A Comment on "Monotone Comparative Statics"

Rabah Amir*& David Rietzke†

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Abstract

Milgrom and Shannon (1994) provide sufficient conditions in parameterized optimization problems that guarantee solution sets are globally monotone in the parameter. We show that these sufficient conditions may be relaxed when focusing on discrete, binary comparisons between solution sets. Our approach relies upon a novel method of embedding a new optimization problem "between" the two original problems of interest. In smooth problems, our sufficient conditions may be verified by elementary differential comparisons, making them well-suited for applied work; we illustrate this with several applications.

Keywords: Monotone comparative statics, quasi-supermodularity, single-crossing property JEL Classifications: C61, C65, D42

*Department of Economics, University of Iowa, 21 E Market St, Iowa City, IA 52242, rabah-amir@uiowa.edu

†Department of Economics, Lancaster University Management School, Bailrigg, Lancaster, LA1 4YX, United Kingdom, d.rietzke@lancaster.ac.uk

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1 Introduction

The theory of monotone comparative statics (MCS) relies solely on two complementarity conditions on the objective function, one between each decision and the parameter and the other between the components of the decision (Topkis, 1978). In the ordinal theory (Milgrom and Shannon, 1994, henceforth, MS), these are respectively the single-crossing property (SCP) and quasi-supermodularity (QSM).[1] Intuitively, it is instructive to decompose the overall effect of a parameter increase on the optimal choice into two distinct channels. First, there is a direct effect reflecting how each component changes due to the parameter change, holding the others constant. Second, there is an indirect effect that captures the mutual interactions between the different choice components. In this perspective, the logic of MS's Monotonicity Theorem is thus: The SCP implies that the direct effect is positive while QSM of the objective in the choice variables ensures that the indirect effect reinforces the direct effect via own complementarities.

It is common in applied theory to compare the behavior of an economic agent in distinct environments (e.g., boom vs. bust), regulatory regimes (e.g., a subsidy vs. a tax) or draw comparisons between the behavior of two distinct decision makers with related objectives (e.g., a firm vs. a social planner). Such comparisons are fundamentally discrete, often binary, and involve comparing optimal behavior under environments that reflect what one might loosely call a regime change. In such cases, a joint parametric representation of the two problems may not even be possible.

While such problems could be handled as special cases of the existing theory by considering a two-parameter space and requiring both complementarity conditions, we show that these conditions may be relaxed. Our approach entails the construction of a new auxiliary problem "between" the two original problems, with a QSM objective function (in the choice variables). The in-betweeness property is defined by the new objective function lying between the original two according to a SCP order (roughly, the higher objective is increasing between any two actions whenever the lower one is). While the SCP and QSM remain central to the theory, the novelty is that QSM is needed only for the auxiliary objective function, not the original two. Constructing this auxiliary problem is then a critical step in any successful application of this result.

In order to provide some insight on how this key in-between problem might be constructed from the original two, we specify a prototypical setting for applications.

Consider two problems of interest of the form $\max\{F(a_1, a_2) + tf(a_1, a_2)\}$ with $t < 0$

[1]Shannon (1995) studies variants of SCP and QSM that yield weaker or stronger MCS than MS.

(e.g., a tax) for one and $t > 0$ (a subsidy) for the other. The question is to compare the argmax (a_1^*, a_2^*) sets under these two distinct regimes. To this end, the in-between problem is the no-policy (or $t = 0$) case. Our main (three-way-comparison) result implies that if F is QSM and f is increasing in (a_1, a_2), the optimal choice set is strong-set higher under a subsidy (than under no policy) than under a tax.[2] The novelty is that no complementarity assumption on f is needed. An ancillary benefit is that, in contrast to existing theory, one need not resort to cardinal complementarity conditions on F to ensure their preservation under sums for additive problems, ubiquitous in economic theory. Thus, our results enhance the usefulness of MS's ordinal approach to MCS in a broad class of economic applications (also see Quah and Strulovici, 2012).

In a nutshell, the new approach pertains to problems of discrete comparative statics involving some form of regime change. In applications, a key step consists of identifying a suitable in-between problem satisfying QSM, for which the above procedure is a prototypical example. Both of these constructions underscore the necessary role of some form of QSM, though not for the original objective functions. With the relaxation of QSM being the main novelty, we closely follow MS in considering necessary and sufficient conditions for MCS, but allowing for simultaneous changes (rather than separate ones in MS) in the choice set and the objective function. Along the way, we develop two binary relations on functions based on QSM and the SCP properties and provide useful comparisons between the two, which are of independent interest.

