Discrete integrable systems associated with deformations of cluster maps

Wookyung Kim

School of Mathematical Sciences

Lancaster University

A thesis submitted for the degree of

Doctor of Philosophy

July 2024

Discrete integrable systems associated with deformations of cluster maps

Wookyung Kim

Abstract

In this thesis, we study a discrete dynamical system provided by the deformation of cluster mutations associated with cluster algebras of finite Dynkin diagrams of type A_{2N} ($N \geq 3$), $\mathrm{C}_2(\cong \mathrm{B}_2)$, B_3, B_4, D_4 and D_6. In the first part of the thesis, we show that the corresponding cluster algebras exhibit a remarkable periodicity phenomenon, known as Zamolodchikov periodicity. We present a particular deformation, which is a novel approach introduced by Hone and Kouloukas. This procedure modifies cluster mutation in such a way that it preserves the natural presymplectic form in cluster algebras. For the cases of type C_2, B_3, D_4, we construct Liouville integrable maps defined by the specific composition of deformed cluster mutations and show that the new cluster algebras emerge by considering Laurentification, which is a lifting of the map into a higher-dimensional space where the Laurent property is exhibited. The corresponding deformed integrable maps are also closely related to Somos-type recurrences.

In particular, we prove the integrability of the birational map defined by a sequence of mutations in cluster algebra of type A_{2N}. We examine the deformation of discrete dynamics in cluster algebra of type A_6 and compare the result associated with the type A_4 case. Following this, we introduce a local expansion operation on quivers, which provides a special family of quivers corresponding to specific types of cluster algebras. We demonstrate that these cluster algebras are obtained by lifting the deformation of type A_{2N} maps via Laurentification. We also show that the discrete dynamics in the new cluster algebras, induced from cluster algebras of type A_{2N}, B_4 and D_6 via deformation, admit the tropical (max-plus) analogue of the system of homogeneous recurrences. This allows us to calculate the exact degree growth of the discrete dynamical system.

Acknowledgements

With many thanks to my supervisor, Jan Grabowski, for providing dedicated support and constant guidance, which gave me confidence and encouragement to endure the hardships of my studies and research. To my co-supervisor, Andrew Hone, for introducing me to the topic of the deformation of cluster mutations and providing a great amount of ideas and insights for my research. His expertise and commitment to the research have been instrumental in shaping the trajectory of my research. Furthermore, to Mase Takafumi for exchanging ideas, which led to broadening my perspective on current research topics. Finally, to my mum for her unwavering support for every step.

This work was funded by EPSRC studentship (Project ref no 2436985) and Mathematics and Statistics at Lancaster University. I'm grateful to them for their financial support during the research project.

Declaration

This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in Section 1.1 . This thesis has not been submitted, either in whole or in part, for a degree at this, or any other university. This thesis does not exceed the maximum permitted word length of 80,000 words.

Wookyung Kim

Contents

Chapter 1

Introduction

Cluster algebras are a class of commutative algebras, constructed as subalgebras of rational function fields, which were introduced by Fomin and Zelevinsky in [1]. These algebras are built differently from many other commutative algebras as cluster algebras are not presented with generators and relations from the beginning. Instead, to define cluster algebras, we start with initial data, given by two objects,

- initial cluster variables, n distinguished generators $\mathbf{x} = (x_1, \ldots, x_n)$ and
- an exchange quiver Q, a finite directed graph with n nodes which does not contain loops or oriented 2-cycles.

The pair of objects $(\mathbf{x}, Q)$ is called an *initial seed*. Then we apply a special iterative process called *mutation* to produce more cluster variables and exchange quivers. Continued application of the mutation process results in constructing the algebra, which is called the associated *cluster algebra*, as the subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

Fomin and Zelevinsky further extended the notion of mutation to define the alternative version known as a *Y-seed pattern* [2, 3]. This introduces *coefficient variables*. Similar to cluster mutation, the coefficient variables have their own dynamics described by the mutation.

Along with the Laurent phenomenon (or Laurent property), which says that every cluster variable can be expressed as a Laurent polynomial in the initial (or indeed any) seed (rather than just being a rational function), the other main result they obtained is the classification of cluster algebras of finite type. A cluster algebra is said to be of finite type if it has only finitely many cluster variables. Such cluster algebras are classified by quivers that are orientations of a Dynkin diagram of a finite dimensional semisimple Lie algebra.

These results raised the importance of the question of studying the recurrence relation which is given by mutations in Y-seeds. Such a relation is equivalent to the difference equation known as a *Y-system*. Y-systems were discovered by Zamolodchikov in [4] and he showed that the solutions of the difference equation are also solutions of the Bethe ansatz equation associated with conformal theories of ADE scattering diagrams. Furthermore, it was observed that the solutions appeared to be periodic with a particular period; this was known as *Zamolodchikov's periodicity conjecture*. In [5], this was investigated via the cluster algebra setting and Fomin and Zelevinsky showed a specific product of mutations exhibited Zamolodchikov periodicity, which we discuss in Section 2.1.5.

In [6], Marsh and Fordy introduced the periodicity phenomenon for particular quivers, which is known as *mutation periodic* that is a feature that the associated quiver remains to be invariant up to permutation of labellings i.e. $\mu_{i_r}\mu_{i_{r-1}}\cdots\mu_{i_1}Q = \rho(Q)$, where ρ is a permutation of the nodes. Following it, Hone and Fordy in [7] (and Nakanishi in [8]) used the notion of mutation periodicity and introduced the birational map between seeds, referred to as the *cluster map*, which induces the new cluster variables but preserves the quiver. There exist unique cases of cluster maps such that the initial seed is fixed under a particular number of iterations of mapping, which is referred to as *periodic cluster maps*. In this thesis, we consider the cluster map which possesses Zamolodchikov periodicity. The notion of cluster maps establishes a connection between cluster algebras and *discrete integrable systems*.

What is an *integrable system*? In the world of physics, physical processes can be described by differential equations whose solutions provide a better understanding of unknown phenomena. Most continuous models using differential equations cannot be solved exactly and require analysis with numerical approximation. However, there exist particular models that allow for direct integration without numerical methods. Models with this special feature are referred to as *integrable systems*.

There are various notions of integrability, for instance: compatibility conditions for the Lax pairs, exhibiting regular motion, and existence of a complete set of conserved quantities (first integrals). Thus there is no general definition for an integrable system as the notion varies depending on the context. Here, we consider the specific case, known as a *Liouville integrable system*, which is, roughly speaking, a dynamical system whose motion is constrained by a sufficient number of conserved quantities which are invariant under the dynamics and are in involution with respect to the associated Poisson bracket. In Section 2.2, we review the definition in the context of Hamiltonian system.

The notion of Liouville integrability can be transferred to a discrete dynamical systems, defined by iteration of mappings. This leads us to the identification of a discrete integrable system (see [9, 7]). From the view of the discrete system, the iteration of cluster maps can be regarded as a discrete dynamical system; one can identify an associated discrete integrable system in the sense of the Liouville integrable map introduced in [9] (see further detail in section 2.2). Over the years, the relationship between cluster algebras and discrete integrable systems has been studied and produced numerous interesting methods and results up to this point.

Recently, Hone and Kouloukas [10] introduced *deformation* of coefficient-free cluster mutation which preserves the symplectic form that is compatible with mutation. They showed that the composition of deformed mutations can be regarded as a symplectic cluster map and thus enables us to discover a family of new discrete integrable systems. However a problem arises upon applying the deformation, that is, it ends up destroying the Laurent property of the cluster map, which is an essential property of cluster algebras. In order to resolve the problem, they took a specific procedure, *Laurentification* (a term coined by Hamad et al. in [11]), that is, lifting the system to a higher dimensional system where the Laurent property exists. This was established by constructing the transformation through the p-adic methods introduced by Kanki in [12] which is analogous to the singularity confinement test [13]. As a result, they presented several examples, including deformed integrable cluster maps associated with Dynkin types A_2, A_3 and A_4, which can be successfully lifted to higher dimensional integrable cluster maps, where Laurent property holds.

1.1 Outline

The main aim of this thesis is to study a special families of discrete integrable systems which emerged from the iteration of Dynkin type cluster maps equipped with periodicity property via the *deformation* procedure of [10].

Chapter 2 reviews definitions and results related to cluster algebras and discrete integrable systems. This chapter provides the necessary background for exploring results in the later chapters. We then review the deformation of cluster mutation introduced in the paper [10]. Following on from this, we consider several examples of integrable deformation of type A_2 and A_4 cluster maps to see how singularity analysis can be used in the process of *Laurentification* which lifts deformed map to cluster algebra of higher rank whose seed is extended by frozen variables.

Chapter 3 is based on the paper [14] which is joint work with Jan Grabowski and Andrew Hone. We begin the analysis of cluster maps associated to type A_{2N} in general and show that the periodic cluster map associated to type A_{2N} is integrable (Theorem 3.1.5). Next we examine the base case for our "inductive" approach, namely type A_6 and compare it with type A_4. We see that the Laurentification of the former is obtained from that of the latter by insertion of a particular quiver, in a form of local expansion. By successive local expansions, we can construct an associated family of quivers that are corresponding to the deformation of type A_{2N} maps via Laurentification. In the final part of the chapter, we examine the degree growth of the deformed type A_{2N} to perform one of the integrability tests, algebraic entropy.

Chapter 4 is based on the paper [15] which is joint work with Andrew Hone and Mase Takafumi. In this chapter we consider an integrable deformation of the C_2 cluster map and show that this lifts to a cluster algebra of rank 5 extended by a single frozen variable. We apply the same procedure to periodic maps of type

B_3 and D_4. We will see that each case admits two distinct deformations that are integrable. This is a novel situation which was not seen in other known cases. We prove that the discrete dynamics, induced by the deformed maps, are closely related to Somos-5 and (a special case of) Somos-7 recurrence

Chapter 5 concerns deformation of periodic cluster maps associated with type B_4 and D_6. We show that these maps can be lifted to cluster maps via Laurentification. We show that the degree growth of these deformed maps is quadratic, which lead us to conjecture that they are integrable.

Chapter 2

Preliminaries

2.1 Cluster algebras

In this section, we will define mutation, which has two aspects: quiver mutation and cluster mutation and introduce an example to see the construction of cluster algebras (see [1, 16, 3, 17] for further details of cluster algebras). We follow by introducing the features of cluster algebras: Laurent phenomenon and finite type cluster algebras; we observe that the features were used to link the cluster algebras with root systems, Zamolodchikov periodicity and mutation periodicity.

2.1.1 Quiver and matrix mutation

Let $Q = (V, E)$ be a quiver with n nodes, $V = \{1, 2, \ldots, n\}$ and directed edges E. We assume that Q does not possess any loops or oriented 2-cycles. However, multiple edges between nodes are allowed and when there are many edges between two vertices we write $\xrightarrow{i}$ as a shorthand for i parallel arrows.

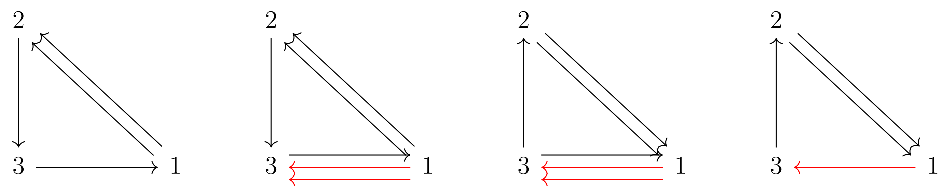
Definition 2.1.1. *Let Q be a quiver. Quiver mutation at node k, to obtain the new quiver $\mu_k(Q)$, is performed by following the steps below:*

1. *For each full subquiver $i \xrightarrow{p} k \xrightarrow{q} j$, insert a (multiple) edge $i \xrightarrow{pq} j$*

2. *Reverse all arrows which are connected to* k*,* $j \xrightarrow{q} k \xrightarrow{p} i$

3. *Remove any 2-cycles which are formed by inserting arrows.*

Example 2.1.2 (Quiver mutation at node 2)**.**



As we assumed that the quiver has no 2-cycles and no loops, then we can identify a $n \times n$ skew-symmetric matrix which corresponds to the quiver.

Remark 2.1.3. *The applet authored by Bernhard Keller, available at* `https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/`*, can be used as a graphical tool to help compute quiver mutations. Throughout this thesis, we have done so and outputs from it appear in e.g. Figure 2.4.*

Definition 2.1.4. *Let* Q *be a quiver with* n *vertices and no 2-cycles and no loops. Then one can encode this quiver in the* $n \times n$ *skew-symmetric integer matrix* $B = B(Q)$ *by setting the matrix entry* b_{ij} *to be the number of arrows* i *to* j*. Then we refer to this matrix* B *as an exchange matrix.*

As the quiver Q can be represented by the exchange matrix $B = B(Q)$, one can formulate the quiver mutation in terms of entries of B, giving $\mu_k(B)$. We refer to this formula as *matrix mutation* (see [17, 3, 16]).

Definition 2.1.5. *Let* B *be an exchange matrix and let* $B' = \mu_k(B)$ *be the new exchange matrix obtained by applying mutation to the exchange matrix* B *in direction* k*. The entries of* B'*,* b'_{ij}*, are given by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise} \end{cases} \tag{2.1}$$

Alongside quiver (correspondingly, matrix) mutation, cluster variables transform under *cluster mutation.*

Definition 2.1.6. *Let $\mathcal{F}$ be the field of rational functions in n independent variables $x_1, \ldots, x_n$ over $\mathbb{C}$ and set $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{F}^n$. The cluster mutation of $\mathbf{x}$ in direction k is $\mu_k(\mathbf{x}) = (x_1, \ldots, x_{k-1}, x'_k, \ldots x_n)$ where x'_k is the element defined by the expression,*

$$\mu_k(x_k) = x'_k = \frac{1}{x_k}\left(\prod_{\substack{j=1 \\ b_{jk}>0}}^{n} x^{b_{jk}} + \prod_{\substack{j=1 \\ b_{jk}<0}}^{n} x^{-b_{jk}}\right) \tag{2.2}$$

This expression is known as a (coefficient free) exchange relation.

Note that we could work over other base fields than $\mathbb{C}$ but we will restrict to this choice to employ geometric methods later.

Given an initial seed $(\mathbf{x}, B)$ of size n, one can apply the mutations in n possible directions. This results in n new seeds. In succession, the mutations can be applied to each such seed in n possible directions again, and so on. It is important to note that consecutive mutations in the same direction do not yield anything new. This is due to the mutation being involutive: $\mu_i^2 = \mathrm{id}$. Then, in this way, we may label the vertices of a rooted n-valent tree by clusters. Note that in general mutations on different vertices do not commute (in the sense that $\mu_i \circ \mu_j \neq \mu_j \circ \mu_i$) unless the vertices are "far apart" (i.e. are at least distance 2 apart in the quiver). Then one can produce the collection of cluster variables which are induced by iterated cluster mutations in all directions; the set of all cluster variables obtained in this way generates the so-called *cluster algebra.*

Definition 2.1.7 (Cluster algebra, [1])**.** *The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is the $\mathbb{C}$-subalgebra of the field $\mathcal{F}$ whose generating set is the set of all cluster variables produced by all possible sequences of mutations applied to the initial seed $(\mathbf{x}, B)$.*

Note that if $\mathbf{x}$ is of size n (so B is an $n \times n$ matrix), we say $(\mathbf{x}, B)$ is of *rank* n.

Example 2.1.8. *Begin with the initial seed $(\mathbf{x}, B)$, where $\mathbf{x} = (x_1, x_2)$ and the exchange matrix*

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The mutation in the direction 1, $\mu_1(\mathbf{x}, B) = (\mathbf{x}', B')$, *where* $\left(x_1', x_2'\right)$

$$\left(x_1', x_2'\right) = \left(\frac{1+x_2}{x_1}, x_2\right), \qquad B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -B$$

Since the mutation returns the initial seed if we apply the mutation in the same direction as previously, the next mutation should be in direction 2. The mutation $\mu_2(\mathbf{x}') = \left(x_1'', x_2''\right)$,

$$x_1'' = x_1'$$

$$x_2'' = \frac{1+x_1'}{x_2'} = \frac{1+\left(\frac{1+x_2}{x_1}\right)}{x_2} = \frac{1+x_1+x_2}{x_1x_2}$$

and

$$\mu_2(B') = B$$

Applying mutation in sequence yields the following cluster variables,

	x_1	x_2	B
$\mu_1(\mathbf{x}, B)$	$\frac{1+x_2}{x_1}$		$-B$
$\mu_2\mu_1(\mathbf{x}, B)$		$\frac{1+x_1+x_2}{x_1x_2}$	B
$\mu_1\mu_2\mu_1(\mathbf{x}, B)$	$\frac{1+x_1}{x_2}$		$-B$
$\mu_2\mu_1\mu_2\mu_1(\mathbf{x}, B)$		x_1	B
$\mu_1\mu_2\mu_1\mu_2\mu_1(\mathbf{x}, B)$	x_2		$-B$

We reach the seed $((x_2, x_1), -B)$ *after a sequence of five mutations. Notice that this seed* $((x_2, x_1), -B)$ *can be obtained from the initial seed* $((x_1, x_2), B)$ *by swapping the labels* 1 *and* 2 *i.e. the entries of* B, $b_{12} \leftrightarrow b_{21}$ *and* $x_1 \leftrightarrow x_2$. *Following on from* $((x_2, x_1), -B)$, *the sequence of mutations generates cluster variables which are* *already found, and the seed will eventually be returning to the initial seed. Hence associated cluster algebra is given by*

$$\mathcal{A}(\mathbf{x}) = \mathbb{C}\left[x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1x_2}\right]$$

Notice from the above example that all the cluster variables are given by Laurent polynomials in initial variables x_i for $i = 1, \ldots n$. This is one of the most significant features of the cluster algebras such that cluster variables, obtained from the sequence of mutations, are expressed as Laurent polynomials in initial cluster variables. This is known as the *Laurent phenomenon*, stated as follows.

Theorem 2.1.9 (Laurent phenomenon (see [1]))**.** *Every cluster variable generated by the cluster mutations is in the Laurent polynomial ring in its initial cluster variables.*

2.1.2 Cluster algebra with coefficients

It is possible to generalize the notion of a cluster algebra by introducing coefficients. In order to define this, let us consider the general class of exchange matrices as follows.

Definition 2.1.10. *An $n \times n$ integer matrix B is called a skew-symmetrizable exchange matrix if there exists an integer diagonal matrix D, which satisfies $(DB)^T = -DB$. We refer to such a matrix D as a skew-symmetrizer.*

Note that D is the identity matrix if B is a skew-symmetric matrix. The notion of cluster algebras can be extended by introducing *frozen variables* in clusters, that are extra variables which do not mutate. We call a cluster $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ an *extended cluster*, formed by mutable cluster variables $x_1, \ldots, x_n$ and the frozen variables $x_{n+1}, \ldots, x_{n+m}$. If the $(n+m) \times n$ matrix $\tilde{B}$ has upper $n \times n$ submatrix a skew-symmetrizable matrix, then we call $\tilde{B}$ an *extended exchange matrix.* Thus we obtain a cluster algebra with initial seed $(\tilde{\mathbf{x}}, \tilde{B})$ generated by cluster variables induced by the sequence of mutations

$\mu_k(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}', \tilde{B}')$ where $\tilde{\mathbf{x}}' = (x_1, \ldots, x'_k, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ and $\tilde{B}'$ are given by the cluster mutation,

$$x'_k x_k = \alpha_k \prod_{\substack{j=1 \\ b_{jk}>0}}^{n} x_j^{b_{jk}} + \beta_k \prod_{\substack{j=1 \\ b_{jk}<0}}^{n} x_j^{-b_{jk}} \tag{2.3}$$

where the coefficients are

$$\alpha_k = \prod_{\substack{j=n+1 \\ b_{j,k}>0}}^{n+m} x_j^{b_{j,k}}, \quad \beta_k = \prod_{\substack{j=n+1 \\ b_{j,k}<0}}^{n+m} x_j^{-b_{j,k}} \tag{2.4}$$

and the matrix mutation is defined by (2.1). Note that the formulas for $\tilde{\mathbf{x}}'$ and $\tilde{B}'$ are the same as previously given except that the range of indices is now from 1 to $n + m$. We refer to these more general cluster algebras as *cluster algebras of geometric type.*

There is an another form of expression for cluster algebra of geometric type. To define it, we need the notion of *tropical semifield.*

Definition 2.1.11. *A semifield is an abelian multiplicative group with the operation $\oplus$ which satisfies the property of commutative, associative and distributive under multiplication i.e. $a(b \oplus c) = ab \oplus ac$*

Definition 2.1.12. *Let P be semifield. Then P is tropical if the operation $\oplus$ of any Laurent polynomials yields following*

$$\prod_i x_i^{a_i} \oplus \prod_j x_j^{b_j} = \prod_k x_k^{\min(a_i, b_i)} \tag{2.5}$$

for $x_i \in P$.

Let P be a tropical semifield defined by generators $x_{n+1}, \ldots, x_{n+m}$ and let $\mathbb{Z}P$ be group ring of P over $\mathbb{Z}$. By setting the $\mathcal{F}$ to be a field of rational functions $\mathbb{Z}P(x_1, \ldots, x_n)$. One can define alternative version of the cluster algebra of geometric type.

Definition 2.1.13 (Coefficient variables)**.** *Let P be tropical semifield and let $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$ be cluster algebra of geometric type with initial seed consisting $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ and $(n+m) \times n$ exchange matrix $\tilde{B}$. The variables*

$$y_i = \prod_{j=n+1}^{n+m} x_j^{b_{ji}} \in P \tag{2.6}$$

are called coefficient variables such that

$$\frac{y_k}{1 \oplus y_k} = \prod_{\substack{j=n+1 \\ b_{j,k}>0}}^{n+m} x_j^{b_{j,k}}, \quad \frac{1}{1 \oplus y_k} = \prod_{\substack{j=n+1 \\ b_{j,k}<0}}^{n+m} x^{-b_{j,k}} \tag{2.7}$$

It turns out that the coefficient variables y_j evolve along the orbits of mutations. We call these as *Y-patterns* (Y-system) stated below.

Definition 2.1.14. *Let* $(\mathbf{y}, B)$ *be the initial seed formed by coefficient variables* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ *and* $(n+m)\times n$ *extended exchange matrix* B*. Then applying mutation* μ_k *on the initial seed produces a new seed* $(\mathbf{y}', B')$ *consisting of new exchange matrix* B' *given by* (2.1) *and*

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j(1 \oplus y_k)^{b_{jk}} & \text{if } j \neq k \text{ and } b_{jk} > 0 \\ y_j y_k^{-b_{jk}}(1 \oplus y_k)^{b_{jk}} & \text{if } j \neq k \text{ and } b_{jk} < 0 \end{cases} \tag{2.8}$$

2.1.3 Cluster algebra of finite type

In Example 2.1.8, we demonstrated that a specific sequence of mutations returned to the initial seed and thus had a finite number of cluster variables. Unlike most cluster algebras, which are formed by infinitely many cluster variables, this is a unique feature that certain types of cluster algebras possess. Such cluster algebras are referred to as *cluster algebras of finite type.*

Definition 2.1.15 (Cluster algebra of finite type)**.** *A cluster algebra is said to be of finite type if it has finitely many seeds.*

Recall that new cluster variables depend on the entries of the exchange matrix. It may be good to think that the characteristics of cluster algebras are determined by the structure of the exchange matrix (or quiver Q). Therefore a cluster algebra $\mathcal{A}(\mathbf{x}_0, B)$ being finite type entirely depends on the form of its exchange matrix B (or quiver Q). One of Fomin and Zelevensky's important results is the characterisation of the exchange matrices, or quivers, corresponding to the cluster algebra of finite type (see Theorem 1.5 in [5]). It turns out that the above case is closely related to the *Cartan matrices* corresponding to the graphs called *Dynkin diagrams* (shown in Figure 2.1) which classifies the finite dimensional semi-simple Lie algebras (Cartan-Killing type).

Definition 2.1.16 (Cartan matrix)**.** *An integer square symmetric matrix $A = (a_{ij})$ is a Cartan matrix if and only if it satisfies the following properties:*

- *all diagonal entries a_{ii} in the matrix are 2*
- *off-diagonal entries $a_{ij} < 0$ for any $i \neq j$*
- *there exists an diagonal matrix $D = diag(d_1, \ldots, d_n)$ with $d_i > 0$, satisfying $(DA)^T = DA$ (skew-symmetrizable)*

Furthermore, if the third condition includes the additional property, $a_{ij}a_{ji} \leq 3$ for $i \neq j$ (Positive definite), then the Cartan matrix is of finite type.

The Cartan matrices are closely related to exchange matrices as each exchange matrix can be encoded to the Cartan matrix, shown below.

Definition 2.1.17. *The Cartan counterpart A of the exchange matrix B is given by*

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -|b_{ij}| & \text{if } i \neq j \end{cases} \tag{2.9}$$

One can see the above relation from the Dynkin diagram in Figure 2.1. For instance, for type A_n, the Cartan matrix can be represented by the connected graph such that edges between i and j in the graph correspond to components $a_{ij} \neq 0$ of the matrix.

Note that there is an equivalence relation between two distinct exchange matrices which are related by sequence of mutations, stated as follows.

Definition 2.1.18 (Mutation equivalent)**.** *The exchange matrices B and B' are mutation equivalent if there exists a sequence of mutations that transforms B to B' and vice versa.*

In addition to the result above, it was shown in [5] that there exists a isomorphism between the cluster algebras whose exchange matrices are mutation equivalent, as stated below.

Theorem 2.1.19 (Theorem 1.7 in [5])**.** *Let B and $\hat{B}$ be exchange matrices such that their associated Cartan counterparts $A(B)$ and $A(\hat{B})$ are Cartan matrices of finite type Dynkin diagrams. Then corresponding cluster algebras, $\mathcal{A}(\mathbf{x}, B)$ and $\mathcal{A}(\mathbf{x}, \hat{B})$ are isomorphic if and only if $A(B)$ and $A(\hat{B})$ are equal up to simultaneous permutation of rows and columns.*

Therefore, based on the fact that there exists a cluster algebra for each Dynkin diagram, Fomin and Zelevensky in [5] found the following result.

Theorem 2.1.20 (Theorem 1.8 in [5])**.** *A cluster algebra is of finite type if and only if it is equipped with an exchange matrix which is mutation equivalent to a matrix whose Cartan counterpart corresponds to the Dynkin diagram.*

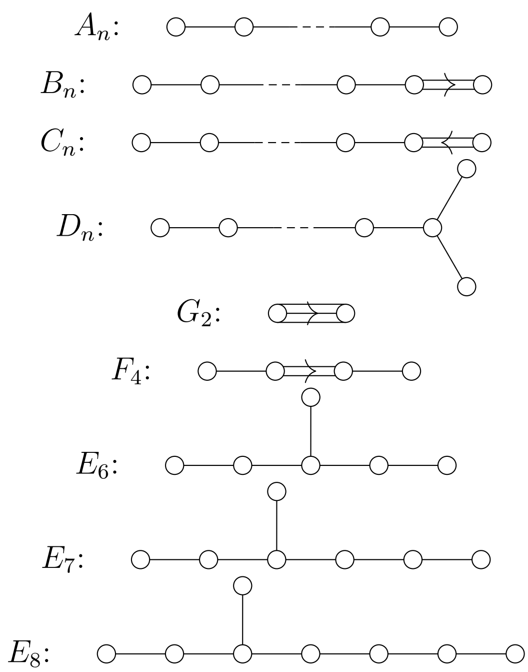


Figure 2.1: Dynkin diagrams

2.1.4 Root systems and cluster algebras

In this section, we will briefly cover the notions of root systems and show how they are related to cluster algebras. We mainly follow the reference [18] written by Marsh.

Definition 2.1.21. *Let V be a finite dimensional vector space, equipped with the bilinear map (inner product) $(\cdot,\cdot) : V \times V \to \mathbb{R}$. Then V is Euclidean space if the map $(\cdot,\cdot)$ satisfies the following properties:*

- *symmetric i.e.* $(\alpha, \beta) = (\beta, \alpha)$
- *positive definite i.e.* $(\alpha, \alpha) > 0$

Let us consider the hyperplane H, that is, an orthogonal complement to one-dimensional subspace of V, $H_\alpha = \{v \in V | (\alpha, v) = 0 \text{ for } \alpha \in V,\ \alpha \neq 0\}$. Here we concentrate on a reflection in the hyperplanes.

Definition 2.1.22 (Reflection)**.** *Let V be a finite dimensional vector space with inner product $(\cdot,\cdot)$. A reflection on V is a linear map $r : V \to V$ such that*

$$r_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)\alpha}{(\alpha, \alpha)}$$

which illustrates the reflection on the hyperplane H_α with α being a normal vector i.e.

- *If $\beta \in H_\alpha$, $r_\alpha(\beta) = \beta$, β is orthogonal to α*
- *If $\beta = \alpha$ then $r_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$, which is on the opposite side of the α*

The group generated by the reflections in V is known as a reflection group, W.

Definition 2.1.23 (Root system)**.** *Let V be a Euclidean space which is equipped with the inner product $(\cdot,\cdot)$. Then a root system $\Phi \subset V$ is a set of vectors (or roots) which satisfy the following conditions:*

- *Φ spans V ;*

- *For $\alpha \in \Phi$, $s \in \mathbb{R}$, $s \cdot \alpha \in \Phi$ if and only if $r = \pm 1$;*

- *For $\alpha, \beta \in \Phi$, $r_\alpha(\beta) \in \Phi$.*

A root system Φ admits a standard basis $\Pi = \{\alpha_i \ : \ i \in I\}$, called a *simple root system*, which spans Φ i.e. $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$. Given the choice of simple root system Π, we call the subset of root system $\Phi_+ \subset \Phi$ a *positive root system* if it contains only positive roots e.g. $+\alpha_i, \alpha_1 + \alpha_2$. Similarly, we refer the subset $\Phi_- \subset \Phi$ a *negative root system* if it consists of negative roots e.g. $-\alpha_i, -\alpha_1 - \alpha_2$.

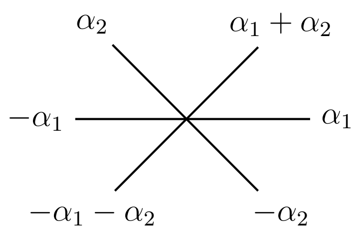
Example 2.1.24 (Type A_2). *Let us consider $\mathbb{R}^3$ equipped with standard basis e_1, e_2, e_3 and inner product satisfying the relation $(e_i, e_j) = \delta_{ij}$ for $i, j \in \mathbb{Z}_4$. Then one can show that the following set*

$$\begin{aligned} \Phi &= \{e_2 - e_1, e_3 - e_2, e_3 - e_1, e_1 - e_2, e_2 - e_3\} \\ &= \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\} \end{aligned} \tag{2.10}$$

together with the simple root system $\Pi = \{\alpha_1, \alpha_2\}$ is a root system. It is clear that the first and second conditions hold; one can show that it also holds for the third condition by direct calculation of the reflection roots,

$$r_{\alpha_1}(\alpha_2) = \alpha_2 - \frac{2(e_2 - e_1, e_3 - e_2)\alpha_1}{(e_2 - e_1, e_2 - e_1)} = \alpha_2 + \alpha_1$$

The reflections: $r_{\alpha_2}(\pm\alpha_1) = \pm\alpha_1 \pm \alpha_2$, $r_{\alpha_2}(\pm\alpha_1) = \pm\alpha_1 \pm \alpha_2$, $r_{\alpha_1}(\alpha_1 + \alpha_2) = \alpha_2$, $r_{\alpha_2}(\alpha_1 + \alpha_2) = \alpha_1$, $r_{\alpha_1}(\pm\alpha_1) = \mp\alpha_1$. Therefore the root system is closed under the reflection. The root system can be depicted as



The corresponding positive root system $\Phi_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ whereas the negative root system $\Phi_- = \{-\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$.

The set of reflections forms a group which is associated with the root system.

Definition 2.1.25 (Weyl group)**.** *Let Φ be a root system. Then the finite group generated by the reflections,*

$$W = \{r_\alpha | \alpha \in \Phi\}$$

is known as the Weyl group (or reflection group) associated with the root system Φ.

Definition 2.1.26 (Coxeter group)**.** *Let W be a Weyl group generated by the reflections $r_1, r_2, \ldots, r_n$. The group W is said to be a Coxeter group if it has a presentation,*

$$W = \left\{r_1, \ldots, r_n | r_i^2 = e,\ (r_i r_{i+1})^2 = e,\ (r_i r_j)^3 = e,\ i \neq j\right\}$$

Example 2.1.27. *The Weyl group of type A_l has presentation,*

$$W(A_l) = \left\{r_1, \ldots, r_n | r_i^2 = e,\ (r_i r_{i+1})^2 = e,\ (r_i r_j)^3 = e,\ i \neq j\right\}$$

Let Φ be the root system associated with a finite type Dynkin diagram and let $\mathcal{A}(\mathbf{x}, B)$ be a cluster algebra of rank n with initial cluster $\mathbf{x}$ and exchange matrix B whose Cartan counterpart is the Cartan matrix of the Dynkin diagram. The root system and cluster algebra are deeply connected as shown in the Theorem below.

Theorem 2.1.28 (Theorem 1.9 in [5])**.** *Let us consider the cluster algebra of finite type $\mathcal{A}(\mathbf{x}, B)$ which consists of cluster variables,*

$$x[\alpha] = \frac{N[\mathbf{x}]}{x^{\mathbf{d}}}$$

Let $\Phi_{\geq -1}$ be an almost positive root system which consists of the roots of the positive root system Φ_+ and negative of a simple system Π i.e. $\Phi_{\geq -1} = \Phi_+ \cup \{-\Pi\}$. Then there is a bijection between almost positive roots $\alpha \in \Phi_{\geq -1}$ and the denominator of cluster variables as follows.

$$x^{\mathbf{d}} = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \longleftrightarrow \alpha = \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n$$

Example 2.1.29 (Type A_2)**.** *Example 2.1.8 showed that the cluster algebra of type A_2 is generated by the following variables.*

$$X = \left\{x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1 x_2}\right\}$$

Furthermore in Example 2.1.24, the positive definite root system $\Phi_{\geq -1}$ yields,

$$\Phi_{\geq -1} = \{-\alpha_1, -\alpha_2, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$

We begin with the associating initial cluster variables with the negative simple roots,

$$x[-\alpha_1] = x_1, \quad x[-\alpha_2] = x_2 \tag{2.11}$$

Then it is clear that there is correspondence between cluster variables and roots as follows:

$$x[\alpha_1] = \frac{1+x_2}{x_1}, \quad x[\alpha_2] = \frac{1+x_1}{x_2}, \quad x[\alpha_1+\alpha_2] = \frac{1+x_1+x_2}{x_1x_2}$$

Therefore there is a bijection $X \longleftrightarrow \Phi_{\geq -1}$

2.1.5 Relation with Zamolodchikov periodicity

In Example 2.1.8, we observed that the specific sequence of mutations takes us back to the initial seed: such composition of mutations is said to be *periodic*. By incorporating the relation between roots and the denominator vector, Fomin and Zelevensky in [19] proved that the periodicity in cluster algebra of finite type can be equivalent to Zamoldchikov periodicity [4]. In other words, there exists a certain sequence of mutations which is periodic with period $h+2$ where h is the Coxeter number. Here we follow the work of [19].

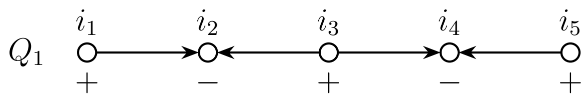
We begin with considering a particular quiver Q whose structure is equivalent to the bipartite graph. Let us define $\epsilon : [1, n] \to \{-1, +1\}$ to partition the set of its vertices I into two subsets,

$$I_+ = \{i \in I : \epsilon(i) = +1\}, \quad I_- = \{i \in I : \epsilon(i) = -1\}$$

where each vertex in I_+ is a source and each vertex in I_- is a sink. Then vertices within in I_+ are not connected by the edges, and this also applies to the I_-. Then the corresponding exchange matrix admits entries b_{ij},

$$b_{ij} > 0 \implies \begin{cases} \epsilon(i) = +1 \\ \epsilon(j) = -1 \end{cases}$$

We refer to corresponding matrix as a *bipartite exchange matrix*. For example



where $i_1, i_3, i_5 \in I_+$ and $i_2, i_4 \in I_-$

Now let us consider the mutation on the bipartite seed. As mentioned above, each vertex in I_+ is incident to only with vertices in I_- and vice versa. Then when we apply mutations, for instance, only on the vertices of I_+, each will reverse the arrows of edges incident to each vertex in I_+, which results in producing the quiver whose edges are reverse of the ones in the original quiver. Thus quiver returns to Q after applying mutations on the other vertices in I_-. Hence there exists a unique sequence of mutations which preserves the structure of the quiver. We can see this from the quiver Q_1 above that the composition mutations $\mu_{i_3}\mu_{i_1}(Q) = -Q$ (arrows reversed), and then applying mutations $\mu_{i_4}\mu_{i_2}$ take us back to the original quiver $\mu_{i_4}\mu_{i_2}(-Q) = Q$. Such composed mutations give discrete dynamics on the bipartite quiver, and it is known as the *bipartite belt.*

Definition 2.1.30 (Bipartite belt)**.** *The composed mutations on the initial seed* $(\mathbf{x}_0, B_0)$ *induce the following iterations,*

$$\cdots \to (\mathbf{x}_0, B_0) \to (\mathbf{x}_0, B_1 = -B_0) \to \cdots \tag{2.12}$$

$$(\mathbf{x}_m, (-1)^m B) = \underbrace{\mu_-\mu_+ \cdots \mu_+\mu_-}_{r factors}(\mathbf{x}_0, B_0) \tag{2.13}$$

In this context, let μ_+ and μ_- be composition of mutations on the vertices I_+ and I_- respectively.

$$\mu_+ = \prod_{\epsilon(i)=+1} \mu_i, \quad \mu_- = \prod_{\epsilon(i)=-1} \mu_i \tag{2.14}$$

Note that the composed mutations μ_- and μ_+ are well defined as there is no edge connected between the nodes in the same set, and thus mutations in the product τ_+ and τ_- commute. Given the $x_{i,m+1} = x_{i,m}$ with $\epsilon(i) = (-1)^m$, the cluster variables under sequence of mutations $\mu_-\mu_+$ can be formulated as

$$x_{j,m+1}x_{j,m-1} = \prod_{\epsilon(i)=-\epsilon(j)} x_{i,m}^{-a_{ij}} + 1 \tag{2.15}$$

with $\epsilon(j) = (-1)^{m-1}$. Such difference equation is known as *T-system*.

Recall Theorem 2.1.28, that for cluster algebra of finite type, the denominators of cluster variables are bijective to the roots $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ in the associated root system $\Phi_{\geq -1}$. This implies there exists a sequence of reflections r_α which is analogous to the sequence of mutations $\mu_+\mu_-$.

Let Φ be an irreducible root system such that the associated Cartan matrix is irreducible (which implies that Cartan matrix can be encoded into Dynkin diagram) and let W be the associated Coxeter group. The indexing set I of a connected Φ can be separated into two disjoint sets I_+ and I_- so that associated sequence of reflections can be separated into

$$\tau_+ = \prod_{\epsilon(i)=+1} r_i, \quad \tau_- = \prod_{\epsilon(i)=-1} r_i \tag{2.16}$$

where r_i are simple reflections and $I_\pm = \{i \in I : \epsilon(i) = \pm 1\}$ i.e. $I = I_+ \cup I_-$. Then the actions of $\tau_\pm$ can be expressed as

$$\tau_\pm(\alpha_j) = \begin{cases} -\alpha_j & \text{if } j \in I_\pm \\ \alpha_j - \sum_{j \neq i} a_{ij}\alpha_i & \text{if } j \in I_\mp \end{cases} \tag{2.17}$$

where α_j are simple roots in Φ and $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. The product of the two compositions of reflections above, $\tau = \tau_-\tau_+$ forms an element in associated Coxeter group, known as the *Coxeter element* which has order h, referred to as the *Coxeter number*.

Let us consider the almost positive root system $\Phi_{\geq -1} = \Phi_+ \cup \{-\Pi\}$, which consists of positive roots Φ_+ and negative simple roots $\{-\Pi\}$. We can define modified version of $\tau_\pm$, that is, $\sigma : \Phi_{\geq -1} \to \Phi_{\geq -1}$ expressed by

$$\sigma_\epsilon(\alpha_i) = \begin{cases} \alpha_i & \text{if } \alpha_i = -\alpha_j \text{ for } j \notin I_\epsilon \\ \tau_\pm(\alpha_i) & \text{otherwise} \end{cases} \tag{2.18}$$

Following on from above, we introduce w_0 expressed as

$$w_0 = \sigma_-\sigma_+ \ldots \sigma_\pm\sigma_\mp$$

. The number of factors of w_0 is the Coxeter number h if w_0 satisfies $w_0(\alpha_i) = -\alpha_k$ with $k \in I$ (see [2] for the details). By using this fact, Fomin and Zelevinsky showed the following result.

Theorem 2.1.31 (Theorem 2.16 in [2]). *The order of $\sigma_-\sigma_+$ is equal to $\frac{h+2}{2}$ if $w_0 = -1$, and equal to $h+2$ otherwise.*

This result can be illustrated by continuing the previous Example 2.1.29.

Example 2.1.32 (Type A_2). *The cartan matrix for type A_2 is given by*

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.19}$$

We set the simple root system for type A_2 by $\Pi = \{\alpha_1, \alpha_2\}$ with indices: $\epsilon(1) = +1$ and $\epsilon(2) = -1$. The associated root system $\Phi_{\geq -1}$ is given by the sequence of reflections as follows.

$$-\alpha_1 \xrightarrow{\sigma_+} \alpha_1 \xrightarrow{\sigma_-} \alpha_1 + \alpha_2 \xrightarrow{\sigma_+} \alpha_2 \xrightarrow{\sigma_-} -\alpha_2$$

$$-\alpha_2 \xrightarrow{\sigma_-} \alpha_2 \xrightarrow{\sigma_+} \alpha_1 + \alpha_2 \xrightarrow{\sigma_-} \alpha_1 \xrightarrow{\sigma_+} -\alpha_1$$

It is clear that

$$w_0(-\alpha_1) = \sigma_+\sigma_-\sigma_+(-\alpha_1) = \alpha_2$$
$$w_0(-\alpha_2) = \sigma_-\sigma_+\sigma_-(-\alpha_2) = \alpha_1$$

Hence iteration returns to simple roots after applying

$$(\sigma_-\sigma_+)^5(-\alpha_i) = -\alpha_i \quad \textit{for } i \in \{1, 2\}$$

which matches with the result in Theorem 2.1.31.

Combining Theorem 2.1.28 and exchange relation (2.15) with Theorem 2.1.31 leads to a result identical to Zamolodchikov periodicity conjecture, which confirms the statement mentioned at the beginning of this section.

Theorem 2.1.33 ([19]). *Let $\mathcal{A}(\mathbf{x}, B)$ be a cluster algebra of finite type together with exchange matrix B whose Cartan counterpart is the Cartan matrix of Dynkin* *diagram. The cluster mutations* $\mu = \mu_-\mu_+$ (2.15) *satisfy the two cases of periodicity such as*

- *period* $\frac{h+2}{2}$
$$\mu^{\frac{h+2}{2}}(\mathbf{x}, Q) = (\mathbf{x}, Q) \tag{2.20}$$
if associated roots α_i *satisfies* $w_0(\alpha_i) = -\alpha_i$

- *period* $h + 2$*,*
$$\mu^{h+2}(\mathbf{x}, Q) = (\mathbf{x}, Q) \tag{2.21}$$
otherwise.

Remark 2.1.34. *The periodicity conjecture holds for Y-seed pattern. In [19], they defined a new piecewise linear function which is analogous to the function* (2.18) *and showed that it is closely related to the Y-seed pattern. This results in showing that coefficient variables* y_i *satisfy the periodicity conjecture. Further detail can be found in [3, 2].*

Remark 2.1.35. *In the literature [5], it was shown that any quivers, which have the orientation of Dynkin diagram, also satisfy the results.*

Remark 2.1.36. *In [8], Nakanishi and the other authors considered the general version of T- and Y-systems and proved the periodicities via several approaches.*

2.1.6 Periodicity in Quiver and Cluster maps

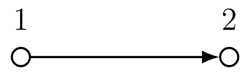
In the previous section, we observed that there exists periodicity in dynamics given by mutations $\mu_-\mu_+$, preserving the structure of the quiver. Such periodicity is essential to define a specific biratinal map between clusters $\mathbf{x}$ and $\mathbf{x}'$, known as a *cluster map*. In this section, we explore the the notion of periodicity in quiver/matrix mutations introduced by Fordy and Marsh [6], and Nakanishi [8].

Definition 2.1.37 (Mutation periodic)**.** *Let* Q *be a quiver with* n *vertices. Then the quiver is mutation periodic with period* m *if there exists a sequence of quiver* *mutations which is equivalent to cyclic permutation of the labelling of the quiver Q i.e.*

$$\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1}(Q) = \rho^m(Q)$$

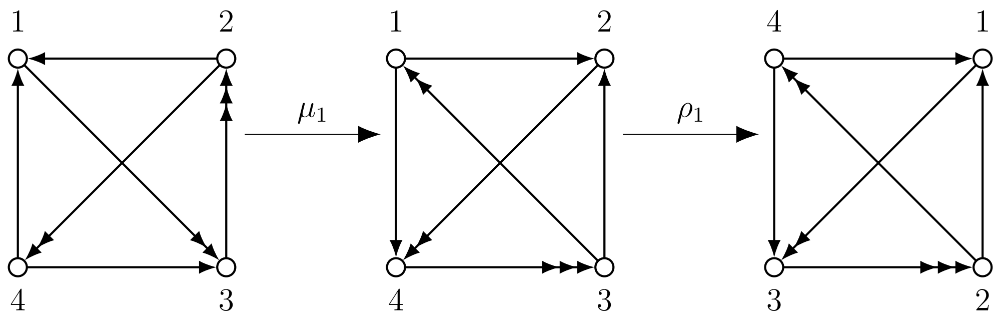
for $N \geq m$, where $\rho : (1, 2, \ldots, N) \to (N, 1, 2, \ldots, N-1)$ is the cyclic permutation.

Example 2.1.38 (Type A_2)**.** *The quiver associated with type A_2 is drawn as*



The mutation μ_1 on the quiver is equivalent to a reversing arrow. Permutating the labels will return the quiver to its original position.

Example 2.1.39 (Somos-4 quiver)**.** *Let us consider the quiver which generates Somos-4 sequence (see formula* (4.22) *in the section 4.1.2 and see [20] for the further detail) . If we apply the mutation μ_1 on the quiver, followed by permutation $\rho_1 = (1, 2, 3, 4)$ and rotating, we recover the initial quiver.*

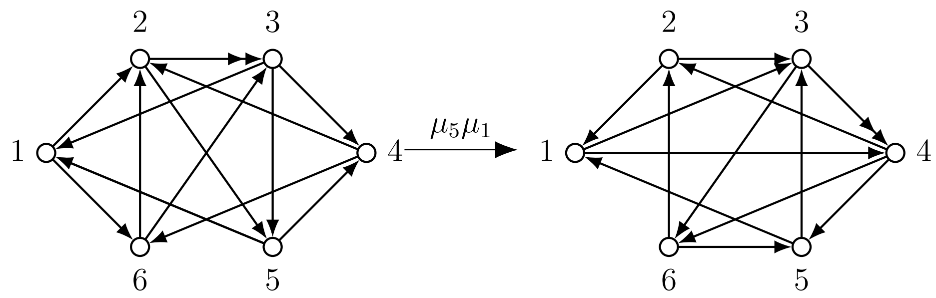


In a more general setting, the periodicity of the exchange matrix was defined by Nakanishi [8] as follows.

Definition 2.1.40. *Let Q be a quiver with n vertices and let $\hat{\rho}$ be permutation of the indices $I = \{1, 2, \ldots, n\}$ of Q. Then the quiver is $\hat{\rho}$-period if there exists sequence of mutations satisfying the relation below*

$$\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1}(Q) = \hat{\rho}(Q)$$

Example 2.1.41. *The sequence of mutations transforms the following quiver,*



If we permute the labellings of the right-hand side quiver by $\rho = (1, 2, 3, 4, 5, 6)$, we retrieve the original quiver.

In the examples above, we have shown that a particular type of quivers satisfies $\mu_{i_r} \cdots \mu_{i_1}(Q) = \rho(Q)$ for some integers $r > 0$ and some permutation ρ. Suppose we define the map $\varphi = \rho^{-1}\mu_{i_r} \cdots \mu_{i_1}$; then the map preserves the structure of the quiver. This induces a birational map that we refer to as a *cluster map.*

Definition 2.1.42 (Cluster map)**.** *Let $(\mathbf{x}, Q)$ be an initial seed with an initial cluster $\mathbf{x} = (x_1, \ldots, x_n)$ and mutation periodic quiver Q i.e. $\mu_{i_m} \cdots \mu_{i_1}(Q) = \rho(Q)$ for some $m > 0$ and some permutation ρ. Then*

$$\varphi = \rho^{-1}\mu_{i_m}\mu_{i_{m-1}} \cdots \mu_{i_2}\mu_{i_1}$$

is a birational map such that

$$\begin{aligned} \varphi \ : \ \mathbb{C}^n \ &\rightarrow \ \mathbb{C}^n \\ \mathbf{x} = (x_1, \ldots, x_n) \ &\mapsto \ \mathbf{x}' = (x_1', \ldots, x_n') \end{aligned} \tag{2.22}$$

where $\mathbf{x}' = (x_1', \ldots, x_n')$ is new cluster given by $\varphi(\mathbf{x})$. Such a map is called a cluster map.

Example 2.1.43. *The cluster map corresponding to quiver in the Example 2.1.38 is given by*

$$\varphi : (x_1, x_2) \rightarrow (x_2, \frac{1 + x_2}{x_1}) \tag{2.23}$$

Remark 2.1.44. *Notice that a single mutation is not identified as a single birational map. This is because the exponent of each cluster variable in the exchange relation changes along the mutations. Furthermore, each mutation cannot be specified simply by the integers labelling a sequence of mutations; it is specified by discrete steps in an n-valent tree. However periodicity enables the sequence of mutations, that return us to the initial exchange matrix, to be identified as the birational mapping.*

2.2 Integrable cluster maps

A discrete dynamical system can be described by the points induced by the finite iteration of mapping (discrete mapping i.e. $x_{n+1} = \varphi(x_n)$). For example, the orbits given by iteration of the cluster map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ can be considered as a discrete dynamical system. Therefore the notion of integrability [9] in discrete systems can be applied to the cluster map by introducing the suitable Poisson bracket that is compatible with the cluster algebra structure.

In this section, we consider the notion of integrability for cluster algebras, introduced in [20, 7]. By mainly following the references [21, 22], we begin with some basic facts regarding the Poisson bracket defined on the manifold and see the rank of the Poisson bracket (or Poisson tensor) forms a connection between the Poisson and the symplectic manifold. This brings us to the notion of Liouville integrable system.

2.2.1 Poisson bracket, symplectic form and symplectic leaves

Definition 2.2.1 (Poisson bracket, Poisson manifold)**.** *Let M be a smooth real manifold and let $f, g, h \in C^\infty(M)$ be smooth functions defined on M. A Poisson bracket is skew-symmetric bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies the properties,*

1. *Leibiniz rule:* $\{fg, h\} = f\{g, h\} + \{f, h\}g$
2. *Jacobi identity:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Then the smooth manifold M with the Poisson structure $\{\cdot,\cdot\}$, satisfying the above properties is known as a Poisson manifold.

In the local coordinates $\mathbf{x} = (x_1, \ldots, x_n)$, the explicit form of the Poisson bracket between smooth functions f, g is written as

$$\{f, g\} = \sum \mathbf{P}_{ij}(\mathbf{x}) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \tag{2.24}$$

where the coefficients

$$\mathbf{P}_{ij}(\mathbf{x}) = \{x_i, x_j\} \tag{2.25}$$

are entries of the $n \times n$ skew-symmetric matrix. Note that the rank of the Poisson structure (or Poisson bracket) is determined by rank of the matrix $(\mathbf{P}_{ij})$. This implies the rank of Poisson structure is equal to the dimension of the manifold if it is full rank.

The explicit form of the Poisson bracket (2.24) and Leibiz rule suggests that we can use a vector field to express the Poisson bracket as follows

$$X_H(\cdot) = \{\cdot, H\} = \mathbf{P}\mathrm{d}H \tag{2.26}$$

Such a vector field is called a *Hamiltonian vector field.* The Poisson bracket (2.24) can be represented as a tensor product of the form

$$\{f, g\} = \sum_{i,j} \left(\mathbf{P}_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \right) (\mathrm{d}f, \mathrm{d}g) \tag{2.27}$$

as $\frac{\partial}{\partial x_i}$ are dual to $\mathrm{d}x_i$. Therefore the Poisson bracket can be represented as

$$\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g) \tag{2.28}$$

where $\pi : T_x^*M \times T_x^*M \to \mathbb{R}$ is called a *Poisson tensor.* Equivalently one can define a linear map $\hat{\pi} : T_x^*M \to T_xM$ such that $\hat{\pi} = \pi(\mathrm{d}f, \cdot)$. Then any Hamiltonian vector field can be written as $X_f = \mathbf{P}\mathrm{d}f = \hat{\pi}(\mathrm{d}f)$. So the map $\hat{\pi}$ can be treated as a Poisson structure.

Next, let us consider the following notion of a map between two Poisson manifolds.

Definition 2.2.2 (Poisson map)**.** *Let* $(M, \{\cdot,\cdot\}_M)$ *and* $(N, \{\cdot,\cdot\}_N)$ *be Poisson manifolds and let* $\mathbf{x} = (x_1, x_2, \dots, x_{2n})$ *be local coordinates of* M*. Then a map* $\varphi : N \to M$ *is a Poisson map if it satisfies*

$$\varphi^*\{x_i, x_j\}_M = \{\varphi^* x_i, \varphi^* x_j\}_N \tag{2.29}$$

Definition 2.2.3 (Poisson submanifold)**.** *A submanifold* $N \subseteq M$ *of a Poisson manifold* M *is called a Poisson submanifold if there exists a Poisson bracket on* N *such that the inclusion map* $N \overset{i}{\hookrightarrow} M$ *is a Poisson map.*

2.2.2 Symplectic manifolds

Suppose the rank of Poisson tensor (the rank of the matrix $P_{ij}(\mathbf{x})$) at any point x on M is maximal i.e. $\text{rank}(\mathbf{P}) = \dim(M) = n$. Then it turns out that the corresponding Poisson manifold can be identified as a *symplectic manifold.*

Definition 2.2.4 (Symplectic manifold)**.** *A smooth manifold $\mathcal{S}$ is a symplectic manifold if it is equipped with bilinear form $\omega : T_xM \times T_xM \to \mathbb{R}$, expressed as*

$$\omega = \sum_{i,j} \omega_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j \tag{2.30}$$

which is a closed ($\mathrm{d}\omega = 0$) and non-degenerate 2-form, called a symplectic form. The coefficient ω_{ij} is given by

$$\omega_{ij} = \omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \tag{2.31}$$

One can define a linear map $\hat{\omega} : T_xM \to T_x^*M$ such that $\hat{\omega} = \omega(X_f, \cdot) = \mathrm{d}f$. This implies that the symplectic form satisfies the following form

$$w(X_f, X_g) = \{f, g\} \tag{2.32}$$

This relation gives rise to

$$\omega(X_f, X_g) = \omega(\mathbf{P}\mathrm{d}f, \mathbf{P}\mathrm{d}g) = \omega(\mathbf{P}(\hat{\omega}(X_f), \mathbf{P}(\hat{\omega}(X_f)) \tag{2.33}$$

Therefore this indicates that the symplectic structure is the inverse of the Poisson structure in non-degenerate case. Similar to the Poisson case, one can define the map on symplectic manifold as following

Definition 2.2.5 (symplectic map)**.** *Let (M, ω_M) and (N, ω_N) be symplectic manifolds of 2m dimension and let $\mathbf{x} = (x_1, x_2, \ldots, x_{2n})$ be local coordinates of M. Then a map $\varphi : N \to M$ is a symplectic map if it satisfies*

$$\varphi^* \omega_M = \omega_N \tag{2.34}$$

For non-degenerate cases, the symplectic and Poisson structures are closely related by (2.32). Following this, the birational map can be either Symplectic or Poisson map depending on its domain and codomain (Symplectic manifold or Poisson manifold).

On the other hand, if rank $\mathbf{P} = 2r < n = \dim(M)$, then there exist $m = n - 2r$ independent functions $\mathcal{C}_k$ such that $\{\mathcal{C}_k, f\} = 0$ for any arbitrary function f, which are called *Casimir functions*. Since the Poisson bracket can be represented as a Hamiltonian vector field (2.26) and we have $X_f(\mathcal{C}_k) = 0$, Casimir functions are constant along the flow of Hamiltonian vector fields i.e. the Hamiltonian vector fields are tangent to Poisson submanifolds

$$S_i = \{\mathbf{x} \in M \ : \ \mathcal{C}_i(\mathbf{x}) = \text{constant}\} \tag{2.35}$$

Each submanifold S_i is known as a *symplectic leaf*, endowed with the symplectic form defined by (2.32).

Definition 2.2.6 (Symplectic leaf)**.** *The symplectic leaf S_p containing the point p is a set of points q which are connected to p by the piecewise curve where each segment is a trajectory of a Hamiltonian vector field.*

Theorem 2.2.7 ([22])**.** *Let M be an n-dimensional Poisson manifold which has rank $m < n$ Poisson tensor. Then each symplectic leaf M_i of M is a Poisson submanifold if* $\text{rank}(P) = \dim(M_i)$.

