Rigid frameworks with dilation constraints

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Abstract

We consider the rigidity and global rigidity of bar-joint frameworks in Euclidean d-space under additional dilation constraints in specified coordinate directions. In this setting we obtain a complete characterisation of generic rigidity. We then consider generic global rigidity. In particular, we provide an algebraic sufficient condition and a weak necessary condition. We also construct a large family of globally rigid frameworks and conjecture a combinatorial characterisation when most coordinate directions have dilation constraints.

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1 Introduction

A *bar-joint framework* (G, p) is the combination of a (finite, simple) graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$ that assigns points in Euclidean d-space to the vertices of G (and hence straight line segments of given length to the edges). These points represent the joints and the straight line segments the bars or the framework. That is, bars do not stretch, compress or bend, and are connected to other bars at the joints, which act as ball joints. Informally, the framework is *rigid* if its lengths locally determine the shape. That is, if every edge-length preserving continuous motion of the vertices arises from a Euclidean isometry.

Determining the rigidity or flexibility (non-rigidity) of a framework is a crucial problem in a variety of practical applications from wireless sensor networks [2] to control of robotic formations [25], and rigidity theoretic tools have recently been used to impact on diverse mathematical problems such as the lower bound theorem for manifolds [7, 18]. However, it is computationally challenging to determine the rigidity of a given framework when $d > 1$ [1]. To get around this issue we sometimes have to focus on generic frameworks; (G, p) is *generic* if the set of coordinates of the points $p(v)$, $v \in V$, are distinct and form an algebraically independent set over $\mathbb{Q}$. (Much weaker variants of genericity are possible but a distraction from the main topic of this paper.) However, it is worth noting here that some of our key results will apply to frameworks with no genericity assumption.

For generic frameworks, rigidity depends only on the underlying graph. That is, if one generic framework (G, q) in $\mathbb{R}^d$ is rigid then *every* generic framework (G, p) in $\mathbb{R}^d$ is rigid [3]. The cornerstone problem of rigidity theory is to determine precisely the class of graphs that are rigid. As far back as Maxwell [20] necessary conditions were known but these are insufficient

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for all $d \geq 3$. When $d = 1$ it is easy to see that a graph is rigid if and only if it is connected. When $d = 2$, a celebrated theorem first obtained by Polaczek-Geiringer [22], and often referred to as Laman's theorem [19], characterises rigid graphs. However for all $d \geq 3$, characterising rigid graphs remains a challenging open problem of key applied and theoretical importance.

Motivated by this, the present article considers the rigidity of bar-joint frameworks in Euclidean space under additional "coordinate dilation" constraints. These frameworks first arose in the context of frameworks on surfaces [17] where characterising rigidity with these dilation constraints was important in understanding stress matrices and global rigidity (defined shortly). This kind of constraint is previously unstudied in the Euclidean context. A somewhat related setting occurred recently in the study of unmanned aerial vehicles [6]. We will describe this at the end of Section 4. Another related setting is the study of frameworks on surfaces [21].

We give precise descriptions of rigidity in d-dimensions under dilation constraints linking them to "ordinary" rigidity in lower dimensions, and hence giving purely combinatorial characterisations in arbitrary dimension provided there are sufficiently many coordinate dilation constraints. As a consequence (Corollary 4.6) we establish that the underlying rigidity matroid of our dilation constraint setting is the union of a smaller dimensional rigidity matroid and the uniform matroid of a specified rank. In this sense our results are similar to recent results obtained in [8] for cylindrical normed spaces and [23] for coordinated edge length motions.

We then investigate variants for global rigidity building on standard results for bar-joint frameworks. Global rigidity of bar-joint frameworks asks, more strongly than rigidity, for the given framework to be unique up to isometries of the space. It follows from a deep result of Gortler, Healy and Thurston [11] that, generically, global rigidity depends only on the underlying graph and the combinatorial difficulty in dimension greater than 2 mirrors the situation for rigidity. In dimension 1 a graph is generically globally rigid if and only if it is 2-connected (see, e.g., [14] for a proof of this folklore fact), in dimension 2 a combination of results due to Hendrickson [13], Connelly [4] and Jackson and Jordán [15] give a complete combinatorial characterisation, whereas when $d \geq 3$ only some partial results are known (see, e.g., [7]).

In Section 5 we give an augmented definition of equilibrium stress and stress matrix and use them to give an analogue of Connelly's sufficient condition [4] that applies to global rigidity in $\mathbb{R}^d$ with dilation constraints. We deduce from this that a well known construction operation (1-extension) preserves global rigidity and then discuss necessary conditions. We conclude the paper, in Section 6, with two open problems on global rigidity.

2 Rigidity theoretic preliminaries

Let $G = (V, E)$ be a graph. Two frameworks (G, p) and (G, q) are said to be *equivalent* if

$$\|p(v) - p(w)\| = \|q(v) - q(w)\| \qquad \text{for all } vw \in E. \tag{1}$$

More strongly, they are *congruent* if Equation (1) holds for any pair of vertices $v, w \in V$. The framework (G, p) in $\mathbb{R}^d$ is *d-rigid* if every equivalent framework (G, q) in a neighbourhood of p, considered as a vector in $\mathbb{R}^{d|V|}$, is obtained from (G, p) by a composition of isometries of $\mathbb{R}^d$. Moreover, (G, p) in $\mathbb{R}^d$ is *globally d-rigid* if every equivalent framework (G, q) in $\mathbb{R}^d$ is obtained from (G, p) by a composition of isometries of $\mathbb{R}^d$. A framework is *minimally d-rigid* if it is d-rigid, but any framework formed by removing edges is not d-rigid.

Differentiating the distance constraints given in Equation (1), we obtain the following. An *infinitesimal motion* of (G,p) is a map $\dot{p}: V \to \mathbb{R}^d$ satisfying the system of linear equations:

$$(p(v)-p(w))\cdot(\dot{p}(v)-\dot{p}(w)) = 0 \qquad \text{for all } vw \in E.$$

The framework (G,p) is *infinitesimally d-rigid* if the only infinitesimal motions arise from isometries of $\mathbb{R}^d$. The *rigidity matrix* $R(G,p)$ of the framework (G,p) is the matrix of coefficients of this system of equations for the unknowns $\dot{p}$. Thus $R(G,p)$ is a $|E|\times d|V|$ matrix, in which: the row indexed by an edge $vw\in E$ has $p(v)-p(w)$ and $p(w)-p(v)$ in the d columns indexed by v and w respectively, and zeros elsewhere. It is straightforward to show that (G,p) is infinitesimally d-rigid if and only if $\operatorname{rank} R(G,p) = d|V| - \binom{d+1}{2}$ whenever G has at least d vertices.

Rigidity and infinitesimal rigidity are linked by the following theorem.