Last but not least, we illustrate the scope of our results, their value added over existing theory, and relative ease of application via two economic applications.

2 Preliminaries

Let X be a non-empty lattice with partial order $\leq$. Let $\mathscr{P}(X) = 2^X \setminus \{\emptyset\}$ denote the set of all non-empty subsets of X and $\mathscr{L}(X)$ the set of all non-empty sublattices of X. For $A, B \in \mathscr{P}(X)$, A is lower than B in the strong set order, $A \leq_s B$, if $a \in A$ and $b \in B$ implies $a \wedge b \in A$ and $a \vee b \in B$. The strong set order, $\leq_s$, is transitive and antisymmetric on $\mathscr{P}(X)$ and also reflexive on $\mathscr{L}(X)$ (Topkis, 1978). We say A is completely lower than B, $A \leq_c B$, if $a \in A$ and $b \in B$ implies $a \leq b$. Clearly, $A \leq_c B \implies A \leq_s B$; moreover, if $A \leq_c B \leq_s C$ or $A \leq_s B \leq_c C$, then $A \leq_c C$.

Let $\mathscr{F}_X$ denote the set of all real-valued functions on X: $\mathscr{F}_X = \{f | f : X \to \mathbb{R}\}$. $f \in \mathscr{F}_X$ is supermodular (SPM) if for all $x', x'' \in X$, $f(x') + f(x'') \leq f(x' \wedge x'') + f(x' \vee x'')$;

[2] More generally, $tf(a_1, a_2)$ may be replaced by $f(a_1, a_2, t)$ as long as $f(a_1, a_2, 0) = 0$.

f is QSM if for all $x', x'' \in X$, $f(x' \wedge x'') \leq (<) f(x') \implies f(x'') \leq (<) f(x' \vee x'')$. If Y is a poset, $f : X \times Y \to \mathbb{R}$ has the SCP in $(x; y)$ if for all $x' < x''$ and $y' < y''$, $f(x', y') \leq (<) f(x'', y') \implies f(x', y'') \leq (<) f(x'', y'')$. For binary comparisons between solution sets of optimization problems, the following relation is needed.

Definition 1. Let $f, g \in \mathscr{F}_X$. g is single-crossing dominated by f ($g \leq_{sc} f$) if, for all $x' < x''$, $g(x') \leq (<) g(x'') \implies f(x') \leq (<) f(x'')$.

We say that g is strictly single-crossing dominated by f ($g <_{sc} f$) if for all $x' < x''$, $g(x') \leq g(x'') \implies f(x') < f(x'')$. Intuitively, whenever g is (weakly) increasing, f is strictly so. If $g \leq_{sc} f$ and $f \leq_{sc} g$ we write $f \sim_{sc} g$. If g is a monotonic transformation of f then $f \sim_{sc} g$. Though not antisymmetric, the relation $\leq_{sc}$ is transitive and reflexive on $\mathscr{F}_X$. Moreover, single-crossing dominance is linked to the SCP in that $g \leq_{sc} f$ if and only if $h = tf + (1-t)g$ has the SCP in $(x; t)$ on $X \times \{0, 1\}$.

For use in applications, we introduce the cardinal analogs. We say g is (strictly) ID-dominated by f, written $g \leq_{id} f$ ($g <_{id} f$), if for all $x' < x''$, $g(x'') - g(x') \leq (<) f(x'') - f(x')$. If $X \subset \mathbb{R}^n$ and f and g are differentiable then $g \leq_{id} f$ if and only if $\nabla g \leq \nabla f$;[3] moreover, $\nabla g \ll \nabla f$ implies $g <_{id} f$. Finally, see that $g \leq_{id} (<_{id}) f \implies g \leq_{sc} (<_{sc}) f$. The next relation plays a key role in our analysis.

Definition 2. Let $f, g : X \to \mathbb{R}$. g is QSM-dominated by f ($g \leq_{qsm} f$) if, for all $x', x'' \in X$, $g(x' \wedge x'') \leq (<) g(x') \implies f(x'') \leq (<) f(x' \vee x'')$.

If $g \leq_{qsm} f$ and $f \leq_{qsm} g$, we write $f \sim_{qsm} g$. We say g is strictly QSM dominated by f, written $g <_{qsm} f$, if, for all $x' \not\leq x''$, $g(x' \wedge x'') \leq g(x') \implies f(x'') < f(x' \vee x'')$.

