COMPACTNESS OF COMPOSITIONS OF STRICTLY SINGULAR OPERATORS ON DIRECT SUMS OF BAERNSTEIN, SCHREIER AND ℓ_p-SPACES

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Abstract. Let X be the direct sum of finitely many Banach spaces chosen from the following three families: (i) the Baernstein spaces B_p for $1 < p < \infty$; (ii) the p-convexified Schreier spaces S_p for $1 \leqslant p < \infty$; (iii) the sequence spaces ℓ_p for $1 \leqslant p < \infty$ (and c_0). We show that the quotient algebra of strictly singular by compact operators on X is nilpotent; that is, there is a natural number k, dependent only on the collections of direct summands from each of the three families, such that:

- every composition of $k+1$ strictly singular operators on X is compact;
- there are k strictly singular operators on X whose composition is not compact.

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1. Introduction

Ever since Kato [6] introduced the class of strictly singular operators nearly 70 years ago as a larger ideal that can play the same role as the ideal of compact operators in the perturbation theory of Fredholm operators, it has been an important task to describe the precise relationship between the strictly singular and compact operators on a given Banach space.

The simplest scenario is that these two ideals are equal; this happens for instance for the classical sequence spaces ℓ_p, $1 \leqslant p < \infty$, and c_0, the quasi-reflexive James spaces J_p, $1 < p < \infty$, and the Tsirelson space T, but is quite rare.

A more common phenomenon is that every composition of two strictly singular operators is compact; Banach spaces with this property include the Lebesgue spaces $L_p[0,1]$ for $1 \leqslant p < \infty$, the continuous functions $C(K)$ on a compact Hausdorff space K, the direct sums $\ell_p \oplus \ell_q$ and $\ell_p \oplus c_0$ for $1 \leqslant p < q < \infty$, and—very importantly for us—the p^{th} Baernstein space B_p for $1 < p < \infty$ and the p-convexified Schreier space S_p for $1 \leqslant p < \infty$ (see [8, Theorem 1.1]; we refer to Section 2 for the precise definitions of the spaces B_p and S_p).

There is nothing special about the number 2 here, of course; for any $k \in \mathbb{N}$, there are examples of Banach spaces X with the property that every composition of $k+1$ strictly singular operators on X is compact, but there are k strictly singular operators whose composition is not compact. In algebraic parlance, this means that the quotient algebra of strictly singular by compact operators on X is nilpotent of index $k+1$. A recent example where this occurs is the Schreier space $X[\mathcal{S}_k]$ induced by the k^{th} Schreier family $\mathcal{S}_k$ (see [5, Theorem 7.6(3)–(4)]; note that this theorem also applies to the Schreier space $X[\mathcal{S}_\xi]$ induced by the Schreier family $\mathcal{S}_\xi$ for a countably infinite ordinal ξ, but the value of the index of nilpotency is harder to state explicitly in those cases). Many other examples exist, including Tarbard's variant $\mathfrak{X}_{k+1}$ of the Argyros–Haydon space [13] and the Tsirelson-like space $\mathfrak{X}^k_{0,1}$ constructed by Argyros, Beanland and Motakis [1].

Our main result builds on several of the above results, as we consider nilpotency of the quotient algebra of strictly singular by compact operators on Banach spaces that are finite direct sums of Baernstein spaces, ℓ_p-spaces and p-convexified Schreier spaces. The precise statement is as follows.

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Theorem 1.1. *Let $L \subset (1,\infty)$ and $M, N \subset [1,\infty)$ be finite sets, not all empty, and define*

$$k = \begin{cases} |L| + |L \cup M| - 1 & \text{if } N = \emptyset \\ |L| + |L \cup M| + |N| & \text{otherwise.} \end{cases} \tag{1.1}$$

Then the Banach space

$$X = \Big(\bigoplus_{p\in L} B_p\Big) \oplus \Big(\bigoplus_{q\in M} \ell_q\Big) \oplus \Big(\bigoplus_{r\in N} S_r\Big) \tag{1.2}$$

satisfies:

(i) *every composition of $k+1$ strictly singular operators on X is compact;*

(ii) *there are k strictly singular operators on X whose composition is not compact.*

Hence, the quotient algebra $\mathscr{S}(X)/\mathscr{K}(X)$ of strictly singular by compact operators on X is nilpotent of index $k+1$.

Remark 1.2. (i) The statement of Theorem 1.1 does not specify which norm we put on the (finite) direct sum (1.2) defining the Banach space X because the conclusions depend only on the isomorphism class of X. For definiteness, we may choose the norm $\|(x_j)_{j=1}^n\|_\infty = \max_{1\leqslant j\leqslant n}\|x_j\|$, where $n = |L| + |M| + |N|$, but emphasize that the theorem is true for any equivalent norm such as $\|(x_j)_{j=1}^n\|_p = \big(\sum_{j=1}^n \|x_j\|^p\big)^{1/p}$ for $1 \leqslant p < \infty$.