2.2.3 Liouville integrable systems and integrable maps

On each symplectic leaf S_i, if we set the variables to be $x_i = \mathcal{C}_i$ for $i = \{2r, \ldots, m\}$, then the Poisson tensor $\mathbf{P}_{ij}$ becomes a $(2r+m) \times (2r+m)$ matrix, which consists of the $2r \times 2r$ non-degenerate Poisson matrix and zero rows and columns. This implies one can lower the dimension of S_i; the local coordinates on each symplectic leaves are $(x_1, \ldots, x_{2r})$ and the Poisson tensor has rank $2r$ which is equal to the dimension of the symplectic leaf. Suppose there exists r independent functions I_i which are in involution with respect to the Poisson bracket $\{I_i, I_j\} = 0$. Then

$$X_{I_i}(I_j) = 0 \tag{2.36}$$

Therefore the level set is tangent to the flow, induced by vector fields X_{I_i}. Furthermore, the vector fields commute which leads to the following definition.

Definition 2.2.8 (Liouville integrable system). *Let M be $2n$ dimensional Poisson manifold with full rank Poisson bracket. If there exist n first integrals which are constant along the Hamiltonian vector fields, satisfying $\{I_i, I_j\} = 0$, then the Hamiltonian system, produced by Hamiltonian vector fields X_{I_i}, is called a Liouville integrable system.*

With the notion of the Poisson map defined above, one can state the discrete version of the Liouville integrable system, introduced in [9].

Definition 2.2.9 (Liouville integrable map). *Let $\mathcal{A}(\mathbf{x}, B)$ be a cluster algebra of rank n where the B is an $n \times n$ exchange matrix and $\mathbf{x}$ is an n-tuple of cluster variables. Suppose $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a cluster map. Given P a Poisson tensor of rank $2r$, the map φ is integrable if there exist*

- *$n - 2r$ Casimir functions $\mathcal{C}_k$, i.e. $\varphi^*(\mathcal{C}_k) = \mathcal{C}_k$ satisfying $\{\mathcal{C}_k, f(\mathbf{x})\} = 0$ for all functions $f(\mathbf{x})$*
- *r first integrals h_j, $j = 1, \ldots, r$ $(\varphi^*(h_j) = h_j)$ such that $\{h_i, h_j\} = 0$.*

2.2.4 Poisson brackets and symplectic forms for cluster algebras

Now we reviewed Poisson and symplectic structure for general manifold, let us consider the specific case of cluster algebras of rank n where B is an $n \times n$ exchange matrix and D is an $n \times n$ diagonal matrix, skew-symmetrizer, satisfying $(DB)^T = -(DB)$. In the case of cluster algebras, we are interested in Poisson brackets taking a particularly nice form on the clusters

$$\{x_i, x_j\} = P_{ij} x_i x_j \tag{2.37}$$

where $P = (P_{ij})$ is a $n \times n$ skew-symmetric matrix, which is referred to as a *Poisson matrix.* The terminology for a bracket of this form is *log-canonical Poisson bracket*, as coined by Gekhtman, Shapiro and Vainshtein [23].

Definition 2.2.10. *For a cluster algebra $\mathcal{A}(\mathbf{x}, B)$, a Poisson bracket $\{\cdot, \cdot\}$ is said to be mutation compatible if the bracket between any clusters is restricted to log-canonical form.*

Let us assume that the exchange matrix B has a maximum rank (non-degenerate matrix $\det(B) \neq 0$). Imposing the condition of mutation compatible, one can obtain the following result (the further details can be found in [23, 24]

Theorem 2.2.11 ([24])**.** *Assume that B is skew-symmetrizable with skew-symmtrizer D and that B is of maximum rank, the Poisson matrix can be written as*

$$P = \lambda D B^{-1}, \quad \lambda \in \mathbb{Q} \tag{2.38}$$

In addition to this, the product PB is mutation invariant.

The fact that the Poisson structure P is directly proportional to the inverse of the exchange matrix B enables us to identify the cluster map as a Poisson map, stated in Definition 2.2.2.

Suppose we wish to adapt the notion of Poisson map to the case of cluster algebras. One can show that single mutation μ_k satisfies the following relations,

$$\mu_k^*\{x_i, x_j\} = P_{ij}\tilde{x}_i\tilde{x}_j$$
$$\{\mu_k^* x_i, \mu_k^* x_j\} = \{\tilde{x}_i, \tilde{x}_j\} = \tilde{P}_{ij}\tilde{x}_i\tilde{x}_j$$

The cluster mutation μ_k is Poisson if and only if the corresponding Poisson matrix is invariant under μ_k. As already noted in the section 2.1.6, this cannot happen for a single mutation μ_k. However, since the cluster map is constructed by the periodicity of the exchange matrix B ($\varphi(B) = B$), then the relation (2.38) suggests that the cluster map also preserves the Poisson matrix P, and hence it is a *Poisson map.*

The above result can be attained if $\text{rank}(B) = n$ (full rank). Conversely, for the $\text{rank}(B) < n$, it turns out the map can be reduced to a symplectic cluster map, defined by cluster algebra of lower rank. The latter will be explained in the next section.

In the same way that there is a Poisson bracket compatible with cluster algebra, an associated symplectic form also exists. This was also shown in [23] that there exists a symplectic form in cluster algebra, which is written in the log-canonical form,

$$\omega = \sum_{i<j} \frac{b_{ij}}{x_i x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j = \sum_{i<j} b_{ij} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j \tag{2.39}$$

It is mutation compatible i.e., the cluster mutation $\mu_k : ((x_1, \ldots, x_N), B) \to ((\tilde{x}_1, \ldots, \tilde{x}_N), \tilde{B})$ yields $\tilde{\omega} = \sum_{i<j} \tilde{b}_{ij} \mathrm{d}\log \tilde{x}_i \wedge \mathrm{d}\log \tilde{x}_j$. Note that if the exchange matrix is degenerate then the bilinear form (2.39) is *pre-symplectic form*; when it is non-degenerate, we have an honest symplectic form. Similar to the Poisson case, the cluster map is *symplectic map* since it preserves the associated exchange matrix that is equivalent to $\varphi^* \omega = \omega$.

2.2.5 Symplectic reduction in cluster algebras

For the non-degenerate case with $\mathrm{rank}(B) = n$, the coefficient matrix of the symplectic form ω is the inverse of the Poisson matrix as $P = B^{-1}$. In the setting above, the cluster map can be either a Poisson or a symplectic map. On the other hand, for the degenerate case $\mathrm{rank}(B) < n$, it is Poisson but not symplectic. Recall that a Poisson manifold equipped with a degenerate Poisson bracket is foliated by symplectic leaves. With the related ideas, there is a canonical way to reduce the Poisson map to a symplectic map via the symplectic form. The following result was shown in the Theorem 2.6 in [7]

Theorem 2.2.12 (Symplectic reduction)**.** *In the degenerate case* $\mathrm{rank}\, B = 2r < N$, *there exists rational map* π,

$$\begin{aligned} \pi : \mathbb{C}^N &\to \mathbb{C}^{2r} \\ \mathbf{x} = (x_1, \ldots, x_N) &\mapsto \mathbf{y} = (y_1, \ldots, y_{2r}) \end{aligned} \tag{2.40}$$

equivalent to

$$y_j = \mathbf{x}^{\mathbf{v}} = \prod_j x_j^{v_j^{(i)}}, \quad \mathbf{v}^{(\mathbf{i})} \in \mathrm{Im}\, B \tag{2.41}$$

which reduces the φ *to the symplectic map* $\hat{\varphi}$ *satisfying the following relation,*

$$\pi \cdot \varphi = \hat{\varphi} \cdot \pi$$

depicted by

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{\varphi} & \mathbb{C}^N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{C}^{2r} & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2r} \end{array}$$

In addition to this, the symplectic form $\hat{\omega}$ associated with $\hat{\varphi}$ is given by $\pi^*\hat{\omega} = \omega$, which ie expressed by following

$$\hat{\omega} = \sum_{i<j} \frac{\hat{b}_{ij}}{y_i y_j} \mathrm{d}y_i \wedge \mathrm{d}y_j \tag{2.42}$$

Example 2.2.13 (Somos-5)**.** *To see the application of Theorem 2.2.12, let us consider one of the examples that has been considered in the paper [7, 20].*

$$B = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix} \tag{2.43}$$

It is periodic $\mu_1 = \rho(B)$ for permutation $\rho = (1,2,3,4,5)$; this gives rise to the cluster map

$$\begin{aligned} \varphi : (x_1, x_2, x_3, x_4, x_5) \to & (x_2, x_3, x_4, x_5, x_6) \\ & = (x_2, x_3, x_4, x_5, \frac{x_2 x_5 + x_3 x_4}{x_1}) \end{aligned} \tag{2.44}$$

where we labeled the new variable as x_6. If we repeat the same procedure over the iteration of cluster maps, we can write non-linear recurrence

$$x_{n+5} x_n = x_{n+1} x_{n+4} + x_{n+2} x_{n+3} \tag{2.45}$$

which is known as Somos-5 recurrence. Since the matrix has rank 2, $\operatorname{Im} B$ *is spanned by the three basis vectors*

$$e_1 = (1, -2, 1, 0, 0)^T, \quad e_2 = (0, 1, -2, 1, 0)^T, \quad e_3 = (0, 0, 1, -2, 1)^T \tag{2.46}$$

We choose

$$\begin{aligned} v_1 &= e_1 - e_2 \\ &= (1, -1, -1, 1, 0)^T, \\ v_2 &= e_2 - e_3 \\ &= (0, 1, -1, -1, 1)^T \end{aligned} \tag{2.47}$$

Then image vectors v_1, v_2 *generates* π,

$$\begin{array}{rcl} \pi \ : & \mathbb{C}^5 & \to \ \mathbb{C}^2 \\ & \mathbf{x} = (x_1, x_2, x_3, x_4, x_5) & \mapsto \ \mathbf{u} = (u_1, u_2) \end{array} \tag{2.48}$$

where the new variables are given by

$$u_1 = \frac{x_1 x_4}{x_2 x_3}, \quad u_2 = \frac{x_2 x_5}{x_3 x_4} \tag{2.49}$$

This induces the following reduced cluster map,

$$\hat{\varphi} : (u_1, u_2) \to (u_1', u_2') = (u_2, \frac{u_2 + 1}{u_1 u_2}) \tag{2.50}$$

Denoting $u_2' = u_3$, *the expression from the second component of* (2.50) *is written as*

$$u_3 u_2 u_1 = u_2 + 1 \tag{2.51}$$

We repeat the process for each step; the subsequent iteration of $\hat{\varphi}$ *gives*

$$u_{n+1} u_n u_{n-1} = u_n + 1 \tag{2.52}$$

Note that since we label new variables as $x_{n+5} = \varphi(x_n)$ *for any* $n > 0$, *the substitutions in* (2.49) *can be written as*

$$u_n = \frac{x_n x_{n+3}}{x_{n+1} x_{n+2}}. \tag{2.53}$$

Then substituting the variable directly into the above expression will give us the recurrence (2.45).

On reduced space, the map $\hat{\varphi}$ *is in fact a Liouville integrable map as there exists an invariant function,*

$$I = u_1 u_2 + \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_1 u_2} \tag{2.54}$$

Remark 2.2.14. *In the example above, the level set defined by the invariant function* I *is a biquadratic curve of genus 1. This suggests that the reduced cluster map* (2.50) *is a special case of map, known as QRT map introduced in [25] (see the Remark 4.1.2 in the section 4.1.1 for brief background of QRT map).*

2.3 Integrability detection

Here we will introduce several algebraic methods which were developed for identifying integrable systems among the discrete dynamical systems. In each section, we will review each method, which is a crucial factor for the results in the later chapters.

2.3.1 Singularity analysis of difference equations

In this section, we introduce a heuristic approach in a discrete system, called the *singularity confinement test*, which was introduced as an algebraic tool detecting integrability. We follow the reference [26] and consider particular examples to demonstrate the process of the test.

The *singularity confinement test* was proposed by Grammaticos, Ramani and Papageorgion in [13] in order to assess the integrability of a discrete dynamical system. The motivation for such criterion came from the local singularity analysis of the solutions of ODEs, called the *Painlevé test*, which was used to detect the ordinary differential equations with the *Painelevé property* (solutions of the ordinary differential equation possessing removable singularities).

In [27], Ablowitz, Ramani and Segur made a conjecture that there exists a connection between non-linear integrable PDEs and non-linear ODEs with the Painlevé property such that any ODE which emerged from the reduction of an integrable PDE is of Painlevé type. Then, passing the Painlevé test gives a necessary condition for the integrability of the system. Thus Grammaticos, Ramani and Papageorgion in [13] adapt these notions for discrete systems to identify the discrete analogue of Painlevé equations by performing the *singularity confinement test*. Let us consider the following examples to see the procedure.

Example 2.3.1 (Lyness recurrence)**.** *The autonomous difference equation*

$$x_{n+2}x_n = ax_{n+1} + b$$

is known as the Lyness recurrence, where a and b are non-zero parameters. It is *clear that one of the potential singularities of the equation is $x_n = 0$. By setting suitable initial data, the iteration of the recurrence reaches the singularity. Let us assume that the step n_0 iteration gives $x_{n_0} = 0$ and $x_{n_0+1} = u$ (finite regular value). Then the further iterations give*

$$\begin{aligned} x_{n_0+2} &= \frac{ax_{n_0+1} + b}{x_{n_0}} = \infty \\ x_{n_0+3} &= \infty + \frac{b}{u} = \infty \\ x_{n_0+4} &= \frac{\infty}{\infty} + \frac{b}{\infty} \end{aligned}$$

As one can see $\frac{\infty}{\infty}$ is an <u>*ambiguity term*</u> *(a true singularity, with loss of information). To resolve it, we study the neighbourhood of the singularity by introducing the small quantity ϵ. Thus let $x_{n_0} = \epsilon$; then*

$$\begin{aligned} x_{n_0} &= \epsilon \\ x_{n_0+1} &= u \\ x_{n_0+2} &= (au + b)\epsilon^{-1} \\ x_{n_0+3} &= \frac{a(au+b)}{u}\epsilon^{-1} + \frac{b}{u} \\ x_{n_0+4} &= \frac{a^2}{u} + O(\epsilon) \\ x_{n_0+5} &= \left(\frac{a^3 + bu}{a(au+b)}\right)\epsilon + O(\epsilon^2) \\ x_{n_0+6} &= \frac{bu}{a^2} + O(\epsilon) \end{aligned}$$

Notice that x_{n_0+4} is no longer an ambiguity term and now it is well defined. As $\epsilon \to 0$, the sequence given by the iteration is

$$(\cdots, \epsilon, u, \epsilon^{-1}, \epsilon^{-1}, a^2/u, \epsilon, b, \dots)$$

$$(\cdots, 0, R, \infty, \infty, R, 0, R, \cdots)$$

The sequence above is referred to as the associated singularity pattern. The iteration enters 0 and then it passes through poles $x_i = \infty$ and zeroes $x_j = 0$, after which the next iteration depends on the initial value u. We call this singularity confined.

Note that the difference equation above can be represented by a two-dimensional complex birational map of the form

$$\psi : \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix} \rightarrow \begin{pmatrix} x_{2,n} \\ \dfrac{b + ax_{2,n}}{x_{1,n}} \end{pmatrix}$$

In [28], Lafortune and Goriely defined the singularity confinement property for discrete mappings.

Definition 2.3.2 (Singularity confinement)**.** *For the class of N-dimensional birational maps of the form*

$$\psi\colon \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{N,n} \end{pmatrix} \rightarrow \begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \\ \vdots \\ x_{N,n+1} \end{pmatrix}$$

a singularity of the map is defined as a point $\mathbf{y} = (y_1, y_2, \dots, y_N)$ where the right-hand side of the map is undefined. This singularity is said to be confined if there exists a positive integer M such that $\lim_{\mathbf{x}\to\mathbf{y}} \psi^M(\mathbf{x}) = \psi_0$ exists.

Now we consider an example of particular relevance to us; we will shortly see that it arises from a deformed A_2 cluster map.

Example 2.3.3. *Consider*

$$\psi\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} \dfrac{1 + a_1x_2}{x_1} \\ \dfrac{x_1 + a_2(1 + a_1x_2)}{x_1x_2} \end{pmatrix} \tag{2.55}$$

This birational map has two singularities $x_1 = 0$ and $x_2 = 0$. We begin singularity analysis with the singularity $x_1 = 0$. Once again we study the singularity by introducing the small quantity $0 < \epsilon < 1$. By setting the initial data $x_1 = u,\ x_2 = \dfrac{-1 + \epsilon u}{a_1}$, the next iteration of the map reaches $\left(\epsilon, \dfrac{(1 + \epsilon a_2)a_1}{\epsilon u - 1}\right)$, where it is at a *singularity:*

$$\begin{pmatrix} -(a_1-1)(a_1+1)\epsilon^{-1} - a_1^2(u+a_2) \\ \dfrac{a_2(a_1-1)(a_1+1)}{a_1}\epsilon^{-1} + \dfrac{ua_2+a_2^2-1}{a_1} \end{pmatrix} \to \begin{pmatrix} -a_2+\epsilon(u+a_2)a_2+O(\epsilon^2) \\ \epsilon\dfrac{-(a_2-1)(a_2+1)a_1}{a_2(a_1-1)(a_1+1)} + O(\epsilon^2) \end{pmatrix}$$

$$\to \begin{pmatrix} -\dfrac{1}{a_2}+O(\epsilon) \\ \dfrac{-a_2^2+u(a_1^2-1)a_2+a_1^2}{a_1(a_2^2-1)} + O(\epsilon) \end{pmatrix}$$

As $\epsilon \to 0$, the sequence becomes

$$\begin{pmatrix} \infty \\ \infty \end{pmatrix} \to \begin{pmatrix} -a_2 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -\dfrac{1}{a_2} \\ \dfrac{-a_2^2+u(a_1^2-1)a_2+a_1^2}{a_1(a_2^2-1)} \end{pmatrix} \tag{2.56}$$

Therefore there exists a limit $\lim_{\mathbf{x}\to\mathbf{y}} \psi^M(\mathbf{x}) \neq 0$, which is non-vanishing. Therefore the singularity is confined.

For the case of the singularity $x_2 = 0$, we take the same procedure with the initial iterates $x_1 = a_2(1+a_1u)/(\epsilon u - 1)$ and then $\epsilon \to 0$ give us the following sequence of iterations,

$$\begin{pmatrix} -a_2 \\ \infty \end{pmatrix} \to \begin{pmatrix} \infty \\ (a_1-a_1a_2^2)/(a_2^2-1) \end{pmatrix} \to \begin{pmatrix} 0 \\ -1/a_1 \end{pmatrix} \to \begin{pmatrix} R \\ R \end{pmatrix} \tag{2.57}$$

where R is some non-zero regular value. Therefore both singularities are confined.

Example 2.3.3 showed that this particular map possesses the confinement property. In addition to this, one can show that the map (2.55) is Liouville integrable, as there exists a first integral that is invariant under the map, given by

$$x_1 + \frac{1}{x_1} + \frac{a_1^2}{x_1} + \frac{a_1}{a_1x_2} + \frac{a_1a_2}{x_2} + \frac{a_1}{x_1x_2} + \frac{a_1x_1}{x_2} + \frac{a_1x_2}{a_2} + \frac{a_1x_2}{x_1} \tag{2.58}$$

From these examples, it may appear that the confinement property leads to finding an integrable map, but in fact, this is not the correct statement. This was shown by Hietarinta and Viallet in [29], who provided a counter-example, passing the confinement property test but non-integrable in the sense of degree growth (see the section 2.3.3).

2.3.2 Singularity pattern via p-adic analysis

There is an alternative procedure to find a singularity confinement pattern by considering birational maps over the completion of $\mathbb{Q}$ of p-adic integers (p is a prime number). In this section, we consider the arithmetic version of singularity confinement, introduced in [12, 30]. We begin with a brief introduction of p-adic integers (following [31, 32]) which is necessary for understanding the process of p-adic analysis.

Any $x \in \mathbb{Q}$ can be uniquely written as

$$x = p^{v(x)} \frac{a}{b} \tag{2.59}$$

where $v(x), a, b \in \mathbb{Z}$ and a, b are coprime integers, which are not divisible by the prime p. The p-adic norm is defined as $\|x\|_p = p^{-v(x)}$.

Definition 2.3.4 (Field of p-adic numbers)**.** *The field of p-adic numbers is the completion $\mathbb{Q}_p$, that is, field $\mathbb{Q}$ endowed with p-adic norm $\|\cdot\|_p$*

Definition 2.3.5 (p-adic expansion)**.** *The expansion of rational number $\alpha \in \mathbb{Q}_p$ in powers of p is called p-adic expansion. i.e.*

$$\alpha = \sum_{n=n_0}^{\infty} \alpha_n p^n \tag{2.60}$$

where $n_0 \in \mathbb{Z} \cup \{\infty\}$ and $0 \leq \alpha_n \leq p-1$

Note that for $n_0 \neq 0$, (2.60) can be written as $p^{n_0}(\sum_{n=0}^{\infty} \alpha_{n_0+n} p^n)$ in the form of (2.59). Hence it is clear that $v_p(\alpha) = n_0$.

Lemma 2.3.6. *Let $\sum_{n=n_0}^{\infty} \alpha_n p^n$ be p-adic expansion of $\alpha \in \mathbb{Q}_p$. Then*

$$\begin{aligned} \alpha &= \sum_{n=n_0}^{k-1} \alpha_n p^n \pmod{p^k} \\ &= \alpha_{n_0} p^{n_0} + \alpha_{n_0+1} p^{n_0+1} + \cdots + \alpha_{k-2} p^{k-2} + \alpha_{k-1} p^{k-1} \pmod{p^k} \end{aligned} \tag{2.61}$$

for all $k > 0$ where the remaining terms vanishes

Example 2.3.7 (5-adic expansion of $\frac{3}{2}$)**.** *Since $\left\|\frac{3}{2}\right\|_5 = 1$, it is 5-adic integer and its 5-adic expansion must be in the following form.*

$$\frac{3}{2} = \alpha_0 + \alpha_1(5) + \alpha_2(5)^2 + \ldots \tag{2.62}$$

To determine the coefficients α_i*, we begin with the case* $k = 1$ *in Lemma 2.3.6,*

$$\frac{3}{2} = \alpha_0 \pmod 5$$

Following congruence is equivalent to $\alpha_0 = \frac{3}{2} \pmod 5 \rightarrow 2\alpha_0 = 3 \pmod 5$*. For the remainder of the division to be 3, the only choice of* α_0 *must be* 4*. If we proceed to* $k = 2$*, the expansion can be written as*

$$\frac{3}{2} = 4 + 5\alpha_1 \pmod{5^2}$$

Once again, rearrange the expression and obtain $-\frac{5}{2} = 5\alpha_1 \pmod{25}$*. Dividing by* 5 *and reordering yield the simplified congruence* $-2\alpha_1 = 1 \pmod 5$*. Therefore fixing* $\alpha_1 = 2$ *solves the expression. Repeating the same procedure for other cases* k*, gives* 5*-adic expansion,*

$$\frac{3}{2} = 4 + 2 \cdot 5 + \cdots \tag{2.63}$$

One can verify that the left-hand side of the expression is indeed $\frac{3}{2}$*.*

Definition 2.3.8 (Ring of p-adic integers)**.**

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p \ : \ |x|_p \leq 1 \iff v(x) \geq 0 \right\}$$

The ring of p-adic integers $\mathbb{Z}_p$ has a unique maximal ideal,

$$\mathfrak{p} = p\mathbb{Z}_p = \{x \in \mathbb{Z}_p \ : \ v_p(x) \geq 1\}$$

Following this, one can define a ring homomorphism between $\mathbb{Z}_p$ and congruence classes $\mathbb{Z}_p/p\mathbb{Z}_p$,

$$\begin{aligned} \hat{\pi} \ : \ \mathbb{Z}_p &\rightarrow \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p \\ x &\mapsto \hat{x} \end{aligned} \tag{2.64}$$

for which the following relation holds:

$$\pi(x \pm y) = \hat{x} + \hat{y}, \quad \pi(x^{\pm} y^{\pm}) = \hat{x}^{\pm} \hat{y}^{\pm} \tag{2.65}$$

This map is called *reduction modulo p*.

We can extend the reduction by replacing the domain with $\mathbb{Q}_p \backslash \{0\}$. Above, we already noted that any elements in $\mathbb{Q}_p$ can be expressed as $x = p^{v_p(x)} x' =$

$p^{v_p(x)} \sum_{i=0} \alpha_i p^i$ by p-adic expansion. Thus the reduction of $x \in \mathbb{Q}_p \backslash \{0\}$ modulo prime $\mathfrak{p}$ is given by

$$x \mapsto \pi(x) = \begin{cases} 0 & v_p(x) > 0 \\ \infty & v_p(x) < 0 \\ \alpha_0 = R & v_p(x) = 0 \end{cases} \tag{2.66}$$

where R is a non-zero regular value.

Example 2.3.9. *Going back to Example 2.3.7, the reduction of* $3/2$ *modulo 5 gives* $\pi(3/2) = 4$*.*

Example 2.3.10 (Reduction modulo 5)**.** *The* 5*-adic expansion of* $1/10$ *yields the following form,*

$$\frac{1}{10} = 3 \cdot \left(\frac{1}{5}\right) + 2 + 2 \cdot (5^2) + \cdots \tag{2.67}$$

The reduction of $1/10$ *modulo* 5 *gives* $\pi(1/10) = \infty$*.*

Now let us consider reduction modulo p of rational mappings $\mathbf{x}_{n+1} = \varphi(\mathbf{x}_n)$ i.e.

$$\varphi\colon \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{N,n} \end{pmatrix} \to \begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \\ \vdots \\ x_{N,n+1} \end{pmatrix} \tag{2.68}$$

where $x_{i,n} \in \mathbb{Q}_p(x_1, \ldots, x_N)$ for $i \in \{1, \ldots N\}$. If we apply the reduction to φ, then we obtain a reduced map $\hat{\mathbf{x}}_{n+1} = \hat{\varphi}(\hat{\mathbf{x}}_n) \in \mathbb{F}_p^N$ which satisfies $\hat{\pi} \cdot \varphi = \hat{\varphi} \cdot \hat{\pi}$ (depicted in the figure 2.2). Such a reduction is known as a *good reduction* in [12].

$$\begin{array}{ccc} \mathbf{x}_{n_0} & \xrightarrow{\varphi} & \mathbf{x}_{n_0+1} \\ \downarrow{\hat{\pi}} & & \downarrow{\hat{\pi}} \\ \hat{\mathbf{x}}_{n_0} & \xrightarrow{\hat{\varphi}} & \hat{\mathbf{x}}_{n_0+1} \end{array}$$

Figure 2.2: Good reduction

However, in particular cases ($\pi(x_{j,n}) = \infty$), the reduction is not well defined. Note that this coincides with the situation of singularity analysis, that is, the iterates $x_{n_0} \in \mathbb{Q}_p$ yielding p-adic norm, $|x_{n_0}|_p > 1$, corresponds to the ϵ^{-1} in the singularity analysis. In [12], the notion analogous to Definition 2.3.2 is introduced as follows

Definition 2.3.11 (Almost good reduction)**.** *Let φ be rational map $\varphi : \mathbb{Q}_p^N \to \mathbb{Q}_p^N$, shown at* (2.68)*. The reduction modulo p is said to be almost good if there exists a positive integer m_0 such that* $\lim_{m \to m_0 - n_0} \hat{\pi}(\varphi^m(\mathbf{x}_{n_0})) = \hat{\varphi}^{m_0 - n_0}(\hat{\mathbf{x}}_{n_0}) = \hat{\mathbf{x}}_{m_0}$

$$\begin{array}{ccccccccc}
\mathbf{x}_{n_0} & \xrightarrow{\varphi} & \mathbf{x}_{n_0+1} & \xrightarrow{\varphi} & \mathbf{x}_{n_0+2} & \overset{\varphi^{m_0-n_0-3}}{\dashrightarrow} & \mathbf{x}_{m_0-1} & \xrightarrow{\varphi} & \mathbf{x}_{m_0} \\
\downarrow \hat{\pi} & & \downarrow \hat{\pi} & & \downarrow \hat{\pi} & & \downarrow \hat{\pi} & & \downarrow \hat{\pi} \\
\hat{\mathbf{x}}_{n_0} & \xrightarrow{\hat{\varphi}} & \hat{\mathbf{x}}_{n_0+1} & \xrightarrow{\hat{\varphi}} & \hat{\mathbf{x}}_{n_0+2} & \overset{\hat{\varphi}^{m_0-n_0-3}}{\dashrightarrow} & \hat{\mathbf{x}}_{m_0-1} & \xrightarrow{\hat{\varphi}} & \hat{\mathbf{x}}_{m_0}
\end{array}$$

Let us again consider the map ψ in Example 2.3.3 to see the process.

Example 2.3.12. *Setting initial clusters $(x_1, x_2) = (1, 1)$ and the parameters to be $a_1 = 2$ and $a_2 = 3$, we consider the orbit given by iteration of the map ψ as shown in the table below,*

n	1	2	3	4	5	6	7
$x_{1,n}$	1	3	7	$\frac{3^3}{5 \cdot 7}$	$\frac{5 \cdot 103}{3^2 \cdot 11}$	$\frac{3 \cdot 11 \cdot 401}{29 \cdot 103}$	$\frac{11 \cdot 29 \cdot 419}{3 \cdot 137 \cdot 401}$
$x_{2,n}$	1	$2 \cdot 5$	$\frac{11}{5}$	$\frac{2^2 \cdot 29}{7 \cdot 11}$	$\frac{7 \cdot 137}{3 \cdot 29}$	$\frac{2 \cdot 3 \cdot 3049}{103 \cdot 137}$	$\frac{17 \cdot 43^2 \cdot 103}{401 \cdot 3049}$

where each iteration is factored into primes.

We can see that for $p = 7, 103, 401$ etc., the p-adic norm for $x_{1,n}$ and $x_{2,n}$ exhibits the patterns

$$\begin{aligned} |x_{1,n}|_p &: 1, p^{-1}, p, 1, 1 \\ |x_{2,n}|_p &: 1, 1, p, p^{-1}, 1 \end{aligned} \tag{2.69}$$

As for the prime $p = 5, 11, 29, 137$ etc., the pattern is

$$\begin{aligned} |x_{1,n}|_p &: 1, 1, 1, p, p^{-1}, 1 \\ |x_{2,n}|_p &: 1, p^{-1}, p, 1, 1, 1 \end{aligned} \tag{2.70}$$

If we take $x_{i,n}$ *modulo* p *then the sequences* (2.69) *and* (2.70) *are analogous to the following singularity pattern:*

$$\begin{pmatrix} \epsilon \\ R \end{pmatrix} \to \begin{pmatrix} \epsilon^{-1} \\ \epsilon^{-1} \end{pmatrix} \to \begin{pmatrix} R \\ \epsilon \end{pmatrix} \to \begin{pmatrix} R \\ R \end{pmatrix} \tag{2.71a}$$

$$\begin{pmatrix} R \\ \epsilon \end{pmatrix} \to \begin{pmatrix} R \\ \epsilon^{-1} \end{pmatrix} \to \begin{pmatrix} \epsilon^{-1} \\ R \end{pmatrix} \to \begin{pmatrix} \epsilon \\ R \end{pmatrix} \to \begin{pmatrix} R \\ R \end{pmatrix} \tag{2.71b}$$

which is identical to the pattern, shown in Example 2.3.3.

2.3.3 Algebraic entropy

Recall that the singularity confinement test was insufficient to detect the integrability of the system as there exists a counter-example which is non-integrable (exhibiting chaotic behaviour) but still passes the test. There is an alternative algebraic test which is a stronger integrability indicator than the singularity confinement test. This test involves an algebraic quantity called *algebraic entropy*, introduced by Bellon and Viallet in [33],

$$\varepsilon = \lim_{n\to\infty} \frac{\log d_n}{n} \tag{2.72}$$

where $d_n = \deg(\varphi^n)$ is given by the maximum degrees of the components of φ^n i.e., the entropy is determined with the growth of the degrees d_n. As we observed in the previous examples, the rational expression becomes more complex as we iterate the map on initial data. If there were no factorisation, then the degree would have grown exponentially. In the iteration, the cancellation of common terms occurs between the numerator and denominator, reducing the complexity of the expression. Thus the factorisation reduces the growth rate of degree d_n. In many cases, it was shown in [33][29],

- Exponential growth represents chaotic behaviour
- Polynomial growth represents regular behaviour

Particularly, the degree with polynomial growths corresponds to zero entropy, $\varepsilon = 0$. From observation of many integrable maps, it is conjectured that zero entropy indicates integrability. Therefore the amount of cancellations in the iteration decides the entropy.

It is well known that the calculation of algebraic entropy is very complicated in general. However, it was shown in [7] that the complexity of calculation is reduced in the cluster setting, which allows us to find the tropical max plus relation analogue of the exchange relation. To comprehend the mechanism, here we follow [7], that is, the exact growth of the degree of cluster maps associated with the period 1 exchange matrix.

Let us consider a cluster algebra $\mathcal{A}(\mathbf{x}, B)$ of rank n. Recall that due to the Laurent phenomenon, any cluster variables generated by cluster mutation are expressed as Laurent polynomials in the initial cluster, which take the form

$$x_{i,n} = \frac{P_n(\mathbf{x})}{\mathbf{x}^{\mathbf{d}_{i,n}}} \tag{2.73}$$

where $P_l(\mathbf{x})$ is a polynomial in the initial cluster $\mathbf{x}_0$ with positive coefficients and $\mathbf{x}^{\mathbf{d}_{i,l}} = \prod_j x_j^{d_{j,l}}$ with *denominator* vector $\mathbf{d}_{i,l}(x_{i,l}) = (d_{1,l}, \dots d_{n,l})$. From exchange relation, we can see that homogeneity degree of numerator $P_n(\mathbf{x})$ is equal to the homogeneity degree of $x_{i,n}$ plus 1. This implies that the d-vector in the monomial is sufficient to determine the growth of the degree of iterates.

To find the d-vector, it is important to remark that the variable arising from the exchange relation is a subtraction-free rational expression due to total positivity of cluster algebras [34]. Thus, upon substituting the (2.73) directly into the exchange relation (2.2) and comparing the denominator of each side of it, we can see that $\mathbf{d}_{i,l}$ satisfies the following relation,

$$\mathbf{d}_{k,l'} + \mathbf{d}_{k,l} = \max\left(\sum_{\substack{i=1 \\ b_{i,k}>0}} b_{ik}\mathbf{d}_{i,l} \ , \ - \sum_{\substack{i=1 \\ b_{i,k}<0}} b_{ik}\mathbf{d}_{i,l} \right) \tag{2.74}$$

which is expressed as (max,+) expression corresponding to (2.2).

Remark 2.3.13. *Simply speaking, the (max,+) algebra is constructed from an ordinary algebra equipped with operations addition and multiplication through* *tropical method in mathematics [35, 36] which replaces the operations as following*

$$a + b \to \max(a, b)$$
$$a \times b \to a + b$$

One of the approaches representing tropical method is ultradiscretization that is process of transforming the difference equations into the (max,+) expression through a limiting procedure i.e setting each dependent or independent variable $x_i = e^{\frac{X_i}{\epsilon}}$ *and using identity of the limit*

$$\lim_{\epsilon \to 0} \epsilon \log\left(e^{\frac{X_1}{\epsilon}} + e^{\frac{X_2}{\epsilon}}\right) = \max(X_1, X_2) \tag{2.75}$$

allow us to rewrite the difference equation as (max,+) expression, which is also known as ultradiscrete equation . This limiting process was introduced by Takahashi and Satsuma in [37] for which they used to show the connection between ultradiscrete KdV equation and specific type of Celluar Automata (CA), the box ball system. Over the years, this procedure has been useful in the various aspects of discrete integrable systems. In this thesis, ultradiscretization is the key for finding the leading order of degree growth which will be seen in the later chapters. If the reader is interested in ultradiscretization in further detail, please have a look [38]

Example 2.3.14 (Algebraic entropy of Somos-5 sequences)**.** *We follow the work in [39]. Recall from Example 2.2.13, the Somos-5 recurrence relation*

$$x_{n+5}x_n = x_{n+1}x_{n+4} + x_{n+2}x_{n+3} \tag{2.76}$$

which can be obtained by the iteration of the period 1 cluster map. By setting the degree vector $\mathbf{d}(x_n) = \mathbf{d}_n = (d_n, d_{n+1}, \ldots, d_{n+5})$*, one can express tropical analogue of* (2.76) *as follows*

$$\mathbf{d}_n + \mathbf{d}_{n+5} = \max\left(\mathbf{d}_{n+1} + \mathbf{d}_{n+4}\ ,\ \mathbf{d}_{n+2} + \mathbf{d}_{n+3}\right) \tag{2.77}$$

If we subtract $\mathbf{d}_{n+4} + \mathbf{d}_{n+1}$ *from both sides of the equation, we obtain*

$$\mathbf{d}_n + \mathbf{d}_{n+5} - \mathbf{d}_{n+4} - \mathbf{d}_{n+1} = \max(\ \mathbf{d}_{n+2} + \mathbf{d}_{n+3}, 0)$$

Following from above, we set variable $U_n = \mathbf{d}_{n+4} + \mathbf{d}_{n+1} - \mathbf{d}_{n+2} - \mathbf{d}_{n+3}$*, which is tropical analogue of the substitution* (2.53)*; the expression becomes*

$$U_{n+1} + U_{n-1} = [U_n]_+ \tag{2.78}$$

where $[U_n]_+ = \max(U_n, 0)$*. Notice that this expression is ultradiscrete version of the recurrence* (2.53) *given by the reduced map in Example 2.2.13.*

With fixing the initial degree vector $\mathbf{d}_1 = -1$ *(equivalent to* $U_0 = -1$*), a direct calculation of first three iterations of* (2.77) *shows by inspection that* $u_n = u_{n+3}$*. Hence* u_n *has period 3, or in other words* U_n *satisfies the relation*

$$(\mathcal{S}^3 - 1)U_n = 0 \tag{2.79}$$

where $\mathcal{S}$ *is the shift operator that sends* $n \to n+1$*. By imposing the condition to* (2.78)*, then we have*

$$\begin{aligned} &(\mathcal{S}^3 - 1)(\mathcal{S} + \mathcal{S}^{-1})U_n \\ &= (\mathcal{S}^3 - 1)(\mathcal{S} + \mathcal{S}^{-1})(\mathcal{S}^4 + \mathcal{S} - \mathcal{S}^2 - \mathcal{S}^3)\mathbf{d}_n \\ &= \mathcal{S}(\mathcal{S}+1)(\mathcal{S}^2+1)(\mathcal{S}^2+\mathcal{S}+1)(\mathcal{S}-1)^3\mathbf{d}_n = 0 \end{aligned} \tag{2.80}$$

By solving the characteristic equation for $\mathcal{S}$*, we obtain the d-vector* $\mathbf{d}_n$ *whose leading order term is given by*

$$\mathbf{d}_n = \alpha n^2 + O(n) \tag{2.81}$$

for some constant $\alpha > 0$*. Since* n *grows faster than* $\log \mathbf{d}_n$ *as* $n \to \infty$*, the algebraic entropy of Somos-5 recurrence relation is 0. Hence the entropy indicates that it is integrable, which matches the result in Example 2.2.13.*

Remark 2.3.15. *In accordance of [40], the complexity of iterations* $x_{n+1} = \varphi(x_n)$*, over field of rational* $\mathbb{Q}$*, can be measured by the height function, that is,*

$$H(x_m) = \max\{|u|, |v|\}$$

where u *and* v *are numerator and denominator of* x_m*, and they are coprime. Then the same author in [40] introduced an alternative version of algebraic entropy, which is known as Diophantine integrability: the discrete dynamics given by* $x_{n+1} = \varphi(x_n)$ *is Diophantine integrable if the logarithmatic of heights* $h(x_m) = \log(H(x_m))$ *grows slower than polynomial in* m*. This is immediate approach to check the growth of complexity.*

2.4 Deformation of cluster mutations

In this section, we will briefly review the deformation theory which preserves the presymplectic form, introduced by the Hone and Kouloukas [10]. Let us consider the exchange relation (2.2) and express it in the following form:

$$x'_j = \begin{cases} x_k^{-1} f_k(M_k^+, M_k^-), & \text{for } j = k \\ x_j, & \text{for } j \neq k \end{cases} \tag{2.82}$$

where $f_k : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a differentiable function and

$$M_k^+ = \prod_{i=1}^{N} x_i^{[b_{ik}]_+}, \qquad M_k^- = \prod_{i=1}^{N} x_i^{[-b_{ik}]_+} \tag{2.83}$$

Note that if $f_k(M_k^+, M_k^-) = M_k^+ + M_k^-$, then the mutation is the ordinary coefficient-free cluster mutation (2.2). In [10], Hone and Kouloukas introduced such functions f in order to extend the definition of cluster mutation μ_k, while still wishing to maintain the property of preserving the pre-symplectic form ω. The first key lemma and theorem in this setting are the following.

Lemma 2.4.1 ([10])**.** *If* $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B')$ *is defined as in* (2.82) *and* (2.1)*, then the symplectic form* ω *is preserved, i.e.*

$$\sum_{i<j} \frac{b'_{ij}}{x'_i x'_j} \mathrm{d}x'_i \wedge \mathrm{d}x'_j = \sum_{i<j} \frac{b_{ij}}{x_i x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \tag{2.84}$$

if and only if

$$f_k(M_k^+, M_k^-) = M_k^+ g_k\left(\frac{M_k^-}{M_k^+}\right) \tag{2.85}$$

for some differentiable function $g_k : \mathcal{F} \to \mathcal{F}$

Theorem 2.4.2 ([10])**.** *Let* $\mu_{i_l}, \ldots, \mu_{i_2}\mu_{i_1}$ *be a sequence of generalised mutations of the form* (2.82) *such that*

$$\mu_{i_l} \cdots \mu_{i_2}\mu_{i_1}(B, \mathbf{x}) = (B, \tilde{\mathbf{x}})$$

with f_{i_j} *being of the form* (2.85)*. Then* $\varphi\colon \mathbf{x} \to \tilde{\mathbf{x}}$ *is such that* $\varphi^*\omega = \omega$*, for* ω *log-canonical as per* (2.39)*.*

Thus exchange matrices that are periodic under a particular sequence of mutations (or more generally, are periodic up to a permutation) give rise to parametric cluster maps that preserve the pre-symplectic form ω. In other words, by adjusting the function f in (2.85), one can generalise the symplectic cluster map. Furthermore, if the map is integrable then one can find a family of deformed integrable maps. Let us consider several examples of the deformed integrable maps.

Example 2.4.3 (Dynkin type A_2)**.** *Let us consider the cluster algebra of type* A_2 *which is constructed by initial data formed by the initial cluster* $\mathbf{x} = (x_1, x_2)$ *and the exchange matrix*

$$B_{\mathrm{A}_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.86}$$

The matrix is invariant under the action of the sequence of mutations $\mu_2\mu_1$*, i.e.* $\mu_2\mu_1(B_{A_2}) = B_{A_2}$*. This composition of mutations enables us define a cluster map* $\varphi_{\mathrm{A}_2} = \mu_2\mu_1$ *such that it preserves the two-form*

$$\omega_{\mathrm{A}_2} = \frac{1}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{2.87}$$

Note that ω_{A_2} *is symplectic form as it is not degenerate. One can confirm that the map* φ_{A_2} *is periodic with period 5 (i.e.* $\varphi^5(\mathbf{x}) = \mathbf{x}$*) by an explicit calculation. Alternatively, one can check by Theorem 2.1.33 (stated in Section 2.1.5), that is,* $h + 2 = 3 + 2 = 5$*, where* $h = 3$ *is the Coxeter number of type* A_2*.*

Since the map satisfies the condition of Theorem 2.4.1, one can define a new symplectic map $\tilde{\varphi}$*, which is constructed by the composition of mutations* $\tilde{\mu}_k$ *for* $1 \leq k \leq 2$*, where* $\tilde{\mu}_k$ *are deformed mutations in the direction* k*,*

$$\tilde{\mu}_k(x_k) = x_k^{-1} M_k^+ g_k\left(\frac{M_k^-}{M_k^+}\right) \tag{2.88}$$

with $M_k^\pm$ *defined in* (2.83)*.*

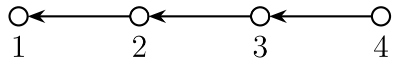
By setting the function $g_k(x) = a_k + b_k x$*, the mutated variables* $\tilde{\mu}_k(x_k)$ *can be written as*

$$\tilde{\mu}_k(x_k) = x_k^{-1}\big(b_k M_k^- + a_k M_k^+\big) \tag{2.89}$$

Furthermore by rescaling each cluster variables, $x_i \to \lambda_i x_i$ *for* $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$*, we fix the parameters* b_1, b_2 *to be 1; we find the deformed cluster map* $\tilde{\varphi}_{A_2} = \tilde{\mu}_2\tilde{\mu}_1$*,* *which takes the form of the map ψ in Example 2.3.3. Due to the theorem 2.4.2, the deformed cluster map $\tilde{\varphi}_{\mathrm{A}_2}$ is symplectic map.*

The deformed cluster map $\tilde{\varphi}_{\mathrm{A}_2}$, which preserves the symplectic form ω_{A_2}, (2.87), *is then ψ as in Example 2.3.3. As mentioned there, the map ψ is integrable as one can find the first integral* (2.58), *which is invariant under the map. Thus deformed map $\tilde{\varphi}_{A_2}$ is integrable.*

Example 2.4.4 (Dynkin type A_4)**.** *The quiver with linear orientation of Dynkin type A_4 can be drawn as*



Let B_{A_4} be exchange matrix associated with type A_4 quiver, which is given by the skew-symmetric matrix

$$B_{A_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.90}$$

This matrix has period 4 with respect to a sequence of cluster mutations:

$$\mu_4\mu_3\mu_2\mu_1(B_{A_4}) = B_{A_4}$$

We define the cluster map φ_{A_4} with the composition of mutations above, $\varphi_{A_4} = \mu_4\mu_3\mu_2\mu_1$. Once again, we fix the function f_k in the cluster mutation as (2.89) *and then apply the rescaling to the cluster, i.e. $x_i \to \lambda_i x_i$, for $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$ to adjust the parameters into $b_i = 1$ and $a_i = 1$ for $i \in \{2, 3\}$. This yields the following parametric map*

$$\tilde{\varphi}_{A_4} : (x_1, x_2, x_3, x_4) \to (x_1', x_2', x_3', x_4')$$

where the mutated variables x_i' are given by the following relations:

$$\begin{aligned} \tilde{\mu}_1 : \quad & x_1 x_1' = b_1 + a_1 x_2 \\ \tilde{\mu}_2 : \quad & x_2 x_2' = 1 + x_1' x_3 \\ \tilde{\mu}_3 : \quad & x_3 x_3' = 1 + x_2' x_4 \\ \tilde{\mu}_4 : \quad & x_4 x_4' = b_4 + a_4 x_3' \end{aligned} \tag{2.91}$$

As the Coxeter number for type A_4 *is* 5*, one can confirm that the original cluster map* φ_{A_4} *is periodic with period 7* ($\varphi^7_{A_4}(\mathbf{x}) = \mathbf{x}$)*. In our case, the periodicity takes an important role in constructing first integrals which are invariant under the type* A_4 *cluster map. This property allows us to define the symmetric functions,*

$$I_1 = \sum_{j=0}^{6} L_j, \quad I_2 = \prod_{j=0}^{6} L_j \tag{2.92}$$

where $L_i = (\varphi^*)^i(x_1)$*. The symmetric functions are first integrals associated with* φ_{A_4} *as they satisfy* $\varphi^*_{A_4}(I_i) = I_i$*. In addition to this, they are in involution with respect to the Poisson bracket, i.e.*

$$\{x_i, x_j\} = P_{ij} x_i x_j \tag{2.93}$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \tag{2.94}$$

With these properties holding, we see that the cluster map φ_{A_4} *is Liouville integrable. In the same fashion, one can show that the deformed map* $\tilde{\varphi}_{A_4}$ *possesses integrability under certain conditions. The most natural candidates for the first integrals are* (2.92)*, as the Poisson structure remains as per* (2.94)*. However, they are not preserved under* $\tilde{\varphi}_{A_4}$*.*

To resolve this problem, we consider expanded first integrals, which are expressed into a sum of monomials and modify them by inserting arbitrary coefficients into each term:

$$\tilde{I}_1 = \sum_i \alpha_i J_i, \quad \tilde{I}_2 = \sum_j \beta_j K_j \tag{2.95}$$

where J_i *and* K_j *are monomials arising from the first integrals* I_1 *and* I_2 *respectively, and* α_i*,* β_j *are arbitrary coefficients. Then by imposing the condition* $\tilde{\varphi}^*_{A_4}(\tilde{I}_i) = \tilde{I}_i$*, one can constrain the coefficients* α_i*,* β_j *and find the necessary and sufficient conditions for integrability with* $\tilde{I}_i$ *being first integrals. Thus if we fix the parameters*

$b_1 = 1 = b_4$, *then* $\tilde{I}_1$ *and* $\tilde{I}_2$ *are first integrals,*

$$\tilde{I}_1 = \frac{1}{x_1x_2x_3x_4}(a_1a_4x_1x_2 + a_1a_4^2x_1x_2x_3 + a_1x_1x_2x_3 + a_1a_4x_1x_2x_3^2 + a_1a_4x_1x_4 + a_1a_4x_1x_2^2x_4$$
$$+ a_1a_4x_3x_4 + a_1a_4x_1^2x_3x_4 + a_4x_2x_3x_4 + a_1^2a_4x_2x_3x_4 + a_4x_1^2x_2x_3x_4 + a_1a_4x_2^2x_3x_4$$
$$+ a_1a_4x_1x_3^2x_4 + a_1a_4x_1x_2x_4^2 + a_1x_1x_2x_3x_4^2)$$

$$\tilde{I}_2 = \frac{(a_1 + x_2)(x_1 + x_3)(x_1 + x_3)(a_4 + x_3)(x_1x_2 + a_4x_1x_2x_3 + x_1x_4 + x_3x_4 + a_1x_2x_3x_4)}{x_1x_2^2x_3^2x_4} \tag{2.96}$$

Furthermore, by explicit calculation, one can show that they commute with respect to the Poisson bracket (2.93). *This allows us to conclude that the deformed map* $\tilde{\varphi}_{A_4}$ *is Liouville integrable.*

The examples above show that $\tilde{\varphi}_{A_2}$ and $\tilde{\varphi}_{A_4}$ are integrable symplectic maps. However, as a result of applying the deformation, the map no longer generates cluster variables, belonging to a Laurent polynomial ring. Thus in general, the deformed map is not a cluster map. To restore the property, we require a process called *Laurentification*, which will be introduced in the next section.

2.5 Laurentification

In this section, we will introduce a specific projectivization that lifts our deformed map, which is not given by a cluster algebra structure, hence we do not have Laurent polynomial expressions for the iterates to a higher dimensional one which is and does. This lifting is called *Laurentification* (coined by Hamad et al. [41]) and was introduced and studied by the authors in ([42],[43]). Thus this procedure helps us to resolve the problem that emerges from the deformation.

To be a little more concrete, recall that one of the key features of a cluster algebra is the Laurent phenomenon, where every variable induced by cluster mutation can be expressed as a Laurent polynomial in the initial cluster variables. This implies that a cluster map, which is composed of certain mutations, has the *Laurent property*, in the following sense.

Definition 2.5.1 (Laurent property)**.** *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ *be an initial cluster and let* $\psi\colon \mathbb{C}^n \to \mathbb{C}^n$ *be an associated cluster map. Then* ψ *is said to have the Laurent property if for all* n*, the* n*th iterates of* ψ *are given by Laurent polynomials in the Laurent polynomial ring* $\mathbb{C}\big[x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm}\big]$

Notice that the deformed map $\tilde{\varphi}_{\mathrm{A}_2}$ in Example 2.4.3 with generic choice of parameters a_1, a_2 is not a cluster map. This is because the iteration of the map, beginning from the initial cluster (x_1, x_2) yields the new variables

$$(\tilde{\varphi}_{\mathrm{A}_2})^2 : \mathbf{x} \to \left(\begin{array}{c} \dfrac{a_1a_2 + a_1x_1 + a_1^2a_2x_2 + x_1x_2}{x_2(1 + a_1x_2)} \\ \dfrac{x_1(a_1a_2^2 + a_1a_2x_1 + x_2 + a_1^2a_2^2x_2 + a_2x_1x_2 + a_1x_2^2)}{(1 + a_1x_2)(a_2 + x_1 + a_1a_2x_2)} \end{array} \right) \tag{2.97}$$

which consists of rational expressions whose denominator is no longer monomial as the parameters prevent the cancellation with the numerator. Thus deformation of the cluster map destroyed the Laurent property of the undeformed counterpart to this map.

To restore the property one must try to lift the map to a higher dimensional space, where the Laurent property is restored.

Definition 2.5.2 (Laurentification)**.** *Let* $\varphi\colon \mathbb{C}^M \to \mathbb{C}^M$ *be a birational map. A birational map* $\psi\colon \mathbb{C}^N \to \mathbb{C}^N$ *(for some* $N \geq M$*) is said to be a* Laurentification *of* φ *if the map* ψ *possesses Laurent property and there exists a rational map* $\pi : \mathbb{C}^N \to \mathbb{C}^M$ *such that* $\varphi \cdot \pi = \pi \cdot \psi$ *holds.*

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{\psi} & \mathbb{C}^N \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{C}^M & \xrightarrow{\varphi} & \mathbb{C}^M \end{array}$$

Figure 2.3: Laurentification

Note that there are several methods which fit the description of Laurentification such as recursive factorization, which was introduced by Hamad and Kamp in [41]. They showed that certain QRT maps, which do not generate the elements of the

Laurent polynomial ring, can be transformed into Somos-4 and Somos-5 recurrence relations with periodic coefficients.

Here we take an approach, that is, finding appropriate rational map π defined by dependent variable transformations, that is, expressed by new variables, *tau-functions* e.g. τ, σ. The following transformation can be identified by the singularity confinement patterns induced by a deformed integrable map. This method was applied to several examples in [43], [10]. To see the significance of this approach, let us consider the Laurentification of the deformed maps $\tilde{\varphi}_{\mathrm{A}_2}$ and $\tilde{\varphi}_{\mathrm{A}_4}$, which are shown in Example 2.5.3 and Example 2.5.5 as below.

