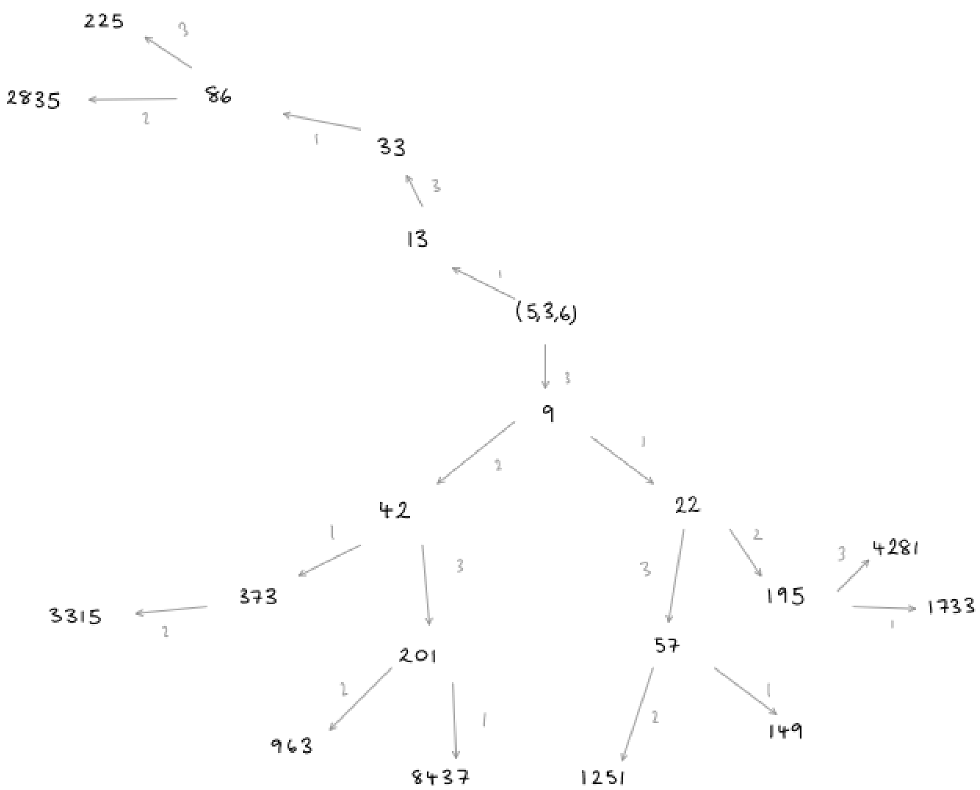


FIGURE 3.4. The 'pruned' exchange tree with initial degree vector (5, 3, 6)

The final possible approach attempted was to define a *slowest growing path*, essentially dualising the notion of Definition 3.2.1. Here, we mutate at the vertex of largest possible degree (without two consecutive mutations in the same direction). This does not appear to provide any useful information. In fact, it appears to depend only upon the value of c in the initial seed—this is illustrated in the examples in Section A.4.

CHAPTER 4

A Partial Order on a Generating Set for $\mathbb{C}[\mathbf{Gr}(k,n)]$

All results from this chapter will remain true in the quantum cluster algebra setting but, to simplify notation, we only explicitly work in the classical setting. We start by setting up some notation for higher degree cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$.

Let $\{a,b,c,d,e,f\}$ be a subset of $\{1,\dots,n\}$, written in increasing order. Define

$$X^{abcdef} := \Delta^{cef}\Delta^{abd} - \Delta^{def}\Delta^{abc}$$

and

$$Y^{abcdef} := \Delta^{bcf}\Delta^{ade} - \Delta^{def}\Delta^{abc}.$$

In what follows, we write $[a,b]$, where $a < b$, to denote the integer interval between a and b and $[a]$ to denote the integer interval $[1,a]$. To simplify notation, we also write $[a,b][c,d]$ to mean $[a,b] \cup [c,d]$, for $a \le b < c \le d$.

We wish to obtain a mutation path in $\mathbb{C}[\mathrm{Gr}(k,n)]$ which behaves 'nicely' with respect to the poset structure defined in Section 2.3. More specifically, we wish to find a path with the following properties:

(P1) The path includes all Plücker coordinates.

(P2) The sequence of Plücker coordinates obtained is monotonically increasing with respect to the partial order $\le_{\mathrm{st}}$ as in (2).

(P3) The path begins (resp. ends) at some well-defined 'minimal' (resp. 'maximal') cluster.

In order to define the minimal and maximal clusters, we use the notion of weak separability.

Definition 4.0.1 (Weak separation, [**Sco00**]). Given two index sets I and J, we write $I \prec J$ if $i < j$ for all $i \in I$ and all $j \in J$. We say I and J are weakly separated if at least one of the following conditions holds:

(i) $|I| \ge |J|$ and $J - I$ can be partitioned into a disjoint union $J - I = J' \sqcup J''$ so that $J' \prec I - J \prec J''$.

(ii) $|J| \geq |I|$ and $I - J$ can be partitioned into a disjoint union $I - J = I' \sqcup I''$ so that $I' \prec J - I \prec I''$.

In what follows, by *maximal weakly separated collection*, we refer to a set $\mathcal{W}$ of pairwise weakly separated Plücker coordinates such that $|\mathcal{W}|$ is as large as possible. In other words, there exist no Plücker coordinate $\Delta^I \notin \mathcal{W}$ such that I is weakly separated from all J such that $\Delta^J \in \mathcal{W}$.

THEOREM 4.0.2 ([**OPS11**, Thm 1.6]). *Let $\mathscr{C}$ be a subset of $\binom{[n]}{k}$. The following are equivalent:*

(i) The set of Plücker coordinates $\{\Delta^I\}_{I\in\mathscr{C}}$ is a cluster in the cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(k,n)]$.
(ii) $\mathscr{C}$ is a maximal weakly separated collection.

Here, $\binom{[n]}{k}$ denotes the set of k-element subsets of $[n]$.

REMARK 4.0.3. *A result by Leclerc and Zelevinsky,* [**LZ98**, Theorem 1.1]*, states that weak separation in the Grassmannian corresponds precisely to quasi-commutation in the quantum setting. Theorem 4.0.2 is therefore analogous to the fact that quantum clusters are maximal quasi-commuting sets.*

DEFINITION 4.0.4 (Minimal/maximal cluster). A minimal cluster for $\mathbb{C}[\mathrm{Gr}(k,n)]$ corresponds to a maximal weakly separated set $\mathscr{W}$ such that for any $I \in \mathscr{W}$, and for any $J \notin \mathscr{W}$ such that $J \leq_{\mathrm{st}} I$, the set $(\mathscr{W}\backslash\{I\}) \cup \{J\}$ is no longer weakly separated. Maximal clusters are defined analogously. We denote these by $\underline{\widetilde{\min}}$ and $\underline{\widetilde{\max}}$ respectively. Note that, by [**OPS11**, Thm 1.6], these are indeed clusters in $\mathbb{C}[\mathrm{Gr}(k,n)]$.

4.1. Minimal and Maximal Clusters for $\mathbb{C}[\mathbf{Gr}(k,n)]$

We now show that there exist well-defined minimal and maximal clusters for $\mathbb{C}[\mathrm{Gr}(k,n)]$. We begin by defining the following subsets of $\Pi_{k,n}$:

$$\mathscr{C}_{\text{frozen}} := \{[1,k-1][n],[1,k-2][n-1,n],\ldots,[n-k+1,n],\\ [n-k,n-1],\ldots,[2,k+1],[1,k]\}$$

$$\mathscr{C}_{\underline{\widetilde{\min}}} := \{I = [1,i_1][i_2,i_3] \in \Pi_{k,n}\} \cup \mathscr{C}_{\text{frozen}}$$

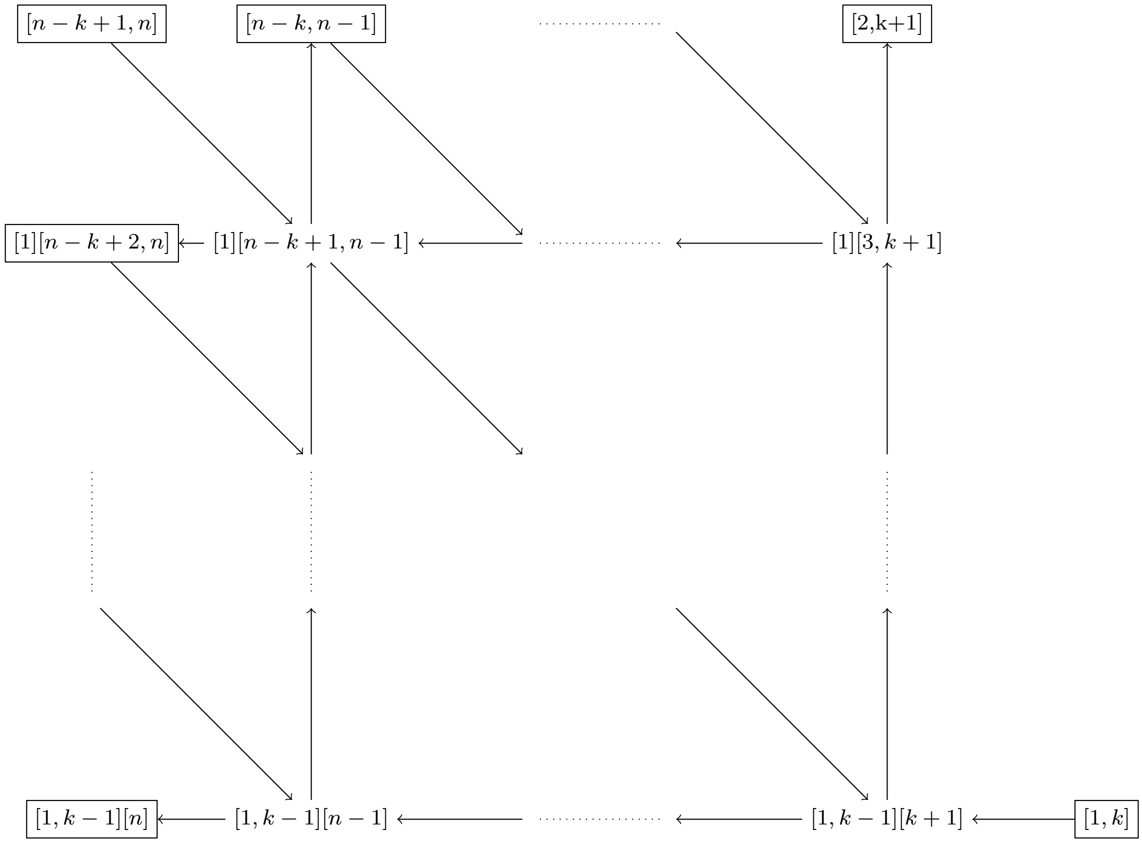


FIGURE 4.1. The exchange quiver $Q_{\widetilde{\underline{\min}}}$. The frozen variables are shown in boxes.

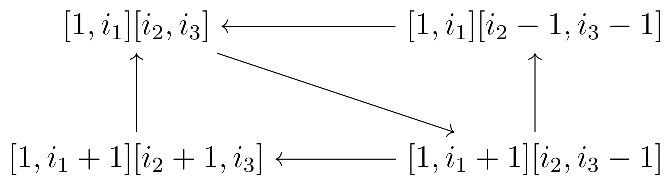
$$\mathscr{C}_{\widetilde{\underline{\max}}} := \{J = [j_1, j_2][j_3, n] \in \Pi_{k,n}\} \cup \mathscr{C}_{\text{frozen}}$$

Where $i_1, i_2, i_3 \in \mathbb{Z}$ are such that $1 \le i_1 < i_2 \le i_3 \le n$, and $j_1, j_2, j_3 \in \mathbb{Z}$ are such that $1 \le j_1 \le j_2 < j_3 \le n$,. We show in Proposition 4.1.2 that the following are minimal and maximal clusters respectively:

$$\widetilde{\underline{\min}} = \{\Delta^I : I \in \mathscr{C}_{\widetilde{\underline{\min}}}\} \quad \text{and} \quad \widetilde{\underline{\max}} = \{\Delta^I : I \in \mathscr{C}_{\widetilde{\underline{\max}}}\} \tag{5}$$

where $\{\Delta^I : I \in \mathscr{C}_{\text{frozen}}\}$ are the frozen variables.

The exchange quiver for the minimal seed $(\widetilde{\underline{\min}}, Q_{\widetilde{\underline{\min}}})$ is shown in Figure 4.1 below, with arrows between mutable vertices as follows.



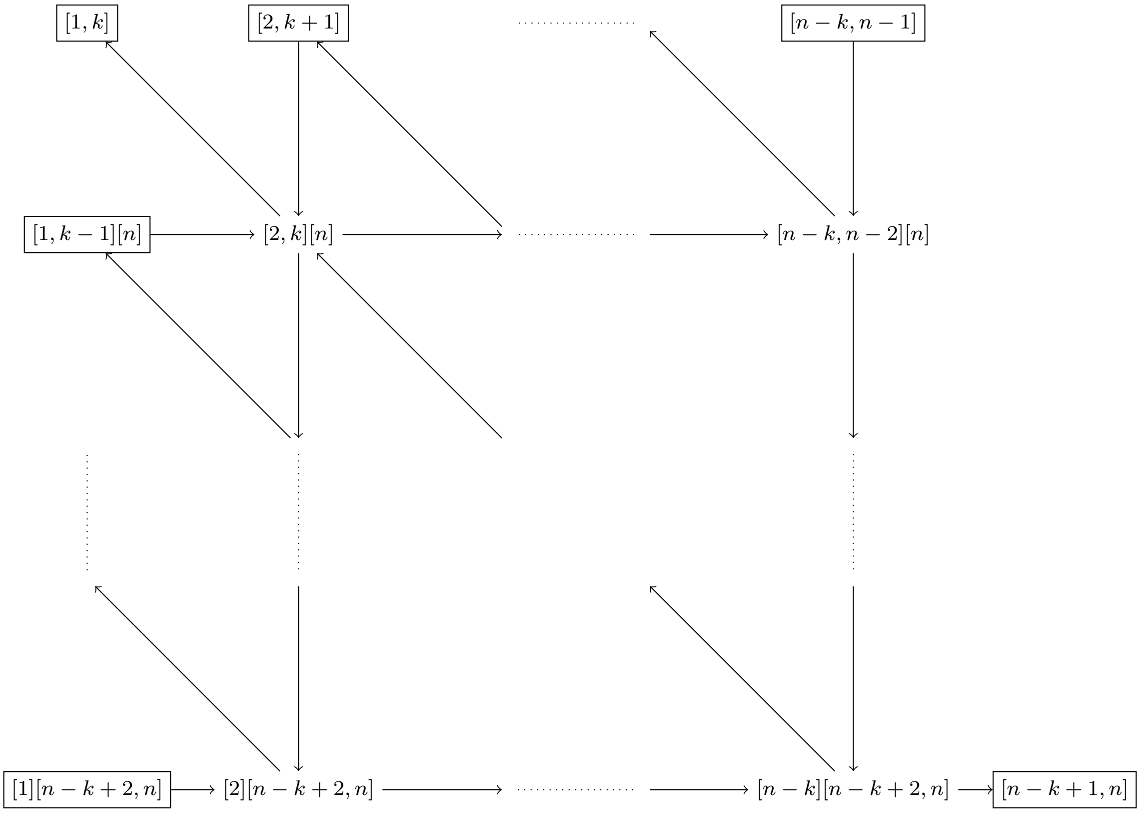
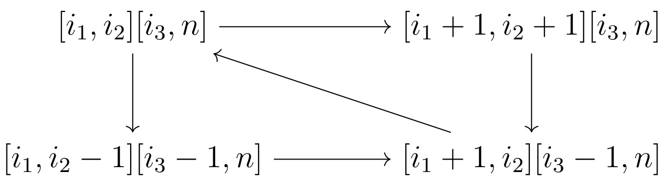


FIGURE 4.2. The exchange quiver $Q_{\widetilde{\underline{\max}}}$. The frozen variables are shown in boxes.

The exchange quiver for the maximal seed $(\widetilde{\underline{\max}}, Q_{\widetilde{\underline{\max}}})$ is shown in Figure 4.2 below. The arrows between mutable vertices are as follows.



PROPOSITION 4.1.1. *The sets* $\widetilde{\underline{\min}}$ *and* $\widetilde{\underline{\max}}$ *defined in* (5) *above occur as clusters in* $\mathbb{C}[\mathrm{Gr}(k,n)]$.

PROOF. Recall that the set $\mathscr{C}_{\text{frozen}}$ corresponds to the set of frozen variables. Since these will appear in all clusters, we begin by showing that each $I \in \mathscr{C}_{\text{frozen}}$ is weakly separated from each $J \in \Pi_{k,n}$.

Let $I = [1,i_1][i_2,n] \in \mathscr{C}_{\text{frozen}}$ and $J \in \Pi_{k,n}$. Then I and J are weakly separated since

$$I' = [1,i_1]\backslash J \prec J\backslash I \prec I'' = [i_2,n]\backslash J.$$

Now, let $I = [i_1, i_2] \in \mathscr{C}_{\text{frozen}}$ and $J \in \Pi_{k,n}$. Then I and J are weakly separated since

$$J' = \{j \in J : j < i_1\} \prec I \backslash J \prec J'' = \{j \in J : j > i_2\}.$$

We now deal with the remaining elements of $\mathscr{C}_{\widetilde{\underline{\min}}}$. Let $I = [1, i_1][i_2, i_3]$, $J = [1, j_1][j_2, j_3] \in \mathscr{C}_{\widetilde{\underline{\min}}}$, and let $J_1 = [1, j_1]$, $J_2 = [j_2, j_3]$. Assume, without loss of generality, that $i_1 \leq j_1$. Then we have the following cases:

Case 1: $1 \leq i_1 < i_2 \leq j_1 \leq i_3 < j_2 \leq j_3 \leq n$.

Case 2: $1 \leq i_1 < i_2 \leq j_1 < j_2 \leq i_3 \leq j_3 \leq n$.

Case 3: $1 \leq i_1 < j_1 \leq i_2 \leq j_2 \leq i_3 \leq j_3 \leq n$,

or $1 \leq i_1 \leq j_1 < i_2 \leq j_2 \leq i_3 \leq j_3 \leq n$.

Case 4: $1 \leq i_1 < j_1 \leq i_2 \leq i_3 < j_2 \leq j_3 \leq n$,

or $1 \leq i_1 \leq j_1 < i_2 \leq i_3 < j_2 \leq j_3 \leq n$.