(ii) Important special cases of Theorem 1.1 occur when two of the three index sets L, M and N are empty. This result is well-known for $X = \bigoplus_{j=1}^m \ell_{p_j}$, where $1 \leqslant p_1 < p_2 < \cdots < p_m < \infty$ (see [14, Theorem 4.7] for details), but the other two cases appear to be new for $m \geqslant 2$, so we state them explicitly for future reference:
- Let $X = \bigoplus_{j=1}^m B_{p_j}$ for some $m \in \mathbb{N}$ and $1 < p_1 < p_2 < \cdots < p_m < \infty$. Then the quotient algebra $\mathscr{S}(X)/\mathscr{K}(X)$ is nilpotent of index $2m$.
- Let $X = \bigoplus_{j=1}^m S_{p_j}$ for some $m \in \mathbb{N}$ and $1 \leqslant p_1 < p_2 < \cdots < p_m < \infty$. Then the quotient algebra $\mathscr{S}(X)/\mathscr{K}(X)$ is nilpotent of index $m+1$.

(As already mentioned, [8, Theorem 1.1] contains these results for $m = 1$; see also Lemma 3.2 below.)

(iii) The reason that the cardinality of the set $L \cup M$ appears in the formula (1.1) for k is that on the one hand, B_p contains a complemented copy of ℓ_p for $1 < p < \infty$, so X contains a complemented copy of ℓ_p for every $p \in L \cup M$, and on the other, $\ell_p \oplus \ell_p \cong \ell_p$, so those p that belong to $L \cap M$ contribute only one copy of ℓ_p to the collection of complemented subspaces of X.

Building on these arguments, we see that since $B_p \cong W \oplus \ell_p$ for some Banach space W, we have $B_p \oplus \ell_p \cong W \oplus \ell_p \oplus \ell_p \cong W \oplus \ell_p \cong B_p$, and therefore $X \cong X \oplus \ell_p$ for $p \in L$. Consequently, we can replace the index set M in the definition (1.2) of X with $M \setminus L$ or $L \cup M$, or any set between them, without affecting the isomorphism class of X.

(iv) As indicated in the abstract, there is a variant of Theorem 1.1 that includes c_0 in the direct sum. To state it concisely, set $Y = X \oplus c_0$, where X is the Banach space defined by (1.2). Then:
- Y is isomorphic to X if $N \neq \emptyset$, so the conclusions of Theorem 1.1 apply verbatim to Y in this case.
- Otherwise $Y = \big(\bigoplus_{p\in L} B_p\big) \oplus \big(\bigoplus_{q\in M} \ell_q\big) \oplus c_0$ for some finite sets $L \subset (1,\infty)$ and $M \subset [1,\infty)$, and we have: Every composition of $|L| + |L \cup M| + 1$ strictly singular operators on Y is compact, but there are $|L| + |L \cup M|$ strictly singular operators on Y whose composition is not compact.

We refer to Remark 3.8 for a detailed justification of these two claims.

2. Preliminaries

All vector spaces (in particular Banach spaces) are over the same scalar field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. We use function notation for sequences, thus writing $x(n)$ for the n^{th} coordinate of a sequence $x \in \mathbb{K}^{\mathbb{N}}$. As usual, c_{00} denotes the subspace of finitely supported elements of $\mathbb{K}^{\mathbb{N}}$, and $(e_n)_{n\in\mathbb{N}}$ is the *unit vector basis* given by $e_n(m) = 1$ if $m = n$ and $e_n(m) = 0$ otherwise.

By an *operator,* we mean a bounded linear map between Banach spaces, and write $\mathscr{B}(X,Y)$ for the set of operators from a Banach space X to a Banach space Y, abbreviated $\mathscr{B}(X)$ when $X = Y$; I_X denotes the identity operator on X. An operator $T \in \mathscr{B}(X,Y)$ is *strictly singular* if no restriction of T to an infinite-dimensional subspace of X is an isomorphic embedding. We write $\mathscr{S}(X,Y)$ and $\mathscr{K}(X,Y)$ for the sets of strictly singular and compact operators from X to Y, respectively, abbreviated $\mathscr{S}(X)$ and $\mathscr{K}(X)$ when $X = Y$. They are closed operator ideals in the sense of Pietsch, and $\mathscr{K}(X,Y) \subseteq \mathscr{S}(X,Y)$ for any Banach spaces X and Y.

Before we can introduce our main objects of interest—the Schreier and Baernstein spaces—we require the following notion, originating in [11]: A *Schreier set* is a finite subset F of the natural numbers $\mathbb{N} = \{1,2,3,\ldots\}$ such that either $F = \emptyset$ or $|F| \leqslant \min F$, where $|F|$ denotes the cardinality of F. As usual, we write $\mathcal{S}_1$ for the family of Schreier sets. Following [3, Section 3], for $1 \leqslant p < \infty$ and $x \in \mathbb{K}^{\mathbb{N}}$, we define

$$\|x\|_{S_p} = \sup\{\mu_p(x,F) : F \in \mathcal{S}_1\} \in [0,\infty], \quad \text{where} \quad \mu_p(x,F) = \begin{cases} 0 & \text{if } F = \emptyset, \\ \Big(\sum_{n\in F}|x(n)|^p\Big)^{1/p} & \text{otherwise.} \end{cases}$$

Then $\mu_p(\,\cdot\,,F)$ is a seminorm on $\mathbb{K}^{\mathbb{N}}$, and $Z_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{S_p} < \infty\}$ is a subspace of $\mathbb{K}^{\mathbb{N}}$ on which $\|\cdot\|_{S_p}$ defines a complete norm. However, the Banach space Z_p fails to be separable (see [3, Corollary 5.6]), so we define the *p-convexified Schreier space*, denoted S_p, as the closure of c_{00} in Z_p. The unit vector basis $(e_n)_{n\in\mathbb{N}}$ is a 1-unconditional, shrinking, normalized basis for S_p by [3, Propositions 3.5 and 3.10 and Corollary 3.12].