Example 2.5.3 (Laurentification of $\tilde{\varphi}_{\mathrm{A}_2}$). *The singularity confinement pattern given by $\tilde{\varphi}_{A_2}$ (shown in Example 2.3.3) defines a rational map,*

$$\pi : (\tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3) \to (x_1, x_2)$$

which is equivalent to the dependent variable transformation

$$x_{1,n} = \frac{\sigma_n \tau_{n+1}}{\sigma_{n+1}\tau_n}, \quad x_{2,n} = \frac{\sigma_{n+3}\tau_{n-1}}{\sigma_{n+2}\tau_n} \tag{2.98}$$

where tau-functions τ and σ represent (2.71a) *and* (2.71b) *respectively. Substituting* (2.98) *gives the deformed map on the space of tau-functions, $\tilde{\psi} = \psi \cdot \pi$, which is equivalent to the following system of equations*

$$\tau_{n+2}\sigma_n = \sigma_{n+2}\tau_n + a_1\sigma_{n+3}\tau_{n-1} \tag{2.99a}$$

$$\sigma_{n+4}\tau_{n-1} = \sigma_{n+2}\tau_{n+1} + a_2\sigma_{n+1}\tau_{n+2} \tag{2.99b}$$

If the relations above are to be regarded as exchange relations, then there must be initial data formed by initial clusters and exchange matrix. The initial clusters can be extracted from (2.99a) *and* (2.99b)*, as follows. Let us denote*

$$(\tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\tilde{x}_1, \tilde{x}_1, \ldots, \tilde{x}_7) \tag{2.100}$$

The presymplectic form $\tilde{\omega}_{A_2}$ on the space of tau-functions can be written as

$$\tilde{\omega}_{A_2} = \pi^*\omega_{A_2} = \sum_{i<j} \tilde{b}_{ij} \mathrm{d}\log\tilde{x}_i \wedge \mathrm{d}\log\tilde{x}_j \tag{2.101}$$

where the $\tilde{b}_{ij}$ *are entries of a new exchange matrix,*

$$\tilde{B}_{A_2} = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \tag{2.102}$$

Let us consider the extended initial cluster $(\tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, a_1, a_2)$ *and extended exchange matrix,*

$$\hat{B}_{A_2} = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{2.103}$$

The cluster mutation in direction 4 acting on the initial clusters (2.100) *gives a new cluster:*

$$\hat{\mu}_4 : (\tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, a_1, a_2) \rightarrow (\tau_{-1}, \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_3, a_1, a_2), \tag{2.104}$$

where the fourth component of the tuple sees σ_0 *replaced by the new variable* τ_2*. Note that* $\hat{\mu}_j$ *indicate cluster mutations in the cluster algebra associated with Laurentification of the deformed map. The mutation* $\hat{\mu}_4$ *induces the following exchange relation*

$$\tau_2\sigma_0 = \tau_{-1}\sigma_3 + a_1\tau_0\sigma_2 \tag{2.105}$$

Further applying the mutation in the direction 1,

$$\mu_1 : (\tau_{-1}, \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_3, a_1, a_2) \rightarrow (\sigma_4, \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_3, a_1, a_2) \tag{2.106}$$

gives another form of exchange relation

$$\sigma_4\tau_{-1} = \sigma_2\tau_1 + a_2\sigma_1\tau_2 \tag{2.107}$$

Notice that equations (2.105) *and* (2.106) *are* (2.99a) *and* (2.99b) *with* $n = 0$ *respectively. Applying the mutations in a similar way consecutively onto the initial seed* $(\tilde{\mathbf{x}}, \hat{B}_{A_2})$*, we see that*

$$\mu_{37}\mu_{26}\mu_{15}\mu_{74}\mu_{63}\mu_{52}\mu_{41}(\tilde{\mathbf{x}}, \hat{B}_{A_2}) = (\tilde{\mathbf{x}}', \hat{B}_{A_2}), \qquad \text{where } \mu_{ij} = \mu_i\mu_j, \tag{2.108}$$

generates a new set of cluster variables $\tilde{\mathbf{x}}' = (\tau_6, \tau_7, \tau_8, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, a_1, a_2)$*, which are in the form of* (2.105) *and* (2.106)*, and moreover the exchange matrix* $\hat{B}_{A_2}$ *is invariant under the sequence of mutations. Recall that the exponents of monomials in exchange relations depend on the entries of the exchange matrix. Thus the mutations* (2.108) *acting on the new seed* $(\tilde{\mathbf{x}}', \hat{B}_{A_2})$ *gives cluster variables which are expressed by* (2.105) *and* (2.106)*. This implies that by the Laurent phenomenon, variables induced by the iteration of the deformed map* $\tilde{\varphi}_{A_2}$ *belong to the Laurent polynomial ring in the initial tau-variables.*

Remark 2.5.4. *Note that deformation of type* A_2 *case was considered in [10] with the cluster map expressed as* $\varphi_{\mathrm{A}_2} = \rho^{-1}\mu_1$ *for permutation of labels* $\rho = (1, 2)$*,*

$$\varphi_{\mathrm{A}_2} : \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix} \to \begin{pmatrix} x_{2,n} \\ \dfrac{b + a x_{2,n}}{x_{1,n}} \end{pmatrix}$$

which corresponds to Lyness recurrence,

$$x_{n+2}x_n = a x_{n+1} + b. \tag{2.109}$$

Following from the process of p-adic analysis, one can find specific pattern (see Example 2.3.1) which suggests the structure of tau functions for each variable

$$x_n = \frac{\tau_{n+5}\tau_n}{\tau_{n+2}\tau_{n+3}}$$

When we take the same procedure as above, we find the exchange matrix $\tilde{B}_{\mathrm{A}_2}$ (2.102) *and another version recurrence of Somos type,*

$$\tau_{n+7}\tau_n = a\tau_{n+6}\tau_{n+1} + b\tau_{n+3}\tau_{n+4} \tag{2.110}$$

which is special case of Somos-7 recurrence

Example 2.5.5 (Laurentification of $\tilde{\varphi}_{A_4}$)**.** *In order to define the associated rational map π, we need to determine the singularity structure for the deformed map $\tilde{\varphi}_{A_4}$. Performing the empirical p-adic method, one can find four types of singularity patterns as follows*

$$(1) : \ldots \to (\epsilon, R, R, R) \to \left(\epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}\right) \to (R, R, R, \epsilon) \to \ldots$$

$$(2) : \ldots \to (R, R, R, \epsilon) \to \left(R, R, R, \epsilon^{-1}\right) \to \left(R, R, \epsilon^{-1}, R\right) \to \left(R, \epsilon^{-1}, R, R\right) \\ \to \left(\epsilon^{-1}, R, R, R\right) \to (\epsilon, R, R, R) \to \ldots$$

$$(3) : \ldots \to (R, \epsilon, R, R) \to \ldots$$

$$(4) : \ldots \to (R, R, \epsilon, R) \to \ldots$$

where the ϵ in the pattern (3) and (4) corresponds to the primes which can be seen only in $x_{3,n}$ and $x_{4,n}$ respectively. Then we define the rational map π,

$$\pi : \tilde{\mathbf{x}}_0 = (q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, p_0) \to \mathbf{x}_0 = (x_1, x_2, x_3, x_4) \tag{2.111}$$

which is equivalent to the following dependent variable transformation:

$$x_{1,n} = \frac{\sigma_n \tau_{n+1}}{\sigma_{n+1}\tau_n} \quad x_{2,n} = \frac{p_n}{\sigma_{n+2}\tau_n} \quad x_{3,n} = \frac{q_n}{\sigma_{n+3}\tau_n} \quad x_{4,n} = \frac{\sigma_{n+5}\tau_{n-1}}{\sigma_{n+4}\tau_n} \tag{2.112}$$

where σ_n, τ_n, p_n, q_n correspond to the singularities in $(1),(2),(3),(4)$ respectively. We define a new initial seed $(\hat{\mathbf{x}}_0, \hat{B}_{A_4})$, where $\hat{\mathbf{x}}_0$ is the extended cluster obtained by adding frozen variables a_1 and a_4,

$$\hat{\mathbf{x}}_0 = (q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, p_0, a_1, a_4)$$

and $\hat{B}_{A_4}$ is the deformed exchange matrix, which is depicted by the quiver in Figure 2.4.

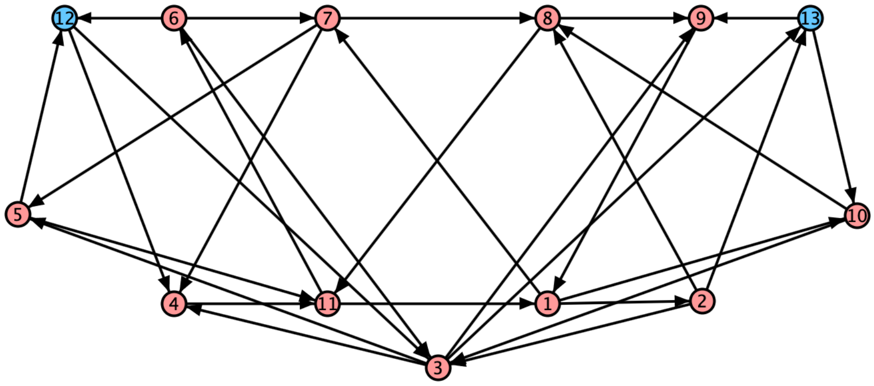


Figure 2.4: Type A_4 deformed quiver

Then the deformed map $\tilde{\varphi}_{A_4}$ *is Laurentified to the cluster map*

$$\psi_{A_4} = \tilde{\varphi}_{A_4}\pi = \hat{\rho}_{A_4}^{-1}\mu_2\mu_1\mu_{11}\mu_5, \quad \textit{for } \hat{\rho}_{A_4} = (2,3,4,5,6,7,8,9,10)$$

on $(\hat{\mathbf{x}}_0, \hat{B}_{A_4})$*, which generates the cluster variables expressed by the following recurrence relations:*

$$\begin{aligned} \tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1 p_n \\ p_{n+1}p_n &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + q_n\sigma_{n+1}\tau_{n+2} \\ q_{n+1}q_n &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + p_{n+1}\sigma_{n+5}\tau_{n-1} \\ \sigma_{n+6}\tau_{n-1} &= \sigma_{n+4}\tau_{n+1} + a_1 q_{n+1} \end{aligned} \tag{2.113}$$

This example follows the working in [10].

Chapter 3

The cluster map of type A_{2N}

The examples above showed that the singularity confinement patterns of type A_2 and A_4 deformed integrable maps allow us to define rational maps π such that $\psi = \tilde{\varphi}\pi$ is a cluster map on the space of tau functions.

Therefore it is natural to ask whether Laurentification can be successfully applied to the defomed integrable maps associated with general type A. We are able to answer this positively for cluster algebras of type A_{2N}.

3.1 Initial analysis

In this section, we use the same argument from [44] to prove that the cluster map associated with cluster algebra of type A_{2N} is Liouville integrable map.

The exchange matrix associated to a linear orientation of a Dynkin diagram of type A_{2N} can be expressed as

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.1}$$

This exchange matrix is mutation periodic with periodicity $2N$ under the particular sequence of mutations $\mu_{2N}\mu_{2N-1}\cdots\mu_2\mu_1$. Let $\varphi_{A_{2N}}$ be the associated birational map; then $\varphi_{A_{2N}}(B) = B$. On the cluster, $\varphi_{A_{2N}} : \mathbf{x} \to \mathbf{x}'$ gives the following exchange relations

$$\begin{aligned} x'_1 x_1 &= 1 + x_2 \\ x'_2 x_2 &= 1 + x'_1 x_3 \\ x'_3 x_3 &= 1 + x'_2 x_4 \\ &\vdots \\ x'_{2N-1} x_{2N-1} &= 1 + x'_{2N-2} x_{2N} \\ x'_{2N} x_{2N} &= 1 + x'_{2N-1} \end{aligned} \tag{3.2}$$

The matrix (3.1) is mutation equivalent to a matrix which represents a bipartite graph. Therefore due to Zamolodchikov periodicity (Theorem 2.1.33), the map φ is periodic with period $2N+3$, i.e. $(\varphi_{A_{2N}})^{2N+3}(\mathbf{x}) = \mathbf{x}$. The sequence of cluster variables, which are generated by $(\varphi^*_{A_{2N}})^i(x_1) = L_i$, are given as follows,

$$\begin{aligned} &L_0 = x_1, \quad L_1 = \frac{1+x_2}{x_1}, \quad L_2 = \frac{x_1+x_3}{x_2}, \; \cdots, \; L_{2N-1} = \frac{x_{2N-2}+x_{2N}}{x_{2N-1}}, \quad L_{2N} = \frac{1+x_{2N-1}}{x_{2N}}, \\ &L_{2N+1} = x_{2N}, \\ &L_{2N+2} = \frac{\prod_{i=1}^{2N-1} x_i + \prod_{i=1}^{2N-2} x_i + \Big(\prod_{i=1}^{2N-3} x_i + \Big(\prod_{i=1}^{2N-4} x_i + \cdots (x_1x_2 + (x_1 + (1+x_2)x_3)x_4)\cdots\Big)x_{2N-1}\Big)x_{2N}}{\prod_{i=1}^{2N} x_i} \end{aligned} \tag{3.3}$$

Let P be the standard Poisson structure for type A_{2N}, given by

$$P_{ij} = (B^{-1})_{ij} = [i-j]_+ \sum_{l=1}^{N}\sum_{k=l}^{N} \delta_{j,2l-1}\delta_{i,2k} - [j-i]_+ \sum_{l=1}^{N}\sum_{k=l}^{N} \delta_{i,2l-1}\delta_{j,2k}$$

where $[i-j]_+ = \max(i-j,0)$. Then the associated Poisson bracket is

$$\{x_i, x_j\} = P_{ij} x_i x_j \tag{3.4}$$

which simplifies to give us the following log-canonical Poisson bracket relations:

$$\{x_{2r-1}, x_{2s}\} = x_{2r-1} x_{2s} \tag{3.5}$$

Using the above, these relations give us the following Poisson bracket relations on the space of functions L_i:

$$\begin{aligned} \{L_0, L_1\} &= L_0 L_1 - 1 \\ \{L_0, L_{2j}\} &= -L_0 L_{2j} \quad \text{for } 1 \leq j \leq N \\ \{L_0, L_{2j+1}\} &= L_0 L_{2j+1} \quad \text{for } 1 \leq j \leq N \end{aligned} \tag{3.6}$$

To find further relations, we can use one of the properties of the cluster map, namely, preservation of the Poisson bracket. For example, let us consider the Poisson bracket $\{L_0, L_1\}$. By the relation above, L_0 and L_1 satisfy

$$\{L_0, L_1\} \circ \varphi_{A_{2N}} = \left\{\varphi^*_{A_{2N}}(L_0), \varphi^*_{A_{2N}}(L_1)\right\}$$

The left-hand side of the equation can be written as

$$\{L_0, L_1\} \circ \varphi = L_1 L_2 - 1$$

Since $\varphi^*_{A_{2N}}$ will shift the index i of L_i by 1, the right hand side is $\{L_1, L_2\}$. Thus altogether, we obtain the bracket relation

$$\{L_1, L_2\} = L_1 L_2 - 1$$

Arguing in this way, the Poisson brackets between the L_i are given by

$$\begin{aligned} \{L_i, L_{i+1}\} &= L_i L_{i+1} - 1 && \text{for } i \geq 0 \\ \{L_i, L_j\} &= (-1)^{i+j+1} L_i L_j && \text{for } i+1 < j \end{aligned}$$

Combining the Poisson relations above, we can represent the Poisson bracket $\{L_i, L_j\}$ as the sum of the two homogenous terms, $\mathbf{P}^{(2)} + \mathbf{P}^{(0)}$. In fact, the two terms give rise individually to Poisson brackets, $\{\cdot,\cdot\}_2$ and $\{\cdot,\cdot\}_0$ as the Jacobi identities are homogeneous.

Lemma 3.1.1. *For $i = 0, \ldots, 2N+2$, the set of functions L_i (3.3) generate a Poisson subalgebra with the brackets*

$$\begin{gathered} \{L_i, L_j\} = \{L_i, L_j\}_2 + \{L_i, L_j\}_0 \\ \mathbf{P}^{(2)} + \mathbf{P}^{(0)} \end{gathered} \tag{3.7}$$

The corresponding $\{\cdot,\cdot\}_2$ and $\{\cdot,\cdot\}_0$ are Poisson brackets, given by

$$\{L_i, L_j\}_2 = C_{ij}^{(2)} L_i L_j, \quad \{L_i, L_j\}_0 = C_{ij}^{(0)}$$

where the skew-symmetric matrices $C^{(2)}$ and $C^{(0)}$ are Toeplitz matrices with their top rows given by $C_{1k}^{(2)} = (0, 1, -1, \ldots, 1, -1)$ and $C_{1k}^{(0)} = (0, -1, 0, \ldots, 0, 1)$ respectively.

Proof. Immediate from the above relations. □

Note that the linear combination of the brackets, $\{\cdot,\cdot\}_2$ and $\{\cdot,\cdot\}_0$, satisfy the properties of Poisson bracket, including Jacobian identities i.e.

$$\begin{aligned} &\big\{L_i, \{L_j, L_k\}_0\big\}_0 + \{L_j, \{L_k, L_i\}_0\}_0 + \big\{L_k, \{L_i, L_j\}_0\big\}_0 \\ &+ \big\{L_i, \{L_j, L_k\}_2\big\}_2 + \{L_j, \{L_k, L_i\}_2\}_2 + \big\{L_k, \{L_i, L_j\}_2\big\}_2 \\ &+ \big\{L_i, \{L_j, L_k\}_0\big\}_2 + \{L_j, \{L_k, L_i\}_0\}_2 + \big\{L_k, \{L_i, L_j\}_0\big\}_2 \\ &+ \big\{L_i, \{L_j, L_k\}_2\big\}_0 + \{L_j, \{L_k, L_i\}_2\}_0 + \big\{L_k, \{L_i, L_j\}_2\big\}_0 \\ &= 0 \end{aligned}$$

Thus the Poisson bracket on the space of functions generated by the L_i can be split into two distinguished Poisson brackets. Moreover, the property of the Poisson bracket holds for the linear combination of the Poisson brackets, which is a compatibility condition for the system to be *bi-Hamiltonian*, introduced by Magri in [45]. In accordance with [45, 46], the bi-Hamiltonian structure leads to the existence of Poisson-commuting first integrals I_j which can be constructed by the so-called *Magri–Lenard scheme*, that is, recursion relations

$$\mathbf{P}^{(0)} \mathrm{d}I_{n+1} = \mathbf{P}^{(2)} \mathrm{d}I_n. \tag{3.8}$$

Since $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_2$ are both degenerate and thus there exist Casimir functions for each Poisson bracket. Starting from the Casimir function I_1 for $\mathbf{P}^{(0)}$, the relation

(3.8) yields the sequence of relations

$$\begin{aligned} \mathbf{P}_0 \boldsymbol{\nabla} I_1 &= 0 \\ \mathbf{P}_0 \boldsymbol{\nabla} I_k &= \mathbf{P}_2 \boldsymbol{\nabla} I_{k-1} \quad \text{for } 2 \leq k \leq N \\ \mathbf{P}_2 \boldsymbol{\nabla} I_N &= 0 \end{aligned} \tag{3.9}$$

which ends at the Casimir function I_N of $\mathbf{P}_2$. One can show that following invariant functions I_1 and I_N expressed as

$$I_1 = \sum_j L_j, \quad I_N = \prod_j L_j, \tag{3.10}$$

are Casimir functions associated with the Poisson structures $\mathbf{P}_0$ and $\mathbf{P}_2$ respectively. Thus by using these Casimir functions above, the relation (3.9) provide N invariant functions I_k (of degree $2k+1$) associated with the cluster map $\varphi_{A_{2N}}$. However, we need to confirm the consistency of system of equations (compatibility) in (3.9). To show this, we need to verify that I_1 satisfies $\boldsymbol{\nabla} I_1 \mathbf{P}_2 \boldsymbol{\nabla} I_{k-1} = 0$. We use the argument mentioned from [44].

Lemma 3.1.2. *The Poisson brackets* $\{\cdot,\cdot\}_0$ *and* $\{\cdot,\cdot\}_2$ *satisfy the relation below*

$$\{I_i, I_j\}_0 = \{I_i, I_{j-1}\}_2 = \{I_{i+1}, I_{j-1}\}_0 \tag{3.11}$$

Proof. Starting from $\{I_i, I_j\}_0$, we find

$$\{I_i, I_j\}_0 = (\boldsymbol{\nabla} I_i)\mathbf{P}^{(0)}(\boldsymbol{\nabla} I_j) = (\boldsymbol{\nabla} I_i)\mathbf{P}^{(2)}(\boldsymbol{\nabla} I_{j-1}) = \{I_i, I_{j-1}\}_2$$

where we used the relation $\mathbf{P}^{(0)}\boldsymbol{\nabla} I_k = \mathbf{P}^{(2)}\boldsymbol{\nabla} I_{k-1}$ in (3.9). Thus if we apply the relations subsequently then

$$\{I_i, I_{j-1}\}_2 = (\boldsymbol{\nabla} I_i)\mathbf{P}^{(2)}(\boldsymbol{\nabla} I_{j-1}) = (\boldsymbol{\nabla} I_{i+1})\mathbf{P}^{(0)}(\boldsymbol{\nabla} I_{j-1}) = \{I_{i+1}, I_{j-1}\}_0$$

□

Lemma 3.1.3. *Casimir function* I_1 *satisfies the following Poisson bracket relation,*

$$\{I_1, I_{k-1}\}_2 = 0 \tag{3.12}$$

for $2 \leq k \leq N$.

Proof. We use the same reasoning in the proof of Lemma 3.1.2 to obtain the result above. We apply the relation (3.11) step by step to reach the particular Poisson bracket, which vanishes due to the anti-symmetric property of the bracket i.e.

$$\{I_1, I_{k-1}\}_2 = \{I_2, I_{k-1}\}_0 = \{I_2, I_{k-2}\}_2 = \cdots = \{I_m, I_m\}_s = 0$$

for some $1 \leq m \leq M$ and $s \in \{0, 2\}$ □

The Poisson bracket (3.12) is $\nabla I_1 \mathbf{P}_2 \nabla I_{k-1} = 0$. Thus Lemma 3.1.3 implies the consistency of recurrence in (3.9).

As mentioned above, by general theory, if the relations (3.9) hold, then the associated first integrals are in involution with respect to the Poisson brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_2$, which we record in following.

Lemma 3.1.4. *Let I_i be the first integrals which are obtained from the sequence of relations* (3.9)*. Then*

$$\{I_i, I_j\}_2 = 0 = \{I_i, I_j\}_0 \tag{3.13}$$

for any i, j.

Proof. We use the same argument used in the proof of Lemma 3.1.3 to show their involution property. Starting from $\{I_i, I_j\}_0$ for $i < j$, we find that

$$\{I_i, I_j\}_0 = \{I_i, I_{j-1}\}_2 = \{I_{i+1}, I_{j-1}\}_0 = \cdots = \{I_m, I_m\}_s = 0.$$

One can show that $\{I_i, I_j\}_2 = 0$ by taking the same steps and hence first integrals I_m are Poisson-commuting with respect to $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_2$. □

This immediately yields the integrability of the cluster map $\varphi_{A_{2N}}$.

Theorem 3.1.5. *The periodic cluster map $\varphi_{A_{2N}}$ associated with type A_{2N} is Liouville integrable.* □

It is important to remark that the same method above has been used in [44]. In this paper, Fordy studied the period 1 cluster map associated with affine type $A_{2N}^{(1)}$ and found the existence of functions periodic under the map. It turns out that the

Poisson structures of those functions take a form which is similar to (3.7). Thus by the Magri–Lenard scheme, the integrability of the map was shown. Soon after, in [7], Fordy and Hone gave explicit formulæ for Poisson-commuting first integrals corresponding to the map:

Theorem 3.1.6 ([7])**.** *In the case of affine type $A_{2N}^{(1)}$, there exists periodic functions $J_{n+p} = J_n$, satisfying Poisson bracket relation*

$$\begin{aligned}\{J_i, J_j\} &= C_{ij}^{(2)} J_i J_j + C_{ij}^{(0)} \\ &= P^{(2)} + P^{(0)}\end{aligned}$$

where $C^{(2)}$ and $C^{(0)}$ are Toeplitz matrices (shown in Lemma 3.1.1). The corresponding Casimir function $\mathcal{K}$ with respect to the Poisson bracket above given by

$$(P^{(2)} + P^{(0)})\boldsymbol{\nabla}\mathcal{K} = 0 \tag{3.14}$$

such that there exist expressions

$$\mathcal{K}_n^{(p+3)} = \mathcal{R}^{(p)}\mathcal{K}^{(p+1)} \quad \text{with } \mathcal{K}^{(2)} = J_n \quad \mathcal{K}^{(3)} = J_n J_{n+1} - 2 \tag{3.15}$$

where $p = 2N - 1$ represents is the period of the function J_n, so $J_n = J_{n+p}$, and

$$\mathcal{R}^{(p)} = -1 + J_{n+p}J_{n+p+1} - J_{n+p}\frac{\partial}{\partial J_n} - J_{n+p+1}\frac{\partial}{\partial J_{n+p-1}} + J_{n+p}J_{n+p+1}\frac{\partial^2}{\partial J_n \partial J_{n+p-1}} \tag{3.16}$$

Moreover, the Poisson matrices are compatible and therefore one can define a bi-Hamiltonian ladder such that $\mathcal{K}$ can be written in the following form:

$$\mathcal{K} = \sum_{j=1}(-1)^j h_j \tag{3.17}$$

with h_j is a homogeneous polynomial of degree $2j - 1$. The h_j are first integrals which are in involution with respect to the Poisson bracket.

Note that when we substitute Casimir function (3.17) directly into (3.14), we can decouple the expression into the system of equations (3.9).

Now we understand that the (undeformed) maps $\varphi_{\mathrm{A}_{2N}}$ of even type A are integrable cluster maps. Next we consider the deformation of the corresponding cluster map. As our method will be inductive, in the next section, we will consider first in detail how the A_6 case may be related that of A_4, which we have seen earlier.

3.2 The periodic type A_6 cluster map

In this section, we consider the deformation of the periodic cluster map of type A_6. The exchange matrix of (linearly oriented) type A_6 is given by

$$B_{\mathrm{A}_6} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.18}$$

The corresponding matrix possesses period 6 with respect to a sequence of cluster mutations:

$$\mu_6\mu_5\mu_4\mu_3\mu_2\mu_1(B) = B$$

Given the initial cluster $\mathbf{x}_0 = (x_1, x_2, x_3, x_4, x_5, x_6)$, let us denote by φ_{A_6} the composition of mutations above i.e. $\varphi_{\mathrm{A}_6} = \mu_6\mu_5\mu_4\mu_3\mu_2\mu_1$. Once more, with $f_k(x) = b_k M_k^- + a_k M_k^+$ in (2.82), we define the modified mutations $\tilde{\mu}_k(x_k) = x_k^{-1}\big(b_k M_k^- + a_k M_k^+\big)$ for $k = 1, \ldots, 6$, which yields deformed map $\tilde{\varphi}_{\mathrm{A}_6} = \tilde{\mu}_6\tilde{\mu}_5\tilde{\mu}_4\tilde{\mu}_3\tilde{\mu}_2\tilde{\mu}_1$ equivalent to the following exchange relations,

$$\begin{aligned} \tilde{\mu}_1 :&\quad x_1 x_1' = b_1 + a_1 x_2 \\ \tilde{\mu}_i :&\quad x_i x_i' = 1 + x_{i-1}' x_{i+1} \quad (2 \le i \le 4) \\ \tilde{\mu}_5 :&\quad x_5 x_5' = b_5 + a_5 x_4' x_6 \\ \tilde{\mu}_6 :&\quad x_6 x_6' = b_6 + a_6 x_5' \end{aligned} \tag{3.19}$$

where the parameters b_i and a_i for $i = 2, 3, 4$ are rescaled to 1 via $x_i \to \lambda_i x_i$ $\lambda_i \in \mathbb{C}^*$ for each cluster variable.

For integrability of the deformed map, we know from Section 3.1, that the first integrals of the original cluster map φ_{A_6} can be found by the relation (3.9). By solving the equations, the corresponding first integrals are Laurent polynomials which are given by following:

$$\begin{aligned} I_1 &= \sum_{j=0}^{8} L_j, \\ I_2 &= \prod_{j=0}^{8} L_j, \\ I_3 &= \sum_{j=0}^{8} L_j L_{j+1}(L_{j+2} + L_{j+4} + L_{j+6}) + \sum_{j=0}^{2} L_j L_{j+3} L_{j+6} \end{aligned} \tag{3.20}$$

where $L_i = (\varphi_{\mathrm{A}_6}*)^i(x_1)$. Recall, as in the Example 2.4.4, the deformed first integrals are found by modifying each term by attaching an arbitrary coefficient and imposing the invariance condition $\tilde{\varphi}^*_{\mathrm{A}_6}(K) = K$ which provides the necessary and sufficient condition.

Adopting the same approach, we can determine the conditions by using computer algebra (MAPLE): we first rewrite the expression $\tilde{\varphi}^*_{\mathrm{A}_6}(K) = K$ as identity between two polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$ and then comparing the coefficients at each degree produces the system of equations in the coefficient. Then we find that the system has a solution if and only if the parameters are fixed as $b_1 = 1 = b_5$ and $b_6 a_5^2 = 1$. Upon the calculations, we obtain the following three independent rational functions:

$$\tilde{I}_1 = \frac{1}{a_5^3 a_6 x_1 x_2 x_3 x_4 x_5 x_6} \begin{pmatrix} a_1 a_5^2 a_6 x_1 x_2 x_3 x_4 + a_1 x_1 x_2 x_3 x_4 x_5 + a_1 a_5^4 a_6^2 x_1 x_2 x_3 x_4 x_5 \\ + a_1 a_5^2 a_6 x_1 x_2 x_3 x_4 x_5^2 + a_1 a_5^3 a_6 x_1 x_2 x_3 x_6 + a_1 a_5^3 a_6 x_1 x_2 x_3 x_4^2 x_6 \\ + a_1 a_5^3 a_6 x_1 x_2 x_5 x_6 + a_1 a_5^3 a_6 x_1 x_2 x_3^2 x_5 x_6 + a_1 a_5^3 a_6 x_1 x_4 x_5 x_6 \\ + a_1 a_5^3 a_6 x_1 x_2^2 x_4 x_5 x_6 + a_1 a_5^3 a_6 x_3 x_4 x_5 x_6 + a_1 a_5^3 a_6 x_1^2 x_3 x_4 x_5 x_6 \\ + a_5^3 a_6 x_2 x_3 x_4 x_5 x_6 + a_1^2 a_5^3 a_6 x_2 x_3 x_4 x_5 x_6 + a_5^3 a_6 x_1^2 x_2 x_3 x_4 x_5 x_6 \\ + a_1 a_5^3 a_6 x_2^2 x_3 x_4 x_5 x_6 + a_1 a_5^3 a_6 x_1 x_3^2 x_4 x_5 x_6 + a_1 a_5^3 a_6 x_1 x_2 x_4^2 x_5 x_6 \\ + a_1 a_5^3 a_6 x_1 x_2 x_3 x_5^2 x_6 + a_1 a_5^4 a_6 x_1 x_2 x_3 x_4 x_6^2 + a_1 a_5^2 x_1 x_2 x_3 x_4 x_5 x_6^2 \end{pmatrix}$$

$$\tilde{I}_2 = (a_1 + x_2)\left(\frac{x_1 + x_3}{x_2}\right)\left(\frac{x_2 + x_4}{x_3}\right)\left(\frac{x_3 + x_5}{x_4}\right)\left(\frac{x_4 + a_5 x_6}{x_5}\right)\left(\frac{x_5 + a_5^2 a_6}{a_5}\right) \cdot \left(\frac{\begin{matrix} a_5^2 a_6 x_1 x_2 x_3 x_4 x_5 + a_1 a_5 x_2 x_3 x_4 x_5 x_6 + a_5 x_1 x_2 x_3 x_6 \\ + a_5 x_1 x_2 x_5 x_6 + a_5 x_1 x_4 x_5 x_6 + a_5 x_3 x_4 x_5 x_6 + x_1 x_2 x_3 x_4 \end{matrix}}{a_5 x_1 x_2 x_3 x_4 x_5 x_6}\right)$$

$$\tilde{I}_3 = \frac{P}{a_1 x_2^2 x_4^2 x_5^2 a_5^3 x_6 x_3^2 x_1 a_6} \tag{3.21}$$

where

$$
\begin{aligned}
P = {} & x_2x_3^2x_6a_6^3(a_1(x_2+x_4)x_3+x_1(a_1+x_2)x_4+a_1x_2x_5)x_5x_4x_1a_5^7+x_3((a_1(x_2+x_4)x_3^2 \\
& +((x_2x_1+a_1(x_1+x_5))x_4+x_2^2x_5)x_3+x_1x_5(x_2+x_4)(a_1+x_2))x_2x_5x_4^2x_1a_6 \\
& +x_6^2((x_5(a_1+x_2)(a_1x_2+1)x_4+a_1x_2x_1)(x_2+x_4)x_3^2+(((a_1^2+2)x_2+2a_1)x_5x_1x_4^2 \\
& +x_2(a_1x_5(x_1+x_5)x_2^2+((a_1^2+1)x_5^2+x_1^2)x_2+a_1(x_1+x_5)^2)x_4+2a_1x_1x_2^2x_5)x_3 \\
& +((a_1x_2^2x_5+a_1x_1+x_2x_1)x_4+a_1x_2x_5)x_5x_1(x_2+x_4)))a_6^2a_5^6 \\
& +x_6a_6^2x_4((x_2+x_4)(x_5(a_1+x_2)(a_1x_2+1)x_4+(x_2x_5^2+2a_1)x_2x_1)x_3^3 \\
& +((a_1x_2^2x_5+(a_1^2x_1+x_5a_1^2+2x_1+x_5)x_2+(2x_1+x_5)a_1)x_5x_4^2+x_2(a_1x_2^2x_5+2x_2x_1 \\
& +a_1(x_5^3+2x_1+3x_5))x_1x_4+x_2^2((x_1x_5+x_5^2+1)x_2+a_1(x_1x_5+2))x_5x_1)x_3^2 \\
& +(((x_5a_1^2+x_1+2x_5)x_2+a_1(x_1+2x_5))x_4 \\
& +x_2((x_1x_5^2+x_1+x_5)x_2+a_1x_1(x_5^2+1)))x_5x_1(x_2+x_4)x_3+x_1^2x_5^2(x_2+x_4)^2(a_1+x_2))a_5^5 \\
& +x_3a_6(x_2(a_1(x_5^2+1)(x_2+x_4)x_3^2+((x_5^2+1)(x_2x_1+a_1(x_1+x_5))x_4 \\
& +(x_5^3+x_5)x_2^2+a_1^2x_2x_5^2)x_3+x_1x_5(x_5^2+1)(x_2+x_4)(a_1+x_2))x_4^2x_1a_6 \\
& +x_6^2x_5((x_5(a_1+x_2)(a_1x_2+1)x_4+a_1x_2x_1)(x_2+x_4)x_3^2 \\
& +(x_1(x_2^2a_1^2+(a_1^2+2)x_5x_2+2a_1x_5)x_4^2 \\
& +x_2(a_1x_5(x_1+x_5)x_2^2+((a_1^2+1)x_5^2+x_1^2)x_2+a_1(x_1+x_5)^2)x_4+2a_1x_1x_2^2x_5)x_3 \\
& +((a_1x_2^2x_5+a_1x_1+x_2x_1)x_4+a_1x_2x_5)x_5x_1(x_2+x_4)))a_5^4+(a_1x_1(x_2+x_4)(a_1x_5+2)x_3^2 \\
& +(a_1x_1x_2(a_1+x_5)x_4^2+(a_1x_5(a_1+x_5)x_2+x_5^2+a_1(x_1^2+1)x_5+2x_1^2)(a_1+x_2)x_4 \\
& +a_1x_1x_2(a_1x_5^2+a_1+3x_5))x_3 \\
& +a_1x_1x_5(x_2+x_4)(x_2x_4+1)(a_1+x_5))x_2x_3x_6a_6x_5x_4a_5^3+x_2x_3((a_1(x_2+x_4)x_3^2 \\
& +((x_2x_1+a_1(x_1+x_5))x_4+x_2(x_2x_5+a_1^2(x_5^2+1)))x_3+x_1x_5(x_2+x_4)(a_1+x_2))a_6 \\
& +a_1^2x_2x_3x_5x_6^2)x_5x_4^2x_1a_5^2+a_1^2x_2^2x_4^2x_5^2x_3^2x_1
\end{aligned}
$$

Then computer-aided calculation (by using MAPLE) enables us to verify the following.

Theorem 3.2.1. *The conditions $b_1 = 1 = b_5$ and $b_6a_5^2 = 1$ on the parameters are necessary and sufficient conditions for $\tilde{I}_1, \tilde{I}_2$ and $\tilde{I}_3$ to be first integrals that are preserved by the type A_6 deformed map, i.e. $\tilde{\varphi}^*_{\mathrm{A}_6}(\tilde{I}_i) = \tilde{I}_i$, and are in involution with respect to the Poisson bracket. Hence $\tilde{\varphi}_{\mathrm{A}_6}$ is a Liouville integrable map whenever these conditions on the parameters hold.* □

The deformed map $\tilde{\varphi}_{\mathrm{A}_6}$ is a Liouville integrable symplectic map, however, it is no longer a cluster map as the generated variables stop being Laurent polynomial after some iterations. Therefore once again, we look to lift the deformed map to a higher dimensional space by Laurentification. Following the same process as in the previous examples, we study the singularity structures of the deformed map $\tilde{\varphi}_{\mathrm{A}_6}$ by observing the prime factorization of iterates defined over $\mathbb{Q}$. Then we observe the following singularity patterns:

$$\begin{aligned}
(1):&\quad \ldots \to (\epsilon, R, R, R, R, R) \to (\epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}) \to (R, R, R, R, R, \epsilon)\\
(2):&\quad \ldots \to (R, R, R, R, R, \epsilon) \to (R, R, R, R, R, \epsilon^{-1}) \to (R, R, R, R, \epsilon^{-1}, R)\\
&\quad \to (R, R, R, \epsilon^{-1}, R, R) \to (R, R, \epsilon^{-1}, R, R, R) \to (R, \epsilon^{-1}, R, R, R, R)\\
&\quad \to (\epsilon^{-1}, R, R, R, R, R) \to (\epsilon, R, R, R, R, R) \to \ldots\\
(3):&\quad \ldots \to (R, \epsilon, R, R, R, R) \to \ldots\\
(4):&\quad \ldots \to (R, R, \epsilon, R, R, R) \to \ldots\\
(5):&\quad \ldots \to (R, R, R, \epsilon, R, R) \to \ldots\\
(6):&\quad \ldots \to (R, R, R, R, \epsilon, R) \to \ldots
\end{aligned} \tag{3.22}$$

By introducing the tau-functions τ_n, σ_n, p_n, r_n, q_n, w_n corresponding to ϵ in each pattern, one can construct the rational map $\pi_{A_6} : \mathbb{C}^{15} \to \mathbb{C}^6$ in the same way as (2.111) in Example 2.5.5.

Definition 3.2.2. *Given the conditions $b_5 = 1 = b_1$ and $b_6 a_5^2 = 1$, the singularity confinement patterns in* (3.22) *enable us to define a rational map π_{A_6}, which is identified as the dependent variable transformation*

$$\begin{gathered}
x_{1,n} = \frac{\sigma_n \tau_{n+1}}{\sigma_{n+1}\tau_n} \quad x_{2,n} = \frac{p_n}{\sigma_{n+2}\tau_n} \quad x_{3,n} = \frac{r_n}{\sigma_{n+3}\tau_n} \quad x_{4,n} = \frac{q_n}{\sigma_{n+4}\tau_n}\\
x_{5,n} = \frac{w_n}{\sigma_{n+5}\tau_n} \quad x_{6,n} = \frac{\sigma_{n+7}\tau_{n-1}}{\sigma_{n+6}\tau_n}
\end{gathered} \tag{3.23}$$

where τ and σ represent patterns (1) and (2) respectively. The variables p, r, q, w correspond to the patterns (3)-(6)

If we substitute these directly into the components (2.91) of $\tilde{\varphi}_{\mathrm{A}_6}$ with the conditions $b_6 a_5^2 = 1$, $b_i = 1 = a_j$ for $i = 1, \ldots, 5$ and $j = 2, 3, 4$ one obtains the following system of equations:

$$
\begin{aligned}
\tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1 p_n \\
p_{n+1}p_n &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + r_n\sigma_{n+1}\tau_{n+2} \\
r_{n+1}r_n &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + q_n p_{n+1} \\
q_{n+1}q_n &= \sigma_{n+5}\sigma_{n+4}\tau_n\tau_{n+1} + w_n r_{n+1} \\
w_{n+1}w_n &= \sigma_{n+6}\sigma_{n+5}\tau_n\tau_{n+1} + a_5\sigma_{n+7}q_{n+1}\tau_{n-1} \\
\sigma_{n+8}\tau_{n-1} &= b_6\sigma_{n+6}\tau_{n+1} + a_6 w_{n+1}
\end{aligned}
\tag{3.24}
$$

Once again, we begin by presenting the initial data as

$$
\begin{aligned}
&(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, , \tilde{x}_{14}, \tilde{x}_{15}) \\
&\qquad = (q_0, w_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, p_0, r_0)
\end{aligned}
$$

Then the corresponding exchange matrix can be found by reading off the coefficients of the new pre-symplectic form, $\pi^*_{\mathrm{A}_6}\omega_{\mathrm{A}_6}$. This matrix is given by

$$
\tilde{B}_{\mathrm{A}_6} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
\tag{3.25}
$$

If this case is similar to that of type A_2 and A_4, then one would expect to be able to find an extended exchange matrix which contains entries corresponding to frozen variables a_1, a_5 and a_6. Then it should be invariant under a certain sequence of mutation, generating cluster variables expressed by the relations (3.24). However, the first few iterations of the recurrence (3.24) give rise to the variable whose denominator is given by

$$a_5^2\sigma_0\tau_{-1}p_0r_0q_0w_0$$

which shows that the denominator contains the frozen variable.

In cluster mutation, the frozen variables are only apparent in the numerator of the fraction in a Laurent expression. This indicates with the condition $b_6a_5^2 = 1$, we cannot generate cluster variables corresponding to the recurrence (3.24). Thus to achieve our goal, one is required to put further constraints on the parameters. The simplest choice is to fix $b_6 = 1 = a_5$, which satisfies $b_6a_5^2 = 1$. By adjusting the parameters, this leads to the following theorem.

Theorem 3.2.3. *The sequence mutations in a cluster algebra defined by* (3.25) *with two frozen variables* a_1, a_6 *generates the sequences of tau functions* (σ_n), (p_n), (r_n), (w_n), (q_n), (τ_n) *satisfying*

$$\begin{aligned}
\tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1p_n \\
p_{n+1}p_n &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + r_n\sigma_{n+1}\tau_{n+2} \\
r_{n+1}r_n &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + q_np_{n+1} \\
q_{n+1}q_n &= \sigma_{n+5}\sigma_{n+4}\tau_n\tau_{n+1} + w_nr_{n+1} \\
w_{n+1}w_n &= \sigma_{n+6}\sigma_{n+5}\tau_n\tau_{n+1} + \sigma_{n+7}q_{n+1}\tau_{n-1} \\
\sigma_{n+8}\tau_{n-1} &= \sigma_{n+6}\tau_{n+1} + a_6w_{n+1}
\end{aligned} \tag{3.26}$$

which are elements of the Laurent polynomial ring

$$\mathbb{Z}_{>0}\big[a_1, a_6, \sigma_0^{\pm}, \sigma_1^{\pm}, \sigma_2^{\pm}, \sigma_3^{\pm}, \sigma_4^{\pm}, \sigma_5^{\pm}, \sigma_6^{\pm}, \sigma_7^{\pm}, \tau_{-1}^{\pm}, \tau_0^{\pm}, \tau_1^{\pm}, p_0^{\pm}, r_0^{\pm}, w_0^{\pm}, q_0^{\pm}\big]$$

Proof. Let us extend the initial data by inserting the frozen variables:

$$\begin{aligned}
&(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14}, \tilde{x}_{15}, \tilde{x}_{16}, \tilde{x}_{17}) \\
&= (q_0, w_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, p_0, r_0, a_1, a_6)
\end{aligned}$$

We add two new rows, whose entries correspond to the frozen variables, to the exchange matrix (3.25) to define the extended exchange matrix

$$\hat{B}_{A_6} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \tag{3.27}$$

which can be depicted by the quiver in Figure 3.1.

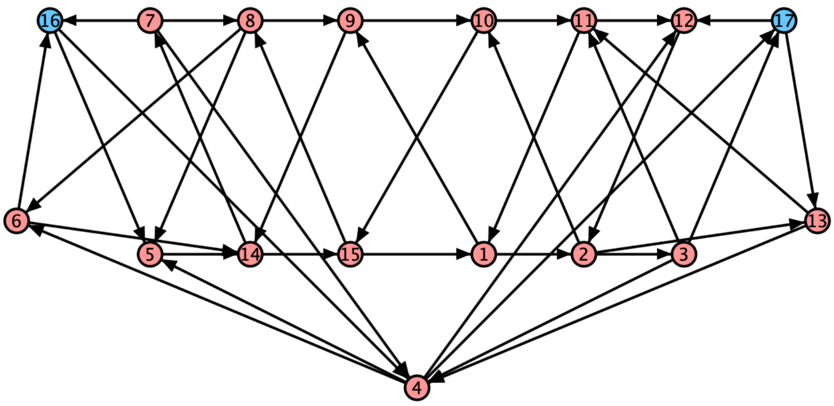


Figure 3.1: Quiver corresponding to $\hat{B}_{\mathrm{A}_6}$

We then apply the mutation sequence $\hat{\mu}_3\hat{\mu}_2\hat{\mu}_1\hat{\mu}_{15}\hat{\mu}_{14}\hat{\mu}_6$ to this quiver. If we arrange the nodes and edges of the mutated quiver as per Figure 3.2, then one can see that this is identical to the initial quiver except that specific labels are shifted by 1. Thus the block mutation $\hat{\mu}_3\hat{\mu}_2\hat{\mu}_1\hat{\mu}_{15}\hat{\mu}_{14}\hat{\mu}_6$ is equivalent to permuting the labels of the nodes in Q_6 and hence

$$\hat{\mu}_3\hat{\mu}_2\hat{\mu}_1\hat{\mu}_{15}\hat{\mu}_{14}\hat{\mu}_6(Q_6) = \rho_6 Q_6 \tag{3.28}$$

where ρ_6 is the permutation

$$\begin{aligned}\rho_6 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 1 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 3 & 14 & 15 & 16 & 17 \end{pmatrix} \\ &= (3,4,5,6,7,8,9,10,11,12,13)\end{aligned} \tag{3.29}$$

Thus if we apply the mutations in the same order as previously, once again the structure of the quiver remains the same except the labels of the nodes are shifted. When we take the inverse of the permutation on each side of (3.28), then we have

$$\psi_{\mathrm{A}_6} := \rho_6^{-1}\hat{\mu}_3\hat{\mu}_2\hat{\mu}_1\hat{\mu}_{15}\hat{\mu}_{14}\hat{\mu}_6(Q_{\mathrm{A}_6}) = Q_{\mathrm{A}_6} \tag{3.30}$$

and it is clear that the composition of mutations on the left-hand side is a cluster map. The first iteration of the map gives rise to a new seed, which contains cluster variables that are expressed by (3.26) with $n = 0$:

$$\begin{aligned}\psi_{A_6} &: (q_0, w_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, p_0, r_0, a_1, a_6) \\ &\to (q_1, w_1, \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, p_1, r_1, a_1, a_6)\end{aligned} \tag{3.31}$$

One can see that a single iteration of the map on the initial values is equivalent to shifting the subscript of the tau functions q, w, τ, σ by 1. Therefore successive applying the map ψ_{A_6} will induce a series of seeds that consist of the cluster variables

$$(q_n, w_n, \tau_{n-1}, \tau_n, \tau_{n+1}, \sigma_n, \sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}, \sigma_{n+4}, \sigma_{n+5}, \sigma_{n+6}, \sigma_{n+7}, p_n, r_n, a_1, a_6) \tag{3.32}$$

for $n \in \mathbb{Z}$, satisfying (3.26). Therefore every tau function, generated by the system of recurrences in (3.26), can be obtained by applying ψ_{A_6} repeatedly. Hence $\tau_n, \sigma_n, p_n, w_n, q_n, r_n$ are in the Laurent polynomial ring by the Laurent phenomenon.

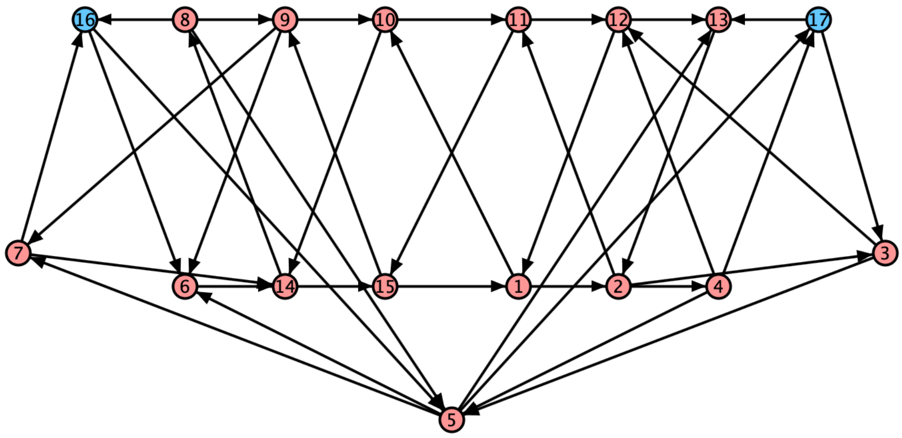


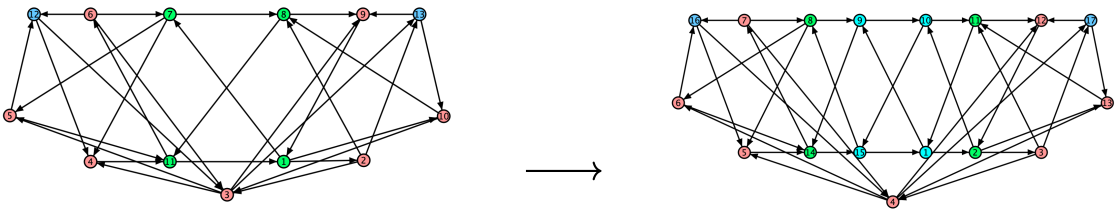
Figure 3.2: Mutated quiver $Q'_{\mathrm{A}_6} = \mu_3\mu_2\mu_1\mu_{15}\mu_{14}\mu_6(Q_{\mathrm{A}_6})$. It has the same structure as Figure 3.1 with permuted labellings.

□

3.3 Local expansion

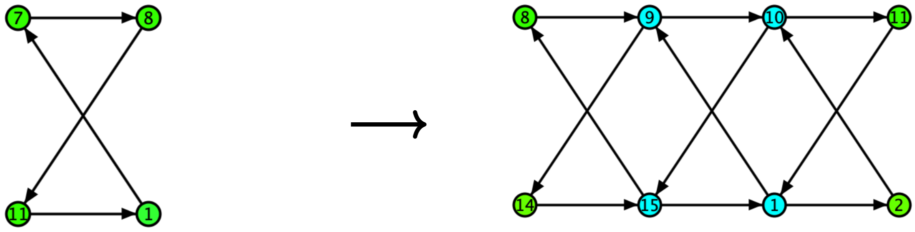
To investigate generalizing the integrable cluster map to arbitrary even rank, we begin by exploring the relation between deformed quivers/exchange matrices of type A_4 and type A_6. Recall that in Example 2.5.5, we showed that Laurentification of deformed type A_4 lead us to new cluster algebra defined by pair of the initial cluster $(q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, p_0, a_1, a_4)$ and the exchange matrix illustrated by Figure 2.4.

Comparison between Q_{A_4} and Q_{A_6} indicates that Q_{A_6} can be obtained from Q_{A_4} by local expansion, as illustrated in Figure 3.3, that is by removing edges between the four-cycle formed by the nodes 1, 7, 8 and 11 in Q_{A_4} and including new nodes and edges in the quiver as per Figure 3.4.



(a) Q_{A_4} (b) Q_{A_6}

Figure 3.3: Extension from Q_{A_4} to Q_{A_6}



(a) Subquiver in Q_{A_4} (b) Subquiver in Q_{A_6}

Figure 3.4: Local expansion of the subquiver in Q_{A_4}

Recall that each node in the deformed quiver corresponds to a tau function, e.g. for Q_{A_4}, the sequence of nodes $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$ corresponds to the sequence of functions $(q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, p_0, a_1, a_4)$. As the figures show, Q_{A_6} can be built from Q_{A_4} by carrying out the local expansion on the four-cycle subquiver with nodes corresponding to the functions σ_3, σ_4, q_0 and p_0. From the cluster point of view, the local expansion is equivalent to relabelling σ_3, σ_4, σ_5 as $\sigma_5, \sigma_6, \sigma_7$ respectively and inserting σ_3, σ_4, p_1 and q_1 in a way that the cluster becomes $(q_1, q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\sigma_6, \sigma_7, p_0, p_1, a_1, a_4)$.

We will show that this pattern continues: one can recursively apply the same local expansion by a four-cycle quiver to obtain the deformed quiver $Q_{\mathrm{A}_{2N}}$ with nodes corresponding to

$$\begin{aligned}(p_1^{N-2} \ldots, p_1^1, p_1^0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{2N+1}, p_2^0, p_2^1, \ldots, p_2^{N-2}, a_1, a_{2N}) \\ = (1, 2, 3, \ldots, 4N+3, 4N+4, 4N+5)\end{aligned}$$

What does the local expansion tell us? The local expansion above gives insight into the structure of the tau functions in the x_i variables. Let us compare the tau functions in type A_4 and type A_6 cases. In the setting $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{11}) = (q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \ldots, \sigma_5, p_0)$, the variables $x_{i,n}$, induced by the deformed map associated to type A_4, are defined as

$$x_{1,n} = \frac{\tilde{x}_5 \tilde{x}_4}{\tilde{x}_6 \tilde{x}_3}, \quad x_{2,n} = \frac{\tilde{x}_{11}}{\tilde{x}_7 \tilde{x}_3}, \quad x_{3,n} = \frac{\tilde{x}_1}{\tilde{x}_8 \tilde{x}_3}, \quad x_{4,n} = \frac{\tilde{x}_{10} \tilde{x}_2}{\tilde{x}_9 \tilde{x}_3}$$

The local expansion above $(q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \ldots, \sigma_5, p_0) \to (q_1, q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \ldots, \sigma_7, p_0, p_1)$ is equivalent to shifting the subscript of the variables $\tilde{x}_i \to \tilde{x}_{i+1}$ for $j = 1, 2, \ldots, 7$ and $\tilde{x}_j \to \tilde{x}_{j+3}$ for $i = 8, 9, 10, 11$ and imposing the new variables

$$x_3 = \frac{\tilde{x}_{15}}{\tilde{x}_9 \tilde{x}_4}, \quad x_4 = \frac{\tilde{x}_1}{\tilde{x}_{10} \tilde{x}_4}$$

Then one obtains the variable transformation in (4.37) whose tau functions are denoted as $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{15}) = (q_1, q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \ldots, \sigma_7, p_0, p_1)$. This suggests that the recursive local expansion constructs the following x_i variables associated to the type A_{2N} deformed map,

$$\begin{aligned} &x_1 = \frac{\tilde{x}_{N+3} \tilde{x}_{N+2}}{\tilde{x}_{N+4} \tilde{x}_{N+1}}, \quad x_2 = \frac{\tilde{x}_{3N+5}}{\tilde{x}_{N+5} \tilde{x}_{N+1}}, \quad x_3 = \frac{\tilde{x}_{3N+6}}{\tilde{x}_{N+6} \tilde{x}_{N+1}}, \ldots, \\ &x_N = \frac{\tilde{x}_{4N+3}}{\tilde{x}_{2N+3} \tilde{x}_{N+1}}, x_{N+1} = \frac{\tilde{x}_1}{\tilde{x}_{2N+4} \tilde{x}_{N+1}}, \quad x_{N+2} = \frac{\tilde{x}_2}{\tilde{x}_{2N+5} \tilde{x}_{N+1}}, \ldots, \\ &x_{2N-1} = \frac{\tilde{x}_{N-1}}{\tilde{x}_{3N+2} \tilde{x}_{N+1}}, \quad x_{2N} = \frac{\tilde{x}_{3N+4} \tilde{x}_N}{\tilde{x}_{3N+3} \tilde{x}_{N+1}} \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} &(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{4N+3}) \\ &= (q_{N-2} \ldots, q_1, q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{2N+1}, p_0, p_1, \ldots, p_{N-2}, a_1, a_{2N}). \end{aligned}$$

The symplectic form associated to A_{2N} is defined by

$$\omega = \sum_{i<j} b_{ij} \mathrm{d} \log x_i \wedge \mathrm{d} \log x_j$$

where b_{ij} are entries of the exchange matrix $B_{A_{2N}}$ (3.1), defined by the following

$$(B_{A_{2N}})_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ -1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$
\begin{aligned}
\pi^*\omega &= \tilde{\omega} \\
&= \mathrm{d}\log\left(\frac{\tilde{x}_{N+3}\tilde{x}_{N+2}}{\tilde{x}_{N+4}\tilde{x}_{N+1}}\right) \wedge \mathrm{d}\log\left(\frac{\tilde{x}_{3N+5}}{\tilde{x}_{N+5}\tilde{x}_{N+1}}\right) \\
&+ \sum_{l=5}^{N+2} \mathrm{d}\log\left(\frac{\tilde{x}_{3N+l}}{\tilde{x}_{N+l}\tilde{x}_{N+1}}\right) \wedge \mathrm{d}\log\left(\frac{\tilde{x}_{3N+(l+1)}}{\tilde{x}_{N+(l+1)}\tilde{x}_{N+1}}\right) \\
&+ \mathrm{d}\log\left(\frac{\tilde{x}_{4N+3}}{\tilde{x}_{2N+3}\tilde{x}_{N+1}}\right) \wedge \mathrm{d}\log\left(\frac{\tilde{x}_{1}}{\tilde{x}_{2N+4}\tilde{x}_{N+1}}\right) \\
&+ \sum_{m=1}^{N-2} \mathrm{d}\log\left(\frac{\tilde{x}_{m}}{\tilde{x}_{2N+3+m}\tilde{x}_{N+1}}\right) \wedge \mathrm{d}\log\left(\frac{\tilde{x}_{m+1}}{\tilde{x}_{2N+4+m}\tilde{x}_{N+1}}\right) \\
&+ \mathrm{d}\log\left(\frac{\tilde{x}_{N-1}}{\tilde{x}_{3N+2}\tilde{x}_{N+1}}\right) \wedge \mathrm{d}\log\left(\frac{\tilde{x}_{3N+4}\tilde{x}_{N}}{\tilde{x}_{3N+3}\tilde{x}_{N+1}}\right)
\end{aligned}
$$

To simplify the calculation, let us define $\alpha_i = \mathrm{d}\log \tilde{x}_i$ and $f_j = \alpha_{3N+j} - \alpha_{N+j}$ and $g_k = \alpha_k - \alpha_{k+2N+3}$. Then the pre-symplectic form can be re-written as

$$
\begin{aligned}
\tilde{\omega} &= (\alpha_{N+3} + \alpha_{N+2} - \alpha_{N+4} - \alpha_{N+1}) \wedge (f_5 - \alpha_{N+1}) \\
&+ \left(\sum_{l=5}^{N+2} f_l \wedge f_{l+1} - f_l \wedge \alpha_{N+1} - \alpha_{N+1} \wedge f_{l+1}\right) + f_{4N+3} \wedge g_1 - \alpha_{N+1} \wedge g_1 \\
&+ \left(\sum_{m=1}^{N-2} g_m \wedge g_{m+1} - g_m \wedge \alpha_{N+1} - \alpha_{N+1} \wedge g_{m+1}\right) \\
&+ (g_{N-1} + \alpha_{N+1}) \wedge (g_N - g_{N+1})
\end{aligned}
$$

Combining and cancelling, we obtain

$$
\begin{aligned}
\tilde{\omega} &= (\alpha_{N+3} + \alpha_{N+2} - \alpha_{N+4}) \wedge (f_5 - \alpha_{N+1}) \\
&+ \left(\sum_{l=5}^{N+2} f_l \wedge f_{l+1}\right) + f_{4N+3} \wedge g_1 + \left(\sum_{m=1}^{N-1} g_m \wedge g_{m+1}\right) \\
&+ g_{N-1} \wedge (\alpha_{3N+4}) + \alpha_{N+1} \wedge (g_N - g_{N+1})
\end{aligned}
$$

Therefore $\tilde{\omega}$ is expressed as

$$
\sum_{r<s} \tilde{b}_{rs} \alpha_r \wedge \alpha_s
$$

whose coefficients are entries of the $(4N+3) \times (4N+3)$ exchange matrix

$$
B_{A_{2N}} = \left(\begin{array}{c|c} \mathbf{A}_{2N} & \mathbf{B}_{2N} \\ \hline -\mathbf{B}_{2N}^T & \mathbf{C}_{2N} \end{array}\right) \tag{3.34}
$$

which is composed of four block skew-symmetric matrices: a $(2N+3)\times(2N+3)$ matrix $\mathbf{A}_{2N}$, a $(2N+3)\times 2N$ matrix $\mathbf{B}_{2N}$ and a $2N\times 2N$ matrix $\mathbf{C}_{2N}$ where

$$\mathbf{A}_{2N}/\mathbf{C}_{2N}=\left(\begin{array}{ccc|cccc|ccc}
0 & 1 & 0 & 0 & & & & \cdots & 0 & 1\\
-1 & \ddots & \ddots & & & & & & & \\
 & \ddots & 0 & 1 & 0 & \cdots & 0 & 0 & & \\
\hline
 & 0 & -1 & & & & & 0 & & \\
 & 0 & 0 & & \multicolumn{2}{c}{\mathbf{A}_4/\mathbf{C}_4} & & 0 & & \\
 & & \vdots & & & & & \vdots & & \\
 & & 0 & & & & & 1 & & \\
\hline
 & & 0 & 0 & 0 & \cdots & -1 & 0 & \ddots & 0\\
0 & & 0 & 0 & 0 & \cdots & 0 & \ddots & \ddots & 1\\
-1 & & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0
\end{array}\right),\quad
\mathbf{B}_{2N}=\left(\begin{array}{ccc|cccc|ccc}
0 & -1 & 0 & 0 & & & & \cdots & 0 & -1\\
1 & \ddots & \ddots & & & & & & & \\
 & \ddots & 0 & -1 & 0 & \cdots & 0 & 0 & & \\
\hline
 & 0 & 1 & & & & & 0 & & \\
 & 0 & 0 & & \multicolumn{2}{c}{\mathbf{B}_4} & & 0 & & \\
 & & \vdots & & & & & \vdots & & \\
 & & 0 & & & & & -1 & & \\
\hline
 & & 0 & 0 & 0 & \cdots & 1 & 0 & \ddots & 0\\
0 & & 0 & 0 & 0 & \cdots & 0 & \ddots & \ddots & -1\\
1 & & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0
\end{array}\right)$$

Here, the block matrix has $(\mathbf{A}_N)_{ij}=(\mathbf{A}_4)_{ij}$ for $N-2<i,j<N+6$, $(\mathbf{B}_N)_{mn}=(\mathbf{B}_4)_{mn}$ for $N<m<N+6$ and $3(N-2)+7<n<3(N-2)+12$ and $(\mathbf{C}_N)_{rs}=(\mathbf{C}_4)_{rs}$ for $3(N-2)+7<r,s<3(N-2)+12$, and

$$\mathbf{A}_4=\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 1 & 1 & -1 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & -1\\
0 & 0 & -1 & 0 & 0 & 0 & -1\\
0 & 0 & 1 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 1 & 1 & -1 & 0
\end{pmatrix},\qquad
\mathbf{B}_4=\begin{pmatrix}
0 & -1 & 1 & 0\\
1 & 0 & 0 & 0\\
0 & 1 & -1 & 0\\
0 & 0 & 0 & 1\\
0 & 0 & 0 & 1\\
0 & 0 & 0 & -1\\
0 & 0 & 0 & 0
\end{pmatrix},\qquad
\mathbf{C}_4=\begin{pmatrix}
0 & 1 & -1 & 0\\
-1 & 0 & 0 & 0\\
1 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}$$

Based on the pattern of local expansion from Q_{A_4} to Q_{A_6}, we introduce frozen variables and extend the matrix by extra rows $\mathbf{b}_1$ and $\mathbf{b}_2$ in which entries are

$$\begin{aligned}
(\mathbf{b}_1)_i &= \delta_{i,N+1}+\delta_{i,N+2}-\delta_{i,N+3}-\delta_{i,N+4}\\
(\mathbf{b}_2)_i &= -\delta_{i,N}-\delta_{i,N+1}+\delta_{i,3N+3}+\delta_{i,3N+4}
\end{aligned}\tag{3.35}$$

so that

$$
\tilde{B}_{A_{2N}} = \left(\begin{array}{ccccccccccccccccccccccc}
0 & 1 & & & & & & & & & & & 1 & 0 & -1 & & & & & & & & -1 \\
-1 & \ddots & & & & & & & & & & & & 1 & \ddots & & & & & & & & \\
 & & 0 & 1 & & & & & & & & & & & & 0 & -1 & & & & & & \\
 & & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & 1 & 0 & -1 & 1 & 0 & & & \\
 & & & -1 & 0 & 1 & 0 & 0 & 0 & 0 & & & & & & & 1 & 0 & 0 & 0 & & & \\
 & & & 0 & -1 & 0 & 1 & 1 & -1 & 0 & & & & & & & 0 & 1 & -1 & 0 & & & \\
 & & & 0 & 0 & -1 & 0 & 0 & 0 & -1 & & & & & & & 0 & 0 & 0 & 1 & & & \\
 & & & 0 & 0 & -1 & 0 & 0 & 0 & -1 & & & & & & & 0 & 0 & 0 & 1 & & & \\
 & & & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & & & & & & 0 & 0 & 0 & -1 & & & \\
 & & & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & & & & & & 0 & 0 & 0 & 0 & -1 & & \\
 & & & & & & & & & -1 & 0 & \ddots & & & & & & & & 1 & 0 & \ddots & \\
 & & & & & & & & & & & & & & & & & & & & & & -1 \\
-1 & & & & & & & & & & & & & & & & & & & & & & 0 \\
0 & \ddots & & & & & & & & & & & & & & & & & & & & & 1 \\
1 & & -1 & & & & & & & & & & & & & 1 & & & & & & & \\
 & & 0 & -1 & & & & & & & & & & & -1 & 0 & 1 & & & & & & \\
 & & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & & & & & & -1 & 0 & 1 & -1 & 0 & & & \\
 & & & 1 & 0 & -1 & 0 & 0 & 0 & 0 & & & & & & & -1 & 0 & 0 & 0 & & & \\
 & & & -1 & 0 & 1 & 0 & 0 & 0 & 0 & & & & & & & 1 & 0 & 0 & 0 & & & \\
 & & & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & & & & & & 0 & 0 & 0 & 0 & 1 & & \\
 & & & & & & & & & 1 & 0 & \ddots & & & & & & & & -1 & 0 & \ddots & \\
 & & & & & & & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & & & & 1 & 0 & -1 & & & & & & & & -1 & 0 \\
 & & & 0 & 0 & 1 & 1 & -1 & -1 & 0 & & & & & & & 0 & 0 & 0 & 0 & & & \\
 & & & 0 & -1 & -1 & 0 & 0 & 0 & 0 & & & & & & & 0 & 1 & 1 & 0 & & &
\end{array}\right) \tag{3.36}
$$

By this calculation and formalisation, we have obtained a description of local expansion in terms of exchange matrices, corresponding to the graphical description in terms of quivers.