Case 5: $1 \leq i_1 < j_1 \leq i_2 \leq j_2 \leq j_3 < i_3 \leq n$,

or $1 \leq i_1 \leq j_1 < i_2 \leq j_2 \leq j_3 < i_3 \leq n$.

Case 6: $1 \leq i_1 \leq j_1 < j_2 < i_2 \leq j_3 \leq i_3 \leq n$.

Case 7: $1 \leq i_1 \leq j_1 < j_2 \leq j_3 < i_2 \leq i_3 \leq n$.

Case 8: $1 \leq i_1 < i_2 \leq j_1 < j_2 < j_3 \leq i_3 \leq n$.

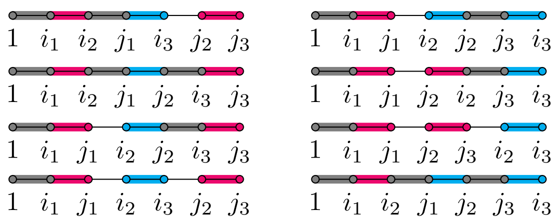
In cases 1-4, I and J are weakly separated since

$$J' = J_1 \backslash I \prec I \backslash J \prec J'' = J_2 \backslash I.$$

In cases 5-8 I and J are weakly separated since

$$J' = J \backslash I \prec I \backslash J \prec J'' = \emptyset.$$

To see this, consider the following diagram, where blue corresponds to $I \backslash J$, pink corresponds to $J \backslash I$, and grey corresponds to $I \cup J$.



Cases 1-4 Cases 5-8

We now turn our attention to the maximal cluster. Let $I = [i_1, i_2][i_3, n]$, $J = [j_1, j_2][j_3, n] \in \mathscr{C}_{\widetilde{\underline{\max}}}$, and let $J_1 = [j_1, j_2]$, $J_2 = [j_3, n]$. Assume that $j_3 \geq i_3$. Then we have the following cases:

Case 1: $1 \leq i_1 \leq j_1 \leq j_2 < j_3 \leq i_2 < i_3 \leq n$.

Case 2: $1 \leq i_1 \leq i_2 \leq j_1 \leq j_2 < j_3 \leq i_3 \leq n$.

Case 3: $1 \leq i_1 \leq j_1 \leq i_2 < j_2 < j_3 \leq i_3 \leq n$.

Case 4: $1 \leq i_1 < j_1 \leq j_2 \leq i_2 < j_3 \leq i_3 \leq n$,

or $1 \leq i_1 < j_1 \leq j_2 \leq i_2 \leq j_3 < i_3 \leq n$.

Case 5: $1 \leq j_1 \leq j_2 \leq i_1 \leq i_2 < j_3 \leq i_3 \leq n$,

or $1 \leq j_1 \leq j_2 \leq i_1 \leq i_2 \leq j_3 < i_3 \leq n$.

Case 6: $1 \leq j_1 \leq i_1 \leq j_2 \leq i_2 < j_3 \leq i_3 \leq n$,

or $1 \leq j_1 \leq i_1 \leq j_2 \leq i_2 \leq j_3 < i_3 \leq n$.

Case 7: $1 \leq j_1 \leq i_1 \leq j_2 < j_3 \leq i_2 < i_3 \leq n$.

Case 8: $1 \leq j_1 \leq j_2 < i_1 \leq j_3 \leq i_2 < i_3 \leq n$.

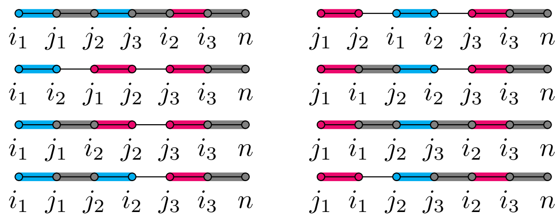
In cases 1-4, I and J are weakly separated since

$$I' = I_1 \backslash J \prec J \backslash I \prec I'' = \emptyset.$$

In cases 5-8, I and J are weakly separated since

$$J' = J_1 \backslash I \prec I \backslash J \prec J'' = J_2 \backslash I.$$

This is illustrated in the following diagram, where blue corresponds to $I \backslash J$, pink corresponds to $J \backslash I$, and grey corresponds to $I \cup J$.



Cases 1-4 Cases 5-8

Note that, in both sets of diagrams above, it is not a problem to have $i_r = j_s$ for some $r, s \in \{1, 2, 3\}$. In this case, the result will be that a 'grey' region becomes a single point, or that a 'blue'/'pink' region becomes empty. Neither of these will change that the two index sets in question are weakly separated. Thus, the sets $\mathscr{C}_{\widetilde{\underline{\min}}}$ and $\mathscr{C}_{\widetilde{\underline{\max}}}$ are weakly separated.

In [**OPS11**, Thm 3.3], it is established that any maximal weakly separated collection must have cardinality $k(n-k)+1$. It is clear, from counting rows and columns of the quivers, that we have $\left|\mathscr{C}_{\widetilde{\underline{\min}}}\right| = \left|\mathscr{C}_{\widetilde{\overline{\max}}}\right| = k(n-k)+1$, and hence these are indeed *maximal* weakly separated sets.

Finally, [**OPS11**, Thm 1.6] states that any maximal weakly separated set corresponds to a cluster in $\mathbb{C}[\mathrm{Gr}(k,n)]$. □

Proposition 4.1.2. *The cluster* $\widetilde{\underline{\min}}$ *(resp.* $\widetilde{\overline{\max}}$*) defined above is minimal (resp. maximal) in the sense of Definition 4.0.4.*

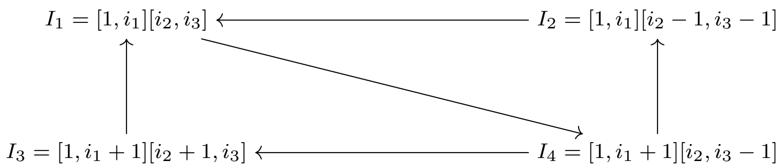
Proof. Let us begin with the minimal cluster. We claim that for any $I \in \mathscr{C}_{\widetilde{\underline{\min}}}$, and for any $J \notin \mathscr{C}_{\widetilde{\underline{\min}}}$ such that $J \leq_{\mathrm{st}} I$, the set $(\mathscr{C}_{\widetilde{\underline{\min}}} \backslash I) \cup \{J\}$ is not weakly separated.

First, note that if I is one of the following:

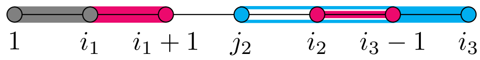
$$[1,k-1][n],[1,k-1][n-1],...,[1,k-1][k+1],$$
$$[1,k],[2,k+1],[1][3,k+1]...,[1,k-2][k,k+1] \quad (6)$$

then it cannot be replaced by anything smaller, since all $J \leq_{\mathrm{st}} I$ are already in $\mathscr{C}_{\widetilde{\underline{\min}}}$.

Now, recall that each square in the mutable part of the exchange quiver is of the following form:



We proceed by induction. Suppose the above claim holds for all $I <_{\mathrm{st}} I_1$ in the maximal cluster. We wish to show that the claim holds for I_1. By assumption, we need only to consider $J \notin \mathscr{C}_{\widetilde{\underline{\min}}}$ such that $I_2, I_3 \leq_{\mathrm{st}} J <_{\mathrm{st}} I_1$. Any such J is of the form $J = [1,i_1][j_2] \cup J' \cup [i_3]$, where $i_1+1 < j_2 < i_2$ and $J' \subset [j_2+1, i_3-1]$. However, elements of this form are not weakly separated from I_4. In order to see this, recall that J and I_4 both have cardinality k; we may therefore deduce that $|J'| < |[i_2,i_3-1]|$. Hence, $[i_2,i_3-1]\backslash J'$ is non-empty, and so there is no choice of J' such that J and I_4 are weakly separated. This is illustrated in the following diagram. Note that $i_2 = i_3-1$ and $1 = i_1$ are the only instances in which points on the diagram coincide, and neither of these have any effect on the argument - the grey or pink regions will simply consist of single points.



It remains to deal with the elements of the form $I_1 = [i, i+k-1]$, i.e. we wish to show that if the claim holds for $I_2 = [i-1, i+k-2], I_3 = [1][i+1, i+k-1], I_4 = [1][i, i+k-2] \in \mathscr{C}_{\underline{\widetilde{\mathrm{min}}}}$, then the claim holds for I_1.

By assumption, we only need to consider $I_2, I_3 \leq_{\mathrm{st}} J <_{\mathrm{st}} I_1$. Any such J is of the form $J = [j] \cup J' \cup [i+k-1]$, where $j < i$ and $J' \subset [j+1, i+k-2]$. However, elements of this form cannot be weakly separated from I_4. In order to see this we first note that J and I_4 are both of cardinality k. Hence, $|J'| < |[i, i+k-2]|$ and so $[i, i+k-2] \backslash J'$ cannot be empty. This means that there is no choice of J' such that J and I_4 are weakly separated.

Hence, the claim holds for all $I \in \mathscr{C}_{\underline{\widetilde{\mathrm{min}}}}$ by induction.

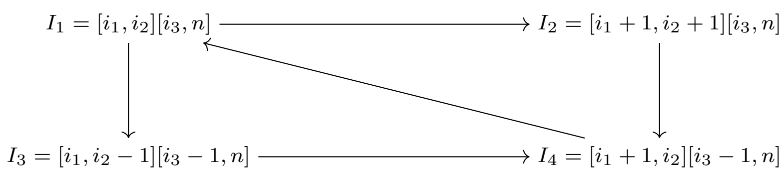
We now turn our attention to the maximal cluster. The proof will be almost identical to that of the minimal case. The claim here is that for any $I \in \mathscr{C}_{\underline{\widetilde{\mathrm{max}}}}$, and for any $J \notin \mathscr{C}_{\underline{\widetilde{\mathrm{max}}}}$ such that $I \leq_{\mathrm{st}} J$, the set $(\mathscr{C}_{\underline{\widetilde{\mathrm{max}}}} \backslash I) \cup \{J\}$ is not weakly separated.

First, notice that if I is one of the following

$$[1][n-k+2, n], [2][n-k+2, n], \ldots, [n-k+1, n], [n-k, n-1], \\ [n-k, n-2][n], \ldots, [n-k, n-k+1][n-k+3, n]$$

then all J such that $I \leq_{\mathrm{st}} J$ are in $\mathscr{C}_{\underline{\widetilde{\mathrm{max}}}}$ already, and so we cannot replace I by anything larger.

Next, recall that the mutable part of the exchange quiver consists of squares of the following form:



Once again, we proceed by induction. Suppose the claim above holds for all I with $I_1 <_{\mathrm{st}} I$. We show that the claim then holds for I_1. By assumption, we need to consider only $J \notin \mathscr{C}_{\underline{\widetilde{\mathrm{max}}}}$ such that $I_1 <_{\mathrm{st}} J \leq_{\mathrm{st}} I_2, I_3$. Any such J is of the form $J = [i_1] \cup J' \cup [j_2][i_3, n]$ where $j_2 > i_2$ and $J' \subseteq [i_1+1, j_2-1]$. However, any such J will not be weakly separated from I_4. To see this, notice that J and I_4 both have cardinality k. We can therefore deduce that $|J'| < |[i_1+1, i_2]|$. Hence, $[i_1+1, i_2]\backslash J'$ must be non-empty, so J and I_4 cannot be weakly separated.

It remains to deal with the elements of $\mathscr{C}_{\underline{\widetilde{\max}}}$ of the form $I_1 = [i, i+k-1]$, i.e. we wish to show that if the claim holds for $I_2 = [i+1, i+k]$, $I_3 = [i, i+k-2][n]$ and $I_4 = [i+1, i+k-1][n]$, then the claim holds for I_1.

By assumption, it suffices to consider $J \in \mathscr{C}_{\underline{\widetilde{\max}}}$ such that $I_1 <_{\mathrm{st}} J \leq_{\mathrm{st}} I_2, I_3$. Any such J must have the form $J = [i] \cup J' \cup [j]$, where $j > i+k-1$ and $J' \subseteq [i+1, j-1]$. However, elements of this form cannot be weakly separated from I_4. To see this, recall that J and I_4 both have cardinality k. Therefore, $|J'| < |[i+1, i+k-1]|$, and so $[i+1, i+k-1]\backslash J'$ is non-empty. Thus, J and I_4 cannot be weakly separated. The claim holds for all $I \in \mathscr{C}_{\underline{\widetilde{\max}}}$ by induction . □

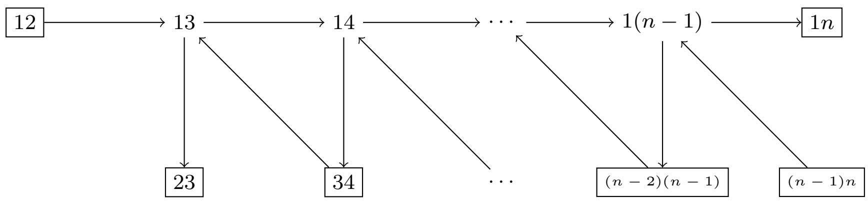
4.2. The Finite Type Case for $\mathbb{C}[\mathbf{Gr}(k,n)]$

We demonstrate below that we may obtain a path satisfying the required properties in the finite type case—it remains to determine whether this will still be the case for infinite type.

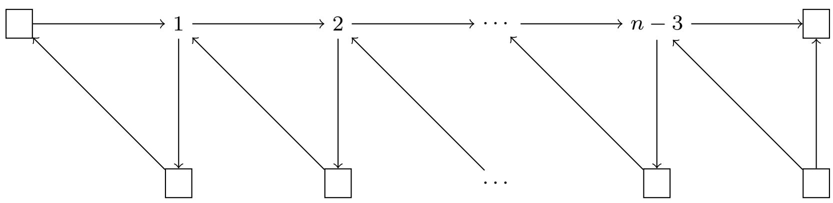
4.2.1. The $\mathbb{C}[\mathbf{Gr}(2,n)]$ Case. We first consider the $\mathrm{Gr}(2,n)$ case. For the initial seed we take the cluster

$$\underline{\widetilde{\min}} = (\Delta^{1n}, \boldsymbol{\Delta}^{\mathbf{1(n-1)}}, \ldots, \boldsymbol{\Delta}^{\mathbf{13}}, \Delta^{12}, \Delta^{23}, \Delta^{34}, \ldots, \Delta^{(n-2)(n-1)}, \Delta^{(n-1)n})$$

together with initial exchange matrix corresponding to the following quiver:



For clarity when dealing with mutation paths, we number the mutable vertices for the initial quiver as follows:



This enumeration is preserved when we mutate; when drawing the mutated quiver, we keep the same vertex positions and change only the arrows.

The maximal cluster is as follows

$$\underline{\widetilde{\max}} = (\Delta^{1n}, \mathbf{\Delta^{2n}}, \dots, \mathbf{\Delta^{(n-2)n}}, \Delta^{12}, \Delta^{23}, \Delta^{34}, \dots, \Delta^{(n-2)(n-1)}, \dots, \Delta^{(n-1)n})$$

and we consider the following mutation path, where the mutable vertices are numbered as above:

$$\begin{aligned}[\underline{p}] &= [1, (2,1), \dots, (n-4, n-5, \dots, 1), (n-3, n-4, \dots, 1)] \\ &= [\underline{p}_{n-3}, ..., \underline{p}_1], \end{aligned} \tag{7}$$

where $\underline{p}_i = (n-2-i, ..., 1)$. We claim that the sequence of new cluster variables obtained after each mutation is as follows:

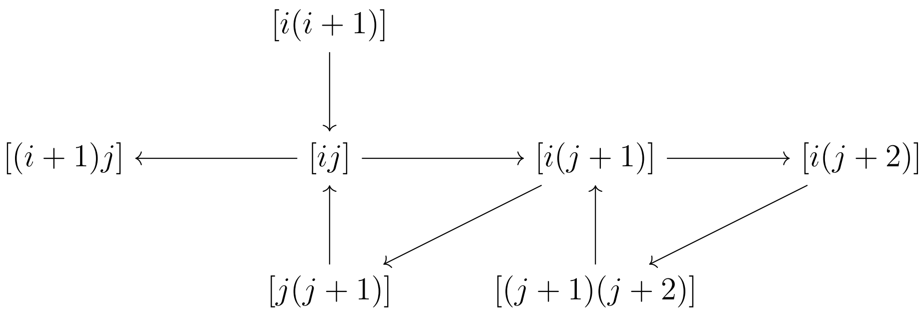
$$\Delta^{24}, \Delta^{25}, \dots, \Delta^{2n}, \Delta^{35}, \Delta^{36}, \dots, \Delta^{3n}, \dots, \Delta^{(n-2)n}$$

Below we prove that this mutation path has the properties (P1)-(P3) as required. In the previous Section, we have shown that $\underline{\widetilde{\min}}$ and $\underline{\widetilde{\max}}$ are indeed clusters in $\mathbb{C}[\mathrm{Gr}(2,n)]$, and that these are minimal/maximal in the sense of Definition 4.0.4.

THEOREM 4.2.1. *The mutation path $[\underline{p}]$ defined above satisfies properties (P1)-(P3).*

PROOF. Note that if $n < 4$ there is nothing to prove. Hence, assume that we have $n \geq 4$. By definition of the mutation path $[\underline{p}]$, we begin at the minimal cluster $\underline{\widetilde{\min}}$ as in (5), verifying the first part of (P3).

To address (P2) we first claim that, when mutating at Δ^{ij} along the path $[\underline{p}]$, the quiver appears locally as



where $[ij]$ is the mutating vertex, corresponding to Δ^{ij} for $1 \leq i < j \leq n-1$. Hence we have

$$\mu_{[ij]}([ij]) = \frac{[i(i+1)][j(j+1)] + [(i+1)j][i(j+1)]}{[ij]} = [(i+1)(j+1)].$$

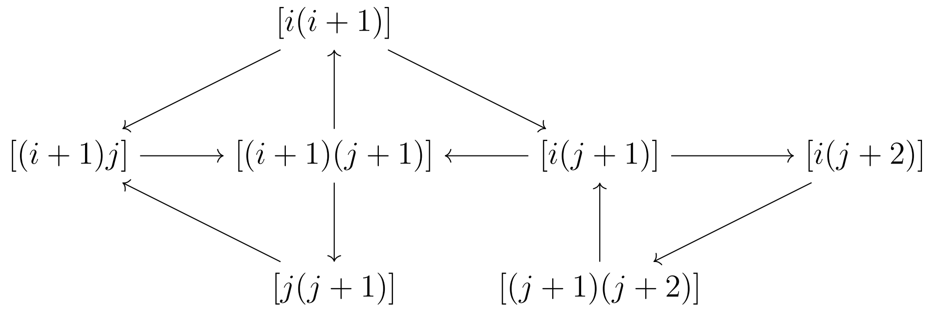
When $j = n-1$ we simply remove the vertices labelled $[i(j+2)]$ and $[(j+1)(j+2)]$.