The analogous definition of the Baernstein spaces involves the notion of a *Schreier chain*, which is a non-empty, finite collection $\mathcal{C}$ of non-empty, consecutive Schreier sets; that is, $\mathcal{C} = \{F_1,\ldots,F_m\}$ for some $m \in \mathbb{N}$ and $F_1,\ldots,F_m \in \mathcal{S}_1 \setminus \{\emptyset\}$ with $\max F_j < \min F_{j+1}$ for $1 \leqslant j < m$. Writing SC for the collection of all Schreier chains, for $1 < p < \infty$ and $x \in \mathbb{K}^{\mathbb{N}}$, we can define

$$\|x\|_{B_p} = \sup\{\beta_p(x,\mathcal{C}) : \mathcal{C} \in \mathrm{SC}\} \in [0,\infty], \quad \text{where} \quad \beta_p(x,\mathcal{C}) = \bigg(\sum_{F\in\mathcal{C}}\Big(\sum_{n\in F}|x(n)|\Big)^p\bigg)^{1/p}.$$

As before, $\beta_p(\,\cdot\,,\mathcal{C})$ is a seminorm on $\mathbb{K}^{\mathbb{N}}$, and $B_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{B_p} < \infty\}$ is a subspace of $\mathbb{K}^{\mathbb{N}}$ on which $\|\cdot\|_{B_p}$ defines a complete norm. In contrast to the Schreier spaces, c_{00} is dense in B_p, which is the p^{th} *Baernstein space*. It is reflexive, and the unit vector basis is a 1-unconditional, normalized basis for it. Baernstein [2] originally defined B_2, while Seifert [12] observed that Baernstein's definition works for general $p > 1$.

We conclude this preliminary section with a standard piece of terminology from algebra that we already used in the statement of Theorem 1.1: A ring $\mathscr{A}$ is *nilpotent* if, for some $k \in \mathbb{N}$, we have $a_1a_2\cdots a_k = 0$ whenever $a_1,\ldots,a_k \in \mathscr{A}$; the smallest value of k for which this identity is satisfied is called the *index of nilpotency* of $\mathscr{A}$.

3. The proof of Theorem 1.1

We present the proofs of the two parts of the theorem separately, beginning with the first. It relies on the matrix representation of operators on a finite direct sum of Banach spaces, defined as follows.

Definition 3.1. Let $X = \bigoplus_{j=1}^n X_j$ for some Banach spaces $X_1,\ldots,X_n$. The *matrix* associated with an operator $T \in \mathscr{B}(X)$ is the operator-valued $(n\times n)$-matrix $(T_{j,k})_{j,k=1}^n$ given by

$$T_{j,k} = Q_jTJ_k \in \mathscr{B}(X_k,X_j) \qquad (j,k \in \{1,\ldots,n\}),$$

where $Q_j\colon X \to X_j$ and $J_k\colon X_k \to X$ denote the j^{th} coordinate projection and k^{th} coordinate embedding, respectively.

It is easy to see that composition of operators corresponds to matrix multiplication:

$$(TU)_{j,m} = \sum_{k=1}^n T_{j,k}U_{k,m} \qquad (T,U \in \mathscr{B}(X),\ j,m \in \{1,\ldots,n\}). \tag{3.1}$$

Furthermore, the identity $T=\sum_{j,k=1}^n J_jT_{j,k}Q_k$ implies that, for any operator ideal $\mathscr{I}$, we have

$$T\in\mathscr{I}(X)\qquad\Longleftrightarrow\qquad T_{j,k}\in\mathscr{I}(X_k,X_j)\text{ for every } j,k\in\{1,\ldots,n\}. \tag{3.2}$$

We shall also require the following result, most of which is known.

Lemma 3.2. (i) *Every strictly singular operator on ℓ_p is compact for $1\leqslant p<\infty$.*
(ii) *Every composition of two strictly singular operators on B_p is compact for $1<p<\infty$.*
(iii) *The composite operator TU is compact whenever $U\colon S_p\to S_p$ and $T\colon S_p\to Y$ are strictly singular, where $1\leqslant p<\infty$ and Y can be any Banach space.*