For the example of A_4 to A_6, the following exchange matrix

$$
\tilde{B}_{A_4} = \left(\begin{array}{c|c} \mathbf{A}_4 & \mathbf{B}_4 \\ \hline -\mathbf{B}_4^T & \mathbf{C}_4 \end{array}\right) \tag{3.37}
$$

represents a quiver in which the edges of a subquiver (four-cycle) have been removed i.e. the beginning and end of off-diagonal entries in each block matrix are set to be zero. The local expansion in Figure 3.3 is equivalent to inserting new columns and rows in each block matrix i.e.

$$\mathbf{A}_6 = \left(\begin{array}{c|cccc|c} 0 & 1 & 0 & \cdots & 0 & 1 \\ \hline -1 & & & & & 0 \\ 0 & & & \mathbf{A}_4 & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & 1 \\ \hline -1 & 0 & 0 & \cdots & -1 & 0 \end{array}\right), \qquad \mathbf{B}_6 = \left(\begin{array}{c|cccc|c} 0 & -1 & 0 & \cdots & 0 & -1 \\ \hline 1 & & & & & 0 \\ 0 & & \mathbf{B}_4 & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & -1 \\ \hline 1 & 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$$\mathbf{C}_6 = \left(\begin{array}{c|cccc|c} 0 & 1 & 0 & \cdots & 0 & 1 \\ \hline -1 & & & & & 0 \\ 0 & & & \mathbf{C}_4 & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & 1 \\ \hline -1 & 0 & 0 & \cdots & -1 & 0 \end{array}\right)$$

Such extension leads to the exchange matrix (3.25), now written as

$$\tilde{B}_{A_6} = \left(\begin{array}{c|c} \mathbf{A}_6 & \mathbf{B}_6 \\ \hline -\mathbf{B}_6^T & \mathbf{C}_6 \end{array}\right) \tag{3.38}$$

3.4 The type A_{2N} deformed periodic cluster map

Earlier we proved that the Laurent property of type A_6 deformed map can be restored by lifting the map into a higher-dimensional cluster map defined on the space of tau functions, which is done by finding the particular sequence of mutations preserving the structure of the quiver up to shifting the labels. In this section, we use a similar procedure to show that there exists a sequence of mutations such that the structure of the candidate deformed quiver $Q_{\mathrm{A}_{2N}}$ (or exchange matrix $\tilde{B}_{\mathrm{A}_{2N}}$, as constructed in the previous section) is preserved and show that the corresponding cluster variables can be produced by a two-parameter family of deformed cluster maps corresponding to type A_{2N}.

Example 3.4.1 (Deformed quiver Q_{A_8})**.** *Let us consider the initial seed formed by the initial cluster* $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{19}) = (q_2, q_1, q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \ldots, \sigma_9, p_0, p_1, p_2, a_1, a_8)$

and exchange matrix

$$
\tilde{B}_{\mathrm{A}_8} = \left(\begin{array}{c|c} \mathbf{A}_8 & \mathbf{B}_8 \\ \hline -\mathbf{B}_8^T & \mathbf{C}_8 \\ \hline \mathbf{b}_8^{(1)} & \mathbf{b}_8^{(2)} \end{array}\right) = \left(\begin{array}{ccccccccccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}\right) \tag{3.39}
$$

where

$$
\mathbf{A}_8/\mathbf{C}_8 = \left(\begin{array}{cc|cccc|cc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \hline
0 & -1 & & & & & 0 & 0 \\
0 & 0 & & \mathbf{A}_4/\mathbf{C}_4 & & & 0 & 0 \\
0 & \vdots & & & & & \vdots & \vdots \\
0 & 0 & & & & & 1 & 0 \\ \hline
0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\
-1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0
\end{array}\right), \qquad
\mathbf{B}_8 = \left(\begin{array}{cc|cccc|cc}
0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\
1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \hline
0 & 1 & & & & & 0 & 0 \\
0 & 0 & & \mathbf{B}_4 & & & 0 & 0 \\
0 & \vdots & & & & & \vdots & \vdots \\
0 & 0 & & & & & -1 & 0 \\ \hline
0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right)
$$

$$
\mathbf{b}_8^{(1)} + \mathbf{b}_8^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

Reading off the exchange matrix, the corresponding deformed quiver Q_{A_8} can be drawn and this is depicted in Figure 3.5.

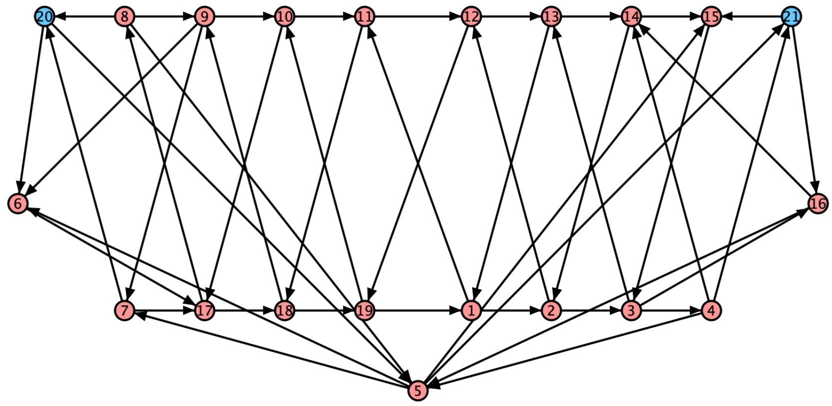


Figure 3.5: (Candidate) deformed quiver Q_{A_8}

Recall that the composition of mutations $\mu_3\mu_2\mu_1\mu_{15}\mu_{14}\mu_6$ maintains the form of the deformed quiver Q_{A_6} except that the particular labellings of the nodes are permuted. Such mutation periodicity was already observed in the type A_4 *case, in which the relevant sequence of mutations is $\mu_2\mu_1\mu_{11}\mu_5$. Comparing the cases, one can deduce the pattern of mutations for the type* A_8*, which is given by $\mu_4\mu_3\mu_2\mu_1\mu_{19}\mu_{18}\mu_{17}\mu_7$. Then performing the iteration of matrix mutations above gives rise to the following exchange matrix:*

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix} \tag{3.40}$$

Let us compare the mutated matrix (3.40) *with* (3.39). *Then one can see that there have been changes in certain regions in the matrix, highlighted above. In the submatrix defined by the orange region, the transformation described by* *mutations is equivalent to cyclic permuting of the matrix, i.e. for the 13-cycle* $\rho = (4,5,6,7,8,9,10,11,12,13,14,15,16)$ *cyclic permutation, the entries satisfy*

$$b_{\rho(i)\rho(j)} = b_{i+1,j+1} \quad \text{for } 4 \leq i,j \leq 16,$$

with

$$b_{i,\rho(16)} = b_{i,1} \ (\text{or } b_{\rho(16),j} = b_{1,j}) \text{ for } i,j \in \{4,5,\dots,16\}. \tag{3.41}$$

As for the green highlighted submatrices, the entries in the upper and lower rows are shifted to the right by 1. For the left and right end column vectors, the entries are moved downwards by 1, i.e letting $I_1 = \{1,2,3\}$ *and* $I_2 = \{17,18,19\}$ *and then*

$$b_{l,\rho(j)} = b_{l,j+1} \qquad \text{for } l \in I_1 \cup I_2, \ \ j \in \{4,\dots,16\}$$

in agreement with $b_{l,\rho(16)} = b_{l,1}$. *Such a transformation is seen to be the cyclic permutation of the labels* $\{4,5,\dots,16\}$ *of the deformed quiver* Q_{A_8}.

Thus by using the same procedure that appeared in the type A_6 *case, one can construct the cluster map* $\psi_{A_8} = \rho_{A_8}^{-1}\mu_4\mu_3\mu_2\mu_1\mu_{19}\mu_{18}\mu_{17}\mu_7$ *which generates the set of cluster variables*

$$\begin{aligned}
\tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1 p_n \\
p_{0,n+1}p_{0,n} &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + p_{1,n}\sigma_{n+1}\tau_{n+2} \\
p_{1,n+1}p_{1,n} &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + p_{2,n}p_{0,n+1} \\
p_{2,n+1}p_{2,n} &= \sigma_{n+5}\sigma_{n+4}\tau_n\tau_{n+1} + q_{2,n}p_{1,n+1} \\
q_{2,n+1}q_{2,n} &= \sigma_{n+6}\sigma_{n+5}\tau_n\tau_{n+1} + q_{1,n}p_{2,n+1} \\
q_{1,n+1}q_{1,n} &= \sigma_{n+7}\sigma_{n+6}\tau_n\tau_{n+1} + q_{1,n}q_{1,n+1} \\
q_{0,n+1}q_{0,n} &= \sigma_{n+8}\sigma_{n+7}\tau_n\tau_{n+1} + \sigma_{n+9}q_{1,n+1}\tau_{n-1} \\
\sigma_{n+9}\tau_{n-1} &= \sigma_{n+7}\tau_{n+1} + a_8 q_{0,n+1}
\end{aligned} \tag{3.42}$$

The exchange matrix (3.39) *can be found by the pullback of the original symplectic form by the rational map, that is, substituting the following variable transformation* (3.43) *into the symplectic form.*

$$\begin{aligned}
&x_{1,n} = \frac{\sigma_n\tau_{n+1}}{\sigma_{n+1}\tau_n} \quad x_{2,n} = \frac{p_{0,n}}{\sigma_{n+2}\tau_n} \quad x_{3,n} = \frac{p_{1,n}}{\sigma_{n+3}\tau_n} \quad x_{4,n} = \frac{p_{2,n}}{\sigma_{n+4}\tau_n} \\
&x_{5,n} = \frac{q_{2,n}}{\sigma_{n+5}\tau_n} \quad x_{6,n} = \frac{q_{1,n}}{\sigma_{n+6}\tau_n} \quad x_{7,n} = \frac{q_{0,n}}{\sigma_{n+7}\tau_n} \quad x_{8,n} = \frac{\sigma_{n+9}\tau_{n-1}}{\sigma_{n+8}\tau_n}
\end{aligned} \tag{3.43}$$

Then by manipulation of equations in (3.42) *and imposing* (3.43)*, we obtain the original two-parameter family of maps* φ_{A_8} *defined on the initial variables* $(x_1, x_2, \ldots, x_8)$*. Hence it has been shown that the candidate deformed quiver/exchange matrix that emerged from the constructive approach allows us to define the cluster map* ψ_{A_8} *which is obtained by considering Laurentification of deformed map* φ_{A_8}*.*

As we have seen from the example, the deformed type φ_{A_8} map admits a lift to a higher dimensional cluster map ψ_{A_8} which preserves the deformed exchange matrix (3.39). We now extend this procedure to show that the deformation of type A_{2N} cluster maps $\varphi_{\mathrm{A}_{2N}}$ can be lifted to a cluster map $\psi_{\mathrm{A}_{2N}}$

Following from the cases of type A_4, A_6 and A_8, there exists a specific sequence of mutation equivalent to a permutation of the vertices:

$$\mu_{\tau_{-1}}\tilde{\mu}\mu_{\sigma_0}(Q_{\mathrm{A}_6}) = \rho(Q_{\mathrm{A}_6})$$

where $\tilde{\mu} = \mu_{q_0}\mu_{q_1}\mu_{p_1}\mu_{p_2}$ and ρ is the inverse cyclic permutation of the vertices $(\tau_{-1}, \tau_0, \tau_1, \sigma_0, \ldots, \sigma_5)$. This begins with a mutation in the direction of σ_0, μ_{σ_0}, followed by the mutations $\tilde{\mu}$, then ends with $\mu_{\tau_{-1}}$. Recall that the local expansion on the deformed quiver introduces the new vertices σ, p_i and q_i and new edges. The position of the other vertices remains the same.

Proposition 3.4.2. *For each deformed quiver* $Q_{A_{2N}}$ *with vertices*

$$(q_{N-2}, \ldots, q_1, q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{2N}, \sigma_{2N+1}, p_0, p_1, \ldots, p_{N-2})$$

we have invariance up to cyclic permutation under mutation:

$$\mu_{\tau_{-1}}\tilde{\mu}\mu_{\sigma_0}(Q_{A_{2N}}) = \rho(Q_{A_{2N}}) \qquad \text{for } \tilde{\mu} = \mu_{q_0}\cdots\mu_{q_{N-2}}\mu_{p_{N-2}}\cdots\mu_{p_0} \tag{3.44}$$

Proof. To see such a phenomenon explicitly, it is convenient for us to approach it from the exchange matrix perspective instead of quivers. As we are aware from the matrix (3.36), the local expansion in the matrix includes new entries on the surrounding block matrices. The non-zero entries b_{ij} in the direction of the first two mutations $\mu_{p_0}\mu_{\sigma_0} = \mu_{3N+5}\mu_{N+3}$ are positioned in the block matrices and their adjacent columns and rows. The matrix mutation replaces old entries with new,

$\mu_k(B)$, if $b_{ik}b_{kj} > 0$. Therefore the the transformation only occurs at the block matrices and their adjacent entries. This implies the first two mutations give the same result as the lower-rank cases (e.g. type A_8).

For the successive mutations, $\mu_{p_1}\mu_{p_0}\mu_{\sigma_0} = \mu_{3N+6}\mu_{3N+5}\mu_{N+3}$, the entries in the direction of the forthcoming mutation, μ_k,

$$b_{i,k} = (\delta_{i,N+1} + \delta_{i,N+2}) + (-\delta_{i,k+1} - \delta_{i,k-1} + \delta_{i,k-2N} + \delta_{i,k-2N+1}) \tag{3.45}$$

and its adjacent entries are written as follows

$$\begin{aligned} b_{i,k-1} &= -(\delta_{i,N+1} + \delta_{i,N+2}) + (\delta_{i,k} + \delta_{i,k-2} - \delta_{i,k-2N-1} - \delta_{i,k-2N}) \\ b_{i,k+1} &= (\delta_{i,k-2N+2} - \delta_{i,k-2N}) + (\delta_{i,k} - \delta_{i,k+2}) \end{aligned} \tag{3.46}$$

From (3.45), one can see there are only two negative entries and the rest of the entries are positive. This implies that $b_{ik}b_{kj} > 0$ if i or j is $k-1$ or $k+1$. Therefore the mutation μ_k does not affect the other entries except the adjacent entries in the direction k. The matrix mutation μ_k can be written as

$$b_{ij}^{(n)} = b_{ij}^{(n-1)} - 2\delta_{j,k}b_{i,j}^{(n-1)} - 2\delta_{i,k}b_{k,j}^{(n-1)} + \mathrm{sgn}(b_{ik})\beta_{ij}^{(n-1)}$$

for

$$\beta_{ij}^{(n-1)} = (\delta_{j,k+1} + \delta_{j,k-1})(\delta_{i,N+1} + \delta_{i,N+2} + \delta_{i,k-2N} + \delta_{i,k-2N+1})$$

and it gives rise to the new columns (and rows) in the $k-1$ and $k+1$ directions. The new column $b'_{i,k+1}$ is $b_{i,k}$ in (3.45) with with the subscript of entries in the second bracket of each column shifted by 1. As for the new entries in $b'_{i,k-1}$, the structure is same as $b_{i,k+1}$ in (3.46) except the entries $(\delta_{i,k-2N+2}) - \delta_{i,k-2N})$ shifted by 1. From (3.36), one can observe that the exchange matrix consists of columns, in which the entries are written as

$$b_{i,v} = \delta_{i,v-2N+1} - \delta_{i,v-2N-1} + \delta_{i,v-1} - \delta_{i,v+1}$$

for $3N + 6 \leq v \leq 4N + 2$. By comparison, the situation of the next mutation is similar to the previous mutation. Thereby applying matrix mutations inductively in the direction $p_{0+l} = (3N + 5) + l$, for $l \in \{1, \ldots, N-3\}$, we obtain the following columns:

$$b_{i,(3N+5+m)} = \delta_{i,N+7+m} - \delta_{i,N+5+m} + \delta_{i,3N+4+m} - \delta_{i,3N+6+m} \tag{3.47}$$

for $m \in \{1, \ldots, l-1\}$. Furthermore, the mutation in direction $4N+3$ brings changes to the last and first columns, which are given by

$$\begin{aligned} b_{i,4N+3} &= -(\delta_{i,N+1} + \delta_{i,N+2}) + (-\delta_{i,2N+3} - \delta_{i,2N+4} + \delta_{i,1} + \delta_{i,4N+2}) \\ b_{i,1} &= (\delta_{i,N+1} + \delta_{N+2}) + (-\delta_{i,2} - \delta_{i,4N+3} + \delta_{i,2N+4} + \delta_{i,2N+5}) \end{aligned} \tag{3.48}$$

Now the mutations take place on the left side of the exchange matrix, which begins from $\mu_{q_{N-2}} = \mu_1$. The columns $b_{i,l}$ in the exchange matrix (3.36) for $l \in \{2, \ldots, N-2\}$ are structured as

$$b_{i,l} = \delta_{i,l+2N+4} - \delta_{i,l+2N+2} + \delta_{i,l-1} - \delta_{i,l+1} \tag{3.49}$$

Under the mutation in the direction of the first column, we have the last column $b_{i,4N+3}$ written in the form of (3.47) with setting $\delta_{i,4N+4} = \delta_{i,1}$, and its subsequent column $b_{i,2}$ is

$$b_{i,2} = (\delta_{i,N+1} + \delta_{i,N+2}) + (-\delta_{i,1} - \delta_{i,3} + \delta_{i,2N+5} + \delta_{i,2N+6}) \tag{3.50}$$

Notice that we have the same situation as in the case of the sequence of the mutations on the right side of the matrix. To be specifically, the entries in $b_{i,2}$ and its adjacent columns are arranged in the same way as (3.45) and (3.46). Computing the matrix mutations $\mu_1 \cdots \mu_{N-2} = \mu_{q_{N-2}} \cdots \mu_{q_0}$ subsequently, we find new columns $(b_{i,s})$ $(1 \leq i \leq 4N+3,\ 1 \leq s \leq N-2)$,

$$b_{i.s} = \delta_{i,s-1} - \delta_{i,s+1} - \delta_{i,s+2N+3} + \delta_{s+2N+5}$$

and the next adjacent column $b_{i,N-1}$ is given by

$$b_{i,N-1} = -\delta_{i,N-2} \underbrace{-\delta_{i,N} + \delta_{i,N+1} + \delta_{i,N+2}}_{(\mathbf{A}_4)_{i,1}} + \underbrace{\delta_{i,3N+2} + \delta_{i,3N+3} - \delta_{i,3N+4}}_{(-\mathbf{B}_4^T)_{i,1}}$$

The mutation in the direction $N-1$ gives the column $b_{i,N}$:

$$b_{i,N} = \underbrace{-\delta_{i,N-1} + \delta_{i,N+2}}_{(\mathbf{A}_4)_{i,2}} + \underbrace{\delta_{i,3N+3}}_{(-\mathbf{B}_4^T)_{i,2}}$$

Therefore, combining all the results above, we can say that the composition of mutations in (3.44) transforms the type A_{2N} deformed exchange matrix (3.34) into the following

$$
\left(\begin{array}{cccccccccccccccccccccc}
0 & 1 & & & & & & & & & & & 1 & 0 & -1 & & & & & & & -1 \\
-1 & \ddots & & & & & & & & & & & & \ddots & & & & & & & & \\
 & & 0 & 1 & & & & & & & & & & & 1 & 0 & -1 & & & & & \\
 & & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & & & & & 1 & 0 & -1 & 0 & & & \\
 & & & -1 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & 0 & 1 & 0 & 0 & & & \\
 & & & -1 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & 0 & 1 & 0 & 0 & & & \\
 & & & 0 & -1 & -1 & 0 & 1 & 1 & -1 & & & & & & 0 & 0 & 1 & 0 & & & \\
 & & & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & & & & & 0 & 0 & 0 & 1 & & & \\
 & & & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & & & & & 0 & 0 & 0 & 1 & & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & & & & 0 & 0 & 0 & -1 & & & \\
 & & & & & & & 1 & 1 & -1 & 0 & 1 & & & & & & & 0 & -1 & & \\
 & & & & & & & & & & -1 & \ddots & & & & & & & 1 & \ddots & & \\
-1 & & & & & & & & & & & & & & & & & & & & & -1 \\
0 & \ddots & & & & & & & & & & & & & 1 & & & & & & & 0 \\
1 & & -1 & & & & & & & & & & & -1 & 0 & 1 & & & & & & 1 \\
 & & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & -1 & 0 & 1 & 0 & 0 & & & \\
 & & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & & & & & & -1 & 0 & 1 & 0 & & & \\
 & & & 1 & 0 & 0 & -1 & 0 & 0 & 0 & & & & & & 0 & 0 & 0 & 0 & & & \\
 & & & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & & & & 0 & 0 & 0 & 0 & 1 & & \\
 & & & & & & & & & & 1 & \ddots & & & & & & & -1 & 0 & \ddots & \\
 & & & & & & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & & & & & 1 & 0 & -1 & & & & & & -1 & 0 \\
 & & & 0 & 0 & 0 & 1 & 1 & -1 & -1 & & & & & & 0 & 0 & 0 & 0 & & & \\
 & & & 0 & 0 & -1 & -1 & 0 & 0 & 0 & & & & & & 0 & 0 & 1 & 1 & & &
\end{array}\right) \tag{3.51}
$$

From the transformed matrix (3.51), one can see the same phenomenon as in Example 3.4.1 such that 1) the components in the $(2N-1)\times(2N-1)$ submatrix which is positioned in the centre of the matrix are cyclically permuted and 2) entries in the middle of the rows and columns, embedded next to the central submatrix, are shifted one downwards and upwards by one respectively. Hence we conclude that the quiver arising from the exchange matrix (3.51) exhibits symmetry such that the structure of the quiver is the same except that the labels $\{N, N-1, \ldots, 3N+5\}$ are cyclically permuted i.e.

$$\mu_N\mu_{N-1}\tilde{\mu}\mu_{3N+5}\mu_{N+3}(Q_{2N}) = \rho(Q_{2N}) \qquad \text{for } \tilde{\mu} = \mu_{N-2}\mu_{N-1}\cdots\mu_1\mu_{4N+3}\mu_{4N+2}\cdots\mu_{3N+7}\mu_{3N+6}$$

where $\rho_{\mathrm{A}_{2N}} = (N, N+1, \ldots, 3N+5)$. □

The statement above enables us to define the cluster map which is associated with the exchange matrix (3.36), that is, we define $\psi_{\mathrm{A}_{2N}} = \rho^{-1}_{\mathrm{A}_{2N}}\mu_N\mu_{N-1}\tilde{\mu}\mu_{3N+5}\mu_{N+3}$ and see that

$$\psi_{\mathrm{A}_{2N}}(B_{\mathrm{A}_{2N}}) = \rho^{-1}_{\mathrm{A}_{2N}}\mu_N\mu_{N-1}\tilde{\mu}\mu_{3N+5}\mu_{N+3}(B_{\mathrm{A}_{2N}}) = B_{\mathrm{A}_{2N}} \tag{3.52}$$

Let us consider the cluster variables generated by the $\psi_{\mathrm{A}_{2N}}$. Recall that the new cluster variables are expressed by the exchange relation, in which the old cluster variables and coefficients appearing in the expression entirely depend on the matrix entries in the direction of mutation.

In the proof of Proposition 3.4.2 above, we see that the transformation given by the first two mutations $\mu_{p_0}\mu_{\sigma_0} = \mu_{3N+5}\mu_{N+3}$ only occurs in the block matrices and their surrounding entries. The corresponding matrix entries are written as

$$\begin{aligned} b_{i,N+3} &= \delta_{i,N+1} + \delta_{i,N+5} - \delta_{i,3N+5} - \hat{\hat{\delta}}_{i,4N+4} \\ b_{i,3N+5} &= \delta_{i,N+1} + \delta_{i,N+2} - \delta_{i,N+3} - \delta_{i,N+4} + \delta_{i,N+5} + \delta_{i,N+6} - \delta_{i,3N+6} \end{aligned} \tag{3.53}$$

where $\hat{\hat{\delta}}_{i,4N+4}$ corresponds to the frozen variable. From the results above, we can see that the column $b_{i,k}$ in the direction of mutation μ_k, for

$$k \in \{1, 2, \dots, N-2, 3N+6, \dots, 4N+3\}$$

are arranged as following

$$b_{i,k} = (\delta_{i,N+1} + \delta_{i,N+2}) + (-\delta_{i,k+1} - \delta_{i,k-1} + \delta_{i,k-2N} + \delta_{i,k-2N+1}) \tag{3.54}$$

At the latter end of the sequence of mutations,

$$\begin{aligned} b_{i,N-1} &= -\delta_{i,N-2} - \delta_{i,N} + \delta_{i,N+2} + \delta_{i,3N+2} + \delta_{i,3N+3} - \delta_{i,3N+4} \\ b_{i,N} &= -\delta_{i,N-1} + \delta_{i,N+2} + \delta_{i,3N+3} - \hat{\hat{\delta}}_{i,4N+5} \end{aligned} \tag{3.55}$$

Thus starting from the initial seed $(\tilde{\mathbf{x}}_0, B_{\mathrm{A}_{2N}})$, where $B_{\mathrm{A}_{2N}}$ is (3.36) and $\tilde{\mathbf{x}}_0$ is the initial cluster

$$(q_{N-2,0}, \dots, q_{1,0}, q_{0,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \dots, \sigma_{2N}, \sigma_{2N+1}, p_{0,0}, p_{1,0}, \dots, p_{N-2,0})$$

the n-th iterate of the cluster map $\psi_{\mathrm{A}_{2N}}$ generates cluster variables defined by the relations

$$\begin{aligned} \tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1 p_{0,n} \\ p_{0,n+1}p_{0,n} &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + p_{1,n}\sigma_{n+1}\tau_{n+2} \\ p_{1,n+1}p_{1,n} &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + p_{2,n}p_{0,n+1} \\ &\vdots \\ p_{N-2,n+1}p_{N-2,n} &= \sigma_{n+N+1}\sigma_{n+N}\tau_n\tau_{n+1} + q_{N-2,n}p_{N-3,n+1} \\ q_{N-2,n+1}q_{N-2,n} &= \sigma_{n+N+2}\sigma_{n+N+1}\tau_n\tau_{n+1} + q_{N-3,n}p_{N-2,n+1} \\ q_{N-3,n+1}q_{N-3,n} &= \sigma_{n+N+3}\sigma_{n+N+2}\tau_n\tau_{n+1} + q_{N-4,n}q_{N-2,n+1} \\ &\vdots \\ q_{0,n+1}q_{0,n} &= \sigma_{n+2N}\sigma_{n+2N-1}\tau_n\tau_{n+1} + \sigma_{n+2N+1}q_{1,n+1}\tau_{n-1} \\ \sigma_{n+2N+2}\tau_{n-1} &= \sigma_{n+2N}\tau_{n+1} + a_{2N}q_{0,n+1} \end{aligned} \tag{3.56}$$

We then impose the variable transformations in (3.33) and we find the exchange relations

$$
\begin{aligned}
x_{1,n}x_{1,n+1} &= 1 + a_1 x_{2,n} \\
x_{2,n}x_{2,n+1} &= 1 + x_{3,n}x_{1,n+1} \\
&\vdots \\
x_{2N-1,n}x_{2N-1,n+1} &= 1 + x_{2N,n}x_{2N-2,n+1} \\
x_{2N,n}x_{2N,n+1} &= 1 + a_N x_{2N-1,n+1}
\end{aligned}
\tag{3.57}
$$

which are induced by the deformed type A_{2N} map $\tilde{\varphi}_{\mathrm{A}_{2N}} = \mu_{2N}\mu_{2N-1}\cdots\mu_1$.

3.5 Tropical dynamics and degree growth for Laurentified deformed map

In the previous section, we observed that upon applying Laurentification, one retrieves a higher dimensional cluster map, possessing Laurent property, from the type A_{2N} deformed map. However, we didn't touch the subject of the integrability of the higher case of deformed maps. The problem is that the calculation for determining the condition for the parameters becomes more complex as we try to use the procedure on each modified first integral in each even case. Thus it is difficult to prove the integrability by constructing invariant functions. Instead, we perform an algebraic entropy test as an alternative to provide strong evidence that deformed maps possess integrability. Here we begin with considering the calculation of degree growth (ref. [7]) of lifted maps, which emerges from the deformation of type A_4; we move onto the higher rank case, type A_{2N}.

3.5.1 Tropical dynamics and degree growth for type A_2 deformed map

Recollecting the example 2.5.3, the rational map π, defined by

$$x_{1,n} = \frac{\sigma_n \tau_{n+1}}{\sigma_{n+1}\tau_n}, \quad x_{2,n} = \frac{\sigma_{n+3}\tau_{n-1}}{\sigma_{n+2}\tau_n}, \tag{3.58}$$

lifts the deformed map $\tilde{\varphi}_{\mathrm{A}_2}$ to higher dimensional cluster map $\psi_{\mathrm{A}_2} = \rho^{-1}\mu_2\mu_1$ which is equivalent to the system

$$\begin{aligned} \tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1\sigma_{n+3}\tau_{n-1} \\ \sigma_{n+4}\tau_{n-1} &= \sigma_{n+2}\tau_{n+1} + a_2\sigma_{n+1}\tau_{n+2} \end{aligned} \tag{3.59}$$

As we are aware that this map possesses Laurent property, we can write the tau functions generated by the system (3.76) as following

$$\tau_n = \frac{N_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}}, \quad \sigma_n = \frac{N_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{e}_n}} \tag{3.60}$$

with $N^{(j)}(\hat{\mathbf{x}}) \in \mathbb{Z}[\hat{\mathbf{x}}]$ being polynomials in the initial cluster variables $\hat{\mathbf{x}} = (\tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, a_1, a_2) = (\tilde{x}_i)_{1\le i\le 9}$, which are not divisible by any of cluster variables $\tilde{x}_i$, and the denominator being monomials whose exponent is d-vector (denominator vector) in $\mathbb{Z}^7$. Note that we are not including the component of d-vector for $\tilde{x}_8, \tilde{x}_9$ as they are frozen variables (they do not emerge in the denominator of cluster variables). We can assemble the d-vectors of initial cluster $\hat{\mathbf{x}}$ to build the 7×7 identity matrix,

$$(\mathbf{d}_{-1}\ \mathbf{d}_0\ \mathbf{d}_1\ \mathbf{e}_0\ \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) = -I \tag{3.61}$$

Using the argument from the section 2.3.3, the components of d-vectors (degree vectors) of the tau functions satisfy the (max,+) tropical version of the exchange relations (2.74). Therefore the d-vectors $\mathbf{d}_n, \mathbf{e}_n$, which arise from (3.59), yields the following system of equations,

$$\begin{aligned} \mathbf{d}_{n+2} + \mathbf{e}_n &= \max(\mathbf{e}_{n+2} + \mathbf{d}_n, \mathbf{e}_{n+3} + \mathbf{d}_{n-1}), \\ \mathbf{e}_{n+4} + \mathbf{d}_{n-1} &= \max(\mathbf{e}_{n+2} + \mathbf{d}_{n+1}, \mathbf{e}_{n+1} + \mathbf{d}_{n+2}), \end{aligned} \tag{3.62}$$

This system permits us to derive the degree growth. To proceed we need to consider the tropical analogue of variables (3.75) given by the rational map π, as shown below,

$$\mathbf{X}_{1,n} = \mathbf{e}_n + \mathbf{d}_{n+1} - \mathbf{e}_{n+1} - \mathbf{d}_n, \qquad \mathbf{X}_{2,n} = \mathbf{e}_{n+3} + \mathbf{d}_{n-1} - \mathbf{e}_{n+2} - \mathbf{d}_n, \tag{3.63}$$

Such quantities leads us to the following result,

Lemma 3.5.1. *Given that d-vectors $\mathbf{d}_n, \mathbf{e}_n$ satisfy the system* (3.62)*, the quantity $\mathbf{X}_{j,n}$ for $j = 1, 2$ in* (3.63) *are induced by the tropical map $\mathbf{X}_{j,n+1} = \varphi_{trop}(\mathbf{X}_{j,n})$, which is specified by*

$$\begin{aligned} \mathbf{X}_{1,n+1} &= [\mathbf{X}_{2,n}]_+ - \mathbf{X}_{1,n}, \\ \mathbf{X}_{2,n+1} &= \big[[\mathbf{X}_{2,n}]_+ - \mathbf{X}_{1,n}\big]_+ - \mathbf{X}_{2,n}. \end{aligned} \tag{3.64}$$

where $[X_{j,n}]_+ = \max(X_{j,n}, 0)$. Given arbitrary initial data $(\mathbf{X}_{1,0}, \mathbf{X}_{2,0})$, the orbit of φ_{trop} is periodic with period 5.

Proof. Firstly, the relations can be found by rearrange (3.62) and rewrite the equations in terms of $\mathbf{X}_{i,n}$ using (3.63). One can confirm that the map φ_{trop} is tropical analogue of original cluster map φ_{A_2} associated with type A_2. This implies that each quantity $\mathbf{X}_{i,n}$ corresponds to each d-vector of cluster variable $x_{i,n}$ defined in cluster algebra of type A_2. Furthermore as already noted that the cluster map is periodic with period 5, and hence the orbits of φ_{trop} is also periodic with period 6. Alternatively, the periodicity also can be found by directly checking case by case analysis of initial data in different sectors of the plane. □

Remark 3.5.2. *For the periodicity of the scalar map, one can directly verify by performing the iteration* (3.64)*. The initial d-vectors* (3.61) *determines the pair of vectors of initial values,*

$$\mathbf{X}_{1,0} = (0, 1, -1, -1, 1, 0, 0), \quad \mathbf{X}_{2,0} = (-1, 1, 0, 0, 0, 1, -1) \tag{3.65}$$

The iteration (3.64) *produces subsequent pair of vectors*

n	$\mathbf{X}_{1,n}$	$\mathbf{X}_{2,n}$
1	$(0, 0, 1, 1, -1, 1, 0)$	$(1, -1, 1, 1, 0, 0, 1)$
2	$(1, 0, 0, 0, 1, -1, 1)$	$(0, 1, -1, -1, 1, 0, 0)$
3	$(-1, 1, 0, 0, 0, 1, -1)$	$(0, 0, 1, 1, -1, 1, 0)$
4	$(1, -1, 1, 1, 0, 0, 1)$	$(1, 0, 0, 0, 1, -1, 1)$
5	$(0, 1, -1, -1, 1, 0, 0)$	$(-1, 1, 0, 0, 0, 1, -1)$

As one can see in the table, the iterates become initial values after 5th iteration of (3.64)*. As another option, when we choose the initial data* $\mathbf{X}_{1,0} = (-1, 0)$ *and* $\mathbf{X}_{2,0} = (0, -1)$*, we have*

$$\begin{aligned} &(-1, 0) \to (1, 0) \to (0, 1) \to (0, -1) \to (1, 1) \to (-1, 0) \\ &(0, -1) \to (1, 1) \to (-1, 0) \to (1, 0) \to (0, 1) \to (0, -1) \end{aligned} \tag{3.66}$$

This corresponds precisely to the sequence of pairs of d-vectors arising from the Zamolodchikov periodicity of the orbit of mutations in cluster algebra of type A_2 *shown in the table in Example 2.1.8(see also Example 2.1.32).*

In fact, the periodicity of the quantities $\mathbf{X}_{1,n}$ and $\mathbf{X}_{2,n}$ is the key to determine the degree growth of d-vectors $\mathbf{d}_n$ and $\mathbf{e}_n$, which is shown in the result below,

Theorem 3.5.3. *The d-vectors* $\mathbf{d}_n$ *and* $\mathbf{e}_n$*, which solve the system of equations* (3.62)*, are elements in the kernel of linear difference operator*

$$\mathcal{L} = (\mathcal{T}^5 - 1)(\mathcal{T}^3 - 1)(\mathcal{T} - 1) \tag{3.67}$$

where $\mathcal{T}$ is shift operator corresponding to $n \to n+1$. For the tau functions generated, the leading order of degree growth of their denominators is given by

$$\begin{aligned}\mathbf{e}_n &= \frac{1}{15}(1,1,1,1,1,1)n^2 + O(n)\\ \mathbf{d}_n &= \frac{1}{15}(1,1,1,1,1,1)n^2 + O(n)\end{aligned} \tag{3.68}$$

Proof. In terms of linear operator $\mathcal{T}$, the tropical relations in (3.63) can be expressed as,

$$\begin{aligned}\mathbf{X}_{1,n} &= (\mathcal{T}-1)\mathbf{d}_n - (\mathcal{T}-1)\mathbf{e}_n\\ \mathbf{X}_{2,n} &= (\mathcal{T}^{-1}-1)\mathbf{d}_n + (\mathcal{T}^3-\mathcal{T}^2)\mathbf{e}_n\end{aligned} \tag{3.69}$$

The sum of $\mathbf{X}_{1,n} + \mathbf{X}_{2,n+1}$ cancels out the terms of $\mathbf{d}_n$ and by the Lemma 3.5.1

$$\mathcal{L}\mathbf{e}_n = (\mathcal{T}^5-1)(\mathcal{T}^4-\mathcal{T}^3-\mathcal{T}+1)\mathbf{e}_n = 0 \tag{3.70}$$

The solution of the characteristic polynomial gives the solution of recurrence relations above

$$\mathbf{e}_n = \mathbf{a}n^2 + O(n) \tag{3.71}$$

for some constant vector $\mathbf{a} \in \mathbb{Z}^7$. Note that the linear operator $\mathcal{L}$ can be rewritten as

$$(\mathcal{T}^5-1)(\mathcal{T}^3-1)(\mathcal{T}-1)\mathbf{e}_n = 0 \iff (\mathcal{T}^5-1)(\mathcal{T}^3-1)\mathbf{e}_{n+1} = (\mathcal{T}^5-1)(\mathcal{T}^3-1)\mathbf{e}_n \tag{3.72}$$

This implies $(\mathcal{T}^5-1)(\mathcal{T}^3-1)\mathbf{e}_n$ is constant in n and it is equal to $30\mathbf{a}$. To derive the constant $\mathbf{a}$ at the leading order term, we need to calculate more terms $\mathbf{e}_n$ by tropical relations (3.62) with initial tropical seed (3.61). This gives rises to sequences

$$\begin{gathered}\mathbf{e}_4 = (1,0,0,1,0,0,0), \quad \mathbf{e}_5 = (1,1,0,1,1,0,0), \quad \mathbf{e}_6 = (2,1,1,2,1,1,0)\\ \mathbf{e}_7 = (3,2,1,3,2,1,1), \quad \mathbf{e}_8 = (3,3,2,4,3,2,1)\end{gathered}$$

Therefore we can calculate the constant term, (3.72),

$$(\mathcal{T}^5-1)(\mathcal{T}^3-1)\mathbf{e}_n = 30\mathbf{a} = (2,2,2,2,2,2,2) \tag{3.73}$$

and fix the vector $\mathbf{a}$. This enable us to determine the leading order behaviour of the sequence and hence it is given by

$$\mathbf{e}_n = \frac{1}{15}(1,1,1,1,1,1)n^2 + O(n)$$

For degree growth of d-vector $\mathbf{d}_n$, we can derive it from the

$$(\mathcal{T}^5 - 1)(\mathcal{T}^3 - 1)X_{1,n} = (\mathcal{T}^5 - 1)(\mathcal{T}^3 - 1)(\mathcal{T} - 1)\mathbf{d}_n - \cancelto{0}{\mathcal{L}\mathbf{e}_n} = 0$$

Thus for $\mathbf{d}_n$ satisfying the relation above, we can write

$$\mathbf{d}_n = \mathbf{b}n^2 + O(n) \tag{3.74}$$

where $\mathbf{b}$ is element of $\mathbb{Z}^7$. We can specify the coefficient $\mathbf{b}$ from the terms, generated by the (3.62),

$$(0,0,0,1,0,0,0), \quad (1,0,0,1,1,0,0), \quad (1,1,0,2,1,1,0), \quad (2,1,1,3,2,1,1),$$
$$(3,2,1,3,3,2,1), \quad (4,3,2,5,3,3,2), \quad (5,4,3,6,5,3,3)$$

As a result, we can fix the constant vector $\mathbf{b}$ from

$$(\mathcal{T}^5 - 1)(\mathcal{T}^3 - 1)\mathbf{d}_n = 30\mathbf{b} = (2,2,2,2,2,2,2,2).$$

Hence the d-vector $\mathbf{d}_n$ yields the following expression,

$$\mathbf{d}_n = \frac{1}{15}(1,1,1,1,1,1)n^2 + O(n)$$

□

Remark 3.5.4. *The result suggests that the algebraic entropy of the cluster map is zero, which hints that the discrete dynamical system given by the deformed cluster map is integrable. This is indeed true as mentioned in Example 2.4.3, the deformed map* $\tilde{\varphi}_{\mathrm{A}_2}$ *preserves the first integral* (2.58) *and hence it is integrable. Thus the result of the algebraic entropy test verifies the integrability of the deformed* A_2 *map.*

3.5.2 Tropical dynamics and degree growth for type A_4 deformed map

Recalling the example 2.5.5, given initial cluster

$$\hat{\mathbf{x}} = (q_0, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, p_0, a_1, a_4) = (\tilde{x}_j)_{1 \le j \le 13},$$

deformed map lifted to higher dimensional cluster map $\psi_{A_4} = \rho^{-1}\mu_2\mu_1\mu_{11}\mu_5$ via the rational map π,

$$x_{1,n} = \frac{\sigma_n\tau_{n+1}}{\sigma_{n+1}\tau_n} \quad x_{2,n} = \frac{p_n}{\sigma_{n+2}\tau_n} \quad x_{3,n} = \frac{q_n}{\sigma_{n+3}\tau_n} \quad x_{4,n} = \frac{\sigma_{n+5}\tau_{n-1}}{\sigma_{n+4}\tau_n} \tag{3.75}$$

which is equivalent to the system,

$$\begin{aligned} \tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1p_n \\ p_{n+1}p_n &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + q_n\sigma_{n+1}\tau_{n+2} \\ q_{n+1}q_n &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + p_{n+1}\sigma_{n+5}\tau_{n-1} \\ \sigma_{n+6}\tau_{n-1} &= \sigma_{n+4}\tau_{n+1} + a_1q_{n+1} \end{aligned} \tag{3.76}$$

Since the map consists of Laurent property, the tau functions, the system of equations in (3.76) produces the sequence of tau functions expressed in the form

$$\tau_n = \frac{N_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}}, \quad \sigma_n = \frac{N_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{e}_n}}, \quad p_n = \frac{N_n^{(3)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{f}_n}}, \quad q_n = \frac{N_n^{(4)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{g}_n}} \tag{3.77}$$

where $N^{(j)}(\hat{\mathbf{x}}) \in \mathbb{Z}[\hat{\mathbf{x}}]$ is polynomials in the initial cluster variables $\hat{\mathbf{x}}$, and $\mathbf{d}_n, \mathbf{e}_n, \mathbf{f}_n, \mathbf{g}_n \in \mathbb{Z}^{11}$ are d-vectors whose initial data is given by 11×11 identity matrix,

$$(\mathbf{g}_0\ \mathbf{d}_{-1}\ \mathbf{d}_0\ \mathbf{d}_1\ \mathbf{e}_0\ \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\ \mathbf{e}_4\ \mathbf{e}_5\ \mathbf{g}_0) = -I \tag{3.78}$$

With the same argument mentioned in previous section, we can find the (max,+) tropical relations of the d-vectors, which arose from (3.76), shown as below

$$\begin{aligned} \mathbf{d}_{n+2} + \mathbf{e}_n &= \max(\mathbf{e}_{n+2} + \mathbf{d}_n, \mathbf{f}_n), \\ \mathbf{f}_{n+1} + \mathbf{f}_n &= \max(\mathbf{e}_{n+2} + \mathbf{e}_{n+3} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{g}_n + \mathbf{e}_{n+1} + \mathbf{d}_{n+2}), \\ \mathbf{g}_{n+1} + \mathbf{g}_n &= \max(\mathbf{e}_{n+3} + \mathbf{e}_{n+4} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{f}_{n+1} + \mathbf{e}_{n+5} + \mathbf{d}_{n-1}), \\ \mathbf{e}_{n+6} + \mathbf{d}_{n-1} &= \max(\mathbf{e}_{n+4} + \mathbf{d}_{n+1}, \mathbf{g}_{n+1}), \end{aligned} \tag{3.79}$$

Let us consider the tropical analogue of variables (3.75) given by the rational map π,

$$\begin{aligned} \mathbf{X}_{1,n} &= \mathbf{e}_n + \mathbf{d}_{n+1} - \mathbf{e}_{n+1} - \mathbf{d}_n, \qquad \mathbf{X}_{2,n} = \mathbf{g}_n - \mathbf{e}_{n+2} - \mathbf{d}_n, \\ \mathbf{X}_{3,n} &= \mathbf{f}_n - \mathbf{e}_{n+3} - \mathbf{d}_n, \qquad \mathbf{X}_{4,n} = \mathbf{e}_{n+5} + \mathbf{d}_{n-1} - \mathbf{e}_{n+4} - \mathbf{d}_n \end{aligned} \tag{3.80}$$

Then one can show the following result,

Lemma 3.5.5. *Given that d-vectors* $\mathbf{d}_n, \mathbf{e}_n, \mathbf{f}_n, \mathbf{e}_n$ *holds the system* (3.79)*, the quantity* $\mathbf{X}_{j,n}$ *for* $1 \leq j \leq 4$ *in* (3.80) *are induced by the tropical map* $\varphi_{\mathrm{A}_4}^{trop}$*, which is specified by*

$$\begin{aligned} \mathbf{X}_{1,n+1} + \mathbf{X}_{1,n} &= [\mathbf{X}_{2,n}]_+, \\ \mathbf{X}_{2,n+1} + \mathbf{X}_{2,n} &= [\mathbf{X}_{1,n+1} + \mathbf{X}_{3,n}]_+, \\ \mathbf{X}_{3,n+1} + \mathbf{X}_{3,n} &= [\mathbf{X}_{2,n+1} + \mathbf{X}_{4,n}]_+, \\ \mathbf{X}_{4,n+1} + \mathbf{X}_{4,n} &= [\mathbf{X}_{3,n+1}]_+. \end{aligned} \tag{3.81}$$

Given arbitrary initial data $(X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0})$*, the orbit of* $\varphi_{\mathrm{A}_4}^{trop}$ *is periodic with period 7.*

Proof. We can use same reasoning as Lemma 3.64. The above system of equations can be derived directly from the (3.79) by using the substitution (3.80). One can see that (3.81) takes form which is identical to (max,+) version of exchange relation given by the original cluster map $\varphi_{\mathrm{A}_4} = \mu_4\mu_3\mu_2\mu_1$ in the example 2.4.4. Hence by the periodicity property of φ_{A_2}, we can see that evolution of $\mathbf{X}_{1,n}$ is periodic with period 7. Alternatively, the periodicity also can be found by directly checking case by case analysis of initial data in different sectors of the plane. □

We can utilise the periodicity of terms (3.80) to obtain the following result.

Theorem 3.5.6. *Let* $\mathcal{T}$ *be linear operator which shifts* $n \to n+1$*. The d-vectors* $\mathbf{d}_n$ *and* $\mathbf{e}_n$*, which solve the system of equations* (3.62)*, are elements in the kernel of linear difference operator*

$$\mathcal{L} = (\mathcal{T}^7 - 1)(\mathcal{T}^5 - 1)(\mathcal{T} - 1) \tag{3.82}$$

where $\mathcal{T}$ *is shift operator corresponding to* $n \to n+1$*. For the tau functions generated, the leading order of degree growth of their denominators is given by*

$$\begin{aligned} \mathbf{e}_n &= \frac{n^2}{35}(2,1,1,1,1,1,1,1,1,1,2) + O(n) \\ \mathbf{d}_n &= \frac{n^2}{35}(2,1,1,1,1,1,1,1,1,1,2) + O(n) \\ \mathbf{q}_n &= \frac{n^2}{35}(4,2,2,2,2,2,2,2,2,2,4) + O(n) \\ \mathbf{p}_n &= \frac{n^2}{35}(4,2,2,2,2,2,2,2,2,2,4) + O(n) \end{aligned} \tag{3.83}$$

Proof. By definition (3.80) , we have

$$\mathbf{X}_{1,n} = (\mathcal{T}-1)\mathbf{d}_n - (\mathcal{T}-1)\mathbf{e}_n$$
$$\mathbf{X}_{4,n+1} = -(\mathcal{T}-1)\mathbf{d}_n + (\mathcal{T}^6-\mathcal{T}^5)\mathbf{e}_n$$

Followed by the periodicity of the terms, the sum $\mathbf{X}_{1,n}+\mathbf{X}_{4,n+1}$ produces the linear difference equation,

$$\begin{aligned}\mathcal{L}\mathbf{e}_n &= (\mathcal{T}^7-1)(\mathcal{T}^6-\mathcal{T}^5-\mathcal{T}+1)\mathbf{e}_n \\ &= (\mathcal{T}^7-1)(\mathcal{T}^5-1)(\mathcal{T}-1)\mathbf{e}_n = 0\end{aligned}$$

Therefore, given that $\mathbf{e}_n$ satisfy $\mathcal{L}\mathbf{e}_n = 0$, from the formula of $\mathbf{X}_{1,n}$ above we find

$$(\mathcal{T}^7-1)(\mathcal{T}^6-\mathcal{T}^5-\mathcal{T}+1)\mathbf{d}_n = 0.$$

Similarly, we can find the relations of d-vectors $\mathbf{g}_n$ and $\mathbf{f}_n$ from the relations $\mathbf{X}_{2,n}$ and $\mathbf{X}_{3,n}$ in (3.80) respectively. Then we have

$$\mathcal{L}\mathbf{g}_n = \mathcal{L}\mathbf{X}_{2,n} + \mathcal{L}\mathbf{e}_{n+2} + \mathcal{L}\mathbf{d}_n = 0$$
$$\mathcal{L}\mathbf{f}_n = \mathcal{L}\mathbf{X}_{3,n} + \mathcal{L}\mathbf{e}_{n+3} + \mathcal{L}\mathbf{d}_n = 0$$

Altogether, solving the linear difference equation above gives the expression for d-vectors whose leading order term is n^2 with some constant coefficient, $\mathbf{e}_n = \mathbf{a}n^2 + O(n)$. We now consider the sequences of d-vectors, emerged from the iteration of (3.79), are given by

$$\begin{aligned}
\mathbf{e}_6 &= (1,1,0,0,1,0,0,0,0,0,1)^T \\
\mathbf{e}_7 &= (1,1,1,0,1,1,0,0,0,0,1)^T \\
\mathbf{e}_8 &= (1,1,1,1,2,1,1,0,0,0,2)^T \\
\mathbf{e}_9 &= (2,1,1,1,2,2,1,1,0,0,3)^T \\
\mathbf{e}_{10} &= (3,2,1,1,2,2,2,1,1,0,3)^T \\
\mathbf{e}_{11} &= (4,3,2,1,3,2,2,2,1,1,4)^T \\
\mathbf{e}_{12} &= (5,3,3,2,4,3,2,2,2,1,5)^T \\
\mathbf{e}_{13} &= (6,4,3,3,5,4,3,2,2,2,7)^T
\end{aligned}$$

Note that from the sequences above, we can see the components of $\mathbf{e}_n$ taking the form

$$\mathbf{e}_n = (e_n^{(1)}, e_{n+2}^{(2)}, e_{n+1}^{(2)}, e_n^{(2)}, e_{n+5}^{(3)}, e_{n+4}^{(3)}, e_{n+3}^{(3)}, e_{n+2}^{(3)}, e_{n+1}^{(3)}, e_n^{(3)}, e_n^{(4)}) \tag{3.84}$$

which holds following relation,

$$(\mathcal{T}^7 - 1)(\mathcal{T}^5 - 1)\mathbf{e}_n = (4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4)^T = 70\mathbf{a}$$

This enables us to fix the constant coefficient $\mathbf{a}$, which lead us to the d-vector

$$\mathbf{e}_n = \mathbf{a}n^2 + O(n) = \frac{n^2}{35}(2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2)^T + O(n)$$

Similarly as (3.84), the sequence of d-vector $\mathbf{d}_n$ takes particular form

$$\mathbf{d}_n = (d_n^{(1)}, d_{n+2}^{(2)}, d_{n+1}^{(2)}, d_n^{(2)}, d_{n+5}^{(3)}, d_{n+4}^{(3)}, d_{n+3}^{(3)}, d_{n+2}^{(3)}, d_{n+1}^{(3)}, d_n^{(3)}, d_n^{(3)})$$

whose components are given by

$$d_n^{(1)} : 0, 0, 0, 0, 0, 1, 1, 2, 3, 3, 4, 5, 7, 8, 9$$
$$d_n : 0, 0, -1, 0, 0, 0, 1, 1, 2, 2, 2, 3, 4, 5, 5$$
$$d_n^{(2)} : 0, 0, 0, 1, 1, 1, 1, 2, 3, 3, 4, 4, 5, 6, 7$$

Then we find leading order quadratic growth of d-vector (3.80)

$$\mathbf{d}_n = \frac{n^2}{35}(2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2)^T + O(n)$$

Following on the result above, we can determine the coefficient of leading order terms for the d-vectors $\mathbf{g}_n$ and $\mathbf{f}_n$ from the formula of $\mathbf{X}_{2,n}, \mathbf{X}_{3,n}$ in (3.80) together with periodicity as following

$$\mathbf{g}_n \sim \mathbf{e}_{n+2} + \mathbf{d}_n \sim 2\mathbf{a}n^2$$
$$\mathbf{f}_n \sim \mathbf{e}_{n+3} + \mathbf{d}_n \sim 2\mathbf{a}n^2$$

with same constant $\mathbf{a}$. We claim the result (5.24).

