

QUANTITATIVE BOUNDS IN THE NONLINEAR ROTH THEOREM

SARAH PELUSE AND SEAN PRENDIVILLE

ABSTRACT. We show that there exists $c > 0$ such that any subset of $\{1, \dots, N\}$ of density at least $(\log \log N)^{-c}$ contains a nontrivial progression of the form $x, x+y, x+y^2$. This is the first quantitatively effective version of the Bergelson–Leibman polynomial Szemerédi theorem for a progression involving polynomials of differing degrees. Our key innovation is an inverse theorem characterising sets for which the number of configurations $x, x+y, x+y^2$ deviates substantially from the expected value. In proving this, we develop the first effective instance of a concatenation theorem of Tao and Ziegler, with polynomial bounds.

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1. INTRODUCTION

Gowers [Gow01a, Problem 11.4] has posed the problem of obtaining quantitative bounds in the polynomial Szemerédi theorem of Bergelson and Leibman [BL96]. This states that given $P_1, \dots, P_m \in \mathbb{Z}[y]$ all having zero constant term, any subset of $\{1, 2, \dots, N\}$ lacking the polynomial progression

$$(1.1) \quad x, x + P_1(y), \dots, x + P_m(y) \quad (y \in \mathbb{Z} \setminus \{0\})$$

has size $o(N)$. Hitherto, all effective versions of this result have been restricted to two-term progressions [Sár78a, Sár78b, PSS88, BPPS94, Sli03, Luc06, Ric19], arithmetic progressions with common difference equal to a perfect power [Gow98, Gow01b, Pre17], or are concerned with the analogous question over finite fields [BC17, Pel18, DLS20, Pel19]. In this paper, we obtain the first bound over the

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integers for a progression of length greater than two and involving polynomials of differing degrees.

Theorem 1.1. *There exists $c > 0$ such that if $A \subset \{1, 2, \dots, N\}$ contains no progression of the form¹*

$$(1.2) \quad x, x + y, x + y^2 \quad (y \neq 0),$$

then²

$$(1.3) \quad |A| \ll N(\log \log N)^{-c}.$$

Remark. Keeping track of exponents in our proof, $c = 1/2^{150}$ is admissible.

Our proof of Theorem 1.1 adapts a strategy of the first author [Pel19] from finite fields to the integer setting. There are multiple issues with applying these ideas in the integers, so the proof of Theorem 1.1 requires several significant modifications and additions. The key insight of [Pel19] is that if one can control the count of an affine independent polynomial progression by the Gowers U^s -norm, then one can use this and an understanding of shorter progressions to prove control of the count by the U^{s-1} -norm. Thus, if one understands shorter progressions and can show control by any U^s -norm, then one can deduce control by the U^1 -seminorm. As the U^1 -seminorm measures correlation with constant functions, this is very powerful information.

Over the integers there are certain ‘local’ issues that preclude effective control by the global U^1 -seminorm. Instead, we control our counting operator by an average of U^1 -seminorms, each localised to a progression of length $N^{1/2}$ and small common difference.

Theorem 1.2 (Inverse theorem for nonlinear Roth). *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a 1-bounded function supported in the interval $[N] := \{1, \dots, N\}$ and let $\delta > 0$. Suppose that*

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in [N^{1/2}]} f(x)f(x+y)f(x+y^2) \right| \geq \delta N^{\frac{3}{2}}.$$

Then either $N \ll \delta^{-O(1)}$ or there exist positive integers $q \ll \delta^{-O(1)}$ and $N' \gg \delta^{O(1)}N^{1/2}$ such that

$$(1.4) \quad \sum_{x \in \mathbb{Z}} \left| \sum_{y \in [N']} f(x+qy) \right| \gg \delta^{O(1)}NN'.$$

To derive our density bound (Theorem 1.1), we use our inverse theorem (Theorem 1.2) to prove that sets lacking the nonlinear Roth configuration have a density increment on a progression with small common difference. In a sequel [PP20] we further bootstrap our inverse theorem (Theorem 1.2) to obtain a larger density increment, and thereby replace the double logarithm in (1.3) with a single logarithm. This bootstrapping procedure follows the (now standard) energy increment procedure of Heath–Brown and Szemerédi [HB87, Sze90], and to avoid obfuscating our argument with further technicalities, we delegate this improvement to a subsequent paper.

¹We call this the *nonlinear Roth* configuration, after Bourgain and Chang [BC17].

²See §1.1 for our conventions regarding asymptotic notation such as ‘ \ll ’.

A variety of perspectives, both ergodic and combinatorial, can be used to establish that the count of nonlinear Roth configurations is controlled by a U^1 -seminorm, in a qualitative sense. The novelty of Theorem 1.2 is that this is demonstrated in a quantitatively effective manner, with polynomial bounds, by avoiding standard tools of higher order Fourier analysis (which give poorer bounds). Indeed, whilst Gowers norms of high degree such as the U^5 -norm play a role in our argument, we completely avoid using the inverse theorem for these norms, the equidistribution theory of nilsequences, or any version of the arithmetic regularity lemma, requiring only Fourier analysis and numerous applications of the Cauchy–Schwarz inequality.

Perhaps the biggest difficulty in adapting the argument of [Pel19] to the integer setting is in first showing that the count of nonlinear Roth configurations is controlled by some global U^s -norm. While this is not too difficult to accomplish in finite fields using Bergelson and Leibman’s PET induction scheme³ [BL96], in the integers such an argument yields control in terms of an average of certain constrained Gowers norms. We must then show, with quantitative bounds, that this average of constrained Gowers norms is controlled by a genuine global U^s -norm. Such ‘concatenation’ results have been proved by Tao and Ziegler [TZ16]; however the quantitative dependence in their argument is (at best) tower-type. We offer a different proof of an instance of their concatenation theorem, one that yields polynomial bounds.

This paper is organised as follows. In §1.2 we give a more detailed outline of the proof of Theorem 1.1. In §2 we derive Theorem 1.1 from a density increment lemma, whose proof is deduced from a generalisation of our inverse theorem (Theorem 1.2) in §8. This generalised inverse theorem (Theorem 7.1) is proved in §7. In §3 we show that our counting operator is controlled by an average of constrained Gowers norms, and in §5 we show that these constrained averages are controlled by a single global Gowers norm of higher degree. This ‘concatenation’ argument uses an arithmetic variant of the box norm inverse theorem, which we state and prove in §4. Finally in §6 we describe our degree lowering procedure, showing how global control of our configuration by the U^s -norm implies global control by the U^{s-1} -norm.

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1.1. Notation.

1.1.1. *Standard conventions.* We use \mathbb{N} to denote the positive integers. For real $X \geq 1$, write $[X] = \{1, 2, \dots, \lfloor X \rfloor\}$. A complex-valued function is *1-bounded* if the modulus of the function does not exceed 1.

We use counting measure on \mathbb{Z} , so that for $f, g : \mathbb{Z} \rightarrow \mathbb{C}$ we have

$$\langle f, g \rangle := \sum_x f(x) \overline{g(x)} \quad \text{and} \quad \|f\|_{L^p} := \left(\sum_x |f(x)|^p \right)^{\frac{1}{p}}.$$

³PET stands for either *polynomial ergodic theorem* or *polynomial exhaustion technique* in various works of Bergelson and Leibman.

Any sum of the form \sum_x is to be interpreted as a sum over \mathbb{Z} . We use Haar probability measure on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, so that for measurable $F : \mathbb{T} \rightarrow \mathbb{C}$ we have

$$\|F\|_{L^p} := \left(\int_{\mathbb{T}} |F(\alpha)|^p d\alpha \right)^{\frac{1}{p}} = \left(\int_0^1 |F(\alpha)|^p d\alpha \right)^{\frac{1}{p}}$$

For $\alpha \in \mathbb{T}$ we write $\|\alpha\|$ for the distance to the nearest integer.

For a finite set S and function $f : S \rightarrow \mathbb{C}$, denote the average of f over S by

$$\mathbb{E}_{s \in S} f(s) := \frac{1}{|S|} \sum_{s \in S} f(s).$$

Given functions $f, g : G \rightarrow \mathbb{C}$ on an additive group with measure μ_G we define their convolution by

$$(1.5) \quad (f * g)(x) := \int_G f(x - y)g(y) d\mu_G,$$

when this makes sense.

We define the Fourier transform of $f : \mathbb{Z} \rightarrow \mathbb{C}$ by

$$(1.6) \quad \hat{f}(\alpha) := \sum_x f(x)e(\alpha x) \quad (\alpha \in \mathbb{T}),$$

again, when this makes sense. Here $e(\alpha)$ stands for $e^{2\pi i \alpha}$.

The *difference function* of $f : \mathbb{Z} \rightarrow \mathbb{C}$ is the function $\Delta_h f : \mathbb{Z} \rightarrow \mathbb{C}$ given by

$$(1.7) \quad \Delta_h f(x) = f(x) \overline{f(x+h)}.$$

Iterating, we set

$$\Delta_{h_1, \dots, h_s} f := \Delta_{h_1} \dots \Delta_{h_s} f.$$

This allows us to define the *Gowers U^s -norm*

$$(1.8) \quad \|f\|_{U^s} := \left(\sum_{x, h_1, \dots, h_s} \Delta_{h_1, \dots, h_s} f(x) \right)^{1/2^s}.$$

When $S \subset \mathbb{Z}$ we define the *localised Gowers U^s -norm*

$$(1.9) \quad \|f\|_{U^s(S)} := \|f1_S\|_{U^s}.$$

Notice that the left-hand side of (1.4) is equal to

$$\sum_x \|f\|_{U^1(x+q \cdot [N'])}.$$

For a function f and positive-valued function g , we write $f \ll g$ or $f = O(g)$ if there exists a constant C such that $|f(x)| \leq Cg(x)$ for all x . We write $f = \Omega(g)$ if $f \gg g$. We sometimes opt for a more explicit approach, using C to denote a large absolute constant, and c to denote a small positive absolute constant. The values of C and c may change from line to line.

1.1.2. *Local conventions.* Up to normalisation, all of the above are well-used in the literature. Next we list notation specific to our paper. We have tried to minimise this in order to aid the casual reader.

The quantity $(N/q)^{1/2}$ appears repeatedly in our arguments, where q is an integer fixed throughout the majority of our paper. Unless otherwise specified, we therefore adopt the convention that

$$(1.10) \quad M := \left\lfloor \sqrt{N/q} \right\rfloor.$$

Define the *counting operator* on the functions $f_i : \mathbb{Z} \rightarrow \mathbb{C}$ by

$$(1.11) \quad \Lambda_q(f_0, f_1, f_2) := \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x+y) f_2(x+qy^2).$$

When the f_i all equal f we simply write $\Lambda_q(f)$.

For a real parameter $H \geq 1$, we use $\mu_H : \mathbb{Z} \rightarrow [0, 1]$ to represent the following normalised Fejér kernel

$$(1.12) \quad \mu_H(h) := \frac{1}{[H]} \left(1 - \frac{|h|}{[H]} \right)_+ = \frac{(1_{[H]} * 1_{[H]})(h)}{[H]^2}.$$

For a multidimensional vector $h \in \mathbb{Z}^d$ we write

$$(1.13) \quad \mu_H(h) := \mu_H(h_1) \cdots \mu_H(h_d).$$

We observe that this is a probability measure on \mathbb{Z}^d with support in the box $(-H, H)^d$.

1.2. An outline of our argument.

1.2.1. *The density increment.* Our proof proceeds via a density increment argument, the same method of proof used by Roth [Rot53] and Gowers [Gow98, Gow01b] to bound the size of sets lacking arithmetic progressions. In Gowers' formulation, if $A \subset [N]$ has density $\delta := |A|/N$ and lacks (say) a 4-term arithmetic progression, then either $N \ll \delta^{-O(1)}$ or there exists a progression $P = a + q \cdot [N']$ of length $N' \gg N^{\delta^{O(1)}}$ on which A has increased density $|A \cap P|/|P| \geq \delta + \delta^{O(1)}$. Consider the rescaled version $A' \subset [N']$ of $A \cap P$ defined by

$$(1.14) \quad A' := \{n \in [N'] : a + qn \in A \cap P\},$$

and note that A' also lacks 4-term arithmetic progressions. We then repeat this process with A' in place of A . This iteration cannot continue indefinitely; indeed since the density cannot exceed one, the procedure must terminate in $O(\delta^{-O(1)})$ steps. The only explanation for termination is that the length N'' of the interval at the final stage of our iteration is too short, $N'' \ll \delta^{-O(1)}$, and since $N'' \geq N^{\exp(-\delta^{O(1)})}$, this allows us to extract a bound on δ .

The success of the above argument relies crucially on the fact that 4-term arithmetic progressions are preserved under translation and scaling, and similarly the argument in [Pre17] relies on the fact that arithmetic progressions with common difference equal to a perfect d^{th} power are preserved under translation and scaling by a perfect d^{th} power. These are very special properties lacked by the vast majority of polynomial progressions, including the nonlinear Roth configuration (1.2).

Indeed, if $A \subset [N]$ has no nontrivial configurations of the form (1.2), then the rescaled set $A' \subset [N']$ defined as in (1.14) has no nontrivial configurations of the

form $x, x + y, x + qy^2$. But if $q > N'$, then every subset of $[N']$ has this property because x and $x + qy^2$ cannot both lie in $[N']$ when $y \neq 0$, and thus there is no hope of continuing the density increment argument in this case. In contrast, the largeness of $q > N'$ does not affect the arguments of [Rot53, Gow98, Gow01b], because these papers consider progressions that are preserved under scaling by q (or q^d in the case of [Pre17]).