Proof. Part (i) is well known; we refer to the sentence after the proof of [9, Proposition 2.c.3] for details. Part (ii) is due to Laustsen and Smith [8, Theorem 1.1]; so is part (iii), but only for $Y=S_p$. However, invoking a theorem of Rosenthal, we can easily extend their proof to the general case. Indeed, suppose that $U\in\mathscr{S}(S_p)$ and $T\in\mathscr{B}(S_p,Y)$ are operators whose composition TU is not compact. Our aim is to prove that T is not strictly singular. Using the elementary observation stated in [8, Lemma 3.4], we can find a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the unit vector basis for S_p such that $\inf_{n\in\mathbb{N}}\|TUu_n\|_Y>0$. Then also $\inf_{n\in\mathbb{N}}\|Uu_n\|_{S_p}>0$, and $(u_n)_{n\in\mathbb{N}}$ is weakly null because the unit vector basis for S_p is shrinking, so [8, Lemma 3.3(ii)] implies that $(u_n)_{n\in\mathbb{N}}$ admits a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ such that $(Uu_{n_j})_{j\in\mathbb{N}}$ is equivalent to the unit vector basis $(d_j)_{j\in\mathbb{N}}$ for c_0; let $V\in\mathscr{B}(c_0,S_p)$ be the operator given by $Vd_j=Uu_{n_j}$ for $j\in\mathbb{N}$. Since

$$\inf_{j\in\mathbb{N}}\|TVd_j\|_Y=\inf_{j\in\mathbb{N}}\|TUu_{n_j}\|_Y>0,$$

a famous result of Rosenthal, originally stated as the first remark following [10, Theorem 3.4], implies that $\mathbb{N}$ contains an infinite subset N for which the restriction of TV to the closed span of $\{d_j : j\in N\}$ is an isomorphic embedding. Hence T is not strictly singular. □

Proof of Theorem 1.1(i). In view of Remark 1.2(iii), we may suppose that the sets L and M are disjoint. Our aim is to prove that the composition of $k+1$ strictly singular operators $R^{(1)},\ldots,R^{(k+1)}$ on X is compact, so by (3.2), we must show that the $(j,m)^{\text{th}}$ entry of the matrix of the composite operator $R^{(k+1)}R^{(k)}\cdots R^{(1)}$ is compact for every $j,m\in\{1,\ldots,n\}$, where $n=|L|+|M|+|N|$. Applying the identity (3.1) repeatedly, we obtain

$$(R^{(k+1)}R^{(k)}\cdots R^{(1)})_{j,m}=\sum_{i_1,\ldots,i_k=1}^{n}R^{(k+1)}_{j,i_k}R^{(k)}_{i_k,i_{k-1}}\cdots R^{(1)}_{i_1,m}.$$

We shall now complete the proof by showing that each of the n^k terms on the right-hand side of this equation is compact. Simplifying the notation, we see that this amounts to verifying that the composite operator $T:=T_{k+1}T_k\cdots T_1$ is compact whenever $T_j\colon X_j\to X_{j+1}$ is a strictly singular operator for every $1\leqslant j\leqslant k+1$ and the Banach spaces $X_1,\ldots,X_{k+2}$ belong to the family $\{B_p : p\in L\}\cup\{\ell_q : q\in M\}\cup\{S_r : r\in N\}$.

For integers $1\leqslant i\leqslant j\leqslant k+2$, set

$$T_{(i\to j)}=\begin{cases}I_{X_j} & \text{if } i=j,\\ T_{j-1}\cdots T_{i+1}T_i\in\mathscr{S}(X_i,X_j) & \text{otherwise.}\end{cases}$$

This notation will allow us to justify the following three observations concisely:

(i) Suppose that $X_h=X_i=X_j=B_p$ for some $p\in L$ and integers $1\leqslant h<i<j\leqslant k+2$. Then $T_{(h\to i)},T_{(i\to j)}\in\mathscr{S}(B_p)$, so their composition is compact by Lemma 3.2(ii), and therefore $T=T_{(j\to k+2)}T_{(i\to j)}T_{(h\to i)}T_{(1\to h)}$ is compact.
(ii) Suppose that $X_i=X_j=\ell_q$ for some $q\in M$ and integers $1\leqslant i<j\leqslant k+2$. Then $T_{(i\to j)}\in\mathscr{S}(\ell_q)$ is compact by Lemma 3.2(i), so $T=T_{(j\to k+2)}T_{(i\to j)}T_{(1\to i)}$ is compact.
(iii) Suppose that $X_i=X_j=S_r$ for some $r\in N$ and integers $1\leqslant i<j\leqslant k+1$. Then $T_{(i\to j)}\in\mathscr{S}(S_r)$ and $T_{(j\to k+2)}\in\mathscr{S}(S_r,X_{k+2})$ because $j<k+2$, so Lemma 3.2(iii) implies that their composition is compact. Hence $T=T_{(j\to k+2)}T_{(i\to j)}T_{(1\to i)}$ is compact.

If $N = \emptyset$, then $k = 2|L| + |M| - 1$ because $L \cap M = \emptyset$. Therefore, choosing $k+2$ spaces $X_1, \ldots, X_{k+2}$ from the family $\{B_p : p \in L\} \cup \{\ell_q : q \in M\}$, we are either in case (i) or (ii), so T is compact.

Otherwise $N \neq \emptyset$, and we have $k = 2|L| + |M| + |N|$. If we are not in cases (i) or (ii), then at least $|N| + 2$ of the spaces $X_1, \ldots, X_{k+2}$ come from the family $\{S_r : r \in N\}$, so we must be in case (iii); hence T is compact. □

In order to streamline the presentation of the proof of the second part of Theorem 1.1, we require a few preparations.