Theorem 2.1 (Asimow and Roth [3])**.** *Let (G,p) be a framework in $\mathbb{R}^d$. If (G,p) is infinitesimally d-rigid then it is d-rigid. Conversely if (G,p) is generic and d-rigid, then it is infinitesimally d-rigid.*

Following from Theorem 2.1, we say that a graph G is *d-rigid* if there exists an infinitesimally d-rigid framework (G,p) (or equivalently, a generic d-rigid framework (G,p)). Likewise, G is *minimally d-rigid* if there exists a generic minimally d-rigid framework (G,p).

For the next result, we recall that a graph $G=(V,E)$ is $(d,\binom{d+1}{2})$*-sparse* if $|E'|\le d|V'| - \binom{d+1}{2}$ for all subgraphs (V',E') on at least d vertices. A $(d,\binom{d+1}{2})$-sparse graph is $(d,\binom{d+1}{2})$*-tight* if $|E| = d|V| - \binom{d+1}{2}$.

Lemma 2.2 (Maxwell [20])**.** *Let G be minimally d-rigid on at least d vertices. Then G is $(d,\binom{d+1}{2})$-tight.*

The converse to Maxwell's lemma holds when $d\le 2$ [22]. However it remains an open problem to determine which $(d,\binom{d+1}{2})$-tight graphs are d-rigid in all higher dimensions. One motivation for this paper is that additional understanding of this problem may be developed by exploring the problem with additional constraints.

An *equilibrium stress* of a framework (G,p) in $\mathbb{R}^d$ is a vector in the cokernel of $R(G,p)$. In other words, a vector $\omega\in\mathbb{R}^{|E|}$ is an equilibrium stress of (G,p) if, for all $v\in V$,

$$\sum_{u\in N_G(v)} \omega_{vu}(p(v)-p(u)) = 0,$$

where $N_G(v)$ denotes the neighbour set of v. Let $n=|V|$. The *stress matrix* $\Omega(\omega)$ is a symmetric $n\times n$-matrix in which the rows and columns are indexed by the vertices and in which the off diagonal entry in row v and column u is $-\omega_{vu}$, and the diagonal entry in row v is $\sum_{u\in V}\omega_{vu}$. Here ω_{vu} is taken to be equal to zero if $vu\notin E$. Equivalently, the stress matrix $\Omega(\omega)$ is the Laplacian matrix of the weighted graph (G,ω).

3 Infinitesimal (d,k)-rigidity

We now introduce our dilation constrained rigidity context. Let (G,p) be a framework in $\mathbb{R}^d$. Fix some $k\in\{1,\dots,d-1\}$. For each coordinate $i\in\{1,\dots,d\}$, let $p_i: V\to\mathbb{R}$ be the restriction of p to the i-th coordinate. Using this notation, we see that $p(v) = (p_1(v), \dots p_d(v))$ for every vertex $v \in V$. With this, we define the map

$$\tilde{p} : V \to \mathbb{R}^{d-k}, \; v \mapsto (p_1(v), \dots, p_{d-k}(v)).$$

Two frameworks (G,p) and (G,q) in $\mathbb{R}^d$ are (d,k)-*equivalent* if they are equivalent and for each $i \in \{d-k+1, \dots, d\}$ there exists a scalar $\alpha_i \neq 0$ such that $p_i = \alpha_i q_i$. Without loss of generality we may fix a vertex $v_0 \in V$ where $p_i(v_0) \neq 0$ for each $i \in \{d-k+1, \dots, d\}$. Then, under the assumption that q is a realisation with $q_i(v_0) \neq 0$ for $i \in \{d-k+1, \dots, d\}$, (d,k)-equivalence can be represented by the following constraint system:

$$\|p(v) - p(w)\| = \|q(v) - q(w)\| \qquad \text{for all } vw \in E, \tag{2}$$

$$\frac{p_i(v)}{p_i(v_0)} = \frac{q_i(v)}{q_i(v_0)} \qquad \text{for all } v \in V \setminus \{v_0\} \text{ and } i \in \{d-k+1, \dots, d\}. \tag{3}$$

We note that the precise choice of constraints in Equation (3) was made for convenience. For most frameworks, constraints corresponding to any connected graph would create precisely the same constraint system as we show in the following simple lemma.

Lemma 3.1. *Let $G = (V,E)$, $H = (V,F)$ be two graphs with the same vertex set, and let $(G,p), (G,q)$ be frameworks in $\mathbb{R}^d$ where $p_i(v) \neq 0$ and $q_i(v) \neq 0$ for all $v \in V$ and $i \in \{d-k+1, \dots, d\}$. If H is connected, then Equation (3) holds if and only if*

$$\frac{p_i(v)}{p_i(w)} = \frac{q_i(v)}{q_i(w)} \qquad \textit{for all } vw \in F \textit{ and } i \in \{d-k+1, \dots, d\}. \tag{4}$$

Proof. Suppose Equation (3) holds. Choose any $vw \in F$ and any $i \in \{d-k+1, \dots, d\}$. Then we have

$$\frac{p_i(v)}{p_i(w)} = \frac{p_i(v)}{p_i(v_0)} \cdot \frac{p_i(v_0)}{p_i(w)} = \frac{q_i(v)}{q_i(v_0)} \cdot \frac{q_i(v_0)}{q_i(w)} = \frac{q_i(v)}{q_i(w)}.$$

Hence, Equation (4) holds.

Suppose Equation (4) holds. Choose any $v \in V \setminus \{v_0\}$ and any $i \in \{d-k+1, \dots, d\}$. Then since H is connected, there exists a path from v to v_0 in H, say $(v, v_1, v_2, \dots, v_t, v_0)$. We have

$$\frac{p_i(v)}{p_i(v_0)} = \frac{p_i(v)}{p_i(v_1)} \frac{p_i(v_1)}{p_i(v_2)} \cdots \frac{p_i(v_t)}{p_i(v_0)} = \frac{q_i(v)}{q_i(v_1)} \frac{q_i(v_1)}{q_i(v_2)} \cdots \frac{q_i(v_t)}{q_i(v_0)} = \frac{q_i(v)}{q_i(v_0)}.$$

Hence, Equation (3) holds. □

As in the standard theory of bar-joint rigidity, we define a framework to be (d,k)-*rigid* if every sufficiently close (d,k)-equivalent framework is congruent. This is equivalent to $q = p$ being the locally unique solution of the system of constraints given by Equations (2) and (3), modulo congruences. Similarly to the standard theory of bar-joint rigidity, solving such a system of equations is computationally challenging. Because of this, we need to linearise the problem by differentiating the constraint system. As is the case with Theorem 2.1, doing so will provide us with a sufficient condition for (d,k)-rigidity that is easier to work with, fast to compute, and also necessary for almost all realisations.

