Structures in the Burnside ring of profinite groups

Ideals, idempotents and $\mathcal{F}$-stable subrings of Burnside rings of profinite groups

Zachary Bede Locock Hall

This thesis is submitted for the degree of

Doctor of Philosophy

Lancaster University

Department of Mathematics and Statistics

May 2024

Abstract

The purpose of this thesis is to take established results and structures for the Burnside ring of finite groups and to create an analogue in the case where we take the Burnside ring of profinite groups. Since every finite group is a profinite group, we create these structures in mind of ensuring that they coincide on the Burnside ring finite profinite groups. The main difference being that in the Burnside ring of profinite groups, we consider almost finite G-spaces, and so we can have infinite series within the Burnside ring representing infinite G-spaces. We begin with taking a pro-fusion system over a pro-p group S and considering the $\mathcal{F}$-stable S-spaces as a subring of $\widehat{B}(S)$. We show $\widehat{B}(\mathcal{F}) \cong \varprojlim_i (B(\mathcal{F}_i)) \cong \widehat{B}(\varprojlim_i \mathcal{F}_i)$and use this to construct a basis for the subring. For prime ideals, we show that there exists an equivalent to the prime ideals in the finite case and that we have prime ideals arising in the infinite case that differ in construction from those in the finite. Finally, we derive expressions for idempotents, showing that they are either finite, and therefore an inflation of an idempotent in $B(G/N)$, or they are infinite.

Acknowledgements

I would like to convey the deepest of thanks to my supervisor, professor Nadia Mazza, for her constant support, help and inspiration throughout this project. I have immense gratitude to the EPSRC as they have been instrumental in this thesis through funding and endorsement. I would also like to thank my office mates for the moments of levity. Finally, I thank my family and friends for their being there every step of the way.

Declaration

I hereby declare that all work contained within this thesis is my own and has not appeared or been submitted for publication or for any other degree.

Contents

Glossary **6**

1 Introduction **9**

1.1 Scope of the Thesis . . . 9

1.2 Burnside rings of pro-fusion systems . . . 10

1.3 Prime Ideals . . . 14

1.4 Idempotents . . . 16

2 Background **19**

2.1 Burnside Rings of finite groups . . . 19

2.2 Burnside Rings of profinite groups . . . 29

2.3 Fusion systems . . . 46

3 Burnside ring of pro-fusion systems **56**

3.1 $\mathcal{F}^o$-stable S-spaces . . . 56

3.2 Burnside ring of a pro-fusion system $\widehat{B}(\mathcal{F})$ . . . 59

3.3 Structure of $\widehat{B}(\mathcal{F})$ . . . 60

3.4 Constructing a basis for $\widehat{B}(\mathcal{F})$ . . . 63

3.5 Burnside ring of $\mathcal{F}_S(G)$ . . . 73

3.6 Induced pro-fusion systems . . . 83

4 Prime Ideals in $\widehat{B}(G)$ **86**

4.1 Topology of $\widehat{B}(G)$ . . . 86

4.2 Open prime ideals . . . 89

4.3 Prime ideals in $\widehat{B}(S)$ . . . 111

4.4 Existence of closed and not open prime ideals in $\widehat{B}(G)$ . . . 114

5 Units and idempotents of Burnside rings **119**

5.1 Units of $\widehat{B}(G)$ . . . 119

5.2 Units of $\widehat{B}(D_{2^\infty})$ . . . 123

5.3 Idempotents 125

5.4 Idempotents of $\widehat{B}(A_5 \times \mathbb{Z}_p)$ 143

5.5 Finite and infinite idempotents 145

5.6 Units from idempotents 153

6 Open Questions **155**

Glossary

$H \leq_o G$ H is an open subgroup of G in the profinite topology. Page 35.

$\mathbb{N}_0$ The natural numbers including 0.

$\mathbb{Z}_p$ The p-adic integers defined by the p-adic completion of the integers.

$H \leq_c G$ H is a closed subgroup of G in the profinite topology. Page 35.

G/H For $H \leq G$ this denotes the equivalence class of the transitive set G/H as an almost finite G-space. Page 20,44.

$B(G)$ The Burnside ring of the profinite group G. Page 38.

$\widehat{B}(G)$ The completed Burnside ring of the profinite group G. Page 45.

$\mathbb{Q}B(G)$ The Burnside algebra of the profinite group G.

$\mathbb{Q}\widehat{B}(G)$ The completed Burnside algebra of the profinite group G. Page 129.

$\mathcal{F}$ A pro-fusion system over a pro-p group S. Page 53.

$\mathcal{F}^o$ The full subcategory of the pro-fusion system $\mathcal{F}$ over a pro-p group S given by the objects the open subgroups of S in the profinite topology. Page 56.

$(G/K)^H = \mathbf{Fix}_H(G/K)$ The points in G/K which are fixed by H-action. Page 27.

$H \sim_G K$ For $H, K \leq G$, we have that there exists $g \in G$ such that $H^g = K$, in the case where the context of the supergroup is clear, we omit the subscript G. Page 20.

$H \sim_{\mathcal{F}} K$ For a pro-fusion system $\mathcal{F}$, $H, K \in \mathrm{ob}(\mathcal{F})$ there exists $\psi \in \mathrm{Mor}(\mathcal{F})$ such that $\psi(H) = K$. Page 49.

$\mathcal{O}(G)$ The open orbit category of the profinite group G.

$\mathbf{Gh}(G)$ The ghost ring of the profinite group G isomorphic to $\mathbb{Z}^{\mathcal{O}(G)} = \prod'_{H \leq_o G} \mathbb{Z}$.

$\sum'_{H \leq G}$ A series taken over a single representative for each G-conjugacy class of subgroups.

X^H The $\mathbb{Z}$-linear extension of the fixed points of the transitive G-spaces within X. This is invariant on equivalence classes for $H \leq_o G$, that is to say $X^H = \sum'_{K \leq_o G} x_K \cdot (G/K)^H$ where $x_K \in \mathbb{Z}$. Page 27.

$\mathbf{res}^G_H$ For $H \leq_o G$, the linear map $\mathrm{res}^G_H : \widehat{B}(G) \to \widehat{B}(H)$ giving the restriction of a G-space to an H-space. $\mathrm{res}^G_H(G/K) = \sum_{g \in [H \backslash G/K]} H/H \cap {}^gK$. Page 46.

φ_H The ring homomorphism $\varphi_H : \widehat{B}(G) \to \mathbb{Z}$, the fixed point map which maps X to $|X^H|$, where the cardinality of the set is taken additively over the basis elements, $\varphi_H(X) = |X^H| = \sum'_{K \leq_o G} x_K \cdot |(G/K)^H|$. Page 46.

φ The ghost map $\varphi : \widehat{B}(G) \to \mathrm{Gh}(G)$ which maps X to $(\varphi_H(X))_{H \leq_o G}$. Page 43.

$\mathbf{ind}^G_H$ For $H \leq_o G$, the linear map $\mathrm{ind}^G_H : \widehat{B}(H) \to \widehat{B}(G)$ which gives the G-space stabilized by the same subgroup. This is defined on the basis elements by $\mathrm{ind}^G_H(H/K) = G/K$. Page 46.

$\mathbf{Inf}^G_{G/N}$ For $N \trianglelefteq_o G$, the linear map given by $\mathrm{Inf}^G_{G/N} : B(G/N) \to \widehat{B}(G)$. This is defined on the basis elements by $\mathrm{Inf}^G_{G/N}((G/N)/(K/N)) = G/KN$. Page 46.

π^G_N The projection map $\pi^G_N : \widehat{B}(G) \to B(G/N)$ which sends X to X^N. Page 33.

$\mathcal{P}_{U,p}$ The prime ideal of $\widehat{B}(G)$ defined by $\mathcal{P}_{U,p} = \{X \in \widehat{B}(G) \mid \varphi_U(X) \equiv 0 \bmod p\}$ for p either prime or 0. Page 98.

c_H For $H \leq_o G$, the map $c_K : \widehat{B}(G) \to \mathbb{Z}$ returns the coefficient of G/H in the series expansion. This is to say $c_H(\sum_{K \leq_o G} x_K \cdot G/K) = x_H$. Page 121.

$\mu(K,H)$ The Möbius function on the poset of subgroups between H and K. It is defined recursively on the poset by $\mu(K,K)=1$ and for $K<H$, we have $\sum_{K\le L\le H}\mu(K,L)=0$. Page 125.

e_H^G The idempotent of $\mathbb{Q}\widehat{B}(G)$ corresponding to $H\le_o G$, defined by the possibly infinite series $e_H^G=\sum_{K\le_o H}\mu(K,H)\frac{|G:N_G(H)|}{|G:K|}G/K$. Page 131.

$H^{(\infty)}$ The subgroup $H^{(\infty)}$ is the minimal closed normal subgroup K of H such that H/K is pro-solvable. Page 134.

H^π The subgroup H^π is the minimal normal subgroup of H such that H/H^π is a pro-solvable pro-π group. Page 141.

f_P^G The idempotent of $\widehat{B}(G)$ corresponding to $P\le_c G$ with $[P,P]=P$. This is defined by the possibly infinite series $f_P^G=\sum_{\substack{H\le_o G\\ H^{(\infty)}=P}}e_H^G$. Page 138.

1 Introduction

1.1 Scope of the Thesis

The broad goal of the thesis was to define the Burnside ring of a pro-fusion system and extend known results to prove analagous results within new settings. Specifically we extend results for finite groups to those for profinite groups. A key motivation behind this is that the theories should be compatible as explained below.

Throughout this section, let G denote a finite group, $B(G)$ denote the Burnside ring of G and $\mathcal{F}$ a finite fusion system over a finite group. We use $\widehat{G}$ to denote a profinite group, $B(\widehat{G})$ denotes the Burnside ring of $\widehat{G}$, $\widehat{B}(\widehat{G})$ to be the completed Burnside ring of $\widehat{G}$ and $\widehat{\mathcal{F}}$ is a pro-fusion system. In the diagram below, let each object denote the theory of all objects of that type, i.e. G below corresponds to the theory of finite groups.

$$\begin{tikzcd}
 & B(G) \arrow[r, hook] \arrow[ddd, "\text{Barsotti, Carman}"'] & \widehat{B}(\widehat{G}) \\
 & G \arrow[r, hook] \arrow[u, "\text{Solomon}"'] \arrow[d, "\text{Puig}"'] & \widehat{G} \arrow[u, "\text{Dress, Siebeneicher}"'] \arrow[d, "\text{Stancu, Symonds}"] \\
 & \mathcal{F} \arrow[r, hook] \arrow[dl, "\text{Reeh}"] & \widehat{\mathcal{F}} \\
B(\mathcal{F}) & &
\end{tikzcd}$$

The relevant known theory and historical results are depicted here. Note that we have inclusions to denote that finite groups are a subset of profinite groups, Burnside rings of finite groups are a subset of the Burnside rings of profinite groups and that fusion systems are a subset of pro-fusion systems. Consequently, we have that the theory of each of the finite cases lies within the theory of the profinite cases.

The first goal was to define the Burnside ring of a pro-fusion system $\widehat{\mathcal{F}}$, $\widehat{B}(\widehat{\mathcal{F}})$, we are once again motivated to ensure that when $\widehat{F}$ is a finite pro-fusion system, then we have that $\widehat{B}(\widehat{\mathcal{F}})$ agrees with the definition established by Reeh for $B(\widehat{F})$.

In doing so, we have another inclusion of theory and so this thesis establishes the following extension to the theory, in the diagram below the historical results are abbreviated and those the methods established by this thesis are marked with a dashed line and labelled Hall.

$$\begin{tikzcd}
 & B(G) \arrow[r, hook] \arrow[dl, "\text{B., C.}"'] & \widehat{B}(\widehat{G}) \arrow[dr, dashed, "\text{Hall}"] & \\
 & G \arrow[u, "\text{So.}"'] \arrow[r, hook] \arrow[d, "\text{P.}"'] & \widehat{G} \arrow[u, "\text{D., Si.}"] \arrow[d, "\text{St., Sy.}"] & \\
 & \mathcal{F} \arrow[r, hook] \arrow[dl, "\text{R.}"] & \widehat{\mathcal{F}} \arrow[dr, dashed, "\text{Hall}"] & \\
B(\mathcal{F}) \arrow[rrr, hook, dashed] & & & \widehat{B}(\widehat{\mathcal{F}})
\end{tikzcd}$$

The second goal of the thesis is to consider the results shown for (non-completed) Burnside rings of profinite groups and prove their analogue for completed Burnside rings of profinite groups. In particular, these results revolve around the prime ideals and idempotents of the respective Burnside rings. Let $\mathrm{Spec}(R)$ denote the prime ideal spectrum of the ring R, and define $\mathrm{Idem}(R)$ to be the set of idempotents of the ring R. Then once again, we use a diagram to show the existing theory and the new advances made by this thesis.

$$\begin{tikzcd}
 & \mathrm{Spec}(B(\widehat{G})) \\
\mathrm{Spec}(B(G)) \arrow[ur, hook, "\text{Dress}"] \arrow[dr, hook, dashed, "\text{Hall}"] & \\
 & \mathrm{Spec}(\widehat{B}(\widehat{G}))
\end{tikzcd}$$

$$\mathrm{Idem}(B(G)) \xhookrightarrow{\text{Hall}} \mathrm{Idem}(\widehat{B}(\widehat{G})).$$

1.2 Burnside rings of pro-fusion systems

In the case of $B(\mathcal{F})$, the Burnside ring of finite saturated fusion system $\mathcal{F}$ over a finite group S, we have that the Burnside ring of $\mathcal{F}$ is defined by Reeh[18] Definition 4.5 to be the $\mathcal{F}$-stable elements of $B(S)$. To combine the theories of the

Burnside ring of a profinite group with that of a pro-fusion system $\widehat{\mathcal{F}}$ over a pro-p group $\widehat{S}$, we establish the $\widehat{\mathcal{F}}$-stable elements of $\widehat{B}(\widehat{S})$. Throughout this thesis, we use the convention of G/H representing the equivalence class of the orbit G/H as an almost finite G-space for $H \leq G$.

Below, we compare and contrast arbitrary elements within their respective Burnside rings, firstly in the Burnside ring of a finite group described by Burnside[6] §184-185 (and subsequently named and defined the Burnside ring by Solomon[20] §1), secondly in the Burnside ring of a profinite group defined by Dress[9] Appendix B §2 and thirdly in the completed Burnside ring of a profinite group defined by Dress and Siebeneicher[10] 2.3. In all cases, the series is taken over representatives of the group conjugacy classes of the group in question.

$$
\begin{aligned}
B(G) &\ni \sideset{}{'}\sum_{H \leq G} x_H \cdot G/H, \\
B(\widehat{G}) &\ni \sideset{}{'}\sum_{\substack{\widehat{H} \leq \widehat{G} \\ |\widehat{G}:\widehat{H}| < \infty}} x_{\widehat{H}} \cdot \widehat{G}/\widehat{H}, \text{ where finitely many of the } x_{\widehat{H}} \neq 0, \\
\widehat{B}(\widehat{G}) &\ni \sideset{}{'}\sum_{\widehat{H} \leq_o G} x_{\widehat{H}} \cdot \widehat{G}/\widehat{H},
\end{aligned}
$$

with each $x_i \in \mathbb{Z}$. In the case where $\widehat{G}$ is a finite group, we have that $B(\widehat{G}) = \widehat{B}(\widehat{G})$. The completed Burnside ring of a profinite group is determined by the open subgroups of $\widehat{G}$. Since $B(S) \supseteq B(\mathcal{F})$, in order to act analogously, the definition must be chosen such that for a pro-fusion system $\widehat{\mathcal{F}}$ over a pro-p group S, we have a corresponding result. As such, it is sufficient to take the summation over the open subgroups of $\widehat{S}$, which is equivalent to considering $\widehat{\mathcal{F}}^o$. It is for this reason that when establishing the Burnside ring of a pro-fusion system $\widehat{\mathcal{F}}$, we need only consider $\widehat{\mathcal{F}}^o$. Reeh also requires the finite fusion system $\mathcal{F}$ to be saturated in the definition of the Burnside ring of $\mathcal{F}$ and so we approach the definition established in this thesis.

Definition 1.1. The Burnside ring of a saturated pro-fusion system $\widehat{\mathcal{F}}$, is the set of elements of $\widehat{B}(\widehat{S})$ which are $\widehat{\mathcal{F}}^o$-stable.

Note that is $\widehat{\mathcal{F}}$ is saturated or pro-saturated, then we have that $\widehat{\mathcal{F}}^o$ is both saturated and pro-saturated. Consequently in the definition above we can replace saturated with 'saturated or pro-saturated'. As an immediate consequence of the motivation of this setup, we have that the Burnside ring of a saturated or pro-saturated fusion system is itself a ring, hence the following result.

Proposition 1.2. *The Burnside ring of a pro-fusion system $\widehat{B}(\widehat{\mathcal{F}})$ is a subring of $\widehat{B}(\widehat{S})$.*

We note that the definition given by Reeh is for saturated fusion systems, and the definition we have given is for pro-fusion systems for which $\widehat{\mathcal{F}}^o$ is pro-saturated. Therefore, there exists an inverse limit of finite fusion systems such that $\widehat{\mathcal{F}}^o \cong \varprojlim_{i\in I} \mathcal{F}_i$ with each $\mathcal{F}_i$ saturated and therefore $B(\mathcal{F}_i)$ is well defined. The following theorem shows that taking the Burnside ring commutes with taking the inverse limit for a pro-fusion system.

Theorem 1.3. *If $\mathcal{F} = \varprojlim_{i\in I} \mathcal{F}_i$ is a pro-saturated fusion system with each $\mathcal{F}_i$ saturated, then $\widehat{B}(\widehat{\mathcal{F}}) \cong \varprojlim_{i\in I} B(\mathcal{F}_i)$.*

Reeh showed that a basis for $B(\mathcal{F})$ is given by the set $\{\alpha_P \mid P \leq S \text{ fully normalized}\}$ where the α_P are a combinatorially defined element of $B(S)$ produced by stabilizing the element

$$\sum_{[P]_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S(P)|}{|N_S(P')|} S/P'.$$

By utlizing the inverse limit of the successive quotients as in the previous theorem, we use this to create a basis for $\widehat{B}(\mathcal{F})$. We show that there is a well defined element in $\mathrm{ob}(\widehat{\mathcal{F}})$ through the inverse limit.

Definition 1.4. The element $\widehat{\alpha}_P$ is defined to be $\widehat{\alpha}_P = (\alpha_{P/N_i})_{i\in I}$ for P fully normalized in the pro-fusion system $\widehat{\mathcal{F}}$.

Theorem 1.5. *The set $\{\widehat{\alpha}_P \mid P \leq_o \widehat{S} \text{ fully normalized}\}$ is a basis for $\widehat{B}(\widehat{\mathcal{F}})$.*

Since this hold for any saturated or pro-saturated pro-fusion system, we have that for the Burnside ring of a pro-fusion system $\widehat{B}(\widehat{\mathcal{F}})$, every element can be expressed in the form

$$X = \sum_{P \leq_o S}' x_P \cdot \widehat{\alpha}_P$$

with the sum taken over $\widehat{\mathcal{F}}$-conjugacy class representatives (a fully normalized representitive taken for each), $x_P \in \mathbb{Z}$. Comparing with the pre-existing definitions of Burnside rings of finite fusion systems, it is justified that this if a reasonable definition by the following properties.

1. Taking $\widehat{\mathcal{F}} = \mathcal{F}$ be a finite fusion system over a finite group $\widehat{S} = S$, where in each case the former expression in each equality is viewed as an inverse limit, we have that the two definitions are equivalent and

$$\widehat{B}(\widehat{\mathcal{F}}) = B(\mathcal{F}) \subseteq B(S) = \widehat{B}(\widehat{S}).$$

2. For any saturated or pro-saturated fusion system $\widehat{\mathcal{F}}$ over a pro-p group $\widehat{S}$, we have that $\widehat{B}(\widehat{\mathcal{F}}) \subseteq \widehat{B}(\widehat{S})$.

3. For a saturated or pro-saturated pro-fusion system $\widehat{\mathcal{F}}$ over a pro-p group $\widehat{S}$, we have that $\widehat{B}(\widehat{\mathcal{F}}) \cong \varprojlim_{i \in I} B(\mathcal{F}_i)$ in a similar to how for a profinite group $\widehat{G}$, we have that $\widehat{B}(\widehat{G}) \cong \varprojlim_{j \in J} B(G_j)$.

Barsotti and Carman[3] Theorem 7.1 showed that using Reeh's definition for the Burnside ring of a fusion system defined by a group G over S, $\mathcal{F}_S(G) = \mathcal{F}$, we have that $\operatorname{res}^G_S(B(G)) = B(\mathcal{F})$ and so that the image of the restriction map is equal to the set of $\mathcal{F}$-stable elements of $B(G)$. Emulating this in the case of the Burnside ring of pro-fusion systems, we use techniques from Barsotti and Carman to prove the following theorems.

Theorem 1.6. *Suppose that $\mathcal{F}_{\widehat{S}}(\widehat{G})$ is a pro-fusion system for $\widehat{S} \leq_o \widehat{G}$, then we have that $\operatorname{res}^G_S(\widehat{B}(\widehat{G})) = \widehat{B}(\widehat{\mathcal{F}})$.*

Theorem 1.7. *The set* $\{\mathrm{res}_S^G(G/P) \mid P \leq_o S$ *fully normalized*$\}$ *is linearly independent.*

1.3 Prime Ideals

The second goal of the thesis is to extend results proved for the Burnside rings of finite groups to the completed Burnside rings of profinite groups. The focus being results on prime ideals and idempotents established by Dress[8], Yoshida[27], Gluck[11] and tom Dieck[23]. The process of finding prime ideals and idempotents is strongly related to the fixed point maps and their image in the ghost ring.

Given any subgroup $H \leq G$ for G a finite group, and any orbit G/K, we can define the points within G/K which are fixed by H-action, which is to say the $x \in G/K$ such that $h \cdot x = x\ \forall h \in H$. We denote the set of all such points by $(G/K)^H$. We then define the fixed point map φ_H on the transitive G-sets by $\varphi_H(G/K) = |(G/K)^H|$, giving entries in $\mathbb{Z}$. By abuse of notation, we extend this map $\mathbb{Z}$-linearly to the Burnside ring of G and so have a function $\varphi_H : B(G) \to \mathbb{Z}$.

Therefore, by taking the preimages of prime ideals of $\mathbb{Z}$ under fixed point maps we can find prime ideals in $B(G)$. Prime ideals in $\mathbb{Z}$ being given by $p\mathbb{Z}$ for p either prime or 0. Dress[8] Proposition 1 then showed that these preimages classify all prime ideals of the Burnside ring of a finite group. Furthermore, taking φ_H for each conjugacy class representative $H \leq G$, we can define the ghost map $\varphi : B(G) \to \prod'_{H \leq G} \mathbb{Z} =: \mathrm{Gh}(G)$ by $\varphi(X) = (\varphi_H(X))_{H \leq G}$. This function is injective and so we have that equivalence classes of virtual finite G-sets are determined uniquely by their image in the ghost ring. This becomes a key property that allows us to distinguish between different elements in the Burnside ring.

There is a similar notion within the (non-completed) Burnside ring of a profinite group discussed by Dress[9] Appendix B §2 in which there are fixed point maps φ_H defined similarly for any $H \leq_c \widehat{G}$. These are well defined as this ring considers only the finite $\widehat{G}$-sets and so there are only finitely many points in each $\widehat{G}$-set which can be fixed by H-action. Therefore we have the maps $\varphi_H : B(\widehat{G}) \to \mathbb{Z}$

which component wise define the ghost map $\varphi : B(\widehat{G}) \to \prod'_{H \leq_c \widehat{G}} \mathbb{Z}$.

For the completed Burnside ring of a profinite group, for each $H \leq_o \widehat{G}$, we can define maps $\varphi_H : \widehat{B}(\widehat{G}) \to \mathbb{Z}$. Since $\widehat{B}(\widehat{G})$ consists of equivalence classes of virtual almost finite $\widehat{G}$-spaces, this ensures that the number of H-fixed points is finite.

An intuitive topology arises when viewing the completed Burnside ring as an inverse limit, namely where an open basis for the topology is given by cosets of the kernels of the projection maps into the Burnside rings of the finite quotients. Equipped with this, the thesis goes on to show that the open prime ideals of $\widehat{B}(\widehat{G})$ are given in a similar way to those given in the finite case, however this does not classify all prime ideals and we have examples of when we can have closed but not open prime ideals of the completed Burnside ring, and therefore proving that the theory diverges.

Theorem 1.8. *The open prime ideals of $\widehat{B}(\widehat{G})$ are all of the form*

$$\mathcal{P}_{\widehat{U},p} = \{X \in \widehat{B}(\widehat{G}) \mid \varphi_{\widehat{U}}(X) \equiv 0 \bmod p\}$$

for p either a prime or 0 and for some $\widehat{U} \leq_o \widehat{G}$.

With the topology we have defined, we see that the open prime ideals do not coincide with the ideals of $B(\widehat{G})$. These prime ideals instead coincide with $B(\widehat{G}) \cap \mathcal{P}_{\widehat{U},p}$ for $\widehat{U} \leq_o \widehat{G}$. In the finite case, all prime ideals are of this form, but this is not the case when you consider a profinite group. Here we see that the theory of the completed Burnside ring of profinite groups diverges from that of Burnside rings of finite groups and (non-completed) Burnside rings of profinite groups.

Proposition 1.9. *There exists a profinite group $\widehat{G}$ such that $\widehat{B}(\widehat{G})$ contains prime ideals which are closed but not open.*

Theorem 1.10. *For a profinite group $\widehat{G} = \widehat{H}\widehat{K}$, where both $\widehat{H}, \widehat{K}$ are both infinite profinite groups closed under conjugation, then there exists closed but not open prime ideals of $\widehat{B}(\widehat{G})$.*

We note that this setup differs from $B(\widehat{G})$ since in the (non completed) Burnside ring of a profinite group, we only consider the finite $\widehat{G}$-sets and so for each closed subgroup, we have a well defined fixed point map to $\mathbb{Z}$ since there are finitely many points in each $\widehat{G}$-set which can be fixed. However, in the completed Burnside ring, we only have such a result for open subgroups of $\widehat{G}$, and therefore we can have (equivalence classes of) infinite almost finite $\widehat{G}$-spaces.

1.4 Idempotents

In order to extend the results of idempotents in both the Burnside algebra and within the completed Burnside ring, this thesis proves a series of lemmas in order to allow us to use equivalent results in the case of profinite groups as we have for finite groups.

An idempotent, e, in a ring is characterised by the property that $e^2 = e$. Since the fixed point maps φ_H are injective group homomorphisms for each $H \leq G$, we have that $\varphi_H(e)^2 = \varphi_H(e)$ and therefore $\varphi_H(e) = 0, 1$ for each $H \leq G$ and this characterises all idempotents in the Burnside ring. We note that not every combination need be possible since the ghost map is not surjective. However, this issue is addressed by considering idempotents within the Burnside algebra, $\mathbb{Q}B(G)$, following the definition given by Solomon[20] §3 where the Burnside algebra is defined to be the set of elements given by the $\mathbb{Q}$-linear extension of the finite G-orbits. The completed Burnside algebra $\mathbb{Q}\widehat{B}(\widehat{G})$ is defined similarly, and we give the definition of arbitrary elements in the respective Burnside algebras below

$$\mathbb{Q}B(G) \ni \sum_{H \leq G}' y_H \cdot G/H$$
$$\mathbb{Q}\widehat{B}(\widehat{G}) \ni \sum_{H \leq_o G}' y_H \cdot G/H$$

where $y_H \in \mathbb{Q}$.

By abuse of notation, we define the map $\varphi_H : \mathbb{Q}B(G) \to \mathbb{Q}$ where we take the

$\mathbb{Q}$-linear extension of the H-fixed points of the G-orbits in each equivalence class of finite G-sets. It was shown by Yoshida[27] Theorem 3.1 that in the Burnside algebra of a finite group, $\mathbb{Q}B(G)$, there exists $e_H^G \in \mathbb{Q}B(G)$ such that $\varphi_K(G/H) = 0$ if $H \not\sim K$ and $\varphi_K(G/H) = 1$ if $H \sim_G K$ and as such the image of the Burnside ring in the algebraic ghost ring $\mathbb{Q}\mathrm{Gh}(G) := \prod'_{H \leq G} \mathbb{Q}$ is equal to the algebraic ghost ring itself, which is to say that the ghost map is surjective. It follows that every element in $\mathbb{Q}\mathrm{Gh}(G)$ has a unique preimage in $\mathbb{Q}B(G)$.

Since $B(G) \subseteq \mathbb{Q}B(G)$ and any idempotents in $B(G)$ are also idempotents in $\mathbb{Q}B(G)$, it is sufficient to find the idempotents in $\mathbb{Q}B(G)$ which have integer coefficients for each G/H, $H \leq G$. We have that ever idempotent in $\mathbb{Q}$ is expressed as $\sum'_{H \leq G} a_H \cdot e_H^G$ with each $a_H \in \{0, 1\}$. We define the idempotents e_H^G as irreducible since every idempotent is a series of the e_H^G, $H \leq G$.

Theorem 1.11. *The Burnside algebra $\mathbb{Q}\widehat{B}(\widehat{G})$ has an irreducible idempotent for each conjugacy class of $H \leq_o \widehat{G}$ of the form*

$$e_H^{\widehat{G}} = \sum_{K \leq_o H} \mu(K, H) \frac{|\widehat{G} : N_{\widehat{G}}(H)|}{|\widehat{G} : K|} \widehat{G}/K$$

where $\mu(K, H)$ denotes the Möbius formula of the subgroup partial ordering.

Here the use of the Möbius function is justified since between any two open subgroups, there are at most finitely many intermediate subgroups. It follows then, that finitely many of the summands in the expression of $\mu(K, H)$ are non zero and so it is genuinely a finite sum. This expression in general can be an infinite series, but since each coefficient is finite, we have that this is an almost finite $\widehat{G}$-space and so has an equivalence class present in the Burnside algebra.

However, even in the finite case these elements need not lie within the completed Burnside ring since the coefficients need not be integers. The characterisation of the idempotents of the completed Burnside ring can be seen as a series of these idempotents of the Burnside algebra. In order to find which idempotents lie within the Burnside ring, we find families $\mathcal{H}$ such that $\sum_{H \in \mathcal{H}} e_H^G$ is an idempotent in $B(G)$

and we call an idempotent irreducible if there is no non empty subset of $T \subseteq \mathcal{H}$ such that $\sum_{H \in T} e_H^G$ is an idempotent.

Theorem 1.12. *The irreducible idempotents of $\widehat{B}(\widehat{G})$ are given by*

$$f_P^{\widehat{G}} = \sideset{}{'}\sum_{\substack{H \leq_o \widehat{G} \\ H^{(\infty)} = P}} e_H^{\widehat{G}} = \sideset{}{'}\sum_{\substack{H \leq_o \widehat{G} \\ H^{(\infty)} = P}} \sum_{K \leq_o H} \mu(K, H) \frac{|\widehat{G} : N_{\widehat{G}}(H)|}{|\widehat{G} : K|} \widehat{G}/K$$

for P a closed perfect subgroup of $\widehat{G}$.

Since each open subgroup K lies within at most finitely many $H \leq_o \widehat{G}$, we have that there is a non zero coefficient of G/K in finitely many $e_H^{\widehat{G}}$ and therefore once again, this is a well defined almost finite $\widehat{G}$-space and therefore its equivalence class indeed lies within the completed Burnside ring of $\widehat{G}$.

Theorem 1.13. *For $\widehat{G}$ a profinite group, the set*

$$\mathrm{Idem}(\widehat{B}(\widehat{G})) = \{f_P^{\widehat{G}} \mid P \leq_c \widehat{G},\ \exists H \leq_o \widehat{G}\}$$

is a complete set of irreducible idempotent representatives of $\widehat{B}(\widehat{G})$.

Note that any idempotent of $B(\widehat{G})$ is also an idempotent of $\widehat{B}(\widehat{G})$ but the converse is not necessarily true since idempotents of $\widehat{B}(\widehat{G})$ may have infinitely many terms. The multiplicative unit of $\widehat{B}(\widehat{G})$ defined by the equivalence class of $\widehat{G}/\widehat{G}$ is the unique element in $\widehat{B}(\widehat{G})$ is the unique element which has image $(1)_{H \leq G}$ under the ghost map. It follows that $1 = \sum'_{H \leq G} e_H^G$ in the Burnside ring of a finite group, and we prove a similar result in the completed Burnside ring of a profinite group. In particular, since this may be an infinite series, we prove that the coefficient of each equivalence class of a transitive $\widehat{G}$-space is finite. This is a result which is not generally possible in the (non completed) Burnside ring of a profinite group.

Theorem 1.14. *The series*

$$\sum_{e \in \mathrm{Idem}(\widehat{B}(\widehat{G}))} e = 1$$

in the completed Burnside ring of $\widehat{G}$.

Now, having found the multiplicative identity we search for other units within the Burnside ring. We note that any unit, u within the Burnside ring must have an image in the ghost ring which is also a unit. However, since the ghost ring is given by copies of $\mathbb{Z}$, we have that we must 1 or -1 in each coordinate, which is to say $\varphi_H(u) = \pm 1$ for each $H \leq G$. It follows that each unit in the Burnside ring is self inverse. Furthermore, we have by definition of the irreducible idempotents of $B(G)$ that $u = \sum'_{H \leq G} u_H \cdot e_H^G$ for some $u_H \in \{\pm 1\}$. In extending this to the completed Burnside ring of a profinite group, we also prove in this thesis that there are suitably finite coefficients.