Lemma 3.3. *Let X be a Banach space, and $\mathscr{I}$ and $\mathscr{J}$ operator ideals. Suppose that, for some $k \in \mathbb{N}$, X contains $k+1$ complemented subspaces $X_1, \ldots, X_{k+1}$ for which there are k operators $R_1 \in \mathscr{J}(X_1, X_2), \ldots, R_k \in \mathscr{J}(X_k, X_{k+1})$ whose composition $R_k R_{k-1} \cdots R_1$ does not belong to $\mathscr{I}(X_1, X_{k+1})$. Then there are operators $T_1, \ldots, T_k \in \mathscr{J}(X)$ whose composition $T_k T_{k-1} \cdots T_1$ does not belong to $\mathscr{I}(X)$.*

Proof. For each $1 \leqslant j \leqslant k+1$, we can take operators $U_j \in \mathscr{B}(X, X_j)$ and $V_j \in \mathscr{B}(X_j, X)$ such that $U_j V_j = I_{X_j}$ because X_j is complemented in X. Set $T_j = V_{j+1} R_j U_j \in \mathscr{J}(X)$ for $1 \leqslant j \leqslant k$. Then we have

$$\begin{aligned} U_{k+1}(T_k T_{k-1} \cdots T_1) V_1 &= (U_{k+1} V_{k+1}) R_k (U_k V_k) R_{k-1} (U_{k-1} V_{k-1}) \cdots (U_2 V_2) R_1 (U_1 V_1) \\ &= R_k R_{k-1} \cdots R_1 \notin \mathscr{I}(X_1, X_{k+1}), \end{aligned}$$

so $T_k T_{k-1} \cdots T_1 \notin \mathscr{I}(X)$ because $\mathscr{I}$ is an operator ideal. □

"Formal inclusion maps" are at the heart of our proof of Theorem 1.1(ii), so our next step is to make this notion precise; we use a definition specifically tailored to the context at hand, where inclusion between Banach spaces is unambiguous because all the Banach spaces we consider consist of scalar sequences, equipped with the coordinatewise vector space operations inherited from $\mathbb{K}^{\mathbb{N}}$.

Definition 3.4. Let Y and Z be vector subspaces of $\mathbb{K}^{\mathbb{N}}$, equipped with norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We say that the pair (Y, Z) *admits a formal inclusion map* if $Y \subseteq Z$ and there is a constant $C_{Y,Z} > 0$ such that

$$\|y\|_Z \leqslant C_{Y,Z} \|y\|_Y \qquad (y \in Y). \tag{3.3}$$

Obviously, the significance of this definition is that when (Y, Z) admits a formal inclusion map, the restriction of the identity operator on $\mathbb{K}^{\mathbb{N}}$ defines a bounded linear map from Y to Z with norm at most $C_{Y,Z}$; we denote this map $R_{Y,Z} \colon Y \to Z$ and call it the *formal inclusion map* from Y to Z, and remark that $R_{Y,Z}$ is simply the identity operator on Y if $Y = Z$.

Lemma 3.5. *Let Y and Z be vector subspaces of $\mathbb{K}^{\mathbb{N}}$, equipped with complete norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively, and suppose that the unit vector basis $(e_n)_{n \in \mathbb{N}}$ is a Schauder basis for both Y and Z. Then (Y, Z) admits a formal inclusion map if (and only if) there is a constant $C_{Y,Z} > 0$ such that*

$$\|x\|_Z \leqslant C_{Y,Z} \|x\|_Y \qquad (x \in c_{00}). \tag{3.4}$$

Proof. To prove the non-trivial implication, suppose that (3.4) is satisfied, and take $y \in Y$. The fact that $(e_n)_{n \in \mathbb{N}}$ is a basis for Y means that $\|y - P_n y\|_Y \to 0$ as $n \to \infty$, where $P_n \colon Y \to c_{00}$ denotes the n^{th} basis projection given by $P_n y = \sum_{j=1}^n y(j) e_j$ for $n \in \mathbb{N}$. In particular, $(P_n y)_{n \in \mathbb{N}}$ is a Cauchy sequence in c_{00} with respect to the norm $\|\cdot\|_Y$ and therefore also with respect to $\|\cdot\|_Z$ by (3.4), so $(P_n y)_{n \in \mathbb{N}}$ converges to some $z \in Z$. Coordinatewise inspection shows that $z = y$, so $y \in Z$, and the inequality (3.3) follows from (3.4) and continuity of the norms. □

The final ingredient we need before we can present the proof of Theorem 1.1(ii) is a relation $\preceq$ defined on the family

$$\mathrm{BSp} = \{B_p : 1 < p < \infty\} \cup \{\ell_p : 1 \leqslant p < \infty\} \cup \{S_p : 1 \leqslant p < \infty\} \cup \{c_0\}$$

of Banach spaces. Its definition is as follows.

