MAXIMA OVER RANDOM TIME INTERVALS FOR HEAVY-TAILED COMPOUND RENEWAL AND LÉVY PROCESSES

SERGEY FOSS, DMITRY KORSHUNOV, AND ZBIGNIEW PALMOWSKI

ABSTRACT. We derive subexponential tail asymptotics for the distribution of the maximum of a compound renewal process with linear component and of a Lévy process, both with negative drift, over *random* time horizon τ that does not depend on the future increments of the process. Our asymptotic results are *uniform* over the whole class of such random times. Particular examples are given by stopping times and by τ independent of the processes. We link our results with random walk theory.
KEYWORDS. uniform asymptotics $\star$ stopping time $\star$ renewal process $\star$ subexponential distribution $\star$ Lévy process $\star$ random walk.

1. INTRODUCTION AND MAIN RESULTS

In this paper we derive subexponential tail asymptotics for the distribution of the maximum of compound renewal processes with linear component and for Lévy processes, both with negative drift, over random time horizon τ (which may be infinite with positive probability) that does not depend on the future increments of X. We focus on obtaining results that are *uniform in a broad class of random times* that is introduced below.

We believe that this is the first work in the context of the subexponential asymptotics in the continuous-time setting where the concepts of random times that do not depend on the future increments of the process are introduced and systematically studied; see Definitions 1 and 2 and Examples 3 and 4 below. To handle this, we propose a new approach based on the creation of some special i.i.d. cycles. This is a crucial step that allows us to obtain the derived results uniformly over the aforementioned random times. We like to underline as well that our approach produces the respective asymptotics for random walks and for general Lévy processes. As such, it seems to be new for the latter class of processes, where the Wiener-Hopf factorisation and ladder process arguments have been used before.

Let us firstly recall some basic definitions related to subexponential distributions. A probability distribution F on the real line has a *heavy (right) tail* if

$$\int_{-\infty}^{\infty} e^{\varepsilon x} F(dx) = \infty \quad \text{for all } \varepsilon > 0,$$

and a *light tail*, otherwise. In this paper we focus on a very important sub-class of heavy-tailed distributions, namely on the class of *strong subexponential* distributions $\mathscr{S}^*$. A distribution F on $\mathbb{R}$ with a finite mean and unbounded support on the right belongs to the class $\mathscr{S}^*$ if

$$\int_0^x \overline{F}(x-y)\overline{F}(y)dy \sim 2a^+\overline{F}(x) \qquad \text{as } x \to \infty,$$

where $\overline{F}(x) = F(x,\infty)$ and $a^+ = \int_0^\infty \overline{F}(y)dy$. By [8, Theorem 3.27], any $F \in \mathscr{S}^*$ is *subexponential*, i.e. satisfies the following two properties: $\overline{F*F}(x) \sim 2\overline{F}(x)$ and F is *long-tailed*, i.e. $\overline{F}(x+1) \sim \overline{F}(x)$. Here '$*$' denotes the convolution operator and, for any two eventually positive functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \to \infty$. We refer to [8] for an overview of the properties of strong subexponential and related distributions.

To define properly the family of (possibly improper) random times of interest, we use the notation $\sigma(\Xi)$ for the sigma-algebra generated by a family of random variables Ξ.

Definition 1. We say that a random time $\tau \in [0,\infty]$ *does not depend on the future increments of the process* $X = \{X_t, t \geq 0\}$ if the sigma-algebras $\sigma\{X_s,\ s \leq t,\ \mathbb{I}\{\tau \leq t\}\}$ and $\sigma\{X_{t+v} - X_t,\ v > 0\}$ are independent, for all $t > 0$.

Equivalently, the independence of the future property may be defined as independence of the sigma-algebras $\sigma\{X_s,\ s < t,\ \mathbb{I}\{\tau < t\}\}$ and $\sigma\{X_{t+v} - X_{t-0},\ v \geq 0\}$, for all $t > 0$. A similar term appeared in [13] and [16] while proving generalizations of Wald's identity and in [4, Section 7] where a random walk was considered.

For a random process X *with independent increments*, there are two important examples of random times that are independent of the future increments of X:

(i) any hitting time τ, and, more generally, any stopping time τ, and

(ii) any random time τ that does not depend on the process X.

Notice that, for a non-deterministic process X, a stopping time τ does not depend on X if and only if τ is a constant time. More examples and comments related to Definition 1 are given in Section 2.

While Definition 1 works well for Lévy processes, it is not so for compound renewal processes, in general. For example, if we follow Definition 1, then a constant $\tau = c$, while being independent of any process, appears to depend on the future increments of a compound renewal process if the process in not compound Poisson. Indeed, the time after c to the next jump (i.e. the overshot of the underlying renewal process at time c) does depend on $\sigma\{X_s,\ s \leq c,\ \mathbb{I}\{\tau \leq c\}\} = \sigma\{X_s,\ s \leq c\}$. Having this observation in mind, we introduce another notion of 'independence of the future', that is with respect to a sequence of (random) embedded epochs.

Definition 2. Given an increasing sequence of random times T_n, we say that $\tau \in [0,\infty]$ is *independent of the future increments of* X *with respect to* $\{T_n\}$ if the following condition holds: for all $n \geq 1$, the two σ-algebras

(1) $\sigma\{X_s,\ s < T_n,\ \mathbb{I}\{\tau < T_n\}\}$ and $\sigma\{X_{T_n+v} - X_{T_n-0},\ v \geq 0\}$ are independent.

Below are two important examples of stochastic processes we are mainly interested in.

Example 3. Let X be a *compound renewal process with linear component*,

$$X_t = \sum_{i=1}^{N_t} Y_i + ct, \quad t \geq 0, \tag{2}$$

where c is a real constant, $\{Y_n\}_{n\geq 1}$ i.i.d. jump sizes with distribution F and finite mean b, and $N = \{N_t, t \geq 0\}$ a renewal process independent of the jump sizes, with jump epochs $0 = T_0 < T_1 < T_2 < \ldots$, where $T_n - T_{n-1} > 0$ are i.i.d. positive random variables with finite mean $1/\lambda$.

If a random time τ does not depend on the future increments of X, then τ satisfies the condition (1) with T_n the jump epochs. Indeed, for any events $A \in \sigma\{X_s,\ s < T_n,\ \tau < T_n\}$ and $B \in \sigma\{X_s - X_{T_n-0},\ s \geq T_n\}$ conditioning on T_n implies a.s. equality

$$\begin{aligned}\mathbb{P}\{AB \mid T_n\} &= \mathbb{P}\{B \mid A, T_n\}\mathbb{P}\{A \mid T_n\} \\ &= \mathbb{P}\{B\}\mathbb{P}\{A \mid T_n\},\end{aligned}$$

where $\mathbb{P}\{B \mid T_n\}$ does not depend on the value of T_n and A due to the renewal structure of X_t and hence

$$\begin{aligned}\mathbb{P}\{AB\} &= \mathbb{E}\mathbb{P}\{AB \mid T_n\}\\ &= \mathbb{P}\{B\}\mathbb{E}\mathbb{P}\{A \mid T_n\}\\ &= \mathbb{P}\{B\}\mathbb{P}\{A\},\end{aligned}$$

so the events A and B are indeed independent.

Example 4. Let X be a Lévy process starting at the origin and let τ not depend on the future increments of X. Fix an $\varepsilon > 0$ and consider the sequence of all jump epochs T_n with jump sizes $|X_{T_n} - X_{T_n-0}| \ge \varepsilon$, so $T_{n+1} - T_n$, $n \ge 0$, are i.i.d. exponentially distributed random variables, hereinafter $T_0 = 0$. Then τ satisfies the condition (1). Indeed, since τ does not depend on the future increments of X, it does not depend on the future jumps of X of size at least ε, which in turn yields (1) by arguments similar to those used above.

In this paper we firstly focus on the compound renewal process X with linear component introduced in (2). We assume throughout that the drift of the process is negative, that is,

$$\mathbb{E}(cT_1 + Y_1) =: -a \quad < \quad 0 \tag{3}$$

(equivalently, $c + b\lambda = -a\lambda < 0$), which implies that the family of distributions of the partial maxima

$$M_t := \max_{u\in[0,t]} X_u \tag{4}$$

is tight and

$$\sup_{t>0} \mathbb{P}\{M_t > x\} \le \mathbb{P}\{M_\infty > x\} \to 0 \quad \text{as } x \to \infty.$$

We are interested in the tail behaviour of M_τ for random times τ. Our first main result here is as follows.

Theorem 5. *Let the drift condition (3) hold. Let either $c \le 0$ or the following condition hold:*

$$c > 0 \quad \text{and} \quad \mathbb{P}\{cT_1 > x\} = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{5}$$

If the distribution F of Y_1 belongs to $\mathscr{S}^*$, then

$$\mathbb{P}\{M_\tau > x\} \quad = \quad \frac{1 + o(1)}{a}\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(y)dy + o(\overline{F}(x)) \quad \text{as } x \to \infty,$$

uniformly for all random times $\tau \in [0, \infty]$ that satisfy (1) in Definition 2.

Hereinafter, we write $f(x,\tau) = o(g(x,\tau))$ as $x \to \infty$ uniformly for all τ if

$$\sup_\tau \left|\frac{f(x,\tau)}{g(x,\tau)}\right| \to 0 \quad \text{as } x \to \infty.$$

Note that a result similar to Theorem 5 has been obtained recently in the case of non-random times.