To shed some light on the two relations on functions at hand, we compare them on Euclidean spaces. If $X = \mathbb{R}$, QSM dominance is equivalent to single-crossing dominance: $g \leq_{qsm} f \iff g \leq_{sc} f$. If $X = \mathbb{R}^2$ then $g \leq_{qsm} f$ if and only if (i) $g \leq_{sc} f$ and (ii) for each k and $x'_{-k} \leq x''_{-k}$, $g(\cdot, x'_{-k}) \leq_{sc} f(\cdot, x''_{-k})$. Intuitively, (i) says that whenever it pays to take a higher action under g, then it pays to take a higher action under f. (ii) says that if it pays to take a higher action in dimension k when the other action is low and the objective function is g, then it pays to take the higher action in dimension k when the other action is high and the objective function is f.

If $X = \mathbb{R}^n$, $n > 2$, QSM dominance implies (i) and (ii), but involves other restrictions. In general, QSM dominance is stronger than single-crossing dominance, but weaker than the combination of single-crossing dominance and QSM; in particular, if f or g is QSM

[3] $\nabla f(x)$ is the vector of first partials of f. We write $\nabla g \leq \nabla f$ if $\nabla g(x) \leq \nabla f(x)$ for all $x \in X$.

and $g \leq_{sc} f$ then $g \leq_{qsm} f$. Hence, if T is a poset and $f : X \times T \to \mathbb{R}$ is QSM on X and has the SCP in $(x; t)$, then for all $t' \leq t''$, $f(\cdot, t') \leq_{qsm} f(\cdot, t'')$.

The next result shows that $\leq_{qsm}$ is transitive on $\mathscr{F}_X$ and formalizes some useful relationships between QSM dominance and QSM/SCP. Proofs are in the Appendix.

Lemma 1. *Let $f, g, h \in \mathscr{F}_X$. (i) If $g \leq_{qsm} f$ then $g \leq_{sc} f$. (ii) If $h \leq_{qsm} g$ and $g \leq_{qsm} f$ then $h \leq_{qsm} f$. (iii) If f or g is QSM and $g \leq_{sc} f$ then $g \leq_{qsm} f$. (iv) $f \sim_{qsm} g$ if and only if $f \sim_{sc} g$ and f and g are QSM. In particular, $f \sim_{qsm} f$ if and only if f is QSM.*

For $S \in \mathscr{P}(X)$ and $f \in \mathscr{F}_X$, we let $D = (S, f)$ denote the problem, $\max_{x \in S} f(a)$, and we let $M(D) = M(S, f)$ denote the set of solutions:

$$M(S, f) = \arg\max_{x \in S} f(a).$$

We restrict attention to the case $M(S, f) \neq \emptyset$. Let $\mathscr{D} \subseteq \mathscr{P}(X) \times \mathscr{F}_X$ denote the set of all problems under consideration: $\mathscr{D} = \{(S, f) \in \mathscr{P}(X) \times \mathscr{F}_X | M(S, f) \neq \emptyset\}$.

We introduce the following relation on $\mathscr{P}(X) \times \mathscr{F}_X$:

Definition 3. Let $D^0, D^1 \in \mathscr{P}(X) \times \mathscr{F}(X)$. We write $D^0 \leq_{\mathscr{D}} D^1$, if $S^0 \leq_s S^1$ and $f^0 \leq_{sc} f^1$. We write $D^0 <_{\mathscr{D}} D^1$ if $S^0 \leq_s S^1$ and $f^0 <_{sc} f^1$.

Since $\leq_s$ is transitive on $\mathscr{P}(X)$ and $\leq_{sc}$ is transitive on $\mathscr{F}_X$, $\leq_{\mathscr{D}}$ is transitive on $\mathscr{P}(X) \times \mathscr{F}(X)$. Furthermore, since $\leq_{sc}$ is reflexive on $\mathscr{F}_X$ and $\leq_s$ is reflexive on $\mathscr{L}(X)$, it follows that $\leq_{\mathscr{D}}$ is reflexive on $\mathscr{L}(X) \times \mathscr{F}(X)$.

3 Results

Our first result complements MS's Monotonicity Theorem.

Theorem 1. *Let $f^0, f^1 \in \mathscr{F}_X$ and restrict attention to problems in $\mathscr{D}$.*

(i) $M(S^0, f^0) \leq_s M(S^1, f^1)$ $\forall S^0 \leq_s S^1$ if and only if $f^0 \leq_{qsm} f^1$.

(ii) If $f^0 <_{qsm} f^1$ then $M(S^0, f^0) \leq_c M(S^1, f^1)$ $\forall S^0 \leq_s S^1$.

To provide a precise comparison of our Theorem 1 with MS's Monotonicity Theorem, let us state their main conclusions in our context:

Theorem (MS). Let $f, f^0, f^1 \in \mathscr{F}_X$ and restrict attention to problems in $\mathscr{D}$.