Definition 3.6. Let $Y, Z \in \mathrm{BSp}$. Then $Y \preceq Z$ if and only if one of the following five mutually exclusive conditions is satisfied:

- $Y = \ell_1$ and $Z \in \mathrm{BSp}$ is arbitrary;
- $Y = B_p$ for some $1 < p < \infty$ and
  $Z \in \{B_q : p \leqslant q < \infty\} \cup \{\ell_q : 1 < q < \infty\} \cup \{S_q : 1 \leqslant q < \infty\} \cup \{c_0\}$;
- $Y = \ell_p$ for some $1 < p < \infty$ and $Z \in \{\ell_q : p \leqslant q < \infty\} \cup \{S_q : p \leqslant q < \infty\} \cup \{c_0\}$;
- $Y = S_p$ for some $1 \leqslant p < \infty$ and $Z \in \{\ell_q : p < q < \infty\} \cup \{S_q : p \leqslant q < \infty\} \cup \{c_0\}$;
- $Y = c_0$ and $Z = c_0$.

In line with standard practice, we write $Y \prec Z$ when $Y \preceq Z$ and $Y \neq Z$.

It is easy to see that $\preceq$ is a linear order, whose definition we can summarize as follows:

$$\ell_1 \prec B_p \prec B_q \prec S_1 \prec \ell_p \prec S_p \prec \ell_q \prec S_q \prec c_0 \qquad (1 < p < q < \infty). \tag{3.5}$$

The next lemma explains its relevance for our purposes.

Lemma 3.7. *Let $Y, Z \in \mathrm{BSp}$ with $Y \prec Z$. Then the pair (Y, Z) admits a formal inclusion map $R_{Y,Z} \colon Y \to Z$ which is strictly singular.*

Proof. Admitting a formal inclusion map is clearly a transitive relation in the sense that if the pairs (X, Y) and (Y, Z) both admit formal inclusion maps, then so does the pair (X, Z), and in this case $R_{X,Z} = R_{Y,Z} R_{X,Y}$, which implies that $R_{X,Z}$ is strictly singular whenever at least one of the formal inclusion maps $R_{X,Y}$ and $R_{Y,Z}$ is.

Hence, in view of (3.5), it suffices to show that each of the following pairs admits a formal inclusion map which is strictly singular:

(i) (ℓ_1, B_p) for $1 < p < \infty$;
(ii) (B_p, B_q) for $1 < p < q < \infty$;
(iii) (B_p, S_1) for $1 < p < \infty$;
(iv) (S_p, ℓ_q) for $1 \leqslant p < q < \infty$;
(v) (ℓ_p, S_p) for $1 < p < \infty$;
(vi) (S_p, c_0) for $1 < p < \infty$.

We begin by explaining why these pairs admit formal inclusion maps. By Lemma 3.5, we must show that there is a constant $C_{Y,Z} > 0$ such that $\|x\|_Z \leqslant C_{Y,Z}\|x\|_Y$ for every $x \in c_{00}$. This is only non-trivial in case (iv), so we leave it last. In the five other cases, we can easily verify that $C_{Y,Z} = 1$ works:

(i) This is simply subadditivity of the B_p-norm together with the fact that the unit vector basis for B_p is normalized.
(ii) This is a consequence of the inequality $\beta_q(x, \mathcal{C}) \leqslant \beta_p(x, \mathcal{C})$ for every Schreier chain $\mathcal{C}$, which in turn follows from the well-known fact that the pair (ℓ_p, ℓ_q) admits a formal inclusion map with constant $C_{\ell_p,\ell_q} = 1$.
(iii) This is immediate from the fact that $\mu_1(x, F) = \beta_p(x, \{F\})$ for every $F \in \mathcal{S}_1$.
(v) This is clear because $\mu_p(x, F) \leqslant \|x\|_{\ell_p}$ for every $F \in \mathcal{S}_1$.
(vi) This follows from the fact that the coordinate functionals $(e_n^*)_{n\in\mathbb{N}}$ on S_p have norm 1.

We address case (iv) by modifying an argument originally due to Graham Jameson for $p = 1$, presented in the first part of the proof of [7, Theorem A.1], and include the details here for the convenience of the reader. Since the S_p- and ℓ_q-norms depend only on the moduli of the coordinates of $x \in c_{00}$, we may suppose that $x(n) \geqslant 0$ for every $n \in \mathbb{N}$. We can find a permutation $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that $x \circ \sigma$ is decreasing because x has finite support, and [7, Lemma A.2] implies that $\|x \circ \sigma\|_{S_p} \leqslant \|x\|_{S_p}$, while $\|x \circ \sigma\|_{\ell_q} = \|x\|_{\ell_q}$. Hence, by replacing x with the decreasing sequence $x \circ \sigma$, we may suppose that x is decreasing. Furthermore, by homogeneity, we may suppose that $\|x\|_{S_p} = 1$.