This claim holds for the first mutation by inspection of the minimal cluster, and we see that

$$\mu_{[13]}([13]) = \frac{[12][34] + [14][23]}{[13]} = [24].$$

Now, assume that the claim holds up to mutation at the variable $[ij]$ along the mutation path $[\underline{p}]$. We will show that it holds for mutation at $[(i)(j+1)]$.

After mutation at $[ij]$ the quiver appears locally as follows, with indices taken modulo n.



The diagram above shows that the next vertex at which we mutate, $[i(j+1)]$, has the required form and

$$\begin{aligned}\mu_{[i(j+1)]}([i(j+1)]) &= \frac{[i(i+1)][(j+1)(j+2)] + [(i+1)(j+1)][i(j+2)]}{[i(j+1)]} \\ &= [(i+1)(j+2)].\end{aligned}$$

We note also that after mutation at $[i(j+1)]$ the vertex $[(i+1)(j+1)]$ will again have the required form.

After applying $\underline{p}_1$, the mutable cluster variables are $\Delta^{24}, ..., \Delta^{2n}$. Note that this is the only time we mutate at vertex $n-3$ along the path $[\underline{p}]$, and hence Δ^{2n} will remain in our final cluster.

In general, after applying $\underline{p}_r$, the mutable vertices are

$$\Delta^{(r+1)(r+3)}, ..., \Delta^{(r+1)n}, ...\Delta^{2n}.$$

The vertices $n-3, ..., n-1-r$ do not appear again in the mutation path, and hence $\Delta^{(r+1)n}, ..., \Delta^{2n}$ will remain in the final cluster.

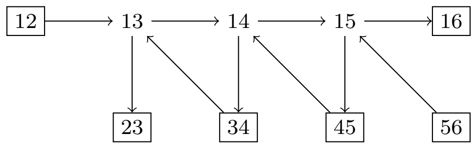
We may now observe that the path produces the variables claimed above, and hence satisfies properties (P1) and (P2). For the second part of (P3) we note that the Plücker coordinates which remain in the final cluster are precisely those in the maximal cluster defined above.

□

EXAMPLE 4.2.2 ($n = 6$). *In the $n = 6$ case, the initial seed consists of the cluster*

$$\underline{\widetilde{\min}} = (\Delta^{16}, \boldsymbol{\Delta^{15}}, \boldsymbol{\Delta^{14}}, \boldsymbol{\Delta^{13}}, \boldsymbol{\Delta^{12}}, \Delta^{23}, \Delta^{34}, \Delta^{45}, \Delta^{56})$$

together with the following quiver:



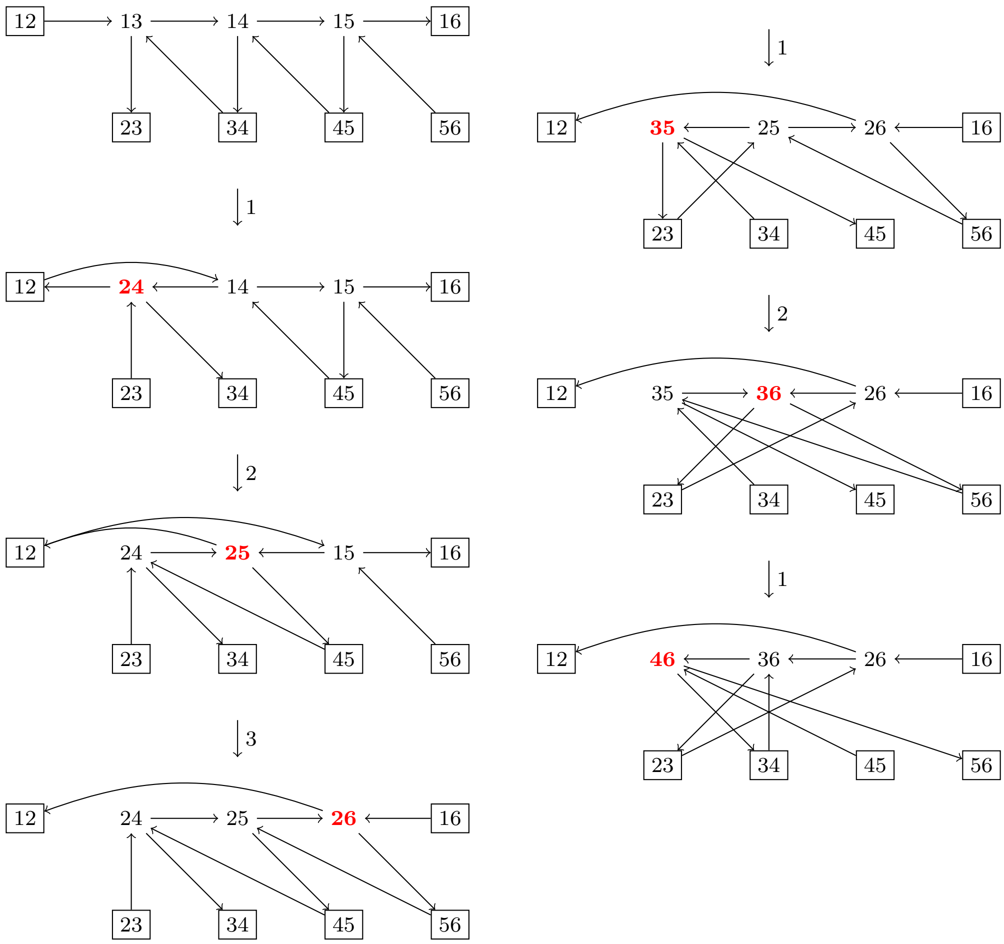


FIGURE 4.3. The mutation path in the Gr(2, 6) case. Mutable vertices are labelled 1-3 from left to right, and the labelled arrows state the vertex at which mutation takes place.

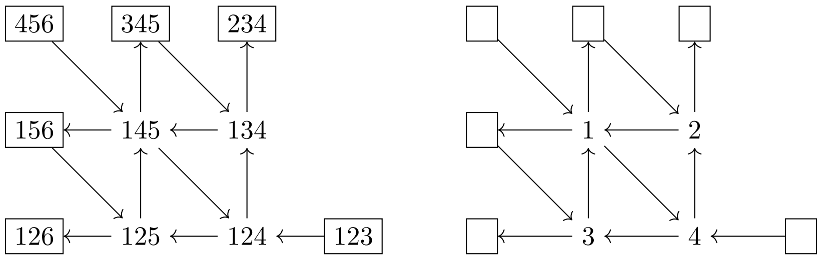
As shown in Figure 4.3, we obtain the following mutation path, with mutable vertices labelled 1-3 from left to right, and sequence of cluster variables:

$$[\underline{p}] = [1, 2, 1, 3, 2, 1], \quad \Delta^{24}, \Delta^{25}, \Delta^{26}, \Delta^{35}, \Delta^{36}, \Delta^{46}.$$

4.2.2. The $\mathbb{C}[\mathbf{Gr}(3,n)]$ Case for $n = 6, 7, 8$. We now deal with the remaining finite-type cases, namely $\mathbb{C}[\mathrm{Gr}(3,n)]$, for $n = 6, 7, 8$.

4.2.2.1. *The $n = 6$ Case.* In the case of $\mathbb{C}[\mathrm{Gr}(3,6)]$, the initial seed is as follows:

$$\underline{\widetilde{\min}} = (\boldsymbol{\Delta}^{\mathbf{145}}, \boldsymbol{\Delta}^{\mathbf{134}}, \boldsymbol{\Delta}^{\mathbf{125}}, \boldsymbol{\Delta}^{\mathbf{124}}, \Delta^{123}, \Delta^{234}, \Delta^{345}, \Delta^{456}, \Delta^{156}, \Delta^{126})$$



For ease when working with mutation paths, we label the mutable vertices as shown on the above—this labelling will be preserved after mutation, by retaining the vertex position and changing only the arrows.

The maximal cluster is the following

$$\underline{\widetilde{\max}} = (\boldsymbol{\Delta}^{\mathbf{236}}, \boldsymbol{\Delta}^{\mathbf{256}}, \boldsymbol{\Delta}^{\mathbf{346}}, \boldsymbol{\Delta}^{\mathbf{356}}, \Delta^{123}, \Delta^{234}, \Delta^{345}, \Delta^{456}, \Delta^{156}, \Delta^{126}).$$

The first few mutations are shown in Figure 4.4, and we obtain the following mutation path

$$[\underline{p}] = [4, 2, 3, 4, 1, 2, 3, 4, 2, 3, 4].$$

This produces the following sequence of cluster variables

$$\Delta^{135}, \Delta^{136}, \Delta^{235}, Y^{123456}, \Delta^{245}, \Delta^{146}, \Delta^{236}, \Delta^{246}, \Delta^{346}, \Delta^{256}, \Delta^{356}.$$

This mutation path satisfies properties (P1)-(P3) as required. Note that precisely half of the quadratic cluster variables appear in this mutation path (see [**GL09**] for a full list of cluster variables)—this will also be the case when $n = 7, 8$.

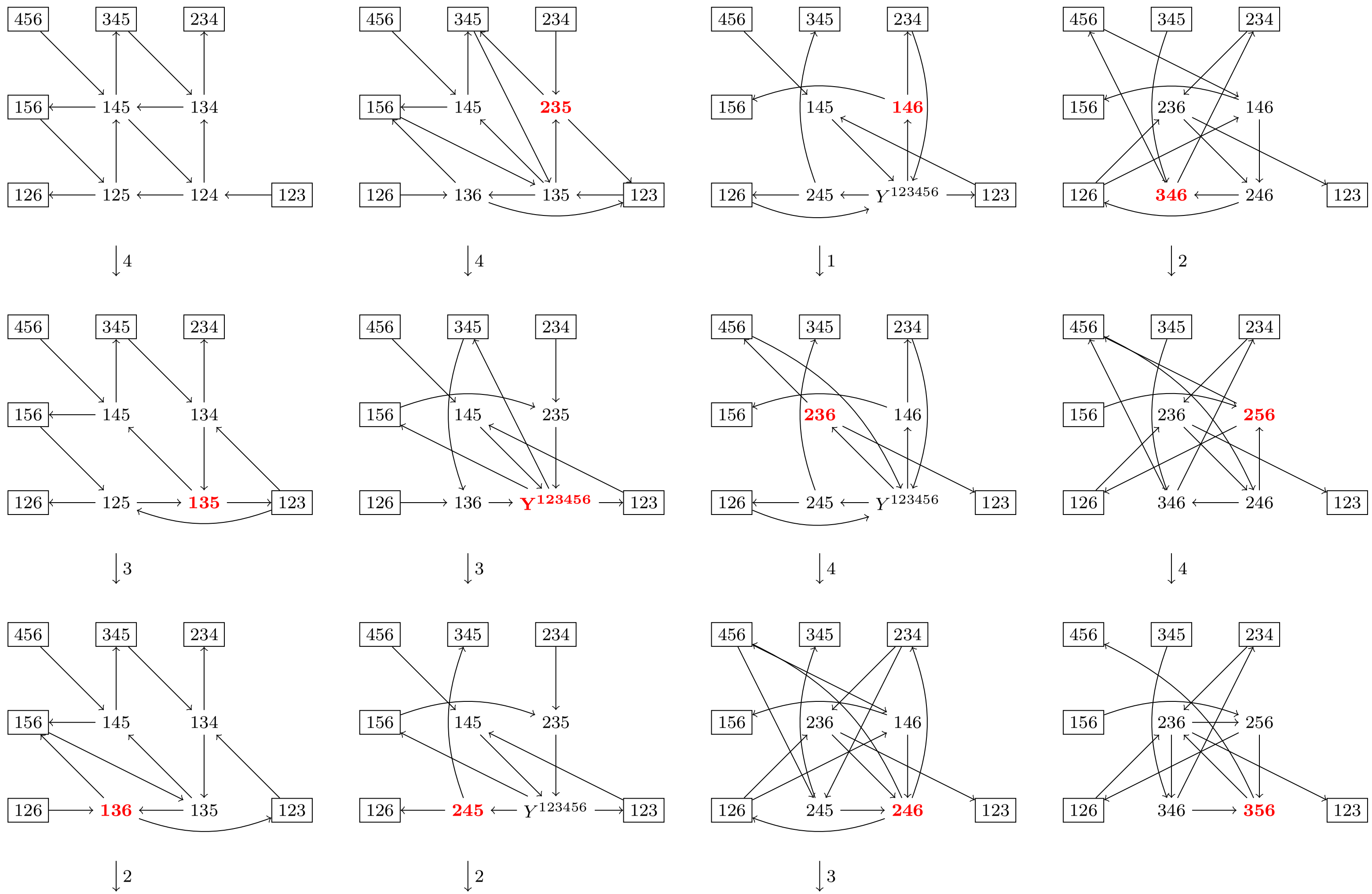
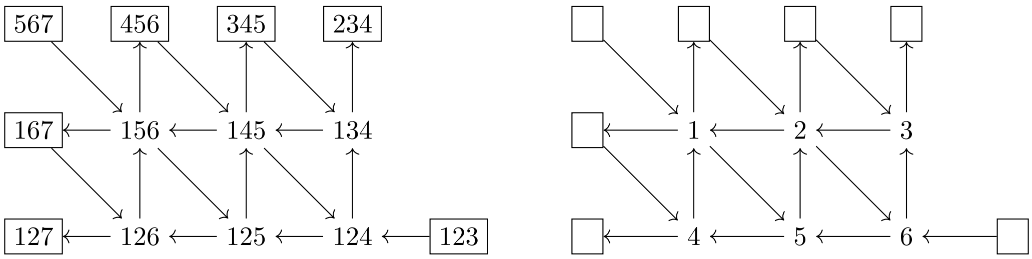


FIGURE 4.4. The mutation path in the $\mathrm{Gr}(3,6)$ case. Mutable vertices are labelled from left to right, and the labelled arrows state at which vertex we mutate.

4.2.2.2. *The $n = 7$ Case.* For the $\mathbb{C}[\mathrm{Gr}(3,7)]$ case, we take the following initial seed:

$$\underline{\widetilde{\min}} = (\boldsymbol{\Delta^{156}}, \boldsymbol{\Delta^{145}}, \boldsymbol{\Delta^{134}}, \boldsymbol{\Delta^{126}}, \boldsymbol{\Delta^{125}}, \boldsymbol{\Delta^{124}},$$

$$\Delta^{123}, \Delta^{234}, \Delta^{345}, \Delta^{456}, \Delta^{567}, \Delta^{167}, \Delta^{127})$$



For mutation paths, we will use the labelling of mutable vertices shown above—this labelling will be preserved after mutation, by retaining the vertex position and changing only the arrows.

The maximal cluster is as follows:

$$\underline{\widetilde{\max}} = (\boldsymbol{\Delta^{237}}, \boldsymbol{\Delta^{347}}, \boldsymbol{\Delta^{457}}, \boldsymbol{\Delta^{267}}, \boldsymbol{\Delta^{367}}, \boldsymbol{\Delta^{467}},$$

$$\Delta^{123}, \Delta^{234}, \Delta^{345}, \Delta^{456}, \Delta^{567}, \Delta^{167}, \Delta^{127})$$

Here, we obtain the following mutation path, in which the numbers correspond to mutable vertices labelled from left to right,

$$[\underline{p}] = [6, 3, 5, 6, 2, 3, 4, 5, 6, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 2, 3, 4, 5, 6, 3, 4, 5, 6].$$

This yields the following sequence of cluster variables

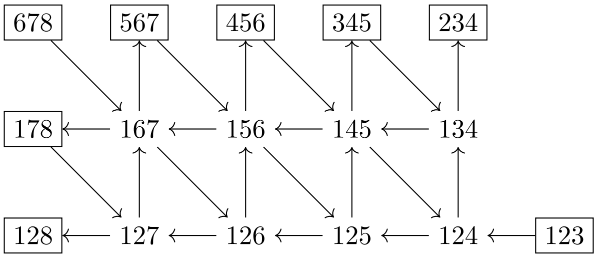
$$\Delta^{135}, \Delta^{136}, \Delta^{137}, \Delta^{235}, Y^{123456}, Y^{123457}, \Delta^{245}, \Delta^{146}, \Delta^{236}, Y^{123467}, \Delta^{246}, \Delta^{346},$$
$$Y^{123567}, \Delta^{147}, \Delta^{237}, Y^{124567}, Y^{134567}, \Delta^{157}, \Delta^{247}, \Delta^{256}, Y^{234567}, \Delta^{257}, \Delta^{267}, \Delta^{356},$$
$$\Delta^{347}, \Delta^{357}, \Delta^{367}, \Delta^{457}, \Delta^{467}.$$

This mutation path satisfies the properties (P1)-(P3) as required.

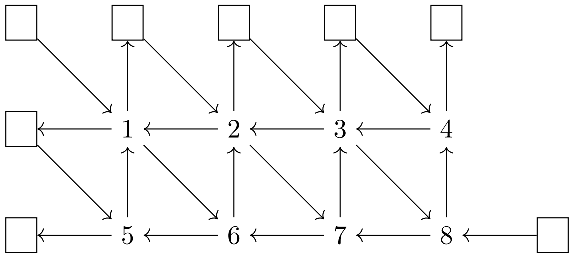
4.2.2.3. *The $n = 8$ Case.* Finally, for the $n = 8$ case, the initial seed is as follows

$$\underline{\widetilde{\min}} = (\boldsymbol{\Delta^{167}}, \boldsymbol{\Delta^{156}}, \boldsymbol{\Delta^{145}}, \boldsymbol{\Delta^{134}}, \boldsymbol{\Delta^{127}}, \boldsymbol{\Delta^{126}}, \boldsymbol{\Delta^{125}}, \boldsymbol{\Delta^{124}},$$

$$\Delta^{123}, \Delta^{234}, \Delta^{345}, \Delta^{456}, \Delta^{567}, \Delta^{678}, \Delta^{178}, \Delta^{128})$$



For mutation paths we will utilise the following labelling of mutable vertices:



This labelling will be preserved after mutation, by retaining the vertex position and changing only the arrows.