Theorem 6 ([15, Theorem 1])**.** *Under the conditions of Theorem 5,*

$$\mathbb{P}\{M_t > x\} \quad \sim \quad \frac{1}{a}\int_0^{a\mathbb{E}N_t} \overline{F}(x+y)dy \quad \text{as } x \to \infty \text{ uniformly for all } t \in (0, \infty].$$

One can see the difference in results of Theorems 5 and 6: in Theorem 5, the expectation of the integral is taken while in Theorem 6, for constant τ, it is simplified to an integral with constant upper limit. We have to comment that, despite the fact that the results look similar, the extension to random times requires a lot of effort in the case where the random time τ depends on the process – say like stopping times. Our approach is based on the introduction of special i.i.d. cycles that allows us to handle random times that do not depend on the future increments. We like to underline as well that our approach produces the respective asymptotics for random walks and for general Lévy processes. Note also that such an extension is really needed in many applications, say in risk or in queueing theory where we often need to know how likely it is to exceed a high level within some time cycle or similar random time interval.

A more general than renewal setting was studied by Tang [19] where the author obtained uniform asymptotics for the supremum over time intervals $[0, t]$ for a counting process that satisfies a law of large numbers.

Let us consider a *random walk* $S_0 = 0$, $S_n := Y_1 + \ldots + Y_n$, $n \ge 1$ and its partial maxima $M_n := \max(S_0, \ldots, S_n)$. Since a random walk may be considered as a particular case of a compound renewal process with constant jump epochs $T_n = n$, we derive from Theorem 5 the following result which replicates and corrects several results from [1, 6, 7, 9, 10, 11, 12]; see also [8] for further references.

Theorem 7. *Let $\mathbb{E}Y_1 =: -a < 0$. If the distribution F of Y_1 belongs to $\mathscr{S}^*$, then*

$$\mathbb{P}\{M_\tau > x\} \quad = \quad \frac{1+o(1)}{a}\mathbb{E}\int_0^{a\tau} \overline{F}(x+y)dy \quad \text{as } x \to \infty,$$

uniformly for all counting random variables $\tau \in [0, \infty]$ that do not depend on the future jumps of S_n.

First results on the asymptotics of randomly stopped sequences with independent increments are due to Greenwood [10] and Greenwood and Monroe [11] where a case of a bounded or regularly varying at infinity stopping time τ and regularly varying at infinity F is considered.

Asmussen [1] obtained this result for the proper hitting time $\tau = \min\{n \ge 1 : S_n \le 0\}$ with $\mathbb{E}\tau = 1/\mathbb{P}\{S_\infty \le 0\}$, where the right hand side is asymptotically equal to $\mathbb{E}\tau\overline{F}(x)$.

In [6], a more general statement than Theorem 7 is presented, it concerns up-crossing of a high level non-linear boundary. However, its proof is based on a key Lemma 1 whose proof contains a gap. Namely, Lemma 1 states some asymptotic result which holds uniformly for all stopping times from a family $\mathscr{T}_\varphi := \{\sigma : 0 \le \sigma \le \varphi\}$ where $\varphi \ge 0$ is a stopping time such that $\mathbb{E}\varphi < \infty$, i.e. the family of stopping times possesses an integrables majorant. Then its proof starts with saying that, without loss of generality, it suffices to assume that $\sigma \ge 1$, and then this assumption is essentially used, together with the existence of an integrable majorant. However, the latter reduction is not supported by any argument. One can see that, in general, the family of σ's conditioned on being greater than 1, may not possess a common integrable majorant if the family of probabilities $\mathbb{P}\{\sigma > 0\}$ is not bounded away from zero. This is exactly the case, in particular, for the family of stopping times $\{\tau' = (\tau - m)^+, m \in \mathbb{N}\}$ that are considered in the proof of Corollary 1 in [6] (denoted σ' there). This questions the proof of Theorem 7 in [6], and it is not clear so far if this result holds true or not in the generality stated there.

Notice that Theorem 7 generalises the following result for partial maxima of a random walk over non-random time intervals.

Theorem 8 ([8, Theorem 5.3])**.** *Under the conditions of Theorem 7,*

$$\mathbb{P}\{M_n > x\} \quad \sim \quad \frac{1}{a}\int_0^{an} \overline{F}(x+y)dy \quad \text{as } x \to \infty \text{ uniformly for all } n \geq 1.$$

Now let X be a *Lévy process* starting at the origin, that is, a càdlàg stochastic process (i.e. almost all its paths are right continuous with existing left limits everywhere) with stationary independent increments, where stationarity means that, for $s < t$, the probability distribution of $X_t - X_s$ depends only on $t - s$ and where the independence of increments means that, for all $k \geq 2$ and for all $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k$, the differences $X_{t_i} - X_{s_i}$, $1 \leq i \leq k$ are mutually independent random variables.

Stemming from Theorem 5 above, by considering the compound Poisson component of X, one can find the correct asymptotic lower bound for the tail of M_τ. However this approach does not work so well for the matching upper bound, because the suprema of the Gaussian component and the compound Poisson component are dependent via τ. For that reason we need to follow an alternative approach to derive the matching upper tail bound. Our main result for a Lévy process is the following theorem.

Theorem 9. *Let $\mathbb{E}X_1 =: -m < 0$. If the distribution of X_1 belongs to $\mathscr{S}^*$, then*

$$\mathbb{P}\{M_\tau > x\} \quad = \quad \frac{1+o(1)}{m}\mathbb{E}\int_x^{x+m\tau} \mathbb{P}\{X_1 > y\}dy + o(\mathbb{P}\{X_1 > x\}) \quad \text{as } x \to \infty,$$

uniformly for all random times $\tau \in [0, \infty]$ that do not depend on the future increments of X.

First results on the asymptotics of randomly stopped Lévy processes go back to Greenwood and Monroe [11] who considered a particular case of regularly varying at infinity distributions F and stopping times τ. The case of independent sampling was considered in Korshunov [15, Theorem 10].

It has been suggested by Asmussen and Klüppelberg [2, Sect. 1.1] and by Asmussen [1, Sect. 2.4] to follow a discrete skeleton argument in order to prove these asymptotics for $\tau = \infty$ when the integrated tail of the Lévy measure is subexponential; notice that this approach requires additional considerations which take into account fluctuations of Lévy processes within time slots.

In Doney et al. [5] the passage time problem has been considered for Lévy processes, emphasising heavy tailed cases; local and functional versions of limit distributions are derived for the passage time itself, as well as for the position of the process just prior to passage, and the overshoot of a high level which is an extension for Lévy processes of corresponding results for random walks, see e.g. Foss et al. [8, Theorem 5.24].

Notice that the conclusions of both Theorems 5 and 9 are not so trivial. For example, one could doubt if they really hold true uniformly for all τ. At first glance, the uniformity may fail for the family of hitting times $\tau(x) = \inf\{t \geq 0 : X_t > x\}$, $x > 0$. However, for such $\tau(x)$, on the one hand

$$\begin{aligned} \mathbb{P}\{M_{\tau(x)} > x\} &= \mathbb{P}\{\tau(x) < \infty\} \\ &= \mathbb{P}\{M_\infty > x\} \sim \frac{1}{m}\int_x^\infty \overline{F}(y)dy \quad \text{as } x \to \infty, \end{aligned}$$

while on the other hand,

$$\mathbb{E}\int_x^{x+m\tau(x)} \overline{F}(y)dy \quad \leq \quad \int_x^\infty \overline{F}(y)dy,$$

and

$$\mathbb{E}\int_x^{x+m\tau(x)} \overline{F}(y)dy \quad\geq\quad \mathbb{P}\{\tau(x)=\infty\}\int_x^\infty \overline{F}(y)dy \sim \int_x^\infty \overline{F}(y)dy,$$

due to $\mathbb{P}\{\tau(x)=\infty\}\to 1$ as $x\to\infty$, so

$$\mathbb{E}\int_x^{x+m\tau(x)} \overline{F}(y)dy \quad\sim\quad \int_x^\infty \overline{F}(y)dy \quad \text{as } x\to\infty.$$

The paper is organised as follows. In Section 2 the property of the independence of the future increments given in Definition 1 is discussed in detail. In Section 3 we prove Theorem 5 on the compound renewal process. For a compound Poisson process, the tail asymptotics may be significantly improved in what concerns the upper limit of the integral, it is done in Section 4. Next, in Section 5 we prove Theorem 9 on Lévy processes. Finally, Appendix includes the proofs of three auxiliary results.

2. INDEPENDENCE OF THE FUTURE INCREMENTS – DISCUSSION AND FURTHER EXAMPLES

We discuss now Definitions 1 and 2. Firstly observe the following properties of independent of the future increments random times:

- while the minimum of two stopping times is a stopping time, the same property for independent of the future increments random times fails, in general;
- the minimum of an independent of the future increments random time and of a stopping time does not depend on the future increments, too;
- for a general compound renewal processes (with a possible linear drift), an independent time (hence e.g. a constant time) depends on the future increments, in the sense of (more-or-less standard) Definition 1. We have eliminated this conceptual problem by introducing a novel Definition 2, that involves embedded time instants. Note that, in particular cases, one can deal with constant times using another approach based on lower and upper bounds, see e.g. Example 6 in [6].