Take $n \in \mathbb{N}_0$, and set $F_n = [2^n, 2^{n+1}) \cap \mathbb{N} \in \mathcal{S}_1$. Then $x(2^n) \geqslant x(j) \geqslant x(2^{n+1})$ for $j \in F_n$ because x is decreasing. The lower of these bounds implies that

$$1 = \|x\|_{S_p}^p \geqslant \mu_p(x, F_n)^p = \sum_{j\in F_n} x(j)^p \geqslant |F_n| \cdot x(2^{n+1})^p = 2^n x(2^{n+1})^p,$$

so $x(2^{n+1}) \leqslant 2^{-n/p}$. Consequently we have

$$\mu_q(x, F_{n+1})^q = \sum_{j\in F_{n+1}} x(j)^q = \sum_{j\in F_{n+1}} x(j)^{q-p}x(j)^p$$
$$\leqslant x(2^{n+1})^{q-p} \sum_{j\in F_{n+1}} x(j)^p \leqslant 2^{\frac{-n(q-p)}{p}} \mu_p(x, F_{n+1})^p \leqslant \big(2^{\frac{p-q}{p}}\big)^n,$$

and therefore

$$\|x\|_{\ell_q}^q = x(1)^q + \sum_{n=0}^{\infty} \mu_q(x, F_{n+1})^q \leqslant 1 + \sum_{n=0}^{\infty} \big(2^{\frac{p-q}{p}}\big)^n = 1 + \frac{1}{1-2^{\frac{p-q}{p}}} = \frac{2^{\frac{q}{p}}-1}{2^{\frac{q-p}{p}}-1}.$$

This proves that the inequality (3.4) is satisfied for some positive constant $C_{S_p,\ell_q} \leqslant \big(\frac{2^{q/p}-1}{2^{(q-p)/p}-1}\big)^{1/q}$, thus completing the proof in case (iv).

To justify that the formal inclusion maps are strictly singular in each of the six cases above, we require two standard notions concerning a pair of infinite-dimensional Banach spaces Y and Z:

- Y and Z are *totally incomparable* if no infinite-dimensional Banach space embeds isomorphically in both Y and Z.
- Y is *saturated* with copies of Z if every closed, infinite-dimensional subspace of Y contains a subspace which is isomorphic to Z.

Trivially, *every* operator between a pair of totally incomparable Banach spaces is strictly singular, and the following three facts imply that each of the first five pairs in the list above are totally incomparable:

- Any pair of distinct spaces from the family $\{\ell_p : 1 \leqslant p < \infty\} \cup \{c_0\}$ are totally incomparable, and every space belonging to this family is saturated with copies of itself (see [9, Proposition 2.a.2, the remark following it, and page 75]).
- B_p is saturated with copies of ℓ_p for $1 < p < \infty$ (see [12, Theorem II.3.3], [4, Theorem 0.15(e)] or [7, Theorem 2.4]).
- S_p is saturated with copies of c_0 for $1 \leqslant p < \infty$ (see [3, Corollary 5.4] or [7, Theorem 2.4]).

Hence, the formal inclusion map $R_{Y,Z}\colon Y \to Z$ is strictly singular in cases (i)–(v).

This argument does not work for the formal inclusion map $R_{S_p,c_0}\colon S_p \to c_0$, but [7, Proposition 6.6] shows that it is strictly singular; alternatively, we can easily deduce this result from case (iv) because $R_{S_p,c_0} = R_{\ell_q,c_0}R_{S_p,\ell_q}$ for $1 \leqslant p < q < \infty$. ☐

Proof of Theorem 1.1(ii). The family

$$\Sigma = \begin{cases} \{B_p : p \in L\} \cup \{\ell_q : q \in L \cup M\} & \text{if } N = \emptyset, \\ \{B_p : p \in L\} \cup \{\ell_q : q \in L \cup M\} \cup \{S_r : r \in N\} \cup \{c_0\} & \text{otherwise} \end{cases} \tag{3.6}$$

consists of complemented subspaces of the Banach space X defined by (1.2) because B_p contains a complemented copy of ℓ_p for $1 < p < \infty$ and S_r contains a complemented copy of c_0 for $1 \leqslant r < \infty$.

Comparing the definitions (3.6) and (1.1), we see that Σ has cardinality $k+1$. Since $\Sigma \subset \mathrm{BSp}$, we can use the linear order $\preceq$ from Definition 3.6 to enumerate its members in increasing order:

$$X_1 \prec X_2 \prec \cdots \prec X_{k+1}.$$

Lemma 3.7 implies that we have a strictly singular formal inclusion map $R_{X_j,X_{j+1}}\colon X_j \to X_{j+1}$ for each $1 \leqslant j \leqslant k$. Their composition

$$X_1 \xrightarrow{R_{X_1,X_2}} X_2 \xrightarrow{R_{X_2,X_3}} X_3 \longrightarrow \cdots \longrightarrow X_k \xrightarrow{R_{X_k,X_{k+1}}} X_{k+1}$$

is simply the formal inclusion map $R_{X_1,X_{k+1}}\colon X_1 \to X_{k+1}$, which is not compact because it maps the unit vector basis for X_1 onto the unit vector basis for X_{k+1}. Now the conclusion follows from Lemma 3.3. ☐

Remark 3.8. The aim of this remark is to justify the two statements made in Remark 1.2(iv). We recall that $Y = X \oplus c_0$, where X is given by (1.2).