We have

$$\underline{\widetilde{\max}} = (\boldsymbol{\Delta}^{\mathbf{238}}, \boldsymbol{\Delta}^{\mathbf{348}}, \boldsymbol{\Delta}^{\mathbf{458}}, \boldsymbol{\Delta}^{\mathbf{568}}, \boldsymbol{\Delta}^{\mathbf{278}}, \boldsymbol{\Delta}^{\mathbf{378}}, \boldsymbol{\Delta}^{\mathbf{478}}, \boldsymbol{\Delta}^{\mathbf{578}},$$

$$\Delta^{123}, \Delta^{234}, \Delta^{345}, \Delta^{456}, \Delta^{567}, \Delta^{678}, \Delta^{178}, \Delta^{128})$$

We obtain the following mutation path, again with numbers corresponding to mutable vertices, labelled as above:

$$[\underline{p}] = [8,4,7,8,3,4,6,7,8,3,4,5,6,7,8,2,3,4,5,6,7,8,2,3,4,5,6,7,8,2,3,$$
$$4,5,6,7,8,2,3,4,5,6,7,8,1,2,3,4,5,6,7,8,2,3,4,5,6,7,8,2,3,4,$$
$$5,6,7,8,3,4,5,6,7,8,4,5,6,7,8].$$

This gives the following sequence of cluster variables

$$\Delta^{135}, \Delta^{136}, \Delta^{137}, \Delta^{138}, \Delta^{235}, Y^{123456}, Y^{123457}, Y^{123458}, \Delta^{245}, \Delta^{146}, \Delta^{236}, Y^{123467},$$
$$Y^{123468}, \Delta^{246}, \Delta^{346}, Y^{123567}, \Delta^{147}, \Delta^{237}, B(8,0), Y^{124567}, Y^{134567}, \Delta^{157}, Y^{123478},$$
$$Y^{123568}, \Delta^{148}, B(3,1), A(3), Y^{123578}, Y^{123678}, Y^{124568}, \Delta^{247}, \Delta^{256}, \Delta^{238}, B(3,0),$$
$$Y^{124578}, Y^{124678}, \Delta^{248}, Y^{234567}, Y^{134568}, \Delta^{347}, B(6,1), A(6), Y^{234568}, \Delta^{356},$$
$$Y^{134578}, \Delta^{257}, \Delta^{158}, B(6,0), Y^{234578}, \Delta^{357}, \Delta^{457}, Y^{125678}, Y^{134678}, \Delta^{267}, B(1,1),$$
$$Y^{135678}, Y^{145678}, \Delta^{168}, Y^{234678}, \Delta^{258}, \Delta^{348}, Y^{235678}, Y^{245678}, \Delta^{268}, \Delta^{278}, \Delta^{358}, \Delta^{367},$$
$$Y^{345678}, \Delta^{368}, \Delta^{378}, \Delta^{467}, \Delta^{458}, \Delta^{468}, \Delta^{478}, \Delta^{568}, \Delta^{578}$$

where $A(i)$ and $B(i,j)$ are the cubic regular functions shown in Table 2 of [**GL09**].

Again, this mutation path satisfies the required properties. Note also that we now obtain precisely one third of the cubic cluster variables.

REMARK 4.2.3. *It is natural at this point to ask whether the mutation paths described above are maximal green sequences. Introduced in* [**Kel11**]*, a maximal green sequence is a certain path in the exchange graph of a cluster algebra, beginning at the unique smallest element and ending at the unique largest element. The existence of these sequences proves useful since, among other things, they provide explicit formulas for a generic basis in the upper cluster algebra. See* [**DK20**] *for more details. A straightforward check shows that the sequences we describe are not maximal green sequences, even in the* $\mathrm{Gr}(2,n)$ *case.*

4.3. The Finite Type Case for $\mathbb{C}[M(k,j)]$

We now wish to apply the theory developed in Section 4 to another related class of (quantum) cluster algebras, namely the coordinate ring of the matrix algebra as in Definition 2.2.10. We will construct a partial order on the generators.

Rather than computing the relevant mutation paths from scratch in these cases, we note that they may be obtained from the corresponding Grassmannian case via the algebra isomorphism α given in [**LR08**]. For convenience, we give the definition of this isomorphism below.

PROPOSITION 4.3.1 ([**GL13**, Prop 6.1]). *Let σ be the automorphism of $\mathbb{K}_q[M(k,n-k)]$ defined by $\sigma(X_{ij}) = qX_{ij}$. The map*

$$\alpha : \mathbb{K}_q[M(k,n-k)][Y^{\pm 1};\sigma] \to \mathbb{K}_q[\mathrm{Gr}(k,n)]\left[[12\dots k]^{-1}\right]$$

defined by

$$\alpha(X_{ij}) = [1\dots \widehat{k-i+1}\dots k(j+k)][1\dots k]^{-1},\ \alpha(Y) = [12\dots k]$$

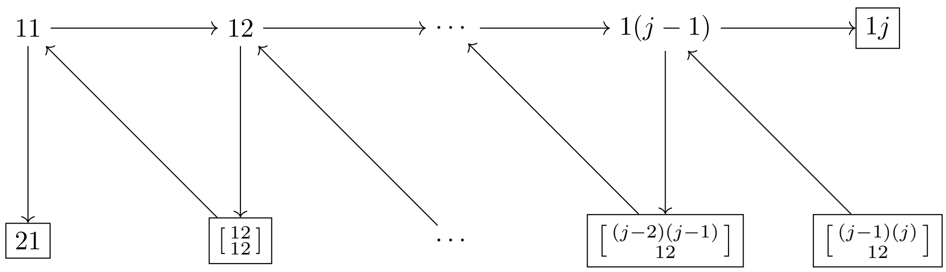
is an algebra isomorphism. Note that $\widehat{a}$ denotes an omitted index.

4.3.1. The $\mathbb{C}[M(2,j)]$ Case.

We first consider the $M(2,j)$ case. For the initial seed we take the cluster

$$\underline{\widetilde{\min}} = \left((21),(11),\dots,(1,j),\left[\begin{smallmatrix}12\\12\end{smallmatrix}\right],\dots,\left[\begin{smallmatrix}(j-1)(j)\\12\end{smallmatrix}\right]\right)$$

together with initial exchange matrix corresponding to the following quiver:



The frozen variables are those in boxes.

Using the mutation path for the corresponding Grassmannian, together with the isomorphism α defined above, we obtain the following mutation path (note that the mutable vertices are labelled from left to right)

$$[\underline{p}] = [(1),(2,1),\ldots,(n-1,n-2,\ldots,1),(n,n-1,\ldots,1)].$$

The sequence of new cluster variables obtained after each mutation is as follows:

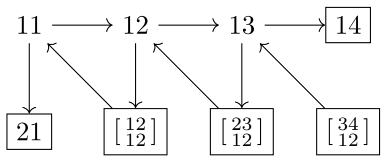
$$(22),\ldots,(2j),\left[\begin{smallmatrix}13\\12\end{smallmatrix}\right],\ldots,\left[\begin{smallmatrix}1j\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}2j\\12\end{smallmatrix}\right],\ldots,\left[\begin{smallmatrix}(j-2)j\\12\end{smallmatrix}\right]$$

Note that this mutation path has the properties (P1)-(P3) as required.

Example 4.3.2 ($j=4$). *In the $j=4$ case, the initial seed consists of the cluster*

$$\widetilde{\underline{\min}} = \big((21),(11),(12),(13),(14),\left[\begin{smallmatrix}12\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}23\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}34\\12\end{smallmatrix}\right]\big)$$

together with the following quiver:



As shown in Figure 4.5, we obtain the following mutation path and sequence of cluster variables:

$$[\underline{p}] = [1,2,1,3,2,1], \quad (22),(23),(24),\left[\begin{smallmatrix}13\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}14\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}24\\12\end{smallmatrix}\right].$$

The maximal cluster in this case is the following:

$$\widetilde{\underline{\max}} = \big((21),\left[\begin{smallmatrix}24\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}14\\12\end{smallmatrix}\right],(24),\left[\begin{smallmatrix}12\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}23\\12\end{smallmatrix}\right],\left[\begin{smallmatrix}34\\12\end{smallmatrix}\right]\big).$$

4.3.2. The $\mathbb{C}[M(3,j)]$ Case.

4.3.2.1. *The $j=3$ Case.* In the case of $\mathbb{C}[M(3,3)]$, the initial seed is as follows:

$$\widetilde{\underline{\min}} = \big(\left[\begin{smallmatrix}12\\12\end{smallmatrix}\right],(21),(12),(11),(31),\left[\begin{smallmatrix}12\\23\end{smallmatrix}\right],\left[\begin{smallmatrix}123\\123\end{smallmatrix}\right],\left[\begin{smallmatrix}23\\12\end{smallmatrix}\right],(13)\big)$$

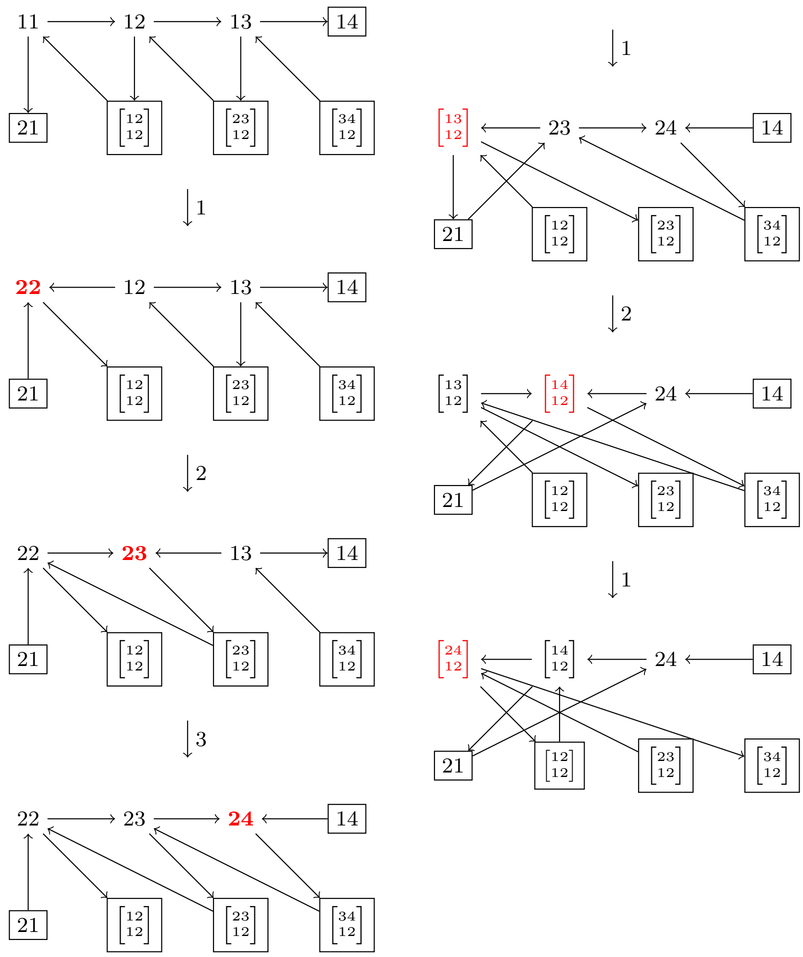
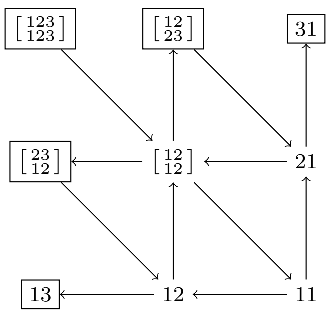


FIGURE 4.5. The mutation path for the $M(2,4)$ case. The labelled arrows correspond to mutable vertices, labelled from left to right and top to bottom.



The maximal cluster is the following

$$\widetilde{\max} = \big((33), \left[\begin{smallmatrix}23\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\23\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\13\end{smallmatrix}\right], (31), \left[\begin{smallmatrix}12\\23\end{smallmatrix}\right], \left[\begin{smallmatrix}123\\123\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\12\end{smallmatrix}\right], (13)\big).$$

We obtain the following mutation path

$$[\underline{p}] = [4, 2, 3, 4, 1, 2, 3, 4, 2, 3, 4].$$

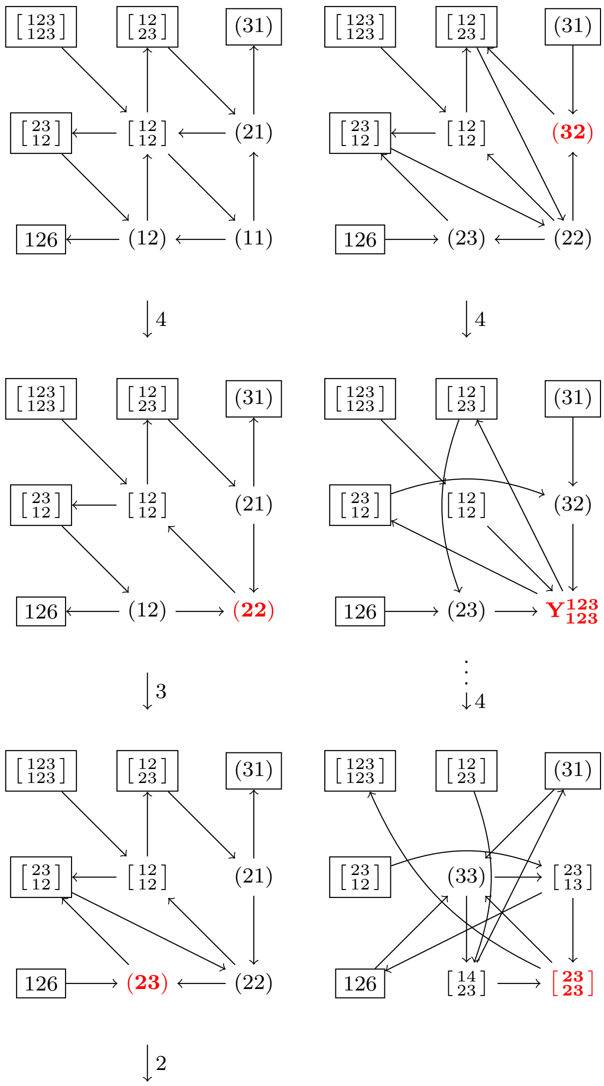


FIGURE 4.6. The mutation path for the $M(3,3)$ case. The labelled arrows correspond to mutable vertices, labelled from left to right.

This produces the following sequence of cluster variables

$$(22), (23), (32), (32)\left[\begin{smallmatrix}13\\12\end{smallmatrix}\right] - (31)\left[\begin{smallmatrix}23\\12\end{smallmatrix}\right], \left[\begin{smallmatrix}12\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\12\end{smallmatrix}\right], (33), \left[\begin{smallmatrix}13\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\23\end{smallmatrix}\right],$$

$$\left[\begin{smallmatrix}23\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\23\end{smallmatrix}\right].$$

The first few steps in this mutation path are shown in Figure 4.6.

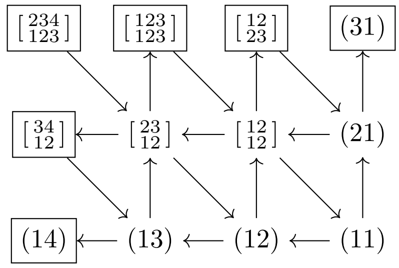
4.3.2.2. *The $j = 4$ Case.*

DEFINITION 4.3.3. Let $\{a, b, c, d, e, f\}$ be a subset of $\{1, \ldots, n\}$ written in increasing order, and define the following:

$$Y^{abc}_{def} = (fb)\left[\begin{smallmatrix}ac\\de\end{smallmatrix}\right] - (fa)\left[\begin{smallmatrix}bc\\de\end{smallmatrix}\right].$$

In the case of $\mathbb{C}[M(3,4)]$, the initial seed is as follows:

$$\underline{\widetilde{\min}} = \Big(\left[\begin{smallmatrix}23\\12\end{smallmatrix}\right], \left[\begin{smallmatrix}12\\12\end{smallmatrix}\right], (21), (13), (12), (11), (31), \left[\begin{smallmatrix}12\\23\end{smallmatrix}\right], \left[\begin{smallmatrix}123\\123\end{smallmatrix}\right], \left[\begin{smallmatrix}234\\123\end{smallmatrix}\right], \left[\begin{smallmatrix}34\\12\end{smallmatrix}\right], (14) \Big)$$



We obtain the following mutation path

$$[\underline{p}] = [6,3,5,6,2,3,4,5,6,2,3,4,5,6,1,2,3,4,5,6,2,3,4,5,6,3,4,5,6].$$

This produces the following sequence of cluster variables

$$(22), (23), (24), (32), Y^{123}_{123}, Y^{124}_{123}, \left[\begin{smallmatrix}12\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\12\end{smallmatrix}\right], (33), Y^{134}_{123}, \left[\begin{smallmatrix}13\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\23\end{smallmatrix}\right],$$

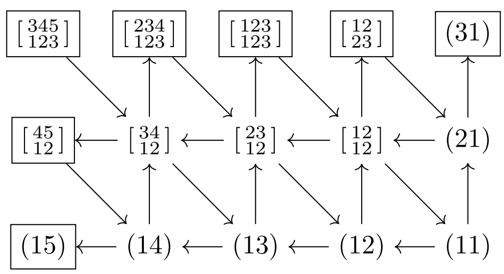
$$Y^{234}_{123}, \left[\begin{smallmatrix}14\\12\end{smallmatrix}\right], (34), (14)Y^{123}_{123}, (24)Y^{123}_{123}, \left[\begin{smallmatrix}24\\12\end{smallmatrix}\right], \left[\begin{smallmatrix}14\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\13\end{smallmatrix}\right], (34)Y^{123}_{123}, \left[\begin{smallmatrix}24\\13\end{smallmatrix}\right],$$

$$\left[\begin{smallmatrix}34\\13\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\23\end{smallmatrix}\right], \left[\begin{smallmatrix}13\\24\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\24\end{smallmatrix}\right], \left[\begin{smallmatrix}34\\23\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\34\end{smallmatrix}\right], \left[\begin{smallmatrix}34\\34\end{smallmatrix}\right].$$

4.3.2.3. *The $j=5$ Case.* The last finite type case to consider is $M(3,5)$, i.e. the equivalent of $\mathrm{Gr}(3,8)$ for the Grassmannian.

In this case, our initial seed is as follows:

$$\underline{\widetilde{\min}} = \Big(\left[\begin{smallmatrix}34\\12\end{smallmatrix}\right], \left[\begin{smallmatrix}23\\12\end{smallmatrix}\right], \left[\begin{smallmatrix}12\\12\end{smallmatrix}\right], (21), (14), (13), \\ (12), (11), (31), \left[\begin{smallmatrix}12\\23\end{smallmatrix}\right], \left[\begin{smallmatrix}123\\123\end{smallmatrix}\right], \left[\begin{smallmatrix}234\\123\end{smallmatrix}\right], \left[\begin{smallmatrix}345\\123\end{smallmatrix}\right], \left[\begin{smallmatrix}45\\12\end{smallmatrix}\right], (15) \Big)$$



Using the mutation path found earlier for $\mathrm{Gr}(3,8)$, together with the isomorphism α defined above, we may obtain the sequence of cluster variables obtained along the relevant mutation path.