Theorem 1.15. *Every unit in the completed Burnside ring of a profinite group is of the form $e_{\mathcal{H}} - e_{\mathcal{H}'}$ where $\mathcal{H}$ and $\mathcal{H}'$ partition the set of conjugacy classes of open subgroups of $\widehat{G}$.*

We note that it is not necessarily true that every element of $\mathbb{Q}\widehat{B}(\widehat{G})$ that has this form is a unit in the completed Burnside ring of a profinite group.

2 Background

2.1 Burnside Rings of finite groups

The Burnside ring of a finite group was first defined by Solomon in his paper "The Burnside Algebra of a Finite group[20]". This paper sought to define the ring as an algebraic structure from foundations discussed in Burnside's "Theory of groups of Finite Order[6]". Burnside discusses the representations of finite groups and shows that each G-set X defines a representation of G in $GL_n(\mathbb{Q})$. In particular, we have that isomorphic G-sets define the same character. Since then, the structure itself has been studied as a means to define an algebraic structure which contains the (finite) sets which are stable under G-action for a finite group G. This gives rise to the following definitions, the majority of which are from Bouc's survey paper[4].

Definition 2.1. Let G be a group, $H, K \leq G$. We write $H \sim_G K$ if there exists $g \in G$ such that $H^g = K$, i.e. that H and K are conjugate. We write $H \lesssim_G K$ if there exists $g \in G$ such that $H^g \leq K$; such a group H is called subconjugate to K.

Group conjugation is a property which is strongly associated with the equivalence of G-sets, particularly transitive G-sets. Lemma 2.3.1 (2) from Bouc's paper[4] gives a characterisation as two transitive G-sets G/H and G/K for $H, K \leq G$ are isomorphic if and only if $H \sim_G K$. Explicitly, the isomorphism class $[G/H] = \{G/K \mid K \sim_G H\}$. Throughout this thesis, we use the convention of G/H representing the isomorphism class of G/H as a G-set. We note that clearly for $H \leq G$ any set of coset representatives G/H is a G-set equipped with the action $g \cdot hH = (gh)H$ for all $g \in G,\ hH \in G/H$. Therefore it is well defined as a G-set and so each isomorphism class is well defined.

Each G-orbit Y is expressible as a left coset G/G_x for $G_x = \{g \in G \mid g \cdot x = x\} \leq G$ for $x \in Y$ any element within Y. Therefore, we have that each G-set can be written $\bigsqcup_{x \in T} G/G_x$ for T a set of representatives for each G-orbit. Note that we have a disjoint union since we can have x, y in different orbits which are isomorphic, i.e. $G/G_x \cong G/G_y$.

Under our convention, isomorphic transitive G-sets are represented by the same symbol and so we can simply count the number of isomorphic G-orbits in a finite G-set. Namely, we can write $a_H \cdot G/H := \bigsqcup_{i=1}^{a_H} G/H$ for $a_H \in \mathbb{N}_0$ and so each finite G-set A can be written as $A = \bigsqcup'_{H \leq G} a_H \cdot G/H$ with each $a_H \in \mathbb{N}_0$ denoting the number of G-orbits in A isomorphic to G/H and here $'$ denotes that we take a single representative for each G-conjugacy class.

Therefore, we have a very usable method to combine G-sets since for two finite G-sets $A = \bigsqcup' a_H \cdot G/H$ and $B = \bigsqcup' b_H \cdot G/H$, we can take their disjoint union $A \sqcup B = \left(\bigsqcup' a_H \cdot G/H\right) \sqcup \left(\bigsqcup' b_H \cdot G/H\right) = \bigsqcup' (a_H + b_H) \cdot G/H$. Clearly there is an additive identity in terms of the empty set $\emptyset$ viewed as a finite G-set. In order to facilitate being able to remove G-orbits from finite G-set, we define what is meant by a virtual G-set.

Definition 2.2. Let G be a finite group. We define a virtual finite G-set to be a formal expression $\bigsqcup' x_H \cdot G/H$ with $x_H \in \mathbb{Z}$ where the disjoint union is taken over H, the representatives of G-conjugacy classes of subgroups.

To avoid confusion, we call a virtual finite G-set, X, an actual finite G-set if it has an expression $X = \bigsqcup x_H \cdot G/H$ with each $x_H \geq 0$. This is to say that it is what we have previously discussed as a finite G-set. With this, we can define the disjoint union of virtual finite G-sets. Given two virtual finite G-sets $X = \bigsqcup' x_H \cdot G/H$ and $Y = \bigsqcup' y_H \cdot G/H$, then we define $X \sqcup Y = \big(\bigsqcup' x_H \cdot G/H\big) \sqcup \big(\bigsqcup' y_H \cdot G/H\big) = \bigsqcup'(x_H + y_H) \cdot G/H$ where we note that $x_H, y_H \in \mathbb{Z}$ can possibly be negative. Consequently, each actual finite G-set, X, has an inverse $-X$ such that $X + (-X) = \emptyset$. It follows that we have an abelian group of isomorphism classes of virtual finite G-sets.

Given actual finite G-sets X and Y, we can define their product by taking the underlying set to be the Cartesian product $X \times Y$ and consider it as an actual finite G-set equipped with the G-action $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$, $x \in X$, $y \in Y$. This carries through to the virtual finite G-sets in the natural way. Taking the virtual finite G-set G/G, we have that for any virutal finite G-set X that $X \cdot G/G \cong G/G \cdot X \cong X$ since the action on G/G is trivial and therefore, viewing the G-sets as representatives of their isomorphism classes, we have $G/G \cdot X = X$ and so G/G is an identity under this product.

Definition 2.3. Bouc[4] 3.1.1 The Burnside Ring $B(G)$ of G is the Grothendieck group of the category G-**set**, for the relations given by decomposition in disjoint union of G-sets. The multiplication on $B(G)$ is induced by the direct product of G-sets.

This definition is equivalent to taking the ring of equivalence classes of virtual finite G-sets given by taking the underlying set of such elements equipped with $X + Y = X \sqcup Y$, $XY = X \times Y$ and $-X$ given by formal negation. As discussed above, we have a multiplicative identity G/G and an additive identity $\emptyset$ when viewed as G-sets. Throughout this thesis, we use the convention of writing $G/G =: 1$ and $\emptyset =: 0$ in the context of the Burnside ring. Similarly, we will use the convention of writing $n := n \cdot G/G$ within the Burnside ring.

Clearly, since $(G/G)^H = G/G$, we have that $\varphi_H(G/G) = 1$ for all $H \leq G$. Therefore $\varphi_H(n \cdot X) = n \cdot \varphi_H(X)$ for each $H \leq G$. Therefore $\varphi(n \cdot X) = (n \cdot \varphi_H(X))_{H \leq G} = n \cdot \varphi(X)$ for each $n \in \mathbb{Z}$. Furthermore, $(\emptyset)^H = \emptyset$ and so $\varphi_H(\emptyset) = 0$ for all $H \leq G$. This aligns with $0 \cdot G/G = \emptyset$ as we would expect. A $\mathbb{Z}$-linear basis can be given for $B(G)$ by considering the set of transitive G-sets $\{G/H \mid H \leq G\}$ since once again each equivalence class of virtual finite G-sets can be expressed as some linear combination of these orbits.

By definition for each G-set, X, there is a G-action defined on X such that $g \cdot x \in X$ for each $g \in G$, $x \in X$. There is an intuitive notion of considering X as an H-set for each $H \leq G$ as we have that there is a well defined H-action for $h \cdot x \in X$ for each $h \in H \leq G$ and $x \in X$. For each virtual finite G-set, we have that the underlying set has finitely many elements, and so considering it as a H-set, we have that it must be a virtual finite H-set. Therefore, taking the representatives in $B(G)$, we have that the restriction to H-action corresponds to a representative in $B(H)$. It follows that we have a well defined functor $\mathrm{res}_H^G : B(G) \to B(H)$ corresponding to viewing the underlying set of a virtual finite G-set as a H-set.

As noted, each virtual finite G-set representative, X, can be seen as a disjoint union $X = \bigsqcup'_{K \leq G} x_K \cdot G/K = \sum'_{K \leq G} x_K \cdot G/K$. Since H is a subgroup of G, then each transitive G-set G/K can be decomposed as a disjoint union of H-orbits. It follows that we can therefore define the restriction on the transitive G-sets and extend it linearly in order to achieve the same map. That is to say, $\mathrm{res}_H^G(X) = \mathrm{res}_H^G(\bigsqcup_{K \leq G} x_K \cdot G/K) = \bigsqcup_{K \leq G} x_K \cdot \mathrm{res}_H^G(G/K)$. We consider the restriction of a transitive G-set to H-action to generalise a formula for the restriction.

In order to decompose a transitive G-set into a disjoint union of transitive H-sets, we note that it is sufficient to take a representative from each H-orbit of the underlying space and then take the disjoint union of these orbits. Considering the transitive G-set G/K, we can take the set of representatives of the H-orbits to be $[H\backslash G/K]$ since this set takes the underlying set G/K and considers the H-orbit by the left H-action. Given $x \in [H\backslash G/K]$, we need to find the expression for this H-orbit as a coset. It corresponds to the set HxK, and therefore we can take H/J where J is the H-stabilizer of any element in the set $HxK = \{h \cdot x \cdot K \mid h \in H\}$.

$$\begin{aligned}
\mathrm{Stab}_H(h \cdot x \cdot K) &= \{g \in H \mid g \cdot h \cdot x \cdot K = h \cdot x \cdot K\} \\
&= \{g \in H \mid (hx)^{-1} g (hx) K = K\} \\
&= \{g \in H \mid (x^{-1}h^{-1}) g (hx) \in K\} \\
&= \{g \in H \mid x^{-1}(h^{-1}gh)x \in K\} \\
&= \{g \in H \mid h^{-1}gh \in {}^{x}K\}.
\end{aligned}$$

Since we can find the stabilizer of any element within the orbit, we are free to choose any $h \in H$, we can in particular choose the element $h = 1$ therefore $\mathrm{Stab}_H(x \cdot K) = \{g \in H \mid g \in {}^{x}K\} = H \cap {}^{x}K$. It follows that the decomposition of G/K into H-orbits is $\mathrm{res}^G_H(G/K) = \bigsqcup_{x \in [H\backslash G/K]} H/(H \cap {}^{x}K)$. We note that for $K_1 \sim_G K_2$ we have that $G/K_1 \cong G/K_2$, and so there exists $\psi : G/K_1 \to G/K_2$ such that $g \cdot \psi(x \cdot K_1) = \psi(g \cdot x \cdot K_1)$ for each $x, g \in G$. In particular, this holds for each $g \in H$ and so the restrictions are isomorphic as H-sets irrespective of the choice of representative.

Let X be a virtual finite G-set, then as previously discussed, there is an expression $X = \bigsqcup_{K \leq G} x_K \cdot G/K$ for some $x_K \in \mathbb{Z}$. Since for any $H \leq G$ we can decompose G/K into a disjoint union of H-orbits, it follows that since the G-orbits themselves are disjoint, we can write $\mathrm{res}^G_H(X) = \bigsqcup_{K \leq G} x_K \cdot \mathrm{res}^G_H(G/K)$. Therefore $\mathrm{res}^G_H(X) = \bigsqcup_{K \leq G} x_K \cdot \left(\bigsqcup_{x \in [H\backslash G/K]} H/(H \cap {}^{x}K)\right)$ up to equivalence as virtual H-sets. Note that since each x_K is finite and G has finitely many subgroups, we have that this is a virtual finite H-sets, and since the restriction agrees regardless of the choice of representatives of virtual finite G-set, we have that the restriction

$\mathrm{res}^G_H : B(G) \to B(H)$ is a well defined map.

Consider any non empty G-orbit of the product $G/H \times G/K$ as we have previously defined, then there is an element (x_1H, x_2K) within this G-orbit. We also have that there exists $x_1^{-1} \in G$ such that $x_1^{-1} \cdot (x_1H, x_2K)$ and therefore we have that the G-orbits $G(x_1H, x_2K) = G(H, x_1^{-1}x_2K)$ are equal. Let $x = x_1^{-1}x_2$, then for $h \in H \leq G$, we have that $h \cdot (H, xK) = (h \cdot H, h \cdot xK) = (H, hxK)$, noting that for $g \notin H$ we have $g \cdot H \neq H$. It follows that we can take the set of all entries in the G-orbit which have H in the first coordinate, namely $\{(H, yK) \mid y \in G\} = \{(H, hxk) \mid h \in H\} = (H, HxK)$.

Suppose that $G(H, x_1K) = G(H, x_2K)$ define the same orbit, then it follows that we have $(H, Hx_1K) = (H, Hx_2K)$ and so $Hx_1K = Hx_2K$. Conversely, suppose that $Hx_1K = Hx_2K$ then we have that $(H, Hx_1K) = (H, Hx_2K)$ and therefore they define the same G-orbit. We conclude that the G-orbit is classified by the double coset representative, and since the double cosets HxK partition the set G, we have that there is exactly one orbit from each double coset representative. It is therefore possible to decompose the product $G/H \times G/K$ into G-orbits indexed by the double coset representatives.

Consider a G-orbit of $G/H \times G/K$ containing the element (H, xK) for some $g \in G$, then the stabilizer of this element is given by

$$\begin{aligned}
\mathrm{Stab}_G(H, xK) &= \{g \in G \mid g \cdot (H, xK) = (H, xK)\} \\
&= \{g \in G \mid (g \cdot H, x^{-1}gxK) = (H, K)\} \\
&= \{g \in G \mid g \in H,\ x^{-1}gx \in K\} \\
&= \{g \in G \mid g \in H,\ g \in {}^xK\} \\
&= H \cap {}^xK.
\end{aligned}$$

Since the G-orbits in $G/H \times G/K$ are indexed by $x \in [H \backslash G/K]$, then we have that it can be decomposed as $G/H \times G/K = \bigsqcup_{x \in [H\backslash G/K]} G/(H \cap {}^xK)$. From this expression, it is evident that there is a correspondence between the multiplication and the restriction, where the disjoint union is indexed by the same set, namely $[H\backslash G/K]$ and the stabilizers are given by the same subgroups, however in the case of the restriction, we have H-orbits and in the multiplication we have G-orbits.

Through restriction, for $H \leq G$ we can define the H-action on the underlying space of a G-set and therefore create a H-set. It will be beneficial to also define an opposing notion which takes H-sets and equips them with a G-action. For ant $K \leq H$, we can consider the transitive H-set stabilized by K, namely H/K. Since we have supposed that $H \leq G$, then it is obvious that $K \leq G$ and so there exists a transitive G-set, G/K, which is stabilized by K. This is well defined on the transitive H-sets since regardless of conjugacy class representative, we have that $K_1 \sim_H K_2$ implies that $K_1 \sim_G K_2$ since $H \leq G$ and so $G/K_1 = G/K_2$. Note that the converse need not be true, for example taking $H = \langle (12), (34) \rangle$, $G = S_4$, then $\langle (12) \rangle \not\sim_H \langle (34) \rangle$ but $\langle (12) \rangle \sim_G \langle (34) \rangle$ and so we can have $G/K_1 = G/K_2$ with $H/K_1 \neq H/K_1$.

Extending this Z-linearly over the transitive H-sets, we define the induction map $\mathrm{ind}_H^G : B(H) \to B(G)$, it is clear that the image of a virtual finite H-set under this map is a virtual finite G-set. By the above justification, we have that the induction map is not necessarily injective, and further if we have that $H \neq G$ then it is also not surjective since G/G cannot be given as a linear combination $\bigsqcup_{K \leq H} x_K \cdot G/K$, a general expression for an element in the image of the induction map. Comparing with the definition of product and restriction, we have that for $H, K \leq G$

$$\begin{aligned} G/H \times G/K = \bigsqcup_{x \in [H\backslash G/K]} G/(H \cap {}^xK) &= \bigsqcup_{x \in [H\backslash G/K]} \mathrm{ind}_H^G \left(H/(H \cap {}^xK) \right) \\ &= \mathrm{ind}_H^G \left(\bigsqcup_{x \in [H\backslash G/K]} H/(H \cap {}^xK) \right) \\ &= \mathrm{ind}_H^G \mathrm{res}_H^G (G/K). \end{aligned}$$

Summarising this, we give the well defined definitions of restriction and induction as discussed in Bouc's "Burnside Rings"[4] §3.1 in a manner which aligns with the intuition for this that we have discussed in this chapter thus far. Note that since we have justified they truly map within the Burnside ring, we use the symbols for addition in the ring rather than that of disjoint union to align with the ring structure.

Definition 2.4. [4]Bouc §3.1 For $H \leq G$, we define the induction and restriction maps $\mathrm{ind}_H^G : B(H) \to B(G)$ and $\mathrm{res}_H^G : B(G) \to B(H)$ by the $\mathbb{Z}$-linear extension of their definition on the transitive H-sets and G-sets respectively as follows. The restriction $\mathrm{res}_H^G(G/K) = \sum_{x \in [H \backslash G/K]} H/(H \cap {}^xK)$, and the induction map $\mathrm{ind}_H^G(H/K) = G/K$. We also have $G/H \times G/K = \sum_{x \in [H \backslash G/K]} G/H \cap {}^xK$.

This definition of the induction map from H to G, whilst always being well defined, does require that the virtual H-set in question first be decomposed into disjoint H-orbits. There is a more explicit method of calculating the induction based on the structure of the H-action on the set and its underlying set. Given a virtual H-set X, consider the set $G \times X$ and equip it with a right H-action defined by $(g, x) \cdot h = (gh^{-1}, h \cdot x)$. If we identify all elements within the equivalence classes of the H-orbits by this right H-action, we have a set $G \times_H X$ that can be considered a virtual G-set through the left G-action $g_1 \cdot [(g_2, x)] = [(g_1 g_2, x)]$ for each $g_1, g_2 \in G, \ x \in X$.

In particular, if we consider this explicit method on a transitive H-set H/K, then we have that the equivalence class under H-action $[(g, xK)]$ for some $x \in K$, $g \in G$. This has explicit form $[(g, xK)] = \{(gh^{-1}, hxK) \mid h \in H\}$ and there is a clear G-set isomorphism given by $[(g, xK)] \mapsto gxK$. Note that this holds regardless of representative since $gh^{-1}hxK = gK$.

Given a G-set X and $H \leq G$, then we can consider the set of elements within X which are fixed by H-action, $X^H = \{x \in X \mid h \cdot x = x, \ \forall h \in H\}$. Consider a transitive G-set G/K, then we can take $(G/K)^H$

$$\begin{aligned}(G/K)^H &= \{gK \in G/K \mid h \cdot gK = gK,\ \forall h \in H\}\\ &= \{gK \in G/K \mid g^{-1}hgK = K\}\\ &= \{gK \in G/K \mid H^g K = K\}\\ &= \{gK \in G/K \mid H^g \leq K\}.\end{aligned}$$

Consequently, it $H \not\lesssim_G K$, we have that there are no H-fixed points in G/K since no conjugate of H lies within K, it follows that $(G/K)^H = \emptyset$. We note by the definition of taking the H-fixed points of a G-set. we have that the H-action on X^H is trivial and so the set X^H can be considered as a $N_G(H)/H$-set since $gH \cdot \{x\} = \{gH \cdot x = g \cdot (H \cdot x) = g \cdot x\}$. We note that the provision that it is a $N_G(H)/H$-set arises since this must be a group in order for the associative group action axiom to be met.

An exceptional case can be considered when we have that $N \trianglelefteq G$ is a normal subgroup. In this case, since N is normal, we have that either every element of a transitive G-set G/K is fixed under N-action or no element is. It follows that either $(G/K)^N = G/K$ viewed as a G/N-set if $N \leq K$ or $(G/K)^H = \emptyset$ if $H \not\leq K$. We once again extend this definition linearly over the transitive G-sets. Clearly, the H-fixed points of a virtual G-set are a virtual finite G/N-set since we have finitely many finite coefficients and so we can define a map $\cdot^N : B(G) \to B(G/N)$ defined by $\left(\sum_{H \leq G} x_H \cdot G/H\right)^N = \sum_{H \leq G} x_H \cdot (G/H)^N$. More generally, for any $H \leq G$ we can view the map $\cdot^H : B(G) \to B(N_G(H)/H)$ in a similar way.

Definition 2.5. [4]Bouc §3.1

Let G be a finite group, $H \leq G$ a subgroup, then we define $\cdot^H : B(G) \to B(N_G(H)/H)$ to be the map taking the H-fixed points. That is to say, for each virtual finite G-set X, we have $X^H = \{x \in X \mid h \cdot x = x\ \forall h \in H\}$.

There is a companion to the fixed points in the form of the H-fixed point ghost map $\varphi_H : B(G) \to \mathbb{Z}$ for each $H \leq G$, this is defined on the transitive finite G-sets

G/K by $\varphi_H(G/K) = |(G/K)^H|$ and extend this $\mathbb{Z}$-linearly, noting that this allows us to have negative numbers. Considering the number of H fixed points becomes a powerful property in classifying the elements of the Burnside ring. The following theorem proven first in [6]Burnside [15, Ch. XII, Theorem 1] shows that we can classify isomorphism classes of finite G-sets by considering just the number of fixed points under each subgroup $H \leq G$.

Theorem 2.6. Bouc 2.3.2 *Let G be a finite group, and X and Y be finite G-sets. Then the following are equivalent:*

1. *The G-sets X and Y are isomorphic.*
2. *For any subgroup H of G, the sets X^H and Y^H have the same cardinality.*

Note that any finite virtual G-set $X = \bigsqcup x_H \cdot G/H$ can be expressed as the formal difference of two actual finite G-sets in the obvious way. This is to say that $X = \bigsqcup_{H \leq G} y_H \cdot G/H - \bigsqcup_{H \leq G} z_H \cdot G/H$ where $y_H = x_H$ if $x_H \geq 0$, and $z_H = -x_H$ if $x_H \leq 0$ and all other coefficients 0. It naturally follows that each $y_H, z_H \in \mathbb{N}_0$ and so we have that X is expressed as a formal difference of two actual finite G-sets as claimed. Consequently, this theorem trivially extends to being able to classify the finite virtual G-sets by the linear extension of the number of fixed points.

Definition 2.7. [4]Bouc §3.1

Let G be a finite group, $H \leq G$ a subgroup, then we define the map $\varphi_H : B(G) \to \mathbb{Z}$ by defining $\varphi_H\left(\sum_{K \leq G} x_K \cdot G/K\right) = \sum_{K \leq G} x_K \cdot \varphi_H(G/K)$, where $\varphi_H(G/K) = |(G/K)^H|$ for each $K \leq G$.

In discussion this map φ_H will be referred to as the number of H fixed points. It may at first appear as though in taking the number of fixed points rather than the fixed points themselves that we are weakening the information we have about a virtual G-set. However, the number of fixed points is sufficient to classify an element within the Burnside ring.

Consider some virtual finite G-set $X = \bigsqcup_{K \leq G} a_K \cdot G/K$, then we can take $X^G = \{x \in X \mid g \cdot x = x \ \forall g \in G\}$. However, by the definition of the fixed point map, we have that X^G is viewed as an element of the Burnside ring $B(G/G) \cong \mathbb{Z}$ since $B(G/G)$ is the $\mathbb{Z}$-linear span of G/G. It follows that $|X^G| = a_G$. Take a proper subgroup H which is maximal up to conjugation, which is to say that there does not exist $g \in G$ such that there exists $K \leq G$ such that $H^g < K < G$. Then, since the H fixed points of any G/K such that H is not subconjugate to K is the empty set, we have that $\varphi_H(X) = a_G + a_H(G/H)$. By the previous justification we have that a_G is already fixed and so we have a unique solution for a_H. Repeating this process, we have that the image within the Ghost ring is unique, therefore we can classify elements in the Burnside ring through their image in the Ghost ring.

2.2 Burnside Rings of profinite groups

In the definition of Burnside rings of finite groups, we have a method of discussing the structure and combination of different equivalence classes of virtual finite G-sets for G a finite group. Dress, in "Notes on the theory of representations of finite groups"[9] Appendix B then took this construction and extended it to profinite groups. This was done in the most obvious way by considering the equivalence classes of virtual finite G-sets where G is a profinite group. This definition uses the underlying structure of a profinite group in order to define parallels to the fixed point maps and establish a basis of transitive G-sets for which the $\mathbb{Z}$-span is the entirety of the Burnside ring of profinite groups.

Extending the theory to cover profinite groups is justified by two key properties. Firstly, that all finite groups are profinite groups and so we have a working model for how we would wish the Burnside ring of profinite groups to function in the case of finite groups. Secondly, through the construction of a profinite group, we have that there are subgroups of finite index, and since the same argument with regards to orbit-stabilizers hold, we have that there are transitive finite G-sets G/H where H is a subgroup of finite index. Since these properties are fundamental in the discussion of the Burnside ring of a finite group, we have that this motivates this choice of definition.

In order to make the definition, we regard the structure of a general profinite group. It harnesses the qualities of having subgroups of finite index without being constricted to the group itself being finite, and indeed it is possibly infinite. This is achieved by an infinite series of projection maps, each projecting finite groups into finite groups. Taking the inverse limit of these we can define the terminal object to be the Burnside ring of a profinite group. We shall go through this procedure in detail as described in [26]Wilson Chapter 1.

Definition 2.8. [7]§1.1 A directed set $(\Lambda, \leq')$ is a set Λ equipped with a relation $\leq'$ such that for any $x, y \in \Lambda$, there exists an element $z \in \Lambda$ such that $x, y \leq' z$.

Given a group G, we can define a directed set on its subgroups by taking $\Lambda_0 = \{H \mid H \leq G\}$ subject to the relation $\leq'$ to be $\geq$. This clearly satisfies the definition for a directed set since for any $H, K \in \Lambda$ we have that $H \cap K \in \Lambda_0$ and $H, K \geq H \cap K$. More presciently, we can define $\Lambda = \{N \leq G \mid N \trianglelefteq G\}$ and again consider $\leq'$ to be $\geq$, then this is a directed set since for any $N_1, N_2 \in \Lambda$, we have that $N_1, N_2 \trianglelefteq G$ and so it follows that $N_1 \cap N_2 \trianglelefteq G$, hence $N_1 \cap N_2 \in \Lambda$ with $N_1, N_2 \geq N_1 \cap N_2$. Equipped with the definition of a directed set, we can define an inverse system.

Definition 2.9. [7]§1.1 Given a directed set $(\Lambda, \leq')$ then we define an inverse system of sets over Λ to be a family of sets $(G_\lambda)_{\lambda \in \Lambda}$ together with a family of maps $(\pi_{\lambda\mu})_{\lambda,\mu \in \Lambda,\ \mu \leq' \lambda}$ satisfying the conditions $\pi_{\lambda\lambda} = id_{G_\lambda}$ and $\pi_{\lambda\mu} \circ \pi_{\mu\sigma} = \pi_{\lambda\sigma}$ whenever $\sigma \leq' \mu \leq' \lambda$.

Further, we can apply this to groups, rings or topological spaces with the corresponding homomorphisms or continuous maps in order to make an inverse system of the alike objects. Maps between similar objects that satisfy the composition condition given in the definition are said to be compatible. In particular, in this way, we can project from one group to another through homomorphisms provided that we can define suitable compatible maps such that we have an inverse system with compatible maps between the groups. By obvious extension, we can apply the same reasoning to a category, with the maps given by the morphisms between objects.

Given an inverse system $(G_\lambda)_{\lambda\in\Lambda}$ of objects of a category $\mathcal{C}$ over a directed set Λ, it follows that naturally you may wish to find some structure, L, which contains all of the information that the inverse system is defined by. Firstly, we would wish to be able to recover G_λ for each $\lambda \in \Lambda$. A natural approach to this is to define that the structure L is equipped with a family of projection maps $(\pi_\lambda)_{\lambda\in\Lambda}$ such that $\pi_\lambda(L) = G_\lambda$ and each projection is a morphism of the requisite type. Secondly, we wish for these projection maps to be compatible, which is to say $\pi_{ji} \circ \pi_i = \pi_j : L \to G_j$ for each $i, j \in \Lambda$. Finally, we wish for L to have a property that for any other structure M with compatible projections $\sigma_\lambda : M \to G_\lambda$, the projection can be factored through L, that is to say that there is a unique morphism $\sigma : M \to L$ such that $\sigma_\lambda = \pi_\lambda \circ \sigma : M \to L \to G_\lambda$ for each $\lambda \in \Lambda$.

This object L is unique up to isomorphism and so we define L to be the inverse limit of the inverse system. In particular, there is one element of the isomorphism class of L that lends itself to explicit expression. Let $\widehat{G}$ denote the subset of $\prod_{\lambda\in\Lambda} G_\lambda$ such that $\widehat{G} = \{(g_i)_{i\in\Lambda} \mid \pi_{ji}(g_i) = g_j\}$. There is an obvious family of projection maps $(\pi_\lambda)_{\lambda\in\Lambda}$ defining the image of $g = (g_\lambda)_{\lambda\in\Lambda} \in \widehat{G}$ to be $\pi_\lambda(g) = g_\lambda$. This satisfies the required conditions by construction and so this is an inverse limit of the inverse system.

Definition 2.10. [5]§7.1 Given an inverse system $(G_\lambda)_{\lambda\in\Lambda}$ of groups over a directed set Λ with compatible maps $(\pi_{ji})_{i\leq j,\ i,j\in\Lambda}$, we define the inverse limit of the inverse system, $\widehat{G} = \varprojlim_{\lambda\in\Lambda} G_\lambda$ to be be a group such that there are group homomorphisms $\pi_\lambda : \widehat{G} \to G_\lambda$ for each $\lambda \in \Lambda$ which are compatible, which is to say that for $i, j \in \Lambda$, $i \leq j$, then we have that $\pi_{ij} \circ \pi_j = \pi_i : \widehat{G} \to G_i$. The homomorphisms π_λ for $\lambda \in \Lambda$ we define to be the projection maps. It must also satisfy that for any group H such that there exist a family of compatible projection maps $\sigma_\lambda : H \to G_\lambda$, then there exists a unique group homomorphism $\sigma : H \to \widehat{G}$ such that $\sigma_\lambda = \sigma \circ \pi_\lambda$ for each $\lambda \in \Lambda$.

With the notation of the definition, this is often depicted and summarised through the following commutative diagram for each $i \leq j$, $i, j \in \Lambda$. Here, the existence of a unique map σ is depicted through the use of $!\sigma$.

$$\begin{tikzcd}
 & H \arrow[d, dashed, "!\sigma"] \arrow[ddl, "\sigma_j"'] \arrow[ddr, "\sigma_i"] & \\
 & \widehat{G} \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
G_j \arrow[rr, "\pi_{ij}"'] & & G_i
\end{tikzcd}$$

As previously previously discussed, this can be defined in greater generality than just in the case of groups provided we have structure preserving morphisms for the projection maps. Similar to inverse systems, the inverse limit can be applied to topological spaces with continuous maps, or rings with ring homomorphisms. If the objects are topological groups, we have that we can similarly define the inverse limit with continuous group homomorphisms. We note, however, that finiteness is not necessarily preserved since the directed set Λ can contain infinitely many elements, even if each element of the inverse system indexed by this directed set is itself finite.

Suppose that we have an inverse system of groups over a directed set Λ with infinitely many elements, and each G_λ, $\lambda \in \Lambda$ is a finite group. Furthermore suppose that there is an infinite chain in Λ, which is to say that there exists $S \subseteq \Lambda$ with the same ordering as on Λ such that for each $\sigma, \mu \in S$ and $\lambda \neq \mu$, then we have either $\sigma < \mu$ or $\mu < \sigma$ and that each of the compatible maps is surjective with non trivial kernel. Since there is no upper limit on the size of the finite groups, we have that the explicit representative of the isomorphism class of the inverse limit $\widehat{G} = \{(g_i)_{i \in \Lambda} \mid \pi_{ji}(g_i) = g_j\}$, we have that there is no upper limit on the cardinality of $\widehat{G}$ and therefore we have that $\widehat{G}$ must be an infinite group.

This occurs explicitly in the case of the p-adic integers $\mathbb{Z}_p$ where we can take $\Lambda = \mathbb{N}_0$ with the usual ordering and the groups $G_i = \mathbb{Z}/p^i\mathbb{Z}$ for each $i \in \Lambda$. The compatible maps are given by defining $\pi_{i,j} : G_j \to G_i$ to be taking the representative x mod p^j and mapping to x mod p^i for $i \leq j$. Since we have that there is a total order on Λ, this defines all of the required compatible maps.

$$\dots \longrightarrow \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{\pi_{2,3}} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi_{1,2}} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\pi_{0,1}} 1$$

We now define the group of p-adic integers to be $\mathbb{Z}_p := \varprojlim_{i\in\mathbb{N}_0} G_i = \varprojlim_{i\in\mathbb{N}_0} \mathbb{Z}/p^i\mathbb{Z}$, noting that this is an additive group. The compatible map π_{ji} can be viewed as multiplication of the representatives of G_j by p^{j-i} followed by an isomorphism $p^{j-i}\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$. It follows that $|\ker(\pi_{ji})| = p^{j-i}$ and so we have that the kernels are non trivial and the surjectivity of the maps is clear. Hence, the reasoning applies to this construction and so $\mathbb{Z}_p$ is infinite, as we may intuitively presume.

Conversely, suppose that we have that Λ is a finite directed set and each G_λ is a finite group. It follows that the explicit isomorphism class representative of the inverse limit $\widehat{G} \subseteq \prod_{\lambda\in\Lambda} G_\lambda$ and since all G_λ are finite and Λ is finite, we must have that $|\prod_{\lambda\in\Lambda} G_\lambda| < \infty$ and so clearly we must have that $\widehat{G}$ is finite. Therefore the inverse limit of inverse system of finite groups can in some cases be finite and some cases can be infinite.