To deal with the poor behavior of the nonlinear Roth configuration under scaling, we prove a stronger density increment lemma that ensures that the arithmetic progression on which we find a density increment has very small step size. Our methods show that if $A \subset [N]$ has density $\delta := |A|/N$ and lacks nontrivial configurations of the form (1.2), then there exists a progression $P = a + q \cdot [N']$ with common difference $q \ll \delta^{-O(1)}$ and length $N' \gg \delta^{O(1)}N^{1/2}$ such that we have the density increment

$$(1.15) \quad \frac{|A \cap P|}{|P|} \geq \delta + \delta^{O(1)}.$$

Defining $A' \subset [N']$ to be the rescaled set as in (1.14), we thus see that A' has increased density in $[N']$ and lacks nontrivial configurations of the form

$$(1.16) \quad x, x + y, x + qy^2.$$

The coefficient q is sufficiently small that the methods employed to treat our original configuration (1.2) still apply to the new configuration (1.16), allowing us to prove a similar density increment result for sets lacking (1.16). We can thus continue the density increment iteration, which terminates in at most $O(\delta^{-O(1)})$ steps. Such an argument yields a density bound of the form

$$\delta \ll (\log \log N)^{-c}$$

for some small absolute constant $c > 0$.

Our general density increment result is stated in Lemma 2.1 and concerns the configuration (1.16). It is a simple deduction from our inverse theorem (Theorem 1.2), or more precisely a generalisation (Theorem 7.1) of our inverse theorem, extending from the nonlinear Roth configuration (1.2) to its dilated analogue (1.16). In the remainder of this section we describe the ideas behind our inverse theorem, Theorem 1.2.

1.2.2. *Quantitative concatenation.* To prove Theorem 1.2, we first prove that our counting operator

$$(1.17) \quad \mathbb{E}_{x \in [N]} \mathbb{E}_{y \in [M]} f_0(x) f_1(x + y) f_2(x + y^2)$$

is controlled by the U^5 -norm of f_2 . The purpose of this subsection is to sketch how we do this with polynomial bounds.

By repeatedly applying both the Cauchy–Schwarz inequality and the van der Corput inequality, we show in §3 that, when $f_0, f_1, f_2 : \mathbb{Z} \rightarrow \mathbb{C}$ are 1-bounded functions supported in the interval $[N]$, largeness of the counting operator (1.17) implies largeness of (a weighted version of) the sum

$$(1.18) \quad \sum_{a, b \in [N^{1/2}]} \sum_{h_1, h_2, h_3 \in [N^{1/2}]} \sum_x \Delta_{ah_1, bh_2, (a+b)h_3} f_2(x),$$

which is always real and nonnegative by [GT10, Appendix B]. This deduction is made following the PET induction scheme of Bergelson and Leibman [BL96]. The gain in working with the counting operator (1.18) over (1.17) is that univariate polynomials such as y^2 , whose image constitutes a sparse set, have been replaced by bilinear forms such as ah_1 , whose image is much denser.

In §§4–5, we show that largeness of (1.18) implies largeness of $\|f_2\|_{U^5}$. If there were no dependence between the coefficients of the h_i in (1.18), then it would be easy to bound (1.18) in terms of $\|f_2\|_{U^s}$ for some $s \ll 1$. We illustrate why this is the case for the sum

$$(1.19) \quad \sum_{a,b,c \in [N^{1/2}]} \sum_{h_1, h_2, h_3 \in [N^{1/2}]} \sum_x \Delta_{ah_1, bh_2, ch_3} f_2(x).$$

The following fact, the formal version of which is Lemma 5.3, is key.

Claim 1.3. *If $\sum_{a, h \in [N^{1/2}]} \sum_x \Delta_{ah} f(x)$ is large then so is $\|f\|_{U^2}$.*

Sketch proof. Apply the Cauchy–Schwarz inequality to double the a and h variables, yielding a bound in terms of

$$(1.20) \quad \sum_{a, a' \in [N^{1/2}]} \sum_{h, h' \in [N^{1/2}]} \sum_x \Delta_{ah - a'h'} f(x).$$

For a random choice of $a, a' \in [N^{1/2}]$, the progression $a \cdot [N^{1/2}] - a' \cdot [N^{1/2}]$ covers a large portion of the interval $(-N, N)$ relatively smoothly. One can make this intuition rigorous using a bit of Fourier analysis and thus deduce largeness of the sum $\sum_{k \in (-N, N)} \sum_x \Delta_k f(x) e(\alpha k)$ for some $\alpha \in \mathbb{T}$. Largeness of $\|f\|_{U^2}$ then follows by another application of the Cauchy–Schwarz inequality. \square

Applying Claim 1.3 three times yields largeness of $\|f_2\|_{U^6}$.

The problem remains of how to handle the dependency between the differencing parameters in (1.18). If we were not concerned with quantitative bounds, we could apply a ‘concatenation’ theorem of Tao and Ziegler [TZ16, Theorem 1.24] to obtain largeness of the U^9 -norm of f_2 . However, the qualitative nature of their result means that it cannot be used to obtain bounds in the nonlinear Roth theorem. In its place we prove Theorem 5.6, which is a special case of [TZ16, Theorem 1.24], using a very different argument that gives polynomial bounds. We spend the remainder of this subsection sketching the argument.

We begin by viewing (1.18) as the average

$$(1.21) \quad \sum_{a, h_1 \in [N^{1/2}]} \|\Delta_{ah_1} f_2\|_a,$$

where

$$(1.22) \quad \|f\|_a^4 := \sum_{b \in [N^{1/2}]} \sum_{h_2, h_3 \in [N^{1/2}]} \sum_x \Delta_{bh_2, (a+b)h_3} f(x).$$

One can view this as an average of 2-dimensional Gowers box norms where, for fixed b , the inner sum corresponds to a box norm in the ‘directions’ b and $a + b$. Note that if we could bound the quantity $\|\Delta_{ah_1} f_2\|_a$ in terms of the U^4 -norm of $\Delta_{ah_1} f_2$ for many pairs (a, h_1) , then by Claim 1.3 we deduce largeness of the U^5 -norm of f_2 . We show that, on average, one can indeed control $\|\cdot\|_a$ in terms of

$\|\cdot\|_{U^4}$, with polynomial bounds. The following can be extracted from the proof of (the more general) Theorem 5.6.

Lemma 1.4. *For each $a \in [N^{1/2}]$ let $f_a : \mathbb{Z} \rightarrow \mathbb{C}$ be a 1-bounded function supported in the interval $[N]$. Suppose that*

$$\mathbb{E}_{a \in [N^{1/2}]} \|f_a\|_a^4 \geq \delta \|1_{[N]}\|_a^4.$$

Then

$$\mathbb{E}_{a \in [N^{1/2}]} \|f_a\|_{U^4}^{16} \gg \delta^{O(1)} \|1_{[N]}\|_{U^4}^{16}.$$

To finish this subsection, we briefly discuss the proof of this key lemma. For most choices of $a, b \in [N^{1/2}]$, the ‘directions’ b and $a + b$ of the box norm

$$(1.23) \quad \sum_{h_2, h_3 \in [N^{1/2}]} \sum_x \Delta_{bh_2, (a+b)h_3} f_a(x)$$

are close to ‘independent’, in the sense that at least one of the directions b and $a + b$ is large and together they have small greatest common divisor. The proof of Lemma 1.4 thus begins by viewing $\|\cdot\|_a$ as an average of box norms

$$(1.24) \quad \|f\|_{\square(X, Y)}^4 := \sum_{x_1, x_2 \in X, y_1, y_2 \in Y} f(x_1, y_1) \overline{f(x_1, y_2) f(x_2, y_1) f(x_2, y_2)}.$$

It is easy to show that largeness of $\|f\|_{\square(X, Y)}$ implies that f correlates with a function of the form $(x, y) \mapsto l(x)r(y)$. We show, analogously, that provided b and $a + b$ are not too small and have greatest common divisor not too large, then largeness of the arithmetic box norm (1.23) implies that f_a correlates with a product $g_b h_{a+b}$ of 1-bounded functions, where g_b is b -periodic and h_{a+b} is almost periodic under shifts by integer multiples of $a + b$. As a consequence, for most $a \in [N^{1/2}]$, largeness of $\|f_a\|_a$ implies largeness of

$$(1.25) \quad \sum_{b \in [N^{1/2}]} \sum_x f_a(x) g_b(x) h_{a+b}(x).$$

In fact, an application of the Cauchy–Schwarz inequality allows us to give an explicit description of h_{a+b} in terms of f_a , namely we may take it to be of the form

$$(1.26) \quad h_{a+b}(x) = \mathbb{E}_{k \in [N^{1/2}]} f_a(x + (a + b)k) g_b(x + (a + b)k).$$

This presentation makes apparent the almost periodicity of h_{a+b} .

Claim 1.5. *Largeness of (1.25) implies that h_{a+b} has large U^3 -norm on average over $b \in [N^{1/2}]$.*

Let us first show why Claim 1.5 in turn implies that f_a has large U^4 -norm, completing our sketch proof of Lemma 1.4. The expression (1.26) and the triangle inequality for Gowers norms together imply that largeness of $\mathbb{E}_{b \in [N^{1/2}]} \|h_{a+b}\|_{U^3}$ implies largeness of $\mathbb{E}_{b \in [N^{1/2}]} \|f_a g_b\|_{U^3}$. Utilising the b -periodicity of g_b we have

$$(1.27) \quad \|f_a g_b\|_{U^3} = \mathbb{E}_{k \in [N^{1/2}]} \|f_a(\cdot) g_b(\cdot + bk)\|_{U^3}.$$

The product $f_a(\cdot) g_b(\cdot + bk)$ resembles a difference function in the direction b . Indeed the Gowers–Cauchy–Schwarz inequality (see [Tao12, Exercise 1.3.19]) shows that if (1.27) is large on average over $b \in [N^{1/2}]$ then so is

$$\mathbb{E}_{b, k \in [N^{1/2}]} \|\Delta_{bk} f_a\|_{U^3}$$

Largeness of $\|f_a\|_{U^4}$ then follows from Claim 1.3.

Finally we sketch the proof of Claim 1.5. The Cauchy–Schwarz inequality allows us to remove the weight $f_a(x)$ from (1.25) and deduce largeness of

$$\sum_x \sum_{b,b' \in [N^{1/2}]} \overline{g_b(x)h_{a+b}(x)} g_{b'}(x) h_{a+b'}(x).$$

Using the periodicity properties of g_b , $g_{b'}$ and h_{a+b} , this is approximately equal to

$$\sum_x \sum_{\substack{b,b' \in [N^{1/2}] \\ k_1, k_2, k_3 \in [N^{1/2}]}} \overline{g_b(x - bk_1)h_{a+b}(x - (a+b)k_2)} g_{b'}(x - b'k_3) h_{a+b'}(x).$$

Changing variables in x , we obtain largeness of the sum

$$\begin{aligned} \sum_x \sum_{\substack{b,b' \in [N^{1/2}] \\ k_1, k_2, k_3 \in [N^{1/2}]}} \overline{g_b(x + (a+b)k_2 + b'k_3)h_{a+b}(x + bk_1 + b'k_3)} \\ g_{b'}(x + bk_1 + (a+b)k_2) h_{a+b'}(x + bk_1 + (a+b)k_2 + b'k_3). \end{aligned}$$

The point here is that all but the last function have arguments depending on at most two of the bilinear forms bk_1 , $(a+b)k_2$ and $b'k_3$. This enables us to employ the Gowers–Cauchy–Schwarz inequality (in the form of Lemma A.4) to deduce largeness of a sum similar to

$$\sum_x \sum_{\substack{b,b' \in [N^{1/2}] \\ k_1, k_2, k_3 \in [N^{1/2}]}} \Delta_{bk_1, (a+b)k_2, b'k_3} h_{a+b'}(x).$$

The utility of this expression is that the directions of the differencing parameters are all ‘independent’ of the direction of periodicity of $h_{a+b'}$. Indeed the approximate $(a+b')$ -periodicity of $h_{a+b'}$ means that one can replace $\Delta_y h_{a+b'}$ with $\mathbb{E}_k \Delta_{y+(a+b')k} h_{a+b'}$ at the cost of a small error. We thereby obtain largeness of

$$(1.28) \quad \sum_x \sum_{b,b' \in [N^{1/2}]} \sum_{\substack{k_1, k_2, k_3 \in [N^{1/2}] \\ k'_1, k'_2, k'_3 \in [N^{1/2}]}} \Delta_{bk_1 + (a+b')k'_1, (a+b)k_2 + (a+b')k'_2, b'k_3 + (a+b')k'_3} h_{a+b'}(x).$$

For a random triple $(a, b, b') \in [N^{1/2}]$ the greatest common divisor of the pairs $(b, a+b)$, $(a+b, a+b')$ and $(b', a+b')$ are all small, and these are the pairs appearing in the differencing parameters of (1.28). The argument used to treat (1.20) may therefore be employed to replace (1.28) with

$$\sum_x \sum_{b' \in [N^{1/2}]} \sum_{k_1, k_2, k_3 \in [N]} \Delta_{k_1, k_2, k_3} h_{a+b'}(x),$$

and thereby yield Claim 1.5.

1.2.3. Degree lowering. After we have shown that the counting operator (1.17) is controlled by the U^5 -norm of f_2 , we carry out a ‘degree lowering’ argument. This technique originated in the work [Pel19] in finite fields. The basic idea is that, under certain conditions, one can combine U^s -control with understanding of

two-term progressions to deduce U^{s-1} -control. Repeating this gives a sequence of implications

$$U^5\text{-control} \implies U^4\text{-control} \implies U^3\text{-control} \implies U^2\text{-control} \implies U^1\text{-control}.$$

Despite the appearance of the U^5 -norm, U^4 -norm, and U^3 -norm, the degree lowering argument, both in [Pel19] and here, does not require the U^s -inverse theorem for any $s \geq 3$. Instead it relies on Fourier analysis in the place of these inverse theorems.