4.4. Perfect Matchings

In what follows, we describe a potential connection with *perfect matchings*. We observe that enhancing the quivers appearing above with certain perfect matchings suggests that one could construct an algorithm to produce the mutation paths from Section 4.2 in a more general setting.

REMARK 4.4.1. *The interested reader may wish to consult* [**ÇKP24**]*,* [**BKM16**] *for other uses of perfect matchings in cluster theory. In* [**ÇKP24**]*, the authors introduce a class of modules for* dimer algebras *which correspond to perfect matchings on the dimer model. This is then used to show that the associated cluster category embeds into that of the appropriate Grassmannian.*

We begin by stating some relevant definitions.

DEFINITION 4.4.2 (Quiver with faces). A *quiver with faces* is a quiver $Q = (Q^0, Q^1)$, together with a set, Q^2, of faces (i.e. oriented cycles). We write $Q = (Q^0, Q^1, Q^2)$.

DEFINITION 4.4.3 ([**BKM16**, Def 2.4]). The quiver $Q(D)$ of a Postnikov diagram D has vertices $Q^0(D) = \mathcal{C}(D)$ given by the labels of the alternating regions of D. The arrows $Q^1(D)$ correspond to intersection points of two alternating regions, with orientations as in [**BKM16**, Figure 4]. The diagram on the right of [**BKM16**, Figure 4] indicates the boundary case. We refer to the arrows between boundary vertices as boundary arrows.

Consider the quiver corresponding to the minimal cluster in the Grassmannian case. We observe that, following the construction in [**BKM16**], the *boundary vertices* are precisely the frozen vertices, and the *boundary arrows* are those between frozen vertices, as shown in Figure 4.7. The remaining arrows are the *internal arrows.* It can be observed that the incidence graphs at each vertex are indeed cycles and hence connected, thus verifying that we have a *dimer model with boundary* as in [**BKM16**, Def 3.2].

We note that $Q_{\widetilde{\underline{\min}}}$, is a quiver with faces since it is a finite dimer model with boundary in a disc. See [**BKM16**, Remark 3.4] for further details. Note also that, although there is an orientation arising from the Postnikov diagram, we do not make use of this in what follows. We choose the set of faces, Q^2, to correspond precisely

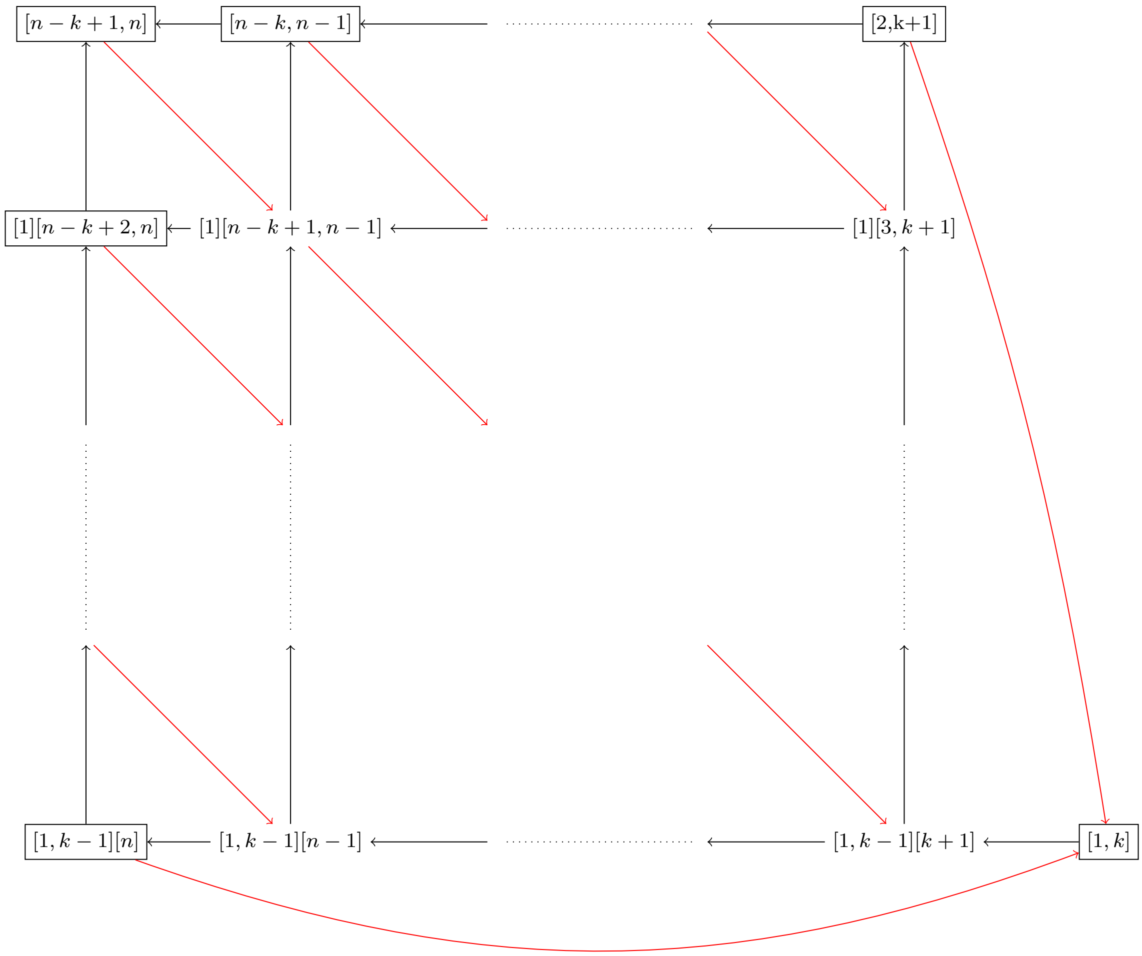
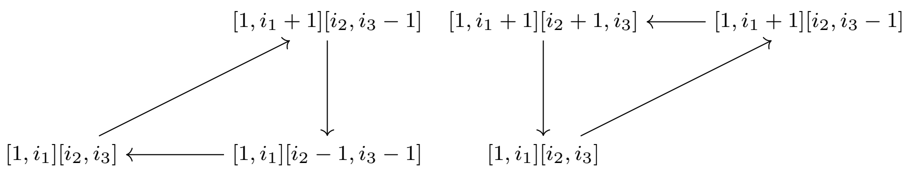
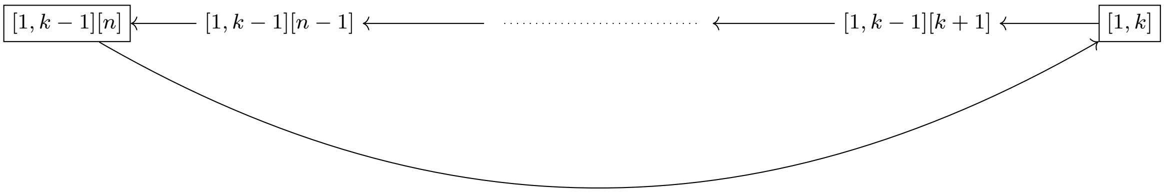


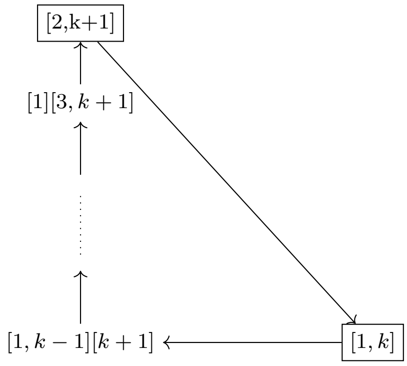
FIGURE 4.7. The red arrows indicate the initial perfect matching on $Q_{\widetilde{\underline{\min}}}$.

to the oriented regions in the Postnikov diagram as in [**ÇKP24**, Def 2.8]. These faces consist of all 3-cycles of the forms shown below:



together with the following larger cycles arising from the addition of boundary arrows:





The mutation rule for boundary arrows is as follows:

(i) Mutate as usual, treating the boundary arrows as ordinary arrows.

(ii) If this results in a 2-cycle of boundary arrows, keep only the new arrow added after mutation. Effectively, this will reverse the direction of the original boundary arrow.

See Figure 4.8 for an example. Note that this is consistent with mutation of Postnikov diagrams, and the same modified form of cluster mutation is detailed in [**BKM16**, Remark 12.3].

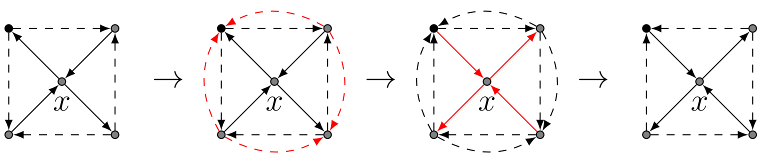


FIGURE 4.8. An example of mutation with boundary arrows. The dashed arrows are the boundary arrows, and we mutate at the vertex x.

DEFINITION 4.4.4 (Perfect matching, [**ÇKP24**, Def 4.1]). Let $Q = (Q^0, Q^1, Q^2)$ be a quiver with faces. A *perfect matching* on Q is a set $\nu \subset Q^1$ such that the boundary of each face in Q^2 contains exactly one arrow in ν.

PROPOSITION 4.4.5. *The red arrows in Figure 4.7 form a perfect matching, which we call $\nu_{\min}$, on the quiver $Q_{\widetilde{\underline{\min}}}$.*

PROOF. We observe that each of the faces described above contains exactly one perfect matching (red) arrow in Figure 4.7. Hence, the set of red arrows does indeed form a perfect matching on $Q_{\underline{\widetilde{\min}}}$. □

DEFINITION 4.4.6. Given a quiver with faces Q and a perfect matching ν, we denote by $Q\backslash\nu = (Q^0, Q^1\backslash\nu)$ the quiver with the same vertex set as Q and with arrow set $Q^1\backslash\nu$.

LEMMA 4.4.7. *The quiver $Q_{\min}\backslash\nu_{\min}$ is connected and acyclic.*

PROOF. Observe from Figure 4.7 that removing the perfect matching arrows will leave only the horizontal and vertical arrows remaining. In fact, what remains is precisely a lattice together with an additional 'initial' vertex. Travelling along these arrows we may only move 'left' or 'up', and hence the quiver is acyclic. We see that the quiver is path connected. □

DEFINITION 4.4.8. We define a partial order, $\leq_{\min}$, on the minimal cluster $\underline{\widetilde{\min}}$ as follows: $y \leq_{\min} y'$ if and only if there is an oriented path from y to y' in $Q_{\min}\backslash\nu_{\min}$.

COROLLARY 4.4.9. *Definition 4.4.8 above defines a partial order on the minimal cluster.*

PROOF. This follows from Lemma 4.4.7. Reflexivity is a consequence of the fact that each vertex is connected to itself via a path of length zero. Antisymmetry follows from acyclicity, since if we had a path from a vertex u to a vertex v and a path from v to u then we must have a cycle $u \to v \to u$. Finally transitivity follows from connectedness; if we have a path from u to v and a path from v to w, we may concatenate these paths to obtain a path from u to w. □

We expect that the additional data of a perfect matching can be utilised together with the mutation paths described in Section 4.2 to construct a partial order on the set of cluster variables. Figure 4.9 below shows the observed behaviour along the mutation path from Section 4.2 when we attempt to find a notion of 'mutation' of the initial perfect matching, obtaining a set of 'red' arrows on the mutated quiver. In the third case, we can choose either of the dashed arrows to be 'red'. This process appears to continue along the paths described above, producing each quiver now enriched with a subset of red arrows.

We have the following conjectures:

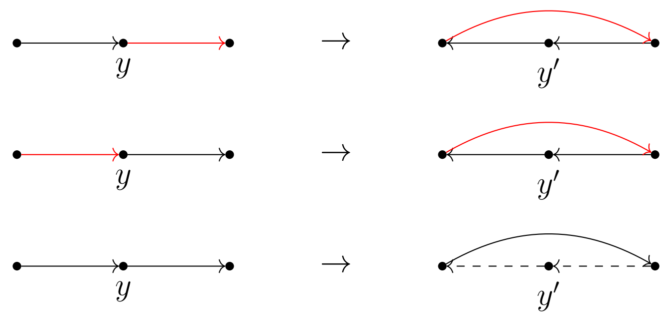


FIGURE 4.9. The observed behaviour of 'mutation' of a perfect matching. The perfect matching arrows are shown in red.

CONJECTURE 4.4.10. *There exists a mutation path for the Grassmannian cluster algebras along which mutation of a perfect matching, following the rules suggested by Figure 4.9, results in a perfect matching on the mutated quiver with respect to an appropriate choice of faces.*

The conjecture above holds in the finite type cases, and we will demonstrate this in Sections 4.4.0.1-4.4.0.4 below. Note that, in order to ensure that what we obtain after mutation might be a perfect matching, we do not mutate at vertices which are both the source and target of red arrows—an example illustrating why this condition is necessary can be found in Figure 4.10. We will call the vertices at which we can mutate *perfect matching source/sinks.* The main difficulty with proving this result in general is that it is not immediately obvious how the faces of a quiver with faces should change under mutation, particularly when the quiver we obtain is no longer planar.

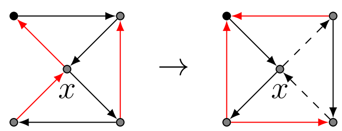


FIGURE 4.10. An example in which mutating at a vertex with both incoming and outgoing perfect matching arrows does not produce a perfect matching. By the rules suggested in Figure 4.9, one of the two dashed arrows must be red, but we can see that neither choice results in a perfect matching on the mutated quiver.

CONJECTURE 4.4.11. *In the setting of Conjecture 4.4.10, for each cluster along the path with quiver Q and perfect matching ν, $Q \backslash \nu$ is connected and acyclic.*

CONJECTURE 4.4.12. *In the setting of Conjecture 4.4.11, there is a partial order on each cluster along the path, with Hasse diagram given by $Q \backslash \nu$. Furthermore,* *there is a consistent extension of these partial orders to a partial order on the union of all clusters along the path:* $\leq_\nu$.

The extent of the challenge of proving the latter part of Conjecture 4.4.12 is clarified by the following observations. Firstly, we note that even if $x_1 \leq_\nu x_2$ is witnessed by a path which does not pass through the vertex at which mutation takes place, and which is hence unaltered by quiver mutation, one must either show that perfect matching mutation is sufficiently local that the path is not interrupted by a perfect matching arrow, or one must see that there is a different such path witnessing $x_1 \leq_{\nu'} x_2$ in the mutated quiver. Now, consider the case in which $x_1 \leq_\nu x_2$ is witnessed by a path $x_1 \to y \to x_2$, where y is the vertex at which we mutate. While it should be helpful that such a mutation yields an arrow $x_1 \to x_2$, one must check carefully that the choice of faces and perfect matching arrows is not such that $x_1 \to x_2$ is in the perfect matching.

Obtaining partial orders for each cluster is not trivial, though there are features for Grassmannian cluster structures which mean that this is plausible. Showing, however, that these partial orders patch together consistently is much more difficult, particularly in the presence of higher degree cluster variables.

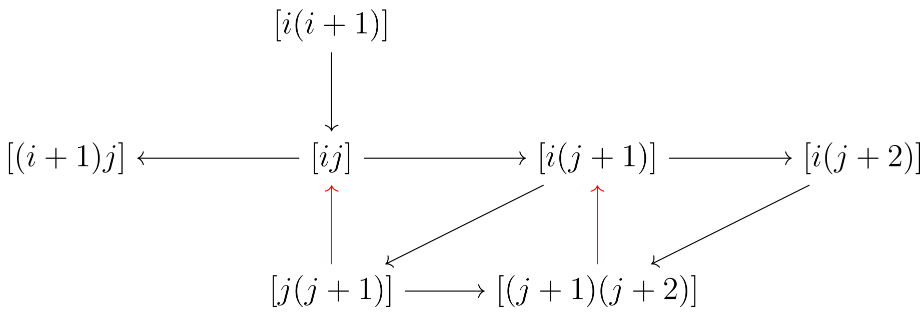
We will observe that such a partial order can be obtained in the finite type cases, and that it coincides with the standard partial order used in [**LR04**].

Definition 4.4.13 (Minimal mutable element). Let $(\underline{x}, Q)$ be a seed. Given a perfect matching ν on the exchange quiver Q such that $Q \backslash \nu$ is connected and acyclic, we say that an element $x \in \underline{x}$ is *minimal mutable* if it is:

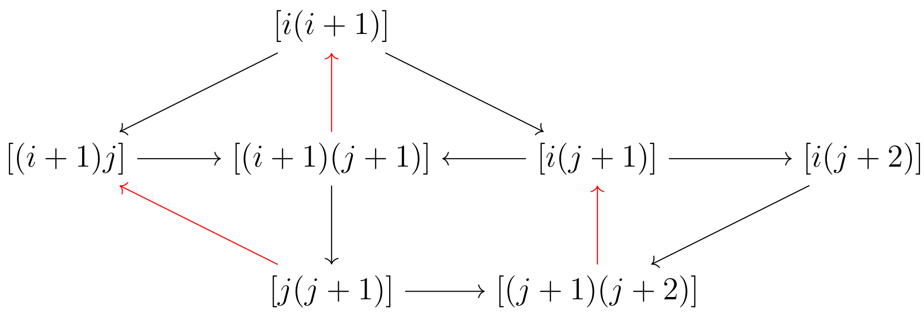
(i) Mutable (i.e. not frozen)
(ii) A minimal element with respect to $\leq_\nu$ restricted to mutable cluster variables.
(iii) A perfect matching sink (i.e. not the source of any perfect matching arrow) with at least one incoming perfect matching arrow.

We will call a vertex corresponding to a minimal mutable element a *minimal mutable vertex.*

4.4.0.1. *Perfect Matchings in the* $\mathbb{C}[Gr(2,n)]$ *Case.* Recall that when we mutate at the vertex $[ij]$ along path $[\underline{p}]$ in the $\mathbb{C}[\mathrm{Gr}(2,n)]$ case, the quiver appears locally as follows, now with 'red' arrows and boundary arrows included.

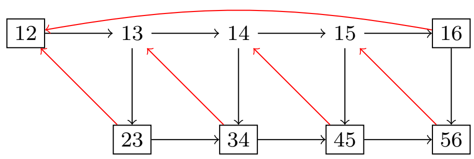


Notice that the vertex $[ij]$ is a minimal mutable element as in Definition 4.4.13. Note also that the red arrows appear to form part of perfect matching on the mutated quiver below, with a suitable choice of faces.