If $\mathcal{C}$ is a category and $\widehat{G}$ is an inverse limit of an inverse system $\widehat{G} = \varprojlim_{\lambda\in\Lambda} G_\lambda$ with each $G_\lambda \in \mathrm{ob}(\mathcal{C})$ and the respective projections $\pi_{\sigma\lambda} \in \mathrm{mor}(\mathcal{C})$ then we say that $\widehat{G}$ is pro-$\mathcal{C}$ and said to be an inverse limit of $\mathcal{C}$. Therefore a profinite group is a group which is an inverse limit of finite groups. As evidenced by the earlier observation, we note that this need not necessarily be a finite group. This approach requires us to have an inverse system of groups, a natural question arises of whether given a group, we can then define an inverse system.

Suppose that G is a finite group and consider the set $\Lambda = \{N \mid N \trianglelefteq G\}$. It follows that this is a directed set with reverse inclusion as previously noted. Motivated by using this directed set to define an inverse system, there is a clear candidate for a group indexed by $N \in \Lambda$ in the form of the factor group G/N and we have a group homomorphism $\pi_N : G \to G/N$ defined by $\pi_N(g) = gN$, giving the coset of N in G that g lies in. Suppose that $N \leq M$ for $M, N \in \Lambda$ we have that we have a group homomorphism $\pi_{M,N} : G/N \to G/M$ given by $\pi_{M,N}(gN) = gM$. This is a well defined map since we have $N \leq M$ and therefore we have that M

can be viewed as a disjoint union of the cosets of N in M and so each coset of M in G is a union of cosets of N in G.

Taking the inverse limit defined by this inverse system, we write in the explicit form $\widehat{G} = \{(gN)_{N \trianglelefteq G} \mid \pi_{M,N}(gN) = gM\}$. Notably, we have that for any group we have that $1 \trianglelefteq G$ for every group G and so the element in $\widehat{G}$ is entirely defined by the image in the projection (abusing notation) $\pi_1 : \widehat{G} \to G/1$. The compatible maps are surjective and so there is an obvious isomorphism between G and $\widehat{G}$. Subsequently if G is a finite group, then we have that $G \cong \widehat{G}$ with the inverse system defined as above. Therefore, every finite group is a profinite group.

The immediate next step would be to consider whether this can be applied to groups which are not finite. Let G be an infinite group, taking the previous reasoning verbatim for G does not give a an inverse system of finite groups since $1 \trianglelefteq G$ and $G/1$ is an infinite group and so the inverse limit cannot be a finite group. Therefore, we take a different definition for the index set Λ which will guarantee that we have an inverse system of finite groups. Let the index set $\Lambda = \{N \mid N \trianglelefteq G,\ |G : N| < \infty\}$ with the usual ordering, it is easily verified that this is a directed set since if we take $N, M \in \Lambda$, then $N \cap M \trianglelefteq G$ and $|G : N \cap M| \leq |G : N||G : M| < \infty$ and so $N \cap M \in \Lambda$.

We can now define an inverse system of finite groups by G/N for $N \in \Lambda$. Since we have an inverse system of finite groups, we can find the inverse limit which will itself be a profinite group. Let $\widehat{G}$ be the inverse limit of this inverse system. By our definition of the inverse limit, we have that there are homomorphisms $\pi_N : \widehat{G} \to G/N$ for each $N \in \Lambda$ and that there is a unique homomorphism $\sigma : G \to \widehat{G}$ such that the required diagram is commutative since clearly we have that the maps $\sigma_N : G \to G/N$ are compatible with the required mapsm this is to say that $\sigma_N = \pi_N \circ \sigma$. In this case, we call $\widehat{G}$ the profinite completion of G.

Definition 2.11. [25]1.2.1 Let G be a group. The profinite completion of G, $\widehat{G}$, is defined to be the inverse limit of the inverse system $(G/N)_{N \in \Lambda}$ over the directed set $\Lambda = \{N \mid N \trianglelefteq G,\ |G : N| < \infty\}$ and the collection of compatible maps

$\pi_{N,M} : G/M \to G/N$ where $\pi_{N,M}(gM) = gN$ for $M \leq N$.

Note that if G is a finite group, the both definitions for Λ agree since $|G : H| < \infty$ for all $H \leq G$. As we have previously, for a finite group, G is isomorphic to its own profinite completion.

It becomes useful to define a topology on the underlying set of a profinite group. We would want the compatible system of maps to be continuous and for the projection maps to be continuous. The most clear way of achieving this is to say that each of the finite groups G/N is equipped with the discrete topology, $\prod_{N \trianglelefteq G,\ |G:N|<\infty} G/N$ with the product topology and $\widehat{G} \subseteq \prod_{\substack{N \trianglelefteq G, \\ |G:N|<\infty}} G/N$ with the subspace topology. By taking the finest topology on the defining objects, we have that this underwrites all of the fundamental properties that we wish to inspect.

Definition 2.12. [25]1.2.12 Let $(G_j)_{j \in J}$ be an inverse system of finite groups. Consider each G_j with the discrete topology and $\prod_{j \in J} G_j$ with the product topology. The induced topology on $\widehat{G}$ with the subspace topology is defined to be the profinite topology.

Since we have that $\pi_N : \widehat{G} \to G/N$ for each $N \in \Lambda$ is a continuous map, we have that $\pi_N^{-1}(1N) = \ker(\pi_N) = N \leq G$ is open since $\{1N\}$ is open in the discrete topology. It follows that for each $N \in \Lambda$ is open, and by a similar argument for any $\{gN\}$, we have that an open base for the topology is given by cosets of the normal subgroups of finite index in $\widehat{G}$. We note that we can conclude that this is an open base for the profinite topology since it is defined by the discrete topology on the finite quotients. Combining the group structure with the profinite topology, we can consider a profinite group as a topological group.

Definition 2.13. [5]III 1, Definition 1 A topological group is a set G which carries a group structure and a topology and satisfies the following two axioms:

1. The group multiplication map $\mu : G \times G \to G$, $\mu(x, y) = xy$ is continuous.

2. The group inverse map $i : G \to G$, $i(g) = g^{-1}$ is continuous.

This is to say that a group is a topological group if it can be equipped with a topology such that the group multiplication and inverse maps are continuous. As discussed, a profinite group is a topological group when considered as a topological space with the profinite topology. When discussing profinite groups, we use this property constantly and therefore we always consider a profinite group to be a topological group under the profinite topology.

Naturally, if G is a profinite group, we use the notation $H \leq_o G$ to denote that H is both an open subset of G and a subgroup of G. This itself is considered a topological group with the induced subspace topology defined by the set of open sets given by $\tau = \{H \cap U \mid U \subseteq_o G\}$. Since the open sets of G in the profinite topology are equivalent to unions of the cosets of the open normal subgroups of G, we have that an open base for the subspace topology on H is given by $\{H \cap gN \mid g \in G,\ N \trianglelefteq_o G\}$.

Note that for each $N \trianglelefteq G$, then we have $N \cap H \trianglelefteq H \leq G$ and $N \cap H \leq H$. In particular, we have that N can be expressed as the union of disjoint cosets of $N \cap H$, that is to say $N = \bigcup_{h \in N/N\cap H} hH \cap N$. Take an element of the open base for the subspace topology $gN \cap H \in \tau$ such that $g \in G$, $N \trianglelefteq G$. It follows that $gN = g(\bigcup_h hH \cap N)$, substituting $M = H \cap N$, we have $gN = g(\bigcup hM)$. Therefore, $gN \cap H = (\bigcup ghM) \cap H$. Since N and H are disjoint unions of cosets of M, it follows that for each $h \in N/N \cap H$, we have that either $ghM \cap H = ghM$ or $ghM = \emptyset$. It follows that each element of the open base of the subspace topology can be expressed as a union of elements in the profinite topology.

Conversely, we know that an open base for the profinite topology is defined by the cosets in H of each open normal subgroup M. Let $M \trianglelefteq_o H$ and consider the element of the open base given by gM with $g \in H$. It is clear since $M \leq H$ that $gM = gM \cap H$ and since $M \subseteq_o G$, we have that this is an element of the open base of the subspace topology. It follows that both topologies are equivalent since each representative of the open bases can be expressed as a union of the other. It therefore follows that H can be viewed as a profinite group with the topology defined by either the subspace topology or the profinite topology since they are equivalent.

As we had previously discussed, the profinite completion of a finite group is isomorphic to the group itself. This is in fact true for all profinite groups and therefore gives an isomorphism that is highly utilised in the subject of profinite groups. Notably, since we have that all the maps involved are continuous, we have that the groups are topologically isomorphic, and we quote a result that proves this.

Proposition 2.14. *[7]1.3 If G is a profinite group, then G is topologically isomorphic to its profinite completion $\widehat{G} = \varprojlim_{N \trianglelefteq_o G}(G/N)$.*

Note that this expression has many advantages, firstly that since $G \cong \widehat{G}$ for G profinite, we have that the profinite completion is isomorphic to $\widehat{G} \cong \varprojlim_{N \trianglelefteq_o \widehat{G}}(\widehat{G}/N)$, and so in order to find an inverse system with limit equal to a given profinite group, it is sufficient to take the inverse system of quotients by the open normal subgroups of G. Secondly, we know that each of the compatible maps and projection maps of the profinite completion are surjective by construction and so each group in the inverse system is entirely structurally described in the inverse limit since $\pi_N : \widehat{G} \to \widehat{G}/N$ is a surjective group homomorphism. Due to these, when we are discussing a profinite group, we use G and $\widehat{G}$ interchangeably since they are isomorphic.

Having now defined profinite groups and remarking that they structurally preserve each of the finite groups in an inverse system, we can now discuss the concept of the Burnside ring of a profinite group. By definition 2.3, we have that for a finite group G the Burnside ring of a finite group $B(G)$ is the Grothendieck group of the category $G-\mathbf{set}$ of finite G-sets. Given a profinite group G, we can define the equivalent of a finite G-set by defining what is meant by left G-action on a set. Since G is a topological group, we ensure that the map is also continuous by requiring that there is a topology on the set to make it a topological space. In general a left action of a topological group on a topological space is defined as follows.

Definition 2.15. [24]§1.1, §1.2 Let G be a topological group and X a topological space. A left action of G on X is a continuous map $\varrho : G \times X \to X$ such that

1. $\varrho(g, \varrho(h, x)) = \varrho(gh, x)$ for $g, h \in G,\ x \in X$
2. $\varrho(e, x) = x$ for $x \in X,\ e \in G$ unit.

A left G-space is a pair (X, ϱ) consisting of a space together with a left action ϱ of G on X.

Throughout we consider left actions and so we omit the word left, referring to left G-spaces as G-spaces. Additionally, we denote $\varrho(g, x)$ by gx and refer to the G-space solely by its underlying topological space X, omitting the action ϱ. In order to define a set X as a topological space, we need to equip it with a topology and therefore a left G-action.

Definition 2.16. [9]Appendix B, §2 Let G be a profinite group. A G-set S is a finite set with discrete topology on which there is a continuous left G-action.

We now have an equivalent for the finite G-sets for G a profinite group which functions in a similar way to that of a finite group. Note that if G is a finite group, then both definitions agree since in a finite group G we have that G is equipped with the discrete topology. Dress, having defined these, then used this to define the Burnside ring of a profinite group.

Definition 2.17. [9]Appendix B §2 For G a profinite group, consider the commutative half ring to be the ring $B^+(G)$ of isomorphism classes of (finite) G-sets formed by taking finite disjoint unions of the finite orbits. Then the Burnside ring of G, $B(G)$ is the corresponding Grothendieck ring.

As in the case of the Burnside ring of a finite group, this becomes the same as considering the ring of isomorphism classes of finite virtual G-sets. Furthermore, since each element is in a finite G-orbit, it must be stabilized by a subgroup of finite index, there is some subgroup $H \leq G$ such that $|G : H| < \infty$ and H stabilizes the orbit. Note that since H is finite, it is the preimage of a closed set since we can consider the continuous map $\mu : G \to G/H$ given by $\mu(g) = gH$, and therefore clearly $H = \mu^{-1}(1H)$.

It follows that since the stabilizer of any orbit is closed and of finite index, then it must be open and so we have that an arbitrary element X in $B(G)$ can be given by an isomorphism class representative $X = \sum'_{H \leq_o G} x_H \cdot G/H$ for some $x_H \in \mathbb{Z}$. Since X is a virtual finite G-set, we have that at most finitely many of the $x_H \neq 0$ since otherwise the underlying set of X contains infinitely many elements. Conversely, we have that since we take the Grothendieck ring of finite G-orbits, every (finite) linear combination of G-orbits is represented by an isomorphism class in the Burnside ring. Subsequently, we have that the this describes all possible elements of the Burnside ring. We adopt the convention of using $X \in B(G)$ to denote the isomorphism class of X.

For H a closed subgroup of finite index, which is to say it is open, we have that there exists an open normal subgroup $N \leq H$, $N \trianglelefteq_o G$. It follows that the action of G on G/H can be viewed as G/N action on G/H since $gN \cdot hH = gh \cdot NH = gh \cdot H$ and so we have that the action is entirely by the coset of N in G. Therefore, the G action on G/H factors through the G/N action on G/H as a finite G/N-set where G/N is a finite group.

By the above reasoning, we have that each element in the Burnside ring of a profinite group can be considered as finitely many finite orbits. Take an element of the Burnside ring $X = \sum'_{H \leq_o G} x_H \cdot G/H \in B(G)$, then we have that at most finitely many of the $x_H \neq 0$. Let S denote the set of representatives of stabilizers of the orbits in X. Since each of these is open, we have that their intersection defined by $K = \cap_{H \in S} H$ is also an open subgroup, and clearly there is an open normal subgroup $N \trianglelefteq_o G$ such that $N \leq K$. It follows that we can view $X \in B(G)$ as a finite G/N-set where G/N is a finite group.

Therefore by a similar reasoning as above, we have that any element in the

Burnside ring of a profinite group can be viewed as a G/N-set for N some open normal subgroup of G. Note that in the case of finite groups, since we have that $1 \leq H$ for each $H \leq G$ we have that the G-action is clearly equivalent to $G/1$-action on each G/H. However, for infinite profinite groups, we do not necessarily have a universal subgroup N such that each transitive finite G-set G/H can be described by G/N-action on G/H. Note that in this way, we see that the N-action on each of these G-sets must be trivial.

We can then construct a map $\pi_N^G : B(G) \to B(G/N)$ for each $N \trianglelefteq_o G$ defined to take the elements in each finite G-set which have a trivial N-action. This is to say the elements in each finite G-set on which the group G/N summarises the action of G. Explicitly, given $X = \sum'_{H \leq_o G} x_H \cdot G/H \in B(G)$, we have that $\pi_N^G(X) = \sum'_{N \leq H \leq_o G} x_H \cdot G/H$ since the N-action on an element is trivial if and only if it lies in an orbit stabilized by H such that $N \leq H$. It is easily verifiable that π_N^G is a ring homomorphism.

We can therefore apply many of the same properties that we have in the case of the Burnside rings of finite groups to the Burnside rings of profinite groups. For $H \leq_o G$, we can define a map $\mathrm{res}_H^G : B(G) \to B(H)$ which as usual restricts the action on a finite G-set to H-action. Furthermore, since the underlying set of a finite G-set is a finite set, it follows that taking any $H \leq_c G$, there are finitely many points in any $X \in B(G)$ which are fixed by H-action. It follows that we can define the H-fixed points for any $H \leq_c G$ in a similar way as we do for the Burnside ring of a finite group, and therefore the following definition is well defined.

Definition 2.18. [9]Appendix B §2 Let G be a profinite group and $H \leq_c G$, then we can define the set of H-fixed points of an element $X \in B(G)$ to be the set $X^H := \{x \in X \mid h.x = x \text{ for each } h \in H\}$. Considering $X = \sum_{K \leq_c G} x_K \cdot G/K$ with the usual expression, we have that we can define the number of fixed points to be the map $\varphi_H : B(G) \to \mathbb{Z}$ given by $\varphi_H(X) = \sum_{K \leq_c G} x_K \cdot |(G/K)^H|$.

Taking the map $\varphi : B(G) \to \prod_{H \leq_c G} \mathbb{Z}$ to be defined by $\varphi(X) = (\varphi_H(X))_{H \leq_c G}$, we have a ghost map which is analogous to the ghost map of the Burnside ring of a finite group. As in the Burnside ring of a finite group, we have that the preimage of any element of the ghost ring in the Burnside ring under the ghost map is unique (up to equivalence). It therefore follows that the Burnside ring of a profinite group is isomorphic to its image in the ghost ring.

Since G is a profinite group, it has a defined inverse system of finite groups $(G/N)_{N \trianglelefteq_o G}$ indexed by the directed set of open normal subgroups of G. We recall that the ordering we impose on this directed set is that of reverse inclusion on the open normal subgroups, then for any $N \leq M$ (with inclusion ordering), we have that there is a projection map $\pi_{G/M}^{G/N} : B(G/N) \to B(G/M)$ as previously defined, by taking the M fixed points. We note that strictly speaking this map should bear the subscript $(G/N)/(M/N)$, but since this is isomorphic to G/M, we choose to write it in this way for ease of notation.

Furthermore, we have that there are canonical projections $\pi_N^G : B(G) \to B(G/N)$ that take the N-fixed points of an element of the Burnside ring of a finite group. If we have $N \leq M$, then we have that all points which are fixed by M must also be fixed by N, and so we have that there is a clear composition of maps $\pi_N^G \circ \pi_{G/M}^{G/N} = \pi_M^G$, showing that the maps themselves are compatible. As previously stated, we have that these are all ring homomorphisms and so there is a system of rings with compatible projection maps indexed by a directed set, and therefore an inverse system of rings. It is natural then to take the completion of this Burnside ring of a profinite group with respect to this inverse system which will itself be a ring.

This ring was introduced by Dress and Siebeneicher, although starting from a more explicit study of the elements of the completed Burnside ring [10]§2.3 and later proven to be isomorphic to the completion of the Burnside ring of a profinite group [10]§2.9.5 with the aforementioned inverse system. We now give the explicit construction as given in the paper of Dress and Siebeneicher, which moves away from the finite sets that we have discussed thus far and instead deals with possibly infinite sets.

By definition of the Burnside ring of the finite quotient G/N, we have that each $B(G/N)$ for $N \trianglelefteq_o G$ has a finite basis given by $\{[G/H] \mid N \leq H \leq G\}$, taking one representative of each equivalence class of G-orbit in $B(G/N)$ and each element of $B(G)$ can be expressed as the inflation of an element of $B(G/N)$ for some $N \trianglelefteq_o G$. However, in the completion of the Burnside ring, we just require that an element is well behaved with respect to the compatible projection maps.

If we assume that G is a profinite group that has a collection of infinitely many open normal subgroups $\{N_i \mid i \in \mathbb{N}\}$ such that $N_i < N_j$ for all $j < i$ and let B_i denote a basis for $B(G/N_i)$, then since $G/N_{i+1} \notin B(G/N_i)$, we have that $B_i \subset B_i \cup \{G/N_{i+1}\} \subseteq B_{i+1}$. Note that $\pi^G_{N_j}(G/N_i) = \emptyset$ for $j < i$ since

$$\begin{aligned}\pi^G_{N_j}(G/N_i) &= \{g \in G/N_i \mid h.gN_i = gN_i \ \forall h \in N_j\} \\ &= \{g \in G/N_i \mid N_j^g \leq N_i\}\end{aligned}$$

but by the assumption we have that $N_i < N_j$ and both are normal and so the set is empty.

It follows that if we consider the infinite series $\sum_{i\in\mathbb{N}} G/N_i$, then we have that this is a well defined element of the inverse limit which is an infinite G-space (with the discrete topology) in the completion of the Burnside ring which cannot be in the Burnside ring. We therefore have that the Burnside ring of the completion of the Burnside ring and the completion of the Burnside ring can be distinct, and in fact always are distinct provided there are infinitely many open subgroups.

By abuse of notation, let π^G_N denote the corresponding projection map $\pi^G_N : \widehat{B}(G) \to B(G/N)$ given by taking the N-fixed points. This is justified since we have that clearly $B(G) \subseteq \widehat{B}(G)$ and the restriction of π^G_N with this definition is equal to the previous projection. This is to say that in the inverse limit construction, the unique map $\sigma : B(G) \to \widehat{B}(G)$ such that the required projections are compatible is given by the inclusion map.

Furthermore, we note that since the completion is defined by the inverse limit of the projections, then for each $H \leq_o G$, we have that there exists $N \trianglelefteq_o G$ such that $N \leq H$. Any H-fixed point must also be fixed by N-action and so we have that $\varphi_{H/N}(\pi_N^G(X)) = \varphi_H(X)$ for each $X \in B(G)$. In a similar abuse of notation as above, taking $X \in \widehat{B}(G)$, we have that $\pi_N^G(X) \in B(G/N)$ and so we define $\varphi_H(X) := \varphi_{H/N}(\pi_N^G(X))$ for $N \leq_o H \leq_o G$, which is clearly in $\mathbb{Z}$ and so we have a well defined function $\varphi_H : \widehat{B}(G) \to \mathbb{Z}$. A natural continuation of this is to define the ghost map $\varphi : \widehat{B}(G) \to \mathrm{Gh}(G)$ as usual where $\varphi(X) = (\varphi_H(X))_{H \leq_o G}$.

Definition 2.19. [10]§2.2 A G-space X is defined to be essentially finite if for any open subgroup $H \leq_o G$, the number of H-fixed points is finite. An essentially finite that is also discrete are defined to be almost finite, this is to say that each element in an almost finite G-space lies within a finite orbit.

Again, by definition of this topological G-space, we have that it can be decomposed into distinct orbits. Suppose that X is a G-space, then we have that $X = \bigsqcup'_{x \in X} G/G_x$ with the disjoint union taken over a single representative of each G-orbit of X. Note that we can have infinitely many orbits, and that the orbits themselves can be infinite for G a profinite group if G is infinite.

Note that as we are currently discussing actual (non-virtual) G-spaces then we have that there is an expression $X = \bigsqcup_{H \leq G} x_H \cdot G/H$, and each $x_H \geq 0$ but possibly infinite. Since G is open in G trivially, we have that $|X^G| = \bigsqcup x_H \cdot |(G/H)^G| = x_G \cdot |G/G| = x_G$. If X is essentially finite, then we have that $x_G \in \mathbb{Z}$ and so in particular there are only finitely many orbits of G/G.

Now suppose that K is a maximal subgroup of the set $\{H \leq_o G\} \backslash \{G\}$, then we have that $|X^K| = \bigsqcup x_H \cdot (G/H)^K = x_K \cdot |(G/K)^K| + x_G |G/G|$ and so if X is essentially finite, we have that $|X^K|$ must be finite and as we have previously shown x_G must be finite, therefore x_K must be finite. Since each $H \leq_o G$ has an open normal subgroup of $N \trianglelefteq_o G$ such that $N \leq H$, we have that G/N has finitely many subgroups, and so there are finitely many subgroups which contain a conjugate of H. Therefore, repeating this process inductively shows that for an essentially finite G-space, we have $x_H \in \mathbb{Z}$ for each $H \leq_o G$.

Conversely, if $x_H \in \mathbb{Z}$ for each $H \leq_o G$, then we have that X must be essentially finite since each orbit containing an open subgroup in its stabilizer is finite and therefore there are finitely many fixed points under the open subgroups of G. It follows that every essentially finite G-space is of this form.

Recall that in the Burnside ring of profinite group, we have that the elements of the Burnside ring can be identified up to equivalence by their fixed points. Since essentially finite G-spaces have finite coefficients x_H for each $H \leq_o G$ as we have discussed above, it follows that we can use a similar justification that the coefficients of G/H with $H \leq_o G$ in the canonical expression are uniquely determined by the number of fixed points under the open subgroups. We define this equivalence relation as follows.

Definition 2.20. [10]§2.2 Let G be a profinite group, and X, Y be two essentially finite G-spaces. We say that X is equivalent to Y if we have that $\varphi_H(X) = \varphi_H(Y)$ for all $H \leq_o G$.

Since every element in an almost finite G-space must lie in a finite orbit, and each stabilizer of an orbit of a G-space is closed, we have that every element is in an orbit stabilized by a closed subgroup of finite index. This is to say that each orbit is stabilized by an open subgroup. It follows that each equivalence class of an almost finite G-space contains a unique almost finite representative (up to isomorphism of orbits) which can be given by $X = \sum_{H \leq_o G} x_H \cdot G/H$. By the definition of the equivalence relation, we can add as many orbits that are stabilized by closed but not open subgroups of G as we may wish to X and stay within the same equivalence class.

Conversely, if we take an essentially finite G-space Y, then we have that there must be an almost finite representative in its equivalence class given by taking the sum $Y = \sum_{H \leq G} y_H \cdot G/H$ and restricting the series to summing only over the open subgroups of G. This is to say the element $X = \sum_{H \leq_o G} y_H \cdot G/H$ is an almost finite representative of the equivalence class, and by the previous discussion this is unique up to isomorphisms of the orbits. It follows that every equivalence class of an essentially finite G-spaces contains a unique almost finite representative (up to isomorphism of the orbits). We can then take these equivalence classes and define what is meant by the completed Burnside ring.

Definition 2.21. [10]§2.3 Let G be a profinite group, then we define the completed Burnside ring of G, denoted by $\widehat{B}(G)$, to be the Grothendieck group of the virtual isomorphism classes of almost finite G-spaces. This is equivalent to taking the Grothendieck group of essentially finite G-spaces.

Since all addition and multiplication within the ring is well behaved on the almost finite representative, which is to say that $[X]+[Y] = [X+Y]$, we shall adopt the convention of using the almost finite representative to denote the equivalence class it represents. Clearly we have a multiplicative identity in this ring in the form of $1 = G/G$ and an additive identity in the form of $0 = \emptyset$. These align with what we expect from a Burnside ring from the previous two definitions. The notation $\widehat{B}(G)$ may seem to clash with that given for the completion of the Burnside ring of a profinite group, but as previously stated it can be shown that the completion is isomorphic to the completed Burnside ring.

Theorem 2.22. *[10]2.9.5 Let G be a profinite group, then the completed Burnside ring $\widehat{B}(G)$ is isomorphic to the completion of the Burnside ring of G. This is to say that $\widehat{B}(G) \cong \varprojlim_{N \trianglelefteq_o G} B(G/N)$.*

We shall use this isomorphism throughout this thesis intuitively. We note that if G is a finite group, then we have that the definition of the Burnside ring of a finite group agrees with the Burnside ring of a profinite group and the completed Burnside ring of the finite group. If G is an infinite profinite group, then we have that the definition of the Burnside ring of a profinite group is contained within the completed Burnside ring of a profinite group. We adopt a convention for the rest of the thesis of using 'Burnside ring' to refer to the completed Burnside ring since this shall be the ring we are considering in most cases. If the non-completed Burnside ring is meant, it shall be made clear.

Definition 2.23. Let G be a profinite group, and $H \leq_o G$, then we have ring homomorphisms res^G_H and φ_H. The map $\mathrm{res}^G_H : \widehat{B}(G) \to \widehat{B}(H)$, as in the previous cases, is given by taking the same underlying set of $X \in \widehat{B}(G)$ and considering it under the restriction to H-action. For $G/K \in \widehat{B}(G)$, we have that the restriction is $\mathrm{res}^G_H(G/K) = \sum_{g \in [H \backslash G/K]} H/H \cap {}^gK$.

For $X = \sum_{K \leq_o G} x_K \cdot G/K$, we have that the map $\varphi_H : \widehat{B}(G) \to \mathbb{Z}$ which is defined by $\varphi_H(X) = \sum'_{K \leq_o G} x_K \cdot |(G/K)^H|$.

Since these are defined on the basis elements, it is sufficient to show how they map the basis elements. In the case of the restriction, we already have an explicit formula. The formula for $\varphi_H(G/K)$ for $K \leq_o G$ is calculated in a similar way as finite groups. That is to say that we have the H-fixed points of G/K are given by $\varphi_H(G/K) = |\{gK \in G/K \mid h.gK = gK \ \forall h \in H\}| = |\{gK \in G/K \mid H^g \leq K\}|$. One immediate consequence of this is that in particular, if we take the H-fixed points of G/H, then $\varphi_H(G/H) = |\{gH \in G/H \mid H^g \leq H\}| = |N_G(H) : H|$.

Definition 2.24. For $H \leq_o G$ and $N \trianglelefteq_o G$, we have that there are linear maps ind^H_G and $\mathrm{Inf}^G_{G/N}$ called the induction and the restriction respectively. The induction map $\mathrm{ind}^G_H : \widehat{B}(H) \to \widehat{B}(G)$ is defined by $\mathrm{ind}^G_H(H/K) = G/K$. The inflation map $\mathrm{Inf}^G_{G/N} : B(G/N) \to \widehat{B}(G)$ is given by $\mathrm{Inf}^G_{G/N}((G/N)/(K/N)) = G/KN$.

2.3 Fusion systems

With the Burnside rings defined above, we have a method of discussing when G-sets or G-spaces have a suitably rigorously defined form of similarity. In the case of Burnside rings of finite groups they are isomorphic and in the case of the (completed) Burnside ring of a profinite group equivalently. However, as shown by [18]Reeh, we can also define G-sets that behave similarly under the action of different subgroups. This is to say we can define an equivalence relation on the subgroups of a group to ensure that the action under any subgroup in an equivalence class is comparable to any other element in the equivalence class on some specific G-sets.

Firstly, we let G be a finite group. Consider the orbit G/H which is stabilized by H-action. We have already stated that this is isomorphic to G/K for K conjugate to H. Furthermore, we have that G-sets can be classified (up to isomorphism) by the Ghost map, recalling that $X \cong Y$ as G-sets if and only if $\varphi(X) = (\varphi_H(X))_{H\leq G} = (\varphi_H(Y))_{H\leq G} = \varphi(Y)$. A reasonable candidate for when H and K act similarly is if $\varphi_H(X) = \varphi_K(X)$ for all X in some collection of G-sets.

Suppose that we have $H, K \leq G$ and that $\varphi_H(X) = \varphi_K(X)$ for each $X \in B(G)$. It follows that in particular, we have that $|N_G(H) : H| = \varphi_H(G/H) = \varphi_K(G/H)$. By a previous observation of the fixed point map, we must have that $K \lesssim H$, that is to say that K is conjugate to some subgroup of H. Conversely, we have that $\varphi_H(G/K) = \varphi_K(G/K) = |N_G(K) : K|$ and so $H \lesssim K$, therefore we must have that H is conjugate to K. It is trivially true that $\varphi_H(X) = \varphi_K(X)$ for all $X \in B(G)$ and so we have an if and only if statement.

We first wish to define an equivalence relation on the subgroups that we wish to act similarly, and from this we can which G-sets they act similarly on. There is a natural condition the we wish to impose, that the subgroups in question be isomorphic. It is natural to define a category with objects the subgroups and morphisms induced by the isomorphisms we wish to consider. By first instinct, we consider the isomorphisms induced by conjugation within a larger group. It is with this in mind that we introduce the definition of a fusion system.

Definition 2.25. [14]Linckelmann 1.1 Let G be a finite group and let S be a Sylow-p-subgroup of G. We define the fusion system of G over S to be the category denoted by $\mathcal{F}_S(G)$ with $\mathrm{ob}(\mathcal{F}_S(G)) = \{P \mid P \leq S\}$ and morphisms given by $\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$ for each $P, Q \in \mathrm{ob}(\mathcal{F}_S(G))$ with the morphisms induced by conjugation in G. That is to say that the set of morphsism between P and Q is $\mathrm{Hom}_G(P, Q) = \{\psi : P \to Q \mid \exists x \in G \text{ such that } \psi(u) = {}^x u \ \ \forall u \in P\}$. Composition of morphisms is given by usual composition of group homomorphisms.

Note that with this definition, if we take $P, Q \leq S$ such that P is S-conjugate to Q, then we have that they lie within the same $\mathcal{F}_S(G)$ isomorphism class since there is clearly a conjugation map induced from S. If $S = G$, then the isomorphism classes of $\mathcal{F}_S(G)$ are precisely the S-conjugacy classes of subgroups of S. It follows that each $\mathcal{F}_S(G)$ isomorphism class of a subgroup of S is given by the disjoint union of S-conjugacy classes of subgroups of S.

Given this, it is naturally motivated that we wish for the action to agree on $\mathcal{F}_S(G)$ conjugacy classes. Since the objects in this category are given by subgroups of S, then it is clear that we study S-sets since the action by objects is already restricted to S-action. However, as we have already noted, if we consider all S-sets, we have that they cannot agree unless the $\mathcal{F}_S(G)$ isomorphism classes agree with the S-conjugacy classes.

We must make rigorous what it means for subgroups to act similarly on an S-set. Suppose that $P, Q \leq S$ lie within the same $\mathcal{F}_S(G)$ isomorphism class, then there is $\psi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q)$ such that $Q = \psi(P)$. By act similarly, as we have expressed above, we mean that if X is an S-set, then P and $\psi(P)$ act isomorphically on X as P-sets. Since we wish for this to be the case for any given $\mathcal{F}_S(G)$ isomorphism class, we have that this must hold for each isomorphism class, and therefore for any subgroup $P \leq S$. This property is summarised as follows.