The Jacobian derivative of the system of constraints given by Equations (2) and (3)[1] is the $(|E|+k(|V|-1)) \times d|V|$ matrix

$$J_{v_0}(G,p) = \begin{pmatrix} R(G,\tilde{p}) & R(G,p_{d-k+1}) & \cdots & R(G,p_d) \\ \mathbf{0} & M_{d-k+1} & & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & & M_d \end{pmatrix},$$

where $R(G,\tilde{p})$ and $R(G,p_i)$, for $d-k+1 \le i \le d$, are the rigidity matrices of the $(d-k)$-dimensional framework $(G,\tilde{p})$ and the 1-dimensional frameworks (G,p_i) respectively, and M_i is the matrix with rows labelled by $V \setminus \{v_0\}$, columns labelled by V and entries

$$M_i(v,w) = \begin{cases} 1/p_i(v_0) & \text{if } w=v, \\ -p_i(v)/p_i(v_0)^2 & \text{if } w=v_0, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2. Let $d=2$, $k=1$, $G=K_3$ and $V(K_3)=\{v_0,v_1,v_2\}$. Then $i=d-k+1=2$, and for illustration we put $\tilde{p}(v_i)=x_i$, $p_2(v_i)=y_i$, so that

$$J_{v_0}(K_3,p) = \begin{pmatrix} x_0-x_1 & x_1-x_0 & 0 & y_0-y_1 & y_1-y_0 & 0 \\ x_0-x_2 & 0 & x_2-x_0 & y_0-y_2 & 0 & y_2-y_0 \\ 0 & x_1-x_2 & x_2-x_1 & 0 & y_1-y_2 & y_2-y_1 \\ 0 & 0 & 0 & -\frac{y_1}{y_0^2} & \frac{1}{y_0} & 0 \\ 0 & 0 & 0 & -\frac{y_2}{y_0^2} & 0 & \frac{1}{y_0} \end{pmatrix}.$$

We now define (G,p) to be *infinitesimally* (d,k)*-rigid* if and only if either G is a complete graph and the set $\{p(v): v \in V\}$ has affine dimension $\min\{d,|V|-1\}$ or G is not complete and $\operatorname{rank} J_{v_0}(G,p) = d|V| - \binom{d-k+1}{2}$.

Using the technique of Asimow and Roth [3], one can show that infinitesimal (d,k)-rigidity implies (d,k)-rigidity, and that the two properties coincide for generic frameworks. Because of this, we define a graph G to be (d,k)*-rigid* if there exists a realisation $p: V \to \mathbb{R}^d$ where (G,p) is infinitesimally (d,k)-rigid. Equivalently, G is (d,k)-rigid if every generic (G,p) in $\mathbb{R}^d$ is infinitesimally (d,k)-rigid.

Suppose that, given (G,p) is a generic framework in $\mathbb{R}^d$, the Jacobian $J_{v_0}(G,p)$ has linearly independent rows and rank $J_{v_0}(G,p) = d|V| - \binom{d-k+1}{2}$. Then

$$|E| + k(|V|-1) = d|V| - \binom{d-k+1}{2} = (d-k)|V| - \binom{d-k+1}{2} + k + k(|V|-1). \quad (5)$$

Hence we say that a graph G is *minimally* (d,k)*-rigid* if it is (d,k)-rigid and $|E| = (d-k)|V| - \binom{d-k+1}{2} + k$.

Example 3.3. Suppose $d=2$, $k=1$ and $G=C_4$ be the cycle with vertex set $\{v_1,v_2,v_3,v_4\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$. Then putting $p(v_1)=(1,1)$, $p(v_2)=(2,1)$, $p(v_3)=(2,2)$ and $p(v_4)=(1,2)$ gives a framework (C_4,p) such that rank $J_{v_0}(C_4,p) = 7 = 2|V| - \binom{2}{2}$ (see Figure 1).

[1] Technically speaking, we are actually taking the derivative of the concatenation of the system of equations given by Equation (2) after squaring then halving both sides, and the system of equations given by Equation (3).

Since $|E| = 4 = 1\cdot 4 - \binom{2}{2} + 1$, (C_4, p) is minimally $(2,1)$-rigid. Since rank is maximised at generic configurations, the same conclusion holds for any generic framework (C_4, q). Suppose on the other hand that $d = 3, k = 2$ and $G = C_4$. Then $J_{v_0}(C_4, p)$ has 10 rows and rank $J_{v_0}(C_4, p) \leq 10 < 3|V| - \binom{2}{2} = 11$ so (C_4, p) is not (infinitesimally) $(3,2)$-rigid for any p.

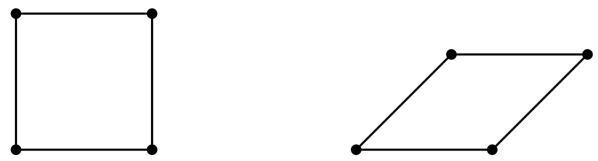

Figure 1: The 2-dimensional framework described in Example 3.3 is depicted on the left. This framework is not 2-rigid since there is a non-trivial continuous deformation taking it to the framework on the right. Nevertheless the framework is $(2,1)$-rigid since the dilation constraints in the y-coordinates prevent any nontrivial motion. The intuition behind this is to first note that translation in the y-direction and rotation evidently break the dilation constraints. Consider now the nontrivial motion depicted. The top left vertex follows the path $\theta \mapsto (1+\sin\theta, 1+\cos\theta)$ and the top right vertex follows the path $\theta \mapsto (2+\sin\theta, 1+\cos\theta)$. As the bottom two vertices have y-coordinate 1, the dilation constraints require that the y-coordinates of the top two vertices – both of which are $1+\cos\theta$ – are constant during the motion, a clear contradiction.

4 Characterising (d,k)-rigidity

To characterise (d,k)-rigidity we will first show that there is a more convenient matrix representation. Recall that the usual d-dimensional (squared) rigidity map $f_{G,d}$ is the map

$$f_{G,d} : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}, \; p \mapsto \left(\|p(v) - p(w)\|^2\right)_{vw\in E}.$$

Given the 1-dimensional rigidity map $f_{G,1}$, we define the $|E| \times (d-k)|V| + k$ matrix

$$DR_k(G,p) := \begin{pmatrix} R(G,\tilde{p}) & f_{G,1}(p_{d-k+1}) & \cdots & f_{G,1}(p_d) \end{pmatrix}.$$

Example 4.1. Let $d = 2$, $k = 1$, $G = K_3$ and $V(K_3) = \{v_0, v_1, v_2\}$. Put $p(v_i) = (x_i, y_i)$ for $0 \leq i \leq 2$. Then we have

$$DR_1(K_3, p) = \begin{pmatrix} x_0 - x_1 & x_1 - x_0 & 0 & (y_0 - y_1)^2 \\ x_0 - x_2 & 0 & x_2 - x_0 & (y_0 - y_2)^2 \\ 0 & x_1 - x_2 & x_2 - x_1 & (y_1 - y_2)^2 \end{pmatrix}.$$

The matrix $DR_k(G,p)$ can be used to determine infinitesimal (d,k)-rigidity.

Theorem 4.2. *Let (G,p) be a framework with a vertex $v_0 \in V$ where $p_i(v_0) \neq 0$ for each $i \in \{d-k+1, \ldots, d\}$. Then (G,p) is infinitesimally (d,k)-rigid if and only if*

$$\operatorname{rank} DR_k(G,p) = (d-k)|V| - \binom{d-k+1}{2} + k,$$

or G is a complete graph and the set $\{p(v) : v \in V\}$ has affine dimension $\min\{d, |V|-1\}$.