□

Remark 3.5.7. *For homogenous degree of left and right hand side of relation* (3.76) *to be consistent, the homogeneous degree for q_n and p_n must be twice the degree of τ_n and σ_n. Hence the degrees of $\mathbf{g}_n$ and $\mathbf{f}_n$ should grow twice as fast as $\mathbf{d}_n$ and $\mathbf{e}_n$, which matches the result above.*

3.5.3 Algebraic entropy of Laurentified cluster map $\psi_{A_{2N}}$

Following the process of which we viewed in the previous section, we set the initial d-vectors

$$\left(\mathbf{g}_0^{(N-2)}, \dots, \mathbf{g}_0^{(0)}, \mathbf{d}_{-1}, \mathbf{d}_0, \mathbf{d}_1, \mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{2N+1}, \mathbf{f}_0^{(0)}, \dots \mathbf{f}_0^{(N-2)}\right) = -I \tag{3.85}$$

associated with initial cluster

$$\tilde{\mathbf{x}} = (\tilde{x}_j) = (q_{N-2,0}, \dots, q_{0,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \dots, \sigma_{2N}, \sigma_{2N+1}, p_{0,0}, \dots, p_{N-2,0})$$

Recall that we have shown that the map $\psi_{A_{2N}}$ given by (3.56) corresponds to the mutations in a cluster algebra. By Laurent property, we can write the tau functions generated by $\psi_{A_{2N}}$ in the form

$$\tau_n = \frac{N_n^{(1)}(\tilde{\mathbf{x}})}{\tilde{\mathbf{x}}^{\mathbf{d}_n}}, \quad \sigma_n = \frac{N_n^{(2)}(\tilde{\mathbf{x}})}{\tilde{\mathbf{x}}^{\mathbf{e}_n}}, \quad p_{j,n} = \frac{\hat{N}_n^{(j)}(\tilde{\mathbf{x}})}{\tilde{\mathbf{x}}^{\mathbf{f}_n^{(j)}}}, \quad q_{j,n} = \frac{\widetilde{N}_n^{(j)}(\tilde{\mathbf{x}})}{\tilde{\mathbf{x}}^{\mathbf{g}_n^{(j)}}}, \tag{3.86}$$

Using the same argument in the previous section above, we find that the corresponding d-vectors satisfy the ultradiscrete version of the system of exchange relations (3.56), which is given by

$$\begin{aligned}
\mathbf{d}_{n+2} + \mathbf{e}_n &= \max(\mathbf{e}_{n+2} + \mathbf{d}_n, \mathbf{f}_n^{(0)}), \\
\mathbf{f}_{n+1}^{(0)} + \mathbf{f}_n^{(0)} &= \max(\mathbf{e}_{n+3} + \mathbf{e}_{n+2} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{f}_n^{(1)} + \mathbf{e}_{n+1} + \mathbf{d}_{n+2}), \\
\mathbf{f}_{n+1}^{(1)} + \mathbf{f}_n^{(1)} &= \max(\mathbf{e}_{n+4} + \mathbf{e}_{n+3} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{f}_n^{(2)} + \mathbf{f}_{n+1}^{(0)}), \\
&\vdots \\
\mathbf{f}_{n+1}^{(N-2)} + \mathbf{f}_n^{(N-2)} &= \max(\mathbf{e}_{n+N+1} + \mathbf{e}_{n+N} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{g}_n^{(N-2)} + \mathbf{f}_{n+1}^{(N-3)}), \\
\mathbf{g}_{n+1}^{(N-2)} + \mathbf{g}_n^{(N-2)} &= \max(\mathbf{e}_{n+N+2} + \mathbf{e}_{n+N+1} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{g}_n^{(N-3)} + \mathbf{f}_{n+1}^{(N-2)}), \\
\mathbf{g}_{n+1}^{(N-3)} + \mathbf{g}_n^{(N-3)} &= \max(\mathbf{e}_{n+N+3} + \mathbf{e}_{n+N+2} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{g}_n^{(N-4)} + \mathbf{g}_{n+1}^{(N-2)}), \\
&\vdots \\
\mathbf{g}_{n+1}^{(0)} + \mathbf{g}_n^{(0)} &= \max(\mathbf{e}_{n+2N} + \mathbf{e}_{n+2N-1} + \mathbf{d}_n + \mathbf{d}_{n+1}, \mathbf{e}_{n+2N+1} + \mathbf{d}_{n-1} + \mathbf{g}_{n+1}^{(1)}), \\
\mathbf{e}_{n+2N+2} + \mathbf{d}_{n-1} &= \max(\mathbf{e}_{n+2N} + \mathbf{d}_{n+1}, \mathbf{g}_{n+1}^{(0)})
\end{aligned} \tag{3.87}$$

For the next step, we introduce the quantities $\mathbf{X}_i$ which are analogous to the tropical version of variable transformation (3.33), as shown below.

$$
\begin{aligned}
X_{1,n} &= \mathbf{e}_n + \mathbf{d}_{n+1} - \mathbf{e}_{n+1} - \mathbf{d}_n, \\
X_{2,n} &= \mathbf{f}_n^{(0)} - \mathbf{e}_{n+2} - \mathbf{d}_n, \\
&\vdots \\
X_{N,n} &= \mathbf{f}_n^{(N-2)} - \mathbf{e}_{n+N} - \mathbf{d}_n, \\
X_{N+1,n} &= \mathbf{g}_n^{(N-2)} - \mathbf{e}_{n+N+1} - \mathbf{d}_n, \\
&\vdots \\
X_{2N-1,n} &= \mathbf{g}_n^{(0)} - \mathbf{e}_{n+2N-1} - \mathbf{d}_n, \\
X_{2N,n} &= \mathbf{e}_{n+2N+1} + \mathbf{d}_{n-1} - \mathbf{e}_{n+2N} - \mathbf{d}_n
\end{aligned}
\tag{3.88}
$$

By defining these quantities, the system (3.87) can be expressed as discrete equations in (3.89), which is identical to the (max,+) relations obtained from the ultradiscretization of the cluster map $\varphi_{\mathrm{A}_{2N}}$. Then this leads to the following result.

Lemma 3.5.8. *The combination of d-vectors defined by* (3.88) *satisfy the tropical analogue of deformed* A_{2N} *map* $\tilde{\varphi}_{A_{2N}}$ (3.57)*, given by the following system of equations,*

$$
\begin{aligned}
X_{1,n+1} + X_{1,n} &= [X_{2,n}]_+ \\
X_{2,n+1} + X_{2,n} &= [X_{1,n+1} + X_{3,n}]_+ \\
&\vdots \\
X_{2N-1,n+1} + X_{2N-1,n} &= [X_{2N-2,n+1} + X_{2N,n}]_+ \\
X_{2N,n+1} + X_{2N,n} &= [X_{2N-1,n}]_+
\end{aligned}
\tag{3.89}
$$

which we specify by the tropical map $\varphi_{A_{2N}}^{trop}$*. Given arbitrary initial data* $(X_{j,0})_{1\leq j\leq 2N}$*, the orbit of the* $\varphi_{A_{2N}}^{trop}$ *is periodic with period* $2N+3$*.*

Proof. The (max,+) relations (3.89) can be directly derived from the structure of quantities (3.88) and the tropical analogue (ultradiscrete) of exchange relations (3.57) which gives (3.87). Notice that (3.89) is indeed a (max,+) relation of undeformed cluster map $\varphi_{A_{2N}}$. As we are aware that the map $\varphi_{A_{2N}}$ possess periodicity with period $2N+3$ due to Zamolodchikov periodicity. Hence the components of $\varphi_{A_{2N}}^{trop}$ is periodic with period $2N+3$ i.e.$(\varphi_{A_{2N}}^{\mathrm{trop}})^{2N+3}\mathbf{X}_n = \mathbf{X_n}$. □

By using the periodicity, one can calculate the degree growth of d-vectors.

Theorem 3.5.9. *The d-vectors* $\mathbf{d}_n, \mathbf{e}_n, \mathbf{g}_n^{(i)}, \mathbf{f}_n^{(i)}$ *satisfying the (max,+) relations* (3.87)*, are solution of following linear difference equation,*

$$(\mathcal{T}^{2N+3}-1)(\mathcal{T}^{2N+1}-1)(\mathcal{T}-1)\mathbf{r}_n = 0 \tag{3.90}$$

for $\mathbf{r}_n = \mathbf{d}_n, \mathbf{e}_n, \mathbf{g}_n^{(i)}, \mathbf{f}_n^{(i)}$*. For the tau functions generated, the leading order of degree growth of their denominators is given by*

$$\begin{aligned} \mathbf{d}_n &= \frac{1}{(8N^2+16N+6)}\mathbf{a}_1 n^2 + O(n), \quad \mathbf{e}_n = \frac{1}{(8N^2+16N+6)}\mathbf{a}_2 n^2 + O(n), \\ \mathbf{g}_n^{(i)} &= \frac{1}{(4N^2+8N+3)}\mathbf{a}_3 n^2 + O(n), \quad \mathbf{f}_n^{(i)} = \frac{1}{(4N^2+8N+3)}\mathbf{a}_4 n^2 + O(n) \end{aligned} \tag{3.91}$$

where $\mathbf{a} = (a_i)_{1\le i\le 4N+3}$ *whose entries* $a_i = 4$ *for* $i = \{1,\ldots,N-1\}\cup\{3N+5,\ldots,4N+3\}$ *and* $a_i = 2$ *for* $i = N-1,\ldots 3N+4$.

Proof. Given the shifting operator $\mathcal{T}: n \to n+1$, the first and last relation in (3.87) can be written as

$$\begin{aligned} X_{1,n} &= (\mathcal{T}-1)\mathbf{d}_n - (\mathcal{T}-1)\mathbf{e}_n, \\ X_{2N,n+1} &= -(\mathcal{T}-1)\mathbf{d}_n + (\mathcal{T}^{2N+2}-\mathcal{T}^{2N+1})\mathbf{e}_n \end{aligned} \tag{3.92}$$

Addition of these two terms, followed by applying the periodicity property i.e. $(\mathcal{T}^{2N+3}-1)(X_{i,n})_{i=1,2} = 0$, produces the linear difference equation,

$$\begin{aligned} &(\mathcal{T}^{2N+3}-1)(\mathcal{T}^{2N+1}-1)(\mathcal{T}-1)\mathbf{e}_n = 0 \\ \implies &(\mathcal{T}-1)^3(\sum_{i=0}^{2N+2}\mathcal{T}^i)(\sum_{i=0}^{2N}\mathcal{T}^i)\mathbf{e}_n = 0 \end{aligned} \tag{3.93}$$

It is clear that the corresponding characteristic polynomial has root $\lambda = 1$ with multiplicity 3, and the other roots are $\lambda = -1, \exp\{2k\pi/(2N+3)\}, \exp\{2l\pi/(2N+1)\}$ for $k \in \mathbb{Z}_{2N+2}$ and $l \in \mathbb{Z}_{2N}$. Thus for constant $\mathbf{a}$, the leading order of $\mathbf{e}_n$ takes the form

$$\mathbf{e}_n = \mathbf{a}n^2 + O(n) \tag{3.94}$$

as $n \to \infty$. Since the linear expression holds for $\mathbf{e}_n$, the constant vector $\mathbf{a}$ satisfies the linear relations,

$$(\mathcal{T}^{2N+3}-1)(\mathcal{T}^{2N+1}-1)\mathbf{e}_n = (8N^2+16N+6)\mathbf{a} \tag{3.95}$$

By inductive method, one can deduce the constant vector $\mathbf{a} = (a_i)_{1\le i\le 4N+3}$ whose entries $a_i = 4$ for $i = \{1,\ldots,N-1\} \cup \{3N+5,\ldots,4N+3\}$ and $a_i = 2$ for $i = N-1,\ldots 3N+4$.

By using the first relation in (3.92) together with periodicity of $\mathbf{X}_{1,n}$, the d-vector $\mathbf{e}_n$ satisfies same linear relation as $\mathbf{d}_n$, which shown by

$$\begin{aligned}(\mathcal{T}^{2N+3}-1)(\mathcal{T}^{2N+1}-1)\mathbf{X}_{1,n} &= \underbrace{(\mathcal{T}^{2N+3}-1)(\mathcal{T}^{2N+1}-1)(\mathcal{T}-1)\mathbf{d}_n}_{=0} \\ &\quad -(\mathcal{T}^{2N+3}-1)(\mathcal{T}^{2N+1}-1)(\mathcal{T}-1)\mathbf{e}_n \\ &= 0\end{aligned}$$

Then

$$\mathbf{d}_n = \mathbf{a}n^2 + O(n) \tag{3.96}$$

We can impose (3.94) and (3.96) into $\mathbf{X}_{i,n}$ for $2 \leq i \leq 2N-1$ and obtain the expression of other d-vectors whose leading order term is polynomial n^2. Then by inductive approach, we can show the required result (3.91). □

Since the growths of degree of the variables is quadratic, the entropy gives zero. Hence the result leads us to conjecture that deformed type A_{2N} cluster map is Liouville integrable.

Chapter 4

Deformation of cluster type: C_2, B_3 and D_4

4.1 An integrable deformation of the C_2 cluster map

In this section, we will consider the deformation of the periodic cluster map, which is constructed by the cluster algebra of type C_2.

4.1.1 Deformed C_2 map

The Cartan matrix for the C_2 root system is

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

and this is the companion to the exchange matrix $B = (b_{ij})$ given by

$$B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}. \tag{4.1}$$

which is obtained from C by removing the diagonal terms and adjusting the signs of the off-diagonal terms appropriately (with the requirement that if $b_{ij} \neq 0$, then

b_{ji} should have the opposite sign). The latter matrix is skew-symmetrizable, since for $D = \mathrm{diag}(1, 2)$ we have that

$$\Omega = BD = (\omega_{ij})$$

is the skew-symmetric matrix. Skew-symmetrizability of B is seen from the fact that $\Omega = BD$ is skew-symmetric, where $D = \mathrm{diag}(1, 2)$ gives

$$\Omega = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \tag{4.2}$$

Starting from an initial cluster $\mathbf{x} = (x_1, x_2)$, we consider a pair of deformed mutations, of the form

$$\begin{array}{lllll} \tilde{\mu}_1 : & (x_1, x_2) \mapsto (x_1', x_2), & x_1' x_1 & = & a_1 x_2^2 + b_1, \\ \tilde{\mu}_2 : & (x_1', x_2) \mapsto (x_1', x_2'), & x_2' x_2 & = & a_2 x_1' + b_2. \end{array} \tag{4.3}$$

One can confirm that, after applying the corresponding pair of matrix mutations, namely μ_1 followed by μ_2, according to the rule (2.1), the exchange matrix (4.1) is mutation periodic under the composition of mutations, that is to say

$$\mu_2 \mu_1(B) = B$$

So, this is similar to the situation for skew form constructed from the exchange matrix in previous cases (e.g.symplectic form (2.87) associated with type A_2 in Example 2.4.3). By a minor variation on Theorem 1.3 in [10] and adjusting the presymplectic structure to the skew-symmetrizable setting (see [24], for instance), the map $\tilde{\varphi}_{\mathrm{C}_2} = \tilde{\mu}_2 \tilde{\mu}_1$ composed from the pair of deformed mutations preserves the log-canonical two-form

$$\omega = \sum_{i<j} \frac{\omega_{ij}}{x_i x_j} \mathrm{d}x_1 \wedge \mathrm{d}x_2 = 2 \mathrm{d} \log x_1 \wedge \mathrm{d} \log x_2 \tag{4.4}$$

The latter is the skew form build from the coefficients of the matrix Ω in (4.2), which is obtained from skew-symmetrization of (4.1). In other words, $\tilde{\varphi}^*_{\mathrm{C}_2}(\omega) = \omega$. In addition to this, the two form (4.4) is non-degenerate as (4.1) is not singular, which suggests that deformed C_2 cluster map is symplectic for arbitrary values of the parameters a_i, b_i. However, note that when these parameters take the generic values, the composition of deformed mutations, $\tilde{\varphi}_{\mathrm{C}_2}$ is not a cluster map, because it does not generate Laurent polynomial in x_1, x_2.

The original undeformed mutations obtained from the exchange matrix (4.1), which generate the cluster algebra of type C_2, are recovered by setting all the parameters to 1. Since the Coxeter number of C_2 is 4, Zamolodchikov periodicity implies that the cluster map $\varphi_{\mathrm{C}_2} = \mu_2\mu_1$ has period $3 = \frac{1}{2}(4+2)$, so that

$$\varphi_{\mathrm{C}_2} = \mu_2\mu_1 \quad (\tilde{\varphi} \text{ with } a_1 = b_1 = a_2 = b_2 = 1) \implies \varphi^3(\mathbf{x}) = \mathbf{x}.$$

Therefore due to the periodicity, for any function $f : \mathbb{C}^2 \to \mathbb{C}$, the associated symmetric function given by the product over an orbit, that is

$$K_f(\mathbf{x}) = \prod_{j=0}^{2} (\varphi^*)^j(f)(\mathbf{x}) = \prod_{j=0}^{2} f((\varphi^*)^j(\mathbf{x})),$$

which are invariant under cluster map φ_{C_2}. Here we consider

$$K = \prod_{j=0}^{2} (\varphi^*)^j(x_2) = x_2 + \frac{2}{x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1x_2}. \tag{4.5}$$

Before proceeding further with the general deformed case, for arbitrary non-zero parameters a_i, b_i, we apply the rescaling $x_i \to \lambda_i x_i$ to each cluster variables with suitable choice of parameters $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$, so that $a_1 = 1 = a_2$ to simplify the calculations. With the remaining parameters b_1, b_2 fixed, the iteration of the deformed map $\tilde{\varphi}_{\mathrm{C}_2}$ is given by a system of recurrences

$$\begin{aligned} x_{1,n+1}x_{1,n} &= x_{2,n}^2 + b_1, \\ x_{2,n+1}x_{2,n} &= x_{1,n+1} + b_2. \end{aligned} \tag{4.6}$$

An invariant function for the deformed map $\tilde{\varphi}_{\mathrm{C}_2}$ can be constructed by following the same procedure, shown in the Section 3.2 (or Example 2.4.4), whereby we modify each Laurent monomial in (4.5) by inserting arbitrary coefficients κ_i in front of each monomial, so that we have

$$\tilde{K} = x_2 + \frac{\kappa_1}{x_2} + \frac{\kappa_2 x_1}{x_2} + \frac{\kappa_3 x_2}{x_1} + \frac{\kappa_4}{x_1x_2} \tag{4.7}$$

Next, we proceed to impose the condition of invariance on $\tilde{K}$, that is $\tilde{\varphi}^*(\tilde{K}) = \tilde{K}$, which constrains the coefficients κ_i and b_i. This gives rise to a necessary and sufficient condition for the deformed map (4.6) to be Liouville integrable, leading to the following result,

Theorem 4.1.1. *The necessary and sufficient condition for a rational function of the form* (4.7) *to be first integral for the map defined by* (4.6) *is*

$$b_1 = b_2 = \beta \tag{4.8}$$

in which case $\tilde{K}$ *is given by*

$$\tilde{K} = x_2 + \frac{1+\beta}{x_2} + \frac{x_2}{x_1} + \frac{\beta}{x_1 x_2} \tag{4.9}$$

Hence the deformed symplectic map $\tilde{\varphi}$ *given by*

$$\begin{array}{rccc} \tilde{\varphi}_{C_2} & : & \mathbb{C}^2 & \rightarrow \mathbb{C}^2 \\ & & \mathbf{x} = (x_1, x_2) & \mapsto \mathbf{x}' = (\frac{(x_2)^2+\beta}{x_1}, \frac{x_1'+\beta}{x_2}) \end{array} \tag{4.10}$$

is Liouville integrable whenever the condition (4.8) *holds.*

Proof. The proof of the above result follow from an explicit calculation, which can be carried out by using a computer algebra package such as MAPLE: assuming that a first integral of the form (4.7) exists, the equation $\varphi^*(\tilde{K}) = \tilde{K}$ can be rewritten as an identity between two polynomials in x_1, x_2, and then comparing coefficients at each degree yields a set of linear equations in the coefficients κ_i; this linear system has a solution if and only if (4.8) holds. □

Remark 4.1.2. *Notice that the family of level sets of the first integral* (4.9), *given by fixing the value* $\tilde{K} = \tilde{\kappa}$,

$$x_1(x_2)^2 + (1+\beta)x_1 + (x_1)^2 + (x_2)^2 + \beta = \tilde{\kappa} x_1 x_2,$$

is a pencil of biquadratic curves, that is, one paramter family of curves. This suggests that we can construct a map of QRT type [47, 25], which is defined by the two involutions :

$$\begin{array}{rl} \textit{Vertical switch:} & i_v : (x_1, x_2) \rightarrow (x_1, x_2^\dagger) \\ \textit{Horizontal switch:} & i_h : (x_1, x_2) \rightarrow (x_1^\dagger, x_2) \end{array} \tag{4.11}$$

where $x_1^\dagger, x_2^\dagger$ *are points on the curve which intersect with horizontal/vertical line, respectively. The QRT map is composed of the switches,*

$$\varphi_{QRT} = i_v \circ i_h \tag{4.12}$$

By using Vieta's formula for the product of roots of a quadratic, the horizontal and vertical switch can be written as

$$\begin{aligned} i_h : x_1 \to x_1^\dagger = \frac{b + x_2^2}{x_1} \\ i_v : x_2 \to x_2^\dagger = \frac{b + x_1}{x_2} \end{aligned} \tag{4.13}$$

Comparison with the formula (4.10)*, shows that in this case, the transformation* μ_1 *is the horizontal switch and* μ_2 *is the vertical switch. Hence the map coincides with the QRT map* φ_{QRT}*. For the full details of QRT map, we direct the reader to [47, 25].*

We have seen that, when the parameters satisfy the constraints (4.8), the symplectic map given by (4.6) is Liouville integrable. However, as mentioned above, general deformed cluster map is not itself a cluster map and this continues to be true for the constrained version (4.10), as the variables generated by the map are not elements of the Laurent polynomial ring. To resolve this issue, we must go to a step further, and try to apply Laurentification, analogously to what was carried out in the previous sections. We once again consider an empirical version of p-adic analysis, which is done by inspecting the prime factorization of the terms given by the iteration. With the choice of values for the initial cluster, $(x_1, x_2) = (1, 1)$, and parameters, $b_1 = 2 = b_2$, we find the prime factorizations of numerators and denominators of successive terms, as in the below:

n	$x_{1,n}$	$x_{2,n}$
1	3	5
2	3^2	$\frac{11}{5}$
3	$\frac{19}{5^2}$	$\frac{3\cdot 23}{5\cdot 11}$
4	$\frac{569}{11^2}$	$\frac{5\cdot 811}{3\cdot 11\cdot 23}$
5	$\frac{17^2\cdot 107}{3^2\cdot 23^2}$	$\frac{11\cdot 8089}{3\cdot 23\cdot 811}$
6	$\frac{139\cdot 3299}{811^2}$	$\frac{3\cdot 7\cdot 23\cdot 23039}{811\cdot 8089}$
7	$\frac{457737691}{8089^2}$	$\frac{13\cdot 173\cdot 811\cdot 3793}{7\cdot 23039\cdot 8089}$
8	$\frac{3\cdot 457\cdot 81689827}{7^2\cdot 23039^2}$	$\frac{5^3\cdot 41\cdot 39461\cdot 8089}{7\cdot 13\cdot 173\cdot 23039\cdot 3793}$

We can see that for each of the primes $p_1 = 5, 11, 23, 8089$ (for instance), the p-adic norm for $x_{1,n}$ and $x_{2,n}$ exhibit the patterns

$$\begin{aligned} &|x_{1,n}|_{p_1} : 1, 1, p^2, 1, 1 \\ &|x_{2,n}|_{p_1} : p^{-1}, p, p, p^{-1} \end{aligned} \tag{4.14}$$

Furthermore the prime $p_2 = 19, 569, 107, 139, 3299$, appear successively as factors in the numerator of $x_{1,n}$ but not in $x_{2,n}$. This suggests that there exists singularity patterns,

$$\begin{aligned} &\text{Pattern 1} : \ldots \to (R, 0) \to (R, \infty^1) \to (\infty^2, \infty^1) \to (R, 0^1) \to \ldots \\ &\text{Pattern 2} : \ldots \to (0^1, R) \to \ldots \end{aligned} \tag{4.15}$$

where R is a regular (non-zero) finite value. Then we introduce two tau-functions, $\tau_n \equiv 0 \pmod{p_1}$ and $\sigma_n \equiv 0 \pmod{p_2}$, which are associated with Pattern 1 and Pattern 2, respectively. Then we define a monomial rational map $\pi : \mathbb{C}^5 \to \mathbb{C}^2$, which is specified by the following transformation of dependent variables:

$$x_{1,n} = \frac{\sigma_n}{\tau_{n+1}^2}, \quad x_{2,n} = \frac{\tau_n \tau_{n+3}}{\tau_{n+1}\tau_{n+2}} \tag{4.16}$$

If the two expressions (4.16) are substituted directly into the components (4.6) of φ with the paramter constrained so that $b_1 = \beta = b_2$, one obtains following system of recurrence relations,

$$\begin{aligned} \sigma_n \sigma_{n+1} &= \beta \tau_{n+1}^2 \tau_{n+2}^2 + \tau_n^2 \tau_{n+3}^2 \\ \tau_n \tau_{n+4} &= \beta \tau_{n+2}^2 + \sigma_{n+1} \end{aligned} \tag{4.17}$$

If we iterate the latter pair of equations with initial values $(\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3) = (1, 1, 1, 1, 1)$ and $\beta = 2$, then we obtain a pair of integer sequences, with the first few terms presented in the following table:

n	0	1	2	3	4	5	6	7
σ_n	3	9	19	569	30923	458561	457737691	111996752817
τ_{n+4}	5	11	69	811	8089	161273	8530457	202237625

This table provides initial evidence that this system (4.17) has the Laurent property. Furthermore we can observe that the primes appearing as isolated factors in each of these integer sequences are the same ones that were identified as factors of numerators and denominators in the preceding table.

The system of recurrence relations (4.17) can be interpreted as iteration of a birational map $\psi : \mathbb{C}^5 \to \mathbb{C}^5$ which is interwined with φ via π, that is,

$$\psi : (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3) \to (\sigma_1, \tau_1, \tau_2, \tau_3, \tau_4), \quad \varphi \cdot \pi = \pi \cdot \psi$$

Then we would like to identify the initial data for the map ψ as an initial cluster in a seed for a cluster algebra of rank 5, so that $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)$. To verify that Laurent property holds when the deformed map $\tilde{\varphi}_{\mathrm{C}_2}$ is lifted to the map ψ on the space of tau functions, we need to find a cluster algebra structure defined by an initial seed $(\tilde{\mathbf{x}}, \tilde{B}_{\mathrm{C}_2})$, for a suitable exchange matrix $\tilde{B}_{\mathrm{C}_2} \in \mathrm{Mat}_5(\mathbb{Z})$. We will then proceed to show that this extends to a seed $(\hat{\mathbf{x}}, \hat{B}_{\mathrm{C}_2})$, where the initial cluster $\hat{\mathbf{x}} = (\tilde{\mathbf{x}}, \beta)$ includes the parameter β as a frozen variable, and $\hat{B}_{\mathrm{C}_2}$ is an extended 6×5 exchange matrix (with an additional row to incorporate the frozen variable).

To start with, we apply the pullback of the symplectic form (4.4) by the rational map π, $\pi^*\omega$,

$$\tilde{\omega} = \pi^*\omega = \sum_{i<j} \tilde{\omega}_{ij},$$

which gives rise to a new skew-symmetric matrix,

$$\tilde{\Omega} = (\tilde{\omega}_{ij}) = \begin{pmatrix} 0 & -2 & 2 & 2 & -2 \\ 2 & 0 & -4 & 0 & 0 \\ -2 & 4 & 0 & -4 & 4 \\ -2 & 0 & 4 & 0 & 0 \\ 2 & 0 & -4 & 0 & 0 \end{pmatrix} \tag{4.18}$$

Similar to the matrix in (4.2), the $\tilde{\Omega}$ can be expressed as a product $\tilde{\Omega} = \tilde{B}_{\mathrm{C}_2}\tilde{D}$ of skew-symmetrizable matrix $\tilde{B}_{\mathrm{C}_2}$ and diagonal matrix $\tilde{D}$. By post-multiplying by the diagonal matrix $\tilde{D}^{-1} = \mathrm{diag}(1, 1/2, 1/2, 1/2, 1/2)$, this gives the exchange matrix

$$\tilde{B}_{\mathrm{C}_2} = \tilde{\Omega}\tilde{D}^{-1} = (\tilde{B}_{ij}) = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -2 & 2 \\ -2 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \end{pmatrix} \tag{4.19}$$

Now observe that if we apply the composition $\hat{\mu}_2\hat{\mu}_1$ for the latter exchange matrix, applying the mutation $\hat{\mu}_1$ associated with index 1 followed by the mutation $\hat{\mu}_2$ associated with index 2, then the initial cluster $\tilde{\mathbf{x}} = (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3)$ gets transformed to $\hat{\mu}_2\hat{\mu}_1(\tilde{\mathbf{x}}) = (\tilde{x}_1', \tilde{x}_2', \tilde{x}_3, \tilde{x}_4, \tilde{x}_5) = (\sigma_1, \tau_4, \tau_1, \tau_2, \tau_3)$, where the new cluster variables σ_1, τ_4 are obtained from a single iteration of each of the recurrences in (4.17), setting $n = 0$ and $\beta = 1$ therein. To generate the general sequence of mutations for tau functions that corresponds to (4.17) with arbitrary β, it is necessary to extend the initial cluster to $\hat{\mathbf{x}} = (\tilde{\mathbf{x}}, \beta)$ by inserting the frozen variable β, and then a further calculation shows that we can define the extended exchange matrix

$$\hat{B} = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -2 & 2 \\ -2 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.20}$$

which is obtained by inserting an extra row at the bottom of (4.19). The form of the recurrence system (4.17) also requires that we permute the cluster variables after applying the two mutations $\hat{\mu}_1$ and $\hat{\mu}_2$.

Theorem 4.1.3. *Let ρ be the permutation* (2345)*. Then $\psi_{\mathrm{C}_2} = \rho^{-1}\hat{\mu}_2\hat{\mu}_1$ is a cluster map that fixes the extended exchange matrix $\hat{B}_{\mathrm{C}_2}$. Iteration of ψ generates two sequences of tau functions (σ_n), (τ_n) satisfying the system* (4.17)*. The tau functions are elements of $\mathbb{Z}_{>0}[\beta, \sigma_0^{\pm 1}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}]$.*

Proof. Consider the cluster algebra with initial cluster $\hat{\mathbf{x}} = (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \beta)$ and extended exchange matrix $\hat{B}_{\mathrm{C}_2}$. One can see that, by applying cluster mutation to $(\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \beta) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)$ in direction 1, mutation $\tilde{\mu}_1$ gives the exchange relation

$$\sigma_1\sigma_0 = \beta\tau_1^2\tau_2^2 + \tau_0^2\tau_3^2,$$

producing the new cluster $\hat{\mu}_1(\hat{\mathbf{x}}) = (\sigma_1, \tau_0, \tau_1, \tau_2, \tau_3, \beta)$ and the mutated exchange matrix $(\hat{B}_{\mathrm{C}_2})_1 = \hat{\mu}_1(\hat{B}_{\mathrm{C}_2})$ given by

$$(\hat{B}_{\mathrm{C}_2})_1 = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & -2 & 0 \\ 2 & -2 & 2 & 0 & -2 \\ -2 & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Following this up with a mutation in direction 2, applying $\tilde{\mu}_2$ gives the new cluster variable τ_4 defined by the following relation:

$$\tau_4\tau_0 = \beta\tau_2^2 + \sigma_1.$$

The new cluster is then $\hat{\mu}_2\hat{\mu}_1(\hat{\mathbf{x}}) = (\sigma_1, \tau_4, \tau_1, \tau_2, \tau_3, \beta)$. Therefore applying the composition of mutations $\hat{\mu}_2$ and $\hat{\mu}_1$ generates this pair of exchange relations, which corresponds to a single iteration of the map ψ_{C_2}, but requires an additional cyclic permutation of the middle 4 variables to obtain $\psi(\hat{\mathbf{x}}) = (\sigma_1, \tau_1, \tau_2, \tau_3, \tau_4, \beta)$. Furthermore, we see that the combination of two matrix mutations is equivalent to a permutation of order 4 acting on the corresponding 4 non-frozen labels, namely $\rho = (2345)$, i.e.

$$\hat{\mu}_2\hat{\mu}_1(\hat{B}_{\mathrm{C}_2}) = P_1\hat{B}_{\mathrm{C}_2}P_2 = \rho(\hat{B}_{\mathrm{C}_2})$$

where P_1 and P_2 are row and column permutation matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.21}$$

Thus we have shown that the extended exchange matrix $\hat{B}$ given by (4.20) is cluster mutation periodic in the generalized sense defined in [8], so that $\psi_{\mathrm{C}_2}(\hat{B}_{\mathrm{C}_2}) = \hat{B}_{\mathrm{C}_2}$, where the cluster map $\psi_{\mathrm{C}_2} = \rho^{-1}\hat{\mu}_2\hat{\mu}_1$ generates two sequences of tau functions satisfying the coupled system (4.17). Hence, by the Laurent phenomenon in the cluster algebra, it follows that iteration of the map ψ on the space of tau functions produces Laurent polynomials that are elements of $\mathbb{Z}_{>0}[\beta, \sigma_0^{\pm1}, \tau_0^{\pm1}, \tau_1^{\pm1}, \tau_2^{\pm1}, \tau_3^{\pm1}]$ (where each monomial that appears has a positive integer coefficient, due to positivity [34, 48]). □

Remark 4.1.4. *Note that Dynkin type* C_2 *and* B_2 *diagrams are isomorphic to each other. Then corresponding two exchange matrices can be obtained from each other by permutation of rows and columns. This implies that the undeformed cluster map, defined in the type* C_2 *case, can be constructed in the cluster algebra of type* B_2*. This leads to the conclusion that, in the case of type* B_2*, we can obtain the same result, presented above.*

Remark 4.1.5. *In a similar manner to the section 3.5, we can measure the algebraic entropy of the discrete dynamical system defined by iteration of* ψ_{C_2}*. By using the periodicity of the original (undeformed) cluster map* φ_{C_2} *and tropical method, one can explicitly show that degree growths of the* ψ_{C_2} *is quadratic (the calculation can be seen [15]), which suggest its algebraic entropy is equal to zero. This confirms that the deformed type* C_2 *map possesses integrability.*

4.1.2 Connection with Somos-5 and (special) Somos-7 recurrences

Recall Remark 2.5.4 in the section 2.5 , it was briefly mentioned that the tau functions generated by the cluster map associated with the deformed type A_2 map satisfy the particular Somos-7 relation. Similarly, there is a close connection between the sequence of tau functions given by (4.17) and the Somos type sequence. Here, we show the Somos-5 and Somos-7 recurrences hold for the sequence of τ_n satisfying (4.17). We start by introducing the general formula of Somos recurrence.

The sequence (x_n) induced by a quadratic recurrence,

$$x_{n+k}x_n = \sum_{j=1}^{\lfloor k/2 \rfloor} \alpha_j x_{n+j} x_{n+k-j} \tag{4.22}$$

is known as the Somos-k sequence. In particular, there are particular cases of Somos-type recurrences which fit into the framework of cluster algebras, namely that the recurrences have s sum of two monomials on the right-hand side of (4.22). One such case is the Somos-5 recurrence relation, which is written in the form,

$$\tau_n \tau_{n+5} = \tilde{\alpha} \tau_{n+1} \tau_{n+4} + \tilde{\beta} \tau_{n+2} \tau_{n+3}$$

It turns out that the Somos-5 recurrence can be reduced to a certain QRT map (see [49], for instance), equivalent to the recurrence,

$$u_{n+1} u_n u_{n-1} = \tilde{\alpha} u_n + \tilde{\beta} \tag{4.23}$$

where u_n takes the form of substitution $x_{2,n}$ in (4.16). We refer to the above iteration as Somos-5 QRT map. The substitution allows us to establish a connection between the system (4.17) and a suitable Somos-5 recurrence relaion.

Theorem 4.1.6. *The sequence of τ_n generated by the iteration of* (4.17) *satisfies Somos-5 relation with coefficients, that are constant along each orbit, given by*

$$\tau_n \tau_{n+5} = \zeta \tau_{n+1} \tau_{n+4} + \theta \tau_{n+2} \tau_{n+3} \tag{4.24}$$

where coefficients are

$$\zeta = 1 - \beta, \quad \theta = \frac{\beta((\beta \tau_1^2 + \sigma_0)\tau_2^2 + \tau_0^2 \tau_3^2)(\tau_1^2 + \sigma_0)}{\sigma_0 \tau_0 \tau_1 \tau_2 \tau_3} = \beta \tilde{K}, \tag{4.25}$$

with $\tilde{K}$ being the value of the first integral (4.9). *Hence $u_n = x_{2,n}$ satisfies the Somos-5 QRT map* (4.23) *with coefficients $\tilde{\alpha} = \zeta$, $\tilde{\beta} = \theta$ as in* (4.25) *along any orbit of the deformed* C_2 *map* (4.10).

Proof. The first three iterations of the Somos-5 sequence can be represented as the matrix form as

$$\underbrace{\begin{pmatrix} \tau_0\tau_5 & \tau_1\tau_4 & \tau_2\tau_3 \\ \tau_1\tau_6 & \tau_2\tau_5 & \tau_3\tau_4 \\ \tau_2\tau_7 & \tau_3\tau_6 & \tau_4\tau_5 \end{pmatrix}}_{M} \begin{pmatrix} 1 \\ -\zeta \\ -\theta \end{pmatrix} = 0 \tag{4.26}$$

As the vector $\mathbf{v} = \Big(1, -\eta, -\theta\Big)^T$ is non-zero, $\det(M) = 0$ is a necessary condition for the tau functions τ_n obtained from (4.17) to satisfy (4.24). With the help of MAPLE software, we can easily confirm that the relation holds. The coefficients ζ and θ can be found by computing the kernel of M, which turns out to be independent under shifting the indices of each tau function ($n \to n+1$): to be precise, $\zeta = 1-\beta$ is just a constant (independent of tau functions), while

$$\theta = \frac{\beta((\beta\tau_1^2+\sigma_0)\tau_2^2+\tau_0^2\tau_3^2)(\tau_1^2+\sigma_0)}{\sigma_0\tau_0\tau_1\tau_2\tau_3},$$

but this is just β times the first integral (4.9) lifted to the space of tau functions. Hence the vector $\mathbf{v}$ is constant along each orbit, and remains in the kernel of the matrix M when the replacement $\tau_n \to \tau_{n+1}$ is made for each tau function appearing therein.

□

We have seen that, subject to the condition $b_1 = \beta = b_2$, the variable $u_n = x_{2,n}$ satisfying one half of the system (4.6), also satisfies the Somos-5 QRT map (4.23) with appropriate coefficients $\tilde{\alpha}, \tilde{\beta}$. This suggests that each invariant curve for the deformed map $\tilde{\varphi}$, given by a level set of (4.9), is birationally equivalent to a corresponding elliptic curve associated with a level set of the Somos-5 QRT map. According to [50], each such curve is also isomorphic to a curve that corresponds to a level set of the Lyness map

$$w_{n+1}w_{n-1} = \tilde{\zeta}w_n + \tilde{\theta} \tag{4.27}$$

which is a birational map of the plane corresponding to the deformed integrable map of type A_2 (forthmentioned at Remark). (for suitable $\tilde{\zeta}$ and $\tilde{\theta}$), which is the integrable deformation of the periodic map of type A_2. Applying the results from [50], it can be shown that the iterates $x_{2,n}$ of the deformed C_2 map $\tilde{\varphi}$ are also associated with the Lyness map, via the transformation

$$w_n = x_{2,n} + \frac{\theta}{\zeta} = \frac{1}{\zeta}\frac{\tau_{n-1}\tau_{n+4}}{\tau_{n+1}\tau_{n+2}}.$$

The above substitution is consistent with the fact that τ_n satisfies the bilinear recurrence

$$\tau_{n+7}\tau_n = \tilde{\zeta}\tau_{n+6}\tau_{n+1} + \tilde{\theta}\tau_{n+3}\tau_{n+4}. \tag{4.28}$$

which a special type of Somos-7 recurrence, namely the same as (2.110) associated with the Lyness map (2.109). This is another type of Somos sequence generated by a sequence of mutations in a cluster algebra of rank 7 (for further detail see [6] and [7]). The same Somos-7 relation will also be seen to appear in the next section.

Remark 4.1.7. *Another way to see that the existence of the special Somos-7 relation (4.28) follows from Theorem 4.1.6, is to apply a result from [49], which says that every Somos-5 sequence also satisfies a Somos-k relation of odd order, for each odd integer $k \geq 7$: so every Somos-5 is also a Somos-7. The converse is not quite true, however: every Somos-7 does satisfy a relation of Somos-5 type, but generically it has one coefficient that is periodic with period 3, rather than having two constant coefficient (the result is proved in Appendix B of [15]).*

4.2 An integrable deformations of the periodic map for B_3

In this section, we consider the deformation of a 3D periodic cluster map which arises from mutations in the cluster algebra of type B_3. The original cluster map in 3 dimensions has period $4 = \frac{1}{2}(6+2)$, which is $\frac{1}{2} \times$ (Coxeter number + 2), and as before our aim is to construct parameter-families of deformations of this map that result in a periodic dynamics that is Liouville integrable. However, in contrast to all the examples previously considered, for the B_3 we find that there is more than one distinct family of deformations that is integrable (in fact, precisely two distinct 1-parameter families, up to obvious equivalence via scaling transformations).

4.2.1 Deformed map B_3

For the B_3 root system, the Cartan matrix is

$$C = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

which is the companion of the skew-symmetrizable exchange matrix,

$$B = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

Skew-symmetrizability of B is seen from the fact that $\Omega = BD$ is skew-symmetric, where $D = \mathrm{diag}(2,1,1)$ gives

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We now consider the sequence of deformed mutations

$$\begin{array}{llrcl} \tilde{\mu}_1: & (x_1,x_2,x_3)\mapsto(x_1',x_2,x_3), & x_1'x_1 & = & b_1+a_1x_2, \\ \tilde{\mu}_2: & (x_1',x_2,x_3)\mapsto(x_1',x_2',x_3), & x_2'x_2 & = & b_2+a_2(x_1')^2x_3, \\ \tilde{\mu}_3: & (x_1',x_2',x_3)\mapsto(x_1',x_2',x_3'), & x_3'x_3 & = & b_3+a_3x_2', \end{array} \tag{4.1}$$

where a_j, b_j are arbitrary parameters. With a generic choice of these parameters, the Laurent property no longer holds for these mutations, so the map $\tilde{\varphi}_{\mathrm{B}_3} = \tilde{\mu}_3 \cdot \tilde{\mu}_2 \cdot \tilde{\mu}_1$ does not have the Laurent property; moreover, it is no longer completely periodic with period 4 (generically, it defines an automorphism of infinite order of the field of rational functions on $\mathbb{C}^3$). However, by a slight extension of Theorem 1.3 in [10] (generalizing from the case of skew-symmetric B to the skew-symmetrizable case), it is not hard to see that this deformed version of $\tilde{\varphi}_{\mathrm{B}_3}$ preserves the same presymplectic form ω.

Before considering the deformed case (4.1) further, there are two ways to simplify the calculations. Firstly, assuming the case of generic parameter values $a_ib_i \neq 0$ for all i, we apply the scaling action of the three-dimensional algebraic torus $(\mathbb{C}^*)^3$, given by $x_i \to \lambda_i\, x_i$, $\lambda_i \neq 0$, and use this to remove three parameters, so that we may set

$$a_i \to 1, \qquad i = 1,2,3,$$

without loss of generality, but keep b_i arbitrary for $i = 1,2,3$. Having simplified the space of parameters, the map $\tilde{\varphi}_{\mathrm{B}_3}$ is equivalent to the iteration of the system of recurrences

$$\begin{array}{rcl} x_{1,n+1}x_{1,n} & = & x_{2,n}+b_1, \\ x_{2,n+1}x_{2,n} & = & x_{1,n+1}^2x_{3,n}+b_2, \\ x_{3,n+1}x_{3,n} & = & x_{2,n+1}+b_3. \end{array} \tag{4.2}$$

Secondly, since we are in an odd-dimensional situation where necessarily $\det(\Omega) = 0$ and ω is degenerate, we can apply Theorem 2.2.12 to reduce the deformed map to

2D symplectic map. By using

$$\ker\Omega =< (1,0,2)^T >, \qquad \mathrm{im}\,\Omega = (\ker\Omega)^\perp =< (0,1,0)^T, (-2,0,1)^T >,$$

we can generate the one-parameter scaling group $(x_1, x_2, x_3) \to (\lambda x_1, x_2, \lambda^2 x_3)$, $\lambda \in \mathbb{C}^*$ (obtained from the null vector field $x_1\partial_{x_1} + 2x_3\partial_{x_3}$ by exponentiation), and the projection $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ onto its monomial invariants,

$$\pi : \qquad y_1 = x_2, \qquad y_2 = \frac{x_3}{x_1^2}.$$

On the (y_1, y_2)-plane, $\tilde{\varphi}_{\mathrm{B}_3}$ induces the reduced map $\hat{\varphi}_{\mathrm{B}_3}$, such that $\pi \cdot \varphi_{\mathrm{B}_3} = \hat{\varphi}_{\mathrm{B}_3} \cdot \pi$, where

$$\hat{\varphi}_{\mathrm{B}_3} : \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1^{-1}((y_1+b_1)^2 y_2 + b_2) \\ y_1^{-1}\left(1 + \frac{b_3 y_1 + b_2}{y_2(y_1+b_1)^2}\right) \end{pmatrix}. \tag{4.3}$$

The reduced map is symplectic, that is to say $\hat{\varphi}^*(\hat{\omega}) = \hat{\omega}$, where the non-degenerate two-form preserved by $\hat{\varphi}$ is

$$\hat{\omega} = \mathrm{d}\log y_1 \wedge \mathrm{d}\log y_2, \qquad \pi^*\hat{\omega} = \omega. \tag{4.4}$$

In the original case where all parameters are 1, the reduced map (4.3) with $b_1 = b_2 = b_3 = 1$ has period 4, because $x_{2,n+4} = x_{2,n}$ and $x_{3,n+4}/x_{1,n+4}^2 = x_{3,n}/x_{1,n}^2$ for all n. In that case we can construct two functionally independent first integrals in the plane, $K^{(i)}, K^{(ii)}$ say. Here we will just focus on one of these, namely

$$\begin{aligned} K^{(i)} &:= \textstyle\sum_{i=0}^{3} (\hat{\varphi}_{\mathrm{B}_3}^*)^i (y_1) \\ &= y_1 y_2 + y_1 + 3y_2 + 3\frac{y_2}{y_1} + \frac{y_2}{y_1^2} + \frac{5}{y_1} + \frac{1}{y_2} + \frac{2}{y_1^2} + \frac{2}{y_1 y_2} + \frac{1}{y_1^2 y_2}, \end{aligned} \tag{4.5}$$

which satisfies $\hat{\varphi}_{\mathrm{B}_3}^*(K^{(i)}) = K^{(i)}$ when $b_1 = b_2 = b_3 = 1$.

Next, we modify $K^{(i)}$ by inserting constant coefficients in front of each of the Laurent monomials in y_1, y_2 that appear, fixing the coefficient of the first term to be 1 without loss of generality, to obtain

$$\hat{K} = y_1 y_2 + c_1 y_1 + c_2 y_2 + c_3\frac{y_2}{y_1} + c_4\frac{y_2}{y_1^2} + \frac{c_5}{y_1} + \frac{c_6}{y_2} + \frac{c_7}{y_1^2} + \frac{c_8}{y_1 y_2} + \frac{c_9}{y_1^2 y_2}. \tag{4.6}$$

If we assume that these modified first integrals are preserved by the deformed map $\hat{\varphi}_{\mathrm{B}_3}$ given by (4.3), then this puts a finite number of constraints on the coefficients c_i and the parameters b_i, which leads to finding necessary and sufficient conditions for the deformed symplectic map to be Liouville integrable. Thus we obtain the following result.

Theorem 4.2.1. *For the deformed symplectic map (4.3) to admit a first integral of the form (4.6), it is necessary and sufficient that the parameters b_i should satisfy either*

$$b_1 = b_2, \quad b_3 = 1, \tag{4.7}$$

or

$$b_2 = b_3 = b_1^2. \tag{4.8}$$

If we fix $b_1 = \beta$, then in the case that the constraint (4.7) holds, the first integral takes the form

$$\hat{K}_1 = y_1 y_2 + y_1 + (2\beta+1)y_2 + \beta(\beta+2)\frac{y_2}{y_1} + \beta^2 \frac{y_2}{y_1^2} + \frac{3\beta+2}{y_1} + \frac{1}{y_2} + \frac{2\beta}{y_1^2} + \frac{2}{y_1 y_2} + \frac{1}{y_1^2 y_2}, \tag{4.9}$$

while in the case that (4.8) holds, the first integral is

$$\hat{K}_2 = y_1 y_2 + y_1 + (2\beta+1)y_2 + \beta(\beta+2)\frac{y_2}{y_1} + \beta^2 \frac{y_2}{y_1^2} + \frac{2\beta^2+2\beta+1}{y_1} + \frac{1}{y_2} + \frac{2\beta^2}{y_1^2} + \frac{\beta^2+1}{y_1 y_2} + \frac{\beta^2}{y_1^2 y_2}. \tag{4.10}$$

Hence the map $\hat{\varphi}_{\mathrm{B}_3}$ given by (4.3) is Liouville integrable whenever either condition (4.7) or (4.8) holds.

Thus we arrive at two 1-parameter families of integrable maps of the plane associated with the deformation of the B_3 cluster map, namely

$$\hat{\varphi}^{(1)}_{\mathrm{B}_3}: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1^{-1}((y_1+\beta)^2 y_2 + \beta) \\ y_1^{-1}\left(1 + \frac{1}{y_2(y_1+\beta)}\right) \end{pmatrix}, \tag{4.11}$$

which has the first integral $\hat{K}_1$ given by (4.9), and

$$\hat{\varphi}^{(2)}_{\mathrm{B}_3}: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1^{-1}((y_1+\beta)^2 y_2 + \beta^2) \\ y_1^{-1}\left(1 + \frac{\beta^2(y_1+1)}{y_2(y_1+\beta)^2}\right) \end{pmatrix}, \tag{4.12}$$

with the first integral $\hat{K}_2$, as in (4.10).

Remark 4.2.2. *Each map has an invariant pencil of genus 1 curves of degree 5 and bidegree $(3,2)$; that is too high for a QRT map, where the bidegree is $(2,2)$ [47].*

Remark 4.2.3. *Clearly the maps coincide for $\beta = 1$ when the map is completely periodic with period 4 (corresponding to a pencil of elliptic curves with 4-torsion). However, it seems that the maps cannot be birationally conjugate to one another for* *other values of β; one way to see this is to look at the j-invariants of the curves in each pencil, which are rational functions of β and the value of the invariant $\hat{K}_j = \kappa$ (for $j = 1, 2$, respectively): the factorizations of the two different j-invariants have polynomial factors in their denominators that appear with quite different degrees, and this could be used to show that there is no automorphism of $\mathbb{C}(\beta, \kappa)$ which transforms one elliptic fibration into the other; or perhaps there is a geometrical way to see this more easily. It is possible to see that the two maps cannot be conjugate to one another more directly, by considering the fixed points: for generic β, in the affine plane $\mathbb{C}^2$ the map (4.11) has three fixed points outside the line $y_1 = 0$ where it is singular, whereas the map (4.12) only has one fixed point outside this line.*

4.2.2 The deformed $\hat{\varphi}^{(1)}_{\mathrm{B}_3}$ for B_3

Let us consider the deformed map (4.11), which is rewritten in the form of recurrences

$$\begin{aligned} y_{1,n+1}y_{1,n} &= (y_{1,n}+\beta)^2 y_{2,n} + \beta \\ y_{2,n+1}y_{2,n}y_{1,n}(y_{1,n}+\beta) &= (y_{1,n}+\beta)y_{2,n} + 1 \end{aligned} \tag{4.13}$$

Following the same process as in the previous section, we study the singularity structures of the deformed map (4.11) by observing the p-adic properties of iterates defined over $\mathbb{Q}$. With the choice of values for parameters $\beta = 5$, the iteration starting from initial data $(x_1, x_2) = (1, 1)$ gives a particular orbit, as shown in the table below:

n	$y_{1,n}$	$y_{2,n}$
1	41	$\frac{7}{2\cdot 3}$
2	$\frac{181}{3}$	$\frac{2^2}{7\cdot 23}$
3	$\frac{127}{3\cdot 23}$	$\frac{3}{2^4\cdot 7}$
4	$\frac{547}{7\cdot 23}$	$\frac{3^2\cdot 23}{59}$
5	$\frac{61\cdot 503}{7\cdot 59}$	$\frac{7\cdot 23^2}{2^3\cdot 3^2\cdot 13^2}$
6	$\frac{73\cdot 3527}{13^2\cdot 59}$	$\frac{7^3\cdot 59}{23^2\cdot 2729}$
7	$\frac{479\cdot 683}{13^2\cdot 2729}$	$\frac{13^2\cdot 59^2}{2\cdot 3\cdot 7^3\cdot 17\cdot 131}$
8	$\frac{43\cdot 1427\cdot 3847}{3\cdot 17\cdot 131\cdot 2729}$	$\frac{2^2\cdot 13^4\cdot 2729}{59^2\cdot 97\cdot 277}$

From the table, we can see that the primes $p_1 = 3, 23, 59$, for instance, provide a specific sequence of p_1-adic norms as follows.

$$\begin{aligned} &|x_{1,n}|_{p_1} : 1, p_1, p_1, 1, 1, 1, 1 \\ &|x_{2,n}|_{p_1} : p_1, 0, p_1^{-1}, p_1^{-2}, p_1^{2}, 0, p \end{aligned} \tag{4.14}$$

, while the prime $p_2 = 41, 43, 61, 73, 181, 127, 479, 503, 547, 683, 1427, 3527, 3847$, appearing only in $y_{1,n}$, corresponds to isolated values of n where $|x_{1,n}|_{p_2} = p_2$. Then we find that there are two singularity patterns for $(y_{1,n}, y_{2,n})$, namely

$$\begin{aligned} &\text{Pattern 1 : } \ldots \to (R, \infty^1) \to (\infty^1, R) \to (\infty^1, 0^1) \to (R, 0^2) \to (R, \infty^2) \to \ldots \\ &\text{Pattern 2 : } \ldots \to (0, R) \to \ldots \end{aligned} \tag{4.15}$$

This indicates that the $y_{1,n}$ and $y_{2,n}$ can be written in terms of tau functions τ_n and η_n as

$$y_{1,n} = \frac{\eta_n}{\tau_{n+2}\tau_{n+3}}, \quad y_{2,n} = \rho_n \frac{\tau_{n+1}^2 \tau_{n+2}}{\tau_n^2 \tau_{n+4}}, \tag{4.16}$$

where the quantity ρ_n is an additional prefactor. However, substituting these expressions directly into (4.13) gives rise to relations between tau functions that are not in the form of cluster exchange relations. So in addition to p-adic analysis, we proceed to consider the singularity patterns more closely via explicit analysis with the introduction of a small quantity ϵ.

