Quantum axion production by a laser wakefield accelerator

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Abstract. Axions are hypothetical, very weakly interacting, low-mass particles that remain popular candidates for dark matter. Most of the effort in the search for axions has focussed on astrophysical sources, although the evolution of high-power laser facilities has generated significant interest in "light shining through wall" experiments where axions are produced in the laboratory. With this in mind, a lower bound on the average number flux of axions produced by a laser wakefield accelerator is calculated from the perspective of quantum theory. The new result is better behaved for very low mass axions than the estimate of the average number flux obtained previously using entirely classical considerations. In particular, it converges in the limit as the axion mass tends to zero. Further calculation suggests the number flux of axions should be tolerable at practical laboratory-scale distances from the source when accounting for dispersion.

1. Introduction

Axions are hypothetical particles that were first proposed nearly 50 years ago in the context of the Strong CP problem. Peccei and Quinn's explanation [1] for CP conservation in the strong interaction relies on a scalar field whose coupling to the quarks has a global chiral U(1) symmetry. However, the scalar field has a non-zero vacuum expectation value, breaking the U(1) symmetry, and the axion is the pseudo-scalar Goldstone boson associated with this broken symmetry [2, 3]. Although the first version of the QCD axion was ruled out by experiment, subsequent theoretical developments [4, 5, 6] led to "invisible" QCD axions whose signatures are outside the realm of high-energy particle colliders. However, their cosmological implications are significant and they remain promising candidates for dark matter [7]. It is also notable

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that low-energy string field theory leads to a plethora of axions [8], including the QCD axion.

Considerable effort has been invested in terrestrial searches for signatures of axions over the last four decades. Most of the search effort has focussed on astrophysical sources of axions; in particular, they should be present in the galactic dark matter halo enveloping the Earth and they should be streaming from the Sun [9]. However, the worldwide development of multi-petawatt laser facilities, such as ELI Beamlines [10] in central Europe and the forthcoming upgrade to the UK's Vulcan laser system [11], offers an alternative paradigm in which axions produced from high-intensity photon collisions can be sought. Proposed experiments in which axions are produced in the overlap of a pair of high-intensity laser beams have been analysed [12, 13]. From a general perspective, such schemes are examples of "light shining through wall" experiments [14, 15], because axions are produced from high-intensity laser beams on one side of a barrier and converted to photons on the other side of the barrier. The ongoing ALPS II experiment [16] is state-of-the-art in traditional realisations of the "light shining through wall" paradigm (see Ref. [17] for a recent summary of the status of ALPS II). A recent discussion of the importance of "light shining through wall" experiments and their potential for future development can be found in Ref. [18]. Other aspects of axion physics that have been explored in the context of high-intensity lasers include the effects of axions on QED vacuum polarization [19] and the implications of a direct coupling between axions and electrons [20].

This article focusses on a variant of the "light shining through wall" approach in which axions are produced inside the wake of a high-intensity laser pulse‖ propagating through a magnetised underdense plasma [22]. In other words, the axions are produced by a laser wakefield accelerator; see Ref. [23] for an overview of the physics of laser wakefield acceleration. The laser pulse, accompanied by axions, exits the plasma and is absorbed by a beam dump. However, the axions couple very weakly to matter; thus, they propagate through the beam dump and enter a region where they are converted to photons. The calculation in Ref. [22] is based on a classical analysis in which the number flux of axions emerging from the plasma was estimated from their momentum flux. The present article improves this estimate by re-examining axion production from the perspective of quantum theory. More precisely, the axion field is regarded as a quantum operator whilst, for simplicity, the quantum aspects of the electromagnetic field are neglected. The electric field of the laser pulse's wake and the field magnetising the plasma are regarded as classical sources¶.

Unless otherwise specified, units are used in which the speed of light satisfies $c = 1$, the reduced Planck constant satisfies $\hbar = 1$ and the permeability of the vacuum satisfies

‖ An alternative to using a high-intensity laser pulse was considered in Ref. [21], where the axions are generated from the interaction of an electron beam with a plasma.

¶ A classical description is sufficient because, for practical purposes, the wake is essentially static in its rest frame over a period given by the reciprocal of the laser photon frequency. Likewise, for practical purposes, the applied magnetic field can be regarded as constant over the same period.

$\mu_0 = 1$.

2. Energy density, momentum flux and number flux

From a general perspective, axions are produced in regions where strong electric and magnetic fields overlap. However, the axion-photon coupling strength is weak and it is reasonable to neglect the impact of the axions on their production when investigating their production. In particular, the density of axions produced by the superposition of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is first-order in the axion-photon coupling strength g, whilst the corrections to the axion density due to the impact of the axions on $\mathbf{E}$ and $\mathbf{B}$ is second-order in g. Thus, one can neglect second-order terms and treat the source $\rho = -g\mathbf{E}\cdot\mathbf{B}$ in the axion field equation as prescribed.

As discussed in Appendix A.2, a quantum treatment of the axion field leads to the conclusion that the total four-momentum $\mathcal{E}^{0\mu}$ of axions produced by a classical source ρ whose duration is finite is[+]

$$\begin{aligned}\mathcal{E}^{0\mu} &= \lim_{\substack{t_\mathrm{i}\to-\infty\\ t_\mathrm{f}\to\infty}} \langle 0, t_\mathrm{i}|\hat{\mathcal{E}}^{0\mu}(t_\mathrm{f})|0, t_\mathrm{i}\rangle \\ &= \int \frac{d^3k}{16\pi^3}\frac{k^\mu}{\omega}\Big|\int dt \int d^3x\, \rho(\mathbf{x}, t)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\Big|^2. \qquad (1)\end{aligned}$$

Furthermore, the total number four-current $\mathcal{E}^{\mu}$ of axions produced by ρ is

$$\begin{aligned}\mathcal{E}^{\mu} &= \lim_{\substack{t_\mathrm{i}\to-\infty\\ t_\mathrm{f}\to\infty}} \langle 0, t_\mathrm{i}|\hat{\mathcal{E}}^{\mu}(t_\mathrm{f})|0, t_\mathrm{i}\rangle \\ &= \int \frac{d^3k}{16\pi^3}\frac{k^\mu}{\omega^2}\Big|\int dt \int d^3x\, \rho(\mathbf{x}, t)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\Big|^2. \qquad (2)\end{aligned}$$

The above integrals are over all space, all time and all k-space, and the wave four-vector k^μ is given by $(\omega, \mathbf{k})$ with $\omega = \sqrt{\mathbf{k}^2 + m_\Psi^2}$. The mass of the axion is denoted m_Ψ. The details of the quantum operators $\hat{\mathcal{E}}^{0\mu}$, $\hat{\mathcal{E}}^{\mu}$ and the quantum state $|0, t_\mathrm{i}\rangle$ are given in Appendix A.2.