Proof. We may suppose that G is not a complete graph. By multiplying each row of $J_{v_0}(G,p)$ corresponding to the vertex $v \in V \setminus \{v_0\}$ and coordinate $i \in \{d-k+1,\dots,d\}$ by $p_i(v_0)$, we obtain the matrix

$$J' = \begin{pmatrix} R(G,\tilde{p}) & R(G,p_{d-k+1}) & \cdots & R(G,p_d) \\ \mathbf{0} & M'_{d-k+1} & & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & & M'_d \end{pmatrix},$$

where

$$M'_i(v,w) = \begin{cases} 1 & \text{if } w = v, \\ -p_i(v)/p_i(v_0) & \text{if } w = v_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let J'_{vw} be the row corresponding to the edge $vw \in E$, and let $J'_{v,i}$ be the row corresponding to the vertex $v \in V \setminus \{v_0\}$ and coordinate $i \in \{d-k+1,\dots,d\}$. We will now form a new matrix from J' by the following row operations:

- For each row vw with $v, w \neq v_0$, $J'_{vw} \mapsto J'_{vw} - \sum_{i=d-k+1}^{d}(p_i(v)-p_i(w))(J'_{v,i} - J'_{w,i})$.
- For each row vv_0, $J'_{vv_0} \mapsto J'_{vv_0} - \sum_{i=d-k+1}^{d}(p_i(v)-p_i(v_0))J'_{v,i}$.
- Shift each column corresponding to the vertex v_0 and coordinate $i \in \{d-k+1,\dots,d\}$ to the right hand side of the matrix and multiply by $p_i(v_0)$.

With this we obtain the following matrix:

$$J'' = \begin{pmatrix} R(G,\tilde{p}) & \mathbf{0} & \cdots & \mathbf{0} & f_{G,1}(p_{d-k+1}) & \cdots & f_{G,1}(p_d) \\ \mathbf{0} & I_{|V\setminus\{v_0\}|} & & \mathbf{0} & b_{d-k+1} & & \mathbf{0} \\ \vdots & & \ddots & & & \ddots & \\ \mathbf{0} & \mathbf{0} & & I_{|V\setminus\{v_0\}|} & \mathbf{0} & & b_d \end{pmatrix},$$

where $I_{|V\setminus\{v_0\}|}$ is the $|V\setminus\{v_0\}| \times |V\setminus\{v_0\}|$ identity matrix, and each b_i is the $|V\setminus\{v_0\}|$-dimensional column vector with coordinates $b_i(v) := -p_i(v)$. We now note that

$$\operatorname{rank} DR_k(G,p) = \operatorname{rank} J'' - k(|V|-1) = \operatorname{rank} J_{v_0}(G,p) - k|V| + k.$$

Since, by definition, (G,p) is infinitesimally (d,k)-rigid if and only if $\operatorname{rank} J_{v_0}(G,p) = d|V| - \binom{d-k+1}{2}$, this and Equation (5) give the desired result. □

We next give a complete description of (d,k)-rigidity for arbitrary pairs d,k in terms of the usual bar-joint rigidity. The characterisation leads to efficient combinatorial algorithms whenever the resulting bar-joint rigidity problem can be solved in such terms; that is, when $d-k \leq 2$.

We will use an inductive argument to prove the following theorem. To this end it is convenient here, and only here, to allow $k=0$ in our definitions. In this case, $(d,0)$-rigidity is precisely d-rigidity.

Theorem 4.3. *A graph $G = (V, E)$ is (d, k)-rigid if and only if either G is a complete graph, or G contains a spanning $(d-k)$-rigid subgraph and $|E| \geq (d-k)|V| - \binom{d-k+1}{2} + k$.*

Proof. We may suppose that G is not a complete graph. If $|E| < (d-k)|V| - \binom{d-k+1}{2} + k$, then G is not (d, k)-rigid by Theorem 4.2. If G does not contain a spanning $(d-k)$-rigid subgraph, then for every realisation $p : V \to \mathbb{R}^d$, the matrix $R(G, \tilde{p})$ has a rank strictly less than $(d-k)|V| - \binom{d-k+1}{2}$. Since $DR_k(G, p)$ is formed from $R(G, \tilde{p})$ by adding k columns, it then must have a rank strictly less than $(d-k)|V| - \binom{d-k+1}{2} + k$. Hence G is not (d, k)-rigid by Theorem 4.2.

Now suppose that G contains a spanning $(d-k)$-rigid subgraph and $|E| \geq (d-k)|V| - \binom{d-k+1}{2} + k$. By deleting edges if necessary, we may suppose that $G = H + \{e_1, \dots, e_k\}$, where $H = (V, F)$ is minimally $(d-k)$-rigid and $e_1, \dots, e_k$ are edges in $E \setminus F$. Define $E_0 = F$, $E_i := F + \{e_1, \dots, e_i\}$, $G_0 := (V, E_0)$ and $G_i := (V, E_i)$ for all $1 \leq i \leq k$. It is immediate that $G_0 = H$ is minimally $(d-k, 0)$-rigid. Suppose, for an inductive argument, that G_j is minimally $(d-k+j, j)$-rigid for some $j \in \{0, \dots, k-1\}$. By the induction hypothesis and Theorem 4.2, there exists a realisation $p : V \to \mathbb{R}^{d-k+j}$ such that

$$\operatorname{rank} DR_j(G_j, p) = (d-k)|V| - \binom{d-k+1}{2} + j.$$

As $DR_j(G_j, p)$ has the highest achievable rank, we have $\operatorname{rank} DR_j(G_{j+1}, p) = \operatorname{rank} DR_j(G_j, p)$. Since $DR_j(G_{j+1}, p)$ contains the rows of $DR_j(G_j, p)^T$ plus one additional row, it follows that there exists a unique (up to scalar multiple) non-zero element of the left kernel of $DR_j(G_{j+1}, p)$. This is equivalent to there existing a unique (up to scalar multiple) non-zero vector $\sigma : E_{j+1} \to \mathbb{R}$ where $\sigma^T R(G, \tilde{p}) = [0 \ \dots \ 0]$ and

$$\sum_{vw \in E_{j+1}} \sigma(vw)(p_i(v) - p_i(w))^2 = p_i^T \Omega(\sigma) p_i = 0 \tag{6}$$

for each $i \in \{d-k+1, \dots, d-k+j\}$, where $\Omega(\sigma)$ is the stress matrix corresponding to σ. Since σ is non-zero, there exists $z \in \mathbb{R}^V$ such that $z^T \Omega(\sigma) z \neq 0$. Fix $p' : V \to \mathbb{R}^{d-k+j+1}$ to be the realisation where $p'_i := p_i$ for all $i \in \{1, \dots, d-k+j\}$, and $p'_{d-k+j+1} := z$. As the left kernel of $DR_{j+1}(G_{j+1}, p')$ is contained in the left kernel of $DR_j(G_{j+1}, p)$, we have $\ker DR_{j+1}(G_{j+1}, p')^T = \{0\}$. By counting edges we see that $|E_j| = (d-k)|V| - \binom{d-k+1}{2} + j + 1$, hence G_{j+1} is minimally $(d-k+j+1, j+1)$-rigid. By induction it now follows that $G = G_k$ is minimally (d, k)-rigid. □

Example 4.4. Let us unpack Theorem 4.3 for some basic special cases. Combining with the folklore 1-dimensional characterisation of rigidity, the theorem implies that a graph is $(2, 1)$-rigid if and only if either it is complete on 1 or 2 vertices or it is a connected graph with at least one cycle. Similarly, a graph $G = (V, E)$ is $(3, 2)$-rigid if and only if either G is complete on at most 3 vertices or G is connected with $|E| \geq |V| + 1$.