CONJECTURE 4.4.14. *For* $\mathbb{C}[Gr(2,n)]$*, applying the mutation path* $[\underline{p}]$ *to the minimal quiver enriched with the perfect matching described above results in the partial order shown in Figure 4.11.*

We may observe that Conjecture 4.4.14 holds when $n = 6$. The minimal quiver is as follows:



The mutation path $[\underline{p}]$ is shown in Figure 4.12, with red arrows determined by the rules from Figure 4.9.

We observe that, with the correct choice of faces, the red arrows do indeed form a perfect matching on each quiver along the path $[\underline{p}]$—we expect that this will be the case for any value of n.

Moreover, we observe that the partial order obtained as described in Conjecture 4.4.12 is precisely the standard partial order used by Lenagan and Rigal. The Hasse diagram is shown in Figure 4.13 below.

4.4.0.2. *Perfect Matchings in the* $\mathbb{C}[Gr(3,6)]$ *Case.* For the $\mathbb{C}[\mathrm{Gr}(3,6)]$ case, we may write down the mutation path $[\underline{p}]$ in its entirety with red arrows as determined

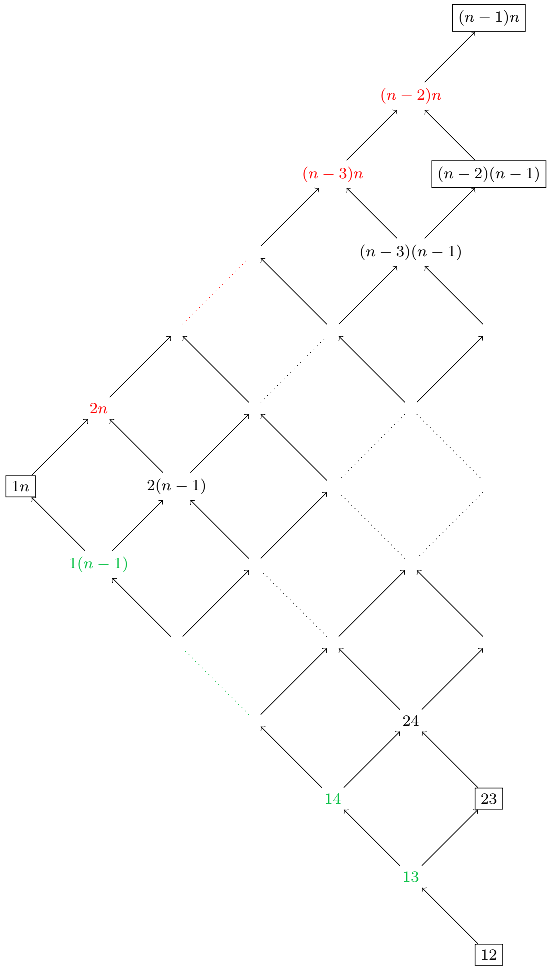


FIGURE 4.11. The poset for the partial order $<_\nu$ in the $\mathrm{Gr}(2, n)$ case. The minimal and maximal clusters are shown in green and red respectively.

by the rules in Figure 4.9. This is shown in Figure 4.14 below. We may observe that, with the correct choice of faces, the red arrows do form a perfect matching on each of the quivers occurring along this path.

We observe also that the resulting partial order, defined as outlined in Conjecture 4.4.12, coincides with the standard partial order in the sense that if $x <_{\mathrm{st}} y$ then we cannot have $x >_\nu y$. Note that we obtain one cluster variable which is not a Plücker coordinate—any cluster variables which are comparable in the ν partial order but not the standard one are connected through this higher degree variable in the Hasse diagram. The Hasse diagram is shown in Figure 4.15 below.

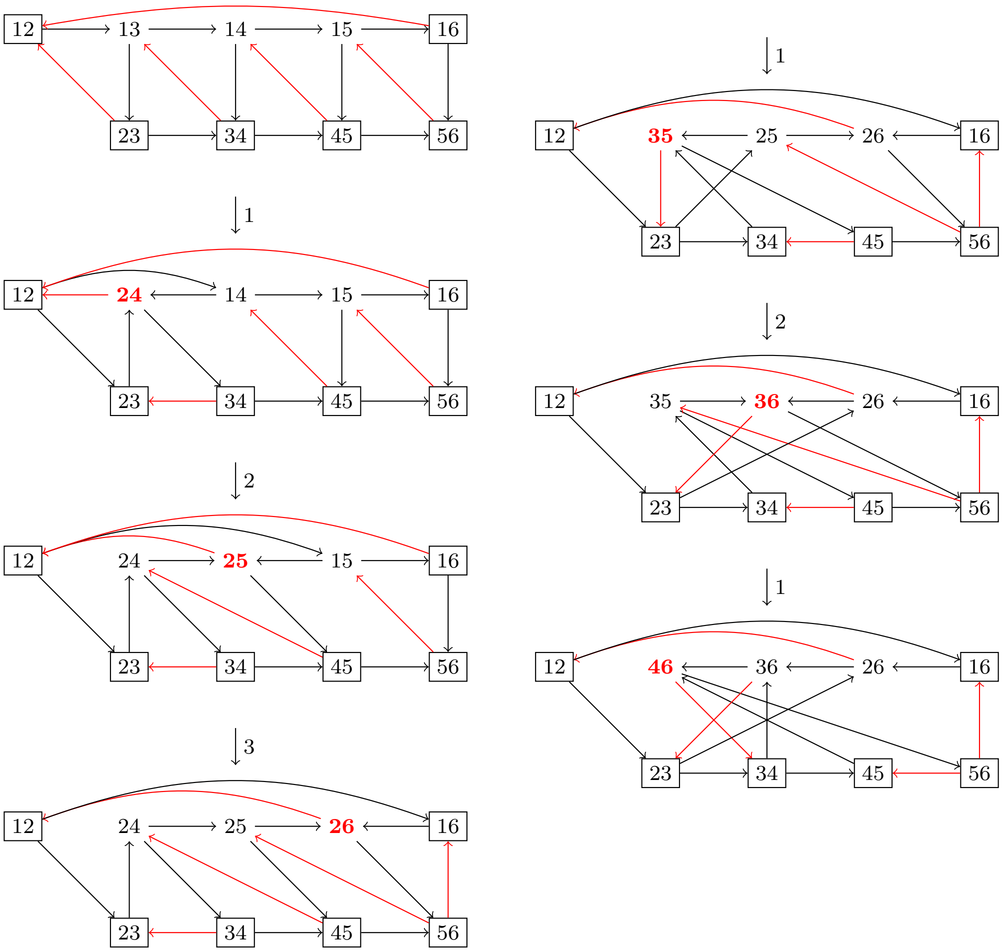


FIGURE 4.12. The Gr(2, 6) case. Mutable vertices are labelled 1-3 from left to right, and the labelled arrows state the vertex at which mutation takes place.

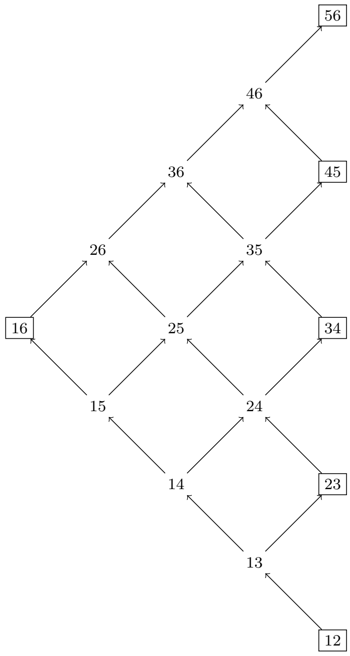


FIGURE 4.13. The poset for the partial order $<_\nu$ in the $\mathrm{Gr}(2,6)$ case.

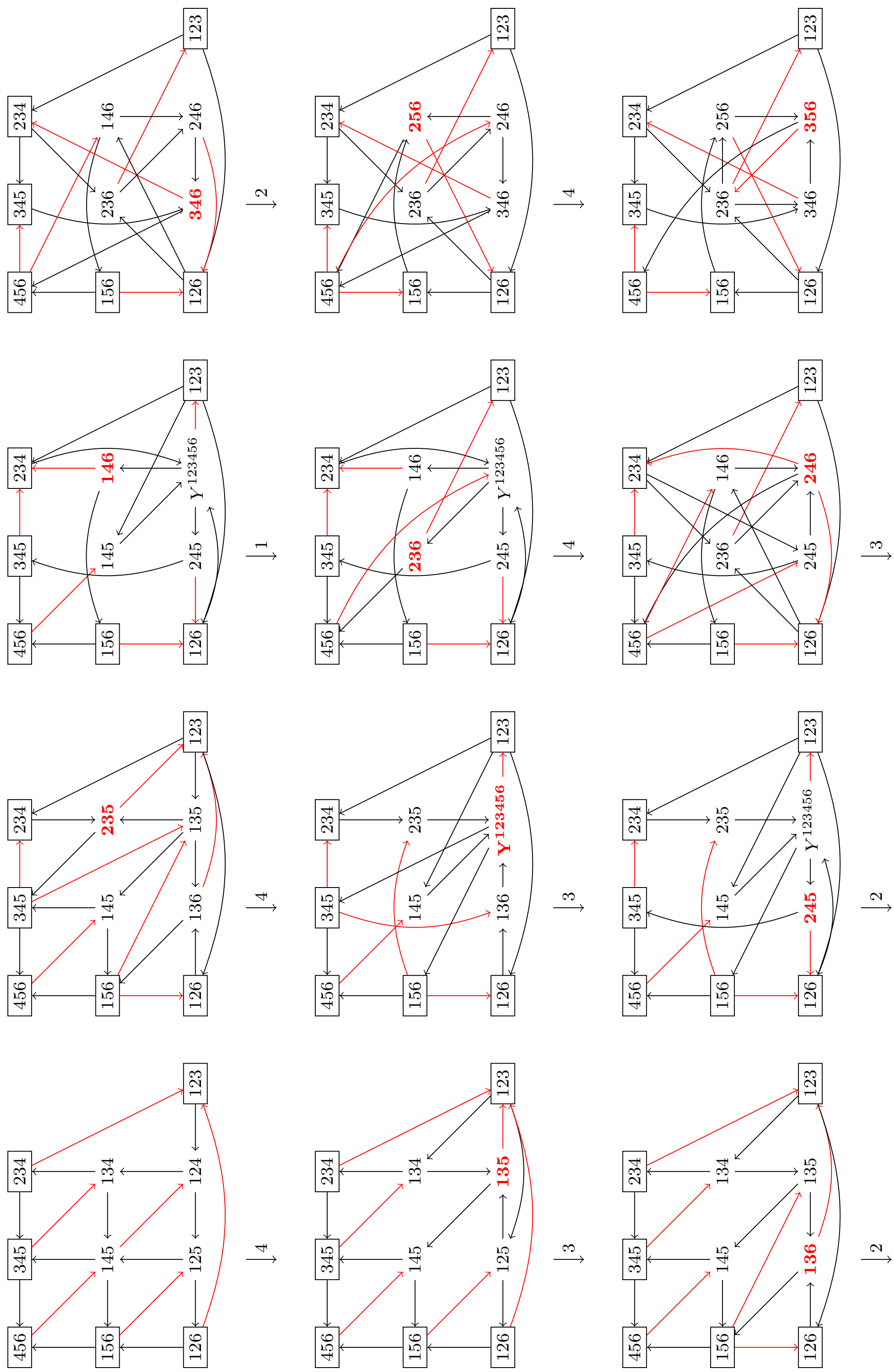


FIGURE 4.14. The Gr(3, 6) case, with perfect matchings shown by red arrows. Mutable vertices are labelled from left to right, and the labelled arrows state at which vertex we mutate.

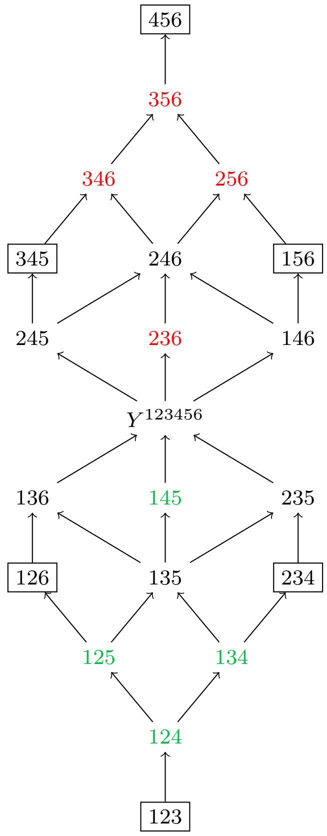


FIGURE 4.15. The poset for the partial order $<_\nu$ in the Gr(3, 6) case. The minimal and maximal clusters are shown in green and red respectively.

4.4.0.3. *Perfect Matchings in the* $\mathbb{C}[Gr(3,7)]$ *Case.* The quivers produced along the path $[\underline{p}]$ in the $\mathbb{C}[\mathrm{Gr}(3,7)]$ case have also been explicitly computed, although we do not include these here as the path is long. What we see, however, is that the set of red arrows produced still forms a perfect matching on the mutated quivers with an appropriate choice of faces. The resulting Hasse diagram is shown in Figure 4.16 below, and we again see that the partial order coincides with the standard partial order, the only difference being that some cluster variables are newly comparable in the ν partial order via a higher degree cluster variable.

4.4.0.4. *Perfect Matchings in the* $\mathbb{C}[Gr(3,8)]$ *Case.* The last finite type case to consider for the Grassmannian is the $\mathbb{C}[\mathrm{Gr}(3,8)]$ case. One can check that the red arrows produced along the path $[\underline{p}]$ again form a perfect matching on the quivers, given an appropriate choice of faces. One may also see that the resulting poset again coincides with the standard one, apart from some cluster variables being newly comparable via a higher degree cluster variable. We do not include the full list of quivers or the Hasse diagram here explicitly, since the path is long and the resulting

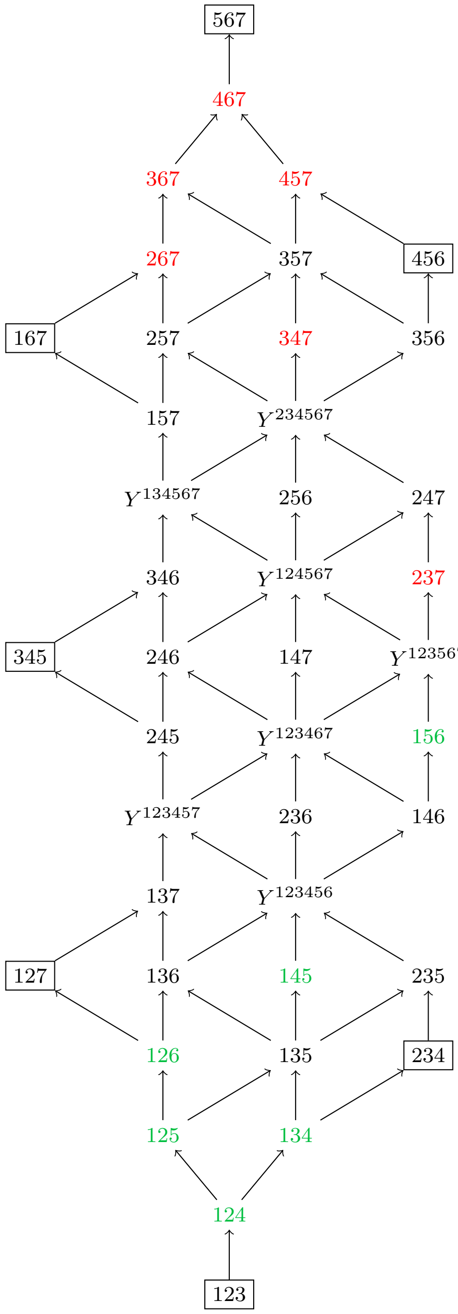


FIGURE 4.16. The poset for the partial order $<_\nu$ in the Gr$(3,7)$ case. The minimal and maximal clusters are shown in green and red respectively.

quivers are fairly complex. The interested reader may reconstruct these by using the path $[\underline{p}]$ stated in Section 4.2 along with the mutation rules for red arrows given in Figure 4.9.

4.4.0.5. *Perfect Matchings in the Matrix Case.* Let us now briefly turn our attention back to the Matrix case. Rather than computing the Hasse diagrams from scratch in these cases, we may again make use of the algebra isomorphism α given in [**LR08**] to obtain these from the corresponding Grassmannian cases. The resulting

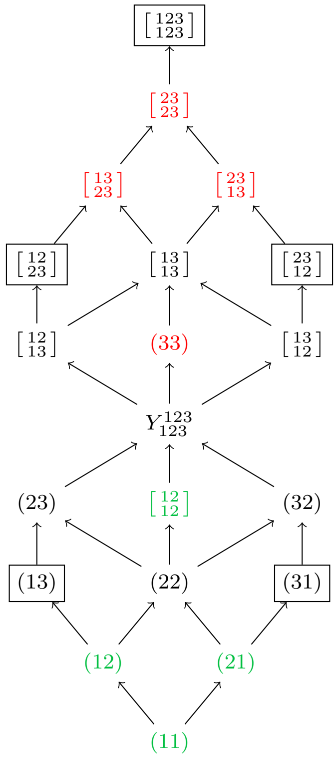


FIGURE 4.17. The poset for the partial order $<_\nu$ in the $M(3,3)$ case. The minimal and maximal clusters are shown in green and red respectively.

Hasse diagrams for the $M(3,3)$ and $M(3,4)$ cases are shown in Figures 4.17 and 4.18 below.