As was mentioned in §1, adapting the degree lowering argument of [Pel19] to the integer setting requires several significant modifications. The first modification is that the U^s -control described above is control in terms of the U^s -norm of the dual function⁴

$$(1.29) \quad F(x) := \mathbb{E}_{y \in [N^{1/2}]} f_0(x - y^2) f_1(x + y - y^2).$$

Thus, to begin the degree lowering argument, we must show that the counting operator (1.17) is controlled by the U^5 -norm of the dual $\|F\|_{U^5}$. To do this, we use the fact that the counting operator is controlled by $\|f_2\|_{U^5}$ together with a simple application of the Cauchy–Schwarz inequality; for details see §7.

We illustrate our degree lowering procedure by sketching how U^3 -control of the dual (1.29) implies U^2 -control, starting from the assumption that

$$\|F\|_{U^3}^8 \geq \delta \|1_{[N]}\|_{U^3}^8.$$

Using the fact that $\|F\|_{U^3}^8 = \sum_h \|\Delta_h F\|_{U^2}^4$ and applying the U^2 -inverse theorem, we deduce the existence of a function $\phi : \mathbb{Z} \rightarrow \mathbb{T}$ such that, for at least $\gg \delta N$ choices of differencing parameter h , we have

$$(1.30) \quad \left| \sum_{x \in [N]} \Delta_h F(x) e(\phi(h)x) \right| \gg \delta N.$$

Note that if, in the above inequality, we could replace the function $\phi(h)$ by a constant $\beta \in \mathbb{T}$ not depending on h , then we could easily deduce largeness of $\|F\|_{U^2}$. Indeed, writing $g(h)$ for the conjugate phase of the sum inside absolute values, this would give

$$\sum_{x,h} \overline{g(h)F(x+h)} F(x) e(\beta x) \gg \delta^{O(1)} N^3,$$

and the usual argument⁵ showing U^2 -control of the equation $x + y = z$ implies that $\|F\|_{U^2}^4 \gg \delta^{O(1)} \|1_{[N]}\|_{U^2}^4$. It thus remains to show that such a β exists.

Expanding the definition of the difference and dual functions in (1.30) and using the Cauchy–Schwarz inequality (as is done in greater generality in the proof of Lemma 6.3), one can show that there exists h' such that for many h satisfying

⁴This terminology comes from the fact that the counting operator (1.17) equals the inner product of f_2 with \overline{F} .

⁵One can either use orthogonality and extraction of a large Fourier coefficient, as in the proof of Lemma A.1, or use two applications of the Cauchy–Schwarz inequality.

(1.30) we have

$$\left| \sum_x \sum_{y \in [N^{1/2}]} \Delta_{h-h'} f_0(x) \Delta_{h-h'} f_1(x+y) e([\phi(h) - \phi(h')][x+y^2]) \right| \gg \delta^{O(1)} N^{3/2}$$

Further application of the Cauchy–Schwarz inequality allows us to remove the difference functions from the above inequality and deduce largeness of the exponential sum

$$\sum_{z \in [N^{1/2}]} \left| \sum_{y \in [N^{1/2}]} e(2[\phi(h) - \phi(h')] yz) \right|.$$

Summing the inner geometric progression and using a Vinogradov-type lemma then shows that $\phi(h) - \phi(h')$ is major arc. There are very few major arcs, so the pigeonhole principle gives the existence of $\beta_0 \in \mathbb{T}$ such that $\phi(h) - \phi(h')$ is very close to β_0 for many $h \in (-N, N)$ that also satisfy (1.30). We may therefore take $\beta = \beta_0 + \phi(h')$ in the argument following (1.30).

2. THE DENSITY INCREMENT

In this section we prove Theorem 1.1 using the following lemma, which is derived from our inverse theorem in §8.

Lemma 2.1 (Density increment). *Suppose that $A \subset [N]$ satisfies $|A| \geq \delta N$ and lacks the configuration*

$$(2.1) \quad x, x+y, x+qy^2 \quad (y \neq 0).$$

Then either $N \ll q^4 \delta^{-O(1)}$ or there exists $q' \ll \delta^{-O(1)}$ and $N' \gg \delta^{O(1)} q^{-3/2} N^{1/2}$ such that

$$(2.2) \quad |A \cap (a + qq' \cdot [N'])| \geq (\delta + \Omega(\delta^{O(1)})) N'.$$

Proof of Theorem 1.1 given Lemma 2.1. Note first that if A lacks the configuration (2.1), then the set

$$\{x : a + qq'x \in A\}$$

lacks configurations of the form

$$x, x+y, x+q^2q'y^2 \quad (y \neq 0).$$

Let $A \subset [N]$ have size δN and lack (1.2). Setting $A_0 := A$, $N_0 := N$ and $q_0 = 1$, let us suppose we have a sequence of tuples (A_i, N_i, q_i) for $i = 0, 1, \dots, n$ that each satisfy the following:

(i) A_i lacks configurations of the form

$$x, x+y, x+q_0^2 q_1^{2^{i-1}} \cdots q_{i-1}^2 q_i y^2 \quad (y \neq 0).$$

(ii) $q_i \ll \delta^{-O(1)}$;

(iii) $A_i \subset [N_i]$ and for $i \geq 1$ we have

$$\frac{|A_i|}{N_i} \geq \delta + \Omega(i\delta^{O(1)});$$

(iv) for $i \geq 1$ we have the lower bound

$$N_i \gg \delta^{O(1)} \left(q_0^{2^{i-1}} \cdots q_{i-1} \right)^{-3/2} N_{i-1}^{1/2}.$$

By Lemma 2.1, at stage n we either have

$$(2.3) \quad N_n \ll \left(q_0^{2^n} q_1^{2^{n-1}} \cdots q_{n-1}^2 q_n \right)^4 \delta^{-O(1)}$$

and the process terminates, or we obtain $(A_{n+1}, N_{n+1}, q_{n+1})$ satisfying conditions (i)–(iv).

The density of A_n on $[N_n]$ cannot exceed 1, so the process must terminate at some $n \ll \delta^{-O(1)}$. At the point of termination, the smallness assumption (2.3) must hold, so that

$$N_n \leq O(1/\delta)^{O(2^n)} \leq \exp \exp(O(\delta^{-O(1)})).$$

On the other hand, iteratively applying the lower bound (iv), we have

$$\begin{aligned} N_n &\geq \frac{N_{n-1}^{1/2}}{\left(q_0^{2^{n-1}} \cdots q_{n-1} \right)^{3/2} \delta^{-O(1)}} \\ &\geq N^{1/2^n} \left[\left(q_0^{2^{n-1}} \cdots q_{n-1} \right)^{3/2} \delta^{-O(1)} \right]^{-(1+\frac{1}{2}+\frac{1}{4}+\cdots+2^{1-n})} \\ &\geq \exp(\log N \exp(-O(\delta^{-O(1)}))) \exp \exp(-O(\delta^{-O(1)})). \end{aligned}$$

Taking logarithms and comparing upper and lower bounds for N_n yields the bound claimed in Theorem 1.1. \square

3. PET INDUCTION

We prove Lemma 2.1 over the course of §§3–8. We begin in §§3–5 by showing how the counting operator $\Lambda_q(f_0, f_1, f_2)$, as defined in (1.11), is controlled by the U^5 -norm of f_2 . This argument starts with the PET induction scheme of Bergelson–Leibman [BL96], which ‘linearises’ a polynomial progression, replacing univariate polynomials such as y^2 with bilinear forms ah . The outcome of this procedure is Lemma 3.2.

For the following, we recall our definition (1.12) of the Fejér kernel μ_H and our definition (1.7) of the difference operator Δ_h .

Lemma 3.1 (van der Corput inequality). *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ and $1 \leq H \leq M$. Then we have the estimate*

$$\left| \sum_{y \in [M]} f(y) \right|^2 \leq (M + H) \sum_h \mu_H(h) \sum_{y \in [M] \cap ([M]-h)} \Delta_h f(y).$$

Proof. This is standard, see for instance [Pre17, Lemma 3.1]. \square

Before embarking on the following, we remind the reader of our conventions (1.10), (1.11), and (1.13) regarding M , Λ_q , and the multivariate Fejér kernel $\mu_H(h)$.

Lemma 3.2 (Linearisation). *Let $f_i : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions with support in $[N]$. Then for any $1 \leq H_1, H_2 \leq M$ we have*

$$\left| \frac{\Lambda_q(f_0, f_1, f_2)}{NM} \right|^{32} \ll \mathbb{E}_{|a|, |b| < H_1} \sum_{h \in \mathbb{Z}^3} \mu_{H_2}(h) \mathbb{E}_{x \in [N]} \Delta_{2q(a+b)h_1, 2qbh_2, 2qah_3} f_2(x) + \frac{H_1 + H_2}{M}.$$

Proof. We repeatedly apply the Cauchy–Schwarz inequality followed by van der Corput’s inequality and a change of variables. We note that for $y \in \mathbb{N}$, if $f_0(x)f_1(x+y)f_2(x+qy^2) \neq 0$ for some x , then $y \in [M]$, since $qy^2 = (x+qy^2) - x \in [N] - [N]$. A first application of this procedure gives

$$\left| \frac{\Lambda_q(f_0, f_1, f_2)}{NM} \right|^2 \ll \frac{1}{NM} \sum_a \mu_{H_1}(a) \sum_x \sum_{y \in [M] \cap ([M] - a)} \Delta_a f_1(x) f_2(x+qy^2 - y) \overline{f_2(x+q(y+a)^2 - y)}.$$

Extending the sum over $y \in [M] \cap ([M] - a)$ to one over all of $[M]$ introduces an error of size

$$\ll \frac{1}{M} \sum_a \mu_{H_1}(a) |a| \ll \frac{H_1}{M}.$$

Thus,

$$\left| \frac{1}{NM} \Lambda_q(f_0, f_1, f_2) \right|^2 \ll \frac{H_1}{M} + \frac{1}{NM} \sum_a \mu_{H_1}(a) \sum_x \sum_{y \in [M]} \Delta_a f_1(x) f_2(x+qy^2 - y) \overline{f_2(x+q(y+a)^2 - y)}.$$

A second application then gives that $\left| \frac{1}{NM} \Lambda_q(f_0, f_1, f_2) \right|^4$ is

$$\ll \frac{H_1}{M} + \frac{1}{NM} \mathbb{E}_{|a| < H_1} \sum_b \mu_{H_1}(b) \sum_x \sum_{y \in [M]} f_2(x) \overline{f_2(x+2qay+qa^2)} \overline{f_2(x+2qby+qb^2-b)} f_2(x+2q(a+b)y+q(a+b)^2-b),$$

where we have used that $\|\mu_{H_1}\|_{L^2}^2 \leq H_1^{-1}$.

Now that we have linearised y^2 , we switch to shifting y by integers of size at most H_2 . A third application therefore gives that $\left| \frac{1}{NM} \Lambda_q(f_0, f_1, f_2) \right|^8$ is

$$\ll \frac{H_1 + H_2}{M} + \frac{1}{NM} \mathbb{E}_{|a|, |b| < H_1} \sum_{h_1} \mu_{H_2}(h_1) \sum_x \sum_{y \in [M]} \Delta_{2qah_1} \overline{f_2(x+qa^2)} \overline{\Delta_{2qbh_1} f_2(x+2q(b-a)y+qb^2-b)} \Delta_{2q(a+b)h_1} f_2(x+2qby+q(a+b)^2-b),$$

a fourth application that $\left| \frac{1}{NM} \Lambda_q(f_0, f_1, f_2) \right|^{16}$ is

$$\ll \frac{H_1 + H_2}{M} + \frac{1}{NM} \mathbb{E}_{|a|, |b| < H_1} \sum_{h_1, h_2} \mu_{H_2}(h_1) \mu_{H_2}(h_2) \sum_x \sum_{y \in [M]} \Delta_{2qbh_1, 2q(b-a)h_2} \overline{f_2(x+qb^2-b)} \overline{\Delta_{2q(a+b)h_1, 2qbh_2} f_2(x+2qay+q(a+b)^2-b)},$$

and a final application that $\left| \frac{1}{NM} \Lambda_q(f_0, f_1, f_2) \right|^{32}$ is

$$\ll \frac{H_1 + H_2}{M} + \frac{1}{N} \mathbb{E}_{|a|, |b| < H_1} \sum_{h \in \mathbb{Z}^3} \mu_{H_2}(h) \sum_x \Delta_{2q(a+b)h_1, 2qbh_2, 2qah_3} f_2(x),$$

after making the change of variables $x \mapsto x - 2qay - q(a+b)^2 - b$. Recalling that f_2 is supported on $[N]$ completes the proof. \square

4. AN INVERSE THEOREM FOR THE ARITHMETIC BOX NORM

The objective in this section is to characterise those 1-bounded functions $f : \mathbb{Z} \rightarrow \mathbb{C}$ with support in $[N]$ for which the quantity

$$(4.1) \quad \sum_{h \in \mathbb{Z}^2} \sum_x \mu_H(h) \Delta_{ah_1, bh_2} f(x)$$

is large. One can think of this as an arithmetic analogue of the two-dimensional ‘box norm’ (1.24), with differencing parameters pointing in the ‘directions’ a and b . In our eventual application we are able to ensure that a and b are a generic pair of integers from the interval $[N^{1/2}]$. In particular, at least one of them has size proportional to $N^{1/2}$ and their greatest common divisor is small. One may think of this as a proxy for linear independence.

We begin by characterising largeness of (4.1) when the directions are coprime.