Next suppose the gap between d and k is 2 and recall that $G = (V, E)$ is a Laman graph if $|E| = 2|V| - 3$ and every edge-induced subgraph (V', E') has $|E'| \leq 2|V'| - 3$. . Then we can use Laman's theorem [19, 22] to deduce that a graph is $(3, 1)$-rigid if and only if G is complete or it is a Laman-plus-one graph (that is, it is obtained from a Laman graph by adding exactly one edge).

When the gap is bigger than 2 we no longer have a combinatorial description of rigidity to rely upon. For example, a graph $G = (V, E)$ is $(4, 1)$-rigid if and only if either G is complete or G contains a spanning 3-rigid subgraph and $|E| \geq 3|V| - 5$. Nevertheless some d-rigidity is understood in some special cases which we can then apply. For example if G is obtained from a triangulation of the sphere by adding some edges then, combining our result with a theorem of Gluck [10], we have that G is $(4, 1)$-rigid.

Remark 4.5. A (d-dimensional) *0-extension* adds a vertex of degree d to a graph. A (d-dimensional) *1-extension* deletes an edge xy and adds a vertex v of degree $d + 1$ adjacent to x and y. It is well known that these operations preserve the rigidity of bar-joint frameworks [24]. We note that it is possible to extend the standard ($(d-k)$-dimensional) 0- and 1-extension arguments to show that (d, k)-rigidity is preserved by these operations. This gives a way to construct large families of (d, k)-rigid graphs. It also gives a combinatorial proof of an interesting special case of Theorem 4.3 when $d = 3$ and $k = 1$. Here one may use Polaczek-Geiringer's [22] characterisation of 2-rigidity to see that the characterisation of minimal $(3, 1)$-rigidity in Theorem 4.3 is equivalent to the graph being a Laman-plus-one graph. Hence, to prove that such graphs are $(3, 1)$-rigid we simply apply a well known recursive construction of Laman-plus-one graphs due to Haas et al [12].

The (d, k)*-rigidity matroid* $\mathcal{R}_{d,k}(G, p)$ of a framework (G, p) is the row matroid of the Jacobian matrix $J_{v_0}(G, p)$. If (G, p) and (G, q) are generic frameworks in $\mathbb{R}^d$, then it is easy to see that their matroids coincide, that is the (d, k)-rigidity matroid depends on d, k and G but, for generic frameworks, it does not depend on the choice of generic realisation. Hence we drop the p and use $\mathcal{R}_{d,k}(G)$ or even $\mathcal{R}_{d,k}$ when the context is clear. We will also use $\mathcal{R}_d$ for the usual d-dimensional rigidity matroid of a graph and $U_k(E)$ for the uniform matroid of rank k on the base set E. When it is clear from the context, we will shorten $U_k(E)$ to U_k.

Let M_1 and M_2 be two matroids with common ground set E. Then the *matroid union* $M_1 \vee M_2$ is the matroid on E with the property that a subset F is independent in $M_1 \vee M_2$ if and only if it has the form $F = F_1 \cup F_2$, where F_i is independent in M_i for $i = 1, 2$.

Corollary 4.6. *We have* $\mathcal{R}_{d,k} = \mathcal{R}_{d-k} \vee U_k$.

Proof. It suffices to show that F is a basis in $\mathcal{R}_{d,k}$ if and only if we can partition F into sets F_1 and F_2 where F_1 is a basis of $\mathcal{R}_{d-k}$ and F_2 is a basis of U_k (equivalently F_2 has size k). This follows from Theorem 4.3, since F is a basis of $\mathcal{R}_{d,k}$ if and only if the graph induced by F is minimally (d, k)-rigid and F_1 is a basis of $\mathcal{R}_{d-k}$ if and only if the graph induced by F_1 is minimally $(d - k)$-rigid. □

It follows from a well known algorithm of Edmond's [9], see for example a similar discussion in [8, Section 5.1], that generic (d, k)-rigidity can be checked efficiently whenever $(d-k)$-rigidity can.

Remark 4.7. There are other types of graph rigidity where the associated rigidity matroid is the matroid union of a rigidity matroid and some other matroid. In [6], Cros et al. investigated the rigidity of unmanned aerial vehicles where there is a time lag between distance information being sent between any two vehicles. They proved that the correct rigidity matroid in this setting (with all vehicles moving in d-dimensional space) was the matroid union of the $(d - 1)$-dimensional rigidity matroid and the graphical matroid. This type of rigidity matroid is identical to that found by Dewar and Kitson in [8]. We also note that Schulze, Serocold and Theran [23] considered rigidity in the context where classes of edges can change in a specific coordinated way and also found a matroid union structure, there with the transversal matroid.

5 Global (d, k)-rigidity

We next consider global (d, k)-rigidity. To do this we first recall the following well known result for bar-joint frameworks.

Theorem 5.1 (Connelly [4]; Gortler, Healy and Thurston [11])**.** *Let (G, p) be a generic framework in $\mathbb{R}^d$. Then (G, p) is globally d-rigid if and only if there exists an equilibrium stress ω of (G, p) such that $\operatorname{rank}\Omega(\omega) = |V| - d - 1$. Furthermore, if (G, q) is also a generic framework in $\mathbb{R}^d$, then (G, q) is globally d-rigid if and only if (G, p) is globally d-rigid.*

This motivates us to work with equilibrium stresses. As noted in the proof of Theorem 4.3, a map $\sigma : E \to \mathbb{R}$ with corresponding stress matrix $\Omega(\sigma)$ is an element of $\ker DR_k(G, p)^T$ if and only if it is an equilibrium stress of $(G, \tilde{p})$ and $p_i^T \Omega(\sigma) p_i = 0$ for each $i \in \{d - k + 1, \ldots, d\}$.