We give a number of examples that clarify our results and the two definitions of the independence of the future. We focus first on the case when X is a compound renewal process defined in (2) and when all the assumptions of Theorem 5 hold true.

Example 10. Let $\{\tau_n\}_{n\geq 1}$ be a family of stopping times independent of X with $p_n := \mathbb{P}\{\tau_n=\infty\}$ and $E_n := \mathbb{E}\{\tau_n \mid \tau_n<\infty\}$. Then,

$$\begin{aligned}\mathbb{P}\{M_{\tau_n}>x\} \quad&=\quad \frac{p_n+o(1)}{a}\int_x^\infty \overline{F}(y)dy \\ &\quad +\frac{1-p_n+o(1)}{a}\int_x^{x+E_n/\lambda} \overline{F}(y)dy + o(\overline{F}(x)) \quad \text{as } x\to\infty,\end{aligned}$$

uniformly for all $n\geq 1$. If additionally $\sup_n E_n<\infty$ and $\inf_n p_n>0$, then, since $\overline{F}(x)=o(\int_x^\infty \overline{F}(y)dy)$, we have

$$\mathbb{P}\{M_{\tau_n}>x\} \quad=\quad \frac{p_n+o(1)}{a}\int_x^\infty \overline{F}(y)dy + o(\overline{F}(x)) \quad \text{as } x\to\infty,$$

uniformly for all $n\geq 1$.

Example 11. In this example we consider a family of stopping times $\{\tau_n\}_{n\in\geq 1}$ with

$$q_n := \mathbb{P}\{\tau_n=0\}\to 1 \quad \text{as } n\to\infty,$$

but the mean value of the conditional distributions given $\tau_n > 0$ tends to infinity, that is, $D_n := \mathbb{E}\{\tau_n \mid \tau_n > 0\} \to \infty$ as $n \to \infty$. Then

$$\mathbb{P}\{M_{\tau_n} > x\} = \frac{1 - q_n + o(1)}{a} \int_x^{x+D_n/\lambda} \overline{F}(y)dy + o(\overline{F}(x)) \quad \text{as } x \to \infty,$$

uniformly for all $n \geq 1$. If additionally we assume that τ_n are bounded by an integrable random variable K, then

$$\mathbb{P}\{M_{\tau_n} > x\} = \frac{\mathbb{E}\tau_n + o(1)}{\lambda a} \overline{F}(x) \quad \text{as } x \to \infty \text{ uniformly for all } n \geq 1.$$

Example 12. As we already mentioned, the constant τ satisfies Definition 1 but not Definition 2. One can consider other examples as well. For example, $\tau = T_k$ for the kth renewal epoch T_k of N is of this type. In this case

$$\mathbb{P}\{M_{T_k} > x\} = \frac{k + o(1)}{\lambda a} \overline{F}(x) \quad \text{as } x \to \infty \text{ uniformly for all } k \geq 1.$$

Example 13. *Assume that $\{T_n - T_{n-1}\}$ are non-degenerate; let $a > 0$ be such that $\mathbb{P}\{T_n - T_{n-1} > a\} \in (0, 1)$. Let $\nu_1 < \nu_2 < \ldots$ be the consecutive times when $T_{\nu_k} - T_{\nu_k - 1} > a$. Let η be an independent non-degenerate counting random variable. Then $\tau = T_{\nu_\eta}$ satisfies Definition 2.*

Now we present a number of examples of independent of the future random times that are not stopping times. We start with examples for random walks. Let $S_n = \sum_{i=1}^n \xi_i$, $n \geq 0$ be a random walk with i.i.d. increments.

Example 14. *Let $\mathbb{P}\{\xi_1 > 0\} > 0$ and $\mathbb{P}\{\xi_1 < 0\} > 0$. Let η be an independent non-degenerate counting random variable and*

$$\tau_1 = \min\{n > 0 : \xi_n < 0\} \;\; \textit{and} \;\; \tau_k = \min\{n > \tau_{k-1} : \xi_n < 0\} \quad \textit{for } k \geq 2.$$

Then a proper random variable $\tau = \tau_\eta$ does not depend on the future of random variables $\{\xi_n\}$, but it is not a stopping time.

Example 15. *Let $B \in \mathbb{B}(\mathbb{R})$. Define hitting times $\tau_1 := \min\{n \geq 1 : S_n \in B\}$ and, for all $k \geq 2$, $\tau_k := \min\{n > \tau_{k-1} : S_n \in B\}$. Let η be an independent counting random variable. Then (possibly improper) random variable τ_η does not depend on the future of random variables $\{\xi_n\}$, but it is not a stopping time.*

Example 16. *Let g be a measurable function on the real line, taking values in $(0, 1)$ and such that $g(x) \neq g(y)$ for $x \neq y$. Let $\{\zeta_z\}_{z \in (0,1)}$ be a family of geometric random variables, with ζ_z having parameter z, that do not depend on the random walk. Then*

$$\tau = 1 + \zeta_{g(\xi_1)}$$

is a random time that does not depend on the future of random variables $\{\xi_n\}$, but is not a stopping time.

Example 17. *Let X_t be a Lévy process. For a fixed $a > 0$, let*

$$\tau_1 = \inf\{t > 0 : \Delta X_t > a\} \;\; \textit{and} \;\; \tau_k = \inf\{t > \tau_{k-1} : \Delta X_t > a\} \quad \textit{for } k \geq 2.$$

Then for an independent non-degenerate counting random variable η, τ_η satsfies Definition 1.

3. PROOF OF THEOREM 5 FOR COMPOUND RENEWAL PROCESS WITH LINEAR DRIFT

The proof of Theorem 5 is split into two parts, the lower bound is considered in Proposition 18 and the upper bound in Proposition 19.

Proposition 18. *Let* $-a = \mathbb{E}(cT_1 + Y_1) < 0$. *If* F *is long-tailed, then*

$$\mathbb{P}\{M_\tau > x\} \quad \ge \quad \frac{1+o(1)}{a}\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(y)dy + o(\overline{F}(x)) \quad \text{as } x \to \infty,$$

uniformly for all random times $\tau \in [0,\infty]$ *that satisfy* (1).

Proof. As the distribution F is long-tailed, there exists a function $h(x) \uparrow \infty$ as $x \to \infty$ such that (see, e.g. [8, Lemma 2.19])

$$\overline{F}(x+h(x)) \quad \sim \quad \overline{F}(x) \quad \text{as } x \to \infty. \tag{6}$$

Since the events $\{M_{T_n-0} \le x, X_{T_n} > x, \tau \ge T_n\}$ are disjoint and each of them implies that $M_\tau > x$, we have

$$\begin{aligned}
&\mathbb{P}\{M_\tau > x\}\\
&\quad \ge \sum_{n=1}^\infty \mathbb{P}\{M_{T_n-0} \le x, X_{T_n} > x, \tau \ge T_n\}\\
&\quad \ge \sum_{n=1}^\infty \mathbb{P}\{M_{T_n-0} \le x, X_{T_n-0} > -(a+\varepsilon)(n-1) - h(x), X_{T_n} > x, \tau \ge T_n\},\\
&\quad \ge \sum_{n=1}^\infty \mathbb{P}\{M_{T_n-0} \le x, X_{T_n-0} > -(a+\varepsilon)(n-1) - h(x),\\
&\qquad\qquad Y_n > x + (a+\varepsilon)(n-1) + h(x), \tau \ge T_n\},
\end{aligned} \tag{7}$$

for any fixed $\varepsilon > 0$. Since $\{\tau \ge T_n\} = \overline{\{\tau < T_n\}}$ and owing to the condition (1), the last series equals to

$$\begin{aligned}
&\sum_{n=1}^\infty \mathbb{P}\{M_{T_n-0} \le x, X_{T_n-0} > -(a+\varepsilon)(n-1) - h(x), \tau \ge T_n\}\overline{F}(x + (a+\varepsilon)(n-1) + h(x))\\
&\quad = \sum_{n=1}^\infty \mathbb{P}\{X_{T_n-0} > -(a+\varepsilon)(n-1) - h(x), \tau \ge T_n\}\overline{F}(x + (a+\varepsilon)(n-1) + h(x))\\
&\qquad - \sum_{n=1}^\infty \mathbb{P}\{M_{T_n-0} > x, X_{T_n-0} > -(a+\varepsilon)(n-1) - h(x), \tau \ge T_n\}\\
&\qquad\qquad \times \overline{F}(x + (a+\varepsilon)(n-1) + h(x))\\
&\quad \ge \sum_{n=1}^\infty \mathbb{P}\{X_{T_n-0} > -(a+\varepsilon)(n-1) - h(x), \tau \ge T_n\}\overline{F}(x + (a+\varepsilon)(n-1) + h(x))\\
&\qquad - \mathbb{P}\{M_\tau > x\}\sum_{n=0}^\infty \overline{F}(x + an).
\end{aligned}$$