Lemma 4.1 (Inverse theorem for the arithmetic box norm). *Let a, b be positive integers with $\gcd(a, b) = 1$. Suppose that $f : \mathbb{Z} \rightarrow \mathbb{C}$ is 1-bounded with support in the interval $[N]$ and satisfies*

$$(4.2) \quad \sum_{h \in \mathbb{Z}^2} \sum_x \mu_H(h) \Delta_{ah_1, bh_2} f(x) \geq \delta N.$$

Then there exist 1-bounded functions $g, h : \mathbb{Z} \rightarrow \mathbb{C}$ such that

- g is a -periodic, in the sense that $g(x+a) = g(x)$ for all x ;
- h is approximately b -periodic, in the sense that for any $\varepsilon > 0$ we have

$$\#\{x \in [N] : h(x+by) \neq h(x) \text{ for some } |y| \leq \varepsilon N/b\} \leq \left(1 + \frac{2\varepsilon N}{b}\right) \left(1 + \frac{N}{a}\right);$$

and furthermore

$$(4.3) \quad \left| \sum_x f(x) g(x) h(x) \right| \geq \delta [H]^2 - 2\left(\frac{H}{a} + \frac{Hb}{N}\right) [H]^2.$$

Remark. In parsing the above inequalities, it may be helpful to keep in mind that in our application a, b and H are of order \sqrt{N} , with H smaller than δa , in which case the lower bound in (4.3) becomes $\Omega(\delta H^2)$.

Proof. The majority of our proof is concerned with manipulating (4.2) until we can interpret it as a genuine box norm (1.24), and thereby apply the box norm inverse theorem. The essential observation is that, since $\gcd(a, b) = 1$, every integer x can be uniquely represented in the form

$$x = ay + bz \quad (y \in \mathbb{Z}, z \in [a]).$$

We note that if $x \in [N]$ then the constraint on z forces y to lie in the range $-b < y < N/a$.

Defining $F : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$ by $F(y, z) := f(ay + bz)$, the left-hand side of (4.2) becomes

$$\sum_{y, y' \in \mathbb{Z}} \sum_{\substack{z \in [a] \\ z' \in \mathbb{Z}}} F(y, z) \overline{F(y', z)} \overline{F(y, z')} F(y', z') \mu_H(y' - y) \mu_H(z' - z).$$

If z' and z contribute to the above sum then $z' \in z + (-H, H) \subset (-H + 1, a + H)$. Hence we can restrict the range of summation of z' to $[a]$, at the cost of perturbing the sum by at most $2\lfloor H \rfloor (\frac{N}{a} + b)$. It follows that

$$\left| \sum_{y, y'} \sum_{z, z' \in [a]} F(y, z) \overline{F(y', z)} \overline{F(y, z')} F(y', z') \mu_H(y' - y) \mu_H(z' - z) \right| \geq \delta N - 2\lfloor H \rfloor \left(\frac{N}{a} + b \right).$$

We remove the Fejér kernels by Fourier expansion:

$$\begin{aligned} & \sum_{\substack{y, y' \\ z, z' \in [a]}} F(y, z) \overline{F(y', z)} \overline{F(y, z')} F(y', z') \mu_H(y' - y) \mu_H(z' - z) = \\ & \int_{\mathbb{T}^2} \sum_{\substack{y, y' \\ z, z' \in [a]}} F(y, z) \overline{F(y', z)} \overline{F(y, z')} F(y', z') \hat{\mu}_H(\alpha) \hat{\mu}_H(\beta) e(\alpha(y' - y) + \beta(z' - z)) d\alpha d\beta \\ & \leq \left(\int_{\mathbb{T}} |\hat{\mu}_H(\alpha)| d\alpha \right)^2 \sup_{\alpha, \beta \in \mathbb{T}} \left| \sum_{\substack{y, y' \\ z, z' \in [a]}} F(y, z) \overline{F_2(y', z)} \overline{F_3(y, z')} F_4(y', z') \right|, \end{aligned}$$

where $F_2(y', z) := F(y', z)e(\beta z)$, $F_3(y, z') := F(y, z')e(\alpha y)$, and $F_4(y', z') := F(y', z')e(\alpha y' + \beta z')$.

We observe that $\hat{\mu}_H(\alpha) = |\hat{1}_{[H]}(\alpha)|^2 / \lfloor H \rfloor^2$, which implies that $\int_{\mathbb{T}} |\hat{\mu}_H(\alpha)| d\alpha = \lfloor H \rfloor^{-1}$. Therefore

$$(4.4) \quad \left| \sum_{\substack{y, y' \\ z, z' \in [a]}} F(y, z) \overline{F_2(y', z)} \overline{F_3(y, z')} F_4(y', z') \right| \geq \delta \lfloor H \rfloor^2 N - 2\lfloor H \rfloor^3 \left(\frac{N}{a} + b \right),$$

for 1-bounded functions $F_i : \mathbb{Z} \times [a] \rightarrow \mathbb{C}$ of the form $F_i(y, z) = f(ay + bz)e(\alpha_1 y + \alpha_2 z)$. Since f is supported on $[N]$, there are at most N pairs $(y', z') \in \mathbb{Z} \times [a]$ for which $F_4(y', z') \neq 0$. Thus, by pigeonholing in y' and z' in (4.4) and setting $L(y) := \overline{F_3(y, z')}$ and $R(z) := \overline{F_2(y', z)} F_4(y', z')$, we get that

$$\left| \sum_y \sum_{z \in [a]} F(y, z) L(y) R(z) \right| \geq \delta \lfloor H \rfloor^2 - 2\lfloor H \rfloor^3 \left(\frac{1}{a} + \frac{b}{N} \right).$$

For each $x \in \mathbb{Z}$, define $l(x) \in \mathbb{Z}$ and $r(x) \in [a]$ by $x = al(x) + br(x)$, and set $g(x) := R \circ r(x)$ and $h(x) := L \circ l(x)$. Then it remains to check the invariance properties of g and h . To see that $g(x) = g(x + ay)$ for all $x, y \in \mathbb{Z}$, just note that $r(x) = r(x + ay)$ for every $x, y \in \mathbb{Z}$.

Finally we establish that, for most $x \in [N]$, we have $h(x) = h(x + bz)$ when $|z| \leq \varepsilon N/b$. First note that $l(x) = l(x + bz)$ whenever $\varepsilon N/b < r(x) \leq a - \varepsilon N/b$. Hence for this to fail, x must lie in one of at most $1 + 2\varepsilon N/b$ congruence classes modulo a . The number of such x lying in the interval $[N]$ is at most

$$\left(1 + \frac{2\varepsilon N}{b} \right) \left(1 + \frac{N}{a} \right).$$

□

The lemma also yields a result in the situation in which $\gcd(a, b) > 1$. In proving this we take the opportunity to smooth out the b -invariance of the h function, whilst also giving an explicit description of h in terms of f . More concretely, we replace h with a projection of fg onto cosets of $b \cdot \mathbb{Z}$.

Lemma 4.2. *There exists an absolute constant $c > 0$ such that on assuming $1 \leq H \leq c\delta^3 N^{1/2}$ and $1 \leq K \leq c\delta^2 H^2 N^{-1/2}$ the following holds. Let $a, b \in [-N^{1/2}, N^{1/2}]$ with $\gcd(a, b) \leq \delta^{-1}$ and $|a| \geq \delta N^{1/2}$. Suppose that $f : \mathbb{Z} \rightarrow \mathbb{C}$ is 1-bounded, supported on the interval $[N]$, and satisfies*

$$\left| \sum_{h \in \mathbb{Z}^2} \sum_x \mu_H(h) \Delta_{ah_1, bh_2} f(x) \right| \geq \delta N.$$

Then there exists a 1-bounded a -periodic function g such that

$$(4.5) \quad \sum_x f(x)g(x) \sum_k \mu_K(k) \overline{f(x+bk)g(x+bk)} \gg \delta^2 H^4 / N.$$

Proof. Since (4.2) and (4.5) are invariant under the transformations $a \mapsto -a$ and $b \mapsto -b$, we may assume that both a and b are positive. Set $q := \gcd(a, b) \leq \delta^{-1}$. For each $u \in [q]$, define a 1-bounded function $f_u : \mathbb{Z} \rightarrow \mathbb{C}$ by $f_u(x) := f(u + qx)$, and let $I_u := \{x : u + qx \in [N]\}$ denote the interval on which f_u is supported. By the pigeon-hole principle, for some u we have

$$\sum_{x, h_1, h_2} \mu_H(h_1) \mu_H(h_2) \Delta_{\frac{a}{q}h_1, \frac{b}{q}h_2} f_u(x) \geq \delta |I_u|.$$

Note that $\gcd(a/q, b/q) = 1$, so by the previous lemma, there exist 1-bounded functions $g_u, h_u : \mathbb{Z} \rightarrow \mathbb{C}$ such that

$$\left| \sum_x f_u(x) g_u(x) h_u(x) \right| \geq \delta [H]^2 - 2 \left(\frac{Hq}{a} + \frac{Hb}{q|I_u|} \right) [H]^2 \gg \delta H^2.$$

Furthermore, g_u is (a/q) -periodic and

$$\begin{aligned} \#\{x \in I_u : h_u(x) \neq h_u(x + yb/q) \text{ for some } |y| \leq \varepsilon |I_u| q/b\} \\ \leq \left(1 + \frac{2q\varepsilon |I_u|}{b}\right) \left(1 + \frac{q|I_u|}{a}\right) \ll \frac{N}{a} + \frac{\varepsilon N^2}{ab}. \end{aligned}$$

Defining $g_{u'}$ and $h_{u'}$ to be identically zero when $u' \neq u$, we set $g(u' + qx) := g_{u'}(x)$ and $h(u' + qx) := h_{u'}(x)$. One can then check that g is a -invariant, that

$$\left| \sum_x f(x) g(x) h(x) \right| \gg \delta H^2,$$

and that

$$\#\{x \in [N] : h(x) \neq h(x + by) \text{ for some } |y| \leq \varepsilon N/b\} \ll \frac{N}{a} + \frac{\varepsilon N^2}{ab}.$$

Taking $\varepsilon := Kb/N$, we may use the latter property to show that, provided $K \geq 1$, we have

$$\left| \sum_x f(x) g(x) h(x) - \sum_x h(x) \mathbb{E}_{y \in [K]} g(x + by) f(x + by) \right| \ll \frac{NK}{a}.$$

Provided that $K \leq c\delta^2 H^2 N^{-1/2}$ we deduce that

$$\left| \sum_x h(x) \mathbb{E}_{y \in [K]} g(x + by) f(x + by) \right| \gg \delta H^2.$$

One can check that, as a function of x , the inner expectation has support contained in $[-2N, 2N]$. Applying the Cauchy–Schwarz inequality and changing variables then gives (4.5). \square

Finally we observe that a function of the form

$$(4.6) \quad h(x) := \sum_k \mu_K(k) f(x + bk)$$

has nice b -periodicity properties.

Lemma 4.3. *If h is defined as in (4.6) for some 1-bounded f , then h is $O(K^{-1})$ -Lipschitz along $b \cdot \mathbb{Z}$, in that for any $x, y \in \mathbb{Z}$ we have $h(x + by) = h(x) + O(|y|/K)$.*

Proof. Recalling the definition (1.12), note that μ_K is $[K]^{-2}$ -Lipschitz, in that $|\mu_K(k + y) - \mu_K(k)| \leq |y|/[K]^2$ for all $k, y \in \mathbb{Z}$. Hence, for $|y| \leq K$, a change of variables gives

$$|h(x + by) - h(x)| \leq \sum_k |\mu_K(k - y) - \mu_K(k)| \leq \frac{|y|}{[K]^2} \sum_{|k| < 2[K]} 1.$$

\square

5. QUANTITATIVE CONCATENATION: CONTROL BY A GLOBAL GOWERS NORM

The endpoint of this section is to show how our counting operator (1.11) is controlled by the U^5 -norm. We begin with four technical lemmas. The first says that convolving Fejér kernels along progressions of coprime common difference covers a substantial portion of an interval in a somewhat regular manner, a fact that can be interpreted Fourier analytically in the following.

Lemma 5.1. *Let $K, L \geq 1$ and let a, b be integers satisfying $|a| \geq \delta L$, $|b| \geq \delta K$ and $\gcd(a, b) \leq \delta^{-1}$. Then*

$$\int_{\mathbb{T}} |\widehat{\mu}_K(a\beta)| |\widehat{\mu}_L(b\beta)| d\beta \ll \frac{\delta^{-4}}{[K][L]}.$$

Proof. Taking complex conjugates inside the absolute values, we may assume that a and b are positive. Expanding Fourier transforms, one can check that

$$\begin{aligned} & \int_{\mathbb{T}} |\widehat{\mu}_K(a\beta)| |\widehat{\mu}_L(b\beta)| d\beta \\ &= [K]^{-2} [L]^{-2} \#\left\{ (x, y) \in [K]^2 \times [L]^2 : a(x_1 - x_2) = b(y_1 - y_2) \right\}. \end{aligned}$$

Writing $d := \gcd(a, b)$, the number of solutions to the equation is at most

$$[K][L] \left(\frac{[K]}{b/d} + 1 \right) \left(\frac{[L]}{a/d} + 1 \right).$$

\square

Our next lemma allows us to discard pairs of integers a, b which are not sufficiently coprime. We exploit this repeatedly.

Lemma 5.2. *For fixed integers $|a_1|, |a_2| \leq M$, the number of pairs (b, c) of integers $|b|, |c| \leq M$ such that $\gcd(a_1 + b, a_2 + c) > \delta^{-1}$ is $\ll \delta M^2$.*

Proof. Notice that if $d = \gcd(a_1 + b, a_2 + c)$ then $d \leq 2M$. Hence

$$\begin{aligned} \sum_{\substack{|b|, |c| \leq M \\ \gcd(a_1 + b, a_2 + c) > \delta^{-1}}} 1 &\leq \sum_{\delta^{-1} < d \leq 2M} \left(\sum_{|m| \leq 2M, d|m} 1 \right)^2 \leq \sum_{\delta^{-1} < d \leq 2M} \left(\frac{4M}{d} + 1 \right)^2 \\ &\ll M^2 \sum_{d > \delta^{-1}} \frac{1}{d^2} \ll \delta M^2. \end{aligned}$$

□

The following lemma says that, as a and h range over $[N^{1/2}]$, the difference function $\Delta_{ah}f$ behaves like $\Delta_k f$ with $k \in [N]$, at least on average.