Example 5.2. In Example 3.3 we showed that C_4 is minimally $(2, 1)$-rigid. From this, it is not hard to then determine that $C_4 + v_1v_3 = K_4 - v_2v_4$ is a circuit in the matroid $\mathcal{R}_{2,1}$. Hence, any generic realisation of $K_4 - v_2v_4$ has a unique equilibrium stress which is non-zero on each edge. If we fix p to be the realisation of $K_4 - v_2v_4$ with $p(v_1) = (0, 0)$, $p(v_2) = (1, 2)$, $p(v_3) = (6, 8)$ and $p(v_4) = (16, 12)$, we have the unique element $\sigma = \begin{pmatrix} 490 & 98 & -8 & 5 & -95 \end{pmatrix}^T \in \ker DR_1(K_4 - v_2v_4, p)^T$ with corresponding stress matrix

$$\Omega(\sigma) = \begin{pmatrix} 400 & -490 & 95 & -5 \\ -490 & 588 & -98 & 0 \\ 95 & -98 & -5 & 8 \\ -5 & 0 & 8 & -3 \end{pmatrix},$$

In this case we have $\operatorname{rank}\Omega(\sigma) = 2 = |V| - d + k - 1$, which is the maximum possible.

5.1 A sufficient condition

We next develop a sufficient condition for global (d, k)-rigidity (Theorem 5.6). This will generalise one direction of Theorem 5.1. We first prove the following results on equilibrium stresses.

Lemma 5.3. *For a graph $G = (V, E)$ with fixed vertex v_0, choose vectors $\sigma \in \mathbb{R}^E$ and $\lambda \in \mathbb{R}^{V \setminus \{v_0\}}$. For any integers $1 \leq k < d$, let $p : V \to \mathbb{R}^d$ be a realisation of G where $p_i(v_0) \neq 0$ for all $i \in \{d - k + 1, \ldots, d\}$. Then $(\sigma, \lambda) \in \ker J_{v_0}(G, p)^T$ if and only if $\sigma \in \ker DR_k(G, p)$ and*

$$\lambda(v) = -p_i(v_0) \left(\sum_{w \in N_G(v)} \sigma(vw)(p_i(v) - p_i(w)) \right) \tag{7}$$

for each $i \in \{d - k + 1, \ldots, d\}$.

Proof. We first note that in either direction of the implication we require that σ is an equilibrium stress of $(G, \tilde{p})$, so we may suppose that this is so throughout the proof. Label the entry of the vector $J_{v_0}(G, p)^T(\sigma, \lambda)$ that corresponds to a vertex $v \in V$ and coordinate $i \in \{1, \ldots, d\}$ by $a(v, i)$. As σ is an equilibrium stress of $(G, \tilde{p})$, it follows that for every $i \in \{1, \ldots, d - k\}$ and for every $v \in V$ we have

$$a(v, i) = \sum_{w \in N_G(v)} \sigma(vw)(p_i(v) - p_i(w)) = 0.$$

Fix $i \in \{d-k+1,\dots,d\}$. If $v \neq v_0$ then

$$a(v,i) = \frac{\lambda(v)}{p_i(v_0)} + \sum_{w \in N_G(v)} \sigma(vw)(p_i(v) - p_i(w))$$

and $a(v,i) = 0$ if and only if Equation (7) holds. Now suppose $v = v_0$. Given $\Omega(\sigma)$ is the stress matrix corresponding to σ, we have

$$\begin{aligned} a(v_0,i) &= \sum_{w \in N_G(v_0)} \sigma(v_0 w)(p_i(v_0) - p_i(w)) - \sum_{v \neq v_0} \frac{\lambda(v) p_i(v)}{p_i(v_0)^2} \\ &= \frac{1}{p_i(v_0)} \left(\sum_{v \in V} \sum_{w \in N_G(v)} \sigma(v_0 w) p_i(v)(p_i(v) - p_i(w)) \right) \\ &= \frac{p_i^T \Omega(\sigma) p_i}{p_i(v_0)}. \end{aligned}$$

Hence, $a(v_0,i) = 0$ if and only if $p_i^T \Omega(\sigma) p_i = 0$. As noted in the proof of Theorem 4.3, an equilibrium stress $\sigma : E \to \mathbb{R}$ of $(G,\tilde{p})$ is an element of $\ker DR_k(G,p)^T$ if and only if $p_i^T \Omega(\sigma) p_i = 0$ for each $i \in \{d-k+1,\dots,d\}$. Combining all of the above, we see that $a(v,i) = 0$ for all $i \in \{1,\dots,d\}$ and all $v \in V$ if and only if $\sigma \in \ker DR_k(G,p)^T$ and Equation (7) holds. This now concludes the proof. □

Lemma 5.4. *Let (G,p) be a generic (d,k)-rigid framework in $\mathbb{R}^d$ and let (G,q) be a (d,k)-equivalent framework. Then $\ker DR_k(G,p)^T = \ker DR_k(G,q)^T$ and, for any choice of $v_0 \in V$, $\ker J_{v_0}(G,p)^T = \ker J_{v_0}(G,q)^T$. Furthermore, if $\sigma \in \ker DR_k(G,p)^T$, $i \in \{d-k+1,\dots,d\}$ and $q_i = \alpha_i p_i$, then*

$$(1-\alpha_i^2)\left(\sum_{w \in N_G(v)} \sigma(vw)(p_i(v) - p_i(w)) \right) = 0. \tag{8}$$

To prove Lemma 5.4 we will use a mild extension of a proposition from Connelly [4]. This extension from Connelly's statement for polynomial functions to rational functions can be proved with the same technique as in [4].

Proposition 5.5. *Suppose that $f : \mathbb{R}^a \to \mathbb{R}^b$ is a function, where each coordinate is a rational function with integer coefficients, $p \in \mathbb{R}^a$ is generic with $f(p)$ well-defined, and $f(p) = f(q)$, for some $q \in \mathbb{R}^a$. Then there are (open) neighbourhoods N_p of p and N_q of q in $\mathbb{R}^a$ and a diffeomorphism $g : N_q \to N_p$ such that for all $x \in N_q$, $f(g(x)) = f(x)$, and $g(q) = p$.*

We will use a dashed arrow to represent that a function is not well-defined at all points. Let $F : \mathbb{R}^{d|V|} \dashrightarrow \mathbb{R}^{|E|+k(|V|-1)}$ be the concatenation of the d-dimensional rigidity map $f_{G,d}$ and the *dilation map* $\Psi : \mathbb{R}^{d|V|} \dashrightarrow \mathbb{R}^{k(|V|-1)}$ such that

$$\Psi(p) = \left(\frac{p_{d-k+1}(v)}{p_{d-k+1}(v_0)} \; , \; \dots \; , \; \frac{p_d(v)}{p_d(v_0)} \right)_{v \in V \setminus \{v_0\}}.$$

Then $J_{v_0}(G,p)$ is the Jacobian of F evaluated at $p \in \mathbb{R}^{d|V|}$.