Therefore, since $\sum_{n=0}^{\infty}\overline{F}(x+an) \le a^{-1}\overline{F}_I(x-a)$, it follows from (7) that

$$\begin{aligned}
&\mathbb{P}\{M_\tau > x\}\\
&\quad\ge \frac{1}{1+a^{-1}\overline{F}_I(x-a)}\sum_{n=1}^{\infty}\mathbb{P}\{X_{T_n-0} > -(a+\varepsilon)(n-1)-h(x), \tau \ge T_n\}\\
&\qquad\qquad \times\overline{F}(x+(a+\varepsilon)(n-1)+h(x))\\
&\quad= (1+o(1))\sum_{n=1}^{\infty}\mathbb{P}\{X_{T_n-0} > -(a+\varepsilon)(n-1)-h(x), \tau \ge T_n\}\\
&\qquad\qquad \times\overline{F}(x+(a+\varepsilon)(n-1)+h(x)) \qquad (8)
\end{aligned}$$

as $x\to\infty$ because $\overline{F}_I(x-a)\to 0$; here $o(1)$ does not depend on τ. Let us decompose the last sum as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}\{\tau \ge T_n\}\overline{F}(x+(a+\varepsilon)(n-1)+h(x))\\
&\qquad -\sum_{n=1}^{\infty}\mathbb{P}\{X_{T_n-0} \le -(a+\varepsilon)(n-1)-h(x), \tau\ge T_n\}\overline{F}(x+(a+\varepsilon)(n-1)+h(x))\\
&=: \Sigma_1 - \Sigma_2. \qquad (9)
\end{aligned}$$

To bound the value of Σ_2, we introduce recursively a sequence $\{\theta_k\}_{k\ge 0}$ of stopping times by letting $\theta_0 = 0$ and, for $k \ge 0$,

$$j_{k+1} := \min\{i : T_i > \theta_k : \ X_{T_i-0} - X_{\theta_k-0} > -(i-j_k)(a+\varepsilon)\},$$

and $\theta_{k+1} = T_{j_{k+1}}$, so that $X_{\theta_k-0} > -j_k(a+\varepsilon)$. By construction, $\theta_k-\theta_{k-1}$, $k\ge 1$, are i.i.d. random variables and

$$\mathbb{E}\theta_1 < \infty. \qquad (10)$$

Introduce the minima over disjoint time intervals

$$L_k := \min_{\theta_k < T_i \le \theta_{k+1}} (X_{T_i-0} - X_{\theta_k-0} + (i-j_k)(a+\varepsilon)), \quad k\ge 0,$$

and notice that

$$\{L_k,\ k\ge 0\} \quad \text{are i.i.d. proper random variables.} \qquad (11)$$

Then, for all $n \ge 1$,

$$\begin{aligned}
&\mathbb{P}\{X_{T_n-0} \le -(a+\varepsilon)(n-1)-h(x), \tau \ge T_n\}\\
&\quad= \sum_{k=0}^{n-1}\mathbb{P}\{\theta_k < T_n \le \theta_{k+1}, X_{T_n-0} \le -(a+\varepsilon)(n-1)-h(x), \tau\ge T_n\}\\
&\quad\le \sum_{k=0}^{n-1}\mathbb{P}\{\theta_k < T_n \le \theta_{k+1}, X_{T_n-0} - X_{\theta_k-0} + (a+\varepsilon)(n-1-j_k) \le -h(x), \tau \ge T_n\}\\
&\quad\le \sum_{k=0}^{n-1}\mathbb{P}\{\theta_k < T_n \le \theta_{k+1}, L_k \le -h(x), \tau \ge \theta_k\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Sigma_2 &\le \sum_{k\ge 0}\sum_{n\ge k+1}\mathbb{P}\{\theta_k<T_n\le\theta_{k+1},L_k\le -h(x),\tau\ge\theta_k\}\overline{F}(x+(a+\varepsilon)(n-1)+h(x))\\
&= \sum_{k\ge 0}\Bigl(\mathbb{E}\mathbb{I}\{\tau\ge\theta_k\}\sum_{n\ge k+1}\mathbb{I}\{\theta_k<T_n\le\theta_{k+1},L_k\le -h(x)\}\Bigr)\overline{F}(x+(a+\varepsilon)k+h(x))\\
&= \sum_{k\ge 0}\mathbb{E}\left[\mathbb{I}\{\tau\ge\theta_k\}(\theta_{k+1}-\theta_k)\mathbb{I}\{L_k\le -h(x)\}\right]\overline{F}(x+(a+\varepsilon)k+h(x)).
\end{aligned}$$

By the condition (1), the event $\{\tau \ge \theta_k\} = \overline{\{\tau<\theta_k\}}$ does not depend on the random variable

$$(\theta_{k+1}-\theta_k)\mathbb{I}\{L_k\le -h(x)\}\in\sigma\{X_t-X_{\theta_k-0},t>\theta_k\},$$

hence we conclude that the last sum is equal to

$$\begin{aligned}
&\sum_{k\ge 0}\mathbb{P}\{\tau\ge\theta_k\}\mathbb{E}\{\theta_{k+1}-\theta_k;\ L_k\le -h(x)\}\overline{F}(x+(a+\varepsilon)k+h(x))\\
&\quad\le \sum_{k\ge 0}\mathbb{P}\{\tau\ge T_k\}\mathbb{E}\{\theta_{k+1}-\theta_k;\ L_k\le -h(x)\}\overline{F}(x+(a+\varepsilon)k+h(x)),
\end{aligned}$$

due to $T_k\le\theta_k$. It follows from (10) and (11) that

$$\sup_k \mathbb{E}\{\theta_{k+1}-\theta_k;\ L_k\le -h(x)\}\to 0 \quad \text{as } x\to\infty,$$

which implies that, as $x\to\infty$,

$$\begin{aligned}
\Sigma_2 &= o(1)\sum_{n\ge 0}\mathbb{P}\{\tau\ge T_n\}\overline{F}(x+(a+\varepsilon)n+h(x))\\
&= o(\overline{F}(x))+o(1)\sum_{n\ge 1}\mathbb{P}\{\tau\ge T_n\}\overline{F}(x+(a+\varepsilon)n+h(x)).
\end{aligned}$$

Thus we derive from (8) and (9) that

$$\begin{aligned}
\mathbb{P}\{M_\tau>x\} &\ge (1+o(1))\sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n\}\overline{F}(x+(a+\varepsilon)(n-1)+h(x))+o(\overline{F}(x))\\
&= (1+o(1))\sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n\}\overline{F}(x+(a+\varepsilon)(n-1))+o(\overline{F}(x)),
\end{aligned}$$

as $x\to\infty$, owing to (6); here $o(1)$ does not depend on τ. The last sum equals

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}\{\tau\in[T_n,T_{n+1})\}\sum_{k=1}^{n}\overline{F}(x+(a+\varepsilon)(k-1))+\mathbb{P}\{\tau=\infty\}\sum_{n=0}^{\infty}\overline{F}(x+(a+\varepsilon)n)\\
&\quad\ge \frac{1}{a+\varepsilon}\Bigl(\sum_{n=1}^{\infty}\mathbb{P}\{\tau\in[T_n,T_{n+1})\}\int_x^{x+(a+\varepsilon)n}\overline{F}(y)dy+\mathbb{P}\{\tau=\infty\}\int_x^{\infty}\overline{F}(y)dy\Bigr)\\
&\quad= \frac{1}{a+\varepsilon}\mathbb{E}\int_x^{x+(a+\varepsilon)N_\tau}\overline{F}(y)dy
\end{aligned}$$

$$\ge \frac{1}{a+\varepsilon}\mathbb{E}\int_x^{x+aN_\tau}\overline{F}(y)dy \tag{12}$$

for all random variables τ that satisfy (1). Since the choice of $\varepsilon>0$ is arbitrary, we conclude the lower bound. □

Proposition 19. *Let $\mathbb{E}(cT_1+Y_1)=-a\le 0$. Assume that either $c\le 0$ or the condition (5) holds. If $F\in\mathscr{S}^*$, then*

$$\mathbb{P}\{M_\tau>x\} \quad\le\quad \frac{1+o(1)}{a}\mathbb{E}\int_0^{aN_\tau}\overline{F}(x+y)dy$$

as $x\to\infty$ uniformly for all random times $\tau\in[0,\infty]$ that satisfy (1).

Proof. The maximum is always attained at a jump associated epoch, either prior to or at the jump epoch.