(a) If f^0 and f^1 are QSM and $f^0 \leq_{sc} f^1$ then $M(S^0, f^0) \leq_s M(S^1, f^1)$ $\forall S^0 \leq_s S^1$.

(b) $M(S^0, f) \leq_s M(S^1, f)$ $\forall S^0 \leq_s S^1$ if and only if f is QSM.

(c) If $M(S, f^0) \leq_s M(S, f^1)$ $\forall S$ then $f^0 \leq_{sc} f^1$.

Part (c) is orthogonal to our results, but, using Lemma 1, we see that parts (a) and (b) are implied by our Theorem 1: (b) follows by choosing $f^0 = f^1 = f$ and noting that $f \leq_{qsm} f$ if and only if f is QSM; (a) follows since QSM dominance is implied by single-crossing dominance and QSM of each objective function. Importantly, QSM dominance does not require QSM of either objective function.

The next result follows by combining parts (ii) and (iii) of Lemma 1:

Lemma 2. *Let $f^0, f^c, f^1 \in \mathscr{F}_X$. If f^c is QSM and $f^0 \leq_{sc} f^c \leq_{sc} f^1$ then $f^0 \leq_{qsm} f^c \leq_{qsm} f^1$. If, in addition, $f^0 <_{sc} f^c$ or $f^c <_{sc} f^1$ then $f^0 <_{qsm} f^1$.*

Thus, a sufficient condition for f^1 to QSM dominate f^0 is the existence of a QSM function that single-crossing dominates f^0 and is single-crossing dominated by f^1.

Combining Theorem 1 and Lemma 2 yields our main result:

Theorem 2. *Let $D^0, D^1 \in \mathscr{D}$. If there exists $D^c = (S^c, f^c)$ in $\mathscr{D}$ such that f^c is QSM and $D^0 \leq_{\mathscr{D}} D^c \leq_{\mathscr{D}} D^1$ then $M(D^0) \leq_s M(D^c) \leq_s M(D^1)$. If, in addition, $D^0 <_{\mathscr{D}} D^c$ or $D^c <_{\mathscr{D}} D^1$ then $M(D^0) \leq_c M(D^1)$.*

Theorem 2 is a powerful and practical result for economic applications, as it provides sufficient conditions for the comparison of two argmax sets in terms of well-known and tractable concepts. If $X \subset \mathbb{R}^n$ and objective functions are differentiable, there are simple sufficient differential conditions. Recall that $\nabla g \leq \nabla f$ implies $g \leq_{sc} f$ and that QSM is implied by SPM. Hence, if one can find a SPM function, f^c, such that $\nabla f^0 \leq \nabla f^c \leq \nabla f^1$, then Theorem 2 applies (provided the choice sets are suitably ordered).

The following examples illustrate the practical value of Theorem 2. In addition to deriving new results of interest for a basic economic model, the first example cements a methodological bridge between MCS and common binary comparisons in economics (involving two distinct agents). While not necessary for our results, for ease of exposition, all objective functions are tacitly assumed to be sufficiently differentiable below.

Example 1A. In this example, we compare the output choices of a multiproduct monopolist with those of a surplus maximizing social planner. Let $q = (q_1, \cdots, q_M)$ denote a vector of quantities, where $q_m \in [0, \overline{q}] \subset \mathbb{R}$. Let $P : [0, \overline{q}]^M \to \mathbb{R}^M$ denote the inverse demand system, which is derived from the choices of a representative consumer with quasilinear utility function, $\tilde{U}(q, q_0) = U(q) + q_0$. P then satisfies the gradient condition, $\nabla U(q) = P(q)$ (see, e.g. Vives, 1999, Ch. 3). Let $C : [0, \overline{q}]^M \to \mathbb{R}$ denote the firm's cost function, and $\pi(q) = q \cdot P(q) - C(q)$ its profit ($x \cdot y$ denotes the dot product of vectors x and y). The firm chooses $q \in [0, \overline{q}]^M$ to maximize profit, while the planner chooses $q \in [0, \overline{q}]^M$ to maximize social welfare, $W(q) = U(q) - C(q)$. We ask: Under what conditions does the firm produce less of each good than the planner?