A discrete dynamical system defined by a birational map can have two types of singularities: the points in phase space at which the map is undefined, and the points where the Jacobian of the map vanishes. From (4.11), one can see that the deformed map $\hat{\varphi}_{\mathrm{B}_3}^{(1)}$ possesses a singularity at $y_1 = -\beta$. Performing singularity analysis by setting $y_{1,n} = -\beta + \epsilon$, we find the confined singularity pattern

$$\begin{pmatrix} -\beta \\ C \end{pmatrix} \to \begin{pmatrix} -1 \\ \infty^1 \end{pmatrix} \to \begin{pmatrix} \infty^1 \\ -1 \end{pmatrix} \to \begin{pmatrix} \infty^1 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0^2 \end{pmatrix} \to \begin{pmatrix} -\beta \\ \infty^2 \end{pmatrix} \to \begin{pmatrix} C' \\ -1/\beta \end{pmatrix}, \tag{4.17}$$

where C, C' are regular values, and when $\epsilon \to 0$ the subsequent terms are not indeterminate (they are generic, regular values). By comparing (4.17) with (4.15), it is clear that (4.17) corresponds to Pattern 1, but with more detail revealed. The detailed form of the singularity pattern suggests another way to relate $y_{1,n}$ to the tau function τ_n, after shifting by the parameter β, expressing it as

$$y_{1,n} = -\beta + \vartheta_n \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}}, \tag{4.18}$$

where ϑ_n is another prefactor. Defining a new variable $w_n = y_1 + \beta$ leads to a system of three recurrence relations, expressed in terms of $w_n, y_{1,n}$ and $y_{2,n}$. Furthermore, subtracting the first relation in (4.13) from w_n times the second and removing a common factor of $y_{1,n}$ results in simplifying the recurrence for $y_{2,n}$, yielding the three equations

$$\begin{aligned} w_n &= y_{1,n} + \beta, \\ y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta, \\ y_{2,n+1}y_{2,n}w_n^2 &= y_{1,n+1} + 1. \end{aligned} \tag{4.19}$$

Theorem 4.2.4. *Under the iteration of* (4.19)*, the quantity w_n satisfies the Lyness recurrence*

$$w_{n+1}w_{n-1} = \tilde{\alpha} w_n + \tilde{\beta}, \tag{4.20}$$

where the coefficients along each orbit of the map $\hat{\varphi}_1$ are $\tilde{\alpha} = 1 - \beta$ and $\tilde{\beta} = \beta\hat{K}_1 + 2\beta^2 + \beta + 1$.

Proof. By following the same approach as used in the proof of Theorem 4.1.6, after setting the prefactor $\vartheta_n \to 1$ in (4.18), one can show that τ_n satisfies a special Somos-7 relation of the same form as (4.28), namely

$$\tau_{n+7}\tau_n = \tilde{\alpha}\tau_{n+6}\tau_{n+1} + \tilde{\beta}\tau_{n+3}\tau_{n+4}. \tag{4.21}$$

where $\tilde{\alpha} = 1 - \beta$, and the coefficient $\tilde{\beta}$ is given as above in terms of β and the conserved quantity $\hat{K}_1$. Then from (4.18) and the first relation in (4.19), it is clear that w_n is given in terms of the tau function τ_n by

$$w_n = \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}}, \tag{4.22}$$

and from this it is an immediate consequence that w_n satisfies the Lyness recurrence (4.20) with these coefficients, which are constant along each orbit. □

As noted in the previous section, the special Somos-7 recurrence (4.21) corresponds to a cluster algebra of rank 7, where (τ_n) is a sequence of cluster variables and the coefficients $\tilde{\alpha}, \tilde{\beta}$ are regarded as frozen variables; and in that setting, the associated exchange matrix has rank 2 (for further details, see [7]). In the discussion of the deformed C_2 map, we already noted that there is a close connection between this special Somos-7 relation and Somos-5. This leads to a related result for the map defined by (4.13).

Theorem 4.2.5. *Under iteration of* (4.13)*, the quantity* $v_n = y_{1,n} + 1$ *satisfies the Somos-5 QRT map, in the form*

$$v_{n+1}v_nv_{n-1} = \hat{\alpha}v_n + \hat{\beta} \tag{4.23}$$

where the coefficients along each orbit of the map $\hat{\varphi}_1$ *are given by* $\hat{\alpha} = \tilde{K}_1 + \beta + 3$ *and* $\hat{\beta} = (\beta - 1)\hat{\alpha}$*.*

Proof. This result, including the above formulae for the coefficients $\hat{\alpha}, \hat{\beta}$, is a consequence of Theorem 1 in [50], which states that each invariant curve of the Lyness map is birationally equivalent to an invariant curve corresponding to the Somos-5 QRT map, and hence there is a direct correspondence between the orbits of the two maps, whenever the parameters of the maps related to each other in a specific way. See also Proposition B.2 in the appendix B section of the paper [15]. □

Recall from the discussion around Theorem 4.1.6 that a substitution of the form

$$v_n = \frac{\hat{\tau}_{n+4}\hat{\tau}_{n+1}}{\hat{\tau}_{n+3}\hat{\tau}_{n+2}} \tag{4.24}$$

relates (4.23) directly to the Somos-5 recurrence, that is

$$\hat{\tau}_{n+5}\hat{\tau}_n = \hat{\alpha}\hat{\tau}_{n+1}\hat{\tau}_{n+4} + \hat{\beta}\hat{\tau}_{n+2}\hat{\tau}_{n+3}. \tag{4.25}$$

Now the substitution (4.24) and the definition of the quantity v_n implies that

$$y_{1,n} = -1 + \frac{\hat{\tau}_{n+4}\hat{\tau}_{n+1}}{\hat{\tau}_{n+3}\hat{\tau}_{n+2}},$$

but in general this is not compatible with the substitution (4.22) that relates w_n to a solution of (4.21), in the sense that the tau functions τ_n and $\hat{\tau}_n$ need not be the same, but rather are related by a gauge factor that depends on n. Rather, the most general way to relate v_n to τ_n is to write

$$v_n = y_{1,n} + 1 = \xi_n \frac{\tau_{n+4}\tau_{n+1}}{\tau_{n+3}\tau_{n+2}}, \tag{4.26}$$

with another prefactor ξ_n that depends on n. It will turn out that, with an appropriate choice of gauge, this quantity is periodic with period 3. (See Theorem 4.2.6 below, and also Appendix B of [15] for the further detail)

Observe that, with the extra variable v added to y_1, y_2 and w, the map defined by (4.13) is equivalent to iteration of a system of four equations, namely

$$\begin{aligned} w_n &= y_{1,n} + \beta, \\ y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta, \\ v_n &= y_{1,n} + 1, \\ y_{2,n+1}y_{2,n}w_n^2 &= v_{n+1}, \end{aligned} \tag{4.27}$$

and upon substituting for y_1, y_2 from (4.16), for w from (4.18), and for v from (4.26) the most general set of relations between the tau functions is found to be the following:

$$\begin{aligned} &\vartheta_n \, \tau_{n+5}\tau_n = \beta \, \tau_{n+3}\tau_{n+2} + \eta_n, \\ &\eta_{n+1}\eta_n = \rho_n(\vartheta_n)^2 \, \tau_{n+5}^2\tau_{n+1}^2 + \beta \, \tau_{n+4}\tau_{n+3}^2\tau_{n+2}, \\ &\xi_n\tau_{n+4}\tau_{n+1} = \tau_{n+3}\tau_{n+2} + \eta_n, \\ &\rho_{n+1}\rho_n(\vartheta_n)^2 = \xi_{n+1}. \end{aligned} \tag{4.28}$$

Theorem 4.2.6. *There is a choice of gauge which fixes $\vartheta_n \to 1$ in the system (4.28), and implies that $\xi_{n+3} = \xi_n$ and $\rho_{n+6} = \rho_n$ for all n, with*

$$\prod_{i=0}^{5} \rho_i = \prod_{j=1}^{3} \xi_j = \hat{K}_1 + \beta + 3. \tag{4.29}$$

In that case, the system corresponds to a lift of the deformed B_3 *map* $\hat{\varphi}_1$ *to a birational map on an extended space of tau functions, that is*

$$\Phi: \ (\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \eta_0, \rho_0, \beta) \mapsto (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \eta_1, \rho_1, \beta), \tag{4.30}$$

where the sequences (τ_n), (η_n) possess the Laurent property, but the periodic coefficients ρ_n do not.

Proof. By definition, in the context of the tau function formulae (4.16), a gauge transformation is any transformation of the tau functions which leaves the variables $y_{1,n}, y_{2,n}$ invariant. If we make the replacement $\tau_n \to g_n \tau_n$, where the dependence of g_n on n is arbitrary, then clearly replacing $\eta_n \to g_{n+2}g_{n+3}\,\eta_n$ leaves $y_{1,n}$ the same, while replacing $\rho_n \to g_n^2 g_{n+4} g_{n+1}^{-2} g_{n+2}^{-1}\,\rho_n$ leaves $y_{2,n}$ unchanged. Now in (4.18), regardless of what non-zero prefactor ϑ_n appears to begin with, we can always make the replacement $\vartheta_n \to g_{n+2}g_{n+3}g_n^{-1}g_{n+5}^{-1}\vartheta_n = 1$; to be precise, this is achieved by specifying any solution of a linear difference equation of order 5 for $\log g_n$. With that choice of gauge, the variable w_n is given in terms of τ_n by (4.22), and the sequence (τ_n) satisfies the special Somos-7 recurrence (4.21), as in the proof of Theorem 4.2.4. However, as already mentioned above, in general the prefactor ξ_n appearing in (4.26) cannot be simultaneously fixed to be 1 (rather, fixing $\xi_n \to 1$, so that τ_n satisfies the Somos-5 relation (4.25), is a *different* gauge choice). Thus, in the "Somos-7 gauge", where $\theta_n = 1$, the system of recurrences (4.28) becomes

$$\begin{aligned} \tau_{n+5}\tau_n &= \beta\,\tau_{n+3}\tau_{n+2} + \eta_n, \\ \eta_{n+1}\eta_n &= \rho_n\,\tau_{n+5}^2\tau_{n+1}^2 + \beta\,\tau_{n+4}\tau_{n+3}^2\tau_{n+2}, \\ \xi_{n+1}\,\tau_{n+5}\tau_{n+2} &= \tau_{n+4}\tau_{n+3} + \eta_{n+1}, \\ \rho_{n+1}\rho_n &= \xi_{n+1} \end{aligned} \tag{4.31}$$

(having shifted $n \to n+1$ in the third relation). In the above, an extended "cluster" of initial data, including the fixed parameter ("frozen variable") β, is given by $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \eta_0, \rho_0, \beta)$, and via (4.16) this fixes initial data $\mathbf{y}_0 = (y_{1,0}, y_{2,0})$ for the map $\hat{\varphi}^{(1)}_{\mathrm{B}_3}$. Now iterating each of the equations (4.31) one by one, in order, starting from $n = 0$, produces in turn $\tau_5, \eta_1, \xi_1, \rho_1$, giving the image of the lifted map Φ as in (4.30). Notice that the intermediate step of finding ξ_1 can be skipped: for each n, by combining the last two relations, we have

$$\rho_{n+1}\rho_n = \frac{\tau_{n+4}\tau_{n+3} + \eta_{n+1}}{\tau_{n+5}\tau_{n+2}}.$$

Hence the first two relations in (4.31) appear like a pair of cluster exchange relations, with one of them having a coefficient ρ_n that is non-autonomous (dependent on n). Upon iteration of the map Φ, we obtain the three sequences (τ_n), (η_n), (ρ_n), which together specify the orbit $\mathbf{y}_n = (\hat{\varphi}^{(1)}_{\mathrm{B}_3})^n(\mathbf{y}_0)$, as well as the sequence (ξ_n) of intermediate values, which appear in the formula (4.26) for the quantities v_n. Now consider the ring of Laurent polynomials

$$\mathcal{R} = \mathbb{Z}[\beta, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}, \tau_4^{\pm 1}, \eta_0^{\pm 1}, \rho_0^{\pm 1}].$$

Direct calculation of three steps of Φ shows by inspection that $\xi_1, \xi_2 \in \mathcal{R}$, and

$$\xi_3 = \frac{\tau_3\tau_2 + \eta_0}{\tau_4\tau_1} = \xi_0 \in \mathcal{R},$$

hence the sequence (ξ_n) has period 3, or in other words $(\mathcal{T}^3 - 1)\,\xi_n = 0$ (where $\mathcal{T}$ denotes the shift operator that sends $n \to n+1$). Then, upon taking logarithms on both sides of the fourth relation in (4.31), we have

$$\begin{aligned} (\mathcal{T} + 1)\, \log \rho_n = \log \xi_{n+1} \implies (\mathcal{T}^3 - 1)(\mathcal{T} + 1)\, \log \rho_n = 0 \\ \implies (\mathcal{T}^6 - 1)\, \log \rho_n = (\mathcal{T}^3 - 1)(\mathcal{T}^3 + 1)\, \log \rho_n = 0, \end{aligned}$$

hence the sequence (ρ_n) has period 6, as required. However, while $\rho_0, \rho_1, \rho_5 \in \mathcal{R}$, we find that $\rho_2, \rho_3, \rho_4 \notin \mathcal{R}$: the latter three terms have non-monomial factors appearing in their denominators, so they cannot be cluster variables. A direct calculation shows that the product of three adjacent ξ_n is

$$\xi_1\xi_2\xi_3 = \hat{K}_1 + \beta + 3 \in \mathcal{R},$$

where here $\hat{K}_1$ is used to denote the value of the invariant along an orbit of the lifted map Φ, considered as a function of $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \eta_0, \rho_0, \beta$; hence $\hat{K}_1 \in \mathcal{R}$, and using the fourth relation in (4.31) once more, we see that the latter product is equal to $\rho_0\rho_1\rho_2\rho_3\rho_4\rho_5$, so (4.29) holds, as required. Next, we claim that $\tau_n, \eta_n \in \mathcal{R}$. To see this, we just need to show that $\tau_n \in \mathcal{R}$ for all n, since if this holds then the first relation in (4.31) implies immediately that $\eta_n = \tau_{n+5}\tau_n - \beta\,\tau_{n+3}\tau_{n+2} \in \mathcal{R}$. So we consider the aforementioned fact that, due to the gauge choice, τ_n satisfies the special Somos-7 recurrence (4.21), which has coefficients $\tilde{\alpha}, \tilde{\beta}$, and there is an associated Lyness invariant quantity (see [50], for instance), which we denote by $\tilde{K}$. Then, by a minor modification of Theorem 3.7 in [51] and its proof, it follows that the Somos-7 recurrence has the strong Laurent property, in the sense that $\tau_n \in \tilde{\mathcal{R}}$ for all $n \geq 0$, where

$$\tilde{\mathcal{R}} = \mathbb{Z}[\tilde{\alpha}, \tilde{\beta}, \tilde{K}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}, \tau_4, \tau_5, \tau_6].$$

(For further details, see Theorem B.5 in the appendix B section of the paper [15]) By inspection of the first two iterates of Φ, we can verify directly that $\tau_5, \tau_6 \in \mathcal{R}$, while from Theorem 4.2.4 we have $\tilde{\alpha} = 1 - \beta$, $\tilde{\beta} = \beta\hat{K}_1 + 2\beta^2 + \beta + 1$ and a short explicit calculation with computer algebra shows that $\tilde{K} = \hat{K}_1 + 2\beta + 2$. Since, as already noted, $\hat{K}_1 \in \mathcal{R}$ on an orbit of Φ, it follows that $\tilde{\alpha}, \tilde{\beta}, \tilde{K} \in \mathcal{R}$, hence $\tilde{\mathcal{R}}$ is a subring of $\mathcal{R}$. Thus we see that $\tau_n \in \mathcal{R}$ for $n \geq 0$, and an analogous argument extends this to $n < 0$ and completes the proof of the theorem. □

Remark 4.2.7. *The first two relations in* (4.31) *resemble exchange relations in a cluster algebra, but the third and fourth relations (which define ρ_n) do not. Thus, the sequence of tau functions cannot be produced by cluster exchange relations with frozen variables alone. Nevertheless, this can be considered an "almost Laurentification" of the deformed map: the tau functions τ_n and η_n are Laurent polynomials, while the periodic quantities ρ_n only contain a finite number of non-monomial factors in their denominators, so this is an example of the extended Laurent property [52], where only a finite extension of the ring $\mathcal{R}$ is required. The coefficients ρ_n are reminiscent of y-variables in a cluster algebra with coefficients, which can be used to generate non-autonomous difference equations, including those of discrete Painlevé type [53, 54]. We have attempted to construct the relations* (4.31) *from a suitable Y-system, by pulling back the two-form* (4.4) *to derive an associated exchange matrix (cf. the formulae* (4.44), (4.45) *and* (4.46) *for the case of the map $\hat{\varphi}_2$ below), but as yet we have not succeeded in doing this in a consistent way.*

4.2.3 The deformed map $\hat{\varphi}^{(2)}_{\mathrm{B}_3}$ for B_3

The action of the deformed map $\hat{\varphi}^{(2)}_{\mathrm{B}_3}$ given by (4.12) is equivalent to iteration of the coupled pair of recurrences

$$\begin{aligned} y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta^2, \\ y_{2,n+1}y_{2,n}y_{1,n}w_n^2 &= w_n^2 y_{2,n} + \beta^2(y_{1,n}+1), \end{aligned} \tag{4.32}$$

where for convenience we made use of the same variable $w_n = y_{1,n} + \beta$ as in the previous discussion of the map $\hat{\varphi}^{(1)}_{\mathrm{B}_3}$. Subtracting the first relation from the second gives rise to a simplified relation for $y_{2,n+1}$, and thus iterates of the map are generated by the system of four relations

$$\begin{aligned} w_n &= y_{1,n} + \beta, \\ y_{1,n+1} y_{1,n} &= w_n^2 y_{2,n} + \beta^2, \\ v_{n+1} &= y_{1,n+1} + \beta^2, \\ y_{2,n+1} y_{2,n} w_n^2 &= v_{n+1}, \end{aligned} \tag{4.33}$$

where, in contrast to (4.27), there is a different definition for the quantity $v_n = y_{1,n} + \beta^2$.

Subject to the parameter value $\beta = 5$ together with initial values $(y_{1,0}, y_{2,0}) = (1, 1)$, the orbits of the system (4.33) provide a specific sequence of rationals shown in the table below:

n	$y_{1,n}$	$y_{2,n}$	w_n	v_n
1	61	$\frac{43}{2\cdot 3^2}$	$2 \cdot 3 \cdot 11$	$2 \cdot 43$
2	$3^2 \cdot 19$	$\frac{2\cdot 7^2}{11^2\cdot 43}$	$2^4 \cdot 11$	$2^2 \cdot 7^2$
3	$\frac{3^2\cdot 17}{43}$	$\frac{307}{2^7\cdot 7^2}$	$\frac{2^4\cdot 23}{43}$	$\frac{2^2\cdot 307}{43}$
4	$\frac{16927}{7^2\cdot 43}$	$\frac{13\cdot 43\cdot 2677}{23^2\cdot 307}$	$\frac{2\cdot 3\cdot 23\cdot 199}{7^2\cdot 43}$	$\frac{2\cdot 13\cdot 2677}{7^2\cdot 43}$
5	$\frac{11\cdot 211\cdot 1283}{7^2\cdot 307}$	$\frac{7^2\cdot 19\cdot 43\cdot 88261}{2\cdot 3^2\cdot 13\cdot 199^2\cdot 2677}$	$\frac{2\cdot 3\cdot 199\cdot 2557}{7^2\cdot 307}$	$\frac{2\cdot 19\cdot 88261}{7^2\cdot 307}$
6	$\frac{3\cdot 9403\cdot 11273}{12\cdot 307\cdot 2677}$	$\frac{2\cdot 7^2\cdot 307\cdot 3401731}{19\cdot 88261\cdot 2557^2}$	$\frac{2^3\cdot 67\cdot 271\cdot 2557}{13\cdot 307\cdot 2677}$	$\frac{2^2\cdot 43\cdot 3401731}{13\cdot 307\cdot 2677}$
7	$\frac{3\cdot 23712415183}{13\cdot 19\cdot 88261\cdot 2677}$	$\frac{13\cdot 307\cdot 853\cdot 9152173\cdot 2677}{2^5\cdot 67^2\cdot 271^2\cdot 3401731}$	$\frac{2^3\cdot 67\cdot 271\cdot 2498599}{13\cdot 19\cdot 88261\cdot 2677}$	$\frac{2^2\cdot 7^2\cdot 853\cdot 9152173}{13\cdot 19\cdot 88261\cdot 2677}$
8	$\frac{537664578593509}{19\cdot 88261\cdot 3401731}$	$\frac{13\cdot 19\cdot 37\cdot 661\cdot 1663\cdot 88261\cdot 27241\cdot 2677}{853\cdot 9152173\cdot 2498599^2}$	$\frac{2\cdot 3^2\cdot 47\cdot 61\cdot 4391\cdot 2498599}{19\cdot 88261\cdot 3401731}$	$\frac{2\cdot 37\cdot 307\cdot 661\cdot 1663\cdot 27241}{19\cdot 88261\cdot 3401731}$

By looking into the prime factorization of the rationals, we observe that p_1-adic norm, for the primes $p_1 = 13, 43, 307, 2677, 88261$ (for instance), follow the pattern

$$\begin{aligned} &|y_{1,n}|_{p_1} : 1, 1, p_1, p_1, 1, 1, 1, 1 \\ &|y_{2,n}|_{p_1} : p_1^{-1}, p_1, 1, p_1^{-1}, p_1^{-1}, 1 \\ &|w_n|_{p_1} : 1, 1, p_1, p_1, 1, 1 \\ &|v_n|_{p_1} : p_1^{-1}, 1, p_1, p_1, 1, p_1^{-1} \end{aligned} \tag{4.34}$$

, whereas for the primes $p_2 = 23, 199, 271, 2557, 2498599$, appearing in $y_{2,n}$ and w_n, give the following pattern

$$\begin{aligned} &|y_{2,n}|_{p_2} : 1, p_2^2, 1 \\ &|w_n|_{p_2} : 1, p_2^{-1}, p_2^{-1}, 1 \end{aligned} \tag{4.35}$$

and the primes $p_3 = 17, 19, 61, 211, 1283, 16927, 9403, 11273, 23712415183$ only emerge in $y_{1,n}$, which gives the p_2-adic norm $|y_{1,n}|_{p_2} = p^{-1}$ for particular values of n. Then we can see the corresponding singularity patterns in the orbits of $(y_{1,n}, y_{2,n}, w_n, v_n)$, which are confined.

$$\begin{aligned}
&\text{Pattern 1}: \ \dots \to (R,0,R,0) \to (R,\infty,R,R) \to (\infty,R,\infty,\infty) \\
&\qquad\qquad \to (\infty,0,\infty,\infty) \to (R,0,R,R) \to (R,R,R,0) \to \dots \\
&\text{Pattern 2}: \ \dots \to (R,R,0,R) \to (R,\infty^2,0,R) \to \dots \\
&\text{Pattern 3}: \ \dots, \to (0,R,R,R), \dots .
\end{aligned} \tag{4.36}$$

We introduce tau functions τ_n, σ_n and η_n which correspond to Patterns 1,2 and 3, respectively, such that $y_{1,n}$, $y_{2,n}$, $w_n = y_{1,n} + \beta$ and $v_n = y_{1,n} + \beta^2$ can be written as

$$y_{1,n} = \frac{\eta_n}{\tau_{n+2}\tau_{n+1}}, \quad y_{2,n} = \frac{\tau_{n+4}\tau_{n+1}\tau_n}{\sigma_n^2\tau_{n+3}}, \quad w_n = \frac{\sigma_{n+1}\sigma_n}{\tau_{n+2}\tau_{n+1}}, \quad v_n = \frac{\tau_{n+4}\tau_{n-1}}{\tau_{n+2}\tau_{n+1}} \tag{4.37}$$

Note that the structure of tau functions in the variable v_n can be verified with fourth equation (4.33) i.e.

$$y_{2,n+1}y_{2,n}w_n^2 = \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}} = y_{1,n+1} + \beta^2 = v_{n+1}$$

Upon inspecting the structure of a particular singularity further by approaching it in the limit of a small parameter $\epsilon \to 0$, one can see that the singularities of $y_{1,n}$ and $y_{2,n}$ in Pattern 1 correspond to the sequence

$$\begin{pmatrix} C \\ -\frac{\beta^2(C+1)}{C^2+2\beta C+\beta^2} \end{pmatrix} \to \begin{pmatrix} -\beta^2 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -1 \\ \infty^1 \end{pmatrix} \to \begin{pmatrix} \infty^1 \\ -1 \end{pmatrix} \to \begin{pmatrix} \infty^1 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -\beta^2 \\ C' \end{pmatrix} \tag{4.38}$$

with C, C' being regular values, which propagates from the point $\left(C, -\frac{\beta^2(C+1)}{C^2+2\beta C+\beta^2}\right)$ where the Jacobian of the deformed map $\hat{\varphi}^{(2)}_{\mathrm{B}_3}$ is zero. Noting that the value $y_1 = -1$ appears in the singularity pattern, we can consider another variable $u_n = y_{1,n} + 1$, and find that

$$u_n = y_{1,n} + 1 = \xi_n \frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}}, \tag{4.39}$$

where the prefactor ξ_n cannot be removed without a change of gauge, which would modify the form of some of the expressions in v_n of (4.37). Notice that the ratios of tau functions in v_n of (4.37) and (4.39) are identical to the substitutions associated with the Lyness map and Somos-5 QRT map, respectively. This suggests that the quantities $v_n = y_{1,n} + \beta^2$ and $u_n = y_{1,n} + 1$ should provide solutions of these maps under iteration, as described by the following statement.

Theorem 4.2.8. *The quantities v_n generated under iteration of the system of recurrences* (4.33) *satisfy the Lyness map which is equivalent to the recurrence*

$$v_{n+1}v_{n-1} = \gamma v_n + \delta, \tag{4.40}$$

where the coefficients along each orbit of the $\hat{\varphi}^{(2)}_{\mathrm{B}_3}$ are specified by $\gamma = 1 - \beta^2$ and $\delta = \beta^2\hat{K}_2 + 2\beta(\beta^3 + 1)$, and the associated sequence of tau functions (τ_n) related via v_n in (4.37) *satisfies the Somos-7 recurrence*

$$\tau_{n+7}\tau_n = \gamma\, \tau_{n+6}\tau_{n+1} + \delta\, \tau_{n+4}\tau_{n+3}. \tag{4.41}$$

The corresponding iterates of $u_n = y_{1,n} + 1$ satisfy the Somos-5 QRT map which is given by the recurrence

$$u_{n+1}u_n u_{n-1} = \hat{\gamma} u_n + \hat{\delta}, \tag{4.42}$$

where $\hat{\gamma} = \hat{K}_2 + 2\beta + 2$ and $\hat{\delta} = (\beta^2 - 1)\hat{\gamma}$.

Proof. This follows from analogous arguments to those used in proving Theorem 4.2.4 and Theorem 4.2.5. For a more detailed explanation of the connection between the Lyness map (4.40) and the Somos-5 QRT map (4.42), see Proposition B.2 in the second appendix of [15]. □

The tau function expressions (4.37) can be substituted directly into (4.33), giving rise to the system of equations

$$\begin{aligned} \sigma_{n+1}\sigma_n &= \beta\tau_{n+2}\tau_{n+1} + \eta_n, \\ \eta_{n+1}\eta_n &= \tau_{n+4}\tau_n\sigma_{n+1}^2 + \beta^2\tau_{n+3}\tau_{n+2}^2\tau_{n+1}, \\ \tau_{n+5}\tau_n &= \beta^2\tau_{n+3}\tau_{n+2} + \eta_{n+1}. \end{aligned} \tag{4.43}$$

Since the three recurrences above are all of the right form for an exchange relation, it appears likely that their iteration can be described by a sequence of cluster mutations in an appropriate cluster algebra. To verify this is the case, we set the initial cluster to be

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7) = (\eta_0, \sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4),$$

and then determine a new exchange matrix via the pullback of the symplectic form (4.4) by the rational map $\tilde{\pi}: \mathbb{C}^7 \to \mathbb{C}^2$ defined by the equations for $(y_{1,0}, y_{2,0}) \in \mathbb{C}^2$ given by setting $n = 0$ in (4.37). As a result, one finds a presymplectic form on the space of tau functions, written in terms of the cluster variables $\tilde{x}_j$ for $1 \leq j \leq 7$ as

$$\tilde{\omega} = \tilde{\pi}^* \omega = \sum_{ij} \tilde{\Omega}_{ij} \mathrm{d} \log \tilde{x}_i \wedge \mathrm{d} \log \tilde{x}_j,$$

where the 7×7 matrix $\tilde{\Omega}$ is given by

$$\tilde{\Omega} = \begin{pmatrix} 0 & -2 & 1 & 1 & 0 & -1 & 1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{4.44}$$

Given the skew-symmetric matrix Ω as above, a skew-symmetrizable exchange matrix $\tilde{B}_{\mathrm{B}_3}$ such that $\tilde{B}_{\mathrm{B}_3} \tilde{D} = \tilde{\Omega}$ can be determined by post-multiplying with the diagonal matrix $\tilde{D}^{-1} = \mathrm{diag}(1, {}^1\!/_2, 1, 1, 1, 1, 1)$, to obtain

$$\tilde{B}_{\mathrm{B}_3} = \tilde{\Omega} \tilde{D}^{-1} = (\tilde{B}_{ij}) = \begin{pmatrix} 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \tag{4.45}$$

The matrix $\tilde{B}_{\mathrm{B}_3}$ above generates a coefficient-free cluster algebra, but both the parameter β and its square appear in front of some of the terms in (4.43). To incorporate this into the exchange relations, we extend the initial cluster by adding the frozen variable $\tilde{x}_8 = \beta$ and adjoining an extra row with entries $(0, 1, 0, 0, 0, 0, -2)$ to the exchange matrix $\tilde{B}_{\mathrm{B}_3}$. Then we can obtain the following statement, which constitutes the Laurentification of the deformed B_3 map $\hat{\varphi}^{(2)}_{\mathrm{B}_3}$.

Theorem 4.2.9. *Let $(\hat{\mathbf{x}}, \hat{B}_{\mathrm{B}_3})$ be given as an initial seed which is composed of the extended initial cluster*

$$\hat{\mathbf{x}} = (\tilde{x}_j)_{1\leq j\leq 8} = (\eta_0, \sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \beta)$$

together with the associated extended exchange matrix

$$\hat{B}_{\mathrm{B}_3} = \begin{pmatrix} 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \tag{4.46}$$

and consider the permutation $\rho = (34567)$. Then the iteration of the cluster map $\psi_{\mathrm{B}_3} = \rho^{-1}\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$ is equivalent to the system of recurrences (4.43)*, and for all $n \in \mathbb{Z}$ the tau functions η_n, σ_n, τ_n are elements of the Laurent polynomial ring $\mathbb{Z}\big[\beta, \eta_0^{\pm1}, \sigma_0^{\pm1}, \tau_0^{\pm1}, \tau_1^{\pm1}, \tau_2^{\pm1}, \tau_3^{\pm1}, \tau_4^{\pm1}\big]$, with positive integer coefficients.*

Proof. Let us consider the seed $(\hat{\mathbf{x}}', \hat{B}'_{\mathrm{B}_3}) = \hat{\mu}_3\hat{\mu}_1\hat{\mu}_2(\hat{\mathbf{x}}, \hat{B}_{\mathrm{B}_3})$ that arises from applying the sequence of mutations $\hat{\mu}_3\hat{\mu}_1\hat{\mu}_2$ to the given initial seed, where (as usual) we use $\hat{\mu}_j$ to denote mutations in the cluster algebra associated with Laurentification of the deformed map. The new cluster is $\hat{\mathbf{x}}' = (\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8)$, where the new cluster variables (with primes) are obtained from the exchange relations

$$\begin{aligned} \tilde{x}'_2\tilde{x}_2 &= \tilde{x}_8\tilde{x}_4\tilde{x}_5 + \tilde{x}_1, \\ \tilde{x}'_1\tilde{x}_1 &= (\tilde{x}_8)^2\tilde{x}_4(\tilde{x}_5)^2\tilde{x}_6 + (\tilde{x}_2{}')^2\tilde{x}_3\tilde{x}_7, \\ \tilde{x}'_3\tilde{x}_3 &= (\tilde{x}_8)^2\tilde{x}_5\tilde{x}_6 + \tilde{x}_1{}', \end{aligned} \tag{4.47}$$

while the mutated exchange matrix $\hat{B}'_{\mathrm{B}_3} = \hat{\mu}_3\hat{\mu}_1\hat{\mu}_2(\hat{B}_{\mathrm{B}_3})$ is given by

$$\begin{pmatrix} 0 & -1 & 1 & 1 & 1 & 0 & -1 \\ 2 & 0 & 0 & 0 & -2 & -2 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.48}$$

For the new cluster variables, if we identify $\tilde{x}_1{}' = \eta_1$, $\tilde{x}_2{}' = \sigma_1$, $\tilde{x}_3{}' = \tau_5$ and replace all variables $\tilde{x}_i$ for $4 \le i \le 8$ with the corresponding tau functions and frozen variable from the original cluster $\hat{\mathbf{x}}$, then we find that the exchange relations (4.47) are equivalent to the recurrence formulae (4.43) for $n = 0$. As for the exchange matrix $\hat{B}'_{\mathrm{B}_3}$, we can rewrite it in the following way:

$$\hat{\mu}_3\hat{\mu}_1\hat{\mu}_2(\hat{B}_{\mathrm{B}_3}) = P_1\hat{B}_{\mathrm{B}_3}P_2 = \rho(\hat{B}_{\mathrm{B}_3}).$$

In the above, the action of the permutation $\rho = (34567)$ is equivalent to applying the row and column permutation matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.49}$$

Hence we see that the cluster map defined by $\psi_{\mathrm{B}_3} = \rho^{-1}\hat{\mu}_3\hat{\mu}_1\hat{\mu}_2$ satisfies $\psi_{\mathrm{B}_3}(\hat{B}_{\mathrm{B}_3}) = B_{\mathrm{B}_3}$, and its action on any cluster is equivalent to the shift $n \to n+1$ on the indices of the tau functions. Since they are cluster variables, these tau functions exhibit the Laurent property. Moreover, the coefficients of the Laurent polynomial cluster variables are positive integers, due to the positivity property [34, 48]. □

Remark 4.2.10. *By adapting the procedure in the section 3.5, we can measure the algebraic entropy of the discrete dynamical system defined by iteration of* ψ_{B_3}*. In [15], we used the fact that the tau function* τ_n *satisfies Somos-7 recurrence, namely* (4.41)*, to simplify the calculation of the degree growth for tau functions. As a result, we showed that the degree growth is quadratic.*

4.3 An integrable deformation of the periodic map for D_4

In this section, we consider the deformation of a 4D cluster map which is composed of mutations in the cluster algebra of type D_4. We will show that there are two essentially different choices of the deformation parameters that yield a discrete integrable system, each of which lifts to a cluster map in higher dimensions via Laurentification.

4.3.1 Deformed D_4 cluster map

The Cartan matrix for the D_4 is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \tag{4.1}$$

The corresponding exchange matrix is

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.2}$$

The deformed mutations with parameters a_j, b_j for $1 \leq j \leq 4$ take the form

$$\begin{aligned}
\tilde{\mu}_1 : (x_1, x_2, x_3, x_4) &\mapsto (x_1', x_2, x_3, x_4), & x_1 x_1' &= b_1 + a_1 x_2 \\
\tilde{\mu}_2 : (x_1', x_2, x_3, x_4) &\mapsto (x_1', x_2', x_3, x_4), & x_2 x_2' &= b_2 + a_2 x_3 x_4 x_1' \\
\tilde{\mu}_3 : (x_1', x_2', x_3, x_4) &\mapsto (x_1', x_2', x_3', x_4), & x_3 x_3' &= b_3 + a_3 x_2' \\
\tilde{\mu}_4 : (x_1', x_2', x_3', x_4) &\mapsto (x_1', x_2', x_3', x_4'), & x_4 x_4' &= b_4 + a_4 x_2'
\end{aligned} \tag{4.3}$$

The deformed map $\tilde{\varphi}_{\mathrm{D}_4} = \tilde{\mu}_4 \tilde{\mu}_3 \tilde{\mu}_2 \tilde{\mu}_1$ reduces to the original cluster map when we fix the parameters $a_i = 1 = b_i$ for all i. The Coxeter number for D_4 is 6, and the periodicity for the cluster map is period $4 = \frac{1}{2}(6+2)$, i.e.

$$\varphi_{\mathrm{D}_4} \cdot (\mathbf{x}, B) = (\varphi_{\mathrm{D}_4}(\mathbf{x}), B) \quad (\text{with } a_j = 1 = b_j) \implies \varphi_{\mathrm{D}_4}^4(\mathbf{x}) = \mathbf{x}. \tag{4.4}$$

As usual, we can reduce the number of parameters in the problem by rescaling each of the cluster variables independently, $x_i \to \lambda_i x_i$, and choose the scalings so that the parameters a_j are removed and the sequence of deformed mutations can be rewritten as

$$\begin{aligned}
x_{1,n+1} x_{1,n} &= x_{2,n} + b_1, \\
x_{2,n+1} x_{2,n} &= x_{3,n} x_{4,n} x_{1,n+1} + b_2, \\
x_{3,n+1} x_{3,n} &= x_{2,n+1} + b_3, \\
x_{4,n+1} x_{4,n} &= x_{2,n+1} + b_4
\end{aligned} \tag{4.5}$$

Since the exchange matrix is skew-symmetric, by the result of Theorem 1.3 in [10] the deformed map preserves the presymplectic form ω given by

$$\omega = \frac{1}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_2 x_3} \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \frac{1}{x_2 x_4} \mathrm{d}x_2 \wedge \mathrm{d}x_4. \tag{4.6}$$

Now since B is degenerate and of rank 2, one can reduce the birational map φ from 4D to a 2-dimensional symplectic map. The null space and image of B are given by

$$\ker(B) =< (1,0,0,1)^T, (1,0,1,0) >, \qquad \mathrm{im}\,(B) =< (0,1,0,0)^T, (-1,0,1,1)^T > . \tag{4.7}$$

Hence the null distribution of the presymplectic form ω is spanned by the two commuting vector fields $\mathbf{v}_1 = x_1 \partial_{x_1} + x_4 \partial_{x_4}$ and $\mathbf{v}_2 = x_1 \partial_{x_1} + x_3 \partial_{x_3}$. The space of leaves of the null foliation has local coordinates

$$y_1 = x_2, \quad y_2 = \frac{x_3 x_4}{x_1} \tag{4.8}$$

Then the rational map defined by

$$\begin{array}{rcrcl} \pi & : & \mathbb{C}^4 & \to & \mathbb{C}^2 \\ & & \mathbf{x}=(x_1,x_2,x_3,x_4) & \mapsto & \mathbf{y}=(y_1,y_2) \end{array} \tag{4.9}$$

reduces the cluster map φ_{D_4} to the 2D symplectic map

$$\begin{array}{rcrcl} \hat{\varphi}_{\mathrm{D}_4} & : & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & & \mathbf{y}=(y_1,y_2) & \mapsto & \left(\frac{(b_1+y_1)y_2+b_2}{y_1}, \frac{(b_4+y_2)y_1+b_1y_2+b_2)((b_3+y_2)y_1+b_1y_2+b_2)}{y_2y_1^2(b_1+y_1)}\right), \end{array} \tag{4.10}$$

which is intertwined with φ_{D_4} via π, so that

$$\hat{\varphi}_{\mathrm{D}_4}\cdot\pi=\pi\cdot\varphi_{\mathrm{D}_4}, \qquad \hat{\varphi}^*_{\mathrm{D}_4}(\hat{\omega})=\hat{\omega},$$

where $\pi^*(\hat{\omega})=\omega$ is the pullback of the symplectic form

$$\hat{\omega}=\frac{1}{y_1y_2}\mathrm{d}y_1\wedge\mathrm{d}y_2 \tag{4.11}$$

under π. When all of the parameters $b_i=1$, the reduced map $\hat{\varphi}_{\mathrm{D}_4}$ has period 4, and one of the first integrals associated with this map takes the form

$$K=\sum_{i=0}^{3}(\hat{\varphi}^*_{\mathrm{D}_4})^i(y_1)=\frac{(1+y_1)^3+(2+5y_1+y_1^3)y_2+(1+y_1)^2y_2^2}{y_1^2y_2} \tag{4.12}$$

By applying the same procedure as in the previous examples, we suppose that there is an analogous first integral that is compatible with the deformed map (4.10), taking the form

$$\tilde{K}=y_1+\alpha_1y_2+\frac{\alpha_2y_1}{y_2}+\frac{\alpha_3y_2}{y_1}+\frac{\alpha_4}{y_2}+\frac{\alpha_5}{y_1}+\frac{\alpha_6y_2}{y_1^2}+\frac{\alpha_7}{y_2y_1}+\frac{\alpha_8}{y_1^2}+\frac{\alpha_9}{y_2y_1^2} \tag{4.13}$$

where α_j are undetermined parameters. Then imposing the requirement that $\tilde{K}$ should be preserved, so that $\hat{\varphi}^*_{\mathrm{D}_4}(\tilde{K})=\tilde{K}$, constrains these parameters and leads us to find necessary and sufficient conditions for the map $\hat{\varphi}_{\mathrm{D}_4}$ to be integrable, as follows.

Theorem 4.3.1. *For the deformed symplectic map $\hat{\varphi}_{\mathrm{D}_4}$ to admit the first integral* (4.13)*, it is necessary and sufficient for the parameters b_i to satisfy one of the following sets of conditions:*

$$(1)\ b_2=b_4=b_1b_3; \tag{4.14a}$$

$$(2)\ b_1 = b_2 = b_3 b_4; \tag{4.14b}$$

$$(3)\ b_2 = b_3 = b_1 b_4. \tag{4.14c}$$

Hence in each of these cases, the deformed map $\hat{\varphi}_{\mathrm{D}_4}$ *given by* (4.10) *is Liouville integrable, preserving the function*

$$\tilde{K} = y_1 + y_2 + \frac{y_1}{y_2} + \frac{(b_1+1)y_2}{y_1} + \frac{b_3+b_4+1}{y_2} + \frac{b_1+b_2+b_3+b_4+1}{y_1} + \frac{b_1 y_2}{y_1^2} + \frac{b_3b_4+b_3+b_4}{y_1y_2} + \frac{2b_2}{y_1^2} + \frac{b_3b_4}{y_1^2y_2} \tag{4.15}$$

Remark 4.3.2. *Observe that the form of the original deformed mutations (4.5) remain invariant under switching* $x_3 \leftrightarrow x_4$*,* $b_3 \leftrightarrow b_4$*, and similarly for the form of the reduced map* $\hat{\varphi}_{\mathrm{D}_4}$ *in* (4.10) *and the first integral* (4.15) *when these last two parameters are switched. Hence cases (1) and (3) are equivalent to one another, and* (1) *and* (2) *are really the only two distinct cases to consider in Theorem 4.3.1.*

For the two essentially distinct cases of the reduced map obtained by deformation of the D_4 cluster map, as identified in the preceding theorem, we have two 2-parameter families of integrable maps given by $\hat{\varphi}^{(1)}_{\mathrm{D}_4}, \hat{\varphi}^{(2)}_{\mathrm{D}_4}$ respectively, where

$$\hat{\varphi}^{(1)}_{\mathrm{D}_4}: \quad (y_1, y_2) \mapsto \left(\frac{(b_1+y_1)y_2+b_1b_3}{y_1}, \quad \frac{[(b_1+y_1)y_2+b_1b_3(y_1+1)]\cdot[y_2+b_3]}{y_1^2y_2} \right) \tag{4.16}$$

$$\hat{\varphi}^{(2)}_{\mathrm{D}_4}: \quad (y_1, y_2) \mapsto \left(\frac{(b_3b_4+y_1)y_2+b_3b_4}{y_1}, \quad \frac{[(b_4+y_2)y_1+b_3b_4(y_2+1)]\cdot[(b_3+y_2)y_1+b_3b_4(y_2+1)]}{b_4y_1^2y_2(b_3b_4+y_1)} \right) \tag{4.17}$$

where the coefficients in each map are fixed in cases (1) and (2), respectively. The corresponding invariant functions $\tilde{K}_1, \tilde{K}_2$ are given by

$$\begin{aligned}\tilde{K}_1 =& y_1 + y_2 + \frac{y_1}{y_2} + \frac{(b_1+1)y_2}{y_1} + \frac{b_3+b_1b_3+1}{y_2} + \frac{b_1+2b_1b_3+b_3+1}{y_1} + \frac{b_1y_2}{y_1^2} \\ &+ \frac{b_3(b_1b_3+b_1+1)}{y_1y_2} + \frac{2b_1b_3}{y_1^2} + \frac{b_1b_3^2}{y_1^2y_2},\end{aligned} \tag{4.18}$$

$$\begin{aligned}\tilde{K}_2 =& y_1 + y_2 + \frac{y_1}{y_2} + \frac{(b_3b_4+1)y_2}{y_1} + \frac{b_3+b_4+1}{y_2} + \frac{2b_3b_4+b_3+b_4+1}{y_1} + \frac{b_3b_4y_2}{y_1^2} \\ &+ \frac{b_3b_4+b_3+b_4}{y_1y_2} + \frac{2b_3b_4}{y_1^2} + \frac{b_3b_4}{y_1^2y_2},\end{aligned} \tag{4.19}$$

which are the particular relevant cases of the function (4.15). The level sets of each of the latter functions gives a pencil of plane curves, of which the generic member has genus 1 and hence corresponds to an elliptic curve. It turns out that the two functions above become equivalent to one another when $b_3 = 1$, upon identifying the remaining parameters b_1 and b_4 in each case, although the associated maps $\hat{\varphi}_1, \hat{\varphi}_2$ remain distinct from one another.

Remark 4.3.3. *Since both sets of curves corresponding to $\tilde{K}_1$ and $\tilde{K}_2$ have bidegree (3,2), they do not correspond to QRT maps, which come from curves of bidegree (2,2) (that is, biquadratic curves).*

4.3.2 The deformed map $\hat{\varphi}^{(1)}_{\mathrm{D}_4}$ for D_4

The iteration of the deformed map $\hat{\varphi}^{(1)}_{\mathrm{D}_4}$ can be written as the following system of recurrence relations:

$$\begin{aligned} y_{1,n+1}y_{1,n} &= (b_1 + y_{1,n})y_{2,n} + b_1 b_3 \\ y_{2,n+1}y_{2,n}y_{1,n}^2 &= ((b_1 + y_{1,n})y_{2,n} + b_1 b_3(y_{1,n} + 1))(y_{2,n} + b_3). \end{aligned} \tag{4.20}$$

Using the p-adic method used in previous cases, we set the values of parameters $b_1 = 2, b_3 = 3$, and initial variables of $\mathbf{y}$ to be $(y_{1,0}, y_{2,0}) = (1, 1)$; observe the orbit of the $\hat{\varphi}^{(1)}_{\mathrm{D}_4}$, that is, a sequence of rational numbers.

n	$y_{1,n}$	$y_{2,n}$
1	3^2	$2^2 \cdot 3 \cdot 5$
2	$2 \cdot 37$	$\frac{2^2 \cdot 7}{3}$
3	$\frac{29}{3}$	$\frac{47}{2^3 \cdot 3 \cdot 7}$
4	$\frac{23}{2^3 \cdot 3}$	$\frac{19 \cdot 167}{2^3 \cdot 47}$
5	$\frac{12149}{2^3 \cdot 47}$	$\frac{3 \cdot 5 \cdot 11 \cdot 17 \cdot 43 \cdot 67}{19 \cdot 47 \cdot 167}$
6	$\frac{3 \cdot 73 \cdot 2069}{47 \cdot 167}$	$\frac{2^4 \cdot 3^3 \cdot 13 \cdot 33347}{11 \cdot 17 \cdot 43 \cdot 67 \cdot 167}$
7	$\frac{2 \cdot 53 \cdot 10247}{43 \cdot 67 \cdot 167}$	$\frac{47 \cdot 1009 \cdot 248309}{3^3 \cdot 13 \cdot 43 \cdot 67 \cdot 33347}$

As one can see from the table, the primes $p_1 = 47, 67, 167, 33347, 248309$ do make appearances in both $y_{1,n}$ and $y_{2,n}$. The p-adic norms for such primes:

$$\begin{aligned} \left|y_{1,n}\right|_{p_1} &= 1, 1, 1, p_1, p_1, 1, 1 \\ \left|y_{2,n}\right|_{p_1} &= 1, p_1^{-1}, p_1, p_1, 1, p_1^{-1}, 1 \end{aligned} \tag{4.21}$$

. There exists primes $p = 17, 19, 1009, 103979$ in $y_{1,n}$, which do not emerge in $y_{2,n}$, and conversely primes $p = 23, 29, 37, 12149, 2067$ which do only show up in $y_{1,n}$. Then we find the following three patterns of singularity in the orbits of $(y_{1,n}, y_{2,n})$:

$$\begin{aligned}
&\text{Pattern 1}: \ \ldots \to (R, 0^1) \to (R, \infty^1) \to (\infty^1, \infty^1) \to (\infty^1, R) \to (R, 0^1) \to (R, R) \to \ldots \\
&\text{Pattern 2}: \ \ldots \to (0^1, R) \to \ldots \\
&\text{Pattern 3}: \ \ldots \to (R, 0^1) \to (R, \infty^1) \to \ldots
\end{aligned} \tag{4.22}$$

By associating a tau function with each pattern, so that τ_n, r_n, σ_n correspond to Patterns 1,2,3 respectively, we are led to the change of variables

$$y_{1,n} = \frac{r_n}{\tau_{n-1}\tau_n}, \quad y_{2,n} = \frac{\sigma_{n+1}\tau_{n-2}\tau_{n+2}}{\sigma_n\tau_n\tau_{n+1}}. \tag{4.23}$$

If we directly substitute these variables into the recurrences (4.22), then we obtain the relations

$$\begin{aligned}
r_{n+1}r_n &= \frac{(b_1\tau_{n-1}\tau_n + r_n)\sigma_{n+1}\tau_{n-2}\tau_{n+2} + b_1b_3\tau_{n-1}\tau_n^2\tau_{n+1}\sigma_n}{\sigma_n} \\
&\sigma_{n+2}\tau_{n+3} \\
&= \frac{[b_1b_3\sigma_n\tau_n\tau_{n+1} + b_1\sigma_{n+1}\tau_{n-2}\tau_{n+2}] \cdot [(b_1\tau_{n-1}\tau_n + r_n)\sigma_{n+1}\tau_{n-2}\tau_{n+2} + b_1b_3\sigma_n\tau_n\tau_{n+1}(\tau_{n-1}\tau_n + r_n)]}{b_1r_n^2\tau_{n-2}\sigma_n}.
\end{aligned} \tag{4.24}$$

To simplify the above relations and decouple them in such a way that they represent exchange relations, it is helpful to observe the full singularity pattern in 4D, which emerges from applying the sequence of deformed mutations (4.5) subject to the conditions $b_2 = b_4 = \beta = b_1b_3$. Once again, we adjust the initial variables $x_{1,0} = x_{2,0} = x_{3,0} = x_{4,0} = 1$ and the parameters $b_1 = 2, b_3 = 3$; we obtain the sequences of $x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n}$ through the iteration, shown as below,

n	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x_{4,n}$
1	3	3^2	$2^2\cdot 3$	$3\cdot 5$
2	$\frac{11}{3}$	$2\cdot 37$	$\frac{7\cdot 11}{2^2\cdot 3}$	$\frac{2^4}{3}$
3	$\frac{2^2\cdot 3\cdot 19}{11}$	$\frac{29}{3}$	$\frac{2^3\cdot 19}{7\cdot 11}$	$\frac{47}{2^4}$
4	$\frac{5\cdot 7\cdot 11}{2^2\cdot 3^2\cdot 19}$	$\frac{23}{2^3\cdot 3}$	$\frac{5\cdot 7\cdot 11}{2^6\cdot 3}$	$\frac{2\cdot 167}{3\cdot 47}$
5	$\frac{3\cdot 19\cdot 71}{2\cdot 5\cdot 7\cdot 11}$	$\frac{12149}{2^3\cdot 47}$	$\frac{2^3\cdot 3\cdot 17\cdot 71}{5\cdot 7\cdot 47}$	$\frac{3\cdot 5\cdot 43\cdot 67}{2^4\cdot 167}$
6	$\frac{5\cdot 7^2\cdot 11\cdot 97}{2^2\cdot 3\cdot 47\cdot 71}$	$\frac{3\cdot 73\cdot 2069}{47\cdot 167}$	$\frac{3^2\cdot 5\cdot 7^2\cdot 13\cdot 97}{2^2\cdot 17\cdot 71\cdot 167}$	$\frac{2^4\cdot 33347}{43\cdot 47\cdot 67}$
7	$\frac{2^2\cdot 3\cdot 17\cdot 23\cdot 71\cdot 109}{5\cdot 7^2\cdot 97\cdot 167}$	$\frac{2\cdot 53\cdot 10247}{43\cdot 67\cdot 167}$	$\frac{2^2\cdot 17\cdot 23\cdot 71\cdot 109\cdot 1009}{3^2\cdot 5\cdot 7^2\cdot 13\cdot 43\cdot 67\cdot 97}$	$\frac{47\cdot 248309}{167\cdot 33347}$
8	$\frac{3^2\cdot 5\cdot 7^2\cdot 13\cdot 97\cdot 1459}{17\cdot 23\cdot 43\cdot 67\cdot 71\cdot 109}$	$\frac{3371\cdot 94513}{43\cdot 67\cdot 33347}$	$\frac{3^2\cdot 5\cdot 7^2\cdot 13\cdot 97\cdot 1459\cdot 103979}{17\cdot 23\cdot 71\cdot 109\cdot 1009\cdot 33347}$	$\frac{5\cdot 7\cdot 167\cdot 544097}{43\cdot 67\cdot 248309}$

From the table, observe that particular primes (for instance $p = 47$, $p = 19$, 29, which correspond to τ_n, σ_n, r_n, respectively) appearing in each of the integer sequences are the same ones that were emerged in the numerators and denominators in the preceding table. In addition to this, we can see there are certain primes that do not appear in the y coordinates, for instance, the primes $p = 11, 71, 97$, arising in the numerator of $x_{1,n}$ and $x_{3,n}$. From this, we can define a transformation on the level of the x-variables via p-adic method, given by

$$x_{1,n} = \rho_n \frac{\sigma_n}{\tau_{n-1}} \quad x_{2,n} = \frac{r_n}{\tau_{n-1}\tau_n} \quad x_{3,n} = \rho_n \frac{\sigma_{n+1}}{\tau_n}, \quad x_{4,n} = \frac{\tau_{n-2}\tau_{n+2}}{\tau_{n-1}\tau_{n+1}}, \tag{4.25}$$

where the extra prefactor ρ_n corresponds to an additional singularity pattern, appearing only on this level. By a short calculation, one can confirm that these new formulae are consistent with the expression for $y_{2,n}$ previously given in (4.23), since we have

$$y_{2,n} = \frac{x_{3,n}x_{4,n}}{x_{1,n}} = \frac{\sigma_{n+1}\tau_{n-1}}{\sigma_n\tau_n} \cdot \frac{\tau_{n-2}\tau_{n+2}}{\tau_{n-1}\tau_{n+1}} = \frac{\sigma_{n+1}\tau_{n-2}\tau_{n+2}}{\sigma_n\tau_n\tau_{n+1}},$$

as required. Thus, with the parameters constrained as in (4.14a), the iteration of the deformed map (4.5) is equivalent to a system of four relations, namely

$$\rho_{n+1}\rho_n\sigma_{n+1}\sigma_n = b_1\tau_{n-1}\tau_n + r_n, \tag{4.26a}$$

$$r_{n+1}r_n = \rho_{n+1}\rho_n\sigma_{n+1}^2\tau_{n+2}\tau_{n-2} + b_1b_3\tau_{n+1}\tau_n^2\tau_{n-1} \tag{4.26b}$$

$$\rho_{n+1}\rho_n\sigma_{n+2}\sigma_{n+1} = b_3\tau_{n+1}\tau_n + r_{n+1} \tag{4.26c}$$

$$\tau_{n+3}\tau_{n-2} = b_1b_3\tau_n\tau_{n+1} + r_{n+1} \tag{4.26d}$$

By incorporating the above relations into (4.24), and eliminating ρ_n, we obtain recurrence relation for σ_{n+2} as shown below,

$$
\begin{aligned}
\sigma_{n+2}\tau_{n+3} &= \frac{[b_1b_3\sigma_n t_n t_{n+1} + b_1\sigma_{n+1}t_{n-2}t_{n+2}]\cdot[\overbrace{(b_1\tau_{n-1}\tau_n + r_n)}^{(4.26a)}\sigma_{n+1}\tau_{n-2}\tau_{n+2} + b_1b_3\sigma_n\tau_n\tau_{n+1}(\tau_{n-1}\tau_n + r_n)]}{b_1r_n^2\tau_{n-2}\sigma_n} \\
&= \frac{[b_1b_3\sigma_n t_n t_{n+1} + b_1\sigma_{n+1}t_{n-2}t_{n+2}]\cdot\left[\rho_{n+1}\rho_n\sigma_{n+1}^2\sigma_n\tau_{n+2}\tau_{n-2} + b_1b_3\sigma_n\tau_{n+1}\tau_n^2\tau_{n-1} + b_1b_3\sigma_n\tau_{n+1}\tau_n r_n)\right]}{b_1r_n^2\tau_{n-2}\sigma_n} \\
&= \frac{[b_1b_3\sigma_n t_n t_{n+1} + b_1\sigma_{n+1}t_{n-2}t_{n+2}]\cdot[\sigma_n\overbrace{(\rho_{n+1}\rho_n\sigma_{n+1}^2\tau_{n+2}\tau_{n-2} + b_1b_3\tau_{n+1}\tau_n^2\tau_{n-1})}^{(4.26b)} + b_1b_3\sigma_n\tau_{n+1}\tau_n r_n]}{b_1r_n^2\tau_{n-2}\sigma_n} \\
&= \frac{[b_1b_3\sigma_n t_n t_{n+1} + b_1\sigma_{n+1}t_{n-2}t_{n+2}]\cdot[\sigma_n r_{n+1}r_n + b_1b_3\sigma_n\tau_{n+1}\tau_n r_n]}{b_1r_n^2\tau_{n-2}\sigma_n} \\
&= \frac{[b_1b_3\sigma_n t_n t_{n+1} + b_1\sigma_{n+1}t_{n-2}t_{n+2}]\cdot[\sigma_n r_n\overbrace{(r_{n+1} + B\tau_{n+1}\tau_n)}^{(4.26d)}]}{b_1r_n^2\tau_{n-2}\sigma_n} \\
&= \frac{[b_1b_3\sigma_n t_n t_{n+1} + b_1\sigma_{n+1}t_{n-2}t_{n+2}]\cdot[\sigma_n r_n\tau_{n+3}\tau_{n-2}]}{b_1r_n^2\tau_{n-2}\sigma_n} \\
&\Longrightarrow\ \sigma_{n+2}r_n = b_3\sigma_n\tau_n\tau_{n+1} + \sigma_{n+1}\tau_{n-2}\tau_{n+2}\ \text{(a)}
\end{aligned}
\tag{4.27}
$$

We take same approach as above to eliminate ρ_n in the second relation (4.26b), which gives

$$
r_{n+1}\sigma_{n+2}r_n = (b_3\tau_n\tau_{n+1} + r_{n+1})\sigma_{n+1}\tau_{n+2}\tau_{n-2} + b_1b_3\sigma_{n+2}\tau_{n+1}\tau_n^2\tau_{n-1}.
$$

Then substituting the $\sigma_{n+2}r_n$ (a) directly into above expression gives rise to the relation as following

$$
\begin{aligned}
&\Longrightarrow\ r_{n+1}\overbrace{\sigma_{n+2}r_n}^{(a)} = (b_3\tau_n\tau_{n+1} + r_{n+1})\sigma_{n+1}\tau_{n+2}\tau_{n-2} + b_1b_3\sigma_{n+2}\tau_{n+1}\tau_n^2\tau_{n-1} \\
&\Longrightarrow\ b_3r_{n+1}\sigma_n\tau_n\tau_{n+1} + r_{n+1}\sigma_{n+1}\tau_{n+2}\tau_{n-2} = (b_3\tau_n\tau_{n+1} + r_{n+1})\sigma_{n+1}\tau_{n+2}\tau_{n-2} + b_1b_3\sigma_{n+2}\tau_{n+1}\tau_n^2\tau_{n-1} \\
&\Longrightarrow\ r_{n+1}\sigma_n = \sigma_{n+1}\tau_{n+2}\tau_{n-2} + b_1\sigma_{n+2}\tau_n\tau_{n-1}
\end{aligned}
$$

Hence we are able to decouple the expressions into a total of three recurrences, which all take the form of exchange relations, given by

$$
\begin{aligned}
\sigma_{n+2}r_n &= b_3\sigma_n\tau_n\tau_{n+1} + \sigma_{n+1}\tau_{n-2}\tau_{n+2} \\
r_{n+1}\sigma_n &= \sigma_{n+1}\tau_{n+2}\tau_{n-2} + b_1\sigma_{n+2}\tau_n\tau_{n-1} \\
\tau_{n+3}\tau_{n-2} &= b_1b_3\tau_n\tau_{n+1} + r_{n+1}.
\end{aligned}
\tag{4.28}
$$

Next, in order to confirm that this gives a cluster map defined on a suitable space of tau functions, we need to build an appropriate exchange matrix which produces

(4.28) via a sequence of mutations. Firstly, let us combine the initial tau functions into a cluster in a seed for a coefficient-free cluster algebra, by setting

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8) = (\sigma_0, \sigma_1, r_0, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2),$$

and let $\tilde{\pi}_1 : \mathbb{C}^8 \to \mathbb{C}^2$ be the rational map defined by (4.23). Then, upon taking the pullback of the symplectic form (4.11) by $\tilde{\pi}_1$, we find

$$\tilde{\omega} = \tilde{\pi}_1^*(\hat{\omega}) = \sum_{1 \le i < j \le 8} \tilde{b}_{ij}^{(1)} \mathrm{d} \log \tilde{x}_i \wedge \mathrm{d} \log \tilde{x}_j \tag{4.29}$$

where the coefficients are combined into the matrix $\tilde{B}_{\mathrm{D}_4}^{(1)} = (\tilde{b}_{ij}^{(1)})$ given by

$$\tilde{B}_{\mathrm{D}_4}^{(1)} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}. \tag{4.30}$$

We proceed to add two extra rows, associated with the frozen variables b_1 and b_3, to the bottom of the exchange matrix (4.30), which will result in the construction of the extended exchange matrix $\hat{B}_{\mathrm{D}_4}^{(1)}$ shown in (4.31) below. Figure 4.1 depicts the quiver associated with the full matrix $\hat{B}_{\mathrm{D}_4}^{(1)}$.