As shown below, the one-dimensional nonlinear model of a laser wakefield accelerator leads to analytically amenable integrals for $\mathcal{E}^{0\mu}$ and $\mathcal{E}^{\mu}$. Consider a laser wakefield accelerator bathed in an ambient uniform magnetic field $\mathbf{B}$ aligned along the direction of propagation of the laser pulse. The field $\mathbf{B}$ is switched on sometime in the distant past and remains constant until it is switched off. For simplicity, all transient effects, such as the electric field induced by the time dependence of the applied magnetic field, are assumed negligible. Only the electric field of the wake trailing the laser pulse will be included in $\mathbf{E}$, and physical effects due to the finite length and width of the wake will not be considered. Furthermore, the quasi-static approximation will be adopted and, thus, the evolution of the wake will be neglected.

[+] All of the integrals in this article are definite. Unless stated otherwise, all integrals are over the full range of the integration variables.

The laser and its wake propagate at speed v along the z-axis. Hence, the source ρ in the axion field equation has the simple form $\rho(\mathbf{x},t) = \theta(-t)\varrho(z - vt)$, where $\varrho(\zeta) = -gBE(\zeta)$ with B constant and $E(\zeta)$ periodic in $\zeta = z - vt$. The phase speed v of the wake is, to first approximation, the group speed of the laser pulse; hence, $0 < v < 1$. The symbol θ denotes the Heaviside function, so the magnetic field is non-zero only for $t < 0$.

The expression $\rho(\mathbf{x},t) = \theta(-t)\varrho(z - vt)$ yields

$$\int dt \int d^3x\, \rho(\mathbf{x},t) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 4\pi^2\delta^{(2)}(\mathbf{k}_\perp)\frac{i\widetilde{\varrho}^*(k_z)}{\omega - k_z v} \tag{3}$$

where the wave vector $\mathbf{k}$ has been decomposed into its projection $\mathbf{k}_\perp = (k_x, k_y)$ in the xy-plane and its component k_z along the z-axis. The complex conjugate of the Fourier transform

$$\widetilde{\varrho}(k_z) = \int d\zeta\, \varrho(\zeta)\, e^{-ik_z\zeta} \tag{4}$$

of $\varrho(\zeta)$ has been introduced, and the 2-dimensional Dirac delta function $\delta^{(2)}(\mathbf{k}_\perp) = \delta(k_x)\delta(k_y)$ emerges from the complex conjugate of the integrals

$$\int dx\, e^{-ik_x x} = 2\pi\delta(k_x), \qquad \int dy\, e^{-ik_y y} = 2\pi\delta(k_y). \tag{5}$$

The integral

$$\int dt\, \theta(-t) e^{-i(\omega - k_z v)t} = \pi\delta(\omega - k_z v) + \frac{i}{\omega - k_z v} \tag{6}$$

involving the Heaviside function is responsible for the denominator in (3). However, the Dirac delta function in (6) does not contribute to (3) because, like the denominator in (3), it is evaluated at $\omega - k_z v$. The angular frequency ω is strictly greater than $k_z v$, because $v < 1$, so the quantity $\omega - k_z v$ is never zero.

Introducing (3) in (1) leads to the divergent result $\int d^2k\, \delta^{(2)}(\mathbf{k}_\perp)^2 = \delta^{(2)}(0)$ which, at first glance, seems alarming. In fact, this divergence is inevitable because the wake is arbitrarily wide in the above model. However, the divergence can be regularised by relaxing the continuum limit (see Appendix A.4) and identifying $4\pi^2\delta^{(2)}(0)$ with the area A of the xy-plane using $4\pi^2\delta^{(2)}(\mathbf{k}_\perp) = \int dx\, dy\, e^{i(k_x x + k_y y)}$.

Likewise, the same approach regularises the divergence emerging from the integral over k_z. The electric field $E(\zeta)$ of the wake is periodic, so we express $\varrho(\zeta)$ as the Fourier series

$$\varrho(\zeta) = \sum_n \varrho_n \exp\left(2\pi i n \frac{\zeta}{l}\right) \tag{7}$$

where the sum is over all integers n. Hence, substituting (7) in (4) gives

$$\widetilde{\varrho}(k_z) = 2\pi \sum_n \varrho_n\, \delta\Big(k_z - \frac{2\pi n}{l}\Big) \tag{8}$$

since $\int d\zeta\, e^{-i(k_z-2\pi n/l)\zeta} = 2\pi\delta(k_z - 2\pi n/l)$. Thus, for example, the z component $\mathcal{E}^{03}$ of the total four-momentum $\mathcal{E}^{0\mu}$ is

$$\begin{aligned}\mathcal{E}^{03} &= A\int \frac{dk_z}{2\pi}\frac{k_z}{2\omega}\frac{1}{(\omega-k_z v)^2}|\widetilde{\varrho}(k_z)|^2 \\ &= 2\pi A\sum_{n,n'}\varrho_n\varrho^*_{n'}\frac{k_\parallel}{2\omega}\frac{1}{(\omega-k_\parallel v)^2}\delta(k_\parallel - k'_\parallel)\end{aligned} \tag{9}$$

where $k_\parallel = 2\pi n/l$ and $k'_\parallel = 2\pi n'/l$ have been introduced for convenience, $\omega = \sqrt{k_\parallel^2 + m_\Psi^2}$ is understood and (1) has been used. However, $2\pi\delta(0)$ is identified with the length L of the z interval; hence, $2\pi A\delta(k_\parallel - k'_\parallel)$ is understood as $V\delta_{nn'}$, where $\delta_{nn'}$ is the Kronecker delta and $V = AL$ is the volume of the spatial domain. The result

$$\mathcal{T}^{03} = \sum_n \frac{k_\parallel|\varrho_n|^2}{2\omega(\omega-k_\parallel v)^2} \tag{10}$$

for $\mathcal{T}^{03} = \mathcal{E}^{03}/V$, i.e. the z-component of the average momentum flux, is well-behaved in the limit as V tends to infinity. A similar calculation, again using (1), leads to the expression

$$\mathcal{T}^{00} = \sum_n \frac{|\varrho_n|^2}{2(\omega-k_\parallel v)^2} \tag{11}$$

for the average energy density $\mathcal{T}^{00} = \mathcal{E}^{00}/V$. Likewise, using (2), the expressions

$$\mathcal{T}^{0} = \sum_n \frac{|\varrho_n|^2}{2\omega(\omega-k_\parallel v)^2}, \tag{12}$$

$$\mathcal{T}^{3} = \sum_n \frac{k_\parallel|\varrho_n|^2}{2\omega^2(\omega-k_\parallel v)^2} \tag{13}$$

emerge for the average number density $\mathcal{T}^0$ of axions and the z-component $\mathcal{T}^3$ of their average number flux. The x-component and y-component of both fluxes are zero because $\int d^2k\, \boldsymbol{k}_\perp \delta^{(2)}(\boldsymbol{k}_\perp)^2 = 0$ using the regularisation described above. Indeed, the pointwise dependence of the source has no preferred direction in the xy-plane (it is invariant under rotation around the z-axis), so it cannot produce a vector whose projection in the xy-plane is non-zero.