For each m and all q, assume $\partial P_m(q)/\partial q_m \leq 0$ and $\sum_{j=1}^M q_j \frac{\partial P_j(q)}{\partial q_m} \leq 0$; equivalently,

$$\frac{\partial \pi(q)}{\partial q_m} = P_m(q) + \sum_{j=1}^{M} q_j \frac{\partial P_j(q)}{\partial q_m} - \frac{\partial C(q)}{\partial q_m} \leq P_m(q) - \frac{\partial C(q)}{\partial q_m} = \frac{\partial W(q)}{\partial q_m}. \tag{1}$$

It follows that, $\nabla \pi \leq \nabla W$. Next, consider the following condition:

$$W \text{ is SPM: } \frac{\partial^2 W(q)}{\partial q_i \partial q_j} = \frac{\partial P_i(q)}{\partial q_j} - \frac{\partial^2 C(q)}{\partial q_i \partial q_j} \geq 0 \text{ for all } q \text{ and } i \neq j. \tag{2}$$

Under conditions (1)-(2), Theorem 2 implies (with W serving as the "middle" objective function in the theorem) that the set of optimal monopoly outputs is below that of the planner. To our knowledge, this is the first general result on this basic question; as it is central in industrial organization, we discuss the economic scope of our conditions.

To this end, (1) is the well-known diagonal dominance property of the Jacobian matrix of the inverse demand function, which is commonly assumed in industrial organization. Intuitively, this condition means that own market effects are "large" compared to cross market effects; i.e., the impact of raising output m on the price of m is large in magnitude compared to the impact on all other prices (see e.g., Vives, 1999, for further details).

Next, for (2) to hold, it is sufficient to have (i) complementary goods in demand, i.e., $\frac{\partial P_i}{\partial q_j} \geq 0$ for $i \neq j$, together with (ii) economies of scope in production, i.e., $\frac{\partial^2 C}{\partial q_i \partial q_j} \leq 0$ for $i \neq j$, although neither of these conditions is necessary, since (2) restricts only the sum.

Example 1B. We continue with the multiproduct monopoly model studied in Example 1A, but we ask another basic and fundamental question: Under what conditions does a reduction in marginal costs yield an increase in all optimal monopoly outputs?

Let C^0 and C^1 be two cost functions, and let $\pi^k(q)$ denote the firm's profit when its cost is C^k. Suppose total revenue is SPM and marginal costs are uniformly ranked:

$$q \cdot P(q) \text{ is SPM: } \frac{\partial P_i}{\partial q_j} + \frac{\partial P_j}{\partial q_i} + \sum_m q_m \frac{\partial^2 P_m}{\partial q_i \partial q_j} \geq 0, \text{ for all } i \neq j, \tag{3}$$

$$\text{There exists } c \in \mathbb{R}^M \text{ s.t. for each } m, \frac{\partial C^1(q)}{\partial q_m} \leq c_m \leq \frac{\partial C^0(q)}{\partial q_m} \text{ for all } q. \tag{4}$$

Let $\pi^c(q) = q \cdot (P(q) - c)$ denote the profit to a monopolist facing constant marginal cost for each good, where c is as given in (4). See that (3) implies π^c is SPM and (4) implies $\nabla\pi^0 \leq \nabla\pi^c \leq \nabla\pi^1$. By Theorem 2, the set of optimal outputs is higher when the firm faces cost function C^1 than when it faces C^0, while the set of optimal outputs for a firm with constant marginal costs is between the two.

A related comparison concerns a firm with economies versus diseconomies of scope. Instead of (4), let C^1 denote a cost function for a production process with economies of scope and C^0 with diseconomies of scope; formally, for each $i \neq j$, $\frac{\partial^2 C^1}{\partial q_i \partial q_j} \leq 0 \leq \frac{\partial^2 C^0}{\partial q_i \partial q_j}$. Abusing notation, let $\frac{\partial C^k(q_m, 0)}{\partial q_m}$ denote marginal cost of product m when q_m units of good m and 0 units of each other good are produced. Suppose for each m and q_m,

$$\frac{\partial C^1(q_m, 0)}{\partial q_m} \leq \frac{\partial C^0(q_m, 0)}{\partial q_m}.$$

It follows that $\nabla\pi^0 \leq \nabla\pi^1$. Moreover, condition (3), together with economies of scope, imply π^1 is SPM. Therefore, by Theorem 2, the set of optimal outputs for the monopolist is higher under economies of scope than under diseconomies of scope.

Example 2. Consider a model of environmental regulation in the spirit of Fischer and Newell (2008): A monopolist operating in a polluting industry chooses both its output, $q \in [0, \overline{q}] \subset \mathbb{R}$, and R&D investment, $a \in [0, \overline{a}] \subset \mathbb{R}$. The firm's product market profit is $\pi(q, a)$ and its emissions are $e(q, a)$. Assume π is QSM in (q, a), which is natural for both product and process innovations (i.e., marginal revenue or marginal cost is increasing in a). Further, assume that e is increasing in q. We compare the firm's behavior under different tax and subsidy policies (fixed exogenously by a government agency).