Definition 2.26. [18]Reeh §1, (1.2) Let X be an S-set and $P \leq S$, then we write ${}_{P,\psi}X$ to denote the P-set with the same underlying set as X with P-action defined by $g.x = \psi(g).x$ with the right hand side using the S-action on X. If $\mathcal{F}$ is a fusion system over a p-group S, we define an S-set X to be $\mathcal{F}$-stable if ${}_{P,\psi}X \cong_{P,incl} X$ as P-sets for all $P \leq S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(P, S)$.

Therefore, we wish to consider all S-sets which are $\mathcal{F}$-stable, building from the definitons of fusion systems. We can, however, define in a more abstract way a fusion system over a p-group S. Note that currently, the only role that G plays is to provide the morphism maps in the category. We can instead derive the properties of the conjugation maps that make them suitable and define the fusion system from them.

Let S be a p-group and suppose that $\mathcal{C}$ is a category with $\mathrm{ob}(\mathcal{C}) = \{P \mid P \leq S\}$, we wish to describe the morphisms of $\mathcal{C}$ such that they somewhat behave like conjugacy maps. Firstly, we note that conjugacy maps are injective and so we wish for the morphisms in $\mathcal{C}$ to be injective since it ensures that the domain of a morphism is isomorphic to its image. Trivially, we wish that the morphisms induced by conjugation by elements of S are included in the morphisms of $\mathcal{C}$. Finally, we wish that for each morphism, ψ, in the category, we have that there is an induced isomorphism in the category to the image of ψ which guarantees that all isomorphic images lie within the same $\mathcal{C}$-isomorphism class.

Definition 2.27. [18]Reeh 2.1 Let S be a finite p-group, then we say that $\mathcal{F}$ is a fusion system over S if it is a category with $\mathrm{ob}(\mathcal{F}) = \{P \mid P \leq S\}$ and morphisms that satisfy the following properties for each $P, Q \in \mathrm{ob}(\mathcal{F})$.

1. $\varphi : P \to Q$ is an injective group homomorphism, where composition of morphisms is the usual composition of group homomorphisms.

2. $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$ where $\mathrm{Hom}_S(P, Q) = \{c_g : P \to Q \mid g \in S\}$ is the set of morphisms induced by conjugation in S.

3. For each $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, then the induced isomorphisms $\hat{\psi} : P \to \psi P$ and $\hat{\psi}^{-1} : \psi P \to P$ are also morphisms in $\mathcal{F}$.

This shall be the ongoing definition we use for a fusion system over a finite group S. It is easily verified that $\mathcal{F}_S(G)$ is also a fusion system by this definition. Furthermore the definition given for $\mathcal{F}$-stability also holds for this definition of a fusion system. Due to the close connection with the conjugacy maps, we write $P \sim_{\mathcal{F}} Q$ if they are isomorphic in $\mathcal{F}$ and we say that they are $\mathcal{F}$-conjugate. Under this equivalence relation, we use $[P]_{\mathcal{F}}$ to denote the equivalence class. We shall use $P \lesssim_{\mathcal{F}} Q$ to denote that P is $\mathcal{F}$-conjugate to a subgroup of Q. If the context is clear, we shall omit the subscript $\mathcal{F}$.

However, this is a weaker definition than that of a fusion system over G since we lose any result gained by S being a Sylow-p-subgroup of G. We therefore define a class of fusion systems that have these properties.

Definition 2.28. [18]Reeh §2, 2.2 Let $\mathcal{F}$ be a fusion system over a finite p-group S. We define $P \in \mathrm{ob}(\mathcal{F})$ to be fully-normalized in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for each Q in the $\mathcal{F}$ isomorphism class of P.

Likewise, we define P to be fully-centralized in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for each Q in the $\mathcal{F}$ isomorphism class of P. A fusion system $\mathcal{F}$ over a p-group S is defined to be saturated if the following are satisfied for each $P \in \mathrm{ob}(\mathcal{F})$:

1. If P is fully-normalized, then $\mathrm{Aut}_S(P)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ and P is fully-centralized.

2. Every homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ where $\psi(P)$ is fully-centralized extends to a homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_\psi, S)$ where $N_\psi = \{x \in N_S(P) \mid \exists y \in S \text{ such that } c_x \circ \psi = \psi \circ c_y\}$.

If $\mathcal{F}$ is a fusion system given by G over S, then we have that $\mathcal{F}$ is saturated. Throughout this thesis, most of the fusion systems we discuss will be saturated. As may be expected, since we wish to discuss the $\mathcal{F}$-stable S-sets, we can define a ring structure which allows us to combine $\mathcal{F}$-stable S-sets as we do in the case of the Burnside rings of finite groups and profinite groups. We do this in the most apparent way by taking all $\mathcal{F}$-stable elements of $B(S)$.

Definition 2.29. [18]Reeh 4.5 Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. We define the Burnside ring of $\mathcal{F}$ to be the $\mathcal{F}$-stable elements of $B(S)$ and is denoted $B(\mathcal{F})$.

It is also shown in [18]Reeh that as a consequence of [18]Reeh 4.11, we have that this is equivalent to taking the Grothendieck group of all actual $\mathcal{F}$-stable S-sets, which is to say that there is a basis of actual $\mathcal{F}$-stable S-sets of this ring. These basis elements of $B(\mathcal{F})$ are given by taking S/P for P fully-normalized and $\mathcal{F}$-stabilizing it through a recursive combinatorial process in order to ensure that the number of fixed points are constant on the $\mathcal{F}$ isomorphism classes.

The process of calculating these basis elements α_P for P fully-normalized is outlined as follows. Firstly, we note that since we try to stabilize the S-orbit S/P

for P fully-normalized. This is to say that we start with S/P and attempt to find a linear combination of elements to add to S/P to create an element α_P such that $\phi_Q(\alpha_P) = \varphi_{Q'}(\alpha_P)$ for all $Q \sim_{\mathcal{F}} Q'$. For $Q \not\lesssim P$, we have that $\varphi_Q(S/P) = 0$ and so we do not need to add S/Q.

It follows that we need only stabilize for $Q \lesssim_{\mathcal{F}} P$. Let $\alpha_P = \sum_{Q\in \mathrm{ob}(\mathcal{F})} x_Q \cdot S/Q$ Suppose that $Q \sim_{\mathcal{F}} P$, then it follows that $\varphi_Q(\alpha_P) = x_Q \cdot |N_S(Q) : Q|$. We note that $\varphi_P(\alpha_P) = |N_S(P) : P|$ and so we wish for $\varphi_Q(\alpha_P) = |N_S(P) : P|$. It follows that $x_Q = \frac{|N_S(P)|}{|N_S(Q)|}$, therefore we have that α_P is the $\mathcal{F}$-stabilization of the element $X := \sum_{[Q]_S\subseteq[P]_{\mathcal{F}}} \frac{|N_S(P)|}{|N_S(Q)|} \cdot S/Q$. Here, it becomes clear as to why we have the condition that P is fully-normalized in order to ensure that all coefficients are integers since $|N_S(P)|$ is maximal in the $\mathcal{F}$-conjugacy class.

Take H a maximal subconjugate subgroup of P which is fully-normalized. This is to say take K a maximal proper subgroup of P and take H a fully-normalized representative of the $\mathcal{F}$ conjugacy class of K. With the motivation that this should be a basis, we wish for x_H to be 0 for any fully-normalized subgroup which is not isomorphic to P. This ensures that they are linearly independent since S/H only appears in the summation α_H.

Let $H' \sim_{\mathcal{F}} H$ but not S-conjugate, it follows that if we consider $\varphi_H(X + x_H \cdot S/H) = \varphi_H(\alpha_P) = \varphi_{H'}(\alpha_P) = \varphi_{H'}(X + x_{H'} \cdot S/H')$. Since we have $x_H = 0$, then $\varphi_H(\alpha_P) = \varphi_H(X) = \varphi_{H'}(X) + \varphi_{H'}(x_{H'} \cdot S/H')$. Therefore, since $\varphi_{H'}(S/H') = |N_S(H') : H'|$, it follows that $x_{H'} := \frac{\varphi_H(X)-\varphi_{H'}(X)}{|N_S(H'):H'|}$ for each $[H']_S \subset [H]_{\mathcal{F}}\backslash[H]_S$ and $x_H = 0$. Take $X_1 := X + \sum_{[H']_S\subseteq[H]_{\mathcal{F}}} x_{H'} \cdot S/H'$, then α_P is the $\mathcal{F}$-stabilization of X_1.

Let $\mathcal{H} = \{[K]_{\mathcal{F}} \mid K \lesssim_{\mathcal{F}} P\}$. We repeat this process for X_1 and some maximal fully-normalized subgroup K of $\mathcal{H}\backslash\{[H]_{\mathcal{F}}\}$, defining the resulting S-set to be stabilized to be X_2 and iterating the process now over the set $\mathcal{H}\backslash\{[H], [K]\}$. We repeat the process until $\mathcal{H}$ is empty, and the resulting S-set, $X_{|\mathcal{H}|}$, is $\mathcal{F}$-stable since the fixed point maps are constant across $\mathcal{F}$-conjugacy classes of subgroups. It follows that $\alpha_P := X_{|\mathcal{H}|}$ and they are linearly independent for each P fully-normalized.

Corollary 2.30. *[18]Reeh 4.11 Let $\mathcal{F}$ be a saturated fusion system over a finite group S, then the set $\{\alpha_P \mid P \in ob(\mathcal{F})$ is fully-normalized$\}$ is a basis for $B(\mathcal{F})$.*

Therefore with a fusion system over a a finite p-group S, we have a method to describe every $\mathcal{F}$-stable virtual S-set and an accompanying Burnside ring. By definition, we have that $B(\mathcal{F}) \subseteq B(S)$ is a subring. Note that in the case when $\mathcal{F} = \mathcal{F}_S(G)$, if we take $X \in B(G)$ then this is a (virtual) G-set with $\varphi_H(X) = \varphi_K(X)$ for each $H \sim_G K$. Since the $\mathcal{F}$-conjugacy class, we have that $\mathrm{res}^G_S(X)$ must be an $\mathcal{F}$-stable S-set since if $H \leq S, G$ then we have $\varphi_H(X) = \varphi_H(\mathrm{res}^G_S(X))$ as the number of fixed points under H action is unchanged.

Furthermore, considering the map $\mathrm{res}^G_S : B(G) \to B(S)$ it then follows that $\mathrm{Im}(\mathrm{res}^G_S) \subseteq B(\mathcal{F}) \subseteq B(S)$. It was claimed by [18]Reeh Example 4.3 that we cannot have equality in the first inclusion since there are $\mathcal{F}$-stable S-sets which cannot be given by the restriction of any actual G-set. However, [3]Barsotti and Carman Theorem 7.1 subsequently proved that this is in fact an equality and that in order to be a surjection, one must consider virtual and not just actual G-sets.

Theorem 2.31. *[3]Barsotti, Carman 7.1 Let G be a finite group and S a Sylow-p-subgroup of G and take the fusion system of G over S, $\mathcal{F}_S(G) = \mathcal{F}$. Then for the map $res^G_S : B(G) \to B(S)$, we have $Im(res^G_S) = B(\mathcal{F})$.*

Thus far in this subsection, all groups have been finite. With the observation that the Burnside ring of a fusion system is a subring of a Burnside ring of a finite group, a natural question arises of if we can establish some subring of the Burnside ring of a profinite group that behaves similarly. In order to do so, we first need an analogue of a fusion system for a profinite group.

Given the way that the profinite groups and the Burnside rings of profinite groups are defined, we have already established machinery for extending the theory from a finite setting to a profinite setting through the inverse limit. It is intuitive that we define an similar structure to fusion systems of a finite group for profinite group by taking the inverse limit of an inverse system of fusion systems over finite groups.

Suppose that each $\mathcal{F}_i$ is a fusion system over S_i. For any inverse system, we must describe a family of compatible maps. That is to say a map $\mathcal{F}_i \to \mathcal{F}_j$ for each $i \leq j$ under some ordering that are compatible. Since each $\mathcal{F}_i$ is a fusion system, in particular we have that each $\mathcal{F}_i$ is a category and so by taking each of these compatible maps to be functors, we have that the inverse limit will itself be a category. Consequently, we must define the mappings of each object and morphism in each fusion system in the inverse system.

Consider $F_{j,i} : \mathcal{F}_i \to \mathcal{F}_j$ a functor and we attempt to define the mappings for each object in $\mathcal{F}_i$. Since the objects in question are subgroups of S_i, and that this is an inverse system, we have that they must be mapped to objects in $\mathcal{F}_j$, in particular to subgroups of S_j. Therefore, we must have compatible maps between the subgroup of the respective p-groups.

If we determine the image of S_i then it must entirely determine the image of each of its subgroups, and since we have that each image of a subgroup must be a subgroup in the image of the functor, it follows that they must be group homomorphisms. It is therefore justified that we have an inverse limit of the S_i as groups. In order to map the morphisms, we can simply consider the induced maps between the image of the objects.

Take $\{f_{j,i} : S_i \to S_j \mid i \leq j\}$ to be a family of group homomorphisms over some directed set I. For any inverse system of fusion systems, we have that there must exist such a family and directed set since they must be compatible. We can therefore define $F_{j,i}(P) = f_{j,i}(P)$ for each $P \in \mathrm{ob}(\mathcal{F}_i)$.

Definition 2.32. [21]Stancu, Symonds 2.7 Suppose that we have an inverse system of fusion systems over finite groups, $\mathcal{F}_i$ over S_i a finite p-group respectively, indexed by a directed set I. By the definition of an inverse system, we have a compatible family of functors $\{F_{j,i} \mid i, j \in I,\ i \leq j\}$, $F_{j,i} : \mathcal{F}_i \to \mathcal{F}_j$ such that $F_{j,i}(P) = f_{j,i}(P)$ for each $P \in \mathrm{ob}(\mathcal{F}_j)$. We set $S = \varprojlim_{i \in I} S_i$ and let $f_i : S \to S_i$ be the induced projections, $N_i = \ker(f_i)$.

Define $\mathcal{F} := \varprojlim_{i \in I} \mathcal{F}_i$ to be a pro-fusion system over S. This is to say the category with $\mathrm{ob}(\mathcal{F}) = \{P \mid P \leq_c S\}$ and morphisms for each $P, Q \in \mathrm{ob}(\mathcal{F})$ given by $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \varprojlim_{i \in I} \mathrm{Hom}_{\mathcal{F}_i}(f_i(P), f_i(Q))$.

In the case of the profinite completion of a group G, we have that we can take the inverse system formed by taking the finite quotients $\{G/N \mid |G : N| < \infty\}$. In this way, we can start with a group and derive an inverse system. In the previous definition, a clear candidate for such an inverse system is given by taking S/N_i since $S/N_i \cong S_i$ is a finite group. In particular, we require that these N_i are closed under the action of morphisms. We also define what it means for a pro-fusion system to be saturated.

Definition 2.33. [21]Stancu, Symonds §2.4 Let S be a pro-p group and $\mathcal{F}$ be a pro-fusion system over S. $Q \leq_c S$ is defined to be strongly closed in S with respect to $\mathcal{F}$ if for all $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, then for $R \leq Q$, we have that $\psi(R) \leq Q$.

Definition 2.34. [21]Stancu, Symonds 2.14, 2.15 Let $\mathcal{F}$ be a pro-fusion system over S. $Q \in \mathrm{ob}(\mathcal{F})$ is defined to be receptive in $\mathcal{F}$ if for each $R \sim_{\mathcal{F}} Q$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$, there exists $\tilde{\psi} \in \mathrm{Hom}_{\mathcal{F}}(N_\psi, N_S(Q))$ such that $\tilde{\psi}|_R = \psi$ where $N_\psi = \{x \in N_S(R) \mid \exists y \in S \text{ such that } c_x \circ \psi = \psi \circ c_y\}$.

Let $K \leq \mathrm{Aut}_{\mathcal{F}}(Q)$, we define Q to be fully K-automized in $\mathcal{F}$ if $\mathrm{Aut}_S^K(Q) := K \cap \mathrm{Aut}_S(Q)$ is a Sylow pro-p subgroup of K.
We define Q to be fully K-normalized in $\mathcal{F}$ if Q is receptive and fully K-automized in $\mathcal{F}$. If $K = \mathrm{Aut}_{\mathcal{F}}(Q)$, then we say that Q is fully $\mathcal{F}$-normalized

In practice, when the context of the pro-fusion system is clear we omit $\mathcal{F}$ and say that Q is fully normalized if it is fully $\mathcal{F}$-normalized. Recall that the definition given for a (finite) fusion system is far less demanding since it only requires $|N_S(Q)|$ to be maximal across the $\mathcal{F}$-conjugacy class of Q. However, we note that the conditions given in the above definition relate to those that we have given for a finite fusion system to be saturated. It then becomes sufficient that with these properties that each $\mathcal{F}$-conjugacy class contains a fully normalized element in order for the pro-fusion system to be saturated. As mentioned, we broadly only use saturated fusion and pro-fusion systems and therefore the definition of fully normalized intuitively agrees in this case.

Definition 2.35. [21]Stancu, Symonds 2.16 A pro-fusion system $\mathcal{F}$ over S is defined to be saturated if each $\mathcal{F}$ isomorphism class contains a fully normalized subgroup of S.

We use $\mathcal{F}^{f.n.}$ to denote the set of all representatives of fully normalized subgroups in $\mathcal{F}$. If $\mathcal{F}$ is saturated then we have a representative in $\mathcal{F}^{f.n.}$ for each $\mathcal{F}$-conjugacy class of subgroups. In the case of saturated fusion systems, it becomes easy to take the quotients and therefore form an inverse limit of the quotients. As previously mentioned, we have a candidate for quotients given by S/N_i and so we need to define the morphisms on S/N_i. The following definition and result show that in the case of a saturated fusion system, this is well behaved.

Definition 2.36. [21]Stancu, Symonds §2.4 Let $\mathcal{F}$ be a fusion system, N a strongly $\mathcal{F}$-closed subgroup of S, then we define $\mathcal{F}/N$ to be the fusion system on S/N with morphisms given by the condition $\psi \in \mathrm{Hom}_{\mathcal{F}/N}(PN/N, S/N)$ if and only if there exists $\tilde{\psi} \in \mathrm{Hom}_{\mathcal{F}}(PN, S)$ such that $\psi(uN) = \psi(u)N$ for all $u \in P$.

Corollary 2.37. *[21]Stancu, Symonds 2.6 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S*

1. *If $F : \mathcal{F} \to \mathcal{G}$ is a morphism of fusion systems, then $F(\mathcal{F})$ is a saturated fusion system isomorphic to $\mathcal{F}/ker(F)$.*
2. *If N is a strongly closed subgroup of S, then $\mathcal{F}/N$ is induced by the subgroups of S that contain N and their morphisms.*

This result states that for a finite fusion system, taking the morphisms between fusion systems is well defined by taking quotients. In particular, we can form an inverse system of these quotients with the strongly closed subgroups being given by $\ker_{j,i}$. This almost naturally translates to pro-fusion system with one notable exception. This result requires that we contain a kernel of a projection map, and so we have that a saturated pro-fusion system projects only onto the finite quotients as saturated fusion systems. Note that by the definition, we also include closed but not open subgroups of S.

Proposition 2.38. *[21]Stancu, Symonds 3.7 For any saturated pro-fusion system $\mathcal{F}$ over a pro-p group S we have that $\mathcal{F} \cong \varprojlim_{N \in \mathcal{N}} \mathcal{F}/N$ where $\mathcal{N}$ is the set of open strongly closed subgroups of S.*

Definition 2.39. [21]Stancu, Symonds 4.1 A pro-fusion system $\mathcal{F}$ is pro-saturated if it is an inverse limit of saturated fusion systems of finite groups.

There is subtlety between the definition of pro-saturated and saturated, and they do not in general coincide, however for the results stated in this thesis, it is regularly sufficient for the pro-fusion systems to be either saturated or pro-saturated. We denote the restriction of the pro-fusion system $\mathcal{F}$ over a pro-p group S to the open subgroups of S by $\mathcal{F}^o$. This is the full subcategory of $\mathcal{F}$ formed by taking $\mathrm{ob}(\mathcal{F}^o) = \{P \mid P \leq_o S\}$. We write $\mathcal{F}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q)$ to be the set of $\mathcal{F}$-homomorphisms from P to Q

3 Burnside ring of pro-fusion systems

3.1 $\mathcal{F}^o$-stable S-spaces

We seek to establish a ring structure for the isomorphism classes of a fusion-stable G-space, much in the same way as the completed Burnside ring does for the isomorphism classes for conjugacy stable G-spaces. In order to explore this notion, we codify being fusion stable as the following and follow a similar process to [18]Reeh, we begin by recalling definition 2.26.

Definition 3.1. [18]Reeh §1, (1.2) Let X be an S-set and $P \leq S$, then we write ${}_{P,\psi}X$ to denote the P-set with the same underlying set as X with P-action defined by $g.x = \psi(g).x$ with the right hand side using the S-action on X. If $\mathcal{F}$ is a fusion system over a p-group S, we define an S-set X to be $\mathcal{F}$-stable if ${}_{P,\psi}X \cong_{P,incl} X$ as P-sets for all $P \leq S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(P, S)$.

This is the finite case and the motivation for us to extend this notion to the profinite case. We do so in the natural way and this results in the following, we make the adjustment that we only consider the open subgroups since we discern isomorphic almost finite G-spaces through the image in the ghost ring under the fixed point map. This is to say that when considering elements within the Burnside ring, we take only the equivalence classes of virtual G-spaces determined by the number of fixed points under the action by the open subgroups. Therefore, in trying to establish a Burnside ring of a pro-fusion system as a subring of the Burnside ring, it suffices for us to only consider stability under the action of open subgroups.

Definition 3.2. Let X be an almost finite S-space and $P \leq_o S$, then we write ${}_{P,\psi}X$ to denote the P-space with the same underlying set as X and P-action defined by $g.x = \psi(g).x$ with the right hand side using the S-action on X. If $\mathcal{F}$ is a pro-fusion system over a pro-p group S, we define a S-space X to be $\mathcal{F}^o$-stable if ${}_{P,\psi}X \cong_{P,incl} X$ as P-spaces for all $P \leq_o S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.

As discussed in the background section, since the Burnside ring is isomorphic to its image in the ghost ring, it is sufficient to consider the number of fixed points of an almost finite G-space under the open subgroups. Therefore, we prove the following results based on fixed points in order to more easily classify almost finite G-spaces. This follows on from the result [18]Reeh Lemma 4.1 for a fusion system of a finite group and so we prove the corresponding result for a pro-fusion system over a pro-p group S.

Proposition 3.3. *The following are equivalent* $\forall X \in \widehat{B}(S)$*;*

(i) ${}_{P,\psi}X \cong_{P,incl} X$ *for each* $\psi \in Hom_{\mathcal{F}}(P, S)$ *and* $P \leq_o S$*,*

(ii) $\varphi_P(X) = \varphi_{\psi P}(X)$ *for each* $\psi \in Hom_{\mathcal{F}}(P, S)$ *and* $P \leq_o S$*,*

(iii) $\varphi_P(X) = \varphi_Q(X)$ *for each* $P, Q \leq_o S$ *with* $P \sim_{\mathcal{F}} Q$.

Proof. We note that the S-spaces in $\widehat{B}(S)$ are by definition almost finite. For $P \leq_o S$, we have the restriction map $\mathrm{res}^S_P : \widehat{B}(S) \to \widehat{B}(P)$ which takes the same underlying space and considers only the P-action on that space. It follows that $\mathrm{res}^S_P(X)$ is a (virtual) almost finite P-space for each $X \in \widehat{B}(S)$. Given that in this case, there is no difference between X and $\mathrm{res}^S_P(X)$ under P-action, by abuse of notation we use in this proof X to also denote $\mathrm{res}^S_P(X)$ when the context of the action is made clear.

Let $\varphi^P : \widehat{B}(P) \mapsto \mathbb{Z}^{O(P)} := \prod_{K \leq_o P} \mathbb{Z}$ be the fixed point map into the ghost ring of P. For $R \leq_o P$, φ^P_R denotes the map into the R coordinate in the ghost ring. If P is open in S, then we have that we have an open basis of open normal subgroups of P of the form $P \cap N$ with $N \trianglelefteq_o S$. It follows that any open subgroup of P must contain some $P \cap N$. In particular, R must contain some $P \cap N$, but P and N are open in S, therefore R is a union of cosets of an open group of S and so must be open in S.

Note that when $R \leq_o P \leq_o S$, we have $\varphi^P_R({}_{P,incl}X) = \varphi_R(X)$ and that if R is open in P, this implies that R is open in S. Now consider the R fixed points $({}_{P,\psi}X)^R = \{x \in_{P,\psi} X \mid \varphi(r).x = x \; \forall r \in R\} = X^{\psi R}$ for any (almost finite) S-space X and all $R \leq_o P$. It then follows that $\varphi^P_R({}_{P,\psi}X) = \varphi_{\psi R}(X)$ for $X \in \widehat{B}(S)$ since the number of R fixed points is not altered by the setting of the larger group.

Assume that property $i)$ holds, then $\varphi_P(X) = \varphi^P_P({}_{P,incl}X) = \varphi^P_P({}_{P,\psi}X) = \varphi_{\psi P}(X)$. This is allowable since if $P \leq_o S$, then there exists some $N_i \leq P$ such that N_i is open and strongly closed subgroup of S. By the definition of strongly closed, it follows that $N_i = \psi(N_i) \leq \psi(P)$ and so $\psi P \leq_o S$ and the ψP-fixed point map is defined for any almost finite S-space. In particular this holds $\forall P \leq_o S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ so we have that $i)$ implies $ii)$.

Assume that property $ii)$ holds and take $P \leq_o S$, $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. By this assumption, we have that $\varphi_{\psi R}(X) = \varphi_R(X)$ for each $R \leq_o P$ for any almost finite S-space X. Consequently, we have $\varphi^P_R({}_{P,\psi}X) = \varphi_{\psi R}(X) = \varphi_R(X) = \varphi^P_R({}_{P,incl}X)$.

However, since the ghost map φ^P is injective, we necessarily have that the almost finite P-spaces ${}_{P,\varphi}X \cong_{P,incl} X$ and so *ii*) implies *i*).

Finally, we have *ii*) and *iii*) are equivalent since $P \sim_{\mathcal{F}} Q$ precisely means that there exists $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $Q = \psi P$. Therefore we have shown that all three properties are equivalent. $\square$

3.2 Burnside ring of a pro-fusion system $\widehat{B}(\mathcal{F})$

Equipped with this notion of $\mathcal{F}^o$-stability, we can now begin to define a Burnside ring structure of a pro-fusion system. Fundamentally, we begin by defining a means of combining actual almost finite S-spaces, in particular we take the induced addition (disjoint union) and multiplication (cartesian product) from the Burnside ring of a profinite group and create a subsemiring in which we do not consider formal negation of S-spaces.

Definition 3.4. Let S be a pro-p group, then we define $\widehat{B}_+(S)$ to be the subsemiring of $\widehat{B}(S)$ consisting of isomorphism classes of actual almost finite G-spaces, which is to say the elements in $\widehat{B}(S)$ with representative $X = \sum_{H \leq_o S} x_H \cdot S/H$ such that $x_H \geq 0$ for each $H \leq_o S$.

For $\mathcal{F}$ a pro-fusion system over the pro-p group S, define $\widehat{B}_+(\mathcal{F}) \subseteq \widehat{B}_+(S)$ to be the set of all isomorphism classes of (non virtual) $\mathcal{F}^o$-stable almost finite S-spaces.

Intuitively, it is clear that we also wish for $\widehat{B}_+(\mathcal{F})$ to also be a subsemiring, that is to say that it should be closed under multiplication and addition and the addition and multiplication are suitably distributive. We already have $0 = \emptyset$ is in $\widehat{B}_+(\mathcal{F})$ since $\varphi_P(\emptyset) = 0$ for all $P \leq_o S$, and therefore $\varphi_P(\emptyset) = \varphi_Q(\emptyset)$ for each $P \sim_{\mathcal{F}} Q$ and so is $\mathcal{F}^o$-stable. Distributivity comes naturally from the multiplication and addition being induced by the operations in $\widehat{B}(S)$.

Take $X, Y \in \widehat{B}_+(\mathcal{F})$ and consider the restriction of the P-fixed point map $\varphi_P : \widehat{B}_+(\mathcal{F}) \to \mathbb{Z}$ for $P \leq_o S$. It is clear that φ_P is a ring homomorphism since it is defined additively on the basis elements of $\widehat{B}(S)$. It follows that $\varphi_P(X + Y) = \varphi_P(X) + \varphi_P(Y) = \varphi_Q(X) + \varphi_Q(Y) = \varphi_P(X + Y)$ for each $P \sim_{\mathcal{F}} Q$ and so

$X + Y$ is also $\mathcal{F}^o$-stable. Similarly, we have $\varphi_P(X \times Y) = \varphi_P(X) \times \varphi_P(Y) = \varphi_Q(X) \times \varphi_Q(Y) = \varphi_Q(X \times Y)$ for each $P \sim_{\mathcal{F}} Q$ and so $X \times Y$ is $\mathcal{F}^o$-stable. Consequently we have that $\widehat{B}_+(\mathcal{F})$ is a subsemiring of $\widehat{B}_+(S)$.

This then gives us methods of combining actual almost finite S-spaces which are $\mathcal{F}^o$-stable. We cannot immediately from this definition state that every every virtual $\mathcal{F}^o$-stable S-space can be expressed as the difference of two actual almost finite $\mathcal{F}^o$ stable S-spaces. That is to say that it may be possible for there to be an element $X - Y \in \widehat{B}(S)$ with $X, Y \in \widehat{B}_+(S)$ such that $X - Y$ is $\mathcal{F}^o$-stable but X and Y themselves are not $\mathcal{F}^o$-stable. With this in mind, we can define a subset of $\widehat{B}(S)$ by taking all virtual almost finite $\mathcal{F}^o$-stable S-spaces. It becomes a ring with the induced ring structure from $\widehat{B}(S)$ with similar reasoning as above.

Definition 3.5. Let $\mathcal{F}$ be a saturated pro-fusion system over S. We define the Burnside ring of $\mathcal{F}$, $\widehat{B}(\mathcal{F})$, to the subring consisting of $\mathcal{F}$-stable elements in $\widehat{B}(S)$.

For the Burnside ring of a (finite) fusion system, [18]Reeh Theorem A showed that $B(\mathcal{F})$ can be generated by taking the Grothendieck group of $B_+(\mathcal{F})$ by finding a basis for $B(\mathcal{F})$ in terms of the elements from $B_+(\mathcal{F})$. This is to say that in fact we do have the property that every virtual finite $\mathcal{F}$-stable S-set is expressible as the difference of two actual finite $\mathcal{F}$-stable S-sets. We seek to show that the same can be done for the Burnside ring of a pro-fusion system. In order to do so, we first examine the structure of the Burnside ring of a pro-fusion system.

3.3 Structure of $\widehat{B}(\mathcal{F})$

As discussed in the definition 2.25, we have that there exists a class of (finite) fusion systems given by $\mathcal{F}_S(G)$ such that S is a Sylow-p-subgroup of G, with morphisms induced by conjugation in G. We note that we can naturally extend a similar structure to a profusion system by taking $\mathcal{F}_S(G)$ where S is a Sylow pro-p subgroup of the profinite group G with the morphisms induced by G-conjugation. Furthermore, [3]Barsotti and Carman theorem 7.1 showed that for a (finite) fusion system $\mathcal{F} = \mathcal{F}_S(G)$, then $B(\mathcal{F}) = \mathrm{Im}(\mathrm{res}^G_S)$. However, for the Burnside ring of a profinite group, the definition of the ring homomorphism $\mathrm{res}_S^G : \widehat{B}(G) \to \widehat{B}(S)$ requires that $S \leq_o G$.

The need for this can be evidenced by taking an infinite profinite group G and considering $X \in \widehat{B}(G)$ where $X = \sum'_{N \trianglelefteq_o G} G/N$. We have shown in the background section that this must be an infinite (actual) almost finite G-set. Note that the number of 1-fixed points in an orbit is equal to the cardinality of the orbit since every point in a topological space is fixed by 1-action. It follows that for X, since this is an actual G-space, that the number of 1-fixed points is equal to the cardinality of X, i.e. $|X|$ since there are no formal negations of orbits. It follows that X cannot be viewed as an almost finite 1-space since it does not have finitely many 1-fixed points and so does not have a representative in $\widehat{B}(1) \cong \mathbb{Z}$.

We first examine the class of pro-fusion systems defined by $\mathcal{F} = \mathcal{F}_S(G)$ for G a profnite group and S a Sylow pro-p subgroup of G such that $S \leq_o G$. We consider the map $\mathrm{res}_S^G : \widehat{B}(G) \to \widehat{B}(S)$ given by restricting the G-action on G-spaces to S-action. Then for $J \leq_o S \leq_o G$ we have $\varphi_J(G/H) = \varphi_J(\mathrm{res}_S^G(G/H))$ since this is the number of fixed points under J action, which remains the same regardless of the which group it is seen as a subgroup of. Since $\varphi_J(G/H) = \varphi_{J^g}(G/H)$ for each $g \in G$, it follows that $\varphi_{J^g}(\mathrm{res}_S^G(G/H)) = \varphi_J(\mathrm{res}_S^G(G/H))$ and so we have that $\mathrm{res}_S^G(G/H)$ is $\mathcal{F}^o$-stable since each of the morphisms in $\mathcal{F}$ are induced by G-conjugation.