3. Source generated by a maximum amplitude plasma wave

The Fourier coefficient ϱ_n is determined by the relationship

$$\varrho_n = gB\frac{m_e}{\gamma e}\frac{2\pi i n}{l}\xi_n \tag{14}$$

following from $\varrho = -gBE$ alongside the expression

$$E = -\frac{m_e}{\gamma e}\frac{d\xi}{d\zeta} \tag{15}$$

for the electric field E of the wake of the laser pulse, where $m_e\xi$ is the relativistic energy of the plasma electrons in the wake, m_e is the rest mass of an electron and e is the elementary charge. The coefficient ξ_n in the Fourier series

$$\xi(\zeta) = \sum_n \xi_n \exp\left(2\pi i n \frac{\zeta}{l}\right) \tag{16}$$

and the period l of the wake when the electric field has its maximum amplitude satisfy

$$\xi_n \approx \begin{cases} \dfrac{4\gamma^2}{3} & \text{for } n = 0, \\ \dfrac{-4\gamma^2}{\pi^2 n^2} & \text{for } n \neq 0, \end{cases} \tag{17}$$

$$l \approx \frac{4\sqrt{2\gamma}}{\omega_p} \tag{18}$$

when $\gamma \gg 1$, which is the situation in laser-driven (or particle beam-driven) wakefield accelerators. The derivation of (15), (17) and (18) can be found in Ref. [22].

4. Analysis of the axion production

As shown in Appendix A.5, (10) coincides precisely with the expression for the cycle-averaged energy flux obtained in Ref. [22]. This agreement follows because the source is periodic, so the average can be calculated by integrating over any whole number of cycles. Furthermore, in Ref. [22], the cycle-averaged number flux was estimated by dividing the cycle-averaged energy flux by the relativistic energy $m_\Psi\gamma$ of an axion co-moving with the plasma wave. However, the quantum theoretic approach adopted here naturally leads to the expression (13) for the average number flux. Like the average momentum flux, the average number flux can be understood as a cycle-averaged quantity; in particular, it is the cycle-averaged number flux.

Substituting (17) and (18) in (12) and (13) yields the approximations

$$\mathcal{T}^0 \approx \frac{16\sqrt{2}g^2 B^2 m_e^2}{e^2\omega_p \pi^5}\gamma^{5/2} \sum_{n\neq 0} \frac{1}{n^2\sqrt{n^2+\sigma^2}(\sqrt{n^2+\sigma^2} - nv)^2}, \tag{19}$$

$$\mathcal{T}^3 \approx \frac{16\sqrt{2}g^2 B^2 m_e^2}{e^2\omega_p \pi^5}\gamma^{5/2} \sum_{n\neq 0} \frac{1}{n(n^2+\sigma^2)(\sqrt{n^2+\sigma^2} - nv)^2} \tag{20}$$

for the average number density $\mathcal{T}^0$ and average number flux $\mathcal{T}^3$, where the variable σ,

$$\sigma = \frac{m_\Psi l}{2\pi} \approx \frac{2\sqrt{2\gamma}\, m_\Psi}{\pi\omega_p}, \tag{21}$$

is the ratio of the wavelength l of the wake and the Compton wavelength $2\pi/m_\Psi$ (since $\hbar = 1$ and $c = 1$) of the axion. The summations in (19) and (20) are over all of the non-zero integers.

The estimate for the cycle-averaged number flux introduced in Ref. [22] diverges in the limit as the axion mass tends to zero. However, (19) and (20) are both well-behaved in this limit. Since

$$\begin{aligned}\lim_{\sigma\to 0}\sum_{n\neq 0}\frac{1}{n^2\sqrt{n^2+\sigma^2}(\sqrt{n^2+\sigma^2}-nv)^2} &= 2(1+v^2)\gamma^4\sum_{n=1}^{\infty}\frac{1}{n^5}\\ &\approx 4\gamma^4\zeta_R(5)\end{aligned}\tag{22}$$

and

$$\begin{aligned}\lim_{\sigma\to 0}\sum_{n\neq 0}\frac{1}{n(n^2+\sigma^2)(\sqrt{n^2+\sigma^2}-nv)^2} &= 4v\gamma^4\sum_{n=1}^{\infty}\frac{1}{n^5}\\ &\approx 4\gamma^4\zeta_R(5)\end{aligned}\tag{23}$$

hold for $\gamma \gg 1$, with ζ_R the Riemann zeta function, it follows

$$\begin{aligned}\lim_{\sigma\to 0}\mathcal{T}^0 &\approx \lim_{\sigma\to 0}\mathcal{T}^3\\ &\approx \frac{(2\gamma)^{13/2}g^2B^2m_e^2}{e^2\omega_p\pi^5}\zeta_R(5)\end{aligned}\tag{24}$$

using (19) and (20). We see that the beam of axions in the low mass limit, with $\gamma \gg 1$, is characterised by a lightlike number four-current.

Although the summation in (20) cannot be immediately expressed in closed form for general σ, upper and lower bounds containing only elementary functions can be readily obtained. Since

$$\sum_{n\neq 0}\frac{1}{n(n^2+\sigma^2)(\sqrt{n^2+\sigma^2}-nv)^2} = 4v\gamma^4\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+\sigma^2}}\frac{1}{(n^2+\gamma^2\sigma^2)^2}\tag{25}$$

and $1/\sqrt{n^2+\sigma^2} < 1$, it follows $\mathfrak{f}_\downarrow < \mathfrak{f} < \mathfrak{f}_\uparrow$ where

$$\mathfrak{f} = \sum_{n\neq 0}\frac{1}{n(n^2+\sigma^2)(\sqrt{n^2+\sigma^2}-nv)^2},\tag{26}$$

$$\mathfrak{f}_\downarrow = 4v\gamma^4\sum_{n=1}^{\infty}\frac{1}{(n^2+\sigma^2)(n^2+\gamma^2\sigma^2)^2},\tag{27}$$

$$\mathfrak{f}_\uparrow = 4v\gamma^4\sum_{n=1}^{\infty}\frac{1}{(n^2+\gamma^2\sigma^2)^2}.\tag{28}$$

The summations in $\mathfrak{f}_\downarrow$ and $\mathfrak{f}_\uparrow$ yield the expressions

$$\mathfrak{f}_\downarrow = \frac{\pi\gamma(1-3\gamma^2)\sigma\coth(\pi\gamma\sigma)-\pi^2\gamma^4v^2\sigma^2\,\mathrm{csch}^2(\pi\gamma\sigma)+2\pi\gamma^4\sigma\coth(\pi\sigma)-2\gamma^4v^4}{\gamma^4v^3\sigma^6}\tag{29}$$

and

$$\mathfrak{f}_\uparrow = v\frac{\pi^2\gamma^2\sigma^2\,\mathrm{csch}^2(\pi\gamma\sigma)+\pi\gamma\sigma\coth(\pi\gamma\sigma)-2}{\sigma^4}.\tag{30}$$

Numerical investigation suggests $\mathfrak{f}_\downarrow$ is a better approximation to $\mathfrak{f}$ than $\mathfrak{f}_\uparrow$ in the parameter range of interest ($\gamma \gg 1$ and $\gamma\sigma \lesssim 10$). Since (17) and (18) are only valid for $\gamma \gg 1$, it is reasonable to substitute $\sigma = s/\gamma$ in (29) and focus on the dominant behaviour in γ. The resulting approximation to the average number flux is

$$\mathcal{T}^3 \approx \frac{(2\gamma)^{13/2} g^2 B^2 m_e^2 c^4}{\mu_0^2 e^2 \omega_p \pi^5} \frac{12 + 2\pi^2 s^2 - 3\pi^2 s^2 \operatorname{csch}^2(\pi s) - 9\pi s \coth(\pi s)}{12\, s^6} \tag{31}$$

where

$$s = \frac{(2\gamma)^{3/2} m_\Psi c^2}{\pi \hbar \omega_p} \tag{32}$$

and the speed of light c, reduced Planck constant $\hbar$ and permeability of the vacuum μ_0 have been restored.