Setting 1: First, suppose emissions depend only on output. Let f^0 denote the firm's objective function under an emissions tax: $f^0(q, a) = \pi(q, a) - \tau(e(q))$, where $\tau(e) \geq 0$ is the tax when emissions are e; assume τ is increasing. Let f^1 denote the firm's objective function under an R&D subsidy: $f^1(q, a) = \pi(q, a) + \sigma(a)$, where $\sigma(a) \geq 0$ is the subsidy when R&D investment is a; assume σ is increasing. Finally, let f^c represent the objective function under the laissez-faire policy: $f^c(q, a) = \pi(q, a)$.

While f^c is QSM, this property is not preserved by summation.[4] But under the assumptions made herein, the key condition $f^0 \leq_{sc} f^c \leq_{sc} f^1$ clearly holds. Hence, The-

[4]Indeed, even with the relatively simple summation we have here (a QSM function plus/minus an increasing function of one of the variables), neither f^0 nor f^1 need be QSM.

orem 2 implies that the set of optimal outputs/investments is higher under the subsidy than the tax (with the set of optimal choices under laissez-faire between the two).

Setting 2: Now suppose emissions depend both on output and R&D investment. Here, we will assume π is SPM in (q, a). Let f^0 represent the firm's problem under a tax, which may now depend on both q and a: $f^0(q, a) = \pi(q, a) - \tau(q, a)$.[5] Let f^1 be the objective function under the subsidy regime considered in the first setting.

Suppose, for each a, $\tau(\cdot, a)$ is increasing and, for each q, $\sigma(\cdot) + \tau(q, \cdot)$ is increasing. Then, $\frac{\partial f^0}{\partial q} - \frac{\partial f^1}{\partial q} = -\frac{\partial \tau}{\partial q} \leq 0$, and $\frac{\partial f^0}{\partial a} - \frac{\partial f^1}{\partial a} = -\left(\sigma' + \frac{\partial \tau}{\partial a}\right) \leq 0$. Hence, $\nabla f^0 \leq \nabla f^1$. Moreover, since π and σ are SPM, f^1 is SPM. By Theorem 2, the set of optimal outputs/investments is greater in the subsidy regime than the tax regime. Note that SPM of π (rather than QSM as in setting 1) is used in this example insofar as it ensures f^1 is SPM, which provides the intermediate problem needed to apply Theorem 2. The same conclusion would be reached if we assumed directly that f^1 (or f^0) is QSM.

Last, we compare two different tax policies – a fixed rate emissions tax, $\tau^0(q, a) = t^0 e(q, a)$, for $t^0 \in \mathbb{R}_+$, and an output tax with R&D tax credits, $\tau^1(q, a) = t^1(q - c(a))$, where $t^1 \in \mathbb{R}_+$ and $c(\cdot)$ is a non-negative increasing function, representing the tax credits. Let $f^k(q, a) = \pi(q, a) - \tau^k(q, a)$ denote the payoff to the firm under tax policy k. If $t^1 \leq t^0 \frac{\partial e}{\partial q}$ and $-t^1 c'(\cdot) \leq t^0 \frac{\partial e}{\partial a}$ then $\nabla f^0 \leq \nabla f^1$. And while f^0 is not necessarily QSM, since π is SPM and τ^1 is additively separable, f^1 is SPM. Theorem 2 implies that the set of optimal output/investment choices is greater under tax τ^1 than τ^0.

For brevity, we limit our exposition to two applications to give an instructive flavor of the novelty and versatility of our results. As these examples suggest, binary comparative statics questions are prevalent in many contexts in economics. Our approach offers a new way to leverage the powerful MCS machinery to address them at a level of generality clearly exceeding that in the extant treatments in the various literature strands.

We next exposit in some detail the links between our approach and MS's results.

Parametric Optimization

To conclude, we discuss how Theorem 2 applies in parameterized optimization problems and further elucidate the relationship between our result and MS's Theorem. Let T be

[5] We do not impose any particular structure on τ, which allows for a host of different tax policies, including, a fixed-rate emissions tax: $\tau(q, a) = te(q, a)$; a variable-rate emissions tax: $\tau(q, a) = g(e(q, a))e(q, a)$; a production tax: $\tau(q, a) = g(q)$; or a production tax with R&D tax credits: $\tau(q, a) = g(q - c(a))$, where $c(a)$ represents the earned tax credits.

a poset, $f : X \times T \to \mathbb{R}$, $S : T \to \mathscr{P}(X)$ and $M(t) = \arg\max_{x \in S(t)} f(x,t)$. Assume that $S(\cdot)$ is increasing in the strong set order and that $M(t) \neq \emptyset$ for each t.