Theorem 4.3.4. *Given the extended initial cluster*

$$\hat{\mathbf{x}} = (\tilde{x}_j)_{1 \le j \le 10} = (\sigma_0, \sigma_1, r_0, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, b_1, b_3),$$

and the permutation $\rho_1 = (123)(45678)$, *the iteration of the cluster map* $\psi_{\mathrm{D}_4}^{(1)} = \rho_1^{-1} \hat{\mu}_4 \hat{\mu}_1 \hat{\mu}_3$ *defined by the extended exchange matrix* $\hat{B}_{\mathrm{D}_4}^{(1)}$ *in* (4.31) *with square submatrix* (4.30) *is equivalent to the system of recurrences* (4.28), *which generates elements of* $\mathbb{Z}_{>0}\big[b_1, b_3, \sigma_0^{\pm 1}, \sigma_1^{\pm 1}, r_0^{\pm 1}, \tau_{-2}^{\pm 1}, \tau_{-1}^{\pm 1}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}\big]$

Proof 4.3.5. *Let us consider the initial seed* $(\hat{\mathbf{x}}, \hat{B}_{\mathrm{D}_4}^{(1)})$ *containing the extended initial*

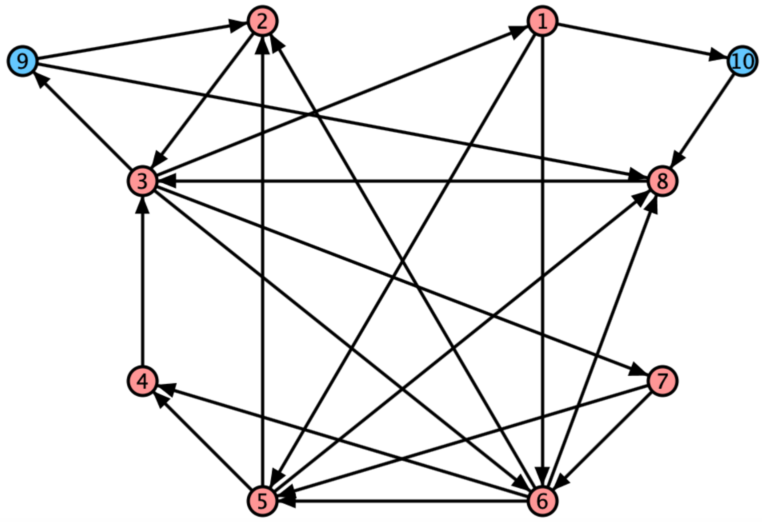


Figure 4.1: Extended quiver associated with the deformed D_4 cluster map $\psi^{(1)}_{\mathrm{D}_4}$

cluster $\hat{\mathbf{x}}$ *as above, with the corresponding extended exchange matrix given by*

$$\hat{B}^{(1)}_{\mathrm{D}_4} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.31}$$

Applying cluster mutation $\hat{\mu}_3$ *at node 3, followed by* $\hat{\mu}_1$ *and* $\hat{\mu}_4$*, will give the exchange relations*

$$\begin{aligned} \tilde{x}_3'\tilde{x}_3 &= \tilde{x}_{10}\tilde{x}_1\tilde{x}_6\tilde{x}_7 + \tilde{x}_2\tilde{x}_4\tilde{x}_8, \\ \tilde{x}_1'\tilde{x}_1 &= \tilde{x}_2\tilde{x}_4\tilde{x}_8 + \tilde{x}_9\tilde{x}_3{}'\tilde{x}_5\tilde{x}_6, \\ \tilde{x}_4'\tilde{x}_4 &= \tilde{x}_9\tilde{x}_{10}\tilde{x}_6\tilde{x}_7 + \tilde{x}_1{}', \end{aligned} \tag{4.32}$$

which have the same form as the expressions in (4.28)*. Under this sequence of mutations, the extended exchange matrix* $\hat{B}^{(1)}_{\mathrm{D}_4}$ *satisfies the relation*

$$\hat{\mu}_4\hat{\mu}_1\hat{\mu}_3(\hat{B}^{(1)}_{\mathrm{D}_4}) = \rho_1(\hat{B}^{(1)}_{\mathrm{D}_4}) = P_1\hat{B}^{(1)}_{\mathrm{D}_4}P_2$$

where the action of the permutation $\rho_1 = (123)(45678)$ *on the rows/columns is represented by the matrices*

$$P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.33)$$

It follows that $\hat{B}^{(1)}_{\mathrm{D}_4}$ *is preserved by the action of the associated cluster map, that is* $\psi^{(1)}_{\mathrm{D}_4}(\hat{B}^{(1)}_{\mathrm{D}_4}) = \hat{B}^{(1)}_{\mathrm{D}_4}$*, where* $\psi^{(1)}_{\mathrm{D}_4} = \rho_1^{-1}\mu_4\mu_1\mu_3$*, and the combination of the inverse permutation with the exchange relations in* (4.32) *precisely corresponds to the shift of index* $n \to n+1$ *acting on the tau functions in each cluster, reproducing the iteration of the system* (4.28)*. Hence this cluster map is a Laurentification of the deformed* D_4 *map* $\hat{\varphi}^{(1)}_{\mathrm{D}_4}$*, generating Laurent polynomials in the initial cluster variables, and positivity for skew-symmetric cluster algebras [34] implies that their coefficients are positive integers.*

Remark 4.3.6. *The subquiver in Figure 4.1 consisting of the 8 unfrozen nodes is mutation equivalent to another particular quiver presented by Okubo, which enables a q-Painlevé VI equation to be constructed from an appropriate combination of coefficient mutations [54].*

4.3.3 The deformed map $\hat{\varphi}^{(2)}_{\mathrm{D}_4}$ for D_4

Let us now consider iteration of the map $\hat{\varphi}^{(2)}_{\mathrm{D}_4}$ given by (4.17), which can be written as the recurrence

$$\begin{aligned} y_{1,n+1}y_{1,n} &= (b_3b_4 + y_{1,n})y_{2,n} + b_3b_4 \\ y_{2,n+1}y_{2,n}y_{1,n}^2 b_4(b_3b_4 + y_{1,n}) &= ((b_4 + y_{2,n})y_{1,n} + b_3b_4(y_{2,n}+1))((b_3 + y_{2,n})y_{1,n} + b_3b_4(y_{2,n}+1)). \end{aligned} \quad (4.34)$$

Repeating the same procedure as in the previous sections,

n	$y_{1,n}$	$y_{2,n}$
1	13	$\frac{2^4\cdot 3\cdot 5}{7}$
2	$\frac{2\cdot 3\cdot 59}{7}$	$\frac{5^2\cdot 23}{7\cdot 19}$
3	$\frac{659}{7\cdot 19}$	$\frac{23\cdot 37}{2\cdot 3^2\cdot 11\cdot 19}$
4	$\frac{41\cdot 157}{2\cdot 3^2\cdot 11\cdot 19}$	$\frac{7\cdot 479\cdot 607}{2\cdot 3^2\cdot 11\cdot 31\cdot 47}$
5	$\frac{1181\cdot 8623}{2\cdot 3^2\cdot 11\cdot 31\cdot 47}$	$\frac{5\cdot 19\cdot 109\cdot 167\cdot 4493}{7\cdot 31\cdot 47\cdot 29009}$
6	$\frac{13\cdot 100445747}{31\cdot 47\cdot 29009}$	$\frac{2^2\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 911\cdot 76379}{19\cdot 29\cdot 58693\cdot 29009}$
7	$\frac{2\cdot 3\cdot 11700760949}{29\cdot 58693\cdot 29009}$	$\frac{2^6\cdot 7\cdot 31\cdot 47\cdot 263\cdot 3623\cdot 2957}{3\cdot 11\cdot 29\cdot 71\cdot 1155961\cdot 58693}$

For the prime $p_1 = 19, 47, 29009, 58693$, the p_1-adic norms follows the pattern:

$$\begin{aligned} &|y_{1,n}|_{p_1} : 1, p_1, p_1, 1 \\ &|y_{2,n}|_{p_1} : p_1, p_1, 1, p_1^{-1}, p_1 \end{aligned} \tag{4.35}$$

There are isolated primes $p_2 = 23, 37, 479, 607$ and $p_3 = 13, 59, 659, 1181$ appearing in numerator of $y_{1,n}$ and $y_{2,n}$, respectively. Thus we find three singularity patterns which arise from orbits of (4.34), namely

$$\begin{aligned} &\text{Pattern 1} : \ldots \to (R, \infty^1) \to (\infty^1, \infty^1) \to (\infty^1, R) \to (R, 0^1) \to (R, \infty^1) \to (R, R) \to \ldots \\ &\text{Pattern 2} : \ \ldots \to (0^1, R) \to \ldots \\ &\text{Pattern 3} : \ \ldots \to (R, 0^1) \to \ldots . \end{aligned} \tag{4.36}$$

By relating the singularities appearing in each pattern with new variables, we define the following variable transformation, in an attempt to Laurentify the deformed map $\hat{\varphi}_2$:

$$y_{1,n} = \frac{\hat{\eta}_n}{\hat{\tau}_{n-1}\hat{\tau}_n}, \quad y_{2,n} = \frac{\rho_n \hat{\tau}_{n-2}}{\hat{\tau}_{n+1}\hat{\tau}_n\hat{\tau}_{n-3}} \tag{4.37}$$

By substituting these expressions into (4.34), we find a rather complicated system of equations: in particular, the resulting expression for the product $\rho_{n+1}\rho_n$ cannot be considered as an exchange relation, as it is not immediately given as a binomial expression in the other variables. To resolve this problem, we follow the same procedure as described in the preceding section (section 4.3.2), where we look at the singularity patterns in the original 4D deformed map (4.5) with $b_1 = b_2 = b_3 b_4$, and introduce variable transformations corresponding to these. The table below represents

n	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x_{4,n}$
1	7	13	$3 \cdot 5$	2^4
2	$\frac{19}{7}$	$\frac{2\cdot3\cdot59}{7}$	$\frac{2^4\cdot23}{3\cdot5\cdot7}$	$\frac{3\cdot5^3}{2^4\cdot7}$
3	$\frac{2^2\cdot3^2\cdot11}{19}$	$\frac{659}{7\cdot19}$	$\frac{3\cdot5^3\cdot37}{2^4\cdot19\cdot23}$	$\frac{2^5\cdot23^2}{3\cdot5^3\cdot19}$
4	$\frac{31\cdot47}{2^2\cdot3^2\cdot7\cdot11}$	$\frac{41\cdot157}{2\cdot3^2\cdot11\cdot19}$	$\frac{2^3\cdot23^2\cdot607}{3^3\cdot5^3\cdot11\cdot37}$	$\frac{5^3\cdot37\cdot479}{2^6\cdot3\cdot11\cdot23^2}$
5	$\frac{2\cdot7\cdot29009}{19\cdot31\cdot47}$	$\frac{1181\cdot8623}{2\cdot3^2\cdot11\cdot31\cdot47}$	$\frac{3\cdot5^4\cdot37\cdot479\cdot4493}{2^4\cdot23^2\cdot31\cdot47\cdot607}$	$\frac{2^5\cdot23^2\cdot109\cdot167\cdot607}{3\cdot5^3\cdot31\cdot37\cdot47\cdot479}$
6	$\frac{19\cdot29\cdot58693}{2^2\cdot3^2\cdot11\cdot29009}$	$\frac{13\cdot100445747}{31\cdot47\cdot29009}$	$\frac{2^4\cdot23^2\cdot109\cdot167\cdot607\cdot76379}{3\cdot5^4\cdot37\cdot479\cdot4493\cdot29009}$	$\frac{3\cdot5^5\cdot7\cdot37\cdot479\cdot911\cdot4493}{2^4\cdot23^2\cdot109\cdot167\cdot607\cdot29009}$
7	$\frac{2^2\cdot3^2\cdot11\cdot71\cdot1155961}{29\cdot31\cdot47\cdot58693}$	$\frac{2\cdot3\cdot11700760949}{29\cdot58693\cdot29009}$	$\frac{2^4\cdot3\cdot5^5\cdot7^2\cdot37\cdot479\cdot911\cdot4493\cdot2957}{23^2\cdot29\cdot109\cdot167\cdot607\cdot76379\cdot58693}$	$\frac{2^4\cdot23^2\cdot109\cdot167\cdot263\cdot607\cdot763793623}{5^5\cdot7\cdot29\cdot37\cdot479\cdot911\cdot4493\cdot58693}$

Similarly to the situation above, one can observe the primes which we recognise from the sequences of rationals in $(y_{1,n}, y_{2,n})$, but there is further detail which can not be seen in the 2D system. Notice that the prime $p = 37$, for instance, begins to appear at the numerator of $x_{3,3}$; it appears again in denominator of the next rational $x_{3,4}$. This situation occurs recursively after $x_{3,4}$. In similar way as preceding statement, the same prime appears in $x_{4,n}$ except the pattern starts at $x_{3,4}$. In other words, in terms p-adic norm, $|x_{3,3}|_p = p^{-1}$ and then for $m > 3$, whenever $|x_{3,m}|_p = p$, then $|x_{4,m}|_p = p^{-1}$ or vice vera. One can also observe similar case happening in $x_{4,n}$. As soon as we move to 2D system, the primes cancel out and only single prime remains throughout the sequence e.g. $p = 37$ exists only in $y_{2,3} = (x_{3,3}x_{4,3})/x_{1,3}$. We associate the primes exhibiting those pattern with new tau function ξ_n. Altogether it suggests that x_i should be expressed as

$$x_{1,n} = \frac{\hat{\tau}_{n+1}\hat{\tau}_{n-3}}{\hat{\tau}_n\hat{\tau}_{n-2}}, \quad x_{2,n} = \frac{\hat{\eta}_n}{\hat{\tau}_n\hat{\tau}_{n-1}}, \quad x_{3,n} = \frac{\hat{r}_n\xi_n}{\hat{\tau}_n}, \quad x_{4,n} = \frac{\hat{\sigma}_n}{\hat{\tau}_n\xi_n}$$

where ξ_n satisfies $\xi_n\xi_{n+1} = \frac{\hat{\sigma}_n}{\hat{r}_n}$. By directly substituting these variables into (4.5), we find a system of equations written as follows:

$$\begin{aligned}
\hat{\tau}_{n+2}\hat{\tau}_{n-3} &= b_3 b_4 \hat{\tau}_n\hat{\tau}_{n-1} + \hat{\eta}_n, \\
\hat{\eta}_{n+1}\hat{\eta}_n &= \hat{r}_n\hat{\sigma}_n\hat{\tau}_{n-2}\hat{\tau}_{n+2} + b_3 b_4 \hat{\tau}_{n+1}\hat{\tau}_n^2\hat{\tau}_{n-1}, \\
\hat{r}_{n+1}\hat{\sigma}_n &= b_3\hat{\tau}_n\hat{\tau}_{n+1} + \hat{\eta}_{n+1}, \\
\hat{\sigma}_{n+1}\hat{r}_n &= b_4\hat{\tau}_n\hat{\tau}_{n+1} + \hat{\eta}_{n+1}.
\end{aligned} \tag{4.38}$$

Also, by observing the singularity pattern for $y_{1,n}$ explicitly from (4.34), one can see that $y_{1,n}$ should satisfy the relation

$$w_{1,n} := y_{1,n} + b_3 b_4 = \frac{\hat{\tau}_{n+2}\hat{\tau}_{n-3}}{\hat{\tau}_n\hat{\tau}_{n-1}}, \tag{4.39}$$

which is in agreement with what is found by combining (4.37) with the first recurrence in (4.38). Furthermore, by setting $\rho_n = \hat{r}_n\hat{\sigma}_n$, the relation for $\rho_{n+1}\rho_n$ obtained from (4.34) follows by taking the product of the last two expressions in (4.38).

Using the above, we can consider $(y_{1,0}, y_{2,0}) = (y_1, y_2)$ and define a rational map $\tilde{\pi}_2 : \mathbb{C}^8 \to \mathbb{C}^2$ by

$$\tilde{\pi}_2 : \qquad y_1 = \frac{\hat{\eta}_0}{\hat{\tau}_{-1}\hat{\tau}_0}, \qquad y_2 = \frac{\hat{\sigma}_0\hat{r}_0\hat{\tau}_{-2}}{\hat{\tau}_1\hat{\tau}_0\hat{\tau}_{-3}}.$$

The exchange matrix describing the cluster dynamics (4.38) is found by pulling back the symplectic form $\hat{\omega}$, as in (4.11), via the rational map $\tilde{\pi}_2$, to obtain the presymplectic form

$$\tilde{\omega} = \tilde{\pi}_2^*(\hat{\omega}) = \sum_{i<j} \frac{\tilde{b}_{ij}^{(2)}}{\tilde{x}_i\tilde{x}_j} \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j.$$

Now if we choose to order the coordinates and identify them with variables in a coefficient-free cluster algebra as

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8) = (\hat{\tau}_{-3}, \hat{r}_0, \hat{\eta}_0, \hat{\sigma}_0, \hat{\tau}_{-1}, \hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_{-2}),$$

then we see that the map $\tilde{\pi}_2$ is equivalent to $\tilde{\pi}_1$ defined by (4.23) in case (1) above, so that

$$y_1 = \frac{\tilde{x}_3}{\tilde{x}_5\tilde{x}_6}, \qquad y_2 = \frac{\tilde{x}_2\tilde{x}_4\tilde{x}_8}{\tilde{x}_1\tilde{x}_6\tilde{x}_7}$$

and the exchange matrix with entries $\tilde{b}_{ij}^{(2)}$ is identical to the one obtained previously, that is

$$\tilde{B}_{\mathrm{D}_4}^{(2)} = \tilde{B}_{\mathrm{D}_4}^{(1)} \tag{4.40}$$

as in (4.30).

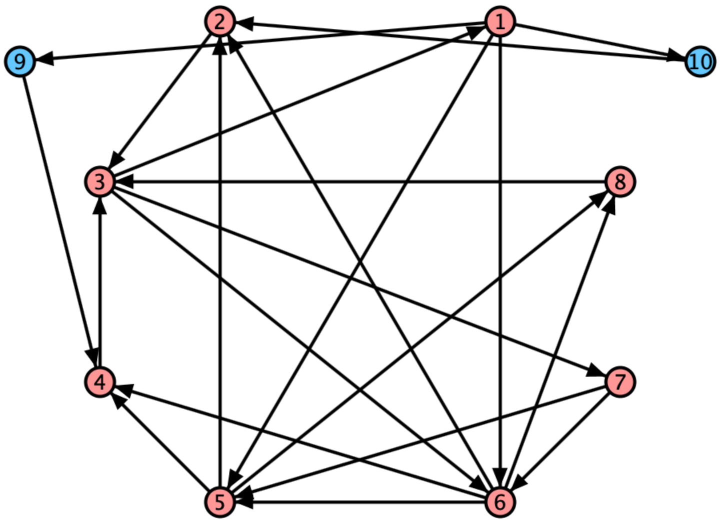


Figure 4.2: Extended quiver associated with the deformed D$_4$ cluster map $\psi_{\mathrm{D}_4}^{(2)}$

To obtain an extended version of $\tilde{B}_{\mathrm{D}_4}^{(2)}$ that includes b_3, b_4 as frozen variables and reproduces (4.38) from a suitable sequence of mutations, we need to construct two extra rows as in (4.41) below. The result of this is represented by the quiver in Figure 4.2, with two frozen nodes.

Theorem 4.3.7. *Given the extended initial cluster*

$$\hat{\mathbf{x}} = (\tilde{x}_j)_{1\leq j\leq 10} = (\hat{\tau}_{-3}, \hat{r}_0, \hat{\eta}_0, \hat{\sigma}_0, \hat{\tau}_{-1}, \hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_{-2}, b_3, b_4),$$

and the permutation $\rho_2 = (24)(18567)$*, the iteration of the cluster map* $\psi_{\mathrm{D}_4}^{(2)} = \rho_2^{-1}\hat{\mu}_2\hat{\mu}_4\hat{\mu}_3\hat{\mu}_1$ *defined by the extended exchange matrix* $\hat{B}_{\mathrm{D}_4}^{(2)}$ *in* (4.41) *with square submatrix* (4.30) *is equivalent to the system of recurrences* (4.38)*, which generates elements of* $\mathbb{Z}_{>0}\big[b_3, b_4, \hat{r}_0^{\pm1}, \hat{\eta}_0^{\pm1}, \hat{\sigma}_0^{\pm1}, \hat{\tau}_{-3}^{\pm1}, \hat{\tau}_{-2}^{\pm1}, \hat{\tau}_{-1}^{\pm1}, \hat{\tau}_0^{\pm1}, \hat{\tau}_1^{\pm1}\big]$.

Proof 4.3.8. *From* (4.40) *we note that the coefficient-free cluster algebra is identical to that specified by the same* 8×8 *exchange matrix as was found in case (1) previously, but we need to extend it in such a way that, once* b_3 *and* b_4 *are included as frozen variables, it is compatible with the four relations in* (4.38) *(whereas in case*

(1) there were only thee relations). In this way, we construct a 10×8 extended exchange matrix $\hat{B}^{(2)}_{\mathrm{D}_4}$ from $\tilde{B}^{(2)}_{\mathrm{D}_4} = \tilde{B}^{(1)}_{\mathrm{D}_4}$, given by

$$\hat{B}^{(2)}_{\mathrm{D}_4} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.41}$$

and we note that the last two rows are different from $\hat{B}^{(1)}_{\mathrm{D}_4}$ *in* (4.31) *(as can be seen by comparing Figures 4.1 and 4.2). Such an extended exchange matrix is invariant under the following*

Applying a sequence of mutations, starting with mutation $\hat{\mu}_1$ *at node 1 and successively mutating at nodes 3,4 and 2, we find that the nodes are permuted by the given permutation* ρ_2*, so that*

$$\hat{\mu}_2\hat{\mu}_4\hat{\mu}_3\hat{\mu}_1(\hat{B}^{(2)}_{\mathrm{D}_4}) = \rho_2(\hat{B}^{(2)}_{\mathrm{D}_4}),$$

which is equivalent to the action of a suitable pair of row/column permutation matrices acting on $\hat{B}^{(2)}_{\mathrm{D}_4}$*. Hence the overall action of* $\psi^{(2)}_{\mathrm{D}_4} = \rho_2^{-1}\hat{\mu}_2\hat{\mu}_4\hat{\mu}_3\hat{\mu}_1$ *leaves* $\hat{B}^{(2)}_{\mathrm{D}_4}$ *invariant, and it is straightforward to check that the corresponding combination of cluster mutations with a permutation is equivalent to one iteration of the relations* (4.38)*. Then as usual, because they are cluster variables, the iterates are elements of the corresponding ring of Laurent polynomials, with positive integer coefficients.*

Remark 4.3.9. *Since the subquiver with 8 unfrozen nodes in Figure 4.2 is the same as that in Figure 4.1, it is also mutation equivalent to the quiver associated with the q-Painlevé VI equation in [54].*

Remark 4.3.10. *The degree growth for the tau functions can be determined in both cases* $\psi^{(1)}_{\mathrm{D}_4}$ *and* $\psi^{(2)}_{\mathrm{D}_4}$ *by proceeding in the same way as in the section 3.5 (the calculation can be found in [15]).*

4.3.4 Connection with special Somos-7 relation

As we have already seen in examples of deformations in type A and B, on fixed level sets the orbits of suitable tau functions satisfy a special Somos-7 relation, which is related to the Lyness map. There is also a corresponding Somos-5 relation, although the details of this are a bit more subtle (see Appendix B in [15]). We find a similar result for the deformed maps of type D_4.

Theorem 4.3.11. *For each integrable case of the deformed* D_4 *map, the variable* $w_n = y_{1,n} + \beta$ *satisfies the Lyness map in the form*

$$w_{n+1}w_{n-1} = (1-\beta)w_n + \delta, \tag{4.42}$$

where in case (1) we have

$$\beta = b_1b_3, \qquad \delta = \beta\tilde{K}_1 + 2\beta^2 + b_1 + b_3,$$

on each level set of the invariant function $\tilde{K}_1$ *given in* (4.18)*, while in case (2) the parameters are specified by*

$$\beta = b_3b_4, \qquad \delta = \beta\tilde{K}_2 + 2\beta^2 + b_3 + b_4,$$

with $\tilde{K}_2$ *as in* (4.19)*. Furthermore, in case (1) we can express* w_n *by the formula*

$$w_n = \frac{\tau_{n+2}\tau_{n-3}}{\tau_n\tau_{n-1}}, \tag{4.43}$$

where the tau function τ_n *satisfies the special Somos-7 relation*

$$\tau_{n+7}\tau_n = (1-\beta)\tau_{n+6}\tau_{n+1} + \delta\tau_{n+4}\tau_{n+3}, \tag{4.44}$$

and for case (2) we have the same expression as (4.43) *except that* τ_n *is replaced by* $\hat{\tau}_n$*, where the latter satisfies the same relation* (4.44) *but with the modified expression for* β *and* δ*, as above. Similarly, in each case the quantity*

$$\hat{w}_n = y_{1,n} + 1$$

satisfies the Somos-5 QRT map, in the form of the recurrence

$$\hat{w}_{n+1}\hat{w}_n\hat{w}_{n-1} = \zeta\hat{w}_n + \theta, \tag{4.45}$$

where, for the appropriate value of β *in each case, the coefficients are given by* $\theta = (\beta - 1)\zeta$ *with*

$$\text{case (1)}: \quad \zeta = \tilde{K}_1 + b_1 + b_3 + 2, \qquad \text{case (2)}: \quad \zeta = \tilde{K}_2 + b_3 + b_4 + 2.$$

The proof of the preceding statements is very similar to what was done before for the other examples, so it is omitted.

Chapter 5

Higher rank type $\mathbb{B}$ and $\mathbb{D}$ cluster maps

Up to this point, we have shown that the cluster maps of initial type B and D admit the deformation. However, as we move onto a higher-order case, the process of finding the integrability conditions of the parameters in a deformed map becomes more complex since we need to build a sufficient number of invariant functions which are in involution with respect to the Poisson bracket defined by the inverse of the skew-symmetric matrix. We have attempted the same procedure used in the type A case by using the periodicity of the map, but as yet, we have not succeeded in finding compatible first integrals. However, it does not imply that one cannot proceed further. From the several successful cases in the previous chapters, we could see that the complexity of iteration of the maps (measured by the height function) reduces significantly when the integrability condition is imposed. On the other hand, the numerator and denominator grow exponentially if we set other parameter conditions. This suggests there is a relatively close relation between integrability conditions and Laurentification.

In this chapter, we consider the deformation of cluster algebra associated with type B_4 and D_6 with certain parameter conditions analogous to the conditions in type B_3 and D_4 cases.

5.1 Deformed B_4 cluster map

The Cartan matrix for the type B_4 is

$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{5.1}$$

which is Cartan counterpart of the following exchange matrix,

$$B_{\mathrm{B}_4} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{5.2}$$

The following sequence of the deformed mutations

$$\begin{aligned} \mu_1 : (x_1, x_2, x_3, x_4) &\mapsto (x_1', x_2, x_3, x_4), & x_1 x_1' &= b_1 + a_1 x_2 \\ \mu_2 : (x_1', x_2, x_3, x_4) &\mapsto (x_1', x_2', x_3, x_4), & x_2 x_2' &= b_2 + a_2 x_3 (x_1')^2 \\ \mu_3 : (x_1', x_2', x_3, x_4) &\mapsto (x_1', x_2', x_3', x_4), & x_3 x_3' &= b_3 + a_3 x_4 x_2' \\ \mu_4 : (x_1', x_2', x_3', x_4,) &\mapsto (x_1', x_2', x_3', x_4'), & x_4 x_4' &= b_4 + a_4 x_3' \end{aligned} \tag{5.3}$$

provides a birational map $\tilde{\varphi}_{\mathrm{B}_4}$ which fixes the exchange matrix B_{B_4}. The original cluster map (deformed map with parameters fixed with 1) is periodic with period 5,

$$\varphi_{\mathrm{B}_4} \cdot (\mathbf{x}, B) = (\varphi_{\mathrm{B}_4}(\mathbf{x}), B) \text{ and } \varphi_{\mathrm{B}_4}^5(\mathbf{x}) = \mathbf{x} \tag{5.4}$$

as the Coxeter number for type B_4 is 8 i.e. $(8+2)/2 = 5$. Both original and deformed maps preserves the following symplectic form,

$$\begin{aligned} \omega = & \frac{2}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_2 x_3} \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ & + \frac{1}{x_3 x_4} \mathrm{d}x_3 \wedge \mathrm{d}x_4 \end{aligned} \tag{5.5}$$

To ease the process of calculation, we rescale the variables i.e. $x_i \to \lambda_i x_i$ in a way that the parameters $a_i = 1$. Then the deformed mutations can be rewritten as

$$\begin{aligned} x_{1,n+1} x_{1,n} &= x_{2,n} + b_1 \\ x_{2,n+1} x_{2,n} &= x_{3,n} x_{1,n+1}^2 + b_2 \\ x_{3,n+1} x_{3,n} &= x_{4,n} x_{2,n+1} + b_3 \\ x_{4,n+1} x_{4,n} &= x_{3,n+1} + b_4 \end{aligned} \tag{5.6}$$

Suppose we impose the condition on the parameters, that is , $b_2 = b_1^2 = b_3 = b_4$, We set initial variables $x_i = 1$ and $b_1 = 5$ and perform the iteration. This produce the iterates which can be written in terms of prime numbers, shown in the table.

n	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x_{4,n}$
1	$2 \cdot 3$	61	$2 \cdot 43$	$3 \cdot 37$
2	11	$3^2 \cdot 19$	$13 \cdot 17$	$\frac{2 \cdot 41}{37}$
3	2^4	331	$\frac{127}{37}$	$\frac{2 \cdot 263}{41}$
4	$3 \cdot 7$	$\frac{2^2 \cdot 43}{37}$	$\frac{3 \cdot 337}{41}$	$\frac{2 \cdot 509}{263}$
5	$\frac{17}{37}$	$\frac{9857}{37 \cdot 41}$	$\frac{3^3 \cdot 733}{537 \cdot 263}$	$\frac{31 \cdot 4243}{37 \cdot 509}$
6	$\frac{2 \cdot 3^3 \cdot 19}{41}$	$\frac{71 \cdot 127 \cdot 239}{41 \cdot 263}$	$\frac{2 \cdot 97 \cdot 223 \cdot 337}{41 \cdot 509}$	$\frac{3^3 \cdot 37 \cdot 559297}{31 \cdot 41 \cdot 4243}$
7	$\frac{2153}{263}$	$\frac{3^5 \cdot 129119}{263 \cdot 509}$	$\frac{89 \cdot 109 \cdot 127 \cdot 977}{31 \cdot 263 \cdot 4243}$	$\frac{2 \cdot 19 \cdot 41 \cdot 653 \cdot 751}{3^2 \cdot 263 \cdot 559297}$
8	$\frac{2^2 \cdot 61^2}{509}$	$\frac{1433 \cdot 5935759}{31 \cdot 509 \cdot 4243}$	$\frac{7 \cdot 149 \cdot 6941549}{3^2 \cdot 509 \cdot 559297}$	$\frac{2 \cdot 263 \cdot 62129 \cdot 6997}{19 \cdot 509 \cdot 653 \cdot 751}$

One can observe from the table that the growth of the number of digits in numerator and denominator is stable which suggests possibility of lifting the $\tilde{\varphi}_{\mathrm{B}_4}$ to the new coordinate system. For the prime $p_1 = 37, 41, 263, 509$, the p-adic norms follows

$$\begin{aligned} |x_{1,n}|_{p_1} &: 1, 1, 1, 1, 1, p_1, 1, 1, 1, 1 \\ |x_{2,n}|_{p_1} &: 1, 1, 1, 1, p_1, p_1, 1, 1, 1, 1 \\ |x_{3,n}|_{p_1} &: 1, 1, 1, p_1, p_1, 1, 1, 1, 1, 1 \\ |x_{4,n}|_{p_1} &: 1, p_1^{-1}, p_1, 1, 1, 1, p_1, p_1^{-1}, 1 \end{aligned} \tag{5.7}$$

The prime $p_2 = 11, 17, 19, 2153$ only emerge at $x_{1,n}$ and not at the other variables . The other prime numbers $p_3 = 19, 61, 331, 9857$ and $p_4 = 13, 17, 127, 337$ appear at $x_{2,n}$, $x_{3,n}$ and $x_{4,n}$ respectively. Let us introduce new variables $\tau \equiv 0 \pmod{p_1}$, $\sigma \equiv 0 \pmod{p_2}$, $\eta \equiv 0 \pmod{p_3}$ and $\xi \equiv 0 \pmod{p_4}$. Altogether suggests that variable transformations (equivalent to the rational map $\pi : \mathbb{C}^9 \to \mathbb{C}^4$) possess the following form,

$$x_{1,n} = \frac{\sigma_n}{\tau_{n+1}}, \quad x_{2,n} = \frac{\eta_n}{\tau_{n+1}\tau_{n+2}}, x_{3,n} = \frac{\xi_n}{\tau_{n+1}\tau_{n+3}}, x_{4,n} = \frac{\tau_n \tau_{n+5}}{\tau_{n+1}\tau_{n+4}} \tag{5.8}$$

This lifts the higher dimensional map ψ_{B_4}, which can be represented as the system of recurrence relation,

$$
\begin{aligned}
\sigma_{n+1}\sigma_n &= \beta\tau_{n+1}\tau_{n+2} + \eta_n \\
\eta_{n+1}\eta_n &= \beta^2\tau_{n+1}\tau_{n+2}^2\tau_{n+3} + \xi_n\sigma_{n+1}^2 \\
\xi_{n+1}\xi_n &= \beta^2\tau_{n+1}\tau_{n+2}\tau_{n+3}\tau_{n+4} + \tau_n\tau_{n+5}\eta_{n+1} \\
\tau_{n+6}\tau_n &= \beta^2\tau_{n+2}\tau_{n+4} + \xi_{n+1}
\end{aligned}
\tag{5.9}
$$

Upon labeling initial variables, $\tilde{\mathbf{x}} = (\xi_0, \sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \eta_0) = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_9)$, we apply pull back of symplectic form ω by the rational map to obtain the new symplectic form, expressed with $\tilde{x}_i$ for $1 \leq i \leq 9$,

$$
\tilde{\omega} = \pi^*\omega = \sum_{ij} \tilde{\Omega}_{ij} \mathrm{d}\log\tilde{x}_i \wedge \mathrm{d}\log\tilde{x}_j
$$

which yields 9×9 skew-symmetric matrix,

$$
\Omega_{D_6} = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 2 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 1 & 0 & 1 & -1 & -1 \\
-1 & 2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & -2 & 0 & 1 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
\tag{5.10}
$$

If we post-multiplying the $\tilde{\Omega}$ with $\tilde{D} = \mathrm{diag}(1, 1/2, 1, 1, 1, 1, 1, 1, 1)$, we obtain the skew-symmetrizable matrix

$$
\tilde{B}_{\mathrm{B}_4} = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 2 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 \\
-1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
\tag{5.11}
$$

One can confirm that this is exchange matrix, which provide coefficient free cluster algebra, whose cluster variables are expressed by the relation (5.9) with $\beta = 1$. We add the extra row (frozen vertices) to the exchange matrix and thus we obtain the following result.

Theorem 5.1.1 (Laurentification of the deformed map). *Let* $(\hat{\mathbf{x}}, \hat{B}_{\mathrm{B}_4})$ *be initial seed which is composed of extended initial cluster*

$$\begin{aligned} \hat{\mathbf{x}} &= (\xi_0, \sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \eta_0, \beta) \\ &= (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}) \end{aligned} \tag{5.12}$$

together with the associated extended exchange matrix

$$\hat{B}_{\mathrm{B}_4} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix} \tag{5.13}$$

and consider the permutation $\rho = (345678)$. *Then the iteration of cluster map* $\psi_{\mathrm{B}_4} = \rho^{-1}\mu_3\mu_1\mu_9\mu_2$ *is equivalent to the recurrence* (5.9)*, and for the tau functions* σ_n*,* η_n*,* ξ_n*,* τ_n *are elements of the Laurent polynomial ring* $\mathbb{Z}_{>0}\big[\beta, \sigma_0^{\pm}, \eta_0^{\pm}, \xi_0^{\pm}, \tau_0^{\pm}, \tau_1^{\pm}, \tau_2^{\pm}, \tau_3^{\pm}, \tau_4^{\pm}, \tau_5^{\pm}\big]$.

Proof. The mutation $\tilde{\mu}_2\tilde{\mu}_9\tilde{\mu}_1\tilde{\mu}_3$ acting on the initial seed $(\hat{\mathbf{x}}, \hat{B}_{\mathrm{B}_4})$ produces the new cluster (mutated variables) $\hat{\mathbf{x}} = (\tilde{x}_1', \tilde{x}_2', \tilde{x}_3', \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9')$, where the new cluster variables are given by following exchange relations

$$\begin{aligned} \tilde{x}_2'\tilde{x}_2 &= \tilde{x}_{10}\tilde{x}_4\tilde{x}_5 + \tilde{x}_9 \\ \tilde{x}_9'\tilde{x}_9 &= \tilde{x}_{10}^2\tilde{x}_4\tilde{x}_6\tilde{x}_7 + \tilde{x}_1(\tilde{x}_2')^2 \\ \tilde{x}_1'\tilde{x}_1 &= \tilde{x}_{10}^2\tilde{x}_4\tilde{x}_5\tilde{x}_6\tilde{x}_7 + \tilde{x}_3\tilde{x}_8\tilde{x}_9' \\ \tilde{x}_3'\tilde{x}_3 &= \tilde{x}_{10}^2\tilde{x}_5\tilde{x}_7 + \tilde{x}_1' \end{aligned} \tag{5.14}$$

along with the mutated exchange matrix

$$\hat{B}_{\mathrm{B}_4} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.15}$$

When we substitute the tau functions (5.12) into the exchange relations (5.14) and replace mutated variables $\tilde{x}'_1$ $\tilde{x}'_2$, $\tilde{x}'_3$, $\tilde{x}'_9$ with ξ_1,σ_1, τ_6, η_1, we obtain (5.9) with $n = 0$.

The mutated exchange matrix can be derived from the exchange matrix via permutating the rows and columns by P_1 and P_2,

$$\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_9\tilde{\mu}_2(\hat{B}_{\mathrm{B}_4}) = \rho(\hat{B}_{\mathrm{B}_4}) = P_1\hat{B}_{\mathrm{B}_4}P_2 \tag{5.16}$$

where permutation $\rho = (345678)$ can be represented by 10×10 and 9×9 matrices,

$$P_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \tag{5.17}$$

Hence the iteration of the cluster map $\psi_{\mathrm{B}_4} = \rho^{-1}\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_9\tilde{\mu}_2$ on the initial cluster $\hat{\mathbf{x}}$ induces cluster variables which are expressed in the form of recursion (5.14).

□

5.2 Tropicalization and degree growth for deformed B_4 map

In the previous example, we have seen that the deformation of type B_4 cluster map can be lifted to higher dimenisonal cluster map via Laurentification with the condition of parameters $b_2 = b_1^2 = b_3 = b_4$. However, we have not yet to prove the integrability of the deformed map. In this section, we consider the calculation of the degree growth by using tropical method which we applied in the type A_{2N} case. This provides us positive indication that the deformed type B_4 map is integrable.

Due to the Laurent property of the cluster map, the sequence of tau-functions τ_n, σ_n, η_n and ξ_n, generated by ψ_{B_4} (5.9), can be expressed as

$$\tau_n = \frac{N_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}}, \quad \xi_n = \frac{N_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{e}_n}}, \quad \sigma_n = \frac{N_n^{(3)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{f}_n}}, \quad \eta_n = \frac{N_n^{(4)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{g}_n}} \tag{5.18}$$

with initial d-vectors of the initial cluster variables (non frozen variables) identified by the 9×9 identity matrix.

$$(\mathbf{e}_0\ \mathbf{f}_0\ \mathbf{d}_0\ \mathbf{d}_1\ \mathbf{d}_2\ \mathbf{d}_3\ \mathbf{d}_4\ \mathbf{d}_5\ \mathbf{g}_0) = -I \tag{5.19}$$

Upon substituting the expression (5.18) into (5.9) and compare the denominator on each side, we obtain the relation which d-vectors holds:

$$\begin{aligned}
\mathbf{f}_{n+1} + \mathbf{f}_n &= \max(\mathbf{d}_{n+1} + \mathbf{d}_{n+2}, \mathbf{g}_n), \\
\mathbf{g}_{n+1} + \mathbf{g}_n &= \max(\mathbf{d}_{n+1} + 2\mathbf{d}_{n+2} + \mathbf{d}_{n+3}, \mathbf{e}_n + 2\mathbf{f}_{n+1}), \\
\mathbf{e}_{n+1} + \mathbf{e}_n &= \max(\mathbf{d}_{n+1} + \mathbf{d}_{n+2} + \mathbf{d}_{n+3} + \mathbf{d}_{n+4}, \mathbf{d}_n + \mathbf{d}_{n+5} + \mathbf{g}_{n+1}), \\
\mathbf{d}_{n+6} + \mathbf{d}_n &= \max(\mathbf{d}_{n+2} + \mathbf{d}_{n+4}, \mathbf{e}_{n+1}),
\end{aligned} \tag{5.20}$$

To determine the degree growths of d-vectors, as usual, we introduce the tropical analogues of the substitutions (5.8), which takes the form,

$$\begin{aligned}
\mathbf{X}_{1,n} = \mathbf{f}_n - \mathbf{d}_{n+1}, \qquad \mathbf{X}_{2,n} = \mathbf{g}_n - \mathbf{d}_{n+1} - \mathbf{d}_{n+2}, \\
\mathbf{X}_{3,n} = \mathbf{e}_n - \mathbf{d}_{n+1} - \mathbf{d}_{n+3}, \qquad \mathbf{X}_{4,n} = \mathbf{d}_n + \mathbf{d}_{n+5} - \mathbf{d}_{n+1} - \mathbf{d}_{n+4}
\end{aligned} \tag{5.21}$$

Then we can use these substitution to derive the tropical version of the expressions (5.6) generated by deformed map $\tilde{\varphi}_{\mathrm{B}_4}$, as stated in below.

Lemma 5.2.1. *The structure of $X_{j,n}$ in* (5.21) *satisfy the tropical analogue of deformed map φ_{B_4} which is given by the following system of (max,+) equations:*

$$\begin{aligned} \mathbf{X}_{1,n+1} + \mathbf{X}_{1,n} &= [\mathbf{X}_{2,n}]_+, \\ \mathbf{X}_{2,n+1} + \mathbf{X}_{2,n} &= [\mathbf{X}_{1,n+1} + \mathbf{X}_{3,n}]_+, \\ \mathbf{X}_{3,n+1} + \mathbf{X}_{3,n} &= [\mathbf{X}_{2,n+1} + \mathbf{X}_{4,n}]_+, \\ \mathbf{X}_{4,n+1} + \mathbf{X}_{4,n} &= [\mathbf{X}_{3,n+1}]_+. \end{aligned} \tag{5.22}$$

where $[a]_+ = \max(a, 0)$. *Given arbitrary initial values* $(Y_{1,0}, Y_{2,0})$, *the quantities* $\mathbf{X}_{j,n}$ *is periodic with period 5.*

As mentioned above the structure of d-vectors in (5.21) and periodicity of $\mathbf{X}_{j,n}$ allow us determine the growths of d-vector, shown below.

Theorem 5.2.2. *Let $\mathcal{T}$ be linear operator which shifts $n \to n+1$. The d-vectors $\mathbf{e}_n$,$\mathbf{d}_n$,$\mathbf{f}_n$ and $\mathbf{g}_n$, which solve the system of equations* (5.20), *are elements of the kernel of linear difference operator*

$$\mathcal{L} = (\mathcal{T}^5 - 1)(\mathcal{T}^4 - 1)(\mathcal{T} - 1) \tag{5.23}$$

where $\mathcal{T}$ is shift operator corresponding to $n \to n+1$. For the tau functions generated, the leading order of degree growth of their denominators is given by

$$\begin{aligned} \mathbf{d}_n &= \frac{n^2}{40}\mathbf{a} + O(n), \quad \mathbf{f}_n = \frac{n^2}{40}\mathbf{a} + O(n) \\ \mathbf{e}_n &= \frac{n^2}{20}\mathbf{a} + O(n), \quad \mathbf{g}_n = \frac{n^2}{20}\mathbf{a} + O(n) \end{aligned} \tag{5.24}$$

where $\mathbf{a} = (2, 2, 1, 1, 1, 1, 1, 1, 2)^T$.

Proof. The expression for $\mathbf{X}_{4,n}$ can be written as

$$(\mathcal{T}^4 - 1)(\mathcal{T} - 1)\mathbf{d}_n = \mathbf{X}_{4,n}.$$

Then by applying Lemma 5.2.1, we have

$$\mathcal{L}\mathbf{d}_n = (\mathcal{T}^5 - 1)(\mathcal{T}^4 - 1)(\mathcal{T} - 1)\mathbf{d}_n = (\mathcal{T}^5 - 1)X_{4,n} = 0$$

which shows $\mathbf{d}_n$ is kernel of linear operator $\mathcal{L}$.

$$\mathcal{L}\mathbf{f}_n = \mathcal{L}\mathbf{X}_{1,n} + \mathcal{L}\mathbf{d}_{n+1} = 0$$

where we used the fact that $\mathbf{X}_{1,n}$ has period 5, from the same lemma. Similarly, the expression $\mathbf{X}_{2,n}$ and $\mathbf{X}_{3,n}$ gives

$$\mathcal{L}\mathbf{g}_n = \mathcal{L}\mathbf{X}_{2,n} + \mathcal{L}\mathbf{e}_{n+2} + \mathcal{L}\mathbf{d}_n = 0$$
$$\mathcal{L}\mathbf{f}_n = \mathcal{L}\mathbf{X}_{3,n} + \mathcal{L}\mathbf{e}_{n+3} + \mathcal{L}\mathbf{d}_n = 0$$

The solution of the difference equations gives the expression for d-vectors whose leading order terms expressed by

$$\mathbf{d}_n = \mathbf{a}_1 n^2 + O(n), \quad \mathbf{e}_n = \mathbf{a}_2 n^2 + O(n),$$
$$\mathbf{f}_n = \mathbf{a}_3 n^2 + O(n), \quad \mathbf{g}_n = \mathbf{a}_4 n^2 + O(n)$$

Now we first consider determining the constant coefficient $\mathbf{a}_1$ in $\mathbf{d}_n$. The expression of linear operator $\mathcal{L}\mathbf{d}_n = 0$ suggests that

$$(\mathcal{T}^5 - 1)(\mathcal{T}^4 - 1)(\mathbf{d}_{n+1} - \mathbf{d}_n) = 0 \iff (\mathcal{T}^5 - 1)(\mathcal{T}^4 - 1)\mathbf{d}_n = \text{constant}$$

Furthermore since difference equation above can be written as $\mathbf{d}_{n+9} - \mathbf{d}_{n+5} - \mathbf{d}_{n+4} + \mathbf{d}_n = \text{constant}$, substituting the $\mathbf{d}_n = \mathbf{a}_1 n^2 + O(n)$ directly into such equation results annihilating the $O(n)$ term. Thus the constant term above is in fact vector $40\mathbf{a}_1$.

To find the $\mathbf{a}_1$, we consider the sequence of d-vectors induced by the relations in (5.20) with the initial data (5.19). We look at the sequence of $\mathbf{d}_n$ given by

$$\begin{aligned}
\mathbf{d}_6 &= (1,2,1,0,0,0,0,0,1)^T \\
\mathbf{d}_7 &= (1,2,1,1,0,0,0,0,2)^T \\
\mathbf{d}_8 &= (2,2,1,1,1,0,0,0,2)^T \\
\mathbf{d}_9 &= (2,2,2,1,1,1,0,0,2)^T \\
\mathbf{d}_{10} &= (3,4,2,2,1,1,1,0,3)^T \\
\mathbf{d}_{11} &= (4,6,3,2,2,1,1,1,5)^T \\
\mathbf{d}_{12} &= (5,6,3,3,2,2,1,1,6)^T
\end{aligned}$$

Based on the observation of $\mathbf{d}_n$ vectors, we can denote each vector components of $\mathbf{d}_n$ as

$$\mathbf{d}_n = (d_n^{(1)}, d_n^{(2)}, d_{n+5}^{(3)}, d_{n+4}^{(3)}, d_{n+3}^{(3)}, d_{n+2}^{(3)}, d_{n+1}^{(3)}, d_n^{(3)}, d_n^{(4)}) \tag{5.25}$$

$$d_n^{(1)} : 0, 0, 0, 0, 0, 0, 1, 1, 2, 2, 3, 4, 5, 6, 7, 8, 10, 11, \dots$$
$$d_n^{(2)} : 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 4, 6, 6, 6, 8, 10, 12, 12, \dots$$
$$d_n^{(3)} : 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 1, 1, 2, 2, 3, 3, \dots$$
$$d_n^{(4)} : 0, 0, 0, 0, 0, 0, 1, 2, 2, 2, 3, 5, 6, 6, 7, 9, 11, 12, \dots$$

For any value, explicit calculation of the difference equation $\mathbf{d}_{n+9} - \mathbf{d}_{n+5} - \mathbf{d}_{n+4} + \mathbf{d}_n$ using the data above gives vector $(2, 2, 1, 1, 1, 1, 1, 1, 2)^T$. Then, by combining this with the other results above, we have

$$(\mathcal{T}^5 - 1)(\mathcal{T}^4 - 1)\mathbf{e}_n = (2, 2, 1, 1, 1, 1, 1, 1, 2)^T = 40\mathbf{a}$$

which fixes the constant vector $\mathbf{a}_1$. Hence we find

$$\mathbf{d}_n = \frac{n^2}{40}(2, 2, 1, 1, 1, 1, 1, 1, 2)^T + O(n)$$

For the other d-vectors, we can take the same procedure since they are in the same situation such that they are elements in kernel of linear operator $\mathcal{L}$. Alternatively, one can use the formula for $\mathbf{X}_{1,n}, \mathbf{X}_{2,n}, \mathbf{X}_{3,n}$ in (5.21) and their periodicity,

$$\begin{aligned}
\mathbf{f}_n &= \mathbf{d}_{n+1} + \mathbf{X}_{1,n} = \mathbf{a}_1 n^2 + O(n), \\
\mathbf{g}_n &= \mathbf{d}_{n+1} + \mathbf{d}_{n+2} + \mathbf{X}_{1,n} = 2\mathbf{a}_1 n^2 + O(n), \\
\mathbf{e}_n &= \mathbf{d}_{n+1} + \mathbf{d}_{n+3} + \mathbf{X}_{3,n} = 2\mathbf{a}_1 n^2 + O(n).
\end{aligned}$$

We obtain the expression for d-vectors whose leading order is quadratic. Hence we obtain the desired outcome.

□

Upon applying the algebraic entropy test, we are led to the conjecture that deformed type B_4 map is Liouville integrable map.