The map res_S^G is a ring homomorphism since the underlying space remains the same under the image, it is just the action on it which changes. It follows that $\mathrm{Im}(\mathrm{res}_S^G)$ is a subring of $\widehat{B}(S)$ which is $\mathcal{F}^o$-stable and so is a subring of $\widehat{B}(\mathcal{F})$. Clearly by the first isomorphism theorem we have that $\mathrm{Im}(\mathrm{res}_S^G) \cong \widehat{B}(G)/\ker(\mathrm{res}_S^G)$ where $\ker(\mathrm{res}_S^G) = \{X \in \widehat{B}(G) \mid \varphi_H(X) = 0 \, \forall H \lesssim_o S\}$. The condition that $S \leq_o G$ is a rather restrictive one, however as we shall note from the following well known result that we prove.

Lemma 3.6. *Suppose that G is an infinite profinite group. Then there is at most one prime p for which the pro-p Sylow subgroup is open in G.*

Proof. Suppose there are $S_1 \leq_o G$ a Sylow pro-p-subgroup and $S_2 \leq_o G$ a pro-q-subgroup respectively for $p \neq q$. Note that since they are both open, then they must be of finite index. It follows that q^k divides $|G : S_1|$ and since S_1 is pro-p, we have that the order of a Sylow pro-q subgroup of G must be q^k. However, this then implies that S_2 is of order q^k and of finite index in G, therefore G must be finite which is a contradiction. □

A consequence of this result is that since for G infinite, there is at most one prime p such that the Sylow pro-p subgroup, S, of G is open, it follows that S (up to conjugacy) is the only subgroup such that there is a fusion system defined over S and has a well defined map $\mathrm{res}_S^G : \widehat{B}(G) \to \widehat{B}(S)$. In essence, this method of defining a pro-fusion system is limiting for this approach to considering some of the $\mathcal{F}^o$-stable element since for each infinite profinite G there is at most one pro-fusion system for which we have a map $\mathrm{res}_S^G : \widehat{B}(G) \to \widehat{B}(\mathcal{F})$.

We note that this differs from the case for a finite fusion system since then we have a well defined fusion system with corresponding restriction map for each prime divisor of the order of $|S|$. Therefore, in the case of a pro-fusion system, it is beneficial to consider other methods. In particular, we note that in a pro-fusion system we have the inverse limit, and so we can use this to make use of results from the finite quotients. Recall that $\widehat{G} \cong \varprojlim_{N \trianglelefteq_o G} G/N$ implies that $\widehat{B}(\widehat{G}) \cong \varprojlim_{N \trianglelefteq_o G} B(G/N)$, it is with this in mind that we pursue a similar result for the Burnside ring of a pro-fusion system.

Theorem 3.7. *Let $\mathcal{F}$ be a saturated pro-fusion system over S given as the inverse limit of the inverse system $\{\mathcal{F}_i \mid i \in I\}$ where each $\mathcal{F}_i$ is a (finite) fusion system over S_i. Define the canonical projection maps $f_i : S \to S_i$ where $N_i = ker(f_i)$, then we have that $\widehat{B}(\mathcal{F}) \cong \varprojlim_{i \in I} B(\mathcal{F}_i)$ with the family of compatible maps given by $\pi_{N_i/N_j}^{S/N_j} : B(\mathcal{F}_j) \to B(\mathcal{F}_i)$ giving the N_i/N_j fixed point set for each $i \leq j$, $i, j \in I$.*

Proof. By considering the family of compatible maps, we have that there is a canonical projection map $\pi_{N_j}^S : \varprojlim_{i \in I} B(\mathcal{F}_i) \to B(\mathcal{F}_j)$. Therefore, taking any element in the inverse limit, $X \in \varprojlim_{i\in I}$, we have that this can be viewed as the element $X = (X^{N_i})_{i\in I}$. From the definition it is clear that X^{N_i} must be $\mathcal{F}_i$-stable. If $N \trianglelefteq_o P$, then we have that $\varphi_P(X) = \varphi_{P/N}(X^N)$ since any element which is fixed by P-action must also be fixed by N-action.

Take $P \in \text{ob}(\mathcal{F}^o)$, recall that since $P \leq_o S$, then we must have that it must contain some $N = N_i$ which is open and strongly closed. Take $\psi \in \text{Hom}_{\mathcal{F}}(P, S)$, then since N is open and strongly closed, we have that $\psi(N) = N \leq \psi(P)$. It follows that each $\mathcal{F}$-conjugate of P can be seen as the inverse image of an element in $\text{ob}(\mathcal{F}_i)$. Since $\text{Hom}_{\mathcal{F}}(P, Q) = \varprojlim_{i\in I} \text{Hom}_{\mathcal{F}_i}(f_i(P), f_i(Q))$, it follows that each morphism from P to S in $\mathcal{F}$ must project into the finite quotient, in particular $\mathcal{F}_i$. Using the fact that X^{N_i} is $\mathcal{F}_i$-stable, we have that $\varphi_P(X) = \varphi_{P/N}(X^N) = \varphi_{\psi P/N}(X^N) = \varphi_{\psi P}(X)$. It follows that X is $\mathcal{F}^o$-stable and so $X \in \widehat{B}(\mathcal{F})$.

Conversely, suppose that $X \in \widehat{B}(\mathcal{F})$. Then by definition we have that $\varphi_P(X) = \varphi_{\psi P}(X)$ for all $\psi \in \text{Hom}_{\mathcal{F}}(P, S)$. It follows that for each $\psi \in \text{Hom}_{\mathcal{F}_i}(P/N_i, S/N_i)$, then there exists $\hat{\psi} \in \text{Hom}_{\mathcal{F}}(P, S)$ such that $F_i(\hat{\psi}) = \psi$. It follows that we have $\varphi_{P/N_i}(X^{N_i}) = \varphi_{\psi P/N_i}(X^{N_i})$ and so X^{N_i} is $\mathcal{F}_i$-stable for each $i \in I$. It follows that it is a well defined element of the inverse limit. □

3.4 Constructing a basis for $\widehat{B}(\mathcal{F})$

A basis (in terms of a linearly independent $\mathbb{Z}$-spanning set) for the Burnside ring of a finite fusion system was given by [18]Reeh 4.11 in terms of the basis elements α_P for P fully normalized. The combinatorial process of defining these α_P is discussed in the background section 2.29-2.30. As previously mentioned, these allow you to prove that there is an equivalence between the Grothendieck group of $B_+(\mathcal{F})$ and $B(\mathcal{F})$ in 3.4-3.5. We wish to emulate this in terms of the Burnside ring of a pro-fusion system. We shall show that the bases of the finite quotients form a well defined inverse system using the inverse limit defined in theorem 3.7.

In order to define the basis elements in the finite quotients, we first require that each of the finite quotients contains a fully normalized subgroup using the definition 2.35. We require that each of the finite quotients is saturated by definition 2.36. We recall a result from [21]Stancu, Symonds that shows that the fully normalized subgroups in a pro-fusion system $\mathcal{F}$ project nicely through the quotients.

Lemma 3.8. *(Stancu, Symonds)[21]Stancu, Symonds 2.23 Let $\mathcal{F}$ be a pro-fusion system over the pro-p group S. Let N be an open and strongly closed subgroup of S and suppose that Q is a subgroup of S that is fully $\mathcal{F}$-normalized. Then we have that Q/N is fully $\mathcal{F}/N$-normalized in $\mathcal{F}/N$.*

This provides us with the information that the image of a fully normalized subgroup under the compatible projection maps is again fully normalized. We show that each $\mathcal{F}_i$-conjugacy class contains at least one subgroup fully normalized in $\mathcal{F}_i$ whose preimage under the projection map is fully normalized in $\mathcal{F}$.

Proposition 3.9. *Let $\mathcal{F}$ be a saturated pro-fusion system over S with $N \trianglelefteq S$ open and strongly closed, then each $\mathcal{F}/N$-conjugacy class $[Q]_{\mathcal{F}/N}$, $Q \leq S/N$ contains a subgroup R fully normalized in $\mathcal{F}/N$ whose preimage under the projection map $f_N : S \to S/N$, $f_N^{-1}(R) = \widehat{R}$ is fully normalized in $\mathcal{F}$.*

Proof. Take $Q \leq S/N$ then there exists $\widehat{Q} \leq S$ the preimage of Q under the map $f_N : S \to S/N$. Since $\mathcal{F}$ is saturated, there exists $\widehat{R}$ fully normalized in $\mathcal{F}$ such that $\widehat{R} \sim_{\mathcal{F}} \widehat{Q}$

$$\Rightarrow f_N(\widehat{R}) \sim_{\mathcal{F}/N} f_N(\widehat{Q}).$$

But $f_N(\widehat{R}) \in [Q]_{\mathcal{F}/N}$ is fully normalized in $\mathcal{F}/N$ by the previous lemma and so $[Q]_{\mathcal{F}/N}$ contains a subgroup fully normalized in $\mathcal{F}/N$ whose preimage under f_N is fully normalized in $\mathcal{F}$. □

The following is a result gives a basis for the Burnside ring of a fusion system over a finite group for which we shall find an analogue for in the profinite, allowing us to construct a basis for the Burnside ring of a pro-fusion system. The process, as discussed in the background section 2.29-2.30, Reeh[18] follows involves the recursive construction of $\mathcal{F}$-stable S-sets for a finite fusion system $\mathcal{F}$ over the finite p-group S. We give an explicit formula for calculating them as follows.

Definition 3.10. [18]Reeh 4.8 For a finite fusion system $\mathcal{F}$ over the finite p-group S, let $P \leq S$ be a fully normalized representative of the $\mathcal{F}$-conjugacy class. Then we define the corresponding $\mathcal{F}$-stable S-set, α_P, as follows. Set $X(P)$ and $H_{Q'}$ for $Q' \leq S$ with fully normalized $\mathcal{F}$-conjugacy class representative $Q \leq S$ by

$$X(P) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S(P)|}{|N_S(P')|} S/P',$$

$$H_{Q'} = \frac{1}{|N_S(Q') : Q'|}(\varphi_Q - \varphi_{Q'}) \left(X(P) + \sum_{Q' < K \leq P} H_K S/K \right)$$

with $\varphi_Q, \varphi_{Q'}$ the fixed point maps in the Burnside ring respectively and the map $(\varphi_Q - \varphi_{Q'}) : B(S) \to \mathbb{Z}$ defined by $(\varphi_Q - \varphi_{Q'})(X) = \varphi_Q(X) - \varphi_{Q'}(X)$,

$$\alpha_P = X + \sideset{}{'}\sum_{\substack{Q \in \mathrm{ob}(\mathcal{F}^{f.n.}) \\ Q \lesssim_{\mathcal{F}} P}} \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} H_{Q'} S/Q'$$

with ' in the first sum denoting to take one fully normalized representative of each $\mathcal{F}$-conjugacy classes.

Reeh[18] showed that if you take one fully normalized representative for each $\mathcal{F}$-conjugacy class, you get a basis and that for any two $P, Q \in [P]_{\mathcal{F}}$ both fully normalized, then we have that $\alpha_P \cong \alpha_Q$. Therefore by abusing notation and allowing α_P to denote the S-set isomorphism class of α_P, we have that we can consider $\alpha : \mathrm{ob}(\mathcal{F}^{f.n.}) \to B(\mathcal{F})$ the map which has $\alpha(P) = \alpha_P$.

Lemma 3.11. *(Reeh)[18]Reeh 4.11 For $\mathcal{F}$ a saturated (finite) fusion system over S, then $\{\alpha_P \mid P$ fully normalized$\}$ is a $\mathbb{Z}$-linear basis for $B(\mathcal{F})$.*

Applying the previous two results we have that the projection from a pro-fusion system to the fusion system given over a finite quotient group under the map $f_N : \mathrm{ob}(\mathcal{F}) \to \mathrm{ob}(\mathcal{F}/N)$ maps the fully normalized subgroups to fully normalized subgroups and that each fully normalized subgroup has a preimage in $\mathcal{F}$ under this map. Since any preimage, P, must contain N, we have that it can be written as $P = PN$ since we can express P as a union of cosets of N.

Corollary 3.12. *For a saturated pro-fusion system $\mathcal{F}$ over S, $P \leq_o S$ the set $\{\alpha_{PN_i/N_i} \mid PN_i/N_i$ fully $\mathcal{F}_i$-normalized$\}$ is a basis for $B(\mathcal{F}_i)$.*

We want to try and construct a basis for $\widehat{B}(\mathcal{F})$ and so a natural place to start is by studying the finite quotients. We consider the following commutative diagram (where we adjust α to the fully normalized element in the $\mathcal{F}$-conjugacy class which exists since $\mathcal{F}$ is saturated). Define a family of compatible maps $\gamma_{i,j} : \mathrm{ob}(\mathcal{F}_j) \to \mathrm{ob}(\mathcal{F}_i)$ to be $\gamma_{i,j}(PN_j/N_j) = PN_i/N_i$ for $P \leq_o S$. As previously mentioned, this will project the fully normalized subgroups to fully normalized subgroups in their respective fusion systems.

Using corollary 3.12, we have that we can define for each $i \in I$ a map from subgroups of S_i to basis elements of $B(\mathcal{F}_i)$ given by $\alpha_i : \mathrm{ob}(\mathcal{F}_i) \to B(\mathcal{F}_j)$ where $Q \mapsto \alpha_P$ where P is a fully normalized representative of the $\mathcal{F}$-conjugacy class of $Q \leq S_i$.

Since we have that for $P \sim_{\mathcal{F}_j} Q$, we know that by [21]Stancu, Symonds Proposition 4.4, then $\gamma_{i,j}(P) \sim_{\mathcal{F}_i} \gamma_{i,j}(Q)$ and therefore belong to the same $\mathcal{F}_i$-conjugacy class. It follows then that we can compose the maps without loss of generality by taking any fully normalized representative since the image of the $\mathcal{F}_j$-conjugacy class is the same regardless of the fully normalized representative chosen.

Since we have that each element in $B(\mathcal{F}_j)$ can be written as a canonical linear combination of the α_P with P fully normalized in $\mathcal{F}_j$, we can express each element $X \in B(\mathcal{F}_j)$ in the form $X = \sum'_{P \in \mathcal{F} f.n.} x_P \cdot \alpha_P$ taking one fully normalized representative from each $\mathcal{F}$-conjugacy class for some $x_P \in \mathbb{Z}$. We can therefore take the preimage of the map α_j to give the the $\mathcal{F}$-conjugacy class. We can extend this map linearly to arrive at a formal element $Y = \sum'_{P \in \mathcal{F} f.n.} x_P \cdot P$. Similarly, we can define $\alpha_j(Y) = X$ by extending the map linearly and applying to the P individually. Extending the maps linearly in this way, we have

$$\begin{tikzcd} \mathrm{ob}(\mathcal{F}_j) \arrow[r, "\gamma_{i,j}"] \arrow[d, "\alpha_j"'] & \mathrm{ob}(\mathcal{F}_i) \arrow[d, "\alpha_i"] \\ B(\mathcal{F}_j) \arrow[r, "\alpha_i\circ\gamma_{i,j}\circ\alpha_j^{-1}"'] & B(\mathcal{F}_i). \end{tikzcd}$$

Define sets $A_i = \{\alpha_i(P) \mid P \in \mathcal{F}_i\}$ to be a basis of $B(\mathcal{F}_i)$ for each $i \in I$, then the maps (set homomorphisms) $f_{i,j} = \alpha_i \circ \gamma_{i,j} \circ \alpha_j^{-1}|_{A_j} : A_j \to A_i$ are compatible, as shown here

$$\begin{aligned} f_{i,j} \circ f_{j,k} &= (\alpha_i \circ \gamma_{i,j} \circ \alpha_j^{-1}) \circ (\alpha_j \circ \gamma_{j,k} \circ \alpha_k^{-1}) \\ &= \alpha_i \circ \gamma_{i,j} \circ \gamma_{j,k} \circ \alpha_k^{-1} \\ &= \alpha_i \circ \gamma_{i,k} \circ \alpha_k^{-1} \\ &= f_{i,k}. \end{aligned}$$

It follows that we can form the inverse system $(A_i)_{i\in I}$ with the family of compatible maps $\{f_{i,j} \mid i \leq j\}$. We can subsequently take the inverse limit of the inverse system given by $A_{\mathcal{F}} := \varprojlim_{i\in I} A_i$. It follows that this is a basis for $\widehat{B}(\mathcal{F})$ since the image in each of the finite quotient is spanning and linearly independent and the inverse limit of fusion systems is defined by the finite quotients. Therefore we have proved the following result.

Theorem 3.13. *For $\mathcal{F}$ a saturated pro-fusion system over S, then the set $A = \varprojlim_{i\in I} A_i$ is a basis for $\widehat{B}(\mathcal{F})$.*

Whilst this has proven that a basis exists, it is not an easily usable definition in order to find the corresponding basis elements. We therefore define a map $\hat{\alpha} : \mathcal{F}^{f.n.} \to \widehat{B}(\mathcal{F})$ which defines a basis element for each fully normalized representative. We wish for this to project onto the finite quotients in a well behaved way. With this in mind, we show that this is possible, however we note that if $N \not\leq P$, then the corresponding element in the projection should be the empty set. We therefore make the definition that $\alpha_i(0) = \emptyset$. We note that since we make no claim that α has any additional properties and is purely a function, we have that this causes no issue in the definition. Likewise, we define for the following proof we set the elements of $\mathrm{ob}(\mathcal{F}_i^{f.n.})$ to be the fully normalized representatives of subgroups of $\mathcal{F}_i$ and a purely formal entry in the form of 0 in order for the following proof to be well defined.

Theorem 3.14. *Let $\mathcal{F}$ be a pro-fusion system over the pro-p group S given by the inverse system $\{\mathcal{F}_i \mid i \in I\}$. Take $\pi^S_{N_i} : S \to S/N_i$ given by $\pi^S_{N_i}(X) = X^{N_i}$ and $h_i : ob(\mathcal{F}) \to ob(\mathcal{F}_i^{f.n})$ defined by $h_i(P) = P/N_i$ if $N_i \leq P$, $h_i(P) = 0$ otherwise, then there exists a map $\hat{\alpha} : \mathcal{F}^{f.n} \to \widehat{B}(\mathcal{F})$ such that for $\pi^S_{N_i} \circ \hat{\alpha} = \alpha_i \circ h_i$ where $i \in I$.*

Proof. We can set $\hat{\alpha}$ to be $(\alpha_i \circ h_i)_{i \in I}$ and so we have that this is trivially true since then we have that $\hat{\alpha}(P) \in \widehat{B}(\mathcal{F})$ for $P \leq_o S$.

$$\begin{tikzcd}
\mathrm{ob}(\mathcal{F}^{\mathrm{f.n.}}) \arrow[r, "h_i"] \arrow[d, "\hat{\alpha}"'] & \mathrm{ob}(\mathcal{F}_i^{\mathrm{f.n.}}) \arrow[d, "\alpha_i"] \\
\widehat{B}(\mathcal{F}) \arrow[r, "\pi^S_{N_i}"'] & B(\mathcal{F}_i).
\end{tikzcd}$$

□

Whilst simple with this setup and trivially true by the definition, we note that the structure of the elements $\hat{\alpha}(P)$ for P fully normalized are descriptive of a basis element. Take $P \leq_o S$, then we claim that $\hat{\alpha}^{N_i} = \emptyset$ for $N_i \not\leq P$ and $\hat{\alpha}^{N_j} = \alpha_{P/N_j}$ for $N_j \leq P$. In particular, we have that the basis elements of the respective fusion systems must project well onto each other and so it is sufficient to describe the basis elements of $\widehat{B}(\mathcal{F})$ by the basis elements of the respective $B(\mathcal{F}_i)$. We prove so in the following results.

Lemma 3.15. *Let $\mathcal{F}$ be a pro-fusion system over the inverse system $\{\mathcal{F}_i \mid i \in I\}$. Consider $\mathcal{F}_j$ and some $N_i \geq N_j$. If $N_i/N_j \not\leq PN_j/N_j$ then $(\alpha_{PN_j/N_j})^{N_i/N_j} = 0$.*

Proof. Since the N_i are strongly closed, we immediately have this since each summand in α_{PN_j/N_j} is of the form S/H where H is $\mathcal{F}_j$-subconjugate to PN_j/N_j and so writing the expression in the form

$$(\alpha_{PN_j/N_j})^{N_i/N_j} = \sum_{H \leq_o S} x_H \cdot (S/H)^{N_i/N_j}$$

for some $X_H \in \mathbb{Z}$. $x_H = 0$ if H is not $\mathcal{F}_j$-subconjugate to PN_j/N_j. Suppose that $x_H \neq 0$, then we have that $H \lesssim_{\mathcal{F}} PN_j/N_j < N_i/N_j$ and so $(S/H)^{N_i/N_j} = 0$. $\square$

Theorem 3.16. *Let $\mathcal{F}$ be a pro-fusion system over the pro-p group S with open basis for the topology given by $\{N_j \mid j \in I\}$, then when $N_i \geq N_j$*

$$(\alpha_{P/N_j})^{N_i/N_j} = \begin{cases} \emptyset & \text{if } N_i \not\leq P \\ \alpha_{P/N_i} & \text{if } N_i \leq P \end{cases}$$

Proof. The first case is given by lemma 3.15. Suppose that $P \geq N_i, N_j$ and $N_j \leq N_i$, consider the element $\alpha_{P/N_j} \in B(\mathcal{F}_j)$. We have a well defined map taking the N_i/N_j-fixed points which takes an $\mathcal{F}_j$-stable element to an $\mathcal{F}_i$-stable element since we have that $Q/N_i \sim_{\mathcal{F}_i} R/N_i$ implies that $Q/N_j \sim_{\mathcal{F}_j} R/N_j$ and so, in particular, we have that for $\varphi_{Q/N_j}(X) = \varphi_{Q/N_i}(X^{N_i/N_j}) = \varphi_{R/N_i}(X^{N_i/N_j}) = \varphi_{R/N_j}(X)$ for each $X \in B(\mathcal{F}_j)$ since any set fixed by Q-action must first be fixed by both N_i and N_j action.

Since $X := \alpha_{P/N_j}^{N_i/N_j}$ is in $B(\mathcal{F}_i$, there must exist some linear combination of basis elements of $B(\mathcal{F}_i)$ equal to X.By [18]Reeh Proposition 4.8, we have that $\varphi_Q(X) = 0$ for all Q not subconjugate to P and so X must be a linear combination of basis elements with representative subgroups subconjugate to Q, but the coefficient of S/Q must be equal to 1 for Q fully normalized and so there is only one possibility in the form of $X = \alpha_{P/N_i}$. $\square$

Proposition 3.17. *Let $\mathcal{F} = \varprojlim_{i\in I} \mathcal{F}_i$ be a pro-fusion system over the pro-p group $S = \varprojlim_{i\in I} S_i$, with $N_i = ker(f_i)$, $f_i : S \to S_i$ for each $i \in I$. Then for N_i strongly closed in $\mathcal{F}$, we have that N_i/N_j is closed in S_j for $i \leq j$.*

Proof. For each $\psi \in \mathcal{F}_i$, there exists $\hat{\psi} \in \mathcal{F}$ such that $F_i(\hat{\psi}) = \psi$, and take $R \leq N_i/N_j$. We can take the preimage under the map f_j to get $f_j^{-1}(R) \leq f_j^{-1}(N_i/N_j) = N_i$. $F_i \circ \hat{\psi}(f_j^{-1}(R)) \leq N_i/N_j$ since $\hat{\psi}(f_j^{-1}(R)) \leq N_i$ since N_i is strongly closed. $\square$

Proposition 3.18. *For $\mathcal{F}$ a pro-fusion system over the pro-p group S $\{\widehat{\alpha}(P) \mid P \in \mathcal{F}^{f.n.}\}$ is linearly independent.*

Proof. Suppose that Q is fully $\mathcal{F}$-normalized and consider the element

$$\widehat{\alpha}(Q) = \sum_{\substack{P \in \mathcal{F}^{f.n.} \\ P \not\sim Q}} x_P \cdot \widehat{\alpha}(P)$$

with $x_P \in \mathbb{Z}$. Suppose that $N_i < Q$ is maximal across the projection kernel subgroups $\{N_i \mid i \in I\}$. For the sake of clarity of expression, we use $\mathrm{Fix}_{N_i} : \widehat{B}(\mathcal{F}) \to B(\mathcal{F}_i)$ to denote the N_i fixed points of any almost finite $\mathcal{F}^o$-stable S-space.

$$\mathrm{Fix}_{N_i}(\widehat{\alpha}(Q)) = \mathrm{Fix}_{N_i}\left(\sum_{\substack{P \in \mathcal{F}^{f.n.} \\ P \not\sim Q}} X_P \widehat{\alpha}(P)\right)$$
$$\Rightarrow \alpha_{Q/N_i} = \sum_{\substack{P \in \mathcal{F}^{f.n.} \\ P \not\sim Q \\ \deg(P) \leq i}} X_P \ \alpha_{P/N_i}$$

But we have that these elements form a basis in the quotient ring and so we must have

$$\alpha_{Q/N_i} = \alpha_{P/N_i}$$

for some P and so we have a contradiction, $\Rightarrow \{\widehat{\alpha}(Q) \mid Q \in \mathcal{F}^{f.n.}\}$ is linearly independent. □

We use the results Dress-Siebeneicher[10]Dress and Siebeneicher 2.7.2-2.7.3 on the Burnside rings of profinite groups to prove the extension of the following from Burnside ring of (finite) fusion systems. This is a generalisation of [18]Reeh theorem B that checks that all of the techniques used in the proof is well defined.

Lemma 3.19. *Let $\mathcal{F}$ be a saturated pro-fusion system over a pro-p group S and let $\widehat{B}(\mathcal{F})$ be the completed Burnside ring of $\mathcal{F}$. We have a short exact sequence*

$$0 \longrightarrow \widehat{B}(\mathcal{F}) \xrightarrow{\varphi} Gh(\mathcal{F}) \xrightarrow{\Psi} \textstyle\prod_{P \in ob(\mathcal{F}^o)} \mathbb{Z}/|N_S(P) : P|\mathbb{Z} \longrightarrow 0$$

where $\varphi = \varphi^S$ is the fixed point map into the ghost ring for the open subgroups of

S,

$$\Psi = \Psi^{\mathcal{F}} : Gh(\mathcal{F}) \to \prod_{P \in ob(\mathcal{F}^o)} \mathbb{Z}/|N_S(P) : P|\mathbb{Z}$$

is a group homomorphism given by the $[P]$-coordinate functions for $P \leq_o S$, for $\xi = (\xi_P)_{P \leq_o S} \in Gh(\mathcal{F})$, $\xi_P \in \mathbb{Z}$,

$$\Psi_P(\xi) = \sum_{\bar{s} \in N_S(P)/P} \xi_{\langle s \rangle P} \mod |N_S P : P|$$

where P is a fully normalized representative of $[P]_{\mathcal{F}}$. $\Psi_P = \Psi_{P'}$ is $P \sim_{\mathcal{F}} P'$ are both fully normalized.

Proof. In Stancu-Symonds[21] they have shown that in a pro-fusion system, if $P \leq_o S$, $Q \sim_{\mathcal{F}}$ then $Q \leq_o S$ and $|S : Q| = |S : P|$. In particular, we have that any two $\mathcal{F}$-conjugacy class representatives of the same $\mathcal{F}$-conjugacy class have the same index. By the Zermelo's well ordering theorem[12], every set can be well ordered. Applying this to $\{P \in \mathrm{ob}(\mathcal{F}) \mid |S : P| = \sigma,\ P \leq_o S\} := \mathrm{ob}_\sigma(\mathcal{F}^o) \subseteq \mathrm{ob}(\mathcal{F})$, we can choose a well ordering of $\mathrm{ob}_\sigma(\mathcal{F})$. In particular a total ordering. Repeating this process for each $\sigma \in \mathbb{N}_0$, we can construct a total order of the $\mathcal{F}$-conjugacy classes $[P], [Q] \in Cl(\mathcal{F}^o)$ by asserting that

$$|S : Q| > |S : P| \Rightarrow [Q] > [P].$$

In particular, in this ordering it holds that

$$Q \sim_{\mathcal{F}} H < P \Rightarrow [P] < [Q].$$

Since we have that the P-coordinate in $\Psi(\xi)$ can be given by

$$\Psi_P(\xi) = \sum_{\bar{s} \in N_S(P)/P} \xi_{\langle s \rangle P} \mod |N_S P : P|,$$

we have that $\Psi_P(\xi)$ is a linear combination of ξ_Q such that $[P] \leq [Q]$ and taking $\bar{s} = 1 \in N_S(P)/P$ we see that $\xi_{\langle s \rangle P} = \xi_P$. With respect to the ordering above, the group homomorphism

$$\Psi : Gh(\mathcal{F}) \to \prod_{P \in \mathrm{ob}(\mathcal{F}^o)} \mathbb{Z}/|N_S(P) : P|\mathbb{Z}$$

can be given as $\Psi(X) = MX$ where M is given by a lower triangular matrix with 1s on the diagonal with the rows and columns ordered with respect to descending order of the ordering as above.

It follows that Ψ is surjective since by taking linear combinations of the elements $\zeta_P = (\delta_{P,Q})_{Q \leq_o S} \in Gh(\mathcal{F})$., we can generate every element of $Gh(\mathcal{F})$ and every element in the codomain of Ψ is an image of some linear combination. φ is injective by Dress-Siebeneicher and $\Psi^S \circ \varphi^S = 0$,

$$\Rightarrow (\Psi)_P = (\Psi^S)_P, \quad P \in \mathcal{F}^{f.n.}$$

and $\varphi = \varphi^S|_{\mathcal{F}}$

$$\Rightarrow \Psi \circ \varphi = 0.$$

It remains to show that $\mathrm{Im}(\varphi) = \ker(\Psi)$ but this is immediate by [10]Dress and Siebeneicher since the formula for Ψ is precisely the classifying congruence relation given in (2.7.2)-(2.7.3) of Dress-Siebeneicher[10]. $\square$

It remains to show that the linearly independent $\hat{\alpha}(P)$ for $P \leq_o S$ fully normalized are indeed spanning. In order to do so, we use techniques from [18]Reeh 4.8 and therefore show that they form a well defined $\mathbb{Z}$-linear basis.

Lemma 3.20. *Let $\mathcal{F}$ be a pro-fusion system over the pro-p group S, then the $\mathbb{Z}$-linear combinations of the $\widehat{\alpha}_P$, $P \in \mathcal{F}^{f.n.}$, $P \leq_o S$ is isomorphic to the Burnside ring of $\mathcal{F}$. That is to say is we take a fully normalized $\mathcal{F}$-conjugacy class representative for each $\mathcal{F}$-conjugacy class, we have* $\mathrm{span}_{\mathbb{Z}}\{\widehat{\alpha}_P \mid P \in ob(\mathcal{F}^{f.n.})\} = \widehat{B}(\mathcal{F})$.

Proof. Let $H = \mathrm{span}_{\mathbb{Z}}\{\widehat{\alpha}_P \mid P \in \mathrm{ob}(\mathcal{F}^{f.n.}\}\}$. Consider the restriction $\varphi|_H$ of the homomorphism $\varphi : \widehat{B}(\mathcal{F}) \to Gh(\mathcal{F})$. Clearly, since the expression for $\hat{\alpha}_P$ contains only orbits of the form S/H such that H is $\mathcal{F}$-subconjugate to P, we have that

$\varphi_Q(\hat{\alpha}_P) = 0$ unless $Q \lesssim_{\mathcal{F}} P$. By [18]Reeh 4.8, we have that the coefficient of S/P in α_{P/N_i} is 1, and therefore since $\hat{\alpha}$ is defined as an inverse limit of the α_{P/N_i}, it follows that $\varphi_P(\widehat{\alpha}_P) = |N_S P : P|$.

Since $\varphi|_H$ is a restriction of φ, we have that every element which maps to 0 under φ must also map to 0 under $\varphi|_H$. However, there may be elements which map to 0 under $\varphi|_H$ which do not map to 0 under φ. It follows that $\ker(\varphi|_H) \supseteq \ker(\varphi)$, therefore if we then take the cokernel, which is to say the codomain of a map factored by its kernel, we have $\operatorname{coker}(\varphi) \leq \operatorname{coker}(\varphi_H) = \prod_{P \in \mathrm{ob}(\mathcal{F})} \mathbb{Z}/|N_S(P) : P|\mathbb{Z}$.

As we have previously shown, we have that $\Psi \circ \varphi = 0$ and so the cokernel of φ must at least contain every element which maps to 0 under Ψ. Notably, we must have $\prod_{P \in \mathrm{ob}(\mathcal{F})} \mathbb{Z}/|N_S(P) : P|\mathbb{Z} \leq \operatorname{coker}(\varphi)$. It follows that we must have equality since we are bounded on both sides by inequalities from the same set. It follows that $\operatorname{Im}(\varphi) = \operatorname{Im}(\varphi|_H) = \ker(\Psi)$. Therefore H is a spanning set since the images agree. □

Theorem 3.21. *$\{\widehat{\alpha}_P \mid [P] \in Cl(\mathcal{F})\}$ is a basis for $\widehat{B}(\mathcal{F})$.*

Proof. This combines the three previous results and is immediate since we have that it is a linearly independent spanning set for the Burnside ring. □

3.5 Burnside ring of $\mathcal{F}_S(G)$

We now turn our attention to the Burnside ring of a fusion system over S given by a group G such that $S \leq_o G$. In particular we are interested in the image under the restriction map from G to S. As we previously discussed before lemma 3.6, we have that $\operatorname{res}^G_S(\widehat{B}(G)) \subseteq \widehat{B}(\mathcal{F})$, and we have quoted [3]Barostti and Carman theorem 7.1 in the background section 2.31.