Let $t' < t''$ and suppose we're interested in comparing $M(t')$ and $M(t'')$. Let $\mathcal{I}(t', t'') = \{t \in T | t' \leq t \leq t''\}$. If f has the SCP in $(x; t)$ on $X \times \mathcal{I}(t', t'')$ and $f(\cdot, t)$ is QSM for each $t \in \mathcal{I}(t', t'')$, then by MS, $M(\cdot)$ is monotone increasing on $\mathcal{I}(t', t'')$, and as a consequence, $M(t') \leq_s M(t'')$. However, monotonicity of $M(\cdot)$ on $\mathcal{I}(t', t'')$ is unnecessary if the aim is only to compare the optimal choices at the two relevant parameters t' and t'', and our results facilitate this comparison under weaker conditions than those of MS.

To spell out the argument, maintain that f has the SCP in $(x; t)$, but suppose there is just *one* parameter value, $t^c \in \mathcal{I}(t', t'')$, such that $f(\cdot, t^c)$ is QSM. Then, for $t^0, t^1 \in \mathcal{I}(t', t'')$ with $t^0 \leq t^c \leq t^1$, the SCP implies $f(\cdot, t^0) \leq_{sc} f(\cdot, t^c) \leq_{sc} f(\cdot, t^1)$. Since $f(\cdot, t^c)$ is QSM, Theorem 2 implies $M(t^0) \leq_s M(t^c) \leq_s M(t^1)$; in particular, $M(t') \leq_s M(t'')$. More generally, if there is any QSM function, $f^c : X \to \mathbb{R}$, such that the critical condition $f(\cdot, t') \leq_{sc} f^c \leq_{sc} f(\cdot, t'')$ holds, then Theorem 2 implies that $M(t') \leq_s M(t'')$.

A Proofs

All the proofs of the results of this paper are provided in this Appendix.

Proof of Lemma 1

Part (i): Suppose $g \leq_{qsm} f$ and let $x' < x''$. Since $g \leq_{qsm} f$, $g(x' \wedge x'') \leq (<) g(x'') \implies f(x') \leq (<) f(x' \vee x'')$, which means $g(x') \leq (<) g(x'') \implies f(x') \leq (<) f(x'')$.

Part (ii): Suppose that $h \leq_{qsm} g$ and $g \leq_{qsm} f$. Let $x', x'' \in X$. See that, $h(x' \wedge x'') \leq (<) h(x') \implies g(x' \wedge x'') \leq (<) g(x') \implies f(x'') \leq (<) f(x' \vee x'')$. The second inequality holds since $h \leq_{sc} g$ by part (i); the third holds since $g \leq_{qsm} f$.

Part (iii). Suppose that $g \leq_{sc} f$ and let $x', x'' \in X$. First, suppose g is QSM. Then, $g(x' \wedge x'') \leq (<) g(x') \implies g(x'') \leq (<) g(x' \vee x'') \implies f(x'') \leq (<) f(x' \vee x'')$. The second inequality holds since g is QSM; the third since $g \leq_{sc} f$. Next, suppose f is QSM. Then, $g(x' \wedge x'') \leq (<) g(x') \implies f(x' \wedge x'') \leq (<) f(x') \implies f(x'') \leq (<) f(x' \vee x'')$. The second inequality holds since $g \leq_{sc} f$ and the third holds since f is QSM.

Part (iv): First suppose that $f \sim_{qsm} g$. By Part (i), $f \sim_{sc} g$. We now show f is QSM; the proof that g is QSM is identical. Let $x', x'' \in X$. See that $f(x' \wedge x'') \leq (<) f(x') \implies g(x'') \leq (<) g(x' \vee x'') \implies f(x'') \leq (<) f(x' \vee x'')$. The second inequality holds since $f \leq_{qsm} g$; the third since $g \leq_{sc} f$. This establishes that f is QSM. Next, suppose that $f \sim_{sc} g$ and f and that g are QSM. By part (iii), $f \sim_{qsm} g$. $\square$