Lemma 5.3. *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a 1-bounded function with support in $[N]$ and $s \geq 1$. Suppose that $\delta N^{1/2} \leq H \leq N^{1/2}$ and*

$$\mathbb{E}_{|a| \leq N^{1/2}} \sum_h \mu_H(h) \|\Delta_{ah}f\|_{U^s}^{2s} \geq \delta \|1_{[N]}\|_{U^s}^{2s}.$$

Then

$$\|f\|_{U^{s+1}}^{2s+1} \gg \delta^{10} \|1_{[N]}\|_{U^{s+1}}^{2s+1}.$$

Proof. Expanding the definition of the U^s -norm

$$\begin{aligned} \mathbb{E}_{|a| \leq N^{1/2}} \sum_h \mu_H(h) \|\Delta_{ah}f\|_{U^s}^{2s} &= \sum_{h_1, \dots, h_s, x} \overline{\Delta_{h_1, \dots, h_s} f(x)} \mathbb{E}_{|a| \leq N^{1/2}} \sum_h \mu_H(h) \Delta_{h_1, \dots, h_s} f(x + ah). \end{aligned}$$

Employing the Cauchy–Schwarz inequality to double the a and h variables gives

$$\mathbb{E}_{|a|, |a'| \leq N^{1/2}} \sum_{h_1, \dots, h_s} \sum_x \sum_{h, h'} \mu_H(h) \mu_H(h') \Delta_{h_1, \dots, h_s, ah - a'h'} f(x) \gg \delta^2 N^{s+1}.$$

By Lemma 5.2 and the pigeon-hole principle, we deduce the existence of $|a|, |a'| \gg \delta^2 N^{1/2}$ with $\gcd(a, a') \ll \delta^{-2}$ such that

$$\sum_{ah - a'h' = h_{s+1}} \mu_H(h) \mu_H(h') \sum_{h_1, \dots, h_s} \sum_x \Delta_{h_1, \dots, h_s, h_{s+1}} f(x) \gg \delta^2 N^{s+1}.$$

We use orthogonality to replace the constraint $ah - a'h' = h_{s+1}$ with a Fourier integral, then extract a large Fourier coefficient to conclude that there exists $\alpha \in \mathbb{T}$ such that the left-hand side above is at most

$$\int_{\mathbb{T}} |\widehat{\mu}_H(a\beta)| |\widehat{\mu}_H(a'\beta)| d\beta \left| \sum_{h_1, \dots, h_{s+1}} \sum_x \Delta_{h_1, \dots, h_s, h_{s+1}} f(x) e(\alpha h_{s+1}) \right|.$$

The result follows on employing Lemma 5.1 and Lemma A.3. □

We now prove a similar lemma, but with $\Delta_{ah}f$ replaced by fg_a where g_a is a -periodic. The moral is that these are similar quantities (on average).

Lemma 5.4. *Let $f, g_a : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions such that g_a is a -periodic and $\text{supp}(f) \subset [N]$. Suppose that*

$$\mathbb{E}_{|a| \leq N^{1/2}} \|fg_a\|_{U^s}^{2^s} \geq \delta \|1_{[N]}\|_{U^s}^{2^s}.$$

Then

$$\|f\|_{U^{s+1}}^{2^{s+1}} \gg \delta^{2^4} \|1_{[N]}\|_{U^{s+1}}^{2^{s+1}}.$$

Proof. Fix $|a| \leq N^{1/2}$. By the periodicity of g_a and a change of variables, we have

$$\sum_{h_1, \dots, h_s} \sum_x \Delta_{h_1, \dots, h_s} g_a(x) \Delta_{h_1, \dots, h_s} f(x) = \sum_{h_1, \dots, h_s} \sum_x \Delta_{h_1, \dots, h_s} g_a(x) \mathbb{E}_{y \in [N^{1/2}]} \Delta_{h_1, \dots, h_s} f(x+ay).$$

Notice that the sum over x is non-zero only if $|x|, |h_i| < N$, hence by the Cauchy–Schwarz inequality and a change of variables

$$\begin{aligned} \left(\mathbb{E}_{|a| \leq N^{1/2}} \|fg_a\|_{U^s}^{2^s} \right)^2 &\ll N^{s+1} \mathbb{E}_{|a| \leq N^{1/2}} \sum_{h_1, \dots, h_s} \sum_x \sum_y \mu_{N^{1/2}}(y) \Delta_{h_1, \dots, h_s, ay} f(x) \\ &= N^{s+1} \mathbb{E}_{|a| \leq N^{1/2}} \sum_y \mu_{N^{1/2}}(y) \|\Delta_{ay} f\|_{U^s}^{2^s} \end{aligned}$$

The result follows on employing Lemma 5.3. \square

We are now ready to give the technical heart of this section. The (somewhat lengthy) assumptions come from our eventual application of Lemma 4.2.

Lemma 5.5. *Fix $a \in \mathbb{Z}$ and let $\delta N^{1/2} \leq K \leq N^{1/2}$. For each b let $f, g_b, h_b : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions such that $\text{supp}(f), \text{supp}(h_b) \subset [N]$ and g_b is b -periodic. Set*

$$(5.1) \quad \tilde{h}_b(x) := \sum_k \mu_K(k) h_b(x + (a+b)k)$$

and suppose that

$$\sum_{\substack{\delta\sqrt{N} \leq |b| \leq \sqrt{N} \\ \gcd(a,b) \leq \delta^{-1}}} \sum_x f(x) g_b(x) \tilde{h}_b(x) \geq \delta N^{3/2}.$$

Then

$$\mathbb{E}_{|b| \leq N^{1/2}} \|h_b\|_{U^3}^8 \gg \delta^{O(1)} \|1_{[N]}\|_{U^3}^8.$$

Proof. We apply the Cauchy–Schwarz inequality to remove the weight $f(x)$ and double the b variable, yielding

$$\sum_{\substack{\delta\sqrt{N} \leq |b|, |b'| \leq \sqrt{N} \\ \gcd(a,b), \gcd(a,b') \leq \delta^{-1}}} \sum_x g_b(x) \tilde{h}_b(x) \overline{g_{b'}(x) \tilde{h}_{b'}(x)} \geq \delta^2 N^2.$$

Employing Lemma 5.2, we may discard those b, b' for which one of $\gcd(b', a+b)$ or $\gcd(a+b', a+b)$ is greater than $C\delta^{-2}$. We may also discard those b, b' for which either $|a+b| \leq c\delta^2\sqrt{N}$ or $|a+b'| \leq c\delta^2\sqrt{N}$. On combining this with the popularity principle, we deduce the existence of $\mathcal{B} \subset [-N^{1/2}, N^{1/2}]$ of size $|\mathcal{B}| \gg \delta^2 N^{1/2}$ such that for each $b \in \mathcal{B}$ there exists $|b'| \leq \sqrt{N}$ with all of $|b|, |b'|, |a+b|, |a+b'| \gg \delta^2\sqrt{N}$ and all of $\gcd(b, a+b), \gcd(b', a+b), \gcd(a+b', a+b)$ at most $O(\delta^{-2})$ and satisfying

$$(5.2) \quad \sum_x g_b(x) \overline{g_{b'}(x) \tilde{h}_{b'}(x)} \tilde{h}_b(x) \gg \delta^2 N.$$

Expanding the definition of $\tilde{h}_{b'}$, using the invariance of g_b and changing variables gives

$$\sum_x \mathbb{E}_{k_1, k_3 \in [K]} \sum_{k_2} \mu_K(k_2) g_b(x + (a+b')k_2 + b'k_3) \overline{h_{b'}(x + bk_1 + b'k_3)} \\ \overline{g_{b'}(x + bk_1 + (a+b')k_2)} \tilde{h}_b(x + bk_1 + (a+b')k_2 + b'k_3) \gg \delta^2 N.$$

Since $h_{b'}$ is supported on $[N]$ and $|b|, |b'|, K \leq N^{1/2}$, there are at most $O(N)$ values of x that contribute to the above sum. Applying Hölder's inequality then gives

$$\sum_x \left(\mathbb{E}_{k_1, k_3 \in [K]} \sum_{k_2} \mu_K(k_2) g_b(x + (a+b')k_2 + b'k_3) \overline{h_{b'}(x + bk_1 + b'k_3)} \right. \\ \left. \overline{g_{b'}(x + bk_1 + (a+b')k_2)} \tilde{h}_b(x + bk_1 + (a+b')k_2 + b'k_3) \right)^8 \gg \delta^{16} N.$$

The sum inside the 8th power corresponds to an integral with respect to three probability measures on \mathbb{Z} , with integrand amenable to Lemma A.4. Combining this with a change of variables gives

$$\sum_x \sum_{k_1, k_2, k_3} \mu_K(k_1) \nu_K(k_2) \mu_K(k_3) \Delta_{bk_1, (a+b')k_2, b'k_3} \tilde{h}_b(x) \gg \delta^{16} N,$$

where we set

$$\nu_K(k) := \sum_{k_1 - k_2 = k} \mu_K(k_1) \mu_K(k_2).$$

By Lemma 4.3 and the definition (5.1), each \tilde{h}_b is $O(K^{-1})$ -Lipschitz along $(a+b) \cdot \mathbb{Z}$. Hence, if $l_i \in [L]$, a telescoping identity shows that

$$\left| \Delta_{h_1+(a+b)l_1, h_2+(a+b)l_2, h_3+(a+b)l_3} \tilde{h}_b(x) - \Delta_{h_1, h_2, h_3} \tilde{h}_b(x) \right| \ll L/K.$$

Taking $L := c\delta^{16}K$ we obtain

$$\sum_x \sum_{k_1, k_2, k_3} \mu_K(k_1) \nu_K(k_2) \mu_K(k_3) \mathbb{E}_{l_1, l_2, l_3 \in [L]} \\ \Delta_{bk_1+(a+b)l_1, (a+b')k_2+(a+b)l_2, b'k_3+(a+b)l_3} \tilde{h}_b(x) \gg \delta^{16} N.$$

We may replace the uniform measure on the l_i by Fejér kernels at the cost of three applications of the Cauchy–Schwarz inequality; this gives

$$\sum_x \sum_{\substack{k_1, k_2, k_3 \\ l_1, l_2, l_3}} \mu_K(k_1) \nu_K(k_2) \mu_K(k_3) \mu_L(l_1) \mu_L(l_2) \mu_L(l_3) \\ \Delta_{bk_1+(a+b)l_1, (a+b')k_2+(a+b)l_2, b'k_3+(a+b)l_3} \tilde{h}_b(x) \gg \delta^{O(1)} N.$$

Write

$$\lambda_1(h) := \sum_{bk+(a+b)l=h} \mu_K(k) \mu_L(l), \quad \lambda_2(h) := \sum_{(a+b')k+(a+b)l=h} \nu_K(k) \mu_L(l), \\ \lambda_3(h) := \sum_{b'k+(a+b)l=h} \mu_K(k) \mu_L(l).$$

Then

$$\sum_x \sum_{h_1, h_2, h_3} \lambda_1(h_1) \lambda_2(h_2) \lambda_3(h_3) \Delta_{h_1, h_2, h_3} \tilde{h}_b(x) \gg \delta^{O(1)} N.$$

By Fourier inversion and extraction of a large Fourier coefficient, there exist $\alpha_i \in \mathbb{T}$ such that

$$\left| \sum_x \sum_{h_1, h_2, h_3} \Delta_{h_1, h_2, h_3} \tilde{h}_b(x) e(\alpha \cdot \underline{h}) \right| \prod_{i=1}^3 \int_{\mathbb{T}} |\hat{\lambda}_i(\beta)| d\beta \gg \delta^{O(1)} N.$$

By our choice of b, b' (see the paragraph preceding (5.2)), together with Lemma 5.1, for each i we have

$$(5.3) \quad \int_{\mathbb{T}} |\hat{\lambda}_i(\alpha)| d\alpha \ll \frac{\delta^{-8}}{KL} \ll \frac{\delta^{-O(1)}}{N},$$

the latter following from the fact that $L \gg c\delta^{16}K$ and $K \geq \delta N^{1/2}$. On combining this with Lemma A.3 we obtain

$$\|\tilde{h}_b\|_{U^3}^8 \gg \delta^{O(1)} N^4.$$

Since \tilde{h}_b is an average of translates of h_b , we may apply the triangle inequality for the U^3 -norm, together with the fact that Gowers norms are translation invariant, and conclude that $\|h_b\|_{U^3}^8 \gg \delta^{O(1)} N^4$. Summing over $b \in \mathcal{B}$ gives our final bound. \square

Finally we synthesise Lemmas 3.2, 4.2 and 5.5.

Theorem 5.6 (Global U^5 -control). *Let $g_0, g_1, f : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions, each supported in $[N]$. Suppose that*

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} g_0(x) g_1(x+y) f(x+qy^2) \right| \geq \delta \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_{[N]}(x) 1_{[N]}(x+y) 1_{[N]}(x+qy^2).$$

Then either $N \ll q$ or

$$\sum_{u \in [q]} \|f\|_{U^5(u+q\mathbb{Z})}^{25} \gg \delta^{O(1)} \sum_{u \in [q]} \|1_{[N]}\|_{U^5(u+q\mathbb{Z})}^{25}.$$

Proof. We recall our conventions (1.10) and (1.11) regarding M and Λ_q , and note that $\Lambda_q(1_{[N]}) \gg NM$ unless $N \ll q$. We begin by applying the linearisation procedure (Lemma 3.2) to deduce that

$$\sum_{a, b \in (-2M, 2M)} \left| \sum_{h \in \mathbb{Z}^3} \mu_H(h) \sum_x \Delta_{q(a+b)h_1, qbh_2, qah_3} f(x) \right| \gg \delta^{32} NM^2$$

provided that $H \ll \delta^{32}M$. Applying Lemma 5.2 we may discard those a, b for which either $\gcd(a, b) > C\delta^{-32}$ or $|b| < c\delta^{32}M$. Partitioning the sum over x into congruence classes $u \bmod q$, the popularity principle gives:

- at least $\Omega(\delta^{32}q)$ residues $u \in [q]$;
- for each of which there is a subset of $h_3 \in (-H, H)$ of μ_H -measure⁶ at least $\Omega(\delta^{32})$;
- for each of which there exist $\Omega(\delta^{32}M)$ values of $a \in (-2M, 2M)$;

⁶i.e. $\sum_{h_3 \in \mathcal{H}} \mu_H(h_3) \gg \delta^{32}$.