It follows that we naturally want to generalise this result for a pro-fusion system. We show that there are well defined notions for each of the properties required in Barsotti and Carman's paper in the context of a pro-fusion system such that $\operatorname{res}^G_S(\widehat{B}(G)) = \widehat{B}(\mathcal{F})$. This holds for any case provided we have $S \leq_o G$ and as discussed in lemma 3.6, for G an infinite profinite group, we have previously noted that there is at most one prime p such that a Sylow pro-p subgroup of G is open.

Theorem 3.22. *Let G be a profinite group and suppose that there exists $S \leq_o G$ a pro-p Sylow subgroup of G and $\mathcal{F} = \mathcal{F}_S(G)$, then $\mathrm{res}^G_S(\widehat{B}(G)) = \widehat{B}(\mathcal{F})$.*

Proof. We seek to define an injective homomorphism $t^G_S : \widehat{B}(\mathcal{F}) \to \widehat{B}(G)$ which is right inverse to res^G_S in order to prove that every element in $\widehat{B}(\mathcal{F})$ has a preimage in $\widehat{B}(G)$. In doing so, we show that $a \in \widehat{B}(\mathcal{F})$ is such that $\mathrm{res}^G_S \circ t^G_S(a) = a$ and therefore, since t^G_S will be injective, we have that the restriction map must be surjective.

Let $H \leq_o G$, then we have that $H \cap S \leq_o G$ is a pro-p group since $S \leq_o G$ is a pro-p group. Consequently, $H \cap S \leq P$ where P is a Sylow pro-p group of H and so $P \leq_o G$. For $a \in \widehat{B}(G),\ \ b \in \widehat{B}(\mathcal{F})$ we define an element $a * b \in \prod_{H \leq_o G} \mathbb{Z} = Gh(G)$ by $|(a * b)^H| = |a^H||b^P|$ where P is a p-Sylow subgroup of H contained in S. We note that $|b^P| \in \mathbb{Z}$ under this construction since we know that $P \leq_o G \Rightarrow P \leq_o S$ and $b \in \widehat{B}(\mathcal{F}) \subseteq \widehat{B}(S)$.

We show that for each $a * b$, there exists $X \in \widehat{B}(G)$ such that the ghost map $\varphi : \widehat{B}(G) \to \mathbb{Z}$, defined as usual by $\varphi = (\varphi_H)_{H \leq_o G}$, has $\varphi(X) = a * b$. Let $x \in \mathrm{Gh}(G) = \prod_{H \leq_o G} \mathbb{Z}$ defined by $x := (x(H))_{H \leq_o G}$ where $x(H) \in \mathbb{Z}$. We recall that in [10]Dress and Siebeneicher 2.7.3, it was shown that $x \in \varphi(\widehat{B}(G))$ if and only if x satisfies

$$\sum_{vU \in V/U} x(\langle vU \rangle) \equiv 0 \bmod |V : U|$$

where $\langle vU \rangle$ is the subgroup of G generated by the coset vU for all pairs $U \trianglelefteq_o V \leq_o G$ such that V/U is a Sylow-q-subgroup of $N_G(U)/U$ for some prime q. That is to say that there exists $X \in \widehat{B}(G)$ such that $\varphi(X) = x$.

If $p \neq q$, and Q/H is a Sylow-q-subgroup of $N_G(H)/H$, then by definition of $a * b$, $|(a * b)^{\langle vH \rangle}| = |a^{vH}||b^P|$ where P is a Sylow pro-p subgroup of $\langle vH \rangle$ contained in S. However, since Q/H is a Sylow-q-subgroup of $N_G(H)/H$ then each Sylow pro-p subgroup of $\langle vH \rangle$ is also a Sylow pro-p subgroup of H for each $vH \in Q/H$.

Take $P \leq S$ a Sylow pro-p subgroup of $H \leq_o G$ then

$$\begin{aligned}\sum_{vH \in Q/H} |(a*b)^{\langle vH \rangle}| &\equiv \sum_{vH \in Q/H} |a^{\langle vH \rangle}||b^P| \\ &\equiv |b^P| \sum_{vH \in Q/H} |a^{\langle vH \rangle}| \\ &\equiv 0 \bmod |Q:H| \text{ since } a \in \widehat{B}(G).\end{aligned}$$

If $p = q$, first note that by [10]Dress and Siebeneicher (2.9.3), we have that for $N \trianglelefteq_o G$

$$\pi_N^G(G/U) = \begin{cases} G/U & \text{if } N \leq U, \\ 0 = \emptyset \text{ otherwise.} \end{cases}$$

Suppose that $N \trianglelefteq_o H \leq_o G$, then for any $H \lesssim K$, we have that $N \leq K$ as K contains some conjugate of H and $N \leq \cap_{g \in G} H^g$. Recall that for $N \trianglelefteq_o G$, $a \in \widehat{B}(G)$, we have $a^N = \pi_N^G(a) \in B(G/N)$. Let $a = \sum_{K \leq_o G} a_K G/K$, $a_K \in \mathbb{Z}$, and suppose that $N \trianglelefteq_o H \leq_o G$, then

$$\begin{aligned}|a^H| &= \left| \left(\sum_{K \leq_o G} a_K G/K \right)^H \right| \\ &= \left| \left(\sum_{H \lesssim K} a_K G/K \right)^H \right| \\ &= \left| \pi_N^G \left(\sum_{K \leq_o G} a_K G/K \right)^{H/N} \right| \\ &= |\pi_N^G(a)^{H/N}|.\end{aligned}$$

Take $\langle vH \rangle \leq_o Q \leq_o G$, then $|a^{\langle vH \rangle}| = |\mathrm{res}_Q^G(a)^{\langle vH \rangle}|$. Let P_Q denote a Sylow pro-p subgroup of Q such that $P_Q \leq_o S$. Since Q/H is a Sylow-p-subgroup of $N_G(H)/H$, we have that $H^g = H \ \forall g \in P_Q$ and so P_Q normalizes H. Define $P_H := P_Q \cap H$, then this is a Sylow pro-p subgroup of H. Since P_Q normalizes both P_Q and H, we have that P_Q normalizes $P_Q \cap H = P_H$, therefore $P_H \trianglelefteq_o P_Q$. Since Q/H is a p-group, then $P_Q H/H$ is a Sylow-p-subgroup of Q/H by [26] 2.2.3, and hence $Q = P_Q H$. By the second isomorphism theorem[22], we have that since $HP_Q = Q$, $H \trianglelefteq_o Q$, $P_Q \leq Q$,

$$Q/H = HP_Q/H \cong P_Q/H \cap P_Q = P_Q/P_H.$$

The isomorphism can be given by

$$\begin{aligned} \theta : P_Q/P_H &\to Q/H \\ xP_H &\mapsto xH. \end{aligned}$$

since θ is a group homomorphism, $\theta(xyP_H) = xyH = xHyH = \theta(xP_H)\theta(yP_H)$. This also has an inverse map $\theta^{-1} : Q/H \to P_Q/P_H$, $\theta^{-1}(xH) = xP_H$. Consequently, $B(Q/H) \cong B(P_Q/P_H)$, let α denote the corresponding ring homomorphism

$$\alpha : B(Q/H) \to B(P_Q/P_H).$$

Note that necessarily we have that for each $X \in B(Q/H)$, $|X^K| = |\alpha(X)^{\theta^{-1}(K)}|$ for each $K \leq Q/H$. Combining the above ring homomorphisms α, $\pi_N^G = \mathrm{Fix}_N^G$, res_H^G for $N \trianglelefteq_o G$, $H \leq_o G$, we obtain two ring homomorphisms

$$\begin{aligned} \alpha \circ \pi_H^Q \circ \mathrm{res}_Q^G &: \widehat{B}(G) \to B(P_Q/P_H), \\ \pi_{P_H}^{P_Q} \circ \mathrm{res}_{P_Q}^S &: \widehat{B}(S) \to B(P_Q/P_H). \end{aligned}$$

Each of these preserves the number of fixed points provided that we take the appropriate isomorphism or quotient group as required for each map. We demonstrate this as follows. Let $a \in \widehat{B}(G)$, $b \in \widehat{B}(\mathcal{F}) \subseteq \widehat{B}(S)$, $\langle vH \rangle$ the subgroup of Q generated by the coset vH of Q/H, and let P_{vH} denote a Sylow pro-p subgroup of

$\langle vH\rangle$ such that $P_H \le P_{vH} \le P_Q$,

$$\begin{aligned}
|(a*b)^{\langle vH\rangle}| &= |a^{\langle vH\rangle}||b^{P_{vH}}| \\
&= |\mathrm{res}^G_Q(a)^{\langle vH\rangle}||\mathrm{res}^S_{P_Q}(b)^{P_{vH}}| \\
&= |\pi^Q_H \circ \mathrm{res}^G_Q(a)^{\langle vH\rangle/H}||\pi^{P_Q}_{P_H} \circ \mathrm{res}^S_{P_Q}(b)^{P_{vH}/P_H}| \\
&= |\alpha \circ \pi^Q_H \circ \mathrm{res}^G_Q(a)^{\theta^{-1}(\langle vH\rangle/H)}||\pi^{P_Q}_{P_H} \circ \mathrm{res}^S_{P_Q}(b)^{P_{vH}/P_H}| \\
&= |\alpha \circ \pi^Q_H \circ \mathrm{res}^G_Q(a)^{P_{vH}/P_H}||\pi^{P_Q}_{P_H} \circ \mathrm{res}^S_{P_Q}(b)^{P_{vH}/P_H}|.
\end{aligned}$$

We define $x := \alpha \circ \pi^Q_H \circ \mathrm{res}^G_Q(a) \in B(P_Q/P_H)$, and $y := \pi^{P_Q}_{P_H} \circ \mathrm{res}^S_{P_Q}(b) \in B(P_Q/P_H)$, then we have that the above can be written as $|(a*b)^{\langle vH\rangle}| = |x^{P_{vH}/P_H}||y^{P_{vH}/P_H}| = |(xy)^{P_{vH}/P_H}|$ since the fixed point map $|(\cdot)^{P_{vH}/P_H}| : B(P_Q/P_H) \to \mathbb{Z}$ is a ring homomorphism. This holds for any $vH \in Q/H$ and so

$$\sum_{vH\in Q/H} |(a*b)^{\langle vH\rangle}| = \sum_{vH\in Q/H} |(xy)^{P_{vH}/P_H}|.$$

Note that $\theta^{-1}(\langle vH\rangle/H) = P_{vH}/P_H = \langle vP_H\rangle/P_H$. Since $xy \in B(P_Q/P_H)$ and $|P_Q/P_H| = |Q : H|$, we have

$$\begin{aligned}
\sum_{vH\in Q/H} |(xy)^{P_{vH}/P_H}| = \sum_{vP_H\in P_Q/P_H} |(xy)^{\langle vP_H\rangle}| &\equiv 0 \bmod (P_Q : P_H) \\
&\equiv 0 \bmod (Q : H).
\end{aligned}$$

We can therefore say that

$$\sum_{vH\in Q/H} |(a*b)^{\langle vH\rangle}| \equiv 0 \bmod (Q : H).$$

This holds for any Sylow-p-subgroup Q/H of $N_G(H)/H$, and therefore by Dress-Siebeneicher[10] (2.7.3), we have that $a * b \in \varphi(\widehat{B}(G))$, since the fixed point map φ is injective, we have that this image is unique and so we shall abuse notation to refer to the the element in the Burnside ring with this image also as

$a * b \in \widehat{B}(G)$. Taking $1 \in \widehat{B}(G)$ we define the map t_S^G as follows,

$$t_S^G : \widehat{B}(\mathcal{F}) \to \widehat{B}(G)$$
$$b \mapsto 1 * b.$$

We now show that the function t_S^G is an injective ring homomorphism. Firstly, we prove injectivity. Suppose that $t_S^G(a) = t_S^G(b)$ for $a, b \in \widehat{B}(\mathcal{F})$, then $1 * a = 1 * b \in \widehat{B}(G) \;\Leftrightarrow\; |(1 * a)^H| = |(1 * b)^H|$ for all $H \leq_o G$. By definition of $*$, let $P_H \leq_o S$ be a Sylow pro-p subgroup of H, then $|a^{P_H}| = |(1*a)^H| = (1*b)^H| = |b^{P_H}|$ for all $H \leq_o G$. Each $P_H \leq_o S$ by definition and for each $K \leq_o S$, there exists $H \leq_o G$ such that $K = P_H$, namely taking $H = K \Rightarrow P_H = P_K = K$. It is therefore sufficient to show $|a^K| = |b^K| \; \forall K \leq_o S$, but this can only be the case if $a \cong b \in \widehat{B}(S)$ as S-spaces and consequently $a \cong b \in \widehat{B}(\mathcal{F}) \subseteq \widehat{B}(S)$.

We now prove it is a ring homomorphism. For $a, b \in \widehat{B}(\mathcal{F})$, we have $t_S^G(a+b) = 1 * (a + b)$, then considering the fixed points under $H \leq_o G$ action, take $H \leq_o S$ since these alone determine the number of fixed points as above, then we have $|(1 * (a + b))^H| = |(a + b)^H| = |(a + b)^H| = |a^H||b^H| = |(1 * a)^H| + |(1 * b)^H| = |((1*a)+(1*b))^H|$ and similarly $|(1*(ab))^H| = |(ab)^H| = |a^H||b^H| = |((1*a)(1*b))|^H|$ for each $H \leq_o S$, which determine the fixed points for each open subgroup of G. Since the fixed point map is injective, we have that $1 * (a + b) = (1 * a) + (1 * b)$ and $1 * (ab) = (1 * a)(1 * b)$.

For $1 \in \widehat{B}(\mathcal{F})$, we have $|(1 * 1)^H| = 1$ for all $H \leq_o G$ and so $1 * 1 = 1 \in \widehat{B}(G)$, and so $t_S^G(1) = 1$. Therefore we have that t_S^G is an injective ring homomorphism. Taking $a \in \widehat{B}(\mathcal{F})$, then for $H \leq_o S$ we have that $|(\mathrm{res}_S^G \circ t_S^G(a))^H| = (1*a)^H = |a^H|$, it follows that $\mathrm{res}_S^G(1 * a) = a \in \widehat{B}(S)$ since the fixed points agree on all open subgroups of S. Hence, t_S^G is right inverse to res_S^G. The map

$$\mathrm{res}_S^G : \widehat{B}(G) \to \widehat{B}(\mathcal{F})$$

is surjective since for each $a \in \widehat{B}(\mathcal{F})$, we have $t_S^G(a) \in \widehat{B}(G)$ has image

$\mathrm{res}_S^G \circ t_S^G(a) = a \in \widehat{B}(\mathcal{F})$. We have shown then that $\mathrm{res}_S^G(\widehat{B}(G)) = \widehat{B}(\mathcal{F})$. $\square$

Since we now have that $\widehat{B}(\mathcal{F}) = \mathrm{res}_S^G(\widehat{B}(G))$ in the case when $\mathcal{F} = \mathcal{F}_S(G)$ a pro-fusion system over $S \leq_o G$, we use this result to define a more usable basis for $\widehat{B}(\mathcal{F})$. Suppose that we have $\mathcal{F}$ a pro-fusion system over S given by $G \geq_o S$. Clearly, since $\{G/H \mid H \leq_o G\}$ is a basis for $\widehat{B}(G)$, we have $\{\mathrm{res}_S^G(G/H) \mid H \leq_o G\}$ is a spanning set for $\widehat{B}(\mathcal{F})$. We then consider the set $R = \{\mathrm{res}_S^G(G/P) \mid P \leq_o S\}$ where we note that by abuse of notation we take the isomorphism class of $G/P \in \widehat{B}(G)$ to be written as G/P. That is to say we have one representative of each G-conjugacy class, or equivalently, each $\mathcal{F}$-conjugacy class. In order to show that R is a linearly independent set, we prove that for each $T \subseteq R$ such that T is a finite set, then we have that T is linearly independent.

Let T be a finite subset of R and define a partial order $\leq$ on T by $\mathrm{res}_S^G(G/P) \leq \mathrm{res}_S^G(G/Q)$ if and only if $P \lesssim_G Q$. Since T is a finite set, it follows that T has maximal elements. Recall that for $Y \in \widehat{B}(G)$, we have that $\mathrm{res}_S^G(Y)$ is defined as the same underlying set but considered under the S-action rather than the G-action. It follows that for $Q \leq_o S \leq_o G$, the Q-action on $\mathrm{res}_S^G(Y)$ and on Y respectively are the same since the S-action is induced by the G action. Therefore we have that $\mathrm{res}_S^G(Y) \cong Y$ as Q-spaces, and notably $\varphi_Q(\mathrm{res}_S^G(Y)) = \varphi_Q(Y)$.

Consider the element $X := \sum'_{P \leq_o S} x_P \cdot \mathrm{res}_S^G(G/P)$ for $x_P \in \mathbb{Z}$ such that $x_P = 0$ if $\mathrm{res}_S^G(G/P) \notin T$. This is to say that X is a linear combination of elements of T. In order to show that T is independent, we want to show that $X = 0$ if and only if x_P=0 for each $P \leq_o S$. By the definition of the equivalence classes of almost finite S-spaces, we have that $X = 0$ in $\widehat{B}(S)$ if and only if $\varphi_Q(X) = \varphi_Q(0) = 0$ for each $Q \leq_o S$. Suppose $X = 0$, we calculate the number of fixed points of X under

$Q \leq_o S$

$$\begin{aligned}
X = \sideset{}{'}\sum_{P \leq_o S} x_P \cdot \mathrm{res}^G_S(G/P) = 0 \\
\Leftrightarrow \varphi_Q\left(\sideset{}{'}\sum_{P \leq_o S} x_P \cdot \mathrm{res}^G_S(G/P)\right) = 0 \ \ \forall Q \leq_o S \\
\Leftrightarrow \sideset{}{'}\sum_{P \leq_o S} x_P \cdot \varphi_Q(\mathrm{res}^G_S(G/P)) = 0 \ \ \forall Q \leq_o S \\
\Leftrightarrow \sideset{}{'}\sum_{P \leq_o S} x_P \cdot \varphi_Q(G/P) = 0 \ \ \forall Q \leq_o S.
\end{aligned}$$

Let $Q \leq_o S$ and consider $\varphi_Q(G/P)$ for $P \leq_o S$. By definition 2.23, we have that this is the number of Q-fixed points in the G-set G/P. By 2.23, we have that $\varphi_Q(G/P) = |\{gP \in G/P \mid Q^g \leq_o P\}|$. It follows that $\varphi_Q(G/P) = 0$ unless $Q \lesssim_G P$. Therefore, $\varphi_Q(X)$ needs to only consider the sum of summands in X of the form $x_P \cdot \varphi_Q(G/P)$ such that $Q \leq_G P$. This is to say we have that the number of Q-fixed points are $\varphi_Q(X) = \sum_{Q \lesssim_G P \leq_o S} x_P \cdot \varphi_Q(G/P)$. Take $\mathrm{res}^G_S(G/Q)$ maximal in T, and consider the Q fixed points of X. By the previous discussion, then we have $\varphi_Q(X) = \sum_{Q \lesssim_G P \leq_o S} x_P \cdot \varphi_Q(G/P) = x_Q \cdot \varphi_Q(G/Q) = 0$ but $\varphi_Q(G/Q) = |N_G(Q) : Q| \neq 0$ so we must have that $x_Q = 0$.

Proceeding recursively on T, define a new set $T' = T \backslash \{\mathrm{res}^G_S(G/Q)\}$, at each step we take one of the maximal elements, Q' of T' and see that $x_{Q'} \cdot \varphi_{Q'}(G/Q') = 0$ implies that $x_{Q'} = 0$ and therefore the set T is linearly independent. Since this holds for any finite subset of R, we have that R must be linearly independent.

In order for R to be a basis as a free $\mathbb{Z}$-module, we must check that it is spanning. Since $\mathbb{Z}$ is not a field, in order to be a $\mathbb{Z}$-basis we must check that every $\mathcal{F}^o$-stable element can be expressed as a linear expression of elements in R. As a consequence of the definition of the restriction map discussed on pages 21-23, we have that $\mathrm{res}^G_S(G/H) = \sum_{g \in [S \backslash G/H]} S/S \cap {}^gH$ and so $S \cap {}^gH \lesssim_G H$.

This implies that the only element in R with a non zero coefficient for S/S is given by $\mathrm{res}^G_S(G/S)$. Note that S/S is $\mathcal{F}^o$-stable since $\varphi_P(S/S) = 1$ for all $P \leq_o S$.

Consider that $\varphi_S(G/S) = \varphi_S(\mathrm{res}^G_S(G/S))$, then it follows that $\varphi_S(\mathrm{res}^G_S(G/S)) = |N_G(S) : S|$. Therefore if $N_G(S) \neq S$, then R is not a $\mathbb{Z}$-spanning set of $\widehat{B}(\mathcal{F})$. Consequently, we show that R forms a $\mathbb{Q}$ basis for $\mathbb{Q}\widehat{B}(\mathcal{F}) := \mathbb{Q} \otimes \widehat{B}(\mathcal{F}$.

In order to show that this is truly a basis, we must show that it is also spanning. We do this by showing that the canonical projections $\pi^S_M : \widehat{B}(\mathcal{F}) \to B(\mathcal{F}/M)$ map R to a basis for $B(\mathcal{F}/M)$ for $M \trianglelefteq_o S$. Suppose that $G \cong \varprojlim_{N \trianglelefteq_o G} G/N$ then since $S \leq_o G$, we have that $S \leq_c G$ and therefore by Wilson[26] Theorem 1.2.5, S can be expressed as $S \cong \varprojlim_{N \trianglelefteq_o G} S/(S \cap N)$. Suppose $N \trianglelefteq_o G$, then $S \cap N \leq_o P$ implies ${}^g(S \cap N) = {}^gS \cap N \leq_o {}^gP$. However, $S \cap N \trianglelefteq_o S \leq_o G$ implies that S lies in the normal core of S in G and so $S \cap N \leq \bigcap_{g \in G} {}^gS$, finally taking intersection of the right hand side with N we have

$$S \cap N \leq \bigcap_{g \in G} {}^gS \cap N \leq S \cap {}^gS \cap N \leq {}^gP \cap S.$$

Hence considering $\mathrm{res}^G_S(G/P) = \sum_{g \in [S \backslash G/P]} S/(S \cap {}^gP)$ we see that either each summand has $S \cap N \leq S \cap {}^gP$ or none of them do. Therefore, the projection can be given as

$$\pi^S_{S\cap N}(\mathrm{res}^G_S(G/P)) = \begin{cases} \sum_{g \in [S \backslash G/P]} S/(S \cap {}^gP) & \text{if } S \cap N \leq P \\ \emptyset & \text{otherwise.} \end{cases}$$

For $N \trianglelefteq_o G$, define $\pi^S_{S\cap N}(R) = \{\pi^S_{S\cap N}(\mathrm{res}^G_S(G/P)) \mid S \cap N \leq P \leq_o S\}$, then clearly since R is linearly independent, we have that $\pi^S_{S\cap N}(R)$ is linearly independent.

Additionally $|\pi^S_{S\cap N}(R)| = |\{\alpha_P \mid P \leq S/S \cap N \text{ fully } \mathcal{F}\text{-normalized}\}|$ since they both take one representative from each G-conjugacy class. It follows that $\pi^S_{S\cap N}(R)$ is a basis for $B(\mathcal{F}/N)$. Define $\pi^S_{S\cap N}(R_0) = \pi^S_{S\cap N}(R) \cup \{\emptyset\}$.

Restricting the canonical projection maps to the sets $\pi^S_{S\cap N}(R_0)$ for $N \trianglelefteq_o G$, we get that there are compatible set morphisms for each $N \leq M$, both open normal in G, $\pi^{S/S\cap N}_{S\cap M/S\cap N} : \pi^S_{S\cap N}(R_0) \to \pi^S_{S\cap M}(R_0)$ to be the restriction of the usual canonical ring homomorphisms to set morphisms between the basis sets, with the empty set added, of $B(\mathcal{F}/(S\cap N))$ and $B(\mathcal{F}/(S\cap M))$ respectively.

Taking the inverse limit with these compatible maps for each $N, M \trianglelefteq_o G$, we get $R_0 := \varprojlim_{N\trianglelefteq_o G}(\pi^S_{S\cap N}(R_0)) = R\cup\{\emptyset\}$, and since $\langle R_0\rangle = \langle R\rangle$, we get that R is a basis of $\mathbb{Q}\widehat{B}(\mathcal{F})$. Recall that $\mathrm{ind}^G_S(S/H) = G/H$ for $H \leq_o S \leq_o G$. Clearly, each element G/H such that $H \leq_o S$ can be written as $\mathrm{ind}^G_S(S/H)$ and so

$$R = \{\mathrm{res}^G_S(G/H) \mid H \leq_o S\} = \{\mathrm{res}^G_S \circ \mathrm{ind}^G_S(S/H) \mid H \leq_o S\}.$$

Consequently taking the obvious $\mathbb{Q}$-linear extension of the maps, we have that the composition $\mathrm{res}^G_S \circ \mathrm{ind}^G_S : \mathbb{Q}\widehat{B}(S) \to \mathbb{Q}\widehat{B}(\mathcal{F})$ is surjective.

By theorem 3.21, we have a $\mathbb{Z}$-basis for $\widehat{B}(\mathcal{F})$ for any pro-fusion system $\mathcal{F}$ given by $\{\hat{\alpha}_P \mid P \in \mathrm{ob}(\mathcal{F}^{f.n.})\} =: A$. We note that in the construction through the inverse limit of basis elements of the Burnside rings of the finite quotients, we have that by [18]Reeh Proposition 4.8, the coefficient of S/P in the expression is 1. Note that by taking $\varphi_P(S/P) = |N_S(P) : P|$, it follows that $\varphi_P(\hat{\alpha}_P) = |N_S(P) : P|$.

If we consider the element $Z_P := \frac{1}{|N_G(P):N_S(P)|} \cdot \mathrm{res}^G_S(G/P) \in \mathbb{Q}\widehat{B}(\mathcal{F})$, then we have that $\varphi_P(Z_P) = \frac{|N_G(P):P|}{|N_G(P):N_S(P)|} = |N_S(P) : P|$. Clearly, we have that $\varphi_Q(Z_P) = \varphi_P(Z_P)$ for any $Q \sim_{\mathcal{F}} P$ since $\varphi_Q(G/P) = \varphi_P(G/P)$ for any $P \sim_{\mathcal{F}} Q$. It follows that if we were to express Z_P as a $\mathbb{Q}$-linear expression of elements of the basis A, then we would have that the coefficient of $\hat{\alpha}_P$ is 1.

Consider that $\mathrm{res}^G_S(G/P) = \sum_{g\in[S\backslash G/P} S/S\cap {}^gP$. In particular, consider the double coset representatives given by $g \in [S\backslash G/P]$. Note that in the expression we take $S/S\cap {}^gP$. If $gP \in N_G(P)/P$, then we have ${}^gP = P$, it follows that the double coset SgP gives the same coset for $g \in S$, and therefore $g \in N_G(P)\cap S$. Therefore we habe that the set of representatives that give distinct double cosets is given by $|N_G(P) : N_S(P)|$. It follows then that we can also write the expression as $\mathrm{res}^G_S(G/P) = |N_G(P) : N_S(P)| \sum_{g\in[S\backslash G/N_G(P)]} S/S\cap {}^gP$. It follows that Z_P is not only a well defined element of $\mathbb{Q}\widehat{B}(\mathcal{F})$, but a well defined element of $\widehat{B}(\mathcal{F})$ since as we have shown, dividing by the given denominator must leave us with integral coefficients.

We have that each orbit in Z_P must be stabilized by some $S \cap {}^gP$ for some $g \in G$. In particular, they must be G-subconjugate to P. Therefore, since the $\hat{\alpha}_P$ form a basis, we have that there is an expression $Z_P = \sum_{Q \lesssim_G P} x_Q \cdot \hat{\alpha}_Q$ with $x_P = 1$. We can rewrite this to say $\hat{\alpha}_P = Z_P - \sum'_{Q<P} x_Q \cdot \hat{\alpha}_Q$ with the series taken over a fully normalized $\mathcal{F}$ conjugacy class representative. We note that this series is well defined since Z_P is a well defined almost finite S-space, as is each $\hat{\alpha}_Q$ and for any $H \leq_o S$, there are only finitely many subgroups U which contain H and so S/H appears in only finitely many $\hat{\alpha}_Q$, therefore the coefficient of each isomorphism class of orbit is finite and therefore it is an almost finite G-space.

We repeat this process, replacing each α_Q with $Z_Q - Y$ for some Y a series of $\hat{\alpha}_H$. It follows that we have an expression for $\hat{\alpha}_P$ which is a linear combination of the Z_Q, the coefficient of each Z_Q being a finite integer. It follows that we have that the set $\{Z_P \| \ P \in \mathrm{ob}(\mathcal{F}^{f.n.})\}$ is $\mathbb{Z}$-spanning, and since it is a linear scaling of R, we have that it must also be linearly independent. It follows that this set is a $\mathbb{Z}$-linear basis of $\widehat{B}(\mathcal{F})$.

Theorem 3.23. *Let S be a Sylow pro-p-subgroup of G and take the pro-fusion system $\mathcal{F} = \mathcal{F}_S(G)$ to be the pro-fusion system of G over S. Then we have a $\mathbb{Z}$-linear basis for $\widehat{B}(\mathcal{F})$ given by $\{Z_P \mid P \in ob(\mathcal{F}^{f.n.})\}$ where*

$$Z_P := \frac{1}{|N_G(P) : N_S(P)|} \cdot res^G_S(G/P).$$

3.6 Induced pro-fusion systems

In the previous subsection, we showed that when we have a fusion system over a pro-p group S such that $S \leq_o G$ is Sylow pro-p subgroup of G, $\mathbb{Q}\widehat{B}(\mathcal{F})$ has a basis given by R, a set of isomorphism classes of restrictions from G-action to S-action of G-spaces. In this section, we shall show that a pro-fusion system $\mathcal{F}$ on a pro-p group S has that all morphisms in $\mathcal{F}$ expressed as G-conjugation followed by inclusion for some profinite group $G \geq S$.

We note that in our current definition of a fusion system of G over S, it is required that S is a Sylow-p-subgroup of G. However, we note that it is possible to consider a fusion system over $P < S$ with the morphisms induced by G-conjugation. In this case we say that the system is induced by G.

Definition 3.24. [16]Park §1 Let P be a (finite) p-group with $P \leq G$, G finite, then we define the fusion system induced by G on P to be the category $\mathcal{F}_P(G)$ with $\mathrm{ob}(\mathcal{F}_P(G)) = \{Q \leq P\}$ and for each $Q, R \leq P$,

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \{\varphi : Q \to R \mid \exists x \in G \text{ such that } \varphi(u) = xux^{-1}\ \forall u \in Q\}.$$

Note that in the case we have that P is the Sylow-p-subgroup of G, we have that this coincides with the fusion system of G over S and therefore there is no conflict with notation since both constructions give the same category. A saturated fusion system $\mathcal{F}$ over a p-group P is said to be realizable if there exists finite G with Sylow-p-subgroup P such that $\mathcal{F} = \mathcal{F}_P(G)$. If no such G exists, then we define $\mathcal{F}$ to be an exotic fusion system[1]. However, results by Puig[17] and Park[16] have shown that every saturated fusion system $\mathcal{F}$ over a p-group P can be a induced by a finite group $G \geq P$.

Theorem 3.25. *[16](Theorem 1) For every saturated fusion system $\mathcal{F}$ on a finite p-group P, there is a finite group G having S as a subgroup such that the fusion system induced by G over S is equal to $\mathcal{F}$, $\mathcal{F} = \mathcal{F}_S(G)$.*

Let $\mathcal{F}$ be a finite exotic fusion system over a p-group P. By this theorem by Park[16] there must be a finite group G such that $\mathcal{F}$ is induced by G over P which is to say $\mathcal{F} = \mathcal{F}_P(G)$. By Reeh[18], there is a basis for $B(\mathcal{F})$ given by $\{\alpha_Q \mid [Q]_{\mathcal{F}},\ Q \leq P\}$, which is to say that there is a 1-1 correspondence between $\mathcal{F}$-conjugacy classes and the basis elements of $B(\mathcal{F})$.

Theorem 3.26. *Let $\mathcal{F}$ be an exotic fusion system over the p-group P, such that $\mathcal{F}$ is induced by a finite group $G \geq P$, then the set $R = \{res^G_P(G/H) \mid H \leq P\}$ is a basis for $B(\mathcal{F})$.*

Proof. Since the set R has equal cardinality to the number of $\mathcal{F}$-conjugacy classes of subgroups of P, it is sufficient to show that this set is linearly independent, which we prove in a similar way as in the previous section. Take the G-space $X = \sum_{H \leq P} X_H \mathrm{res}^G_P(G/H)$, $X_H \in \mathbb{Z}$ then assume that $X = 0$.

$$
\begin{aligned}
X = & \sum_{H \leq P} X_H \mathrm{res}^G_P(G/P) = 0 \\
\Leftrightarrow & \varphi_K \left(\sum_{H \leq P} X_H \mathrm{res}^G_P(G/H) = 0 \right) \quad \forall K \leq P \\
\Leftrightarrow & \sum_{H \leq P} X_H \varphi_K(G/H) = 0 \ \ \forall K \leq P.
\end{aligned}
$$

By maximality, we get that $X_P = 0$. Repeating for all maximal subgroups H such that X_H has not yet shown to be zero, gives $X_H = 0$ and so the only solution is given when $X_H = 0$ for all $H \leq P$, which is to say $X = 0$ and therefore R is linearly independent. $\square$

Let $\mathcal{F}$ be a saturated pro-fusion system over a pro-p group S. By Stancu-Symonds[21] (2.24) we have that if N an open strongly closed subgroup of S, then $\mathcal{F}/N$ is a saturated fusion system on S/N. Namely, it is a saturated fusion system on a finite group. Applying the theorem above, we have that for any saturated pro-fusion system $\mathcal{F}$ on a pro-p group S, N open strongly $\mathcal{F}$-closed in S, then there exists a finite group $G^N \geq S/N$ such that $\mathcal{F}/N = \mathcal{F}_{S/N}(G^N)$.