Firstly consider the case where $c\le 0$ and hence the maximum is attained at a jump epoch. Define $\theta_0=0$. By the strong law of large numbers, for any $\varepsilon\in(0,a)$ there exists an $A<\infty$ such that the stopping time

$$\theta_1 \quad:=\quad \inf\{T_n:\ X_{T_n}>n(-a+\varepsilon)+A\}$$

is finite with a small probability,

$$\mathbb{P}\{\theta_1<\infty\} \quad\le\quad \varepsilon.$$

As $X_t\le A$ for all $t<\theta_1$, by the total probability law, for $x>A$,

$$\begin{aligned}\mathbb{P}\{X_{\theta_1\wedge\tau}>x\} &= \sum_{n=1}^{\infty}\mathbb{P}\{\theta_1\wedge\tau\ge T_n,\ X_{T_{n-1}}\le(n-1)(-a+\varepsilon)+A,\ X_{T_n}>x\}\\ &\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n,\ X_{T_{n-1}}\le n(-a+\varepsilon)-c_1,\ X_{T_n}>x\}\\ &\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n,\ Y_n>x+n(a-\varepsilon)+c_1\},\end{aligned}\tag{13}$$

due to $c\le 0$, where $c_1:=-a+\varepsilon-A$. Since $\{\tau\ge T_n\}=\overline{\{\tau<T_n\}}$ and τ satisfies (1),

$$\begin{aligned}\mathbb{P}\{X_{\theta_1\wedge\tau}>x\} &\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n\}\overline{F}(x+n(a-\varepsilon)+c_1)\\ &= \mathbb{E}\sum_{n=1}^{N_\tau}\overline{F}(x+n(a-\varepsilon)+c_1).\end{aligned}$$

Since $\overline{F}$ is a decreasing function,

$$\mathbb{P}\{X_{\theta_1\wedge\tau}>x\} \quad\le\quad \frac{1}{a-\varepsilon}\mathbb{E}\int_0^{aN_\tau}\overline{F}(x+c_1+y)dy.$$

Taking into account that $M_{\theta_1\wedge\tau}\le\max(A,X_{\theta_1\wedge\tau})$, we conclude an upper bound, for $x>A$,

$$\mathbb{P}\{M_{\theta_1\wedge\tau}>x\} \quad\le\quad \mathbb{P}\{X_{\theta_1\wedge\tau}>x\}\ \le\ \overline{G}_\tau(x),\tag{14}$$

where the distribution G_τ on $[A,\infty)$ defined by its tail as

$$\overline{G}_\tau(x) \quad:=\quad \min\Big(1,\frac{1}{a-\varepsilon}\mathbb{E}\int_0^{aN_\tau}\overline{F}(x+c_1+y)dy\Big),\ x\ge A,$$

is long-tailed because F is so. Hence

$$\overline{G}_\tau(x) \quad\sim\quad \frac{1}{a-\varepsilon}\mathbb{E}\int_0^{aN_\tau}\overline{F}(x+y)dy \quad \text{as } x\to\infty,\tag{15}$$

uniformly for all $\tau \ge 0$. For $k \ge 2$, define recursively stopping times θ_k as follows: on the event $\{\theta_{k-1} < \infty\}$,

$$j_k \quad := \quad \inf\{n : T_n > \theta_{k-1} : \ X_{T_n} - X_{\theta_{k-1}} > (n - j_{k-1})(-a+\varepsilon) + A\}$$

and $\theta_k = T_{j_k}$. Then, by the renewal structure of the process,

$$\mathbb{P}\{\theta_k < \infty \mid \theta_{k-1} < \infty\} \ = \ \mathbb{P}\{\theta_1 < \infty\} \quad \le \quad \varepsilon. \tag{16}$$

Similar to (14) we deduce, for all k,

$$\mathbb{P}\{M_{\theta_k \wedge \tau} - X_{\theta_{k-1}\wedge\tau} > x \mid \theta_{k-1} < \infty\} \quad \le \quad \overline{G}_\tau(x). \tag{17}$$

Since

$$\begin{aligned} M_\tau \quad &= \quad \sum_{k=1}^{\infty}(M_{\theta_k \wedge \tau} - M_{\theta_{k-1}\wedge\tau}) \\ &\le \quad \sum_{k=1}^{\infty}(M_{\theta_k \wedge \tau} - X_{\theta_{k-1}\wedge\tau}), \end{aligned}$$

it follows from (16) and (17) that

$$\mathbb{P}\{M_\tau > x\} \quad \le \quad \sum_{k=1}^{\infty} \overline{G_\tau^{*k}}(x)\varepsilon^{k-1}.$$

Now we need the following fact (a distant analogue of the well-known Kesten's lemma) which is proved in Appendix.

Lemma 20. *Let a, $\varepsilon > 0$ and let $F \in \mathscr{S}^*$. Then, for any $\delta > 0$, there exists a $C(\delta) < \infty$ such that $\overline{G_\tau^{*n}}(x) \le C(\delta)(1+\delta)^n \overline{G}_\tau(x)$ for all $\tau \in [0,\infty]$, $x > 0$, and $n \ge 1$.*

As the distribution F is strong subexponential, by Lemma 20 above, uniformly for all $\tau \in [0,\infty]$ satisfying (1),

$$\begin{aligned} \mathbb{P}\{M_\tau > x\} \quad &\le \quad \overline{G}_\tau(x)(1+o(1))\sum_{k=1}^{\infty} k\varepsilon^{k-1} \\ &\sim \quad \frac{1}{(a-\varepsilon)(1-\varepsilon)^2}\mathbb{E}\int_0^{aN_\tau} \overline{F}(x+y)dy \quad \text{as } x \to \infty, \end{aligned}$$

due to (15). Since we can choose $\varepsilon > 0$ as small as we please, this concludes the proof of Proposition 19 in the case $c \le 0$.

Now proceed with the proof in the case $c > 0$ under the condition (5). We need to slightly modify the starting point, because in the case $c > 0$ it is not any longer true that the maximum can be only attained at a jump epoch T_n, instead it can be attained just prior to that, at time epoch $T_n - 0$. We define θ_1 as

$$\theta_1 \quad := \quad \inf\{T_n : \ X_{T_n - 0} > n(-a+\varepsilon) + A\}$$

and choose A sufficiently large that

$$\mathbb{P}\{\theta_1 < \infty\} \quad \le \quad \varepsilon.$$

Hence we observe that, for $x > A$,

$$\begin{aligned}\mathbb{P}\{X_{\theta_1\wedge\tau} > x\} &\le \sum_{n=1}^{\infty}\mathbb{P}\{\theta_1\wedge\tau \ge T_n,\ X_{T_n-0} \le (n-1)(-a+\varepsilon)+A,\\ &\qquad\qquad X_{T_{n+1}-0} > x\}\\ &\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau \ge T_n,\ X_{T_n-0} \le n(-a+\varepsilon)-c_1,\ X_{T_{n+1}-0} > x\}\\ &\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau \ge T_n,\ Y_n + c(T_{n+1}-T_n) > x + n(a-\varepsilon)+c_1\},\end{aligned}$$

where $c_1 = -a+\varepsilon-A$. Since $\{\tau \ge T_n\} = \overline{\{\tau < T_n\}}$ and since τ satisfies (1),

$$\mathbb{P}\{X_{\theta_1\wedge\tau} > x\} \le \sum_{n=1}^{\infty}\mathbb{P}\{\tau \ge T_n\}\mathbb{P}\{Y_n + c(T_{n+1}-T_n) > x + n(a-\varepsilon)+c_1\}.$$

Due to the condition (5),

$$\mathbb{P}\{Y_n + c(T_{n+1}-T_n) > y\} \sim \overline{F}(y) \quad \text{as } y\to\infty,$$

which allows us to conclude the proof of Proposition 19 in the case $c > 0$ along the lines of the earlier proof in the case $c \le 0$. □

4. APPLICATION TO COMPOUND POISSON PROCESS WITH LINEAR COMPONENT

For a *compound Poisson process with linear component* X where $N = \{N_t, t \ge 0\}$ is a homogeneous Poisson process with intensity of jumps λ, we have $\mathbb{E}N_t = t\lambda$ and $\mathbb{P}\{cT_1 > x\} = e^{-\lambda x/c} = o(\overline{F}(x))$ provided F is heavy-tailed. The following result holds.

Theorem 21. *Let X be a compound Poisson process with linear component. If $\mathbb{E}(c/\lambda + Y_1) =: -a < 0$ and the distribution F of Y_1^+ is strong subexponential, then*

$$\begin{aligned}\mathbb{P}\{M_\tau > x\} &= \frac{1+o(1)}{a}\mathbb{E}\int_x^{x+aN_\tau}\overline{F}(v)dv + o(\overline{F}(x))\\ &= \frac{1+o(1)}{a}\mathbb{E}\int_x^{x+a\lambda\tau}\overline{F}(v)dv + o(\overline{F}(x))\\ &= \frac{1+o(1)}{|\mathbb{E}X_1|}\mathbb{E}\int_x^{x+|\mathbb{E}X_1|\tau}\mathbb{P}\{X_1 > v\}dv + o(\overline{F}(x))\end{aligned}$$

as $x\to\infty$ uniformly for all random times $\tau\in[0,\infty]$ that do not depend on the future increments of X.

Proof. The first equivalence in the theorem follows directly from the discussion on the condition (1) in Introduction and from Theorem 5. We prove the second equivalence now, doing so in several steps.

We start from the following analogue of the Wald–Kolmogorov–Prokhorov identity (see [13]) that holds for Lévy processes, which has independent own interest.

Lemma 22. *Let X be a Lévy process with finite drift $m := \mathbb{E}X_1$, and let τ be a random time that does not depend on the future increments of X. If $\mathbb{E}\tau < \infty$ then*

$$\mathbb{E}X_\tau = m\mathbb{E}\tau.$$

In addition, if $\sigma^2 = \mathbb{V}ar X_1 < \infty$, then

$$\mathbb{E}(X_\tau - m\tau)^2 = \sigma^2\mathbb{E}\tau.$$

Based on Lemma 22, one can prove the following asymptotic equivalence.