5.3 Deformed D_6 cluster map

The Cartan matrix for the type D_6 is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \tag{5.26}$$

which is corresponding to the following exchange matrix,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{5.27}$$

Let us consider the deformation of the composition of cluster mutations $\mu_6\mu_5\mu_4\mu_3\mu_2\mu_1$ which is shown as below.

$$\begin{array}{lrcl}
\mu_1 : (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1', x_2, x_3, x_4, x_5, x_6), & x_1x_1' & = & b_1 + a_1x_2 \\
\mu_2 : (x_1', x_2, x_3, x_4, x_5, x_6) \mapsto (x_1', x_2', x_3, x_4, x_5, x_6), & x_2x_2' & = & b_2 + a_2x_3x_1' \\
\mu_3 : (x_1', x_2', x_3, x_4, x_5, x_6) \mapsto (x_1', x_2', x_3', x_4, x_5, x_6), & x_3x_3' & = & b_3 + a_3x_4x_2' \\
\mu_4 : (x_1', x_2', x_3', x_4, x_5, x_6) \mapsto (x_1', x_2', x_3', x_4', x_5, x_6), & x_4x_4' & = & b_4 + a_4x_5x_6x_3' \\
\mu_5 : (x_1', x_2', x_3', x_4', x_5, x_6) \mapsto (x_1', x_2', x_3', x_4', x_5', x_6), & x_5x_5' & = & b_5 + a_5x_4' \\
\mu_6 : (x_1', x_2', x_3', x_4, x_5', x_6) \mapsto (x_1', x_2', x_3', x_4', x_5', x_6'), & x_6x_6' & = & b_6 + a_6x_4'
\end{array} \tag{5.28}$$

As usual, the deformed map $\tilde{\varphi}_{\mathrm{D}_6} = \tilde{\mu_6}\tilde{\mu_5}\tilde{\mu_4}\tilde{\mu_3}\tilde{\mu_2}\tilde{\mu_1}$ transforms back to original cluster map $\varphi_{\mathrm{D}_6} = \mu_6\mu_5\mu_4\mu_3\mu_2\mu_1$ when we fix the parameters $a_i = 1 = b_i$ for all $i = 1, \ldots, 4$. Furthermore, since Coxeter number h for the type D_6 is 10, the periodicity for the cluster map is period 6,

$$\varphi_{\mathrm{D}_6} \cdot (\mathbf{x}, B) = (\varphi_{\mathrm{D}_6}(\mathbf{x}), B) \quad \text{and} \quad \varphi^6(\mathbf{x}) = \mathbf{x} \tag{5.29}$$

The symplectic form ω, that is invariant under the deformed map, takes a following form,

$$\begin{aligned}\omega =&\frac{1}{x_1x_2}\mathrm{d}x_1\wedge\mathrm{d}x_2+\frac{1}{x_2x_3}\mathrm{d}x_2\wedge\mathrm{d}x_3\\&+\frac{1}{x_4x_5}\mathrm{d}x_4\wedge\mathrm{d}x_5+\frac{1}{x_4x_6}\mathrm{d}x_4\wedge\mathrm{d}x_6\end{aligned}\tag{5.30}$$

Like several previous examples, we can rescale each cluster variables (i.e. $x_i \to \lambda_i x_i$) to adjust the parameters in a way that the (5.28) can be rewritten as

$$\begin{aligned}x_{1,n+1}x_{1,n} &= x_{2,n}+b_1,\\ x_{2,n+1}x_{2,n} &= x_{3,n}x_{1,n+1}+b_2,\\ x_{3,n+1}x_{3,n} &= x_{4,n}x_{2,n+1}+b_3\\ x_{4,n+1}x_{4,n} &= x_{5,n}x_{6,n}x_{3,n+1}+b_3\\ x_{5,n+1}x_{5,n} &= x_{4,n+1}+b_5\\ x_{6,n+1}x_{6,n} &= x_{4,n+1}+b_6\end{aligned}\tag{5.31}$$

From the exchange matrix B_{D_6}, (5.27), one can see that it is degenerate and possess rank 4. Thus we consider the null and column space of B_{D_6} given by following

$$\begin{aligned}\ker(B) &= \left\{(1,0,1,0,0,1)^T,(1,0,1,0,1,0)^T\right\},\\ \mathrm{im}\,(B) &= \left\{(0,1,0,0,0,0)^T,(-1,0,1,0,0,0)^T,(0,-1,0,1,0,0)^T,(0,0,-1,0,1,1)^T\right\}\end{aligned}\tag{5.32}$$

The vectors in $\mathrm{im}\,(B)$ leads to constructing new variables y_1, y_2, y_3, y_4 i.e.

$$y_1 = x_2,\quad y_2=\frac{x_3}{x_1},\quad y_3=\frac{x_4}{x_2},\quad y_4=\frac{x_5x_6}{x_3}\tag{5.33}$$

which reduces the deformed map into the 4D symplectic map i.e.

$$\hat{\varphi}:(y_1,y_2,y_3,y_4)\to(y_1',y_2',y_3',y_4')\tag{5.34}$$

where the produced variables are written as

$$\begin{aligned}y_1' &= \frac{(b_1+y_1)y_2+b_2}{y_1}\\ y_2' &= \frac{y_3(b_1+y_1)y_2+b_2y_3+b_3}{y_2(b_1+y_1)}\\ y_3' &= \frac{y_3(b_1+y_1)y_2y_4+b_2y_3y_4+b_3y_4+b_4}{y_3(b_2+(b_1+y_1)y_2)}\\ y_4' &= \frac{((b_1y_2+y_1y_2+b_2)y_3y_4+b_6y_1y_3+b_3y_4+b_4)((b_1y_2+y_1y_2+b_2)y_3y_4+b_5y_1y_3+b_3y_4+b_4)}{y_1^2y_3^2y_4((b_1y_2+y_1y_2+b_2)y_3+b_3)}\end{aligned}\tag{5.35}$$

Along the map, the exchange matrix is also reduced to the 4×4 skew-symmetric matrix of the form,

$$\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Here we take same step as shown in the type B_4 case, that is, we impose certain parameter condition which yields iterations where the number of factors in both each numerator and denominator grows slower than exponential growth. Suppose we impose the condition on the parameters, that is , $b_1 = b_2 = b_3 = b_4 = b_5 b_6$. We set initial variables $y_i = 1$ and $b_5 = 2$ and $b_6 = 3$ and perform the iteration. This gives the iterates which can be written in terms of prime numbers, shown in the table.

n	$y_{1,n}$	$y_{2,n}$	$y_{3,n}$	$y_{4,n}$
1	13	$\frac{19}{7}$	$\frac{5^2}{13}$	$\frac{2^2 \cdot 3^3 \cdot 7}{19}$
2	$\frac{31}{7}$	$\frac{43}{19}$	$2 \cdot 3 \cdot 7$	$\frac{7 \cdot 47}{43}$
3	$\frac{127}{19}$	$\frac{2^3 \cdot 3 \cdot 7 \cdot 23}{73}$	$\frac{977}{127}$	$\frac{5 \cdot 11 \cdot 29}{2^2 \cdot 3 \cdot 19 \cdot 23}$
4	$\frac{2 \cdot 3^3 \cdot 137}{73}$	$\frac{13109}{7 \cdot 241}$	$\frac{2 \cdot 23 \cdot 467}{3^3 \cdot 19 \cdot 137}$	$\frac{3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13}{73 \cdot 13109}$
5	$\frac{5 \cdot 2797}{7 \cdot 241}$	$\frac{47 \cdot 499}{2 \cdot 3 \cdot 19 \cdot 653}$	$\frac{3 \cdot 7 \cdot 36263}{5 \cdot 73 \cdot 2797}$	$\frac{2^4 \cdot 17 \cdot 19 \cdot 443}{47 \cdot 241 \cdot 499}$
6	$\frac{94307}{2 \cdot 3 \cdot 19 \cdot 653}$	$\frac{3 \cdot 7 \cdot 257501}{73 \cdot 24107}$	$\frac{2^2 \cdot 19 \cdot 23 \cdot 19259}{241 \cdot 94307}$	$\frac{73 \cdot 13457 \cdot 5197}{3^2 \cdot 653 \cdot 257501}$
7	$\frac{3 \cdot 13117691}{73 \cdot 24107}$	$\frac{2^4 \cdot 19 \cdot 211 \cdot 30559}{7 \cdot 241 \cdot 269 \cdot 2011}$	$\frac{5^2 \cdot 73 \cdot 2711 \cdot 62311}{3^2 \cdot 653 \cdot 13117691}$	$\frac{2^5 \cdot 7 \cdot 23 \cdot 41 \cdot 181 \cdot 241 \cdot 881}{211 \cdot 30559 \cdot 24107}$
8	$\frac{2 \cdot 199 \cdot 980249}{241 \cdot 269 \cdot 2011}$	$\frac{2^3 \cdot 13 \cdot 73 \cdot 149 \cdot 1053713}{3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 653 \cdot 10289}$	$\frac{83 \cdot 229 \cdot 241 \cdot 6317 \cdot 8681}{199 \cdot 980249 \cdot 24107}$	$\frac{3^2 \cdot 11 \cdot 179 \cdot 653 \cdot 98597 \cdot 14281}{2^3 \cdot 13 \cdot 149 \cdot 269 \cdot 1053713 \cdot 2011}$
9	$\frac{19 \cdot 252533 \cdot 45127}{3^3 \cdot 7 \cdot 11 \cdot 653 \cdot 10289}$	$\frac{59 \cdot 97 \cdot 241 \cdot 605239213}{73 \cdot 7712933 \cdot 24107}$	$\frac{3^2 \cdot 11 \cdot 13 \cdot 109 \cdot 653 \cdot 74857 \cdot 31627}{19 \cdot 269 \cdot 252533 \cdot 45127 \cdot 2011}$	$\frac{5 \cdot 1543 \cdot 24107 \cdot 6089 \cdot 280759}{2 \cdot 3 \cdot 7 \cdot 59 \cdot 97 \cdot 605239213 \cdot 10289}$

From the table, we see that the number of prime factors increases as polynomial growths. One can see that the prime numbers $p = 73, 241, 653$ appears in all variables. The corresponding p-adic norm gives the sequence

$$\begin{aligned} |y_{1,n}|_p &= 1,1,p,1,1,p,1 \\ |y_{2,n}|_p &= 1,p,1,1,p,1,p^{-1},p,1 \\ |y_{3,n}|_p &= 1,1,1,p,1,p^{-1},1,1 \\ |y_{4,n}|_p &= 1,1,p,1,p^{-1},1,1 \end{aligned} \tag{5.36}$$

There are particular values which emerge in two variables. For $p = 31, 137$, $|y_{1,n}|_p = p^{-1}$ and $|y_{3,n}|_p = p$. For $p_3 = 19, 23, 43, 47, 499, 13109$, $|y_{2,n}|_p = p^{-1}$ and $|y_{4,n}|_p = p$. There are unique primes which do not appear in the other variables. e.g primes $p = 31, 137$ emerge only in the y_1.

For each variable, there are unique primes which do not appear in the other variables. e.g primes $p = 31, 137$ emerge only in the y_1. Thus one can observe the pattern in the iterations in $(y_{1,n}, y_{2,n}, y_{3,n}, y_{4,n})$

$$\begin{aligned}
&\text{Pattern 1} : \ldots \to (R, \infty^1, R, R) \to (\infty^1, R, R, \infty^1) \to (R, R, \infty^1, R) \to (R, \infty^1, R, 0^1) \\
&\qquad\qquad \to (\infty^1, R, 0^1, R) \to (R, 0^1, R, R) \to (R, \infty^1, R, R) \to \ldots \\
&\text{Pattern 2} : \ldots \to (0^1, R, \infty, R) \to \ldots \\
&\text{Pattern 3} : \ldots \to (R, 0^1, R, \infty) \to \ldots \\
&\text{Pattern 4} : \ldots \to (R, R, 0^1, R) \to \ldots \\
&\text{Pattern 5} : \ldots \to (R, R, R, 0^1) \to \ldots
\end{aligned} \tag{5.37}$$

Given tau-functions specified by $\tau \equiv 0 \pmod{p_1}$, $P \equiv 0 \pmod{p_2}, Q \equiv 0 \pmod{p_3}$, $R \equiv 0 \pmod{p_4}, V \equiv 0 \pmod{p_5}$, $J \equiv 0 \pmod{p_6}$, each symplectic coordinates can be written as

$$y_{1,n} = \frac{P_n}{\tau_{n+2}\tau_{n+5}}, \quad y_{2,n} = \frac{\tau_{n+1}Q_n}{\tau_n\tau_{n+3}\tau_{n+6}}, \quad y_{3,n} = \frac{\tau_{n+2}R_n}{\tau_{n+4}P_n}, \quad y_{4,n} = \frac{\tau_{n+3}V_nJ_n}{\tau_{n+5}Q_n} \tag{5.38}$$

If we substitute the variables into exchange relations, we obtain long expressions which do not match with exchange relation. Instead, we look at view of original system in terms of x-variables. We fix the initial cluster variables $x_i = 1$ for $i = 1, 2, 3, 4, 5, 6$ and set the same paramter values as above. Then iteration yields sequences shown in the table below,

n	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x_{4,n}$
1	7	13	19	5^2
2	$\frac{19}{7}$	$\frac{31}{7}$	$\frac{43}{7}$	$2 \cdot 3 \cdot 31$
3	$\frac{73}{19}$	$\frac{127}{19}$	$\frac{2^3 \cdot 3 \cdot 7 \cdot 23}{19}$	$\frac{977}{19}$
4	$\frac{241}{73}$	$\frac{2 \cdot 3^3 \cdot 137}{73}$	$\frac{13109}{7 \cdot 73}$	$\frac{2^2 \cdot 23 \cdot 467}{19 \cdot 73}$
5	$\frac{2^2 \cdot 3 \cdot 653}{241}$	$\frac{5 \cdot 2797}{7 \cdot 241}$	$\frac{2 \cdot 47 \cdot 499}{19 \cdot 241}$	$\frac{3 \cdot 36263}{73 \cdot 241}$
6	$\frac{24107}{2^2 \cdot 3 \cdot 7 \cdot 653}$	$\frac{94307}{2 \cdot 3 \cdot 19 \cdot 653}$	$\frac{257501}{2^2 \cdot 73 \cdot 653}$	$\frac{2 \cdot 23 \cdot 19259}{3 \cdot 241 \cdot 653}$
7	$\frac{2 \cdot 7 \cdot 269 \cdot 2011}{19 \cdot 24107}$	$\frac{3 \cdot 13117691}{73 \cdot 24107}$	$\frac{2^5 \cdot 211 \cdot 30559}{241 \cdot 24107}$	$\frac{5^2 \cdot 2711 \cdot 62311}{3 \cdot 653 \cdot 24107}$
8	$\frac{3^2 \cdot 7 \cdot 11 \cdot 19 \cdot 10289}{2 \cdot 73 \cdot 269 \cdot 2011}$	$\frac{2 \cdot 199 \cdot 980249}{241 \cdot 269 \cdot 2011}$	$\frac{2^2 \cdot 13 \cdot 149 \cdot 1053713}{3 \cdot 269 \cdot 653 \cdot 2011}$	$\frac{2 \cdot 83 \cdot 229 \cdot 6317 \cdot 8681}{269 \cdot 24107 \cdot 2011}$

n	$x_{5,n}$	$x_{6,n}$
1	3^3	$2^2 \cdot 7$
2	$\frac{2^2 \cdot 47}{3^3}$	$\frac{3^3}{2^2}$
3	$\frac{3^3 \cdot 5 \cdot 7 \cdot 29}{2^2 \cdot 19 \cdot 47}$	$\frac{2^3 \cdot 11 \cdot 47}{3^3 \cdot 19}$
4	$\frac{2^3 \cdot 11^2 \cdot 47}{5 \cdot 29 \cdot 73}$	$\frac{3^3 \cdot 5^3 \cdot 13 \cdot 29}{2^3 \cdot 11 \cdot 47 \cdot 73}$
5	$\frac{5^3 \cdot 13 \cdot 29 \cdot 443}{2^3 \cdot 11^2 \cdot 47 \cdot 241}$	$\frac{2^8 \cdot 11^2 \cdot 17 \cdot 47}{5^3 \cdot 13 \cdot 29 \cdot 241}$
6	$\frac{2^6 \cdot 11^2 \cdot 17 \cdot 47 \cdot 13457}{3 \cdot 5^3 \cdot 13 \cdot 29 \cdot 443 \cdot 653}$	$\frac{5^3 \cdot 13 \cdot 29 \cdot 443 \cdot 5197}{2^8 \cdot 3 \cdot 11^2 \cdot 17 \cdot 47 \cdot 653}$
7	$\frac{5^3 \cdot 13 \cdot 23 \cdot 29 \cdot 41 \cdot 443 \cdot 881 \cdot 5197}{2^6 \cdot 11^2 \cdot 17 \cdot 47 \cdot 13457 \cdot 24107}$	$\frac{2^{1}6 \cdot 7 \cdot 11^2 \cdot 17 \cdot 47 \cdot 181 \cdot 13457}{5^3 \cdot 13 \cdot 29 \cdot 443 \cdot 24107 \cdot 5197}$
8	$\frac{2^{15} \cdot 3 \cdot 7 \cdot 11^3 \cdot 17 \cdot 47 \cdot 181 \cdot 98597 \cdot 13457}{5^3 \cdot 13 \cdot 23 \cdot 29 \cdot 41 \cdot 269 \cdot 443 \cdot 881 \cdot 5197 \cdot 2011}$	$\frac{5^3 \cdot 13 \cdot 23 \cdot 29 \cdot 41 \cdot 179 \cdot 443 \cdot 881 \cdot 5197 \cdot 14281}{2^{1}6 \cdot 7 \cdot 11^2 \cdot 17 \cdot 47 \cdot 181 \cdot 269 \cdot 13457 \cdot 2011}$

The prime numbers, appeared in the y variables, once again can be spotted in the x-variables. e.g $p = 73, 241, 653$ corresponds to τ, $p = 31, 137$ associated with P , $p = 19, 23, 43, 47, 499$ corresponds to Q. $p = 467, 977, 36263$ corresponds to R. There are specific primes in x_5 and x_6, which cancels out in $y_4 = x_5 x_6 / x_3$. As we witnessed in the type D_4, we define the new tau function σ_n which is associated such primes. Hence by observing singularity pattern, we can construct the tau-function expressions,

$$\begin{aligned} x_{1,n} &= \frac{\tau_{n+6}\tau_n}{\tau_{n+1}\tau_{n+5}}, \quad x_{2,n} = \frac{P_n}{\tau_{n+2}\tau_{n+5}}, \quad x_{3,n} = \frac{Q_n}{\tau_{n+3}\tau_{n+5}} \\ x_{4,n} &= \frac{R_n}{\tau_{n+4}\tau_{n+5}}, \quad x_{5,n} = \frac{V_n \sigma_n}{\tau_{n+5}}, \quad x_{6,n} = \frac{J_n}{\tau_{n+5}\sigma_n} \end{aligned} \tag{5.39}$$

where $\sigma_n \sigma_{n+1} = \frac{J_n}{V_n}$. Then directly inputting the variables into the deformed mutations produce the following relations,

$$\begin{aligned}
\tau_{n+7}\tau_n &= b_5 b_6 \tau_{n+5}\tau_{n+2} + P_n \\
P_{n+1}P_n &= b_5 b_6 \tau_{n+2}\tau_{n+3}\tau_{n+5}\tau_{n+6} + Q_n\tau_{n+1}\tau_{n+7} \\
Q_{n+1}Q_n &= b_5 b_6 \tau_{n+3}\tau_{n+4}\tau_{n+5}\tau_{n+6} + R_n P_{n+1} \\
R_{n+1}R_n &= b_5 b_6 \tau_{n+4}\tau_{n+5}^2\tau_{n+6} + V_n J_n Q_{n+1} \\
V_{n+1}J_n &= b_5 \tau_{n+5}\tau_{n+6} + R_{n+1} \\
J_{n+1}V_n &= b_6 \tau_{n+5}\tau_{n+6} + R_{n+1}
\end{aligned} \tag{5.40}$$

which we denote the relations as iteration of birational maps

$$\psi_{\mathrm{D}_6} : (\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,,\tau_5,,\tau_6,P_0,Q_0,R_0,V_0,J_0) \to (\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,,\tau_6,\tau_7,P_1,Q_1,R_1,V_1,J_1) \tag{5.41}$$

Let us set initial tau functions

$$(\tilde{x}_1,\tilde{x}_2,\tilde{x}_4,\tilde{x}_5,\tilde{x}_6,\tilde{x}_7,\tilde{x}_8,\tilde{x}_9,\tilde{x}_{10},\tilde{x}_{11},\tilde{x}_{12}) = (\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,,\tau_6,P_0,Q_0,R_0,V_0,J_0)$$

and let $\pi : \mathbb{C}^{12} \to \mathbb{C}^4$ defined by (5.39) . By repeating the same process as in the previous sections, we apply pull back of symplectic form ω by the rational map π; we construct the exchange matrix, which is essential in defining ψ_{D_6}. In addition to this, we inserts extra rows, $(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 0)^T$ and $(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1)^T$, at the bottom of the matrix, which results in building extended exchange matrix $\tilde{B}_{\mathrm{D}_6}$, illustrated in Figure 5.1 As a result of the insertion, the parameters b_5, b_6

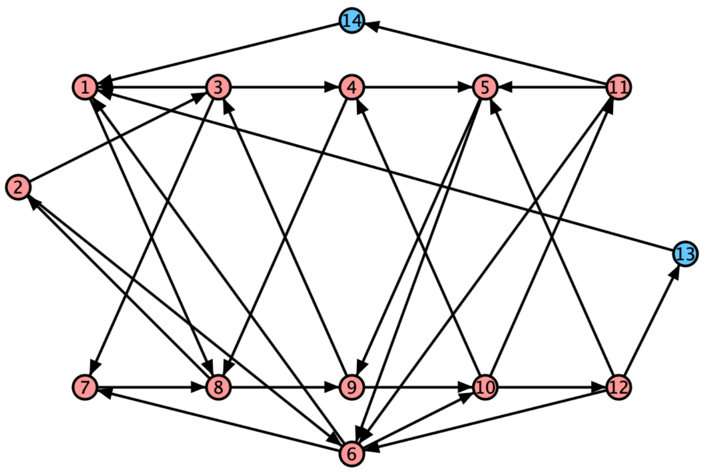


Figure 5.1: Extended quiver associated with the deformed D_6

appears in the cluster variables generated by mutations in cluster algebra consisting the pair: matrix $\tilde{B}_{\mathrm{D}_6}$ and initial cluster

$$\begin{aligned}\hat{\mathbf{x}} &= (\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, P_0, Q_0, R_0, V_0, J_0, b_5, b_6) \\ &= (\tilde{x}_1, \tilde{x}_2, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14}))\end{aligned}$$

Thus we reaches to the following result.

Theorem 5.3.1 (Laurentification of the deformed map)**.** *Let* $(\hat{\mathbf{x}}, \hat{B}_{\mathrm{D}_6})$ *be initial seed which is composed of extended initial cluster*

$$\begin{aligned}\hat{\mathbf{x}} &= (\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, P_0, Q_0, R_0, V_0, J_0, b_5, b_6) \\ &= (\tilde{x}_1, \tilde{x}_2, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14})\end{aligned} \tag{5.42}$$

together with the associated extended exchange matrix

$$\tilde{B}_{D_6} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0. & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.43}$$

and consider the permutation $\rho = (1234567)(11, 12)$ *. Then the iteration of cluster map* $\psi = \rho^{-1}\mu_{12}\mu_{11}\mu_{10}\mu_9\mu_8\mu_1$ *is equivalent to the recurrence* (5.40), *and for the tau functions* τ_n*,* P_n*,* Q_n*,* R_n*,* V_n*,* J_n *are elements of the Laurent polynomial ring* $\mathbb{Z}_{>0}\big[b_5, b_6, P_0^{\pm}, Q_0^{\pm}, R_0^{\pm}, V_0^{\pm}, J_0^{\pm}, \tau_0^{\pm}, \tau_1^{\pm}, \tau_2^{\pm}, \tau_3^{\pm}, \tau_4^{\pm}, \tau_5^{\pm}, \tau_6^{\pm}\big]$.

5.4 Tropicalization and degree growth for deformed D_6

In section 3.5.3 and section 5.2, we showed that the degree growth of iteration of cluster maps (constructed by Laurentification) is quadratic which led us to conjecture that the deformed type B_4 map is integrable. In this section, we proceed with the same method to find the expression for the d-vectors of cluster variables generated by the system (5.40) (ψ_{D_6}) and determine associated algebraic entropy.

As the Laurent property of the cluster map ψ_{D_6} holds, we can write the sequence of tau-functions τ_n, σ_n, η_n and ξ_n as Laurent polynomial in initial cluster $\hat{\mathbf{x}}$ (5.42), shown below

$$\begin{array}{lll} \tau_n = \dfrac{N_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}}, & P_n = \dfrac{N_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{p}_n}}, & Q_n = \dfrac{N_n^{(3)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{q}_n}}, \\ R_n = \dfrac{N_n^{(4)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{r}_n}}, & V_n = \dfrac{N_n^{(4)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{v}_n}}, & J_n = \dfrac{N_n^{(4)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{j}_n}} \end{array} \tag{5.44}$$

where d-vectors $\mathbf{d}_n, \mathbf{d}_n, \mathbf{d}_n, \mathbf{d}_n$ is associated with the tau-functions assoicated with unfrozen variables $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, P_0, Q_0, R_0, V_0, J_0)$ and have initial data given by a 12×12 identity matrix.

$$(\mathbf{d}_0\ \mathbf{d}_1\ \mathbf{d}_2\ \mathbf{d}_3\ \mathbf{d}_4\ \mathbf{d}_5\ \mathbf{d}_6\ \mathbf{p}_0\ \mathbf{q}_0\ \mathbf{r}_0\ \mathbf{v}_0\ \mathbf{j}_0) = -I \tag{5.45}$$

As usual, a direct substitution (5.44) into (5.40) yields the (max,+) relations for the d-vectors,

$$\begin{aligned} \mathbf{d}_{n+7} + \mathbf{d}_n &= \max(\mathbf{d}_{n+5} + \mathbf{d}_{n+2}, \mathbf{p}_n), \\ \mathbf{p}_{n+1} + \mathbf{p}_n &= \max(\mathbf{d}_{n+2} + \mathbf{d}_{n+3} + \mathbf{d}_{n+5} + \mathbf{d}_{n+6}, \mathbf{q}_n + \mathbf{d}_{n+1} + \mathbf{d}_{n+7}), \\ \mathbf{q}_{n+1} + \mathbf{q}_n &= \max(\mathbf{d}_{n+3} + \mathbf{d}_{n+4} + \mathbf{d}_{n+5} + \mathbf{d}_{n+6}, \mathbf{r}_n + \mathbf{p}_{n+1}), \\ \mathbf{r}_{n+1} + \mathbf{r}_n &= \max(\mathbf{d}_{n+4} + 2\mathbf{d}_{n+5} + \mathbf{d}_{n+6}, \mathbf{v}_n + \mathbf{j}_n + \mathbf{q}_{n+1}), \\ \mathbf{v}_{n+1} + \mathbf{j}_n &= \max(\mathbf{d}_{n+5} + \mathbf{d}_{n+6}, \mathbf{r}_{n+1}), \\ \mathbf{j}_{n+1} + \mathbf{v}_n &= \max(\mathbf{d}_{n+5} + \mathbf{d}_{n+6}, \mathbf{r}_{n+1}), \end{aligned} \tag{5.46}$$

Next we introduce quantities which is analogous to the tropical version of (5.39) as following,

$$\mathbf{X}_{1,n} = \mathbf{d}_n + \mathbf{d}_{n+6} - \mathbf{d}_{n+1} - \mathbf{d}_{n+5}, \quad \mathbf{X}_{2,n} = \mathbf{p}_n - \mathbf{d}_{n+2} - \mathbf{d}_{n+5},$$
$$\mathbf{X}_{3,n} = \mathbf{q}_n - \mathbf{d}_{n+3} - \mathbf{d}_{n+5}, \quad \mathbf{X}_{4,n} = \mathbf{r}_n - \mathbf{d}_{n+4} - \mathbf{d}_{n+5}, \tag{5.47}$$
$$\mathbf{X}_{5,n} = \mathbf{v}_n + \mathbf{s}_n - \mathbf{d}_{n+5}, \quad \mathbf{X}_{6,n} = \mathbf{j}_n - \mathbf{d}_{n+5} - \mathbf{s}_n,$$

along with quantities corresponding to symplectic coordinates y_j (5.38), shown below.

$$\mathbf{Y}_{1,n} = \mathbf{p}_n - \mathbf{d}_{n+2} - \mathbf{d}_{n+5}, \qquad \mathbf{Y}_{2,n} = \mathbf{d}_{n+1} + \mathbf{q}_n - \mathbf{d}_n - \mathbf{d}_{n+3} - \mathbf{d}_{n+6},$$
$$\mathbf{Y}_{3,n} = \mathbf{d}_{n+2} + \mathbf{r}_n - \mathbf{d}_{n+4} - \mathbf{p}_n, \qquad \mathbf{Y}_{4,n} = \mathbf{d}_{n+3} + \mathbf{v}_n + \mathbf{j}_n - \mathbf{d}_{n+5} - \mathbf{q}_n \tag{5.48}$$

Then we can see that $\mathbf{X}_{i,n}$ satisfies ultradiscretized expression of original type D_6 ((5.31) with all $b_i = 1$) from the (max,+) equations (5.46) as stated in below,

Lemma 5.4.1. *The structure of* $\mathbf{X}_{j,n}$ *in* (5.47) *satisfy the tropical analogue of deformed map* φ_{B_4} *which is given by the following system of (max,+) equations:*

$$\begin{aligned}
\mathbf{X}_{1,n+1} + \mathbf{X}_{1,n} &= [\mathbf{X}_{2,n}]_+, \\
\mathbf{X}_{2,n+1} + \mathbf{X}_{2,n} &= [\mathbf{X}_{1,n+1} + \mathbf{X}_{3,n}]_+, \\
\mathbf{X}_{3,n+1} + \mathbf{X}_{3,n} &= [\mathbf{X}_{2,n+1} + \mathbf{X}_{4,n}]_+, \\
\mathbf{X}_{4,n+1} + \mathbf{X}_{4,n} &= [\mathbf{X}_{5,n} + \mathbf{X}_{6,n} + \mathbf{X}_{3,n+1}]_+. \\
\mathbf{X}_{5,n+1} + \mathbf{X}_{5,n} &= [\mathbf{X}_{4,n+1}]_+. \\
\mathbf{X}_{6,n+1} + \mathbf{X}_{6,n} &= [\mathbf{X}_{4,n+1}]_+.
\end{aligned} \tag{5.49}$$

where $[a]_+ = \max(a, 0)$*. Given arbitrary initial values* $(\mathbf{X}_{1,0}, \mathbf{X}_{2,0}, \mathbf{X}_{3,0}, \mathbf{X}_{4,0}, \mathbf{X}_{5,0}, \mathbf{X}_{6,0})$*, the quantities* $\mathbf{X}_{j,n}$ *and* $\mathbf{Y}_{i,n}$ *are periodic with period 6 for* $1 \le i \le 4$ *and* $1 \le j \le 6$.

Proof. By similar argument in the proof of Lemma 3.5.1, the statement above holds. □

It important to remember from the previous sections that the periodicity of $\mathbf{X}_{i,n}$ (or $\mathbf{Y}_{j,n}$) for all i, is essential in finding the degree growth of d-vectors of tau functions. By using the periodicity, we obtain the following result.

Theorem 5.4.2. *Let* $\mathcal{T}$ *be linear operator which shifts* $n \to n+1$*. The d-vectors* $\mathbf{e}_n$*,*$\mathbf{d}_n$*,*$\mathbf{f}_n$ *and* $\mathbf{g}_n$*, which solve the system of equations* (5.20)*, satisfy the following* *linear difference equations*

$$\mathcal{L}\mathbf{r}_n = (\mathcal{T}^6 - 1)(\mathcal{T}^5 - 1)(\mathcal{T} - 1)\mathbf{r}_n = 0 \tag{5.50}$$

where $\mathcal{T}$ *is shift operator corresponding to* $n \to n+1$ *and* $\mathbf{r}_n = \mathbf{e}_n, \mathbf{d}_n, \mathbf{f}_n, \mathbf{g}_n$*. For the tau functions generated, the leading order of degree growth of their denominators is given by*

$$\begin{aligned} \mathbf{d}_n &= \frac{n^2}{60}\mathbf{a} + O(n), \quad \mathbf{v}_n = \frac{n^2}{60}\mathbf{a} + O(n), \quad \mathbf{j}_n = \frac{n^2}{60}\mathbf{a} + O(n) \\ \mathbf{p}_n &= \frac{n^2}{30}\mathbf{a} + O(n), \quad \mathbf{q}_n = \frac{n^2}{30}\mathbf{a} + O(n), \quad \mathbf{r}_n = \frac{n^2}{30}\mathbf{a} + O(n), \end{aligned} \tag{5.51}$$

where $\mathbf{a} = (1,1,1,1,1,1,1,2,2,2,1,1)^T$*.*

Proof. Let us consider the quantity $\mathbf{X}_{1,n}$. With the periodicity feature mentioned in Theorem 5.4.1, we find the expression for $\mathbf{d}_n$ vectors as follows.

$$\begin{aligned} &\mathbf{X}_{1,n} = (\mathcal{T}^6 - \mathcal{T}^5 - \mathcal{T} + 1)\mathbf{d}_n \\ &\implies (\mathcal{T}^6 - 1)\mathbf{X}_{1,n} = (\mathcal{T}^6 - 1)(\mathcal{T}^5 - 1)(\mathcal{T} - 1)\mathbf{d}_n = 0 \end{aligned} \tag{5.52}$$

By a similar argument above, one can show that the d-vectors $\mathbf{p}_n$, $\mathbf{q}_n$ and $\mathbf{r}_n$ satisfy the recurrence relation above. As for the rest of d-vectors $\mathbf{v}_n$ and $\mathbf{j}_n$, we first look at the last two (max,+) relations in (5.46). Subtracting these relations gives rise to expression

$$(\mathcal{T} - 1)\mathbf{v}_n = (\mathcal{T} - 1)\mathbf{j}_n. \tag{5.53}$$

Now applying periodicity in Lemma 5.4.1 to $\mathbf{Y}_{4,n}$, followed by using the recurrence of $\mathbf{d}_n$ in (5.52), we have

$$(\mathcal{T}^6 - 1)(\mathcal{T}^5 - 1)(\mathbf{v}_{n+1} + \mathbf{j}_{n+1} - \mathbf{v}_n - \mathbf{j}_n) = 0 \tag{5.54}$$

Then substituting the identity (5.53), we find that both $\mathbf{v}_n$ and $\mathbf{j}_n$ satisfy the recurrence in (5.52).

For the rest of proof (determining the coefficients of leading order terms of d-vectors), we can apply the similar arguments in the proof of Theorem 5.2.2 and attain the result. □

Since the degree growth of each variable is quadratic, associated algebraic entropy vanishes, which leads to the conjecture that the deformed type D_6 map is a Liouville integrable map.

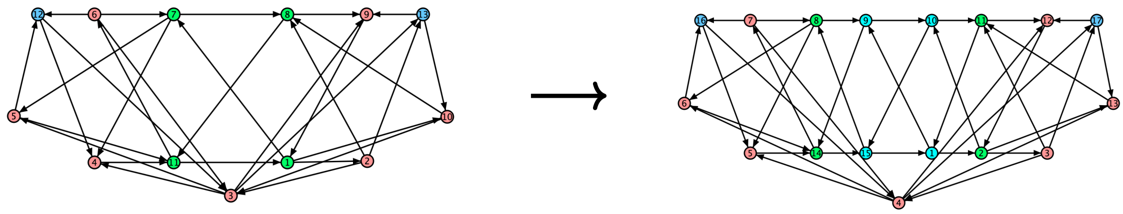
Chapter 6

Conclusion

In this thesis, we considered a special class of discrete dynamical systems which are described by periodic (Zamolodchikov periodicity) cluster maps associated with Dynkin type A, B and D. This is an extension from the work introduced by Hone and Kouloukas in [10].

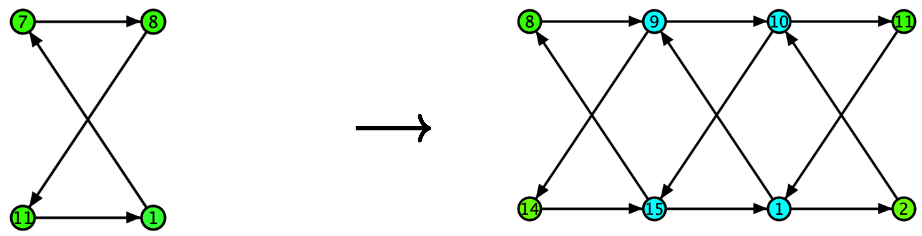
In chapter 2, we began with reviewing background of cluster algebra and discrete integrable system, which is necessary materials for the later chapters of the thesis. We were led to one of the main results produced by Fomin and Zelevensky in [19], which forms a strong link between cluster algebra of finite type and Zamolodchikov periodicity conjecture [4]. We then studied another notion of periodicity, that is, mutation periodicity in quiver. The composition of mutations, which fixes the quiver, can be regarded as a birational map between cluster seeds, which we call cluster map. This discussion led us to the theory of discrete integrable system in cluster algebra which is described by iteration of Liouville integrable map (introduced in [9, 7]). We introduced the singularity confinement test, algebraic entropy, and p-adic analysis, which are algebraic methods used in detecting the integrability of the system. In final part of this chapter, we gave a summary of the deformation of cluster mutations introduced by Hone and Kouloukas in [10].

In chapter 3, we considered the construction of a 2-parameter of deformations of type A_{2N} cluster maps for all $N \geq 2$. We began with considering the cluster algebra of type A_{2N} and proved that the specific cluster map is the Liouville integrable by using Bi Hamiltonian theorem [45] and the results in [7]. Then we showed that the periodic cluster map associated with type A_6 has a 2-parameter integrable deformation, which can be lifted to a cluster map that is identified by the particular quiver Q_{A_6} consisting 2 frozen nodes and 15 mutable nodes. In comparison with the quiver Q_{A_4} in the type A_4 case, we observed that inserting a particular quiver (shown in Figure 6.5) into the Q_{A_4} gives rise to the Q_{A_6}. The local expansion gave insight into the structure of the tau functions in the variable transformations, which led us to construction of class of quivers arose from the pull-back of symplectic form associated type A_{2N}, for $N \geq 2$. Followed by an inductive approach, we showed that the



(a) Q_{A_4} (b) Q_{A_6}

Figure 6.1: Extension from Q_{A_4} to Q_{A_6}



(a) Subquiver in Q_{A_4} (b) Subquiver in Q_{A_6}

Figure 6.2: Local expansion of the subquiver in Q_{A_4}

particular structure of this expansion indeed gives Laurentification of the deformed type A_{2N} cluster map. For the integrability of the map, we used most trusted integrability detector, algebraic entropy. To use this, we needed to calculate the degree growth of the Laurentified cluster map (obtained by Laurentification). Due to the Laurent property and total positivity of cluster algebras, we only needed to concern the total degree of monomial in each cluster variable for measure the growth.

To find exact measurements, we used the Zamolodchikov periodicity property of cluster algebra of finite type. This enabled us to show that the growth of the degree is quadratic. As a result, its algebraic entropy vanishes. This makes us confident that the periodic cluster map associated with type A_{2N} admits integrable deformation.

In the chapter 4, we extended the analysis of deformation to other root systems Dynkin type C_2, B_3 and D_4. Firstly, with the same procedure used in type A cases, we found an integrable deformed type C_2 map, which can be lifted to the 1-parameter family of cluster map on space of tau functions. In addition to the above, we showed that tau functions, generated by the iteration of the latter map, satisfy the Somos-5 recurrence relation. As a result of linking it to Somos-5, we were able to show that it is closely related to a specific QRT map [25], which provides a special family of discrete dynamical systems. It is important to note that the deformation of type C_2 cluster map gives the same result as the case of type C_2 since their Dynkin diagrams are isomorphic to each other. Next, we studied the case of type B_3 and D_4. Compared with the results in type A_2, A_4, A_6, C_2, each cluster map associated with type B_3 and D_4 can be deformed to an integrable map in more than one distinct way. Nevertheless, there seem to be very close connections between the two cases obtained for B_3; the close connections between the case (1) and case (2) deformations for D_4 are even more apparent, given that the underlying coefficient-free cluster algebra is the same for both.

In chapter 5, we investigated the next higher rank of type B and D. In contrast to the cases in the previous chapter, the number of required first integrals (invariant function) rises as we move onto the higher rank. Unlike the case of type A_{2N}, we have not yet found the appropriate first integrals that commute with respect to the associated log canonical Poisson bracket. From the cases we considered so far, we observed that the complexity of rational sequences (the size of digits in numerator and denominator of rationals) of corresponding deformed maps, reduces considerably when we set the particular conditions for the corresponding deformed map to be integrable. We were able to see the same phenomenon when we fixed the parameters analogous to the integrability conditions that appeared in type B_3 and D_4, respectively. Following from this, we found the several distinct singularity patterns which allowed us to perform Laurentification on the type B_4 and D_6. This resulted in finding the new cluster map in enlarged space for each case. By carrying out the same method, used in section 3.5, we showed that the degree growth of each latter map is quadratic and thus algebraic entropy vanishes. This provides strong evidence that the deformed maps for type B_4 and D_6 possess integrability.

In recent studies, we successfully performed Laurentification in the case of type B_6 and D_8 , subject to the parameter conditions analogous to the ones in type B_4 and D_6 respectively. As a result of the procedure, we built an exchange matrix corresponding to type B_6

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and the matrix corresponding to type D_8, which can be depicted by the quiver in Figure 6.3. In comparison with the quiver in Figure 5.1, we can see that the quiver in Figure 6.3 can be constructed by the local expansion procedure (introduced in section 3.3) as shown in fig 6.4. Then recursive local expansion gave rise to a family of quivers associated with type D_{2N} represented by the exchange matrix in the figure 6.6. Similarly, for type B case, the expansion, which transforms the exchange matrix $\tilde{B}_{\mathrm{B}_6}$ to $4N+2 \times 4N+1$ skew-symmetrizable matrix, was achieved via inserting the rows and columns, taking the form of the matrix in Figure 6.7. The future work will be aiming at the complete description of integrable deformation corresponding to type A, B and D, including odd rank case A_{2N+1} B_{2N+1} and D_{2N+1}. We hope this investigation finds a systematic procedure that enables us to find Poisson commuting first integrals which admit deformation.

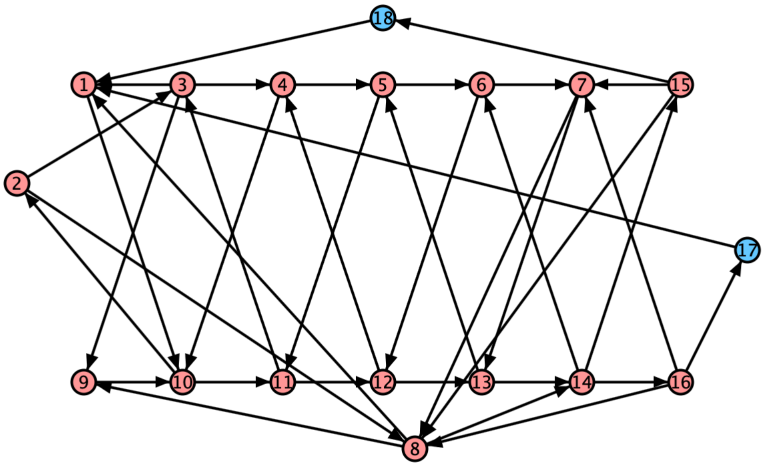
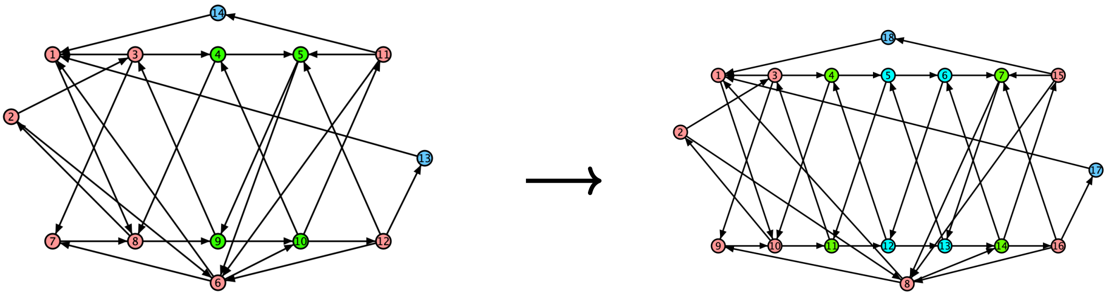
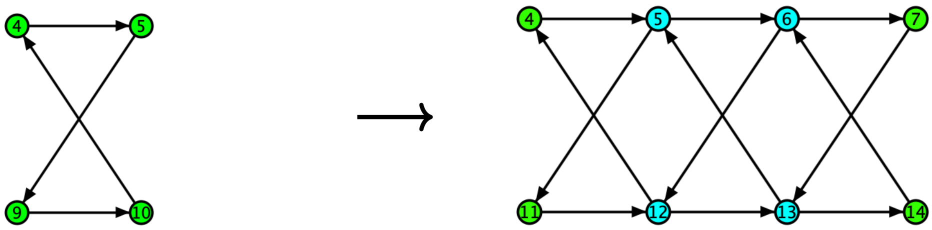


Figure 6.3: Extended quiver associated with the deformed D_8 cluster map ψ_2



(a) Q_{D_6} (b) Q_{D_8}

Figure 6.4: Extension from Q_{D_6} to Q_{D_8}



(a) Subquiver in Q_{D_6} (b) Subquiver in Q_{D_8}

Figure 6.5: Local expansion of the subquiver in Q_{D_8}

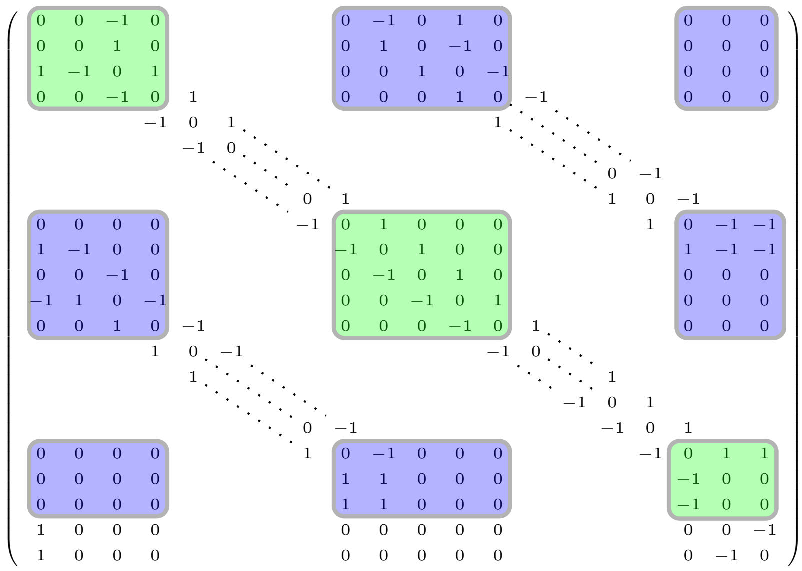


Figure 6.6: Exchange matrix associated with type D_{2N} for $N \geq 3$, built from recursive local expansion. The coloured regions in the matrix are submatrices of the matrix (5.43)

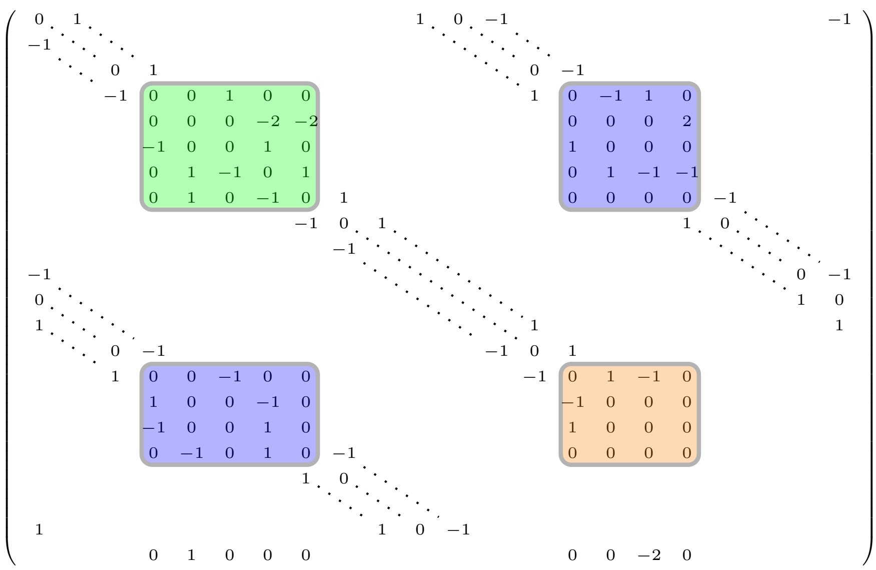


Figure 6.7: Exchange matrix associated with type B_{2N} for $N \geq 2$, built from recursive local expansion. The coloured regions in the matrix are submatrices of the matrix (5.13)

Bibliography

[1] Sergey Fomin and Andrei Zelevinsky. “Cluster algebras I: Foundations”. In: *Journal of the American Mathematical Society* 15 (2001).

[2] Sergey Fomin and Andrei Zelevinsky. “Y-Systems and Generalized Associahedra”. In: *Annals of Mathematics* 158.3 (2003), pp. 977–1018.

[3] Lauren Williams. “Cluster algebras: An introduction”. In: *Bulletin of the American Mathematical Society* 51 (2012).

[4] Al.B. Zamolodchikov. “On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories”. In: *Physics Letters B* 253.3 (1991), pp. 391–394.

[5] Sergey Fomin and Andrei Zelevinsky. “Cluster algebras II: Finite type classification”. In: *Inventiones mathematicae* 154.1 (2003), pp. 63–121.

[6] Allan P. Fordy and Bethany R. Marsh. “Cluster mutation-periodic quivers and associated Laurent sequences”. In: *Journal of Algebraic Combinatorics* 34.1 (2010), pp. 19–66.

[7] Allan P. Fordy and Andrew Hone. “Discrete Integrable Systems and Poisson Algebras From Cluster Maps”. In: *Communications in Mathematical Physics* 325.2 (2013), pp. 527–584.

[8] Tomoki Nakanishi. “Periodicities in cluster algebras and dilogarithm identities”. In: *Representations of algebras and related topics* 5 (2011), p. 407.

[9] Shigeru Maeda. “Completely integrable symplectic mapping”. In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 63.6 (1987), pp. 198–200.

[10] Andrew N. W. Hone and Theodoros E. Kouloukas. "Deformations of cluster mutations and invariant presymplectic forms". In: *Journal of Algebraic Combinatorics* 57.3 (2023), pp. 763–791.

[11] Khaled Hamad et al. "QRT maps and related Laurent systems". In: *Advances in Applied Mathematics* 96 (2017).

[12] M Kanki et al. "Discrete Painlevé II equation over finite fields". In: *Journal of Physics A: Mathematical and Theoretical* 45.34 (2012), p. 342001.

[13] Basile Grammaticos, Alfred Ramani, and Vassilios G. Papageorgiou. "Do integrable mappings have the Painlevé property?" In: *Physical review letters* 67 14 (1991), pp. 1825–1828.

[14] Jan E. Grabowski, Andrew N. W. Hone, and Wookyung Kim. *Deformed cluster maps of type A_{2N}*. 2024. arXiv: `2402.18310 [nlin.SI]`.

[15] Andrew N. W. Hone, Wookyung Kim, and Takafumi Mase. *New cluster algebras from old: integrability beyond Zamolodchikov periodicity.* 2024. arXiv: `2403.00721 [nlin.SI]`.

[16] Max Glick and Dylan Rupel. "Introduction to Cluster Algebras". In: *Symmetries and Integrability of Difference Equations: Lecture Notes of the Abecederian School of SIDE 12, Montreal 2016*. Springer International Publishing, 2017, pp. 325–357.

[17] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras. Chapters 1-3*. 2021. arXiv: `1608.05735 [math.CO]`.

[18] B.R Marsh. *Lecture Notes on Cluster Algebras.* Zurich lectures in advanced mathematics. European Mathematical Society, 2013.

[19] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras IV: Coefficients". In: *Compositio Mathematica* 143.1 (2007), pp. 112–164.

[20] Andrew N. W. Hone, Philipp Lampe, and Theodoros E Kouloukas. "Cluster algebras and discrete integrability". In: *Nonlinear Systems and Their Remarkable Mathematical Structures* (2019).

[21] Allan P Fordy. "Periodic cluster mutations and related integrable maps". In: *Journal of Physics A: Mathematical and Theoretical* 47.47 (2014), p. 474003.

[22] Alan Weinstein. "The local structure of Poisson manifolds". In: *Journal of differential geometry* 18.3 (1983), pp. 523–557.

[23] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. *Cluster algebras and Poisson geometry*. 167. American Mathematical Soc., 2010.

[24] Rei Inoue and Tomoki Nakanishi. "Difference equations and cluster algebras I: Poisson bracket for integrable difference equations". In: *RIMS Kokyuroku Bessatsu B* 28 (2011), pp. 63–88.

[25] G. Quispel, J.A.G. Roberts, and C.J. Thompson. "Integrable mappings and soliton equations". In: *Physics Letters A* 126 (Jan. 2007), pp. 419–421.

[26] J. Hietarinta, N. Joshi, and F. W. Nijhoff. *Discrete Systems and Integrability*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2016.

[27] M. J. Ablowitz, A. Ramani, and H. Segur. "Nonlinear evolution equations and ordinary differential equations of painlevè type". In: *Lettere al Nuovo Cimento (1971-1985)* 23.9 (1978), pp. 333–338.

[28] S. Lafortune and A. Goriely. "Singularity confinement and algebraic integrability". In: *Journal of Mathematical Physics* 45.3 (2004), pp. 1191–1208.

[29] Jarmo Hietarinta and Claude Viallet. "Singularity Confinement and Chaos in Discrete Systems". In: *Physical Review Letters* 81.2 (1998), pp. 325–328.

[30] Masataka Kanki. *Studies on the discrete integrable equations over finite fields*. 2014.

[31] Alexa Pomerantz. "AN INTRODUCTION TO THE p-ADIC NUMBERS". In: 2020. URL: https://api.semanticscholar.org/CorpusID:225101559.

[32] N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. Graduate Texts in Mathematics. Springer New York, 2012.

[33] M. P. Bellon and C.-M. Viallet. "Algebraic Entropy". In: *Communications in Mathematical Physics* 204.2 (July 1999), pp. 425–437.

[34] Kyungyong Lee and Ralf Schiffler. "Positivity for cluster algebras". In: *Annals of Mathematics* 182 (2015).

[35] David Speyer and Bernd Sturmfels. "Tropical Mathematics". In: *Mathematics Magazine* 82 (June 2009), pp. 163–173.

[36] Bart De Schutter and Ton van den Boom. "Max-plus algebra and max-plus linear discrete event systems: An introduction". In: June 2008, pp. 36–42.

[37] Daisuke Takahashi and Junkichi Satsuma. "A Soliton Cellular Automaton". In: *Journal of the Physical Society of Japan* 59.10 (1990), pp. 3514–3519.

[38] Asma Al-Ghassani. "A note on Ultra-discrete equations". In: *Journal of Interpolation and Approximation in Scientific Computing* 2016 (Feb. 2016), pp. 14–24.

[39] Allan Fordy and Andrew Hone. "Symplectic Maps from Cluster Algebras". In: *Symmetry Integrability and Geometry Methods and Applications* 7 (Sept. 2011), p. 091.

[40] R G Halburd. "Diophantine integrability". In: *Journal of Physics A: Mathematical and General* 38.16 (Apr. 2005), p. L263.

[41] Khaled Hamad and Peter H. van der Kamp. "From discrete integrable equations to Laurent recurrences". In: *Journal of Difference Equations and Applications* 22.6 (2016), pp. 789–816.

[42] Andrew N.W. Hone. "Laurent Polynomials and Superintegrable Maps". In: *Symmetry, Integrability and Geometry: Methods and Applications* (2007).

[43] Andrew N.W. Hone, Theodoros E. Kouloukas, and G.R.W. Quispel. "Some integrable maps and their Hirota bilinear forms". In: *Journal of Physics A: Mathematical and Theoretical* 51.4 (2017).

[44] Allan P. Fordy. "Mutation-periodic quivers, integrable maps and associated Poisson algebras". In: *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 369.1939 (2011), pp. 1264–1279.

[45] Franco Magri. "A Simple model of the integrable Hamiltonian equation". In: *Journal of Mathematical Physics* 19 (1978), pp. 1156–1162.

[46] Peter J. Olver. "Canonical Forms for Bihamiltonian Systems". In: Boston, MA: Birkhäuser Boston, 1993, pp. 239–249.

[47] J.J. Duistermaat. *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*. Springer Monographs in Mathematics. Springer New York, 2010.

[48] Mark Gross et al. “Canonical bases for cluster algebras”. In: *J. Am. Math. Soc.* 31.2 (2018), pp. 497–608.

[49] Andrew Hone. “Sigma function solution of the initial value problem for Somos 5 sequences”. In: *Transactions of the American Mathematical Society* 359 (Mar. 2005).

[50] Andrew N. W. Hone. “ECM Factorization with QRT Maps”. In: *Advances in Software Engineering, Education, and e-Learning*. Cham: Springer International Publishing, 2021, pp. 395–409.

[51] Andrew N. W. Hone and Christine Swart. “Integrality and the Laurent phenomenon for Somos 4 and Somos 5 sequences”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 145 (2008), pp. 65–85.

[52] Takafumi Mase. “Investigation into the role of the Laurent property in integrability”. In: *Journal of Mathematical Physics* 57.2 (2016).

[53] Andrew N W Hone and Rei Inoue. “Discrete Painlevé equations from Y-systems”. In: *Journal of Physics A: Mathematical and Theoretical* 47.47 (2014), p. 474007.

[54] Naoto Okubo. “Bilinear equations and q-discrete Painlevé equations satisfied by variables and coefficients in cluster algebras”. In: *Journal of Physics A: Mathematical and Theoretical* 48.35 (2015), p. 355201.