Definition 3.27. Let P be a pro-p group, $S \leq G$, G profinite, we define the pro-fusion induced by G on P, $\mathcal{F}_P(G)$, by $\mathrm{ob}(\mathcal{F}_P(G)) = \{P \leq_c S\}$ and

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \{\varphi : Q \to R \mid \exists x \in G \text{ such that } \varphi(u) = xux^{-1} \ \forall u \in Q\}.$$

Let S be a Sylow pro-p subgroup of G a profinite group where S is not necessarily open in G. Suppose that $\mathcal{F} = \mathcal{F}_S(G)$ is a pro-fusion system over S and that we have $N \leq S$ an open strongly $\mathcal{F}$-closed subgroup in S. Consider the map $\mathcal{F} \to \mathcal{F}/N$ then there exists $G^N \geq S/N$ such that $\mathcal{F}/N = \mathcal{F}_{S/N}(G^N)$, then we have a basis of $\mathbb{Q}B(\mathcal{F}_{S/N}(G^N))$ is given by $T_N := \{\mathrm{res}^{G^N}_{S/N}(G^N/P) \mid N \leq P \leq S\}$.

Suppose that $M, N \leq S$ are both open strongly $\mathcal{F}$-closed subgroups such that $M \leq N$ and take G^M to be a finite group such that $\mathcal{F}/M = \mathcal{F}_{S/M}(G^M)$. Since N is open strongly $\mathcal{F}$-closed then for $\varphi(N/M) = N/M$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}/M}(N/M, -)$. Then we can take the projection map $f_{M,N} : G^M \to G^M/N$ and therefore we have $\mathcal{F}/N = \mathcal{F}_{S/N}(G^M/N)$.

4 Prime Ideals in $\widehat{B}(G)$

4.1 Topology of $\widehat{B}(G)$

Suppose that G is a profinite group, we define a topology on $\widehat{B}(G)$ by taking an open base for the topology to be cosets of the kernels of the projection maps

$$\pi^G_N : \widehat{B}(G) \to B(G/N)$$

for $N \trianglelefteq_o G$. Therefore, an open base for the topology is given by

$$A := \{X + \ker(\pi^G_N) \mid \pi^G_N,\ N \trianglelefteq_o G,\ X \in \widehat{B}(G)\}.$$

We prove this is truly a topology by showing it adheres to the topology axioms. Let τ denote the set generated by union of elements in A. Taking the empty collection from the open base, we see that $\emptyset \in \tau$. For $N \leq_o G$, we can write $\widehat{B}(G) = \bigcup_{X \in \widehat{B}(G)} X + \ker(\pi^G_N) \in \tau$. Suppose that $T \subseteq \tau$, then we have that $\bigcup_{t \in T} t \in \tau$ by definition since we have defined τ to be generated by unions.

Recall from §2.18-§2.19 that for $Z \in \widehat{B}(G)$, then we can write $Z = \sum_{H \leq_o G} z_H \cdot G/H$ such that $z_H \in \mathbb{Z}$. Take $M \trianglelefteq_o G$, then we have that $\pi^G_M(G/H) = 0$ if $M \not\leq_o H$, therefore we must have that $G/H \in \ker(\pi^G_M)$ if $M \not\leq_o H$. It follows that for $X, Z \in \widehat{B}(G)$, $Z + \ker(\pi^G_M) = X + \ker(\pi^G_M)$ if and only if $z_H = x_H$ for all $H \geq M$ with the x_H defined similarly to the z_H.

Suppose we have $X + \ker(\pi^G_N)$, $Y + \ker(\pi^G_K) \in A$ for $X, Y \in \widehat{B}(G)$. By above, without loss of generality we can write elements of A by the coset representatives $X = \sum_{H \geq N} x_H \cdot G/H$ and $Y = \sum_{H \geq K} y_H \cdot G/H$. Consider the intersection $I = (X + \ker(\pi_N^G)) \cap (Y + \ker(\pi_K^G))$. For $H \geq N, K$, we have that $I = \emptyset$ unless $x_H = y_H$ since G/H is in neither kernel.

Suppose that $H \geq N$ but $H \not\geq K$ and take $Z \in I$ then $z_H = x_H$ and similarly if $H \not\geq N$ but $H \geq K$ then $z_H = y_H$. If $H \not\geq N, K$ then $G/H \in \ker(\pi_N^G), \ker(\pi_K^G)$ and so $G/H \in \ker(\pi_N^G) \cap \ker(\pi_K^G)$. Since $\ker(\pi_{N \cap K}^G) \subseteq \ker(\pi_N^G) \cap \ker(\pi_K^G)$, we can choose a set of coset representatives of $\ker(\pi_{N\cap K}^G)$ given by $T \subseteq A$ such that $\ker(\pi_N^G) \cap \ker(\pi_K^G) = \cup_{t \in T} t + \ker(\pi_{N \cap K}^G)$. For $W \in \widehat{B}(G)$,

$$\begin{aligned} W + \ker(\pi_N^G) \cap \ker(\pi_K^G) &= W + \bigcup_{t \in T} t + \ker(\pi_{N\cap K}^G) \\ &= \bigcup_{t \in T} W + t + \ker(\pi_{N \cap K}^G). \end{aligned}$$

We seek to define a representative of a coset of $\ker_{N \cap K}^G$ to write I as a coset. If we can define $z_H \in \mathbb{Z}$, for each $H \leq_o G$ up to conjugacy, we therefore define the required representative in $\widehat{B}(G)$. We define $z_H := x_H = y_H$ for $H \geq N, K$ since we have that they must agree. For the remaining entries, if we have $H \not\geq N$ but $H \geq K$, we have that $G/H \in \ker(\pi_N^G)$ but $G/H \notin \ker(\pi_K^G)$ and so the representative of the intersection must have $z_H := y_H$. Similarly if $H \geq N$ but $H \not\geq K$, we define $z_H := x_H$. Finally, define $Z := \sum_{H \leq_o G} z_H \cdot G/H$. It follows that

$$I = \begin{cases} \emptyset \text{ if } x_H \neq y_H \text{ for some } H \geq N, K \\ \bigcup_{t \in T} Z + t + \ker(\pi_{N \cap K}^G) \text{ otherwise.} \end{cases}$$

Since $N, K \trianglelefteq_o G$, we have that their intersection $N \cap K \trianglelefteq_o G$ and so $I \in \tau$ in either case. Take $a, b \in \tau$ such that $a = \cup_i a_i,\ b = \cup_j b_j,\ a_i, b_j \in A$. Then by the usual algebra of set operations we have $a \cap b = (\cup_i a_i) \cap (\cup_j b_j) = \cup_i \cup_j (a_i \cap b_j) = \cup_{i,j}(a_i \cap b_j) \in \tau$ since we can express the sum as a union of pairwise intersections of elements of A, which we have shown also lie in τ. Therefore we have shown that this is a well defined topology on the underlying space $\widehat{B}(G)$.

In order to say that $\widehat{B}(G)$ is a topological ring, we show that the ring operations of addition, additive inverse and multiplication are continuous maps. Let $\iota : \widehat{B}(G) \to \widehat{B}(G)$ denote the map $\iota : X \mapsto -X$, that is to say sends an element in the Burnside ring to its additive inverse. For $X \in \ker(\pi_N^G)$, $N \trianglelefteq_o G$, we have that $\iota(X) = -X \in \ker(\pi_N^G)$ since π_N^G is a ring homomorphism and so $\pi_N^G(\iota(X)) = \pi_N^G(-X) = -\pi_N^G(X) = 0$. It follows that $\iota(\ker(\pi_N^G) = \ker(\pi_N^G)$. Let $Y + \ker(\pi_N^G) \in A$ and consider $\iota(Y + \ker(\pi_N^G)) = \iota(Y) + \iota(\ker(\pi_N^G)) = -Y + \ker(\pi_N^G)$. A is an open base for the topology and $\iota = \iota^{-1}$, therefore we have shown that the preimage of every open set is open and so ι is continuous.

Let $\sigma : \widehat{B}(G) \times \widehat{B}(G) \to \widehat{B}(G)$ be the addition map given by $\sigma(X, Y) = X + Y$ for $X, Y \in \widehat{B}(G)$. By definition, we must have $\sigma(X - Z, Z) = X$ for all $X, Z \in \widehat{B}(G)$. Since $\ker(\pi_N^G)$ is closed under addition, we have that for $X \in \widehat{B}(G)$,

$$\{(X - Z + \ker(\pi_N^G), Z + \ker(\pi_N^G)) \mid Z \in \widehat{B}(G)\} = \sigma^{-1}(X + \ker(\pi_N^G)).$$

Since $\cup_{Z \in \widehat{B}(G)} Z + \ker(\pi_N^G) \in \tau$, we have that the above subset of the preimage is open in the product topology therefore the preimage itself is open. Since this holds for any $a \in A$, we have for any collection $\{a_t \mid t \in T\} \subseteq A$, the preimage $\sigma^{-1}(\cup_{t \in T} a_t) = \cup_{t \in T} \sigma^{-1}(a_t)$ is also a union of the elements in A and therefore lies in τ. Since the preimage of any open set is open, we have that σ is a continuous map.

Let $\mu : \widehat{B}(G) \times \widehat{B}(G) \to \widehat{B}(G)$ denote the multiplication map given by $\mu(X, Y) = X \times Y$ for $X, Y \in \widehat{B}(G)$. By the multiplication map, we have that for $H, K \leq_o G$, then we have that $G/H \times G/K = \sum_{g \in [H \backslash G/K]} G/H \cap {}^gK$. In particular, we have that $N_1 \leq H \cap {}^gK$ where $N_1 \trianglelefteq_o K$ is a maximal normal subgroup of K. Since this multiplication is commutative, we can similarly deduce that $N_2 \leq H \cap {}^gK$ where $N_2 \trianglelefteq_o H$ is a maximal normal subgroup of H.

Therefore, considering the preimage $\mu^{-1}(X + \ker(\pi_N^G))$, we have that the representative $X = \sum_{H \leq_o G} x_H \cdot G/H$ where we can without loss of generality say that $x_H = 0$ for $N \not\leq H$. The set of elements (Y, Z) such that $\mu(Y, Z) = X$ must have that the entries $y_H, z_H = 0$ for $N \not\leq H$. Since $\ker(\pi_N^G) = \operatorname{span}_{\mathbb{Z}}\{G/H \mid N \not\leq H\}$, we have that

$$\begin{aligned}\mu^{-1}(X + \ker(\pi_N^G)) &= \{(Y, Z) \mid \mu(Y, Z) \in X + \ker(\pi_N^G)\} \\ &= \bigcup_{\mu(Y,Z)=X} \{(Y + \ker(\pi_N^G), Z + \ker(\pi_N^G))\}\end{aligned}$$

and so is a union of open sets in the product topology. Therefore we have that the preimage of any open set is open and so μ is continuous. Consequently, we have that $\widehat{B}(G)$ is a well defined topological ring with this topology.

4.2 Open prime ideals

This section examines results proven by [8]Dress Proposition 1 on the prime ideals of Burnside rings of finite groups and seeks to identify an analogue to them in the Burnside rings of profinite groups.

A key aspect of the proof for the Burnside ring of finite groups is being able to identify a minimal transitive G-set for a finite group G which is not in the prime ideal. The proof proceeds by considering the preimage of a prime ideal in $\mathbb{Z}$ under the H-fixed point map. By finding the minimal transitive G-set G/H which is not in the prime ideal (under an ordering), we find the minimal H such that the preimage of a prime ideal classifies the prime ideal under the preimage of φ_H

The argument in question is built upon the well known result that for R, S rings, $\theta : R \to S$ a ring homomorphism, then if $P \subseteq S$ is a prime ideal of S then $\theta^{-1}(P) \subseteq R$ is a prime ideal of R. This follows from the argument that if $a, b \in R$ such that $ab \in \theta^{-1}(P)$ then $\theta(ab) = \theta(a)\theta(b) \in P$ but since P is a prime ideal then we have that $\theta(a) \in P$ or $\theta(b) \in P$ and therefore $a \in \theta^{-1}(P)$ or $b \in \theta^{-1}(P)$ and so θ^{-1} is prime in R.

In particular, for G a finite group, $H \leq G$ we have the usual fixed point ring homomorphisms $\varphi_H : B(G) \to \mathbb{Z}$, $X \mapsto |X^H|$. Then for each $H \leq G$ we have a class of prime ideals can be defined by the preimages of the prime ideals in $\mathbb{Z}$

since the preimage of a prime ideal under a ring homomorphism is itself a prime ideal. The prime ideals of $\mathbb{Z}$ are given by $\{0\}$ and $p\mathbb{Z}$. The class of prime ideals characterised by the preimage of these ring homomorphisms can be expressed as $\varphi_H^{-1}(\{0\}) = \ker(\varphi_H) = \{X \in B(G) \mid \varphi_H(X) = 0\}$ and

$$\varphi_H^{-1}(p\mathbb{Z}) = \{X \in B(G) \mid \varphi_H(X) \in p\mathbb{Z}\} = \{X \in B(G) \mid \varphi_H(X) \equiv 0 \bmod p\}.$$

Dress[8] Proposition 1 showed that this class of prime ideals is in fact all the prime of ideals of $B(G)$ for G a finite group. This result is obtained by showing each $\mathcal{P} \subseteq B(G)$ a prime ideal, there is a unique minimal transitive G-set in $B(G)$ which is not in $\mathcal{P}$ under the ordering $G/H \leq G/K \Leftrightarrow H \lesssim_G K$. Taking the stabilizer subgroup, H of the G-orbit G/H corresponds to the defining fixed point map in question $\varphi_H^{-1}(p\mathbb{Z}) = \mathcal{P}$ for some p either prime or 0. The existence of this minimal element is dependent on the subgroup lattice of a finite group being finite, a condition that we do not have in general for profinite groups.

We can, however, define an equivalent class of prime ideals of $\widehat{B}(G)$ for a profinite group G. Recall from definition 2.19 that since each $X \in \widehat{B}(G)$ is almost finite by definition, it must necessarily be essentially finite and so for each $H \leq_o G$, we have $X^H < \infty$. Therefore $\varphi_H := |(\cdot)^H| : \widehat{B}(G) \to \mathbb{Z}$ is a well defined ring homomorphism and $\varphi_H^{-1}(\{0\})$, $\varphi_H^{-1}(p\mathbb{Z})$ for p a prime are the preimages of prime ideals and therefore prime ideals themselves.

Define $\widehat{B}(G)$ to have the topology defined in the previous subsection and equip $\mathbb{Z}$ with the the discrete topology and consider the ring homomorphism $\varphi_H : \widehat{B}(G) \to \mathbb{Z}$. Since $\mathbb{Z}$ has the discrete topology, each subset of $\mathbb{Z}$ is open. In particular, the singleton sets $\{q\} \subseteq \mathbb{Z}$ are open. G itself is a topological group under the profinite topology, that is to say that the product and inverse maps $p(x, y) = xy$ and $c(x) = x^{-1}$ are continuous. For $g \in G$, we can define a constant map $f : G \to G,\ x \mapsto g$, suppose that $U \subseteq_o G$ then $f^{-1}(U) = G$ if $g \in U$, $f^{-1}(U) = \emptyset$ otherwise and therefore f is continuous.

Let $\mathrm{id}_G : G \to G,\ x \mapsto x$, clearly this is continuous. Consider $G \times G$ with the product topology, then we have that a map $h : G \to G \times G,\ x \mapsto (h_1(x),\ h_2(x))$ is continuous if and only if h_1, h_2 are both continuous. Define h by $h_1(x) = f(x) = g$ and $h_2(x) = \mathrm{id}_G(x) = x$ then this is a continuous map. Finally, we compose this with the function p and get $p \circ h : G \to G,\ x \mapsto gx$ is a continuous function. Since it is continuous, the preimage of any open set is open. For $U \subseteq_o G$ we have $(p \circ h)^{-1}(U) = g^{-1}U$, this process can be repeated for any $g \in G$ and so each coset of U is open. A similar argument for right multpilication by g^{-1} shows that each G-conjugate of U is open.

Consider $\varphi_H^{-1}(q) = \{X \in \widehat{B}(G) \mid \varphi_H(X) = q\}$. If $H \leq_o G$, we have that $|G : H| < \infty$, and therefore $|G : N_G(H)|$ and so there are finitely many conjugates of H in G. Each H^g is open, therefore the normal core in G of H, defined as $K_G(H) := \bigcap_{g \in G} H^g$ is open in G since it is the intersection of finitely many open subgroups and normal since it is the intersection of all the conjugates of H.

Suppose that $K \leq_o G$, we have that $\varphi_H(G/K) = |\{gK \mid H^g \leq K\}|$. Since $K_G(H) = \bigcap_{g \in G} {}^gH \leq H^g$, then if $K_G(H) \not\leq K$, we have $\varphi_H(G/K) = 0$. Take $Y := \sum_{K \leq_o G} y_K \cdot G/K \in \widehat{B}(G)$ then

$$\begin{aligned}
\varphi_H(Y) = \varphi_H\left(\sum_{K \leq_o G} y_K \cdot G/K\right) &= \sum_{K \leq_o G} y_K \cdot \varphi_H(G/K) \\
&= \sum_{K_G(H) \leq K \leq G} y_K \cdot \varphi_H(G/K) \\
&= \varphi_H\left(\sum_{K_G(H) \leq K \leq G} y_K \cdot G/K\right).
\end{aligned}$$

Since the value $\varphi_H(Y)$ has no dependence on the coefficient Y_K in the expression of Y if $K \not\geq K_G(H)$ and so we are free to choose any $y_K \in \mathbb{Z}$ for such $K \leq_o G$. As previously discussed for $K_G(H) \not\leq K$, we have $\varphi_{K_G(H)}(G/K) = 0$ and therefore $G/K \in \ker(\pi^G_{K_G(H)})$. It follows then that for some collection of $X = \sum_{K_G(H) \leq K \leq G} x_K \cdot G/K$, we have $\varphi_H^{-1}(q) = \bigcup_{\varphi_H(X) = q} X + \ker(\pi^G_{K_G(H)})$ and therefore $\varphi_H^{-1}(q)$ is open for all $q \in \mathbb{Z}$. Since the preimage of each singleton set is open, we have that φ_H is a continuous map, therefore each prime ideal of the form $\varphi_H^{-1}(p\mathbb{Z})$

for some $H \leq_o G$, p a prime or 0 is open.

Definition 4.1. Define $I(G) = \{G/H| \ H \leq_o G\}$ be a $\mathbb{Z}$-spanning set for $\widehat{B}(G)$. By abuse of notation, we use this notation to denote a single representative from each almost finite transitive G-space by taking one representative H from each G-conjugacy class of subgroups. We define a partial order on this set by $G/H \preceq G/K \Leftrightarrow H \lesssim_G K$, noting that this is well defined since if we have $H, K \leq_o G$ such that $H \sim_G K$, $L \leq K$, then there exists $g \in G$ such that $H = K^g$, $K \geq L \Leftrightarrow H = K^g \geq L^g$ and therefore $G/L \preceq G/H$.

With this ordering, we see a divergence from the theory in the case of prime ideals of Burnside rings of finite groups where, as previously discussed, they can always guarantee that there is a minimal transitive G-set which is not in the ideal. We prove that if there are minimal transitive G-spaces that are not in a prime ideal of $\widehat{B}(G)$ for G a profinite group, then there is a unique minimum to that set.

As usual, we define the character of a ring to be the least $n \in \mathbb{N}$ such that $n \times 1 = 0$, if no such n exists, then we take the character to be defined to be 0. We denote the chatacter of R by $\operatorname{char}(R)$. In the case of the Burnside ring, we have that $1 := G/G$ and so we have that $\operatorname{char}(\widehat{B}(G)) = 0$ for the Burnside ring, but we can also take the character of the factor ring $\operatorname{char}\widehat{B}(G)/\mathcal{P}$ which will either be p a prime or 0. This corresponds to whether $pG/G \in \mathcal{P}$ for some prime p or not.

Proposition 4.2. *Let $\mathcal{P}$ be a prime ideal in $\widehat{B}(G)$. Then the set*

$$I(G)\backslash I(G) \cap \mathcal{P} = \{G/H|H \leq_o G, G/H \notin \mathcal{P}\}$$

contains at most one minimal element. In the case a minimum exists, we shall call the unique minimum $T_{\mathcal{P}} = G/U$ and for $p = \operatorname{char}\widehat{B}(G)/\mathcal{P}$, we have

$$\mathcal{P} = \{X \in \widehat{B}(G) \mid \varphi_U(X) \equiv 0 \bmod p\}.$$

Proof. Suppose that $I(G)\backslash I(G) \cap \mathcal{P} = \emptyset$, then for all $H \leq_o G$, we have $G/H \in \mathcal{P}$.

$\mathcal{P}$ is a prime ideal and so is closed under addition, therefore since $I(G)$ is $\mathbb{Z}$-spanning set for $\widehat{B}(G)$, we must have that $\langle G/H \mid H \leq_o G \rangle = \widehat{B}(G) \subseteq \mathcal{P}$ which contradicts $\mathcal{P}$ being prime. Now suppose that $I(G) \backslash I(G) \cap \mathcal{P}$ has no minimal element, then we are done. Finally, suppose that $I(G) \backslash I(G) \cap \mathcal{P}$ has minimal elements.

Let $G/H, G/K \in I(G) \backslash I(G) \cap \mathcal{P}$ and assume they are both minimal.

$$\Rightarrow G/H \times G/K = \sum_{g \in [H \backslash G/K]} G/H \cap {}^gK \notin \mathcal{P}$$

since $\mathcal{P}$ is a prime ideal and so the product being in the prime ideal would imply either $G/H \in \mathcal{P}$ or $G/K \in \mathcal{P}$ and so a contradiction. Assume $H \not\sim_G K$, then $H \lesssim_G H, K$ but $H \cap {}^gK \not\sim_G H, K$, then we must have $G/H \cap {}^gK \prec G/H, G/K$ for each $g \in G$, namely that each summand above is strictly less than G/H and G/K in the ordering on $I(G)$. Since this sum is not in $\mathcal{P}$ and $\mathcal{P}$ is closed under addition, there must be at least one of these not in the prime ideal which contradicts the minimality of $G/H, G/K$. We are left to conclude that $H \sim_G K$ and so $G/H \cong G/K$. Since this holds for any two minimal elements, we have that they must all coincide and so if a minimum exists, it is unique.

Suppose that $I(G) \backslash I(G) \cap \mathcal{P}$ has a minimal element G/H and take $X = \sum_{K \leq_o G} X_K \; G/K$, $X_K \in \mathbb{Z}$. Recall for any $K \leq_o G$, $g \in G$, we have $G/H \cap {}^gK \prec G/H$. Let $Y = \sum_{L \leq_o G} Y_H \; G/L = G/H \times X$, then

$$\begin{aligned} Y = G/H \times X = G/H \times \sum_{K \leq_o G} X_K \; G/K &= \sum_{K \leq_o G} X_K \sum_{g \in [H \backslash G/K]} G/H \cap {}^gK \\ &= \sum_{L \lesssim_G H} Y_L \; G/L. \end{aligned}$$

Since $\varphi_H(G/K) = 0$ for $H \not\lesssim_G K$, we have

$$\varphi_H(Y) = \sum_{L \lesssim_G H} Y_L \; \varphi_H(G/L) = Y_H \varphi_H(G/H).$$

Consequently, we have $\varphi_H(G/H)\varphi_H(X) = \varphi_H(Y) = Y_H\varphi_H(G/H)$ and so it follows $\varphi_H(X) = Y_H$. By the expression of $Y = \sum_{L \lesssim_G H} Y_L \; G/L$, and by the minimality of G/H, we have that each other summand for $L \not\sim_G H$ is in $\mathcal{P}$. Therefore $X \in \mathcal{P}$ if and only if $\varphi_H(X) \in \mathcal{P}$, which is to say that $\varphi_H(X) \equiv 0 \bmod \mathrm{char}\widehat{B}(G)/\mathcal{P}$, and since $\mathcal{P}$ is prime, it must either have prime characteristic or 0 characteristic, using the convention that for ease of notation $\varphi_H(X) \equiv 0 \bmod 0 \Leftrightarrow \; \varphi_H(X) = 0$. $\square$

We show that both cases can occur, which is to say that there exists a profinite group G such that the prime ideal $\mathcal{P}_1$ of the Burnside ring of G for which $I(G)\backslash I(G) \cap \mathcal{P}$ has a minimal element, and there exists a profinite group H such that the prime ideal $\mathcal{P}_2$ of the Burnside ring of H for which $I(H)\backslash I(H) \cap \mathcal{P}_2$.

Example 4.3. Take $G = S_3$, since $I(G)\backslash I(G) \cap \mathcal{P}$ is finite, it has a minimal element. This condition holds for any finite group, all of which are profinite by definition. As previously discussed, we can get a prime ideal of $\widehat{B}(G)$ by taking the preimage of a prime ideal under the fixed point map. Let $H = \langle (12) \rangle$, then there is a prime ideal defined by $\varphi_H^{-1}(0) =: \mathcal{P}$. Consider the fixed points of H action on the transitive G-sets, then for $K \leq G$ we have $\varphi_H(G/K) = |\{g \in G/K \mid H^g \leq K\}|$. We have then that $\mathcal{P}$ contains both $S_3/1, \; S_3/\langle (123) \rangle$ since neither contain a conjugate of H and so $\varphi_H(G/1), \varphi_H(G/\langle 123 \rangle) = 0$. Furthermore, we see trivially that $\varphi_H(S_3/S_3) = 1$ and $\varphi_H(S_3/H) = 1$ since H is self normalizing. Consequently, we have that $\mathcal{P} = \langle S_3/1, \; S_3/\langle (123) \rangle \;, S_3/S_3 - S_3/H \rangle$ since linear combinations of these are the only methods of achieving 0 fixed points under H-action. As we can see, G/H is the minimal element of $\mathcal{P}$ in this case.

This approach precisely coincides with the method of taking prime ideals of Burnside rings of finite groups discussed in Dress' paper[8], for in a finite group we have that the number of conjugacy classes of subgroups is always finite and so by [8]Dress Proposition 1, we have that there is always a minimal element not in the prime ideal since each poset of transitive G-sets which are not in the prime ideal is finite.

We shall prove that for the (completed) Burnside ring of a profinite group, there are prime ideals which are not given in this way. In order to show that we can have a prime ideal $\mathcal{P}$ of the Burnside ring of a profinite group G for which $I(G)\backslash I(G)\cap\mathcal{P}$ has no minimal element then, we must have that that G is an infinite profinite group. This alone is not sufficient, however, as we demonstrate with the following examples.

Example 4.4. Let $G=\mathbb{Z}_p$ be the p-adic integers and suppose that $\mathcal{P}\subseteq\widehat{B}(G)$ is a prime ideal such that $I(G)\backslash I(G)\cap\mathcal{P}$ has no minimum. Consider $\mathbb{Z}_p$ as a ring, since it is a principal ideal domain, we have that all ideals can be generated by a single element and so additive subgroups of G as a group are of the form $\langle p^i\rangle$ for some $i\in\mathbb{N}_0$ and so form a total chain.

$$\langle 1\rangle\supseteq\langle p\rangle\supseteq\cdots\supseteq\langle p^i\rangle\supseteq\ldots.$$

If $I(G)\cap\mathcal{P}$ is non empty, then we have that there must be a non negative integer i such that $G/\langle p^i\rangle\in\mathcal{P}$, $G/\langle p^{i+1}\rangle\notin\mathcal{P}$ since otherwise $I(G)\cap\mathcal{P}$ is either empty or $I(G)\backslash I(G)\cap\mathcal{P}$ is finite.

Assume that we have $G/\langle p^i\rangle\in\mathcal{P}$ and $G/\langle p^{i+1}\rangle\notin\mathcal{P}$. Clearly the product $G/\langle p^i\rangle\times G/\langle p^{i+1}\rangle\in\mathcal{P}$ since $\mathcal{P}$ is closed by supermultiplication,

$$\Rightarrow G/\langle p^i\rangle\times G/\langle p^{i+1}\rangle=\sum_{g\in\langle p^i\rangle\backslash G/\langle p^{i+1}\rangle}G/\langle p^i\rangle\cap{}^{g}\langle p^{i+1}\rangle.$$

However, since G is abelian, $|H\backslash G/K|=|G/HK|$ for all $H,K\leq G$ and ${}^{g}\langle p^{i+1}\rangle=\langle p^{i+1}\rangle$. Using that these subgroups form a total chain, we know that $\langle p^i\rangle\cap\langle p^{i+1}\rangle=\langle p^{i+1}\rangle$. Consequently, we have $G/\langle p^i\rangle\times G/\langle p^{i+1}\rangle=|G/\langle p^{i+1}\rangle|\;G/\langle p^{i+1}\rangle\in\mathcal{P}$. By assumption we have $G/\langle p^{i+1}\rangle\notin\mathcal{P}$ and so $p^{i+1}=|G/\langle p^{i+1}\rangle|\in\mathcal{P}$. Consider $G/\langle p^{i+1}\rangle\times G/\langle p^{i+1}\rangle=|G/\langle p^{i+1}\rangle|\;G/\langle p^{i+1}\rangle=p^{i+1}\;G/\langle p^{i+1}\rangle$, then this is in $\mathcal{P}$ since $p\in\mathcal{P}$, but that would imply $G/\langle p^{i+1}\rangle\in\mathcal{P}$ and so we have a contradiction. Namely, that there we cannot have $G/\langle p^i\rangle\in\mathcal{P}$ when $G/\langle p^{i+1}\rangle\notin\mathcal{P}$.

Hence we have contradiction and so no such prime ideal exists, which is to say that all prime ideals of $\widehat{B}(\mathbb{Z}_p)$ must have a minimal isomorphism class of a transitive G-space not in the ideal.

Example 4.5. Consider the profinite completion of the integers, that is to say $\widehat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \prod_{p \text{ prime}} \mathbb{Z}_p =: G$. Let p be a prime and consider the subgroup $H_i = p^i\mathbb{Z}_p \times \prod_{q\neq p} \mathbb{Z}_q \leq_o G$, $i \in \mathbb{N}$. Then we have a prime ideal defined by $\varphi_{H_i}^{-1}(p\mathbb{Z})$. Since $\varphi_{H_i}(G/H_i) = p^i$, we have that $G/H \in \mathcal{P}$ and so $\mathcal{P} \neq \emptyset$. Consider that the H_i fixed points of the identity must be 1 and so $\varphi_{H_i}(G/G) = 1$, therfore $G/G \notin \mathcal{P}$.

Define a map for each prime q, $\pi_q : G \to \mathbb{Z}_q$ to be the projection into that coordinate of the direct product. Choose a prime $r \neq p$ and take a subgroup $K := \prod K_q \leq_o G$ where $K_r <_o \mathbb{Z}_r$, $K_q = \mathbb{Z}_q$ otherwise. Then $\varphi_{H_i}(G/K) = 0$ since H_i is not contained in any conjugate of K since G is abelian and $\pi_r(K) \neq \mathbb{Z}_q = \pi_r(H_i)$. Therefore, $G/K \in \mathcal{P}$. Consider $\varphi_{H_i}(G/H_j) = p^j$ for $i \geq j$ and so $G/H_j \in \mathcal{P}$, therefore the only element of $I(G)\backslash I(G) \cap \mathcal{P}$ is G/G and so $I(G)\backslash I(G) \cap \mathcal{P} = \{G/G\}$. Since this does not depend on our choice of $i \in \mathbb{N}$ for H_i, we have that this holds for any prime ideal defined in such a way.

Proposition 4.6. *For G a profinite group, $\mathcal{P}$ a prime ideal of $\widehat{B}(G)$, if $T = I(G)\backslash I(G) \cap \mathcal{P}$ is infinite, then there does not exist a minimal element of T.*

Proof. Suppose $G/H \in T$ is the minimal element of T and take $G/K \in T\backslash\{G/H\}$. Since $G/H \prec G/K$, we have $H \lesssim_G K$ and so the normal core of H lies within the normal core of K, $\bigcap_{g\in G} H^g \leq \bigcap_{g\in G} K^g$. Since this must hold for each $G/K \in T\backslash\{G/H\}$, we have $\bigcap_{g\in G} H^g \leq \bigcap_{G/K\in T} \bigcap_{g\in G} K^g$. Take a collection of open normal subgroups of G, $\mathcal{N} = \{N \trianglelefteq_o G \mid \exists G/K \in T \text{ such that } \bigcap_{g\in G} K^g = N\}$.