Lemma 23. *Let $T > 0$ and $\mathscr{T}_T = \{\tau : \tau \le T\}$ be a family of random times that do not depend on the future increments of N. If F is long-tailed then*

$$\mathbb{E}\int_0^{aN_\tau} \overline{F}(x+y)dy \;\sim\; \mathbb{E}\int_0^{a\lambda\tau} \overline{F}(x+y)dy \quad\sim\quad a\lambda\mathbb{E}\tau\overline{F}(x)$$

as $x \to \infty$ uniformly for all $\tau \in \mathscr{T}_T$.

The proofs of Lemmas 22 and 23 are given in Appendix.

Then the following result is straightforward.

Lemma 24. *Let X be a Lévy process with finite drift $m := \mathbb{E}X_1$ and diffusion coefficient $\sigma^2 = \mathbb{V}ar X_1$. Then, for any fixed $\varepsilon > 0$,*

$$\sup_\tau \mathbb{E}(X_\tau - (m+\varepsilon)\tau) \quad<\quad \infty.$$

Proof. The Lévy process $Y_t = X_t - (m+\varepsilon)t$, $t \ge 0$ is negatively driven and its jumps are square integrable, so

$$\mathbb{E}\sup_{t>0} Y_t \quad<\quad \infty,$$

hence the result follows. □

We are ready to prove the second equivalence in the statement of Theorem 21. For a general τ, fix an $\varepsilon > 0$ and a $T < \infty$, and consider the following decomposition and the upper bound

$$\begin{aligned}
\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(v)dv \quad &= \quad \mathbb{E}\int_x^{x+aN_{\tau\wedge T}} \overline{F}(v)dv + \mathbb{E}\Big(\int_{x+aN_T}^{x+aN_\tau} \overline{F}(v)dv;\ \tau > T\Big) \\
&\le \quad \mathbb{E}\int_x^{x+aN_{\tau\wedge T}} \overline{F}(v)dv + \mathbb{E}\Big(\int_x^{x+a(N_\tau-N_T)} \overline{F}(v)dv;\ \tau > T\Big) \\
&= \quad \mathbb{E}\int_x^{x+aN_{\tau\wedge T}} \overline{F}(v)dv + \mathbb{E}\int_x^{x+a(N_\tau-N_T)^+} \overline{F}(v)dv \\
&=: \quad E_1 + E_2. \qquad (18)
\end{aligned}$$

Fix an $\varepsilon > 0$. Then

$$\begin{aligned}
E_2 \quad &\le \quad \mathbb{E}\int_x^{x+a(\lambda+\varepsilon)(\tau-T)^+} \overline{F}(v)dv + \overline{F}(x)a\mathbb{E}[(N_\tau - N_T)^+ - (\lambda+\varepsilon)(\tau-T)^+]^+ \\
&\le \quad \mathbb{E}\int_x^{x+a(\lambda+\varepsilon)(\tau-T)^+} \overline{F}(v)dv \\
&\qquad\qquad +\overline{F}(x)a\mathbb{E}\Big\{\sup_{s\ge 0}\{(N_{T+s}) - N_T) - (\lambda+\varepsilon)s\}; \tau > T\Big\}.
\end{aligned}$$

We recall that the process $\widehat{N}_s = N_{T+s} - N_T$, $s \ge 0$, is independent of the event $\tau > T$. Hence by Lemma 24,

$$E_2 \quad\le\quad \mathbb{E}\int_x^{x+a(\lambda+\varepsilon)(\tau-T)^+} \overline{F}(v)dv + c(\varepsilon)\mathbb{P}\{\tau > T\}\overline{F}(x),$$

where the constant $c(\varepsilon)$ depends on neither τ nor T. Further,

$$\mathbb{E}\int_x^{x+a(\lambda+\varepsilon)(\tau-T)^+} \overline{F}(v)dv \quad\le\quad \frac{\lambda+\varepsilon}{\lambda}\mathbb{E}\int_x^{x+a\lambda(\tau-T)^+} \overline{F}(v)dv,$$

because the tail function $\overline{F}(x)$ is decreasing. Hence,

$$\begin{aligned} E_2 &\le (1+\varepsilon/\lambda)\mathbb{E}\Big(\int_x^{x+a\lambda(\tau-T)}\overline{F}(v)dv;\ \tau>T\Big)+\mathbb{P}\{\tau>T\}c(\varepsilon)\overline{F}(x) \\ &\le (1+\varepsilon/\lambda)\mathbb{E}\Big(\int_x^{x+a\lambda\tau}\overline{F}(v)dv;\ \tau>T\Big)+\mathbb{P}\{\tau>T\}c(\varepsilon)\overline{F}(x). \end{aligned}$$

Therefore, due to the long-tailedness of F, there exists an $\widehat{x}=\widehat{x}(T)$ such that, for $x\ge\widehat{x}$,

$$E_2 \le (1+\varepsilon/\lambda)\mathbb{E}\Big(\int_x^{x+a\lambda\tau}\overline{F}(v)dv;\ \tau>T\Big)+\mathbb{P}\{\tau>T\}2c(\varepsilon)\overline{F}(x+a\lambda T).$$

By Lemma 23, for all τ and for all sufficiently large x,

$$\begin{aligned} E_1 &\le (1+\varepsilon)\mathbb{E}\int_x^{x+a\lambda(\tau\wedge T)}\overline{F}(y)dy \\ &= (1+\varepsilon)\mathbb{E}\Big(\int_x^{x+a\lambda\tau}\overline{F}(y)dy;\ \tau\le T\Big)+(1+\varepsilon)\mathbb{P}\{\tau>T\}\overline{F}(x+a\lambda T). \end{aligned}$$

Thus, for all τ and sufficiently large x,

$$E_1+E_2 \le (1+\widehat{\varepsilon})\mathbb{E}\int_x^{x+a\lambda\tau}\overline{F}(v)dv+\mathbb{P}\{\tau>T\}\widehat{c}\overline{F}(x+a\lambda T),$$

where $\widehat{\varepsilon}=\varepsilon\max(1,1/\lambda)$ and $\widehat{c}=1+\varepsilon+2c(\varepsilon)$. Since

$$\mathbb{P}\{\tau>T\}\overline{F}(x+a\lambda T) \le \frac{1}{a\lambda T}\mathbb{E}\Big(\int_x^{x+a\lambda\tau}\overline{F}(v)dv;\ \tau>T\Big),$$

we conclude an upper bound

$$E_1+E_2 \le (1+\widehat{\varepsilon}+\widehat{c}/a\lambda T)\mathbb{E}\int_x^{x+a\lambda\tau}\overline{F}(v)dv.$$

Firstly letting $T\to\infty$ and then $\varepsilon\downarrow 0$, we derive from (18) that

$$\mathbb{E}\int_x^{x+aN_\tau}\overline{F}(v)dv \le (1+o(1))\mathbb{E}\int_x^{x+a\lambda\tau}\overline{F}(v)dv$$

as $x\to\infty$ uniformly for all τ that do not depend on the future increments of N_t.

To get a matching lower bound we start with the inequality

$$\begin{aligned} \mathbb{E}\int_x^{x+aN_\tau}\overline{F}(v)dv &\ge \mathbb{E}\int_x^{x+aN_{\tau\wedge T}}\overline{F}(v)dv \\ &\quad +\mathbb{E}\Big(\int_{x+aN_T}^{x+aN_\tau}\overline{F}(v)dv;\ \tau>T, N_T\le(\lambda-\varepsilon)T\Big); \end{aligned}$$

further arguments are quite similar to that used for the analysis of the right hand side terms in the upper bound (18). □

5. PROOF OF THEOREM 9 FOR LÉVY PROCESS

Given the distribution of X_1 is infinitely divisible, recall the Lévy–Khintchine formula for its characteristic exponent $\Psi(\theta):=\log\mathbb{E}e^{i\theta X_1}$, for all $\theta\in\mathbb{R}$,

$$\begin{aligned} \Psi(\theta) &= \Big(i\alpha\theta-\frac{1}{2}\sigma^2\theta^2\Big)+\int_{0<|x|<1}(e^{i\theta x}-1-i\theta x)\Pi(dx)+\int_{|x|\ge 1}(e^{i\theta x}-1)\Pi(dx) \\ &=: \Psi_1(\theta)+\Psi_2(\theta)+\Psi_3(\theta); \end{aligned}$$

see, e.g. Kyprianou [17, Sect. 2.1]. Here Π is the Lévy measure concentrated on $\mathbb{R}\setminus\{0\}$ and satisfying $\int_{\mathbb{R}}(1\wedge x^2)\Pi(dx)<\infty$. Let $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ be independent processes given in the Lévy–Itô decomposition $X_t \overset{d}{=} X_t^{(1)}+X_t^{(2)}+X_t^{(3)}$, $t\ge 0$, where $X^{(1)}$ is a drifted Brownian motion with characteristic exponent given by $\Psi^{(1)}$, $X^{(2)}$ a square integrable martingale with an almost surely countable number of jumps on each finite time interval which are of magnitude less than unity and with characteristic exponent given by $\Psi^{(2)}$, and $X^{(3)}$ a compound Poisson process with intensity

$$\lambda := \Pi(\mathbb{R}\setminus(-1,1))$$

and jump distribution

$$F(dx)=\Pi(dx)/\lambda$$

concentrated on $\mathbb{R}\setminus(-1,1)$. It is known—see, e.g. Kyprianou [17, Theorem 3.6] or Sato [18, Theorem 25.17]—that the process $Z_t := X_t^{(1)}+X_t^{(2)}$, $t\ge 0$ possesses all exponential moments finite which allows us to show the following result, see e.g. Corollary 8 in Korshunov [15].