Proof of Lemma 5.4. As p is generic and q is equivalent to p, both $F(p)$ and $F(q)$ are well-defined. By applying Proposition 5.5 to the map F, there exist open neighbourhoods N_p of p and N_q of q in $\mathbb{R}^{d|V|}$ and a diffeomorphism $g : N_q \to N_p$ such that $g(q) = p$ and, for all $q' \in N_q$, $F(g(q')) = F(q')$. Taking differentials at q, we obtain $J_{v_0}(G, q) = J_{v_0}(G, p)L$ where L is the Jacobian matrix of g at q. If $(\sigma, \lambda) \in \ker J_{v_0}(G, p)^T$ we have

$$J_{v_0}(G, q)^T(\sigma, \lambda) = L^T J_{v_0}(G, p)^T(\sigma, \lambda) = L^T\mathbf{0} = \mathbf{0}.$$

Hence $(\sigma, \lambda) \in \ker J_{v_0}(G, q)^T$. It now follows by symmetry that $\ker J_{v_0}(G, p)^T = \ker J_{v_0}(G, q)^T$. Hence, by Lemma 5.3, we have $\ker DR_k(G, p)^T = \ker DR_k(G, q)^T$. Furthermore, given our previous choice of σ and any $i \in \{d - k + 1, \dots, d\}$, we have

$$p_i(v_0)\left(\sum_{w\in N_G(v)} \sigma(vw)(p_i(v) - p_i(w))\right) = q_i(v_0)\left(\sum_{w\in N_G(v)} \sigma(vw)(q_i(v) - q_i(w))\right).$$

We now obtain Equation (8) from the above equation by substituting $q_i = \alpha_i p_i$, dividing both sides by $p_i(v_0)$ and rearranging. □

We now prove that a similar stress rank condition to Theorem 5.1 is sufficient for global (d, k)-rigidity.

Theorem 5.6. *Let (G, p) be a generic framework in $\mathbb{R}^d$. If there exists $\sigma \in \ker DR_k(G, p)^T$ such that $\operatorname{rank}\Omega(\sigma) = |V| - d + k - 1$, then (G, p) is globally (d, k)-rigid.*

Proof. Fix a vertex $v_0 \in V$ and choose any (d, k)-equivalent framework (G, q). By Lemma 5.4, Equation (8) holds for each $i \in \{d - k + 1, \dots, d\}$. If $\sum_{w\in N_G(v)} \sigma(vw)(p_i(v) - p_i(w)) = 0$ for some $i \in \{d - k + 1, \dots, d\}$ then $\operatorname{rank}\Omega(\sigma) < |V| - d + k - 1$ (since its kernel already contains the all 1's vector and $p_1, \dots, p_{d-k}$), hence $\alpha_i = \pm 1$ for each $i \in \{d - k + 1, \dots, d\}$. By applying reflections to q, we may suppose that $\alpha_i = 1$, and hence $q_i = p_i$, for all $i \in \{d - k + 1, \dots, d\}$. Hence for each $vw \in E$ we have

$$\|p(v) - p(w)\|^2 - \|q(v) - q(w)\|^2 = \|\tilde{p}(v) - \tilde{p}(w)\|^2 - \|\tilde{q}(v) - \tilde{q}(w)\|^2$$

and thus $(G, \tilde{p})$ is equivalent to $(G, \tilde{q})$. Since σ is an equilibrium stress of $(G, \tilde{p})$, $\tilde{q}$ is congruent to $\tilde{p}$ by Theorem 5.1. If we choose any two (possibly non-adjacent) vertices $v, w \in V$, then

$$\begin{aligned}
\|q(v) - q(w)\|^2 &= \|\tilde{q}(v) - \tilde{q}(w)\|^2 + \sum_{i=d-k+1}^{d} (q_i(v) - q_i(w))^2 \\
&= \|\tilde{p}(v) - \tilde{p}(w)\|^2 + \sum_{i=d-k+1}^{d} (p_i(v) - p_i(w))^2 \\
&= \|p(v) - p(w)\|^2,
\end{aligned}$$

hence q is congruent to p. □

We next show that we can use the theorem to recursively construct a large class of globally (d, k)-rigid graphs. The proof techniques of the following two lemmas are sufficiently well used in the literature that we provide only sketches.

Lemma 5.7. *Suppose (G, p) is a generic framework in $\mathbb{R}^d$ on at least $d - k + 1$ vertices. Let $G' = (V', E')$ be obtained from G by deleting an edge $e = v_1v_2$ and adding a new vertex v_0 and new edges $v_0v_1, v_0v_2, \ldots, v_0v_{d-k+1}$. Then there exists a map $q : V' \to \mathbb{R}^d$ such that $\operatorname{rank} J_{v_0}(G', q) = \operatorname{rank} J_{v_0}(G, p) + d$. Furthermore, if σ is an equilibrium stress of (G, p) with corresponding stress matrix $\Omega(\sigma)$ and $\sigma_e \neq 0$, then there exists an equilibrium stress σ' for (G', q) such that $\operatorname{rank} \Omega(\sigma') = \operatorname{rank} \Omega(\sigma) + 1$.*

Sketch of proof. Define (G', q) by putting $q(v) = p(v)$ for all $v \in V$ and $q(v_0) = \frac{1}{2}(p(v_1)+p(v_2))$. It is now straightforward to use the standard "collinear triangle" technique (see, for example, [24]) to show that $\operatorname{rank} J_{v_0}(G', q) = \operatorname{rank} J_{v_0}(G, p) + d$.

Let σ' be the equilibrium stress of (G', q) defined by putting $\sigma'_f = \sigma_f$ for all $f \in E - e$, $\sigma'_{v_0v_1} = 2\sigma_e$, $\sigma'_{v_0v_2} = 2\sigma_e$ and $\sigma_f = 0$ otherwise. It is straightforward to verify that σ' is an equilibrium stress. We may now manipulate the stress matrix $\Omega(\sigma')$ to see that $\operatorname{rank} \Omega(\sigma') = \operatorname{rank} \Omega(\sigma) + 1$. □

To see the explicit matrix manipulations (in a different context) in the final part of the proof see [16, Lemma 6.1].

Lemma 5.8. *Suppose (G, p) is a (d, k)-rigid framework in $\mathbb{R}^d$ and $\operatorname{rank} \Omega(\sigma) = |V| - d + k - 1$ for some equilibrium stress σ of (G, p). Then (G, q) has an equilibrium stress σ' with $\operatorname{rank} \Omega(\sigma') = |V| - d + k - 1$ for all generic $q \in \mathbb{R}^{d|V|}$. In addition, σ' can be chosen so that $\sigma'_e \neq 0$ for all $e \in E$.*

Sketch of proof. The first conclusion uses the technique of Connelly and Whiteley [5, Theorem 5]. In essence, we can define a function f from the set of frameworks with maximal rank (d, k)-rigidity matrix to the space $\mathbb{R}^E$ such that $f(q)$ is an equilibrium stress of (G, q), each entry of f is a rational function, and $f(p) = \omega$. By interpreting a stress matrix having full rank as an algebraic condition regarding determinants one can see that every generic framework must also have a full rank stress matrix.