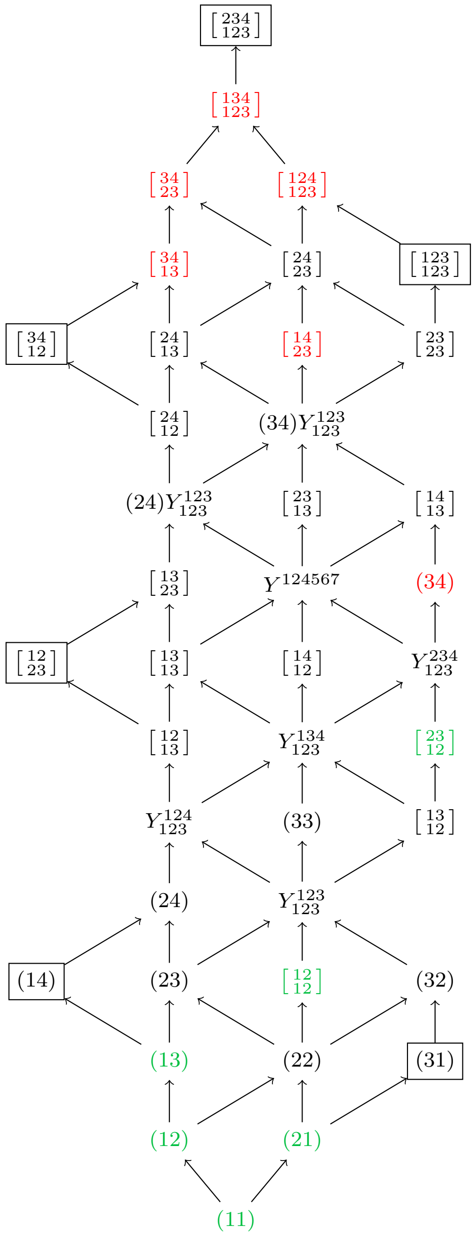


FIGURE 4.18. The poset for the partial order $<_\nu$ in the $M(3,4)$ case. The minimal and maximal clusters are shown in green and red respectively.

CHAPTER 5

Segre Products of Graded Cluster Algebras

The following Chapter is joint work with Jan E. Grabowski - see [**GH24**].

5.1. The Segre Product

The map $\sigma : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{n+m+nm}$ of projective spaces defined by

$$\sigma((x_0 \colon \dots \colon x_n),(y_0 \colon \dots \colon y_m)) = (x_0y_0 \colon x_0y_1 \colon \dots \colon x_iy_j \colon \dots \colon x_ny_m)$$

is known as the *Segre embedding*—it is injective and its image is a subvariety of $\mathbb{P}^{n+m+nm}$. We may then define the Segre product of two projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ as the image of $X \times Y$ with respect to the Segre embedding. We denote the Segre product by $X \bar{\otimes} Y \stackrel{\text{def}}{=} \sigma(X \times Y)$.

In what follows, rather than the geometric setting described above, we will be interested in the dual notion of the Segre product of graded algebras. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ and $B = \bigoplus_{i \in \mathbb{N}} B_i$ be $\mathbb{N}$-graded $\mathbb{K}$-algebras. Then their Segre product $A \bar{\otimes} B$ is the $\mathbb{N}$-graded algebra

$$A \bar{\otimes} B \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{N}} A_i \otimes_{\mathbb{K}} B_i \tag{8}$$

with the usual tensor product algebra multiplication. Letting X and Y be projective varieties with homogeneous coordinate rings A and B respectively, the Segre product $A \bar{\otimes} B$ is the homogeneous coordinate ring of $X \bar{\otimes} Y$.

In all known examples when a cluster algebra is the coordinate algebra of a projective variety, we have a compatible grading on the cluster algebra, with all cluster variables being homogeneous. Such cluster algebras are naturally called *graded cluster algebras* and the general theory of these is set out in work of the first author ([**Gra15**]).

In this chapter, inspired by [**Pre23**, Remark 4.14], we define a cluster algebra structure on the Segre product of graded cluster algebras. This generalises the particular case arising in [**Pre23**] in the study of cluster algebra structures on positroid varieties and in doing so, we are able to clarify the required input data to be able to form a Segre product.

We show that from the point of view of cluster algebras, forming the Segre product is given by a gluing operation on suitable frozen variables. We also record some simple observations on the preservation or otherwise of cluster-algebraic properties under taking Segre products.

5.2. Segre Products of Graded Cluster Algebras

It was shown by Galashin and Lam in [**GL19**] that coordinate rings of positroid varieties in the Grassmannian have cluster algebra structures. This class is closed under Segre product and in [**Pre23**], Pressland shows how the Galashin–Lam cluster structure on the product is related to that on the factors.

In what follows, we aim to generalise this construction to the case of graded skew-symmetric cluster algebras: we start with two graded cluster algebras and show that their Segre product has a natural cluster structure. For coordinate rings of positroid varieties, Pressland's result shows that the Galashin–Lam cluster structure on the product is equal to that obtained by the Segre product construction we give here.

We start by establishing some notation; readers unfamiliar with graded cluster algebras may wish to refer to [**Gra15**] for further details and examples.

First, let $\mathscr{A}_i = (\underline{\tilde{x}}_i, \underline{x}_i, B_i, G_i)$ be (skew-symmetric) graded cluster algebras, for $i \in \{1, 2\}$, such that

- $\underline{\tilde{x}}_1 = \{x_1, \dots, x_{n_1}\}$ and $\underline{\tilde{x}}_2 = \{y_1, \dots, y_{n_2}\}$ are the respective initial clusters;
- $\underline{x}_i \subsetneq \underline{\tilde{x}}_i$ is the set of mutable cluster variables;
- every frozen variable (i.e. those elements in $\underline{\tilde{x}}_i \setminus \underline{x}_i$) is invertible;
- B_i is an exchange matrix (with rows indexed by $\underline{\tilde{x}}_i$ and columns by $\underline{x}_i$) with skew-symmetric principal part;
- $G_i \in \mathbb{Z}^{n_i}$ is a grading vector, i.e. a vector such that $B_i^T G_i = 0$.

Throughout, we will work over a field $\mathbb{K}$, so that our cluster algebras are $\mathbb{K}$-algebras and we take all tensor products to be over $\mathbb{K}$. As we will see, the underlying field plays essentially no role in our construction.

Let $\underline{\tilde{x}}$ be a cluster with x a cluster variable and B the exchange matrix associated to $\underline{\tilde{x}}$. We denote by B^x the row of B indexed by x and by $\widehat{B}^x$ the matrix obtained from B by removing the row B^x. If x is frozen, $\widehat{B}^x$ is again an exchange matrix.

REMARK 5.2.1. *In the above we require at least one frozen cluster variable in each cluster algebra—this will be important when defining a cluster structure on their Segre product since this will involve 'gluing' at frozen variables.*

We have also asked that every frozen variable is invertible, which is a common but not universal assumption in cluster theory. In fact, an examination of our construction shows that this assumption can be weakened to only asking that the glued frozen variables are invertible, which may be a more appropriate assumption for geometric applications.

We wish to define a cluster algebra structure on the Segre product $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$. Following the approach of [**Pre23**], we aim to construct a new cluster algebra from $\mathscr{A}_1$ and $\mathscr{A}_2$ by gluing at frozen variables of the same degree, which we will show coincides with the Segre product under suitable further conditions.

5.2.1. A Gluing Construction. Fix $x \in \underline{\tilde{x}}_1 \setminus \underline{x}_1$ and $y \in \underline{\tilde{x}}_2 \setminus \underline{x}_2$ such that $(G_1)_x = (G_2)_y$. That is, x and y are frozen variables in their respective clusters and their degrees are equal. We will identify the frozen variables x and y, denoting a new proxy variable replacing both of these by z.

The initial data for our new cluster algebra is as follows. For the initial cluster, we take

$$\underline{\tilde{x}}_1 \square \underline{\tilde{x}}_2 \stackrel{\text{def}}{=} (\underline{\tilde{x}}_1 \setminus \{x\}) \cup (\underline{\tilde{x}}_2 \setminus \{y\}) \cup \{z\}.$$

The mutable variables are $\underline{x}_1 \cup \underline{x}_2$, and for the initial exchange matrix, we form the block matrix

$$B_1 \square B_2 \stackrel{\text{def}}{=} \begin{bmatrix} \widehat{B}_1^x & 0 \\ 0 & \widehat{B}_2^y \\ B_1^x & B_2^y \end{bmatrix}.$$

Finally, for the initial grading vector we take

$$G_1 \square G_2 \stackrel{\text{def}}{=} \begin{bmatrix} \widehat{G}_1^x \\ \widehat{G}_2^y \\ G_1^z \end{bmatrix}$$

where $\widehat{G}_1^x$ is the grading vector G_1 with the entry indexed by x removed (and similarly for $\widehat{G}_2^y$) and $G_1^z \stackrel{\text{def}}{=} (G_1)_x = (G_2)_y$. We can now define a cluster algebra

$$\mathscr{A}_1 \square \mathscr{A}_2 = \mathscr{A}(\underline{\tilde{x}}_1 \square \underline{\tilde{x}}_2, \underline{x}_1 \cup \underline{x}_2, B_1 \square B_2, G_1 \square G_2)$$

from this initial data.

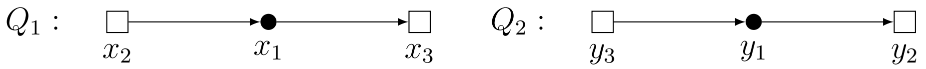
Let us extend the above notation to write

$$\underline{\tilde{x}}'_1 \Box \underline{\tilde{x}}'_2 = (\underline{\tilde{x}}'_1 \setminus \{x\}) \cup (\underline{\tilde{x}}'_2 \setminus \{y\}) \cup \{z\},$$

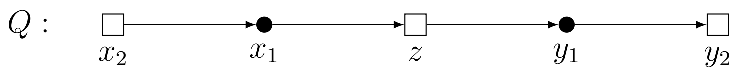
where $\underline{\tilde{x}}'_1$, $\underline{\tilde{x}}'_2$ are now allowed to be any clusters of $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively, and say that $\underline{\tilde{x}}'_1 \Box \underline{\tilde{x}}'_2$ is obtained by gluing x and y. This is well-defined since x and y are frozen. Similarly, we extend the notation $B_1 \Box B_2$ and $G_1 \Box G_2$ to any appropriate input matrices/vectors.

The process of gluing at frozen variables with matching degree is illustrated in the example below. Here and elsewhere, $\mathbb{1}$ denotes the vector $(1, \ldots, 1)^T$.

Example 5.2.2. *Let* $\mathscr{A}_1 = (\underline{\tilde{x}}_1 = \{x_1, x_2, x_3\}, \underline{x}_1 = \{x_1\}, Q_1, G_1 = \mathbb{1})$ *and* $\mathscr{A}_2 = (\underline{\tilde{x}}_2 = \{y_1, y_2, y_3\}, \underline{x}_2 = \{y_1\}, Q_2, G_1 = \mathbb{1})$ *be cluster algebras with exchange quivers as follows:*



The quiver obtained by 'gluing' at the frozen variables x_3 *and* y_3 *is shown below—we denote the new variable by* z.



The cluster algebra $\mathscr{A}_1 \Box \mathscr{A}_2$ *is then given by the initial data*

$$(\underline{\tilde{x}} = \{x_1, x_2, y_1, y_2, z\}, \underline{x} = \{x_1, y_1\}, Q, G = \mathbb{1}).$$

We will show in Theorem 5.2.7 that this gives a cluster structure on the Segre product $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$.

We record some straightforward observations about the cluster algebra $\mathscr{A}_1 \Box \mathscr{A}_2$.

Lemma 5.2.3. *Let* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *be graded cluster algebras. Fix* $x \in \underline{\tilde{x}}_1 \setminus \underline{x}_1$ *and* $y \in \underline{\tilde{x}}_2 \setminus \underline{x}_2$ *such that* $(G_1)_x = (G_2)_y$. *Then the cluster algebras* $\mathscr{A}_1 \Box \mathscr{A}_2$ *and* $\mathscr{A}_2 \Box \mathscr{A}_1$ *are isomorphic as cluster algebras.*

Proof. This is clear from comparing the initial data for $\mathscr{A}_1 \Box \mathscr{A}_2$ and $\mathscr{A}_2 \Box \mathscr{A}_1$ and in particular noting that the two initial clusters are equal up to permutation of the entries. $\Box$

LEMMA 5.2.4. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be graded cluster algebras. Fix $x \in \underline{\tilde{x}}_1 \setminus \underline{x}_1$ and $y \in \underline{\tilde{x}}_2 \setminus \underline{x}_2$ such that $(G_1)_x = (G_2)_y$.*

(i) *Every cluster variable of $\mathscr{A}_1 \square \mathscr{A}_2$ is naturally identified with a cluster variable of $\mathscr{A}_1$, a cluster variable of $\mathscr{A}_2$ or is equal to z.*

(ii) *There is a bijection between pairs of clusters $(\underline{\tilde{x}}_1', \underline{\tilde{x}}_2')$ and clusters of $\mathscr{A}_1 \square \mathscr{A}_2$ given by gluing, i.e. sending $(\underline{\tilde{x}}_1', \underline{\tilde{x}}_2')$ to $\underline{\tilde{x}}_1' \square \underline{\tilde{x}}_2'$ for a cluster $\underline{\tilde{x}}_1'$ of $\mathscr{A}_1$ and $\underline{\tilde{x}}_2'$ of $\mathscr{A}_2$.*

PROOF. This follows from observing that our gluing process does not introduce any new arrows between mutable vertices. Since mutation is a local phenomenon and concentrated on mutable vertices, it is straightforward to see that mutating at vertices indexed by $\underline{x}_1$ is independent of mutating at vertices indexed by $\underline{x}_2$ and the (mutable) variables obtained are exactly as if the gluing had not been carried out. The frozen variables of $\mathscr{A}_1 \square \mathscr{A}_2$ are those of $\mathscr{A}_1$ and $\mathscr{A}_2$ excluding x and y, along with the glued frozen z.

For the second part, note that the same argument shows that there is a similar bijection for the clusters of $\mathscr{A}_1 \times \mathscr{A}_2$, where the latter denotes the "disconnected" product of cluster algebras, where one simply takes the union of clusters and direct sum of exchange matrices. Now there is evidently a bijection between the clusters of $\mathscr{A}_1 \times \mathscr{A}_2$ and those of $\mathscr{A}_1 \square \mathscr{A}_2$, given by $\underline{\tilde{x}}_1' \cup \underline{\tilde{x}}_2' \mapsto \underline{\tilde{x}}_1' \square \underline{\tilde{x}}_2'$, from which the claim follows. □

COROLLARY 5.2.5. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be graded cluster algebras. Fix $x \in \underline{\tilde{x}}_1 \setminus \underline{x}_1$ and $y \in \underline{\tilde{x}}_2 \setminus \underline{x}_2$ such that $(G_1)_x = (G_2)_y$.*

Then

(i) *$\mathscr{A}_1 \square \mathscr{A}_2$ is of finite type if and only if $\mathscr{A}_1$ and $\mathscr{A}_2$ are;*

(ii) *writing $\kappa(\mathscr{A})$ for the number of cluster variables of a cluster algebra $\mathscr{A}$, we have $\kappa(\mathscr{A}_1 \square \mathscr{A}_2) = \kappa(\mathscr{A}_1) + \kappa(\mathscr{A}_2) - 1$ when these numbers are all finite; and*

(iii) *writing $K(\mathscr{A})$ for the number of clusters of $\mathscr{A}$, we have $K(\mathscr{A}_1 \square \mathscr{A}_2) = K(\mathscr{A}_1)K(\mathscr{A}_2)$ when these numbers are all finite.*

PROOF. These are now immediate from the previous lemma. Note that there is an overall reduction of one in the number of cluster variables because we have glued two previously distinct frozen variables; this highlights the difference between this construction and the disconnected product. □

REMARK 5.2.6. *One might hope that this construction extends straightforwardly to graded quantum cluster algebras (cf.* [**GL13**]*). However, computation in small examples shows that this is not the case.*

For if one tries the naïve approach in which initial quantum cluster variables from $\mathscr{A}_1$ commute *with those from* $\mathscr{A}_2$*, one rapidly finds situations in which after performing a mutation, the new variable does not quasi-commute with the rest of its cluster. For it to do so requires the compatibility condition between the exchange and quasi-commutation matrices for the glued data and this imposes a collection of "cross-term" requirements between* B_1 *and* L_2 *(respectively,* B_2 *and* L_1*) in respect of the glued frozen variables.*

5.2.2. Relationship with the Segre product. Our main result is the following theorem, showing that the cluster algebra construction $\mathscr{A}_1 \square \mathscr{A}_2$ induces a cluster algebra structure on the Segre product. The isomorphism we will use is directly analogous to the map δ^{src} defined in [**Pre23**].

THEOREM 5.2.7. *Let* $\mathscr{A}_i = (\underline{\tilde{x}}_i, \underline{x}_i, B_i, G_i)$*,* $i = 1, 2$ *be graded cluster algebras such that there exist* $x \in \underline{\tilde{x}}_1 \setminus \underline{x}_1$ *and* $y \in \underline{\tilde{x}}_2 \setminus \underline{x}_2$ *both of degree 1.*

Then the map $\varphi : \mathscr{A}_1 \square \mathscr{A}_2 \to \mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$ *given on initial cluster variables by*

$$
\begin{aligned}
\varphi(x_j) &= x_j \otimes y^{\deg x_j} \quad \text{for } x_j \in \underline{\tilde{x}}_1 \setminus \{x\}, \\
\varphi(y_j) &= x^{\deg y_j} \otimes y_j \quad \text{for } y_j \in \underline{\tilde{x}}_2 \setminus \{y\} \text{ and} \\
\varphi(z) &= x \otimes y
\end{aligned}
$$

is a graded algebra isomorphism, with the property that the above formulæ hold for any cluster of $\mathscr{A}_1 \square \mathscr{A}_2$*.*

PROOF. Recalling that we set

$$\underline{\tilde{x}}_1 \square \underline{\tilde{x}}_2 = (\underline{\tilde{x}}_1 \setminus \{x\}) \cup (\underline{\tilde{x}}_2 \setminus \{y\}) \cup \{z\},$$

let φ denote the algebra homomorphism $\varphi \colon \mathbb{K}(\underline{\tilde{x}}_1 \square \underline{\tilde{x}}_2) \to \mathbb{K}(\underline{\tilde{x}}_1) \otimes \mathbb{K}(\underline{\tilde{x}}_2)$ obtained from the above specification on generators of the domain. This map is injective since the elements $\varphi(x_j)$, $\varphi(y_j)$ and $\varphi(z)$ are algebraically independent.

Now let φ denote the restriction of the above map to $\mathscr{A}_1 \square \mathscr{A}_2$. We first claim that the restricted map φ takes values in the subalgebra $\mathscr{A}_1 \otimes \mathscr{A}_2$. To prove this, we proceed by induction on the number of mutation steps from the initial cluster.

We may take as base case that of zero mutations from the initial cluster: there is nothing to do, since we see immediately that $\varphi(x_j)$, $\varphi(y_j)$ and $\varphi(z)$ lie in $\mathscr{A}_1 \otimes \mathscr{A}_2$ by definition.

Now assume that the result holds $r-1$ mutations from the initial cluster $\underline{\tilde{x}} = \underline{\tilde{x}}_1 \Box \underline{\tilde{x}}_2$ (for $r \geq 1$) of $\mathscr{A}_1 \Box \mathscr{A}_2$. That is, let $\underline{y} = \mu_{k_{r-1}} \mu_{k_{r-2}} \cdots \mu_{k_1}(\underline{\tilde{x}})$. Set $B = \mu_{k_{r-1}} \mu_{k_{r-2}} \cdots \mu_{k_1}(B_1 \Box B_2)$.