5. Estimate of the axion flux downstream of the plasma

In the experimental set-up proposed in Ref. [22], the axions produced by the laser wakefield accelerator exit the plasma and, for the purpose of detection, convert into photons in a transverse static magnetic field downstream of the plasma. The axions are accompanied by the laser pulse and ordinary matter so, for the experiment to be viable, the laser pulse and ordinary matter must be filtered out. Hence, a beam dump should be located between the laser wakefield accelerator and the transverse static magnetic field. The interaction between the axions and the beam dump is negligible, ensuring the axions readily propagate into the region containing the transverse static magnetic field.

Although (31) is reasonable inside the plasma, it only remains applicable close to where the axions emerge from the plasma. The waist radius w (the radius of the transverse cross-section) of the axion beam will be similar to the waist radius w_{wake} of the wake when the axions are inside the plasma, but will increase as the axions propagate through free space and the beam dump. The axion beam inherits the wavelength of the wake, so a simple estimate of the waist radius is given by $w(z) = w_{\text{wake}}\sqrt{1 + z^2/z_R^2}$, where z is the distance from the end of the plasma and $z_R = \pi w_{\text{wake}}^2/l$ is the Rayleigh length of a Gaussian beam with wavelength l. The axion number is a conserved quantity in free space, so an estimate of the axion number flux at z is given by $\Phi = \mathcal{T}^3 w_{\text{wake}}^2/w^2$.

The parameters $\omega_p = 2\pi \times 10^{13}\,\text{rad}\,\text{s}^{-1}$, $\gamma = 100$ and $w_{\text{wake}} = 31\,\mu\text{m}$ are representative of a typical laser wakefield accelerator [23]. Laser-driven systems exist for generating a magnetic field of strength up to $B = 800\,\text{T}$ whose spatial and temporal behaviour compared to the properties of the wake are approximately constant [25]. Representative values of the axion-photon coupling strength and axion mass are $g = 0.66 \times 10^{-10}\,\text{GeV}^{-1}$ and $m_\Psi = 1 \times 10^{-5}\,\text{eV}/c^2$, respectively; those values are on the boundary of the parameter region excluded by the CAST helioscope. Noting $l \approx 4\sqrt{2\gamma}\,c/\omega_p$, the above parameter choices lead to the expression $\Phi \approx 10^{13}\,(1\,\text{m}/z)^2\,\text{cm}^{-2}\,\text{s}^{-1}$ for the axion flux at distance z from the end of the plasma.

The above considerations suggest Φ should be tolerable at practical laboratory-scale distances from the source. The solar axion flux at the Earth is expected to be $g_{10}^2\, 3.75 \times 10^{11}\,\text{cm}^{-2}\,\text{s}^{-1}$, where $g_{10} = g\, 10^{10}\,\text{GeV}$, so we conclude Φ may be greater than the solar axion flux at $z \lesssim 8\,\mathrm{m}$. This distance compares well with the length ($\sim 9\,\mathrm{m}$) of the dipole magnet used to convert solar axions into photons in the CAST experiment. Although, unlike the continuous flux of axions from the Sun, the flux of axions produced by a laser wakefield accelerator is pulsed, it is notable that the L3-HAPLS laser system at ELI Beamlines [10] can, in principle, produce a train of pulses suitable for driving wakefields with a repetition rate of 10 Hz. Even so, it is inevitable that the time-integrated signal at a detector will be lower than for the photons produced from solar axions. However, unlike the flux of solar axions, the flux of axions from a laser wakefield accelerator can be readily controlled by altering the strength of the magnetic field within the plasma.

In general, the response of the axion flux Φ to changes in the laser-plasma parameters and the axion mass is complicated by the non-polynomial dependence of Φ on the dimensionless parameter s. However, since $s = 0.22$ for the above choice of parameters, insight into the response of Φ to small deviations in those parameters can be readily gained by truncating the Maclaurin expansion

$$\frac{12 + 2\pi^2 s^2 - 3\pi^2 s^2 \operatorname{csch}^2(\pi s) - 9\pi s \coth(\pi s)}{12\, s^6} = \frac{\pi^6}{945} - \frac{\pi^8}{4725} s^2 + \mathcal{O}(s^4). \tag{33}$$

The estimate

$$\Phi \approx \frac{(2\gamma)^{13/2} g^2 B^2 m_e^2 c^4}{\mu_0^2 e^2 \omega_p \pi^5} (1.0 - 2.0 s^2) \frac{z_R^2}{z^2} \tag{34}$$

for the axion flux at $z \gg z_R$ follows by discarding $\mathcal{O}(s^4)$ terms in (33). The numerical coefficients in the series in (34) are accurate to one decimal place, and it is clear that, when $s = 0.22$, the axion flux is close to its maximum at $s = 0$. As such, the axion flux can be considered constant for $m_\Psi \lesssim 1 \times 10^{-5}\,\mathrm{eV}/c^2$ with the above choice of laser-plasma parameters. However, for larger values of s, it is best to retain the exact expression

$$\Phi \approx \frac{(2\gamma)^{13/2} g^2 B^2 m_e^2 c^4}{\mu_0^2 e^2 \omega_p \pi^5} \frac{12 + 2\pi^2 s^2 - 3\pi^2 s^2 \operatorname{csch}^2(\pi s) - 9\pi s \coth(\pi s)}{12\, s^6} \frac{z_R^2}{z^2} \tag{35}$$

for the estimate of the axion flux. For example, Φ is lowered by 75% when $m_\Psi = 4.7 \times 10^{-5}\,\mathrm{eV}/c^2$ (which corresponds to $s = 1.03$). More broadly, increasing s alone always reduces Φ. Excluding s, the remaining parameter dependences in (35) are power laws; for example, a 75% reduction in Φ follows from a 50% reduction in the applied magnetic field B.

6. Conclusion

Key differences between the results of a quantum-theoretical analysis of axion production by a laser wakefield accelerator and those of an entirely classical approach [22] have been uncovered. Unlike the results in Ref. [22], the analytical approximation to the average number flux developed here converges in the limit as the axion mass tends to zero. The estimate of the average number flux in Ref. [22] arises from the ratio of the average energy flux and the relativistic energy of a massive classical axion propagating at the same velocity as the wake; as such, the estimate diverges as the rest mass of the axion tends to zero. However, the estimate calculated here emerges from a sum over contributions corresponding to momentum states. Each term in the sum is finite, even if the axion is massless, and the sum converges.