Proof of Theorem 1

Part (i): Suppose $f^0 \leq_{qsm} f^1$ and let $S^0 \leq_s S^1$ be such that $(S^0, f^0), (S^1, f^1) \in \mathscr{D}$. Let $x^0 \in M(S^0, f^0)$ and $x^1 \in M(S^1, f^1)$. Then $S^0 \leq_s S^1 \implies x^0 \wedge x^1 \in S^0$ and $x^0 \vee x^1 \in S^1$. By definition of x^0, $f^0(x^0 \wedge x^1) \leq f^0(x^0)$. Since $f^0 \leq_{qsm} f^1$, this implies, $f^1(x^1) \leq f^1(x^0 \vee x^1)$. But since $x^0 \vee x^1 \in S^1$, by definition of x^1, $x^0 \vee x^1 \in M(S^1, f^1)$. Similarly, $x^0 \vee x^1 \in S^1 \implies f^1(x^0 \vee x^1) \leq f^1(x^1) \implies f^0(x^0) \leq f^0(x^0 \wedge x^1)$; so, by definition of x^0, $x^0 \wedge x^1 \in M(S^0, f^0)$. Thus, $M(S^0, f^0) \leq_s M(S^1, f^1)$.

Now suppose that $M(S^0, f^0) \leq_s M(S^1, f^1)$ for all $S^0 \leq_s S^1$ with $(S^0, f^0), (S^1, f^1) \in \mathscr{D}$. Choose any $x', x'' \in X$. Let $S^0 = \{x', x' \wedge x''\}$ and $S^1 = \{x'', x' \vee x''\}$. Clearly, $(S^0, f^0), (S^1, f^1) \in \mathscr{D}$ and $S^0 \leq_s S^1$, hence $M(S^0, f^0) \leq_s M(S^1, f^1)$. If $f^0(x' \wedge x'') \leq f^0(x')$, then, since $M(S^0, f^0) \leq_s M(S^1, f^1)$, we must have $x' \vee x'' \in M(S^1, f^1)$ and therefore, $f^1(x'') \leq f^1(x' \vee x'')$. Moreover, if $f^1(x' \vee x'') \leq f^1(x'')$, then since $M(S^0, f^0) \leq_s M(S^1, f^1)$, we must have $x' \wedge x'' \in M^0(S^0, f^0)$, which means $f^0(x') \leq f^0(x' \wedge x'')$. This shows that $f^0 \leq_{qsm} f^1$ and thus completes the proof of part (i).

We now show part (ii). Let $x^0 \in M(S^0, f^0)$ and $x^1 \in M(S^1, f^1)$. Towards eventual contradiction, suppose that $x^0 \nleq x^1$. Then, $S^0 \leq_s S^1$ implies $x^0 \wedge x^1 \in S^0$ and $x^0 \vee x^1 \in S^1$. Hence, by definition of x^0, $f^0(x^0 \wedge x^1) \leq f^0(x^0)$. Since $f^0 <_{qsm} f^1$, this implies that $f^1(x^1) < f^1(x^0 \vee x^1)$, which contradicts the definition of x^1. $\square$

Proof of Lemma 2

Let $f^0 \leq_{sc} f^c \leq_{sc} f^1$ where f^c is QSM. It is immediate from Lemma 1(iii) that $f^0 \leq_{qsm} f^c$ and $f^c \leq_{qsm} f^1$; by Lemma 1(ii), $f^0 \leq_{qsm} f^1$. Next, suppose $f^0 <_{sc} f^c$ or $f^c <_{sc} f^1$; in particular, suppose $f^0 <_{sc} f^c$; the proof for the case $f^c <_{sc} f^1$ is similar. Let $x' \nleq x''$ and see that $f^0(x' \wedge x'') \leq f^0(x') \implies f^c(x' \wedge x'') < f^c(x') \implies f^c(x'') < f^c(x' \vee x'') \implies f^1(x'') < f^1(x' \vee x'')$. The second inequality holds since $f^0 <_{sc} f^c$; the third since f^c is QSM; and the last inequality since $f^c \leq_{sc} f^1$. $\square$

We are now ready for the proof of the main result of this note.

Proof of Theorem 2

By definition, $D^0 \leq_{\mathscr{D}} D^c \leq_{\mathscr{D}} D^1$ means $S^0 \leq_s S^c \leq_s S^1$ and $f^0 \leq_{sc} f^c \leq_{sc} f^1$. Since f^c is QSM, Lemma 2 implies that $f^0 \leq_{qsm} f^c \leq_{qsm} f^1$. By Theorem 1(i), $M(D^0) \leq_s M(D^c) \leq_s M(D^1)$. If, in addition, $D^0 <_{\mathscr{D}} D^c$ or $D^c <_{\mathscr{D}} D^1$ then this means that $f^0 <_{sc} f^c$ or $f^c <_{sc} f^1$; by Lemma 2, $f^0 <_{qsm} f^1$, and finally by Theorem 1(ii), $M(D^0) \leq_c M(D^1)$. This concludes the proof of Theorem 2. $\square$

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