- for each of which there are $\Omega(\delta^{32}M)$ values of $b \in (-2M, 2M)$ satisfying $\gcd(a, b) \ll \delta^{-32}$ and $|b| \gg \delta^{32}M$;

and together these satisfy

$$\left| \sum_{h_1, h_2} \mu_H(h_1, h_2) \sum_x \Delta_{(a+b)h_1, bh_2, ah_3} f(qx - u) \right| \gg \delta^{32}M^2.$$

For fixed u, h_3, a write $\tilde{f}(x) := \Delta_{ah_3} f(qx - u)$, so that \tilde{f} has support in the interval $[(2M)^2]$ and

$$\left| \sum_{h_1, h_2} \mu_H(h_1, h_2) \sum_x \Delta_{(a+b)h_1, bh_2} \tilde{f}(x) \right| \gg \delta^{32}M^2.$$

Set

$$(5.4) \quad H := c\delta^{96}M \quad \text{and} \quad K := c^3\delta^{256}M,$$

with c sufficiently small to ensure that we may apply Lemma 4.2. This gives the existence of a 1-bounded b -periodic function g_b such that on setting

$$(5.5) \quad \tilde{h}_b(x) := \sum_k \mu_K(k) \overline{\tilde{f}(x + (a+b)k)g_b(x + (a+b)k)}$$

we have

$$\sum_x \tilde{f}(x)g_b(x)\tilde{h}_b(x) \gg \delta^{448}M^2.$$

Setting $\eta := c\delta^{480}$ for some small absolute constant $c > 0$, we may sum over our set of permissible b (taking g_b to be zero if b is not permissible) to deduce that

$$\sum_{\substack{\eta M \leq |b| \leq 2M \\ \gcd(a, b) \leq \eta^{-1}}} \sum_x \tilde{f}(x)g_b(x)h_b(x) \geq \eta M^3.$$

The hypotheses of Lemma 5.5 having been met, we conclude that

$$\mathbb{E}_{|b| \leq 2M} \|\tilde{f}g_b\|_{U^3}^8 \gg \delta^{O(1)} \|1_{[M^2]}\|_{U^3}^8.$$

Applying Lemma 5.4 then gives

$$\|\tilde{f}\|_{U^4}^{16} \gg \delta^{O(1)} \|1_{[M^2]}\|_{U^4}^{16}.$$

Recalling that $\tilde{f}(x) = \Delta_{ah_3} f_u(x)$ where $f_u(x) := f(qx - u)$, we may integrate over the set of permissible h_3 and a , utilising positivity to extend the range of summation, and deduce that

$$\mathbb{E}_{|a| \leq 2M} \sum_h \mu_H(h_3) \|\Delta_{ah_3} f_u\|_{U^4}^{16} \gg \delta^{O(1)} \|1_{[M^2]}\|_{U^4}^{16}$$

Using Lemma 5.3 and summing over the permissible range of u we get that

$$\mathbb{E}_{u \in [q]} \|f_u\|_{U^5}^{32} \gg \delta^{O(1)} \|1_{[M^2]}\|_{U^5}^{32},$$

and the result follows. \square

6. DEGREE LOWERING

So far, we have shown that $\Lambda_q(f_0, f_1, f_2)$ is controlled by $\mathbb{E}_{u \in [q]} \|f_2\|_{U^5(u+q\mathbb{Z})}^{2^5}$ whenever f_0, f_1 , and f_2 are 1-bounded complex-valued functions supported on the interval $[N]$. The next step in our argument is to bound $\Lambda_q(f_0, f_1, f_2)$ in terms of the $U^5(u+q\mathbb{Z})$ -norm of the dual function

$$(6.1) \quad F(x) := \mathbb{E}_{y \in [M]} f_0(x - qy^2) f_1(x + y - qy^2).$$

We postpone this deduction until §7. In this section we show how U^5 -control of the dual implies U^2 -control.

Our argument combines three simple lemmas: Weyl's inequality; what we call 'dual-difference interchange', which allows us to replace the difference function of the dual by the dual of the difference functions; and the fact that a function whose difference functions correlate with 'low rank' Fourier coefficients must have a large uniformity norm of lower degree.

The following log-free variant of Weyl's inequality can be found in [GT08, Lemma A.11].

Lemma 6.1 (Weyl's inequality). *Let $\alpha, \beta \in \mathbb{T}$, $\delta \in (0, 1)$ and let $I \subset \mathbb{Z}$ be an interval. Suppose that*

$$|\mathbb{E}_{y \in I} e(\alpha y^2 + \beta y)| \geq \delta.$$

Then either $|I| \ll \delta^{-O(1)}$ or there exists a positive integer $q \ll \delta^{-O(1)}$ such that

$$\|q\alpha\| \ll \delta^{-O(1)} |I|^{-2}.$$

This has the following consequence, which does not necessarily assume our convention (1.10) regarding M .

Lemma 6.2. *Suppose that for $\alpha \in \mathbb{T}$ there are 1-bounded functions $g_0, g_1 : \mathbb{Z} \rightarrow \mathbb{C}$ supported on the interval $[N]$ such that*

$$\left| \sum_x \sum_{y \in [M]} g_0(qx) g_1(qx + y) e(\alpha y^2) \right| \geq \delta MN/q.$$

Then either $M \ll q\delta^{-O(1)}$ or there exists a positive integer $q' \ll \delta^{-O(1)}$ such that $\|q'q^2\alpha\| \ll \delta^{-O(1)} q^2/M^2$.

Proof. We split the sum over $y \in [M]$ into arithmetic progressions modulo q and split the sum over x into intervals of length M/q . Hence, by the pigeon-hole principle, there exists $u \in [q]$ and an integer m such that on rounding the sum over y we have

$$\left| \sum_{x, y \in [M/q]} g_0(q(m+x)) g_1(u + q(m+x+y)) e(\alpha(u+qy)^2) \right| \gg \delta(M/q)^2.$$

Define the functions

$$\begin{aligned} h_0(x) &:= g_0(q(m+x)) 1_{[M/q]}(x), & h_1(x) &:= g_1(u + q(m+x)) 1_{[2M/q]}, \\ h_2(x) &:= e(\alpha(u+qx)^2) 1_{[M/q]}(x) \end{aligned}$$

Then by orthogonality, extraction of a large Fourier coefficient and Parseval's identity we have

$$\begin{aligned} \delta M^2/q^2 &\ll \left| \sum_{x,y} h_0(x)h_1(x+y)h_2(y) \right| = \left| \int_{\mathbb{T}} \hat{h}_0(\beta)\hat{h}_1(-\beta)\hat{h}_2(\beta)d\beta \right| \\ &\ll \|\hat{h}_2\|_{\infty} \|\hat{h}_0\|_{L^2} \|\hat{h}_1\|_{L^2} \ll \|\hat{h}_2\|_{\infty} M/q. \end{aligned}$$

It follows that there exists $\beta \in \mathbb{T}$ such that

$$\left| \sum_{x \in [M/q]} e(\alpha(u+qx)^2 + \beta x) \right| \gg \delta M/q.$$

Applying Weyl's inequality, we deduce the existence of $q' \ll \delta^{-O(1)}$ such that $\|q'q^2\alpha\| \ll \delta^{-O(1)}/(M/q)^2 = \delta^{-O(1)}q^2/M$. \square

Lemma 6.3 (Dual-difference interchange). *For each $y \in [M]$, let $F_y : \mathbb{Z} \rightarrow \mathbb{C}$ be a 1-bounded function, all with support contained in the same interval of length N . Set*

$$F(x) := \mathbb{E}_{y \in [M]} F_y(x).$$

Then for any function $\phi : \mathbb{Z}^s \rightarrow \mathbb{T}$ and finite set $\mathcal{H} \subset \mathbb{Z}^s$ we have

$$\begin{aligned} \left(N^{-s-1} \sum_{\underline{h} \in \mathcal{H}} \left| \sum_x \Delta_{\underline{h}} F(x) e(\phi(\underline{h})x) \right| \right)^{2^s} &\ll_s \\ N^{-2s-1} \sum_{\underline{h}^0, \underline{h}^1 \in \mathcal{H}} \left| \sum_x \mathbb{E}_{y \in [M]} \Delta_{\underline{h}^0 - \underline{h}^1} F_y(x) e(\phi(\underline{h}^0; \underline{h}^1)x) \right|, \end{aligned}$$

where

$$\phi(\underline{h}^0; \underline{h}^1) := \sum_{\omega \in \{0,1\}^s} (-1)^{|\omega|} \phi(\underline{h}^\omega) \quad \text{and} \quad \underline{h}^\omega := (h_1^{\omega_1}, \dots, h_s^{\omega_s}).$$

Proof. We proceed by induction on $s \geq 0$, the base case being an identity. Suppose then that $s \geq 1$. For $\underline{h} \in \mathbb{Z}^{s-1}$ and $h \in \mathbb{Z}$, we note that

$$(6.2) \quad \Delta_{(\underline{h}, h)} F(x) = \Delta_{\underline{h}} \left(\mathbb{E}_{y, y' \in [M]} F_y(x) \overline{F_{y'}(x+h)} \right).$$

Furthermore, if $(\underline{h}, h) \in \mathcal{H}$ contributes a non-zero expression of the form (6.2) then $(\underline{h}, h) \in (-N, N)^s$, since the support of F is contained in an interval of length N . Hence by the induction hypothesis

$$\begin{aligned} \left(N^{-s-1} \sum_h \sum_{\substack{\underline{h} \\ (\underline{h}, h) \in \mathcal{H}}} \left| \sum_x \Delta_{(\underline{h}, h)} F(x) e(\phi(\underline{h})x) \right| \right)^{2^s} &\ll_s \\ \left(N^{-2s} \sum_h \sum_{\substack{\underline{h}^0, \underline{h}^1 \\ (\underline{h}^i, h) \in \mathcal{H}}} \left| \sum_x \mathbb{E}_{y, y' \in [M]} \Delta_{\underline{h}^0 - \underline{h}^1} F_y(x) \Delta_{\underline{h}^0 - \underline{h}^1} \overline{F_{y'}(x+h)} e(\phi(\underline{h}^0; \underline{h}^1; h)x) \right| \right)^2, \end{aligned}$$

where

$$\phi(\underline{h}^0; \underline{h}^1; h) := \sum_{\omega \in \{0,1\}^{s-1}} (-1)^{|\omega|} \phi(\underline{h}^\omega, h).$$

Letting $e(\psi(\underline{h}^0; \underline{h}^1; h))$ denote the conjugate phase of the inner absolute value, we take the sum over h inside and apply the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} & \left(\sum_{\underline{h}^0, \underline{h}^1, x} \mathbb{E}_{y, y' \in [M]} \sum_{\substack{h \\ (\underline{h}^i, h) \in \mathcal{H}}} \Delta_{\underline{h}^0 - \underline{h}^1} F_y(x) \Delta_{\underline{h}^0 - \underline{h}^1} \overline{F_{y'}(x+h)} e(\phi(\underline{h}^0; \underline{h}^1; h)x + \psi(\underline{h}^0; \underline{h}^1; h)) \right)^2 \\ & \ll_s N^{2s-1} \sum_{\underline{h}^0, \underline{h}^1} \sum_{\substack{h^0, h^1 \\ (\underline{h}^i, h^j) \in \mathcal{H}}} \\ & \left| \sum_x \mathbb{E}_{y \in [M]} \Delta_{\underline{h}^0 - \underline{h}^1} F_y(x) \Delta_{\underline{h}^0 - \underline{h}^1} \overline{F_y(x+h^0-h^1)} e\left(\left(\phi(\underline{h}^0; \underline{h}^1; h^0) - \phi(\underline{h}^0; \underline{h}^1; h^1)\right)x\right) \right|. \end{aligned}$$

The result follows. \square

If $\phi(h_1, \dots, h_{s-1})$ is a function of $s-1$ variables we write $\phi(h_1, \dots, \hat{h}_i, \dots, h_s) := \phi(h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_s)$. We say that $\phi(h_1, \dots, h_s)$ is *low rank* if there exist functions $\phi_i(h_1, \dots, h_{s-1})$ such that

$$\phi(h_1, \dots, h_s) = \sum_{i=1}^s \phi_i(h_1, \dots, \hat{h}_i, \dots, h_s).$$

From the definition of the Gowers norm together with the U^2 -inverse theorem (Lemma A.1), one can show that largeness of the U^{s+2} -norm is equivalent to the existence of $\phi : \mathbb{Z}^s \rightarrow \mathbb{T}$ such that

$$\sum_{h_1, \dots, h_s} \left| \sum_x \Delta_h f(x) e(\phi(h)x) \right| \gg N^{s+1}.$$

The following lemma says that if ϕ is low-rank, then the U^{s+1} -norm must also be large.