It follows that the estimates presented here are more reliable than those obtained in Ref. [22] when applied to very low mass axions. A lower bound on the average number flux of axions, valid for general values of the axion mass, was determined, and the response of the average number flux to changes in those parameters was described. In principle, the one-dimensional quasi-static model of the wake used here could be replaced by a more realistic model describing a three-dimensional dynamical structure [26, 27, 28] of finite extent containing trapped electrons. However, accurately determining the integral (2) corresponding to such a model would probably be a challenging task.

The development of multi-petawatt laser facilities offers new opportunities in the search for experimental signatures of extensions to the Standard Model of particle physics. Axions are of particular interest in this context because their very weak coupling to ordinary matter ensures they are inaccessible to conventional particle collider experiments, and their low mass is well-suited to interaction with electromagnetic fields. Although the significance of laser wakefield acceleration in the development of new high-energy particle accelerators has been firmly established [29], the variant of the "light shining through wall" class of experiments discussed here highlights the broader role they could play at the forefront of particle physics.

Appendix A. Preliminary considerations

Before turning to quantum theory, it is useful to initially adopt a fully classical perspective in order to establish key concepts that are common to classical and quantum approaches.

Appendix A.1. Classical theory

A classical axion field Ψ satisfies

$$\partial_t^2 \Psi - \nabla^2 \Psi + m_\Psi^2 \Psi = \rho \tag{A.1}$$

where the source is given by $\rho = -g\mathbf{E} \cdot \mathbf{B}$, with m_Ψ the mass of the axion and g the axion-photon coupling. Rather than introducing Fourier integral representations from the outset, it is convenient to restrict the fields to a box and impose periodic boundary conditions at the walls of the box. Fourier transforms emerge as the size of the box tends to infinity. We begin with the Fourier series

$$\Psi(\mathbf{x}, t) = \sum_{\mathbf{k}} \frac{1}{\sqrt{V}} \widetilde{\Psi}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A.2}$$

$$\rho(\mathbf{x}, t) = \sum_{\mathbf{k}} \frac{1}{\sqrt{V}} \widetilde{\rho}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A.3}$$

for Ψ and ρ, where V is the volume of the box containing the fields.

Inspection of (A.1) shows that the coefficients $\widetilde{\Psi}_{\mathbf{k}}$ and $\widetilde{\rho}_{\mathbf{k}}$ in (A.2) and (A.3) satisfy the ordinary differential equation

$$\ddot{\widetilde{\Psi}}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 \widetilde{\Psi}_{\mathbf{k}} = \widetilde{\rho}_{\mathbf{k}}, \tag{A.4}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m_\Psi^2}$. The solution to (A.4) satisfying the conditions $\widetilde{\Psi}_{\mathbf{k}}(t_{\mathrm{i}}) = 0$ and $\dot{\widetilde{\Psi}}_{\mathbf{k}}(t_{\mathrm{i}}) = 0$ at the initial time t_{i} is

$$\widetilde{\Psi}_{\mathbf{k}}(t) = \int_{t_{\mathrm{i}}}^{t} dt' \, \frac{\sin\left(\omega_{\mathbf{k}}(t - t')\right)}{\omega_{\mathbf{k}}} \, \widetilde{\rho}_{\mathbf{k}}(t'). \tag{A.5}$$

The total four-momentum $\mathcal{P}^\mu$ of the axion field according to observers at rest in the laboratory frame is

$$\mathcal{P}^\mu = \int d^3x \, T_0{}^\mu \tag{A.6}$$

where the integral is over the interior of the box only and $T_{\mu\nu}$ is the stress tensor

$$T_{\mu\nu} = \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2}(\eta^{\lambda\omega} \partial_\lambda \Psi \partial_\omega \Psi - m_\Psi^2 \Psi^2) \eta_{\mu\nu}. \tag{A.7}$$

Greek indices are lowered and raised using the metric $\eta_{\mu\nu}$ and the inverse metric $\eta^{\mu\nu}$, respectively, with signature -2, i.e.

$$\eta_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu = 0, \\ -1 & \text{for } \mu = \nu \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{A.8}$$

Thus, the total energy $\mathcal{E}$ and total three-momentum $\boldsymbol{\mathcal{P}}$ at time t are

$$\mathcal{E}(t) = \int d^3x \, \frac{1}{2}\Big((\partial_t \Psi)^2 + (\boldsymbol{\nabla} \Psi)^2 + m_\Psi^2 \Psi^2\Big) \tag{A.9}$$

$$\boldsymbol{\mathcal{P}}(t) = -\int d^3x \, \partial_t \Psi \boldsymbol{\nabla} \Psi. \tag{A.10}$$

Substituting the expressions

$$\partial_t \Psi = \sum_{\mathbf{k}} \frac{1}{\sqrt{V}} \dot{\widetilde{\Psi}}_{\mathbf{k}} \, e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \boldsymbol{\nabla} \Psi = \sum_{\mathbf{k}} \frac{1}{\sqrt{V}} i\mathbf{k} \widetilde{\Psi}_{\mathbf{k}} \, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A.11}$$

into (A.9) and (A.10), and using

$$\int d^3x \exp\left(i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{x}\right) = V \delta_{\mathbf{k}\mathbf{k}'} \tag{A.12}$$

where $\delta_{\mathbf{k}\mathbf{k}'}$ is the Kronecker delta, gives

$$\mathcal{E} = \sum_{\mathbf{k}} \frac{1}{2}\Big(\dot{\widetilde{\Psi}}_{\mathbf{k}} \dot{\widetilde{\Psi}}_{-\mathbf{k}} + \omega_\Psi^2 \widetilde{\Psi}_{\mathbf{k}} \widetilde{\Psi}_{-\mathbf{k}}\Big), \tag{A.13}$$

$$\boldsymbol{\mathcal{P}} = \sum_{\mathbf{k}} \frac{i\mathbf{k}}{2}\Big(\dot{\widetilde{\Psi}}_{\mathbf{k}} \widetilde{\Psi}_{-\mathbf{k}} - \dot{\widetilde{\Psi}}_{-\mathbf{k}} \widetilde{\Psi}_{\mathbf{k}}\Big), \tag{A.14}$$

respectively. Note that the domain of each component of $\mathbf{k}$ is symmetric, and the sum over $\mathbf{k}$ includes all values of every component. Thus, only the symmetric part of any summand contributes to the overall result. This observation is responsible for the form of the summand shown in (A.14).