- Suppose that $N \neq \emptyset$, and take $r \in N$. Arguing as in Remark 1.2(iii), we write $S_r \cong W \oplus c_0$ for some Banach space W, which in combination with the fact that $c_0 \oplus c_0 \cong c_0$ implies that $S_r \oplus c_0 \cong W \oplus c_0 \oplus c_0 \cong W \oplus c_0 \cong S_r$, so $Y \cong X$ in this case.
- Now suppose that $N = \emptyset$, so that $Y = \big(\bigoplus_{p\in L} B_p\big) \oplus \big(\bigoplus_{q\in M} \ell_q\big) \oplus c_0$ for some finite sets $L \subset (1,\infty)$ and $M \subset [1,\infty)$, and set $k = |L| + |L \cup M|$. The proof of Theorem 1.1(i) that we gave above in the case $N = \emptyset$ carries over almost verbatim because Lemma 3.2(i) applies to c_0, too; that is, every strictly singular operator on c_0 is compact.

  It is also easy to modify the above proof of Theorem 1.1(ii) to the present context: Simply define $\Sigma = \{B_p : p \in L\} \cup \{\ell_q : q \in L \cup M\} \cup \{c_0\}$ and argue as before.

According to Pitt's Theorem [9, Proposition 2.c.3], every operator from ℓ_p to ℓ_q is compact for $1 \leqslant q < p < \infty$. Seifert claimed in his dissertation [12, Corollary II.3.4] that the analogous result is true for the Baernstein spaces, that is, every operator from B_p to B_q is compact for $1 < q < p < \infty$. However, using Lemma 3.7, we can easily show that Seifert's claim is false. The origin of this error appears to be [12, Lemma II.3.2]; although Seifert does not explicitly cite this lemma in his proof of [12, Corollary II.3.4], he uses it implicitly. We refer to [7, the paragraph below Theorem 2.4] for a corrected version of [12, Lemma II.3.2].

Unfortunately, Seifert's incorrect claim has gained much wider publicity than one would usually expect for a result contained in an unpublished PhD thesis because it was reproduced (without proof) in the lecture notes [4, Theorem 0.15(f)]. Hence, we shall state our correction formally.

Corollary 3.9. *There are strictly singular, non-compact operators from B_p to B_q for every pair $p, q \in (1,\infty)$.*

Proof. Definition 3.6 shows that $B_p \prec \ell_q$, so by Lemma 3.7, we have a formal inclusion map $R_{B_p,\ell_q}\colon B_p \to \ell_q$ which is strictly singular. It is not compact because it maps the unit vector basis for B_p onto the unit vector basis for ℓ_q. Since B_q contains a complemented subspace that is isomorphic to ℓ_q, we can find operators $U\colon B_q \to \ell_q$ and $V\colon \ell_q \to B_q$ such that $I_{\ell_q} = UV$. It follows that $VR_{B_p,\ell_q}\colon B_p \to B_q$ is a strictly singular, non-compact operator. □

The realization that there are non-compact operators from B_p to B_q for $1 < q < p < \infty$ raises the question whether some, possibly very rapidly increasing, subsequence of the unit vector basis for B_p dominates a subsequence of the unit vector basis for B_q. We conclude by showing that this is impossible.

Proposition 3.10. *Let $p, q \in (1,\infty)$. Then the unit vector basis for B_p admits a subsequence which dominates a subsequence of the unit vector basis for B_q if and only if $p \leqslant q$.*

Proof. For clarity, we denote the unit vector bases for B_p and B_q by $(e^p_n)_{n\in\mathbb{N}}$ and $(e^q_n)_{n\in\mathbb{N}}$, respectively.

The implication $\Leftarrow$ is clear because, for $1 < p \leqslant q < \infty$, the formal inclusion map $B_p \to B_q$ is bounded, which means that $(e^p_n)_{n\in\mathbb{N}}$ dominates $(e^q_n)_{n\in\mathbb{N}}$.

Conversely, suppose that $(e^p_{m_j})_{j\in\mathbb{N}}$ dominates $(e^q_{n_j})_{j\in\mathbb{N}}$ for some integers $1 \leqslant m_1 < m_2 < \cdots$ and $1 \leqslant n_1 < n_2 < \cdots$. Then

$$x_k = \frac{1}{2^{k-1}} \sum_{j=2^{k-1}}^{2^k-1} e^p_{m_j} \in B_p \qquad \text{and} \qquad y_k = \frac{1}{2^{k-1}} \sum_{j=2^{k-1}}^{2^k-1} e^q_{n_j} \in B_q \qquad (k \in \mathbb{N})$$

are unit vectors because their supports $\{m_j : 2^{k-1} \leqslant j < 2^k\}$ and $\{n_j : 2^{k-1} \leqslant j < 2^k\}$ are Schreier sets. By hypothesis, the block basic sequence $(x_k)_{k\in\mathbb{N}}$ dominates $(y_k)_{k\in\mathbb{N}}$. Since $\|x_k\|_\infty = 1/2^{k-1} \to 0$ as $k \to \infty$, [7, Proposition 2.14] implies that $(x_k)_{k\in\mathbb{N}}$ admits a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ which is dominated by the unit vector basis for ℓ_p. Furthermore, being a normalized block basic sequence of the unit vector basis for B_q, $(y_{k_j})_{j\in\mathbb{N}}$ dominates the unit vector basis for ℓ_q by [7, Lemma 2.10]. In conclusion, it follows that the unit vector basis for ℓ_p dominates the unit vector basis for ℓ_q, which is possible only if $p \leqslant q$. □

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