By Lemma 5.2.4(ii), we have that $\underline{y} = \underline{y}_1 \Box \underline{y}_2$ for some clusters $\underline{y}_1$, $\underline{y}_2$ of $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Moreover, there is a decompostion

$$\{k_1, \ldots, k_{r-1}\} = \{l_1, \ldots, l_s\} \sqcup \{m_1, \ldots, m_t\}$$

such that $\underline{y}_1 = \mu_{l_s} \cdots \mu_{l_1}(\underline{\tilde{x}}_1)$ and $\underline{y}_2 = \mu_{m_t} \cdots \mu_{m_1}(\underline{\tilde{x}}_2)$.

Let $\underline{y}_1 = \{x_1, \ldots, x_{n_1}\}$ and $\underline{y}_2 = \{y_1, \ldots, y_{n_2}\}$, so that

$$\underline{y} = \underline{y}_1 \Box \underline{y}_2 = (\{x_1, \ldots, x_{n_1}\} \setminus \{x\}) \sqcup (\{y_1, \ldots, y_{n_2}\} \setminus \{y\}) \sqcup \{z\}$$

Let $C = \mu_{l_s} \cdots \mu_{l_1}(B_1)$, $D = \mu_{m_t} \cdots \mu_{m_1}(B_2)$, $H = \mu_{l_s} \cdots \mu_{l_1}(G_1)$ and $K = \mu_{m_t} \cdots \mu_{m_1}(G_2)$. Then in particular $B = C \Box D$ and $H_j = \deg x_j$ and $K_j = \deg y_j$. We also set $[n]_+ = \max\{n, 0\}$ and $[n]_- = \max\{-n, 0\}$.

We then compute φ for one further mutation in direction $k_r = k$. We first consider the case in which $x_k \in \underline{y}_1$ is mutable.

We have

$$\begin{aligned}
\varphi(\mu_k(x_k)) &= \varphi\Bigg(\frac{1}{x_k}\Bigg[\Bigg(\prod_{x_j\in \underline{y}_1\setminus\{x\}} x_j^{[B_{x_j,x_k}]_+}\Bigg)\Bigg(\prod_{y_j\in \underline{y}_2\setminus\{y\}} y_j^{[B_{y_j,x_k}]_+}\Bigg) z^{[B_{z,x_k}]_+} \\
&\quad + \Bigg(\prod_{x_j\in \underline{y}_1\setminus\{x\}} x_j^{[B_{x_j,x_k}]_-}\Bigg)\Bigg(\prod_{y_j\in \underline{y}_2\setminus\{y\}} y_j^{[B_{y_j,x_k}]_-}\Bigg) z^{[B_{z,x_k}]_-}\Bigg]\Bigg) \\
&= \varphi\Bigg(\frac{1}{x_k}\Bigg[\Bigg(\prod_{x_j\in \underline{y}_1\setminus\{x\}} x_j^{[B_{x_j,x_k}]_+}\Bigg) z^{[B_{z,x_k}]_+} + \Bigg(\prod_{x_j\in \underline{y}_1\setminus\{x\}} x_j^{[B_{x_j,x_k}]_-}\Bigg) z^{[B_{z,x_k}]_-}\Bigg]\Bigg) \\
&= \varphi\Bigg(\frac{1}{x_k}\Bigg[\Bigg(\prod_{x_j\in \underline{y}_1\setminus\{x\}} x_j^{[C_{x_j,x_k}]_+}\Bigg) z^{[C_{x,x_k}]_+} + \Bigg(\prod_{x_j\in \underline{y}_1\setminus\{x\}} x_j^{[C_{x_j,x_k}]_-}\Bigg) z^{[C_{x,x_k}]_-}\Bigg]\Bigg) \\
&= \frac{1}{x_k\otimes y^{\deg x_k}}\Bigg[\prod_{x_j\in \underline{y}_1\setminus\{x\}}\Bigg(x_j^{[C_{x_j,x_k}]_+}\otimes y^{[C_{x_j,x_k}]_+\deg x_j}\Bigg) x^{[C_{x,x_k}]_+}\otimes y^{[C_{x,x_k}]_+} \\
&\quad + \prod_{x_j\in \underline{y}_1\setminus\{x\}}\Bigg(x_j^{[C_{x_j,x_k}]_-}\otimes y^{[C_{x_j,x_k}]_-\deg x_j}\Bigg) x^{[C_{x,x_k}]_-}\otimes y^{[C_{x,x_k}]_-}\Bigg] \\
&= \frac{1}{x_k\otimes y^{\deg x_k}}\Bigg[\prod_{x_j\in \underline{y}_1} x_j^{[C_{x_j,x_k}]_+}\otimes y^d + \prod_{x_j\in \underline{y}_1} x_j^{[C_{x_j,x_k}]_-}\otimes y^d\Bigg] \\
&= \frac{1}{x_k}\Bigg(\prod_{x_j\in \underline{y}_1} x_j^{[C_{x_j,x_k}]_+} + \prod_{x_j\in \underline{y}_1} x_j^{[C_{x_j,x_k}]_-}\Bigg)\otimes y^{d-\deg x_k} \\
&= \mu_k(x_k)\otimes y^{d-\deg x_k} \\
&= \mu_k(x_k)\otimes y^{\deg \mu_k(x_k)}
\end{aligned}$$

where

$$\begin{aligned}
d = \sum_{x_j}[C_{x_j,x_k}]_+\deg x_j &= \sum_{C_{x_j,x_k}>0} C_{x_j,x_k}H_{x_j} \\
= \sum_{C_{x_j,x_k}<0} -C_{x_j,x_k}H_{x_j} &= \sum_{x_j}[C_{x_j,x_k}]_-\deg x_j
\end{aligned}$$

noting that the third equality holds since $C^T H = 0$. Also, we use that $\deg \mu_k(x_k) = d - \deg x_k$.

Note that the fifth equality is where the assumption that $\deg x = 1$ is used: without it, the claimed equality of d with the other stated quantities need not hold.

An analogous argument shows that $\varphi(\mu_k(y_k)) = x^{\deg \mu_k(y_k)} \otimes \mu_k(y_k)$ for $y_k \in \underline{y}_2$ mutable, noting that this time, it is $\deg y = 1$ that is required.

Since we have $\deg x = \deg y = 1$, the above tells us that for any cluster variable x' of $\mathscr{A}_1 \square \mathscr{A}_2$, we either have $\varphi(x') = x' \otimes y^{\deg x'}$ or $\varphi(x') = x^{\deg x'} \otimes x'$ and hence $\varphi(x') \in (\mathscr{A}_1)_{\deg x'} \otimes (\mathscr{A}_2)_{\deg x'}$. That is, the image of φ is contained in the Segre product $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$ without any further constraints and the map φ is a graded map.

It remains to check surjectivity. Note that a generating set for $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$ is given by taking the elementary tensors with components in generating sets for $\mathscr{A}_1$ and $\mathscr{A}_2$, i.e.

$$\{z_1 \otimes z_2 | z_1 \in (\mathscr{A}_1)_d, z_2 \in (\mathscr{A}_2)_d \text{ cluster variables}, d \in \mathbb{Z}\}$$

Now

$$z_1 \otimes z_2 = (z_1 \otimes y^d)(x^d \otimes z_2)(x^{-d} \otimes y^{-d}) = \varphi(z_1)\varphi(z_2)\varphi(z)^{-d}.$$

Hence, $\operatorname{Im} \varphi$ contains a generating set for $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$, and so φ is surjective onto $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$. The claim follows. □

REMARK 5.2.8. *One might be tempted to try changing the specification of the map ϕ to*

$$\begin{aligned} \varphi(x_j) &= x_j^{\deg y} \otimes y^{\deg x_j} \quad \text{for } x_j \in \underline{\tilde{x}}_1 \setminus \{x\}, \\ \varphi(y_j) &= x^{\deg y_j} \otimes y_j^{\deg x} \quad \text{for } y_j \in \underline{\tilde{x}}_2 \setminus \{y\} \text{ and} \\ \varphi(z) &= x^{\deg y} \otimes y^{\deg x} \end{aligned}$$

in an attempt to avoid the $\deg x = \deg y = 1$ *assumption. Note that one should however ask for* $\deg x$ *and* $\deg y$ *strictly positive, to avoid issues with needing inverses of arbitrary cluster variables.*

While this does indeed fix the issue with d that occurs in the calculation in the above proof for $x_k \in \underline{y}_1$, the appearance of $x^{\deg y}$ in the first tensor factor means that we do not obtain $\mu_k(x_k)$ unless $\deg y = 1$.

More explicitly, following the same approach as in the previous proof, one would arrive at

$$\frac{1}{x_k^{\deg y} \otimes y^{\deg x_k}} \left[\prod_{x_j \in \underline{y}_1} x_j^{[C_{x_j,x_k}]_+ \deg y} \otimes y^d + \prod_{x_j \in \underline{y}_1} x_j^{[C_{x_j,x_k}]_- \deg y} \otimes y^d \right]$$

but this is not equal to

$$\frac{1}{x_k^{\deg y}}\left(\prod_{x_j\in \underline{y}_1} x_j^{[C_{x_j,x_k}]_+} + \prod_{x_j\in \underline{y}_1} x_j^{[C_{x_j,x_k}]_-}\right)^{\deg y} \otimes y^{d-\deg x_k}$$

if $\deg y \neq 1$.

By symmetry, the other case tells us that we also need $\deg x = 1$. *That is, the degree 1 assumption is unavoidable.*

REMARK 5.2.9. *Notice that in proving surjectivity, we required* $\varphi(z) = x \otimes y$, *and hence* x *and* y *themselves, to be invertible, but no other frozen variables needed to be invertible for the proof to hold.*

REMARK 5.2.10. *Via Lemma 5.2.4, we see that the cluster structure on* $\mathscr{A}_1 \square \mathscr{A}_2$ *and hence that on* $\mathscr{A}_1 \overline{\otimes} \mathscr{A}_2$ *is independent of the choices of initial seeds. Therefore the only requirements to obtain a cluster structure on the Segre product are the existence of a frozen variable of degree 1 for each factor.*

Graded cluster algebras with at least one frozen variable of degree one are, perhaps surprisingly, ubiquitous. Many examples arising geometrically have this property: coordinate rings of Grassmannians and more generally partial flag varieties and their cells ([**GLS11b**]*), double Bruhat cells (*[**BFZ03**]*) and, as motivated this work, positroid varieties (*[**GL19**]*).*

Note too that the claims on the cluster structure of $\mathscr{A}_1 \square \mathscr{A}_2$ in Corollary 5.2.5 therefore also apply to the induced cluster structure on the Segre product.

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APPENDIX A

Growth in Graded Cluster Algebras of Rank 3: Examples

In this Appendix we collect, and comment on, some of the raw data behind Chapter 3.

A.1. Fastest Growing Paths

We begin by illustrating the behaviour of the fastest growing paths as in Definition 3.2.1. By Proposition 3.2.9, we expect that taking logs yields a Fibonacci type sequence, with the ratio of terms tending to the golden ratio. This behaviour can be seen in Tables 1-4 below.

Mutation Radius	ln (Highest Degree)	Ratio
1	0.477	-
2	0.477	1
3	0.699	1.465
4	1.114	1.594
5	1.792	1.609
6	2.907	1.622
7	4.696	1.615
8	7.600	1.618
9	12.295	1.618
10	19.895	1.618

Table 1. The fastest growing path for the initial degree seed $(1,3,1)$

Mutation Radius	ln (Highest Degree)	Ratio
1	0.845	-
2	1.256	1.485
3	2.090	1.665
4	3.344	1.600
5	5.434	1.625
6	8.777	1.615
7	14.211	1.619
8	22.989	1.618
9	37.200	1.618
10	60.188	1.618

Table 2. The fastest growing path for the initial degree seed $(3,3,7)$

Mutation Radius	ln (Highest Degree)	Ratio
1	0.602	-
2	1.114	1.850
3	1.681	1.509
4	2.792	1.661
5	4.473	1.602
6	7.266	1.624
7	11.739	1.616
8	19.005	1.619
9	30.744	1.618
10	49.750	1.618

Table 3. The fastest growing path for the initial degree seed $(4, 3, 4)$

Mutation Radius	ln (Highest Degree)	Ratio
1	0.602	-
2	0.845	1.404
3	1.415	1.674
4	2.250	1.590
5	3.665	1.628
6	5.915	1.614
7	9.580	1.620
8	15.495	1.617
9	25.075	1.618
10	40.570	1.618

Table 4. The fastest growing path for the initial degree seed $(2, 1, 4)$

The cluster algebras corresponding to Tables 1 and 2 above are examples of mutation-acyclic cluster algebras, Tables 3 and 4 show mutation-cyclic cluster algebras. We see that these do not exhibit noticeably different behaviour.

A.2. Averaging Degrees

In Remark 3.2.11, we stated that it is reasonable to ask what happens if we look at the average (absolute value of) degree at each radius of mutation, rather than looking at one individual mutation path. The examples below show that this does not give significantly different results when compared to the fastest growing paths above. It seems likely that the fastest growing path eventually dominates sufficiently so that they still tend to φ, just somewhat slower.

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	2.000	0.693	-
2	2.333	0.847	1.222
3	4.667	1.540	1.818
4	13.000	2.565	1.665
5	88.917	4.488	1.750

Table 5. Average degree for initial degree seed $(1, 1, 3)$

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	3.667	1.299	-
2	8.000	2.079	1.600
3	34.000	3.526	1.696
4	435.000	6.075	1.723
5	38620.229	10.562	1.738

Table 6. Average degree for initial degree seed $(2, 1, 4)$

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	10.333	2.335	-
2	34.667	3.546	1.518
3	533.167	6.279	1.771
4	13438.583	9.506	1.514
5	9290199.042	16.044	1.688

Table 7. Average degree for initial degree seed $(4, 3, 4)$

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	16.333	2.793	-
2	76.333	4.335	1.552
3	985.917	6.894	1.590
4	71235.333	11.174	1.621
5	113512388.792	18.547	1.660

Table 8. Average degree for initial degree seed $(5, 3, 6)$

We note also that this is much more difficult computationally, since we are required to calculate *all* degrees at a given mutation radius, rather than just one.

A.3. The ‘Pruned’ Exchange Tree

In Section 3.3, we defined a ‘pruned’ version of the exchange tree, with the hope of eliminating the fastest growing path starting at each point. Tables 10-12 below show the results we obtain in this case. Again, this does not seem to achieve what we require—cluster variable growth is still very fast, and there appears to be no way of distinguishing between different initial conditions.

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	24.000	3.178	-
2	19.333	2.926	0.932
3	34.000	3.526	1.191
4	147.778	4.996	1.417
5	2284.000	7.734	1.548
6	249760.424	12.428	1.607

TABLE 9. Average degree of pruned exchange tree for initial degree seed $(4, 2, 20)$

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	2.000	0.693	-
2	1.667	0.511	0.737
3	2.200	0.788	1.543
4	3.000	1.099	1.393
5	3.571	1.273	1.159
6	4.720	1.552	1.219

TABLE 10. Average degree of pruned exchange tree for initial degree seed $(2, 1, 4)$

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	8.000	2.076	-
2	20.000	2.996	1.441
3	52.000	3.951	1.319
4	136.000	4.913	1.243
5	356.000	5.875	1.196
6	932.000	6.837	1.164

TABLE 11. Average degree of pruned exchange tree for initial degree seed $(4, 3, 4)$

Mutation Radius	Average Degree	ln (Average Degree)	Ratio
1	11.000	2.398	-
2	32.333	3.476	1.450
3	182.400	5.206	1.498
4	2576.556	7.854	1.509
5	209716.059	12.254	1.560
6	294765443.485	19.502	1.592

TABLE 12. Average degree of pruned exchange tree for initial degree seed $(5, 3, 6)$

A.4. Slowest Growing Paths

The final approach attempted was to determine a *slowest growing path*. Tables 13-19 below show that this again fails in general to distinguish between the mutation-cyclic and mutation-acyclic cases, although the degree growth is notably smaller along this path when compared to the fastest growing paths. In fact, this seems only to depend on the value of c in the initial degree seed.

Mutation Radius	Degree	Ratio
1	9	-
2	23	2.556
3	60	2.609
4	157	2.617
5	411	2.618
6	1076	2.618
7	2817	2.618
8	7375	2.618
9	19308	2.618
10	50549	2.618

TABLE 13. Slowest growing path for initial degree seed $(4, 3, 4)$

Mutation Radius	Degree	Ratio
1	4	-
2	4	1.000
3	8	2.000
4	20	2.500
5	52	2.600
6	136	2.615
7	356	2.618
8	932	2.618
9	2440	2.618
10	6388	2.618

TABLE 14. Slowest growing path for initial degree seed $(4, 3, 8)$

Mutation Radius	Degree	Ratio
1	3	-
2	8	2.667
3	29	3.652
4	108	3.724
5	403	3.731
6	1504	3.732
7	5613	3.732
8	20948	3.732
9	78179	3.732
10	291768	3.732

Table 15. Slowest growing path for initial degree seed $(4, 4, 13)$

Mutation Radius	Degree	Ratio
1	9	-
2	22	2.444
3	57	2.591
4	149	2.614
5	390	2.617
6	1021	2.618
7	2673	2.618
8	6998	2.618
9	18321	2.618
10	47965	2.618

Table 16. Slowest growing path for initial degree seed $(5, 3, 6)$

Mutation Radius	Degree	Ratio
1	-2	-
2	-1	0.500
3	2	2.000
4	3	1.500
5	1	0.333
6	-2	2.000
7	-1	0.500
8	2	2.000
9	3	1.500
10	1	0.333

Table 17. Slowest growing path for initial degree seed $(1, 1, 3)$. Note that this path follows a cycle in the exchange graph, and hence repeats with period 5 - this cyclic behaviour will always occur when $c = 1$.

Mutation Radius	Degree	Ratio
1	4	-
2	5	1.250
3	6	1.200
4	7	1.167
5	8	1.143
6	9	1.125
7	10	1.111
8	11	1.100
9	12	1.091
10	13	1.083

TABLE 18. Slowest growing path for initial degree seed $(3, 2, 2)$

Mutation Radius	Degree	Ratio
1	2	-
2	3	1.500
3	7	2.333
4	18	2.571
5	47	2.611
6	123	2.617
7	322	2.618
8	843	2.618
9	2207	2.618
10	5778	2.618

TABLE 19. Slowest growing path for initial degree seed $(3, 3, 7)$