Introducing the notation

$$[f_{\mathbf{k}}](t) = \int_{t_{\mathrm{i}}}^{t} dt'\, f_{\mathbf{k}}(t')\, e^{-i\omega_{\mathbf{k}} t'} \tag{A.15}$$

allows (A.5) to be written as

$$\widetilde{\Psi}_{\mathbf{k}} = \frac{1}{2i\omega_{\mathbf{k}}}(e^{i\omega_{\mathbf{k}} t}[\widetilde{\rho}_{\mathbf{k}}] - e^{-i\omega_{\mathbf{k}} t}[\widetilde{\rho}^{*}_{\mathbf{k}}]^{*}). \tag{A.16}$$

Furthermore,

$$\dot{\widetilde{\Psi}}_{\mathbf{k}} = \int_{t_{\mathrm{i}}}^{t} dt'\, \cos\left(\omega_{\mathbf{k}}(t - t')\right) \widetilde{\rho}_{\mathbf{k}}(t') \tag{A.17}$$

follows from (A.5), which can be written as

$$\dot{\widetilde{\Psi}}_{\mathbf{k}} = \frac{1}{2}(e^{i\omega_{\mathbf{k}} t}[\widetilde{\rho}_{\mathbf{k}}] + e^{-i\omega_{\mathbf{k}} t}[\widetilde{\rho}^{*}_{\mathbf{k}}]^{*}). \tag{A.18}$$

Substituting (A.16) and (A.18) into (A.13) yields

$$\mathcal{E} = \sum_{\mathbf{k}} \frac{1}{4}([\widetilde{\rho}^{*}_{\mathbf{k}}]^{*}[\widetilde{\rho}_{-\mathbf{k}}] + [\widetilde{\rho}_{\mathbf{k}}][\widetilde{\rho}^{*}_{-\mathbf{k}}]^{*}) \tag{A.19}$$

where $\omega_{-\mathbf{k}} = \omega_{\mathbf{k}}$ has been used. However, the identity

$$\sum_{\mathbf{k}} [\widetilde{\rho}_{\mathbf{k}}][\widetilde{\rho}^{*}_{-\mathbf{k}}]^{*} = \sum_{\mathbf{k}} [\widetilde{\rho}_{-\mathbf{k}}][\widetilde{\rho}^{*}_{\mathbf{k}}]^{*} \tag{A.20}$$

follows because, as noted previously, only the symmetric part of any summand contributes to the final result. Furthermore, the source ρ is real so $\tilde{\rho}_{-\mathbf{k}} = \tilde{\rho}^{*}_{\mathbf{k}}$, and so (A.19) can be expressed as

$$\mathcal{E} = \sum_{\mathbf{k}} \frac{1}{2}|[\widetilde{\rho}^{*}_{\mathbf{k}}]|^{2}. \tag{A.21}$$

Likewise, substituting (A.16) and (A.18) in (A.14) gives

$$\boldsymbol{\mathcal{P}} = \sum_{\mathbf{k}} \frac{\mathbf{k}}{4\omega_{\mathbf{k}}}([\widetilde{\rho}^{*}_{\mathbf{k}}]^{*}[\widetilde{\rho}_{-\mathbf{k}}] - [\widetilde{\rho}_{\mathbf{k}}][\widetilde{\rho}^{*}_{-\mathbf{k}}]^{*}). \tag{A.22}$$

which can be reduced to

$$\boldsymbol{\mathcal{P}} = \sum_{\mathbf{k}} \frac{\mathbf{k}}{2\omega_{\mathbf{k}}}|[\widetilde{\rho}^{*}_{\mathbf{k}}]|^{2} \tag{A.23}$$

because the domain of the sum is symmetric and $\widetilde{\rho}_{-\mathbf{k}} = \widetilde{\rho}^{*}_{\mathbf{k}}$.

Inspection of (A.21) and (A.23) reveals the result

$$\mathcal{P}^{\mu} = \sum_{\mathbf{k}} \frac{k^{\mu}}{2\omega_{\mathbf{k}}}|[\widetilde{\rho}^{*}_{\mathbf{k}}]|^{2} \tag{A.24}$$

for the total four-momentum $\mathcal{P}^{\mu}$ of the axion field according to observers at rest in the laboratory frame, where the wave four-vector k^{μ} is $(\omega_{\mathbf{k}}, \mathbf{k})$.

Note that $\mathcal{P}^\mu$ precisely agrees with $\mathcal{E}^{0\mu}$, where $\mathcal{E}^{\mu\nu}$ is given by

$$\mathcal{E}^{\mu\nu} = \sum_{\mathbf{k}} \frac{k^\mu k^\nu}{\omega_{\mathbf{k}}} \frac{|[\widetilde{\rho}^*_{\mathbf{k}}]|^2}{2\omega_{\mathbf{k}}}. \tag{A.25}$$

However, the purely spatial components of $\mathcal{E}^{\mu\nu}$ do not coincide with the purely spatial components of the integral of $T^{\mu\nu}$. The integral of $T^{\mu\nu}$ over all space evaluated at time t includes extra terms that are, in general, non-zero. But if ρ is zero at, and for all time after, some instant t_{f}, then those extra terms are sinusoidal. As such, their average over t, where $t \geq t_{\mathrm{f}}$, is zero.

Appendix A.2. Quantum theory

The quantum field operator $\hat{\Psi}(\mathbf{x}, t)$ and its canonical momentum $\hat{\Pi}(\mathbf{x}, t)$ satisfy the equal-time commutation relations

$$[\hat{\Psi}(\mathbf{x}, t), \hat{\Pi}(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{A.26}$$

$$[\hat{\Psi}(\mathbf{x}, t), \hat{\Psi}(\mathbf{x}', t)] = [\hat{\Pi}(\mathbf{x}, t), \hat{\Pi}(\mathbf{x}', t)] = 0 \tag{A.27}$$

with $\delta^{(3)}$ the 3-dimensional Dirac delta function. Their dynamics is determined by the Hamiltonian

$$\hat{H} = \int d^3x : \frac{1}{2}\Big(\hat{\Pi}^2 + (\boldsymbol{\nabla}\hat{\Psi})^2 + m_\Psi^2 \hat{\Psi}^2 - \rho\hat{\Psi}\Big) : \tag{A.28}$$

and the Heisenberg equation

$$i\dot{\hat{\mathcal{O}}} = [\hat{\mathcal{O}}, \hat{H}] \tag{A.29}$$

where the normal ordering in (A.28) is undertaken with respect to the ladder operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ introduced in the series

$$\hat{\Psi}(\mathbf{x}, t) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}} V}} \Big(\hat{a}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}\Big), \tag{A.30}$$

$$\hat{\Pi}(\mathbf{k}, t) = -i \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}}{2V}} \Big(\hat{a}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} - \hat{a}^\dagger_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}\Big). \tag{A.31}$$

The operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ satisfy the equal-time commutation relations

$$[\hat{a}_{\mathbf{k}}(t), \hat{a}^\dagger_{\mathbf{k}'}(t)] = \delta_{\mathbf{k}\mathbf{k}'}, \tag{A.32}$$

$$[\hat{a}_{\mathbf{k}}(t), \hat{a}_{\mathbf{k}'}(t)] = [\hat{a}^\dagger_{\mathbf{k}}(t), \hat{a}^\dagger_{\mathbf{k}'}(t)] = 0 \tag{A.33}$$

where $\delta_{\mathbf{k}\mathbf{k}'}$ is the Kronecker delta.