The second conclusion follows from the same argument as in [16, Lemma 6.3]. Briefly, suppose $\sigma'_e = 0$ for some $e \in E$. Then σ' is an equilibrium stress of the (d, k)-rigid framework $(G - e, p)$. Hence there is another equilibrium stress σ^* obtained by adding e which is non-zero on e. Then for sufficiently small ε the matrix $\Omega(\sigma' + \varepsilon\sigma^*)$ will have the same rank as $\Omega(\sigma')$. □

Corollary 5.9. *Let $G = (V, E)$, let (G, p) be a generic framework in $\mathbb{R}^d$ and suppose there exists $\sigma \in \ker DR_k(G, p)^T$ such that $\sigma_e \neq 0$ for all $e \in E$ and $\operatorname{rank} \Omega(\sigma) = |V| - d + k - 1$. Let H be a graph obtained from G by a sequence of $(d - k)$-dimensional 1-extensions and edge additions. Then any generic framework (H, q) is globally (d, k)-rigid.*

Proof. By Theorem 5.6 it now suffices to show that a sequence of $(d-k)$-dimensional 1-extensions and edge additions preserves a full rank stress matrix. This in turn follows from Lemma 5.7 and the fact that we may simply choose $\sigma_e = 0$ for any edge addition e provided that we can show that any edge f we perform a 1-extension on has $\sigma_f \neq 0$. This was proved in Lemma 5.8. □

5.2 A necessary condition

Hendrickson proved the following natural necessary conditions for global rigidity of bar-joint frameworks.

Theorem 5.10 (Hendrickson [13])**.** *Let (G,p) generic framework in $\mathbb{R}^d$. If (G,p) is globally d-rigid then either $|V(G)| \leq d+1$ and G is complete or $|V(G)| \geq d+2$ and G is $(d+1)$-connected and $G-e$ is d-rigid for any edge e of G.*

We can obtain a weak version of a similar result for arbitrary frameworks by linking global (d,k)-rigidity to global $(d-k)$-rigidity. In the generic case we suspect a stronger statement may be possible.

Proposition 5.11. *Let (G,p) a framework in $\mathbb{R}^d$ and suppose (G,p) is globally (d,k)-rigid. Then $(G,\tilde{p})$ is globally $(d-k)$-rigid.*

Proof. Suppose that $(G,\tilde{p})$ is not globally $(d-k)$-rigid. Then there exists a realisation $\tilde{q} : V \to \mathbb{R}^{d-k}$ of G where $(G,\tilde{q})$ is equivalent to, but not congruent to, $(G,\tilde{p})$. Define $q : V \to \mathbb{R}^d$ to be the realisation of G where $q(v) = (\tilde{q}(v), p_{d-k+1}(v), \ldots, p_d(v))$ for each $v \in V$. For each pair $v, w \in E$ we have

$$\|p(v)-p(w)\|^2 - \|q(v)-q(w)\|^2 = \|\tilde{p}(v)-\tilde{p}(w)\|^2 - \|\tilde{q}(v)-\tilde{q}(w)\|^2,$$

hence (G,q) is (d,k)-equivalent to, but not congruent to, (G,p), i.e., (G,p) is not globally (d,k)-rigid. □

Observe that the resulting necessary connectivity condition for global (d,k)-rigidity, by depending on k, can be much lower than that needed for global d-rigidity. The following simple lemma illustrates that globally (d,k)-rigid graphs need not be highly connected.

Lemma 5.12. *Let G_1 and G_2 be graphs on at least $d-k+2$ vertices and with at least $d-k+1$ vertices in common. Let (G,p) be a generic realisation of $G = G_1 \cup G_2$ and let $p^i = p|_{G_i}$. Suppose that (G_i, p^i) is globally (d,k)-rigid for $i = 1,2$. Then (G,p) is globally (d,k)-rigid.*

Proof. Let (G,q) be a (d,k)-equivalent framework to (G,p). By applying a suitable isometry to q we may assume that $p(u) = q(u)$ for all $u \in V(G_1) \cap V(G_2)$. Since (G_1,p^1) is globally (d,k)-rigid, $q|_{G_1} = \iota \circ p^1$ for some isometry ι. Since (G_2,p^2) is globally (d,k)-rigid, there is a unique equivalent realisation of G_2 which maps u to $p(u)$ for all $u \in V(G_1) \cap V(G_2)$. Since both $(G_2, q|_{G_2})$ and $(G_2, \iota \circ p^2)$ have this property, $q|_{G_2} = \iota \circ p^2$. Hence $q = \iota \circ p$ and (G,p) is congruent to (G,q). □

6 Concluding remarks

We conclude the paper with two conjectures on generic global (d,k)-rigidity. The first is that a full stress rank is necessary as well as sufficient, analogous to Theorem 5.1.

Conjecture 6.1. *Let (G,p) be a generic framework in $\mathbb{R}^d$. If (G,p) is globally (d,k)-rigid and G is not complete, then there exists $\sigma \in \ker DR_k(G,p)^T$ such that $\operatorname{rank}\Omega(\sigma) = |V| - d + k - 1$. Moreover, if (G,q) is also a generic framework in $\mathbb{R}^d$, then (G,q) is globally (d,k)-rigid if and only if (G,p) is globally (d,k)-rigid.*

We know from Proposition 5.11 and Theorem 5.1 that if (G,p) is globally (d,k)-rigid, then there exists an equilibrium stress σ of $(G,\tilde{p})$ such that $\operatorname{rank}\Omega(\sigma) = |V| - d + k - 1$. For $\sigma \in \ker DR_k(G,p)^T$, we also require that $p_i^T \Omega(\sigma) p_i = 0$ for all $i \in \{d-k+1, \ldots, d\}$. It seems non-trivial to prove this.

We conclude the paper with an alternative conjectured characterisation motivated by the characterisation in [15]. A framework (G, p) is *redundantly* $(d-k)$*-rigid* if it is still $(d-k)$-rigid after deleting any one edge.

Conjecture 6.2. *Suppose positive integers* d, k *are chosen so that* $d - k \leq 2$ *and* (G, p) *is generic. Then the following are equivalent:*

(i) (G, p) *is globally* (d, k)*-rigid;*

(ii) there exists a set F *of* k *edges such that* $(G - F, \tilde{p})$ *is* $(d-k+1)$*-connected and redundantly* $(d-k)$*-rigid; and*

(iii) there exists a set F *of* k *edges such that* $(G - F, \tilde{p})$ *is globally* $(d-k)$*-rigid.*

Since $d - k \leq 2$, it follows from either a folklore result (when $d - k = 1$) or [15] (when $d - k = 2$) that (ii) and (iii) are equivalent. We suspect that (i) implies (iii) could be proved by a Hendrickson-type argument [13]. If this were true it would suffice to prove that (ii) implies (i). We verify this in the case when $d = 2$ and $k = 1$.

Lemma 6.3. *Let* (G, p) *be generic. Suppose* G *contains an edge* e *such that* $G - e$ *is 2-connected. Then* (G, p) *is globally* $(2, 1)$*-rigid.*

Proof. Let H denote the graph obtained from K_4 by deleting an edge. It is easy to see that every graph G such that $G - e$ is 2-connected can be obtained from H by subdividing edges and adding new edges. In Example 5.2 we constructed a nowhere zero full rank stress of a specific (d, k)-rigid realisation of H. Hence the result follows from Lemma 5.8 and Corollary 5.9 by induction. □

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