Proposition 25. (i) *The distribution of X_1 is long-tailed if and only if the distribution of $X_1^{(3)}$ is so. In both cases, $\mathbb{P}\{X_1>x\}\sim\mathbb{P}\{X_1^{(3)}>x\}$ as $x\to\infty$.*

(ii) *The distribution of X_1 is strong subexponential if and only if the distribution F is so. In both cases, $\mathbb{P}\{X_1>x\}\sim\Pi(x,\infty)$ as $x\to\infty$.*

Proof of Theorem 9. Since X_1 is assumed to be strong subexponential, by Proposition 25 the distribution F is strong subexponential too and $\overline{\Pi}(x)\sim\mathbb{P}\{X_1>x\}\sim\lambda\overline{F}(x)$ as $x\to\infty$.

The proof is split into two parts, where we obtain matching lower and upper bounds. We start with the lower bound.

Consider the sequence of jump epochs T_n of the compound Poisson component $X^{(3)}$, that is when $Y_n := X_{T_n}-X_{T_n-0}\in\mathbb{R}\setminus(-1,1)$. Then, as discussed in Introduction, the τ satisfies the condition (1). The distribution F of Y is long-tailed by Proposition 25.

For the lower bound for $\mathbb{P}\{M_\tau>x\}$, we can literally follow the lines of the proof of Proposition 18 with $a:=|\mathbb{E}X_1|/\lambda$, $\lambda:=\Pi(\mathbb{R}\setminus(-1,1))$, and then we arrive at the lower bound, as $x\to\infty$,

$$\mathbb{P}\{M_\tau>x\} \quad\ge\quad \frac{1+o(1)}{a}\mathbb{E}\int_x^{x+aN_\tau}\overline{F}(y)dy+o(\overline{F}(x))$$

for all random variables τ that do not depend on the future increments of X, where N is the Poisson process with intensity λ that counts the number of jumps of $X^{(3)}$. As explained in Section 4, this implies the required lower bound,

$$\mathbb{P}\{M_\tau>x\} \quad\ge\quad \frac{1+o(1)}{m}\mathbb{E}\int_x^{x+m\tau}\mathbb{P}\{X_1>y\}dy+o(\overline{F}(x)) \quad \text{as } x\to\infty. \tag{19}$$

Now let us proceed with a matching upper bound. Similar to the proof for a renewal process, we define $\theta_0=0$ and a stopping time

$$\begin{aligned}\theta_1 &:= \inf\{t:\ X_t>n(-a+\varepsilon)+A \text{ for some } n \text{ and } t<T_n\}\\ &= \inf\{t:\ X_t>n(-a+\varepsilon)+A \text{ for some } n \text{ and } t\in[T_{n-1},T_n)\}.\end{aligned}$$

By the strong law of large numbers for X_t, for any $\varepsilon\in(0,a)$, there exists an $A<\infty$ such that

$$\mathbb{P}\{\theta_1<\infty\} \quad\le\quad \varepsilon.$$

The upcrossing of the level $n(-a+\varepsilon)+A$ by the Gaussian component $X^{(1)}$ results in zero overshoot, while the square-integrable martingale $X^{(2)}$ leads to an overshoot of size at most unity. Hence, for $A>1$ and $x>A$, the event $X_{\theta_1}>x$ can only occur due to a jump of the compound Poisson process $X^{(3)}$. Therefore, by the total probability law, for $A>1$ and $x>A$,

$$\begin{aligned}
&\mathbb{P}\{X_{\theta_1\wedge\tau}>x\}\\
&\quad\le \sum_{n=1}^{\infty}\mathbb{P}\{\theta_1\wedge\tau\ge T_n,\ X_{T_n-0}\le n(-a+\varepsilon)+A,\ X_t>x \text{ for some } t\in[T_n,T_{n+1})\}\\
&\quad\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n,\ Y_n+\Delta_n>x+n(a-\varepsilon)-A\},
\end{aligned}$$

where $\Delta_n:=\sup(X_t-X_{T_n},\ t\in[T_n,T_{n+1})$. Since $\{\tau\ge T_n\}=\overline{\{\tau<T_n\}}$ and since τ satisfies (1) due to its independence of the future increments of X and the discussion in Introduction,

$$\begin{aligned}
\mathbb{P}\{X_{\theta_1\wedge\tau}>x\} &\le \sum_{n=1}^{\infty}\mathbb{P}\{\tau\ge T_n\}\mathbb{P}\{Y_n+\Delta_n>x+n(a-\varepsilon)-A\}\\
&= \mathbb{E}\sum_{n=1}^{N_\tau}\mathbb{P}\{Y_n+\Delta_n>x+n(a-\varepsilon)-A\}.
\end{aligned}$$

Since only $X^{(1)}$ and $X^{(2)}$ contribute to the value of Δ_n and since the exponentially distributed time interval $[T_n,T_{n+1})$ is independent of them, the distribution of the random variable Δ_n is light-tailed and hence

$$\mathbb{P}\{Y_n+\Delta_n>y\}\quad\sim\quad\overline{F}(y)\quad\text{as } y\to\infty.$$

Therefore, as $x\to\infty$, uniformly for all τ,

$$\begin{aligned}
\mathbb{P}\{X_{\theta_1\wedge\tau}>x\} &\le (1+o(1))\mathbb{E}\sum_{n=1}^{N_\tau}\overline{F}(x+n(a-\varepsilon)-A)\\
&\sim \mathbb{E}\sum_{n=1}^{N_\tau}\overline{F}(x+n(a-\varepsilon))
\end{aligned}$$

because F is long-tailed. Since $\overline{F}$ is a decreasing function,

$$\mathbb{E}\sum_{n=1}^{N_\tau}\overline{F}(x+n(a-\varepsilon))\quad\le\quad\frac{1}{a-\varepsilon}\mathbb{E}\int_0^{aN_\tau}\overline{F}(x+y)dy.$$

Taking into account that $M_{\theta_1\wedge\tau}\le\max(A,X_{\theta_1\wedge\tau})$, we conclude the upper bound, for $x\ge x_0$,

$$\mathbb{P}\{M_{\theta_1\wedge\tau}>x\}\quad\le\quad\mathbb{P}\{X_{\theta_1\wedge\tau}>x\}\ \le\ \overline{G}_\tau(x), \tag{20}$$

where $x_0>A$ is sufficiently large and the distribution G_τ on $[x_0,\infty)$ is defined by its tail as

$$\overline{G}_\tau(x)\quad:=\quad\min\Big(1,\ \frac{1+\varepsilon}{a-\varepsilon}\mathbb{E}\int_0^{aN_\tau}\overline{F}(x+y)dy\Big),\quad x\ge x_0.$$

For $k \geq 2$, let us recursively define stopping times θ_k as follows: on the event $\{\theta_{k-1} < \infty\}$,

$$\theta_k \quad := \quad \inf\{t > \theta_{k-1} :\ X_t - X_{\theta_{k-1}} > (n - j_{k-1})(-a+\varepsilon)+A \text{ for some } n \text{ and } t < T_n\}$$

where j_{k-1} is such that $\theta_{k-1} \in [T_{j_{k-1}}, T_{j_{k-1}+1})$. By the renewal properties of the Lévy process X,

$$\mathbb{P}\{\theta_k < \infty \mid \theta_{k-1} < \infty\} \ = \ \mathbb{P}\{\theta_1 < \infty\} \quad \leq \quad \varepsilon. \tag{21}$$

Similar to (20) we deduce, for all k,

$$\mathbb{P}\{M_{\theta_k \wedge \tau} - X_{\theta_{k-1} \wedge \tau} > x \mid \theta_{k-1} < \infty\} \quad \leq \quad \overline{G}(x). \tag{22}$$

Since

$$\begin{aligned} M_\tau \quad &= \quad \sum_{k=1}^{\infty} (M_{\theta_k \wedge \tau} - M_{\theta_{k-1} \wedge \tau}) \\ &\leq \quad \sum_{k=1}^{\infty} (M_{\theta_k \wedge \tau} - X_{\theta_{k-1} \wedge \tau}), \end{aligned}$$

it follows from (21) and (22) that

$$\mathbb{P}\{M_\tau > x\} \quad \leq \quad \sum_{k=1}^{\infty} \overline{G_\tau^{*k}}(x) \varepsilon^{k-1}.$$

As the distribution F is strong subexponential, by Lemma 20, uniformly for all τ,

$$\begin{aligned} \mathbb{P}\{M_\tau > x\} \quad &\leq \quad \overline{G}_\tau(x)(1+o(1)) \sum_{k=1}^{\infty} k\varepsilon^{k-1} \\ &\sim \quad \frac{1+\varepsilon}{(a-\varepsilon)(1-\varepsilon)^2} \mathbb{E} \int_0^{aN_\tau} \overline{F}(x+y)dy \quad \text{as } x \to \infty, \end{aligned}$$

due to (15). Letting $\varepsilon \downarrow 0$, we conclude the desired upper bound which together with the lower bound (19) implies the required asymptotics. □