Lemma 6.4 (Low rank correlation implies lower degree). *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a 1-bounded function with support in $[N]$. Then for $\phi_1, \dots, \phi_m : \mathbb{Z}^{s-1} \rightarrow \mathbb{T}$ with $m \leq s$ we have*

$$(6.3) \quad \frac{1}{N^{s+1}} \sum_{h_1, \dots, h_s} \left| \sum_x \Delta_h f(x) e\left(\sum_{i=1}^m \phi_i(h_1, \dots, \hat{h}_i, \dots, h_s)x\right) \right| \ll_s \left(\frac{\|f\|_{U^{s+1}}^{2^{s+1}}}{N^{s+2}} \right)^{2^{-m-1}}.$$

Proof. We proceed by induction on $m \geq 0$, the base case corresponding to the Cauchy–Schwarz inequality. Suppose then that $m \geq 1$ and the result is true for smaller values of m . Letting $e(\psi(h))$ denote the conjugate phase of the inner-most

sum, the left-hand side of (6.3) is equal to

$$\frac{1}{N^{s+1}} \sum_{h_2, \dots, h_s, x} \Delta_{h_2, \dots, h_s} f(x) e(\phi_1(h_2, \dots, h_s)) \sum_{h_1} \Delta_{h_2, \dots, h_s} \overline{f(x+h_1)} \\ e\left(\sum_{i=2}^m \phi_i(h_1, \dots, \hat{h}_i, \dots, h_s) x + \psi(h_1, \dots, h_s)\right).$$

By the Cauchy–Schwarz inequality, the square of this is at most

$$\ll_s \frac{1}{N^{s+2}} \sum_{h_2, \dots, h_s} \sum_{h_1, h'_1 \in (-N, N)} \left| \sum_x \Delta_{h_1-h'_1, h_2, \dots, h_s} f(x) e\left(\sum_{i=2}^m (\phi_i(h_1, \dots, \hat{h}_i, \dots, h_s) - \phi_i(h'_1, \dots, \hat{h}_i, \dots, h_s)) x\right) \right|.$$

Taking a maximum over $h'_1 \in (-N, N)$ and changing variables in h_1 , the latter is at most an absolute constant times

$$\frac{1}{N^{s+1}} \sum_{h_1, h_2, \dots, h_s} \left| \sum_x \Delta_{h_1, h_2, \dots, h_s} f(x) e\left(\sum_{i=2}^m (\phi_i(h_1 + h'_1, h_2, \dots, \hat{h}_i, \dots, h_s) - \phi_i(h'_1, h_2, \dots, \hat{h}_i, \dots, h_s)) x\right) \right|.$$

This phase has lower rank than the original, hence we may apply the induction hypothesis to yield the lemma. \square

Lemma 6.5 (Degree lowering). *Let $f_0, f_1 : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions with support in $[N]$ and, writing $M := \lfloor \sqrt{N/q} \rfloor$, define the dual function*

$$F(x) := \mathbb{E}_{y \in [M]} f_0(x - qy^2) f_1(x + y - qy^2).$$

If, for $s \geq 3$, we have

$$\sum_{u \in [q]} \|F\|_{U^s(u+q\mathbb{Z})}^{2^s} \geq \delta \sum_{u \in [q]} \|1_{[N]}\|_{U^s(u+q\mathbb{Z})}^{2^s},$$

then either $N \ll_s q^4 \delta^{-O_s(1)}$ or

$$\sum_{u \in [q]} \|F\|_{U^{s-1}(u+q\mathbb{Z})}^{2^{s-1}} \gg_s \delta^{O_s(1)} \sum_{u \in [q]} \|1_{[N]}\|_{U^{s-1}(u+q\mathbb{Z})}^{2^{s-1}}.$$

Proof. Given $u \in [q]$ set $F_u(x) := F(u + qx)$, a function with support in the interval $[2N/q]$. Applying the popularity principle, there exists a set of $\Omega_s(\delta q)$ residues $u \in [q]$ for which $\|F_u\|_{U^s}^{2^s} \gg_s \delta (N/q)^{s+1}$. Expanding the definition of the U^s -norm (1.8) we have

$$\sum_{h_1, \dots, h_{s-2}} \|\Delta_{h_1, \dots, h_{s-2}} F_u\|_{U^2}^4 \gg_s \delta (N/q)^{s+1}.$$

Applying the U^2 -inverse theorem (Lemma A.1), there exists $\mathcal{H} \subset (-2N/q, 2N/q)^{s-2}$ of size $|\mathcal{H}| \gg_s \delta (N/q)^{s-2}$ and a function $\phi : \mathbb{Z}^{s-2} \rightarrow \mathbb{T}$ such that for every $\underline{h} \in \mathcal{H}$

we have

$$(6.4) \quad \left| \sum_x \Delta_{\underline{h}} F_u(x) e(\phi(\underline{h})x) \right| \gg_s \delta N/q.$$

Set $T := \lceil C_s \delta^{-1} N/q \rceil$, with C_s an absolute constant taken sufficiently large to ensure that, on rounding $\phi(\underline{h})$ to the nearest fraction of the form t/T , the validity of (6.4) remains. Summing over $\underline{h} \in \mathcal{H}$ and applying Lemma 6.3, we deduce that

$$\sum_{\underline{h}^0, \underline{h}^1 \in \mathcal{H}} \left| \sum_x \mathbb{E}_{y \in [M]} \Delta_{\underline{h}^0 - \underline{h}^1} f_0(u + qx - qy^2) \Delta_{\underline{h}^0 - \underline{h}^1} f_1(u + qx + y - qy^2) e(\phi(\underline{h}^0; \underline{h}^1)x) \right| \gg_s \delta^{O_s(1)} (N/q)^{2s-3}.$$

Applying the pigeon-hole and popularity principle, there exists $\mathcal{H}' \subset \mathcal{H}$ of size $\gg_s \delta^{O_s(1)} (N/q)^{s-2}$ and $\underline{h}^1 \in \mathcal{H}$ such that for every $\underline{h}^0 \in \mathcal{H}'$ we have

$$\left| \sum_x \sum_{y \in [M]} \Delta_{\underline{h}^0 - \underline{h}^1} f_0(u + qx - qy^2) \Delta_{\underline{h}^0 - \underline{h}^1} f_1(u + qx + y - qy^2) e(\phi(\underline{h}^0, \underline{h}^1)x) \right| \gg_s \delta^{O_s(1)} MN/q.$$

By Lemma 6.2, for each $\underline{h}^0 \in \mathcal{H}'$ there exists $q' \ll \delta^{-O_s(1)}$ such that

$$\|q' q^2 \phi(\underline{h}^0, \underline{h}^1)\| \ll \delta^{-O_s(1)} q^3/N.$$

Notice that $q^2 \phi(\underline{h}^0, \underline{h}^1)$ is an element of the additive group $\{t/T : t \in [T]\} \subset \mathbb{T}$. Moreover, for any Q we have the inclusion

$$\{\alpha \in \mathbb{T} : \exists q' \leq Q \text{ with } \|q'\alpha\| \leq Qq^3/N\} \subset \bigcup_{1 \leq a \leq q' \leq Q} \left[\frac{a}{q'} - \frac{q^3 Q}{N}, \frac{a}{q'} + \frac{q^3 Q}{N} \right].$$

By a volume packing argument, the number of t/T lying in this union of intervals is at most $Q^2(1 + \frac{2q^3 QT}{N})$. When $Q \ll_s \delta^{-O_s(1)}$, we have $Q^2(1 + \frac{2q^3 QT}{N}) \ll_s \delta^{-O_s(1)}$ (unless $N \ll_s q^4 \delta^{-O_s(1)}$, which we are permitted to assume does not happen). It therefore follows from the pigeon-hole principle that there exists $\mathcal{H}'' \subset \mathcal{H}'$ of size $\gg_s \delta^{O_s(1)} (N/q)^{s-2}$ and $t_0 \in [T]$ such that for any $\underline{h}^0 \in \mathcal{H}''$ we have $\phi(\underline{h}^0, \underline{h}^1) = t_0/T$. In particular, when restricted to the set \mathcal{H}'' , the function ϕ satisfies

$$\phi(\underline{h}^0) = t_0/T - \sum_{\omega \in \{0,1\}^{s-2} \setminus \{0\}} (-1)^{|\omega|} \phi(\underline{h}^\omega).$$

The right-hand side of this identity is *low rank* according to the terminology preceding Lemma 6.4.

Summing over $\underline{h} \in \mathcal{H}''$ in (6.4), we deduce the existence of a low rank function $\psi : \mathbb{Z}^{s-2} \rightarrow \mathbb{T}$ such that

$$\sum_{\underline{h}} \left| \sum_x \Delta_{\underline{h}} F_u(x) e(\psi(\underline{h})x) \right| \gg_s \delta^{O_s(1)} (N/q)^{s-1}.$$

Employing Lemma 6.4 then gives

$$\|F_u\|_{U^{s-1}}^{2s-1} \gg_s \delta^{O_s(1)} (N/q)^s.$$

Summing over permissible u , then extending to the full sum over $u \in [q]$ by positivity, we obtain the bound claimed in the lemma. \square

7. THE INVERSE THEOREM FOR OUR COUNTING OPERATOR

In this section we show how U^5 -control of the final function in our counting operator, as proved in Theorem 5.6, also yields U^5 -control of the dual function. Combining this with the degree lowering of §6, we deduce that the dual is controlled by the U^1 -norm. This allows us to deduce the following key inverse theorem for our counting operator.

Theorem 7.1 (Inverse theorem for nonlinear Roth). *Let $f_0, f_1, f_2 : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions, each with support in $[N]$. Suppose that*

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x+y) f_2(x+qy^2) \right| \geq \delta \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_{[N]}(x) 1_{[N]}(x+y) 1_{[N]}(x+qy^2).$$

Then either $N \ll q^4 \delta^{-O(1)}$, or there exists $q' \ll \delta^{-O(1)}$ such that for each $i = 0, 1, 2$ there exists a 1-bounded function $\phi_i : \mathbb{Z} \rightarrow \mathbb{C}$ that is $\ll \delta^{-O(1)} q^{3/2} N^{-1/2}$ Lipschitz along $q'q \cdot \mathbb{Z}$, in that

$$(7.1) \quad |\phi_i(x+q'qy) - \phi_i(x)| \ll \delta^{-O(1)} q^{3/2} N^{-1/2} |y| \quad (\forall x, y \in \mathbb{Z}),$$

and for which

$$\left| \sum_x f_i(x) \phi_i(x) \right| \gg \delta^{O(1)} N.$$

Remark. Inspection of the following proof reveals that for $i = 1$ one can in fact ensure that the function ϕ_1 is $\ll \delta^{-O(1)} q^{1/2} N^{-1/2}$ Lipschitz along $q' \cdot \mathbb{Z}$. This stronger property is not needed in our present application.

Proof. Define the dual function

$$(7.2) \quad F_2(x) := \mathbb{E}_{y \in [M]} f_0(x - qy^2) f_1(x + y - qy^2).$$

Either $N \ll q$ or we have

$$\delta NM \ll |\Lambda_q(f_0, f_1, f_2)| = M \left| \sum_x F_2(x) f_2(x) \right|.$$

Since f_2 is supported on $[N]$, the Cauchy–Schwarz inequality gives

$$\Lambda_q(f_0, f_1, \overline{F_2}) = M \sum_x F_2(x) \overline{F_2(x)} \gg \delta^2 NM.$$

Since the functions $f_0, f_1, \overline{F_2}$ all have support contained in $[2N]$, we may apply Theorem 5.6 to deduce that

$$\sum_{u \in [q]} \|F_2\|_{U^5(u+q\mathbb{Z})}^{2^5} \gg \delta^{O(1)} \sum_{u \in [q]} \|1_{[2N]}\|_{U^5(u+q\mathbb{Z})}^{2^5}.$$

We now apply Lemma 6.5 three times to obtain

$$\sum_{u \in [q]} \|F_2\|_{U^2(u+q\mathbb{Z})}^4 \gg \delta^{O(1)} \sum_{u \in [q]} \|1_{[2N]}\|_{U^2(u+q\mathbb{Z})}^4.$$

By the popularity principle, there are at least $\gg \delta^{O(1)}q$ values of $u \in [q]$ for which $\|F_2\|_{U^2(u+q\mathbb{Z})}^4 \gg \delta^{O(1)}\|1_{[2N]}\|_{U^2(u+q\mathbb{Z})}^4$. The inverse theorem for the U^2 -norm then gives the existence of $\phi(u) \in \mathbb{T}$ for which

$$(7.3) \quad \left| \sum_x F_2(u+qx)e(\phi(u)x) \right| \gg \delta^{O(1)}N/q.$$

Set $T := \lceil C\delta^{-C}N/q \rceil$, with C an absolute constant taken sufficiently large to ensure that, on rounding $\phi(u)$ to the nearest fraction of the form t/T , the inequality (7.3) remains valid.

By Lemma 6.2, for each u satisfying (7.3), there exists a positive integer $q' \ll \delta^{-O(1)}$ such that $\|q'q^2\phi(u)\| \ll \delta^{-O(1)}q^3/N$. By a volume packing argument similar to that given in the proof of Lemma 6.5, the function ϕ is constant on a proportion of at least $\gg \delta^{O(1)}$ of the residues $u \in [q]$ satisfying (7.3). Summing over these u , then extending the sum to all of $[q]$, we deduce the existence of $\alpha \in \mathbb{T}$ such that

$$(7.4) \quad \sum_{u \in [q]} \left| \sum_x F_2(u+qx)e(\alpha x) \right| \gg \delta^{O(1)}N.$$

Expanding the dual function, there is a 1-bounded function $\psi(u \bmod q)$ such that the left-hand side of the above is equal to

$$(7.5) \quad \begin{aligned} \sum_{u \in [q]} \psi(u \bmod q) \sum_{x \equiv u(q)} \mathbb{E}_{y \in [M]} f_0(x - qy^2) f_1(x + y - qy^2) e(\alpha x/q) \\ = \sum_x f_0(x) \psi(x \bmod q) e(\alpha x/q) \mathbb{E}_{y \in [M]} f_1(x + y) e(\alpha y^2). \end{aligned}$$

Applying Lemma 6.2 with $q = 1$, there exists $q' \ll \delta^{-O(1)}$ such that $\|q'\alpha\| \ll \delta^{-O(1)}/M^2 = \delta^{-O(1)}q/N$.