Introducing (A.30), (A.31) and (A.3) in (A.28) gives

$$\hat{H} = \sum_{\mathbf{k}} \Big(\omega_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} - \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (\widetilde{\rho}^*_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \widetilde{\rho}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}})\Big) \tag{A.34}$$

and so, using (A.29), the equation of motion for the annihilation operator $\hat{a}_{\mathbf{k}}$ is

$$i\dot{\hat{a}}_{\mathbf{k}} = \omega_{\mathbf{k}} \hat{a}_{\mathbf{k}} - \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \widetilde{\rho}_{\mathbf{k}}. \tag{A.35}$$

Hence,

$$\hat{a}_{\mathbf{k}}(t) = \hat{a}_{\mathbf{k}}(t_{\rm i})e^{-i\omega_{\mathbf{k}}(t-t_{\rm i})} + \frac{ie^{-i\omega_{\mathbf{k}}t}}{\sqrt{2\omega_{\mathbf{k}}}}\int_{t_{\rm i}}^{t} dt' e^{i\omega_{\mathbf{k}}t'}\widetilde{\rho}_{\mathbf{k}}(t') \tag{A.36}$$

and, likewise,

$$\hat{a}^{\dagger}_{\mathbf{k}}(t) = \hat{a}^{\dagger}_{\mathbf{k}}(t_{\rm i})e^{i\omega_{\mathbf{k}}(t-t_{\rm i})} - \frac{ie^{i\omega_{\mathbf{k}}t}}{\sqrt{2\omega_{\mathbf{k}}}}\int_{t_{\rm i}}^{t} dt' e^{-i\omega_{\mathbf{k}}t'}\widetilde{\rho}^{*}_{\mathbf{k}}(t') \tag{A.37}$$

follows for the creation operator $\hat{a}^{\dagger}_{\mathbf{k}}$.

Thus, introducing the vacuum state $|0, t_{\rm i}\rangle$ annihilated by the operator $\hat{a}_{\mathbf{k}}(t_{\rm i})$ leads to the result

$$\langle 0, t_{\rm i}|\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}|0, t_{\rm i}\rangle = \frac{|[\widetilde{\rho}^{*}_{\mathbf{k}}]|^2}{2\omega_{\mathbf{k}}} \tag{A.38}$$

for the expected number of axions with three-momentum $\mathbf{k}$ at time t. Equation (A.15) has been used to write (A.38) in a compact manner.

The operator $\hat{n}_{\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$ represents the number of axions at time t, and this leads to the expressions

$$\hat{\mathcal{E}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\hat{n}_{\mathbf{k}}, \tag{A.39}$$

$$\hat{\boldsymbol{\mathcal{P}}} = \sum_{\mathbf{k}} \mathbf{k}\,\hat{n}_{\mathbf{k}} \tag{A.40}$$

for the operators $\hat{\mathcal{E}}$ and $\hat{\boldsymbol{\mathcal{P}}}$ representing the total energy and total three-momentum of the axions, respectively. Inspection of (A.38) reveals that the expectations of $\hat{\mathcal{E}}$ and $\hat{\boldsymbol{\mathcal{P}}}$ in the state $|0, t_{\rm i}\rangle$ agree with the classical results (A.21) and (A.23). Ultimately, this result is a consequence of the fact that $|0, t_{\rm i}\rangle$ is an eigenstate of the annihilation operator $\hat{a}_{\mathbf{k}}$, as inspection of (A.36) shows. The classical source ρ produces a coherent state of axions.

The above considerations also suggest the introduction of the total number operator $\hat{\mathcal{N}}$ and total number three-current operator $\hat{\boldsymbol{\mathcal{J}}}$ given by

$$\hat{\mathcal{N}} = \sum_{\mathbf{k}} \hat{n}_{\mathbf{k}}, \tag{A.41}$$

$$\hat{\boldsymbol{\mathcal{J}}} = \sum_{\mathbf{k}} \frac{\mathbf{k}}{\omega_{\mathbf{k}}}\,\hat{n}_{\mathbf{k}}. \tag{A.42}$$

Collectively, the above results motivate the introduction of the family

$$\hat{\mathcal{E}}^{\mu\ldots\nu} = \sum_{\mathbf{k}} \frac{k^{\mu}\ldots k^{\nu}}{\omega_{\mathbf{k}}}\hat{n}_{\mathbf{k}}. \tag{A.43}$$

of symmetric tensor operators since

$$\hat{\mathcal{E}}^{00} = \hat{\mathcal{E}}, \quad \hat{\mathcal{E}}^{01} = \hat{\mathcal{P}}^{1}, \quad \hat{\mathcal{E}}^{02} = \hat{\mathcal{P}}^{2}, \quad \hat{\mathcal{E}}^{03} = \hat{\mathcal{P}}^{3}, \tag{A.44}$$

$$\hat{\mathcal{E}}^{0} = \hat{\mathcal{N}}, \quad \hat{\mathcal{E}}^{1} = \hat{\mathcal{J}}^{1}, \quad \hat{\mathcal{E}}^{2} = \hat{\mathcal{J}}^{2}, \quad \hat{\mathcal{E}}^{3} = \hat{\mathcal{J}}^{3}. \tag{A.45}$$

In addition to the above reasoning, the series (A.39) and (A.40) emerge from the expression

$$\hat{\mathcal{E}}^{0\mu} = \int d^3x : \hat{T}^{0\mu} : \tag{A.46}$$

where

$$\hat{T}^{00} = \frac{1}{2}\Big(\hat{\Pi}^2 + (\boldsymbol{\nabla}\hat{\Psi})^2 + m_\Psi^2\hat{\Psi}^2\Big), \tag{A.47}$$

$$\hat{T}^{01} = -\hat{\Pi}\,\partial_x\hat{\Psi}, \quad \hat{T}^{02} = -\hat{\Pi}\,\partial_y\hat{\Psi}, \quad \hat{T}^{03} = -\hat{\Pi}\,\partial_z\hat{\Psi} \tag{A.48}$$

are components of the quantum stress-energy-momentum tensor $\hat{T}^{\mu\nu}$. However, the remaining (i.e. purely spatial) components of $\hat{\mathcal{E}}^{\mu\nu}$ do not agree with the integral of the normal ordering of

$$\hat{T}^{ab} = \eta^{a\mu}\eta^{b\nu}\partial_\mu\hat{\Psi}\partial_\nu\hat{\Psi} - \frac{1}{2}(\hat{\Pi}^2 - (\boldsymbol{\nabla}\hat{\Psi})^2 - m_\Psi^2\hat{\Psi}^2)\eta^{ab}, \tag{A.49}$$

where $a, b = 1, 2, 3$. In general, the spatial integral of the normal ordering of any term quadratic in $\hat{\Pi}$, $\hat{\Psi}$, or their spatial derivatives, will yield terms containing $\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}$ as well as $\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}$. Even so, akin to the classical analysis, if ρ is zero for all $t \geq t_{\mathrm{f}}$ then inspection of (A.36) shows $\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \propto e^{-2i\omega_{\mathbf{k}}t}$ (and $\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \propto e^{2i\omega_{\mathbf{k}}t}$) when $t \geq t_{\mathrm{f}}$. In this case, the desired expression for $\hat{\mathcal{E}}^{ab}$ is recovered by averaging $\int d^3x : \hat{T}^{ab} :$ over $t \geq t_{\mathrm{f}}$. This result can be formally expressed as

$$\hat{\mathcal{E}}^{\mu\nu} = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} dt' \int d^3x : \hat{T}^{\mu\nu}(\mathbf{x}, t') : \tag{A.50}$$

subject to the condition $\rho(\mathbf{x}, t') = 0$ for all $t' > t$.