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APPENDIX

Proof of Lemma 20. By Fubini's Theorem,

$$\begin{aligned} \frac{1}{a-\varepsilon} \mathbb{E} \int_0^{aN_\tau} \overline{F}(x+y)dy \quad &= \quad \frac{1}{a-\varepsilon} \int_0^{\infty} \mathbb{P}\{N_\tau \in dz\} \int_0^{az} \overline{F}(x+y)dy \\ &= \quad \frac{1}{a-\varepsilon} \int_0^{\infty} \overline{F}(x+y)dy \mathbb{P}\{N_\tau > y/a\} \\ &= \quad \int_0^{\infty} \overline{F}(x+y)\mu_\tau(dy), \end{aligned}$$

where the measure

$$\mu_\tau(dy) = \frac{1}{a-\varepsilon}\mathbb{P}\{N_\tau > y/a\}dy$$

satisfies the condition

$$\mu_\tau(x,x+1] \le 1/(a-\varepsilon) \quad \text{for all } x \text{ and } \tau.$$

In addition, the distribution F is strong subexponential, so Corollary 3.36 from [8] or Lemma 9 from [3] are applicable and we conclude the result. □

Proof of Lemma 22. For any fixed T, the random variable $\tau \wedge T$ also does not depend on the future increments of X as τ, hence

$$\begin{aligned} \mathbb{E}(X_T - X_{\tau\wedge T}) &= \mathbb{E}\mathbb{E}\{X_T - X_{\tau\wedge T} \mid \tau\} \\ &= m\mathbb{E}(T-\tau\wedge T). \end{aligned}$$

Therefore,

$$\begin{aligned} mT &= \mathbb{E}X_T \\ &= \mathbb{E}X_{\tau\wedge T} + \mathbb{E}(X_T - X_{\tau\wedge T}) \\ &= \mathbb{E}X_{\tau\wedge T} + mT - m\mathbb{E}(\tau\wedge T), \end{aligned}$$

which implies $\mathbb{E}X_{\tau\wedge T} = m\mathbb{E}(\tau\wedge T)$ for all T. Hence, by the Lebesgue monotone convergence theorem applied to $\tau\wedge T$,

$$\mathbb{E}X_{\tau\wedge T} \to m\mathbb{E}\tau \quad \text{as } T\to\infty.$$

On the other hand, $\mathbb{E}X_{\tau\wedge T} \to \mathbb{E}X_\tau$ because the random variable X_τ is integrable. Indeed,

$$|X_\tau| \le \sum_{n=1}^{[\tau]+1} Y_n,$$

where the random variables

$$Y_n := \sup_{s\in(0,1]} |X_{n-s} - X_{n-1}|, \quad n\ge 1,$$

are i.i.d. with finite mean value because X is a Lévy process with finite drift. Since $[\tau]+1$ does not depend on the future of the sequence $\{Y_n\}$, by the Kolmogorov–Prokhorov equality,

$$\mathbb{E}\sum_{n=1}^{[\tau]+1} Y_n = (\mathbb{E}[\tau]+1)\mathbb{E}Y_1 < \infty,$$

so the proof of the first statement is complete.

For the second statement, firstly notice that, for a bounded τ, that is $\tau\le T$ for some $T<\infty$,

$$\begin{aligned} \sigma^2 T &= \mathbb{E}(X_T - mT)^2 \\ &= \mathbb{E}(X_\tau - m\tau)^2 + 2\mathbb{E}(X_\tau - m\tau)(X_T - X_\tau - m(T-\tau)) \\ &\qquad + \mathbb{E}(X_T - X_\tau - m(T-\tau))^2. \end{aligned}$$

Conditioning on τ implies that

$$\begin{aligned} &\mathbb{E}(X_\tau - m\tau)(X_T - X_\tau - m(T-\tau)) \\ &= \mathbb{E}\mathbb{E}\{(X_\tau - m\tau)(X_T - X_\tau - m(T-\tau) \mid \tau\} \\ &= 0, \end{aligned}$$

because $X_T - X_\tau - m(T-\tau)$ is independent of $X_\tau - m\tau$ given τ. Similarly,

$$\begin{aligned}\mathbb{E}(X_T - X_\tau - m(T-\tau))^2 &= \mathbb{E}\mathbb{E}\{(X_T - X_\tau - m(T-\tau))^2 \mid \tau\} \\ &= \sigma^2 \mathbb{E}(T-\tau).\end{aligned}$$

Combining the last three equalities we conclude that

$$\begin{aligned}\mathbb{E}(X_\tau - m\tau)^2 &= \sigma^2 T - \mathbb{E}(X_T - X_\tau - m(T-\tau))^2 \\ &= \sigma^2 \mathbb{E}\tau\end{aligned}$$

for any bounded τ. For an unbounded τ, we apply it to $\tau \wedge T$ and then let $T \to \infty$ as in the proof of the first statement. □

Proof of Lemma 23. As we know from Lemma 22, $\mathbb{E}N_\tau = \lambda\mathbb{E}\tau$ and $\mathbb{E}(N_\tau - \lambda\tau)^2 = \lambda\mathbb{E}\tau$. Therefore,

$$\begin{aligned}\mathbb{E}N_\tau^2 &= \mathbb{E}(N_\tau - \lambda\tau)^2 + 2\lambda\mathbb{E}N_\tau\tau - \lambda^2\mathbb{E}\tau^2 \\ &\le \mathbb{E}(N_\tau - \lambda\tau)^2 + 2\lambda T\mathbb{E}N_\tau.\end{aligned}$$

due to $\tau \le T$. Thus,

$$\begin{aligned}\mathbb{E}N_\tau^2 &\le \lambda\mathbb{E}\tau + 2\lambda T\lambda\mathbb{E}\tau \\ &= (2T\lambda^2 + \lambda)\mathbb{E}\tau,\end{aligned}$$

which in turn yields, for all $A > 0$,

$$\begin{aligned}\mathbb{E}\{N_\tau;\ N_\tau > A\} &\le \frac{\mathbb{E}N_\tau^2}{A} \\ &\le (2T\lambda^2 + \lambda)\frac{\mathbb{E}\tau}{A},\end{aligned}$$

so, owing to Lemma 22 again,

$$\begin{aligned}\mathbb{E}\{N_\tau;\ N_\tau \le A\} &= \mathbb{E}N_\tau - \mathbb{E}\{N_\tau;\ N_\tau > A\} \\ &\ge \Big(\lambda - \frac{2T\lambda^2 + \lambda}{A}\Big)\mathbb{E}\tau. \qquad (23)\end{aligned}$$

Further, on the one hand,

$$\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(v)dv \le \overline{F}(x)\mathbb{E}(aN_\tau) = a\lambda\mathbb{E}\tau\overline{F}(x). \qquad (24)$$

On the other hand, for any fixed A,

$$\begin{aligned}\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(v)dv &\ge \mathbb{E}\Big(\int_x^{x+aN_\tau} \overline{F}(v)dv;\ N_\tau \le A\Big) \\ &\ge \overline{F}(x+aA)\mathbb{E}\{aN_\tau;\ N_\tau \le A\} \\ &\ge \Big(a\lambda - \frac{2Ta\lambda^2 + \lambda}{A}\Big)\mathbb{E}\tau\overline{F}(x+aA),\end{aligned}$$

by (23). Thus, for any fixed $\varepsilon > 0$, we can choose a sufficiently large A such that

$$\begin{aligned}\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(v)dv &\ge (a\lambda - \varepsilon/2)\mathbb{E}\tau\overline{F}(x+aA) \\ &\ge (a\lambda - \varepsilon)\mathbb{E}\tau\overline{F}(x) \qquad (25)\end{aligned}$$

for all sufficiently large x, due to the long-tailedness of the distribution F. Combining the bounds (24) and (25), we conclude the first uniform asymptotics stated in the lemma,

$$\mathbb{E}\int_x^{x+aN_\tau} \overline{F}(v)dv \quad \sim \quad a\lambda\mathbb{E}\tau\overline{F}(x) \quad \text{as } x \to \infty.$$

Taking into account that

$$a\lambda\mathbb{E}\tau\overline{F}(x+T) \quad \leq \quad \mathbb{E}\int_x^{x+a\lambda\tau} \overline{F}(v)dv \;\leq\; a\lambda\mathbb{E}\tau\overline{F}(x),$$

we also conclude the second uniform equivalence of the lemma,

$$\mathbb{E}\int_x^{x+a\lambda\tau} \overline{F}(v)dv \quad \sim \quad a\lambda\mathbb{E}\tau\overline{F}(x) \quad \text{as } x \to \infty.$$

□

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ACTUARIAL MATHEMATICS AND STATISTICS, HERIOT-WATT UNIVERSITY, EDINBURGH EH14 4AS, SCOTLAND, UK

SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, RUSSIA
Email address: S.Foss@hw.ac.uk

DEPARTMENT OF MATHEMATICS AND STATISTICS, LANCASTER UNIVERSITY, UK
Email address: d.korshunov@lancaster.ac.uk

FACULTY OF PURE AND APPLIED MATHEMATICS, WROCŁAW UNIVERSITY OF SCIENCE AND TECHNOLOGY, WYB. WYSPIAŃSKIEGO 27, 50-370 WROCŁAW, POLAND
Email address: zbigniew.palmowski@gmail.com