Using this, let us demonstrate the correlation of f_0 with a suitably Lipschitz function; the case of f_1 is similar (in fact simpler) and the case of f_2 is dealt with shortly. Setting

$$\phi_0(x) := \psi(x \bmod q) e(\alpha x/q) \mathbb{E}_{y \in [M]} f_1(x + y) e(\alpha y^2),$$

we have $\sum_x f_0(x) \phi_0(x) \gg \delta^{O(1)}N$. Our aim is to show that ϕ_0 is $\ll \delta^{-O(1)}q^{3/2}N^{-1/2}$ Lipschitz along $q'q \cdot \mathbb{Z}$.

For any $x, z \in \mathbb{Z}$ we have

$$\begin{aligned} |\phi_0(x) - \phi_0(x + q'qz)| &\leq |1 - e(\alpha q'z)| \\ &\quad + \left| \mathbb{E}_{y \in [M]} f_1(x + y) e(\alpha y^2) - \mathbb{E}_{y \in [M]} f_1(x + y + q'qz) e(\alpha y^2) \right|. \end{aligned}$$

The first term satisfies

$$|1 - e(\alpha q'z)| \ll \|q'\alpha\| |z| \ll \delta^{-O(1)}q|z|/N \ll \delta^{-O(1)}q^{3/2}N^{-1/2}|z|.$$

Changing variables in y , the second term is at most

$$\begin{aligned} & q'q|z|M^{-1} + \left| \mathbb{E}_{y \in [M]} g(x+y)e(\alpha y^2) - \mathbb{E}_{y \in [M]} g(x+y)e(\alpha(y-q'qz)^2) \right| \\ & \ll q'q|z|M^{-1} + \mathbb{E}_{y \in [M]} |e(\alpha y^2) - e(\alpha(y-q'qz)^2)| \\ & \ll q'q|z|M^{-1} + \left\| \alpha(q'qz)^2 \right\| + \mathbb{E}_{y \in [M]} \|\alpha 2yq'qz\| \\ & \ll \delta^{-O(1)} q^{3/2} |z| N^{-1/2} + \delta^{-O(1)} q^3 |z|^2 N^{-1}. \end{aligned}$$

We thereby obtain the required Lipschitz inequality (7.1) in the case that $|z| \leq N^{1/2} q^{-3/2}$. In the remaining case the Lipschitz inequality is trivial, since ϕ_0 is 1-bounded.

Having proved the inverse theorem for f_0 and f_1 , we now focus on f_2 . As before, define the dual function

$$F_1(x) := 1_{[N]}(x) \mathbb{E}_{y \in [M]} f_0(x-y) f_2(x+qy^2-y),$$

so that

$$\left| \sum_x F_1(x) f_1(x) \right| \gg \delta N.$$

Applying the Cauchy–Schwarz inequality gives

$$\Lambda_q(f_0, \overline{F_1}, f_2) = M \sum_x F_2(x) \overline{F_2(x)} \gg \delta^2 NM,$$

and the inverse theorem then yields $q_1 \ll \delta^{-O(1)}$ such that there exists a 1-bounded function $\phi_1 : \mathbb{Z} \rightarrow \mathbb{C}$ which is $\ll \delta^{-O(1)} q^{3/2} N^{-1/2}$ Lipschitz along $q_1 q \cdot \mathbb{Z}$ and for which

$$|\Lambda_q(f_0, \phi_1, f_2)| = |\Lambda_q(f_0, \phi_1 1_{[N]}, f_2)| = M \left| \sum_x F_1(x) \phi_1(x) \right| \gg \delta^{O(1)} NM.$$

Repeating the above procedure, we obtain $q_0 \ll \delta^{-O(1)}$ such that there exists a 1-bounded function $\phi_0 : \mathbb{Z} \rightarrow \mathbb{C}$ which is $\ll \delta^{-O(1)} q^{3/2} N^{-1/2}$ Lipschitz along $q_0 q \cdot \mathbb{Z}$ and for which

$$|\Lambda_q(\phi_0 1_{[N]}, \phi_1, f_2)| \gg \delta^{O(1)} NM.$$

We may replace $1_{[N]}$ in the above by a continuous function ψ which is 1 on $[c\delta^C N, (1-c\delta^C)N]$, 0 on $\mathbb{Z} \setminus [N]$ and linear everywhere else. Since the linear parts have gradient $\ll \delta^{-O(1)}/N$, we deduce that ψ is $\ll \delta^{-O(1)}/N$ Lipschitz along \mathbb{Z} . As a consequence, the product $\phi_0 \psi$ is $\ll \delta^{-O(1)} q^{3/2}/N^{1/2}$ Lipschitz along $q_0 q \cdot \mathbb{Z}$. Finally, we observe that

$$\delta^{O(1)} N \ll \max_{y \in [M]} \left| \sum_x (\phi_0 \psi)(x-yy^2) \phi_1(x+y-yy^2) f_2(x) \right|,$$

and the the function $x \mapsto (\phi_0 \psi)(x-yy^2) \phi_1(x+y-yy^2)$ is $\ll \delta^{-O(1)} q^{3/2}/N^{1/2}$ Lipschitz along $q_0 q_1 q \cdot \mathbb{Z}$. \square

Corollary 7.2 (Local correlation with constant functions). *Let $f_0, f_1, f_2 : \mathbb{Z} \rightarrow \mathbb{C}$ be 1-bounded functions, each with support in $[N]$. Suppose that*

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x+y) f_2(x+qy^2) \right| \geq \delta \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_{[N]}(x) 1_{[N]}(x+y) 1_{[N]}(x+qy^2).$$

Then either $N \ll q^3 4\delta^{-O(1)}$, or there exists $q' \ll \delta^{-O(1)}$ and $N' \gg \delta^{O(1)} q^{-3/2} N^{1/2}$ such that for each $i = 0, 1, 2$ we have

$$\sum_{x \in \mathbb{Z}} \left| \sum_{y \in [N']} f_i(x + q'qy) \right| \gg \delta^{O(1)} NN'.$$

Proof. Let ϕ_i denote the function guaranteed by Theorem 7.1. By the Lipschitz property of ϕ_i we have

$$|\phi_i(x) - \mathbb{E}_{y \in [N']} \phi_i(x + q'qy)| \ll \delta^{-O(1)} q^{3/2} N' N^{-1/2}.$$

Hence taking $N' = c\delta^C N^{1/2} q^{-3/2}$ for $c > 0$ sufficiently small and C sufficiently large, we deduce that

$$\left| \sum_x f_i(x) \mathbb{E}_{y \in [N']} \phi_i(x + q'qy) \right| \gg \delta^{O(1)} N.$$

Changing variables in x and applying the triangle inequality yields the result. \square

8. THE DENSITY INCREMENT LEMMA

In this section we prove Lemma 2.1.

Proof of Lemma 2.1. We either have $N \ll q$ or

$$\sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_{[N]}(x) 1_{[N]}(x + y) 1_{[N]}(x + qy^2) \gg N^{3/2} q^{-1/2}.$$

Therefore

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_A(x) 1_A(x + y) 1_A(x + qy^2) - \delta^3 1_{[N]}(x) 1_{[N]}(x + y) 1_{[N]}(x + qy^2) \right| \gg \delta^3 N^{3/2} q^{-1/2}.$$

By a telescoping identity, there exist 1-bounded functions $f_0, f_1, f_2 : \mathbb{Z} \rightarrow \mathbb{R}$ all with support in $[N]$ and at least one of which is equal to $1_A - \delta 1_{[N]}$ such that

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + y) f_2(x + qy^2) \right| \gg \delta^3 N^{3/2} q^{-1/2}.$$

Applying our inverse theorem (Corollary 7.2) we deduce that there exists $q' \ll \delta^{-O(1)}$ and $N' \gg \delta^{O(1)} q^{-3/2} N^{1/2}$ such that

$$\sum_{x \in \mathbb{Z}} \left| \sum_{y \in [N']} (1_A - \delta 1_{[N]})(x + q'qy) \right| \gg \delta^{O(1)} NN'.$$

Since the corresponding sum without absolute values is equal to zero, we are able to find x such that

$$\sum_{y \in [N']} (1_A - \delta 1_{[N]})(x + q'qy) \gg \delta^{O(1)} N'.$$

Writing $x_1 + q'q \cdot [N_1]$ for $[N] \cap (x + q'q \cdot [N'])$ we have $N_1 \gg \delta^{O(1)} N' \gg \delta^{O(1)} q^{-3/2} N^{1/2}$ and

$$|A \cap (x_1 + q'q \cdot [N_1])| \geq (\delta + \Omega(\delta^{O(1)})) N_1.$$

□

APPENDIX A. BASIC THEORY OF THE GOWERS NORMS

Lemma A.1 (Inverse theorem for the U^2 -norm). *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a 1-bounded function with support in $[N]$. Then there exists $\alpha \in \mathbb{T}$ such that*

$$\|f\|_{U^2}^4 \leq N \left| \sum_x f(x) e(\alpha x) \right|^2.$$

Proof. Using the definition of the Fourier transform (1.6), together with orthogonality of additive characters, we have

$$\|f\|_{U^2}^4 = \int_{\mathbb{T}} |\hat{f}(\alpha)|^4 d\alpha \leq \|\hat{f}\|_{\infty}^2 \int_{\mathbb{T}} |\hat{f}(\alpha)|^2 d\alpha \leq \|\hat{f}\|_{\infty}^2 N.$$

□

For each $\omega \in \{0, 1\}^s$, let $f_{\omega} : \mathbb{Z} \rightarrow \mathbb{C}$ be a function with finite support. Then we define the *Gowers inner product* by

$$[f_{\omega}]_{U^s} := \sum_{x, h_1, \dots, h_s} \prod_{\omega \in \{0, 1\}^s} \mathcal{C}^{|\omega|} f_{\omega}(x + \omega \cdot h).$$

Here \mathcal{C} denotes the operation of complex conjugation. Notice that $[f]_{U^s} = \|f\|_{U^s}^{2^s}$.

Lemma A.2 (Gowers–Cauchy–Schwarz inequality). *For each $\omega \in \{0, 1\}^s$, let $f_{\omega} : \mathbb{Z} \rightarrow \mathbb{C}$ be a function with finite support. Then we have*

$$[f_{\omega}]_{U^s} \leq \prod_{\omega \in \{0, 1\}^s} \|f_{\omega}\|_{U^s}.$$

Proof. See [Tao12, Exercise 1.3.19].

□

Lemma A.3 (Phase invariance for $s \geq 2$). *Let $L \in \mathbb{R}[x, h_1, \dots, h_s]$ be a linear form, with $s \geq 2$ and let $f : \mathbb{Z} \rightarrow \mathbb{C}$. Then*

$$\left| \sum_{x, h_1, \dots, h_s} \Delta_{h_1, \dots, h_s} f(x) e(L(x, h_1, \dots, h_s)) \right| \leq \|f\|_{U^s}^{2^s}.$$

Proof. The linear form may be written as

$$L(x, h_1, \dots, h_s) = \alpha x + \beta_1(x + h_1) + \dots + \beta_s(x + h_s),$$

for some real α and β_i . Write $f_0(x) := f(x)e(\alpha x)$, $f_{e_i}(x) := f(x)e(-\beta_i x)$ for $i = 1, \dots, s$, and for $\omega \in \{0, 1\}^s \setminus \{0, e_1, \dots, e_s\}$ set $f_{\omega} := f$. Then by Gowers–Cauchy–Schwarz we have

$$\left| \sum_{x, h_1, \dots, h_s} \Delta_{h_1, \dots, h_s} f(x) e(L(x, h_1, \dots, h_s)) \right| \leq \prod_{\omega} \|f_{\omega}\|_{U^s}.$$

It therefore suffice to prove that for a phase function $e_\alpha : x \mapsto e(\alpha x)$ we have $\|fe_\alpha\|_{U^s} = \|f\|_{U^s}$. The latter follows on observing that

$$\Delta_{h_1, \dots, h_s}(fe_\alpha) = (\Delta_{h_1, \dots, h_s}f)(\Delta_{h_1, \dots, h_s}e_\alpha),$$

and for any x, h_1, \dots, h_s with $s \geq 2$ we have $\Delta_{h_1, \dots, h_s}e_\alpha(x) = 1$. □

Lemma A.4 (Box Gowers–Cauchy–Schwarz). *Let μ_1, μ_2, μ_3 be probability measures on \mathbb{Z} with the discrete sigma algebra. If F_1, F_2, F_3 are 1-bounded function on \mathbb{Z}^2 and F is a 1-bounded function on \mathbb{Z}^3 then*

$$\left| \sum_{x \in \mathbb{Z}^3} F_1(x_2, x_3) F_2(x_1, x_3) F_3(x_1, x_2) F(x_1, x_2, x_3) \mu_1(x_1) \mu_2(x_2) \mu_3(x_3) \right|^8 \\ \leq \sum_{x^0, x^1 \in \mathbb{Z}^3} \prod_{\omega \in \{0,1\}^3} C^{|\omega|} F(x_1^{\omega_1}, x_2^{\omega_2}, x_3^{\omega_3}) \mu_1(x_1^0) \mu_1(x_1^1) \mu_2(x_2^0) \mu_2(x_2^1) \mu_3(x_3^0) \mu_3(x_3^1).$$

Proof. This follows from three applications of the Cauchy–Schwarz inequality, each application doubling one of the variables x_i . □

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DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, USA
Email address: `speluse@stanford.edu`

SCHOOL OF MATHEMATICAL SCIENCES, LANCASTER UNIVERSITY, UK
Email address: `s.prendiville@lancaster.ac.uk`