Finally, using (A.38) and (A.43), we note that the expectation of $\hat{\mathcal{E}}^{\mu\ldots\nu}$ in the state $|0, t_{\mathrm{i}}\rangle$ is

$$\langle 0, t_{\mathrm{i}}|\hat{\mathcal{E}}^{\mu\ldots\nu}|0, t_{\mathrm{i}}\rangle = \sum_{\mathbf{k}} \frac{k^\mu \ldots k^\nu}{\omega_{\mathbf{k}}} \frac{|[\widetilde{\rho}^*_{\mathbf{k}}]|^2}{2\omega_{\mathbf{k}}}. \tag{A.51}$$

Appendix A.3. The continuum limit

It is convenient in applications to express (A.51) in the continuum limit. Furthermore, we only require (A.51) at times after the source ρ is no longer active. Since (A.51) is constant for all $t \geq t_{\mathrm{f}}$, it is sufficient to calculate (A.51) at $t = t_{\mathrm{f}}$. We begin by noting

$$\widetilde{\rho}^*_{\mathbf{k}}(t_{\mathrm{f}}) = \frac{1}{\sqrt{V}}\int d^3x\,\rho(\mathbf{x}, t_{\mathrm{f}})\,e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A.52}$$

follows from (A.3) and the fact ρ is real (so $\widetilde{\rho}^*_{\mathbf{k}} = \widetilde{\rho}_{-\mathbf{k}}$). Hence

$$[\widetilde{\rho}^*_{\mathbf{k}}](t_{\mathrm{f}}) = \frac{1}{\sqrt{V}}\int_{t_{\mathrm{i}}}^{t_f} dt \int d^3x\,\rho(\mathbf{x}, t)\,e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}}t)} \tag{A.53}$$

using (A.15). The continuum limit is obtained using the replacement

$$\sum_{\mathbf{k}}\frac{1}{V} \to \int\frac{d^3k}{8\pi^3} \tag{A.54}$$

and so (A.51) yields

$$\langle 0,t_{\rm i}|\hat{\mathcal{E}}^{\mu\ldots\nu}(t_{\rm f})|0,t_{\rm i}\rangle \to \int \frac{d^3k}{8\pi^3}\frac{k^\mu\ldots k^\nu}{\omega_{\mathbf{k}}}\frac{1}{2\omega_{\mathbf{k}}}\bigg|\int_{t_{\rm i}}^{t_f} dt \int d^3x\,\rho(\mathbf{x},t)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\bigg|^2. \quad \text{(A.55)}$$

Equation (A.55) is the starting point for the calculations in the main text.

It is straightforward to verify (A.54) is correct by noting the substitution

$$\widetilde{f}_{\mathbf{k}} \to \frac{\widetilde{f}(\mathbf{k})}{\sqrt{V}} \quad \text{(A.56)}$$

emerges from a comparison of the expression

$$\widetilde{f}(\mathbf{k}) = \int d^3x f(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \text{(A.57)}$$

for the Fourier transform of $f(\mathbf{x})$ and the coefficient

$$\widetilde{f}_{\mathbf{k}} = \frac{1}{\sqrt{V}}\int d^3x f(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \text{(A.58)}$$

in the Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{V}}\widetilde{f}_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}}. \quad \text{(A.59)}$$

Hence, a comparison of the result

$$\int d^3x f(\mathbf{x})h(\mathbf{x}) = \sum_{\mathbf{k}} \widetilde{f}_{\mathbf{k}}\,\widetilde{h}_{-\mathbf{k}} \quad \text{(A.60)}$$

with the consequence

$$\int d^3x f(\mathbf{x})h(\mathbf{x}) = \int \frac{d^3k}{8\pi^3}\widetilde{f}(\mathbf{k})\,\widetilde{h}(-\mathbf{k}) \quad \text{(A.61)}$$

of Parseval's theorem confirms (A.54).

Appendix A.4. Regularisation of divergent integrals

The spatial integrals in (1) and (2) are divergent because, for mathematical convenience, we chose the wake in our model to occupy the entire box introduced in Appendix A.1. Divergences are removed by suitably identifying the formal expressions $\delta(0)$, $\delta^{(2)}(0)$ and $\delta^{(3)}(0)$ with finite quantities, and dividing out the volume of the box.

The details of the substitutions can be obtained by choosing the box to be a cube of length L. Prior to taking L to infinity, we have the following result:

$$\int_{-L/2}^{L/2} dx\,e^{-2\pi inx/L} = L\,\delta_{n0}$$

where n is an integer, with similar results for the y and z directions. However, the Fourier representation of the Dirac delta function ensures

$$\lim_{L\to\infty}\int_{-L/2}^{L/2} dx\,e^{-ik_xx} = 2\pi\delta(k_x)$$

and, thus, we can return to the finite-sized box by making the replacement

$$2\pi\delta(k_x) \to L\,\delta_{n0}$$

where $k_x \to 2\pi n/L$ is understood. Likewise, the formal expressions $4\pi^2\delta^{(2)}(0) = (2\pi\delta(0))^2$ and $8\pi^3\delta^{(3)}(0) = (2\pi\delta(0))^3$ correspond to the cross-sectional area $A = L^2$ and volume $V = L^3$ of the box, respectively.

Appendix A.5. Comparison with a previous result

It is straightforward to show that the quantity $\mathcal{T}^{03}$ given in (10) is the cycle-averaged energy flux previously calculated in Ref. [22]. The sum in (10) is over all integer n and, as such, only the symmetric part of the summand contributes to the final result. It follows

$$\begin{aligned} \mathcal{T}^{03} &= \sum_n \frac{k_\parallel |\varrho_n|^2}{4\omega}\Big(\frac{1}{(\omega - k_\parallel v)^2} - \frac{1}{(\omega + k_\parallel v)^2}\Big) \\ &= \sum_n \frac{k_\parallel^2 v |\varrho_n|^2}{(\omega^2 - k_\parallel^2 v^2)^2} \end{aligned} \tag{A.62}$$

because $|\varrho_n|^2$ and $\omega = \sqrt{k_\parallel^2 + m_\Psi^2}$ are even in n, whilst $k_\parallel = 2\pi n/l$ is odd in n. The symmetry of $|\varrho_n|^2 = \varrho_n^* \varrho_n$ follows immediately from $\varrho_n^* = \varrho_{-n}$, which is a consequence of ϱ being real. Thus

$$\mathcal{T}^{03} = \sum_n \frac{4\pi^2 n^2 l^2 \gamma^4 v}{(4\pi^2 n^2 + m_\Psi^2 \gamma^2 l^2)^2} |\varrho_n|^2 \tag{A.63}$$

where the Lorentz factor $\gamma = 1/\sqrt{1 - v^2}$ of the wake of the laser pulse has been introduced. Substituting ϱ_n using (14) yields

$$\mathcal{T}^{03} = \sum_n \frac{16\pi^4 n^4 g^2 B^2 \gamma^2 v}{(4\pi^2 n^2 + m_\Psi^2 \gamma^2 l^2)^2} \frac{m_e^2}{e^2} |\xi_n|^2 \tag{A.64}$$

which is identical to the expression obtained in Ref. [22] for the cycle-averaged energy flux.

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