If $\mathcal{N}$ is finite, then there are at most $|G/\bigcap_{N\in\mathcal{N}} N|$ elements in T, but since $\mathcal{N}$ is finite, we have that $\bigcap_{N\in\mathcal{N}} N$ is an open subgroup and so $|G/\bigcap_{N\in\mathcal{N}} N|$ is finite. Consequently, we have that $\mathcal{N}$ must be infinite by assumption that T is infinite. Note that $\bigcap_{N\in\mathcal{N}} N = \bigcap_{G/K\in T} \bigcap_{g\in G} K^g \geq \bigcap_{g\in G} H^g$, and any normal subgroup of H must be contained in $\bigcap_{g\in G} H^g$, but $|G : \bigcap_{N\in\mathcal{N}} N| = \infty$ implies that $|G : \bigcap_{g\in G} H^g| = \infty$. It follows that there does not exist $M \trianglelefteq_o G$ such that $M \leq H$ and so $H \leq G$ is not open, hence $G/H \notin I(G)$. □

Corollary 4.7. *For G a profinite group, $\mathcal{P}$ a prime ideal of $\widehat{B}(G)$, then $T = I(G)\backslash I(G) \cap \mathcal{P}$ is finite if and only if T has a minimal element.*

Example 4.8. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ where $\mathbb{Z}_p$ is the p-adic integers, and take the collection $\mathcal{C} = \{\mathbb{Z}_p \times \mathbb{Z}_p/(\mathbb{Z}_p \times p^i\mathbb{Z}_p) \mid i \in \mathbb{N}_0\} \subseteq \widehat{B}(G)$. $\mathcal{C}$ is infinite, we wish to find a prime ideal $\mathcal{P} \subseteq \widehat{B}(G)$ such that $\mathcal{C} \cap \mathcal{P} = \emptyset$, in doing so we will have demonstrated a prime ideal which has no minimal element by Proposition 4.6 since we will have infinitely many transitive G-space which are not in $\mathcal{P}$. In order to do this, we form a multiplicative set in $\widehat{B}(G)$.

Take $X = \{\prod_{c \in D} c^{k_c} \mid k_c \in \mathbb{N},\ D \subseteq \mathcal{C}\}$ to be the set of product of elements of $\mathcal{C}$. By definition this is a multiplicative set since $G/G = 1 \in \mathcal{C}$ and for any $x, y \in X$, we have $xy \in X$. By Krull's separation lemma[2], if we can find an ideal which is disjoint from X, the non empty multiplicative set, then is it contained in a prime ideal $\mathcal{P}$ which is disjoint from X.

Consider the set $Y = \{\mathbb{Z}_p \times \mathbb{Z}_p/(p^j\mathbb{Z}_p \times p^k\mathbb{Z}_p) \mid j \neq 0,\ k \in \mathbb{N}_0\}$ and the ideal it generates $I = \langle Y \rangle$. Define $S(Y) = \{H \leq_o G$ such that $G/H \in Y\}$ and the projection map $\pi_1 : G \to \mathbb{Z}_p$ to be the projection into the first coordinate. By definition of Y we have that for any $H \in S(Y)$, $\pi_1(H) < \mathbb{Z}_p$. For any $G/K \in \mathcal{C}$, we have $\pi_1(H) = \mathbb{Z}_p$ and so $Y \cap \mathcal{C} = \emptyset$.

Take $G/K \in Y$, $Z \in \widehat{B}(G)$ such that $Z = \sum_{H \leq_o G} z_H \cdot G/H$, $z_H \in \mathbb{Z}$ and consider the product

$$\begin{aligned} G/K &= \sum_{H \leq_o G} z_H \cdot G/K \times G/H \\ &= \sum_{H \leq_o G} z_H \sum_{g \in [K \backslash G/H]} G/K \cap {}^gH. \end{aligned}$$

Since each $K \cap {}^gH \leq K$ for all $g \in G$, $H \leq_o G$, we have $\pi_1(K \cap {}^gH) < \mathbb{Z}_p$ and so no summand of rZ can be written as a linear combination of elements in $\mathcal{C}$.

Take $G/L,\ G/M \in \mathcal{C}$ then

$$G/L \times G/M = \sum_{g\in[L\backslash G/M]} G/L \cap {}^gM = \sum_{g\in[L\backslash G/M]} G/L \cap M$$

since G is abelian. Since $\pi_1(L) = \pi_1(M) = \mathbb{Z}_p$, $\pi_2(L \cap M) = \min\{\pi_2(L), \pi_2(M)\}$ and $L \cap M = \pi_1(L \cap M) \times \pi_2(L \cap M) = \mathbb{Z}_p \times \min\{\pi_2(L), \pi_2(M)\}$, we get that $G/L \cap M$ is also in $\mathcal{C}$, in fact $G/L \cap M = \min\{G/L, G/M\}$. Therefore $G/L \times G/M$ can be expressed as a linear combination of elements in $\mathcal{C}$. Clearly, any finite product of elements in $\mathcal{C}$ can be written as a linear combination of elements of $\mathcal{C}$ and so every element of X can be written as a linear combination of elements of $\mathcal{C}$.

Taking any $s \in I$, we see that for some index set J,

$$s = \sum_{\substack{r_i\in Y,\\ Z_i\in\widehat{B}(G)\\ i\in J}} r_i Z_i \notin X$$

since no summand of each $r_i Z_i$ can be expressed as an element of $\mathcal{C}$. Therefore $I \cap X = \emptyset$ and so by Krull's separation lemma, we have that there exists a prime ideal $\mathcal{P} \supseteq I$ such that $\mathcal{P} \cap X = \emptyset$. We have thus shown the existence of a prime ideal such that $\mathcal{C} \subseteq I(G)\backslash I(G) \cap \mathcal{P}$ is infinite.

Since we have shown there are profinite groups G such that there is a prime ideal $\mathcal{P} \subseteq \widehat{B}(G)$ with a minimal element of $I(G)\backslash I(G)\cap\mathcal{P}$, and a profinite group H such that there is a prime ideal $\mathcal{Q} \subseteq \widehat{B}(H)$ with no minimal element of $I(H)\backslash I(H) \cap \mathcal{Q}$, we have proved the existence of both cases. A natural question arises, given the topology we have specified on $\widehat{B}(G)$, which prime ideals of the Burnside ring of a profinite group G are open, and which are closed but not open. As we have already stated, the prime ideals $\varphi_H^{-1}(p\mathbb{Z})$ for p either 0 or prime, $H \leq_o G$, are open, we will show that these are precisely the open ideals in the topology of $\widehat{B}(G)$ are given by these ideals.

Definition 4.9. We define the ideal $\mathcal{P}_{U,p}$ to be the prime ideal of elements whose number of U-fixed points for $U \leq_o G$ is congruent to 0 mod p for p either prime or 0. That is to say the set $\mathcal{P}_{U,p} = \{X \in \widehat{B}(G) | \varphi_U(X) \equiv 0 \text{ mod } p\}$. This is certainly a prime ideal since it can be written as $\varphi_U^{-1}(p\mathbb{Z})$, noting that we use the convention that in the case $p = 0$, the congruence $\varphi_U(X) \equiv 0 \Leftrightarrow \varphi_U(X) = 0$.

For Burnside rings of finite groups, Dress has shown that this class of prime ideals is exhaustive[8]. This can be seen by applying Proposition 4.2 to a finite group. Consequently we have that for a finite group G, $\mathcal{P}$ a prime ideal of $\widehat{B}(G) = B(G)$ then $I(G) \backslash I(G) \cap \mathcal{P}$ has a minimal element since it is finite. Call this element G/U then by 4.2, we have that $\mathcal{P} = \mathcal{P}_{U,p}$ for p either prime or 0. We note that it is clear that $\varphi_H^{-1}(p\mathbb{Z}) = \mathcal{P}_{H,p}$ for $H \leq_o G$ for G a profinite group by definition 4.9.

Theorem 4.10. *For G a profinite group, then every open prime ideal $\mathcal{P}$ of $\widehat{B}(G)$ is of the form $\mathcal{P} = \mathcal{P}_{U,p}$ for some $U \leq_o G$, p either a prime or 0.*

Proof. Suppose that G is a profinite group, and that $\mathcal{P}$ is an open prime ideal of $\widehat{B}(G)$. Since $\mathcal{P}$ is open, we have that there exists a set $X = \{x\} \subseteq \widehat{B}(G)$ such that $\mathcal{P} = \bigcup_{x \in X} x + \ker(\pi^G_{N_x})$ for some $N_x \trianglelefteq_o G$. However, $\mathcal{P}$ is a prime ideal and so $x + \ker(\pi^G_{N_x}) \subseteq \mathcal{P}$ implies that $-(x + \ker(\pi^G_{N_x})) = -x - \ker(\pi^G_{N_x}) \subseteq \mathcal{P}$ and since $-\ker(\pi^G_{N_x}) = \ker(\pi^G_{N_x})$, we have that $-x + \ker(\pi^G_{N_x}) \subseteq \mathcal{P}$. As $\mathcal{P}$ is closed under addition, we have $-x + x + \ker(\pi^G_{N_x}) = \ker(\pi^G_{N_x}) \subseteq \mathcal{P}$.

Therefore the prime ideal contains the isomorphism classes of transitive G-spaces which are in the kernel of $\pi^G_{N_x}$, $\mathcal{P} \supseteq \ker(\pi^G_{N_x}) \supseteq \{G/K \mid N \not\leq K \leq_o G\}$. It follows that $I(G) \backslash I(G) \cap \mathcal{P} \subseteq \{G/H \mid N \leq H\}$, and since G/N is finite, we have that $\{G/H \mid N \leq H\}$ is finite and so has a minimal element. By Proposition 4.2 we have that $\mathcal{P} = \mathcal{P}_{U,p}$ for some $U \leq_o G$, p either prime or 0.

□

Corollary 4.11. *For G a profinite group, then $\mathcal{P}$ a prime ideal of $\widehat{B}(G)$ is open if and only if $I(G) \backslash I(G) \cap \mathcal{P}$ is finite.*

Note that with the definition above, we now have a method of expressing the kernel of the restriction map. Dress-Siebeneicher[10] (2.10.6) states that for

$U \leq_o G$, $\ker(\mathrm{res}^G_U) = \{X \in \widehat{B}(G) \mid \varphi_V(X) = 0,\ \forall V \leq_o G,\ V \lesssim_G U\}$. In particular, each $X \in \ker(\mathrm{res}^G_S)$ must also lie in each $X \in \mathcal{P}_{V,0}$ for each $V \lesssim_G S$, and so we have $\ker(\mathrm{res}^G_S) = \bigcap_{V \lesssim_G S} \mathcal{P}_{V,0}$. Since res^G_S is a ring homomorphism, we use the first isomorphism theorem and the following result is immediate by $\widehat{B}(\mathcal{F}) = \mathrm{res}^G_S(\widehat{B}(G))$ as proved in the previous chapter, in Theorem 3.22.

Proposition 4.12. *For $\mathcal{F} = \mathcal{F}_S(G)$ the pro-fusion system of G over S, $S \leq_o G$, then we have*

$$\mathrm{res}^G_S(\widehat{B}(G)) = \widehat{B}(\mathcal{F}) \cong \widehat{B}(G)/ \bigcap_{H \lesssim_O S} \mathcal{P}_{H,0}.$$

Proof. By Theorem 3.22, we have that $\mathrm{res}^G_S : \widehat{B}(G) \to \widehat{B}(\mathcal{F})$ is a surjective ring homomorphism. Subsequently, applying the first isomorphism theorem for rings, we have that $\widehat{B}(\mathcal{F}) \cong \widehat{B}(G)/\ker(\mathrm{res}^G_S)$. Note that the kernel of res^G_S is the set of elements of $\widehat{B}(G)$ which map to $0 \in \widehat{B}(\mathcal{F})$. This is equivalent to the elements $X \in \widehat{B}(G)$ which have $\varphi_P(\mathrm{res}^G_S(X)) = 0$ for all $P \leq_o S$ since the image in the ghost ring is injective.

Since X and $\mathrm{res}^G_S(X)$ denote the same underlying set perceived with G-action or the S-action induced by G respectively, we have that they are equivalent under P-action for $P \leq_o S$. It therefore follows that we have $\ker(\mathrm{res}^G_S) = \{X \in \widehat{B}(G) \mid \varphi_H(X) = 0 \quad \forall H \leq_o S\}$. By definition 4.9, we have that $\mathcal{P}_{U,0} = \{X \in \widehat{B}(G) \mid \varphi_U(X) = 0\}$ and so $\ker(\mathrm{res}^G_S) = \bigcap_{H \leq_o S} \mathcal{P}_{H,0}$. □

We have now classified all open prime ideals, however, there is no explicit requirement that $\mathcal{P}_{U,p} \neq \mathcal{P}_{V,q}$ for some $U, V \leq_o G$, $U \not\sim_G V$, p, q prime or 0 and so we wish to investigate when two prime ideals defined in this way are equal. We know that both of these expressions must share a minimal isomorphism class of a transitive G-space but there is no claim that this need be the class of G/U or G/V. We begin by showing that it can be the case that two prime ideals, each of the form $\mathcal{P}_{U,p}$ defined by different subgroups need not be distinct. We subsequently prove Proposition 4.14 which is a generalisation of [8]Dress Proposition 1.b) on when prime ideals are subsets of each other.

Example 4.13. In example 4.5 we took the profinite completion of the integers, given by the expression $G = \prod_{p \text{ prime}} \mathbb{Z}_p$, and showed that for $H_i = p^i\mathbb{Z}_p \times \prod_{P \neq q} \mathbb{Z}_q \leq_o G_i$, $i \in \mathbb{N}$, we have that the prime ideal given by $\varphi_{H_i}^{-1}(p\mathbb{Z}) = \mathcal{P}_{H_i,p} =: \mathcal{P}_i$ has a minimal element of $I(G)\backslash I(G) \cap \mathcal{P}_i$ given by G/G. By Proposition 4.2, this is a defining element of $\mathcal{P}_i$ and so we also have $X \in \mathcal{P}_i$ if and only if $\varphi_G(X) \equiv 0 \mod p$. Since this holds for any $i \in \mathbb{N}$, we have that $\mathcal{P}_i = \mathcal{P}_j$ for all $i, j \in \mathbb{N}$ and so we see that they need not be distinct prime ideals despite $H_i \neq H_j$ for $i \neq j$.

Proposition 4.14. *Let G be a profinite group, $U, V \leq_o G$ and p, q primes or 0, then we have that*

$$\mathcal{P}_{U,p} \subseteq \mathcal{P}_{V,q} \Leftrightarrow \begin{cases} p = q & \mathcal{P}_{U,p} = \mathcal{P}_{V,q}, \\ p = 0, q \neq 0 & \mathcal{P}_{U,q} = \mathcal{P}_{V,q}. \end{cases}$$

Therefore $\mathcal{P}_{U,0}$ is minimal and $\mathcal{P}_{U,p}$, $p \neq 0$ is maximal.

This result is equivalent to saying that if we have two prime ideals defined by the number of U-fixed points and the number of V-fixed points respectively such that $\mathcal{P}_{U,p} \subseteq \mathcal{P}_{V,q}$, then we have that either $p = q$ and the prime ideals are the same and every element, or we have that for $q \neq 0$, $\mathcal{P}_{U,0} \subset \mathcal{P}_{V,q} = \mathcal{P}_{U,q}$.

Proof. Suppose that p, q are both primes with $p \neq q$, then we have by definition that $pG/G \in \mathcal{P}_{U,p}$ and $qG/G \in \mathcal{P}_{V,q}$ since $\varphi_U(pG/G) = p\varphi_U(G/G) = p$ and $\varphi_V(qG/G) = q\varphi_V(G/G) = q$. Assume that $\mathcal{P}_{U,p} \subseteq \mathcal{P}_{V,p}$, it follows that $pG/G, qG/G \in \mathcal{P}_{V,q}$ and by immediate consequence of Bezout's lemma we have that there exist $m, n \in \mathbb{Z}$ such that $mp + nq = 1 = G/G$ and so by $\mathcal{P}_{V,q}$ being closed under addition, we have $1 \in \mathcal{P}_{V,q}$. Taking any $X \in \widehat{B}(G)$ it follows that $1 \cdot X = X \in \mathcal{P}_{V,q}$ and so $\mathcal{P}_{V,q} = \widehat{B}(G)$ which is a contradiction and therefore we cannot have p and q distinct primes.

Now consider $p = q$ a prime, and consider the sets $T_1 = I(G)\backslash I(G) \cap \mathcal{P}_{U,p}$, $T_2 = I(G)\backslash I(G) \cap \mathcal{P}_{V,p}$, then we have $T_2 \subseteq T_1$ since $\mathcal{P}_{U,p} \subseteq \mathcal{P}_{V,p}$. By Proposition 4.2 there is a minimal element to both T_1 and T_2 respectively, define G/H to be the minimal element of T_1 and G/K to be the minimal element of T_2. Suppose that $T_2 = T_1$ then they share a minimal element and by Proposition 4.2 we have that this minimal element defines both ideas by the relation $X \in \mathcal{P}_{U,p}$ if and only if $\varphi_H(X) \equiv 0 \text{ mod } p \Leftrightarrow \ \varphi_K(X) \equiv 0 \text{ mod } p$ if and only if $X \in \mathcal{P}_{V,p}$ since we must have $H \sim K$. In this case we get equality of the prime ideals $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$.

Suppose that $\mathcal{P}_{U,p} \subset \mathcal{P}_{V,p}$, then in particular we have $G/H \prec G/K$. Consider the fixed points $\varphi_K(G/K) = |N_G(K) : K| \not\equiv 0 \text{ mod } p$, we have $\varphi_H(G/K) = |\{gK \in G/K \mid H^g \leq V\}| = |N_G(K) : K||\{g \in G/N_G(K) \mid H^g \leq K\}|$. If p divides $|\{G/N_G(K) \mid H^g \leq K\}|$ then $G/K \in \mathcal{P}_{U,p}, G/K \notin \mathcal{P}_{V,p}$ and so we contradict out assumption. If p does not divide $|\{G/N_G(K) \mid H^g \leq K\}|$, then it has an inverse in $\mathbb{Z}/p\mathbb{Z}$, and so we can select $r \in \mathbb{Z}$ such that $G/K - rG/H \equiv 0 \text{ mod } p$, in which case we have $G/K - rG/H \in \mathcal{P}_{U,p}$ but $G/K - rG/H \notin \mathcal{P}_{V,p}$, once again we have a contradiction and so we conclude that we cannot have $\mathcal{P}_{U,p} \subset \mathcal{P}_{V,p}$ for any prime p.

If $p = q = 0$ then both prime ideals are defined by the same unique element $G/H \in I(G)\backslash I(G) \cap \mathcal{P}_{U,p}$ since otherwise we can follow a similar process to above to contradict the inclusion, and therefore the only possibility is given by $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$. If p is prime but $q = 0$ then we have a contradiction since there are elements in $\mathcal{P}_{U,p}$ which are not in $\mathcal{P}_{V,q}$, namely we can take the element pG/K, since $\varphi_K(pG/K) = p|N_G(K) : K| \neq 0$ but $\varphi_H(pG/K) \equiv 0 \text{ mod } p$.

Finally, we suppose that $p = 0$, q a prime. We have $\mathcal{P}_{U,0} \subseteq \mathcal{P}_{U,q}$ since we have that $\varphi_H(X) = 0 \Rightarrow \varphi_H(X) \equiv 0 \text{ mod } q$. Any prime ideal with characteristic q which contains $\mathcal{P}_{U,0}$ must also contain $\mathcal{P}_{U,p}$ since by Proposition 4.2 we have shown that this defines the minimal ideal with this property and so $\mathcal{P}_{U,q} \subseteq \mathcal{P}_{V,q}$. By the rest of the Proposition, we have already shown that in this case we must have equality and so we have proved all required statements. It naturally follows that any open prime ideal defined with characteristic 0 is minimal and any prime ideal with prime characteristic is maximal.

□

The following few Propositions now cover the final result on prime ideals as proved in Dress' paper[8]. The result in question concerns the case when we have equality of prime ideals $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$ and gives a condition for when they coincide. This is namely that (for the Burnside ring of a finite group) $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$ if and only if $U^p \sim V^p$ where U^p denotes the smallest normal subgroup of U such that U/U^p is a p-group. Difficulty arises with profinite groups since the natural analogue would be to take the minimal normal subgroup, N, of an open subgroup U of a profinite group G such that U/N is a pro-p group, the problem being that N may not be open and so any argument based on almost finite G-spaces becomes difficult since there is no condition that the number of fixed points be finite for non open subgroups.

Indeed, if for example we take the p-adic integers $\mathbb{Z}_p$ or any infinite pro-p group, we have that there is never an open normal subgroup minimal such that the factor group is a p-group. In appendix B of Dress' notes on representation[9], he considers the Burnside ring of profinite groups defined by the Grothendieck ring of isomorphism of finite G-sets for a profinite group G, this guarantees that the number of fixed point shall always be finite under the action of any closed subgroups since there are only finitely many fixed points. We seek to find a useful replacement for taking U^p in the context of profinite groups.

We first begin with a technical lemma that shall prove useful and then follow with the most simple case, that is when this minimal element does exist and is open.

Lemma 4.15. *Let U be a profinite group and suppose that $W \trianglelefteq_o U$ is an open normal subgroup of U such that U/W is a p-group, then for $X \in \widehat{B}(U)$, we get $\varphi_U(X) \equiv \varphi_W(X)$ mod p and $\mathcal{P}_{U,p} = \mathcal{P}_{W,p}$.*

Proof. Suppose $W \trianglelefteq_o U, U/W$ a p-group and take $X \in \widehat{B}(U)$. Recall that the set $X^W = \{x \in X \mid W.x = x\}$ denotes the elements in X which are invariant under W-action. Since $W \leq U$, we have that those elements which are invariant under U action can be seen as a subset of $X^W \supseteq X^U$. We can therefore deconstruct the set as $X^W = X^U + (X^W - X^U) = X^U + X^W \backslash X^U$ with addition and negation induced by the Burnside ring to be set unions with multiplicity and formal negation respectively. Take $x \in X^W \backslash X^U$, then we have by definition $U.x \neq x$ and $W.x = x$. It follows that we can express the W-action on $X^W \backslash X^U$ by allowing W to act trivially on $X^W \backslash X^U$. Therefore, any U action on $X^W \backslash X^U$ can be expressed as U/W action on it.

It follows that there exist $K \leq U/W$, $a_K \in \mathbb{Z}$ such that

$$X^W \backslash X^U = \sum_{K \leq U/W} a_K((U/W)/K)^W$$

as a U-space since U action in this case is equivalent to U/W-action. Additionally, we have that in the sum, $K \neq U/W$ since in this case the elements would also be invariant under U-action and so lie within X^U. It follows that as a U-space we can write the number of fixed point of X under W action as the sum $X^W = X^U + X^W \backslash X^U = X^U + \sum_{K<U/W} a_K((U/W)/K) = X^U + \sum_{K<U/W} a_K(U/WK)$. Note that U/W a p-group implies that p divides U/WK for all K such that $U/WK \neq U/W$. Take the W fixed points of both sides and we have

$$\begin{aligned}
\varphi_W(X) = |X^W| = |X^U| + \sum_{K<U/W} a_K|(U/WK)^W| \\
\equiv |X^U| \bmod p \\
\equiv \varphi_U(X) \bmod p.
\end{aligned}$$

$$\mathcal{P}_{U,p} = \{X \in \widehat{B}(G) | \varphi_U(X) \equiv \varphi_W(X) \equiv 0 \bmod p\} = \mathcal{P}_{W,p}$$

□

Example 4.16. Let $G = GL_n(\mathbb{Z}_p)$ for $p \geq 7$ a prime, then we have a system of open normal subgroups given by $N_k = \{x \in G \mid x \equiv I_n \bmod p^k\}$ for $k \in \mathbb{N}$ and consider the open subgroups $N := N_k,\ M := N_l$ for some $k, l \in \mathbb{N}$. For $K \trianglelefteq_o G$ and any $H \leq_o G$, we have that $\varphi_K(G/H) = |\{gH \in G/H \mid K^g \leq H\}| = |\{gH \in$

$G/H \mid K \leq H\}|$, this then splits into two cases, if $K \leq H$, $\varphi_K(G/H) = |G : H|$ otherwise we get $\varphi_K(G/H) = 0$. Take $\mathcal{P} = \mathcal{P}_{N,p} = \{X \in \widehat{B}(G) \mid \varphi_N(X) \equiv 0 \bmod p\}$.

If $N \not\leq H$, we have that $\varphi_N(G/H) = 0$ and so $G/H \in \mathcal{P}$, suppose that $N \leq H$ and that p divides $|G : H|$ then $G/H \in \mathcal{P}$. If $N, M \leq H$ then $\varphi_N(G/H) = |G : H| = \varphi_M(G/H)$ and if $M \leq H$, $N \not\leq H$ then $\varphi_N(G/H) = 0$, $\varphi_M(G/H)$ and vice versa. Since the open normal subgroups make a chain, we have $\varphi_M(G/H) = |G : H| \not\equiv 0$ but $\varphi_N(G/H) \equiv 0$ implies that $H \geq N$ but $H \not\geq M$ and therefore $\varphi_N(G/H) = 0$. Assume that $M \leq N$, then we clearly have by the fixed points of the transitive G-spaces that for $X \in \widehat{B}(G)$ written as $X = \sum_{H \leq_o G} X_H \; G/H$, then

$$\begin{aligned}\varphi_N(X) = \varphi_N\left(\sum_{H \leq_o G} X/H \; G/H\right) &= \sum_{H \leq_o G} X_H \; \varphi_N(G/H) \\ &\equiv \sum_{p \nmid \varphi_N(G/M)} X_H \varphi_N(G/H) \bmod p\end{aligned}$$

$$\varphi_M(X) \equiv \sum_{\substack{p \nmid \varphi_N(G/H) \\ p \nmid \varphi_M(G/H)}} X_H \varphi_M(G/H) + \sum_{\substack{p \nmid \varphi_M(G/H) \\ \varphi_M(G/H)=0}} X_H \varphi_M(G/H).$$

Clearly, since $M \leq N$, we have that the $p \nmid \varphi_N(G/H)$ implies $p \nmid \varphi_M(G/H)$ since if $N \leq H$, we must have that $M \leq H$. Assume that p does not divide $|G : H| = \varphi_M(G/H)$, then we have $|G : M| = |G : H||H : M|$, and so the multiplicity of p in $|G : M|$ is equal to the multiplicity of p in $|G : H|$. Taking the group G/M, we see that H/M is a Sylow p-subgroup of this and so $N/M \leq H/M$ since N/M is normal in G/M. Subsequently we have that we have a contradiction and so the latter sum in the expression of $\varphi_M(X)$ is congruent to 0 mod p.

It follows that $\varphi_N(X) \equiv \varphi_M(X) \bmod p$ for all $X \in \widehat{B}(G)$. Therefore we have $\mathcal{P}_{N,p} = \mathcal{P}_{M,p}$.

Proposition 4.17. *Let U be a profinite group and p be a prime. Then if there is a minimal open normal subgroup $U^p \leq U$ such that U/U^p is a p-group, then $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$ if and only if $U^p \sim V^p$.*

Proof. Assume that $U^p \sim V^p$, then we have $\varphi_{U^p}(X) = \varphi_{V^p}(X)$ for every $X \in \widehat{B}(G)$. Applying the previous lemma to U and V respectively, we have $\varphi_{U^p}(X) \equiv \varphi_U(X) \equiv \varphi_V(X) \equiv \varphi_{V^p}(X) \bmod p$ for each $X \in \widehat{B}(G)$. Therefore, we have $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$.

Conversely, assume that $\mathcal{P}_{U,p} = \mathcal{P}_{V,p} = \mathcal{P}$ and that the minimal element of $I(G)\backslash I(G) \cap \mathcal{P}$ is given by G/W. Then a prime ideal $\mathcal{P}_{H,p}$ is equivalent to $\mathcal{P}$ if and only if $\varphi_H(X) \equiv \varphi_W(X) \equiv 0 \bmod p$ for all $X \in \mathcal{P}$. Now consider

$$\begin{array}{ccc} N_G(U^p) & \longrightarrow & N_G(U^p)/U^p \\ | & & | \\ U_p & \longrightarrow & S \\ | & & | \\ U^p & \longrightarrow & 1 \end{array}$$

where U_p is the preimage of a p-Sylow subgroup, S of $N_G(U^p)/U^p$ under the quotient map. Now applying that U^p is the minimal open normal subgroup of U such that U/U^p is a p-group, therefore $(U_p)^p = U^p$ as otherwise there would be some smaller $H \trianglelefteq_o U^p$ such that U/H is a p-group. U^p is characteristic in U_p and so $N_G(U_p) \subseteq N_G(U^p)$ and $p \nmid |N_G(U_p) : U_p|$ since U_p is the preimage of a p-Sylow subgroup.

$$\Rightarrow \varphi_{U_p}(X) \equiv \varphi_U(X) \equiv \varphi_{U^p}(X) \bmod p$$

by applying the previous lemma and noting that $U^p \trianglelefteq U_p$. Repeating this processes for V and using that the respective U_p, V_p are preimages of Sylows,

$$\Rightarrow U_p \sim V_p$$

$$\Rightarrow (U_p)^p = U^p \sim V^p = (V_p)^p.$$

$\square$

Noting that in the above argument we see that $U_p \notin \mathcal{P}$, we in particular get the following.

Corollary 4.18. *In the above Proposition, with $\mathcal{P} = \mathcal{P}_{U,p}$ with definition 4.9, then $T_{\mathcal{P}} = G/U_p$.*

Example 4.19. Let $G = GL_2(\mathbb{Z}_5)$ to be the general linear group over the ring of p-adic integers, then this is virtually a pro-5 group, which is to say that there exists a subgroup $H \leq_o G$ such that H is a pro-5 group. Namely, the open normal subgroup $N = \{x \in GL_2(\mathbb{Z}_5) \mid x \equiv I_2 \bmod 5\}$ is pro-5, although not maximal pro-5 since we have $G/N \cong GL_2(5)$ and this has subgroups of order 5, for example the subgroup given by $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. Taking S to be a Sylow-3-subgroup of G/N, and taking the preimeage under the projection map ${\pi_N^G}^{-1}(S) = SN$, then the minimal open subgroup of G, M, such that SN/M is a 3-group, we have that $S^3 = N$ since N is a pro-5 group and so no subgroup has index 3.

We have thus shown that it is possible to have a subgroup which adheres to having a minimal open subgroup such that U/U^p is a p-group. However, Proposition 4.17 also states that all such minimal open subgroups are conjugate in G. It's important to note that $U^p \trianglelefteq_o U \leq_o G$ but that U^p is not necessarily open in G, although we do have that in this case.

When dealing with an infinite profinite group, U, it is possible that there is an infinite set $T = \{N \mid N \trianglelefteq_o U,\ U/N \text{ a } p \text{ group}\}$, for example $\mathbb{Z}_p$. In this case, the subgroup $M = \cap_{N \in T} N$ gives a minimal open subgroup of U such that U/N is a pro-p group. Since this group M is closed and not open, it is not certain that there are finitely many fixed points under M action on almost finite U-spaces. We show that this can be expressed as an inverse limit as shown in Dress' notes[9].

Let $G = \varprojlim_{N \trianglelefteq_o G} G/N$, then for $U \leq_o G$, $N \trianglelefteq_o G$, we have that $(f_N(U))^p \leq f_N(U)$ is the minimal (finite) normal subgroup such that $f_N(U)/(f_N(U))^p$ is a p-group. Take $M \trianglelefteq_o U$ the minimal open normal subgroup of U such that U/M is a pro-p group. Consider the projection $f_N(M) = f_N(NM)$, then NM is an open subgroup of U containing M, therefore U/NM is a p-group since U/M is a pro-p group and so $f_N(M)$ is a normal subgroup such that $f_N(U)/f_N(M)$ is a p-group. It follows that $(f_N(U))^p \subseteq f_N(M)$. It also holds that $K := {f_N}^{-1}(f_N(U)^p) \cap U$ is a normal subgroup of U such that U/K is a p-group and so $f_N(M) \subseteq (f_N(U)^p)$. Consequently we have that $U^p = \varprojlim_{N \trianglelefteq_o G} f_N(U)^p$.

We now note that this is equivalent to stating $U^p = \varprojlim_{N \trianglelefteq_o U} (U/N)^p$. We note that since $U^p \leq U$, we have that $U \leq N_G(U^p)$ and so $N_G(U^p)$ is open in G. Since U/U^p is a pro-p group, there is a Sylow pro-p group of $N_G(U^p)/U^p$ containing it, denote this subgroup by S. Taking the preimage of S in $N_G(U^p)$, we define this element to be U_p. Consider the map $f_N : G \to G/N$, then we have that $f_N(U_p) = U_p/N$. Now $|N_G(U^p)/N : U_p/N|$ must be coprime to p, and so we get that $U_p = \varprojlim_{N \trianglelefteq_o G} (U/N)_p$ and so is a well defined open subgroup of G for G open.

Proposition 4.20. *Let G be a profinite group, $U, V \leq_o G$, then we have $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$ for a prime p if and only if $U^p \sim V^p$.*

Proof. Assume that $U^p \sim V^p$ and that U^p is closed but not open since otherwise we have already proved by Proposition 4.17. It follows that the set $T = \{N \mid N \trianglelefteq_o U,\ U/N \text{ a } p\text{-group}\}$ is an infinite set since otherwise, $U^p = \cap_{N \in T} N$ is the intersection of finitely many open subgroups and therefore open. Take $M \in T$, then we have $M \leq_o U$ and $U^p \leq M$. U/M is a p-group and M/U^p is a pro-p group since U/U^p is a pro-$p$group. Since M is normal in G, M/U^p must be normal within the quotient of $N_G(U^p)/U^p$ and so in particular lies within every Sylow pro-p subgroup of $N_G(U^p)/U^p$ since it is also a pro-p group. It follows then that $M \leq U_p$ such that U/M is a p-group and U_p/M is a p-group, it follows that $\varphi_U(X) \equiv \varphi_M(X) \equiv \varphi_{U_p}(X) \bmod p$ for all $X \in \widehat{B}(G)$. In particular,we note that $\varphi_U(G/U_p) \equiv \varphi_{U_p}(G/U_p) \not\equiv 0 \bmod p$ and so we have that G/U_p is the defining element for the prime ideal.

Repeating the process with V, we see that $U^p \sim V^p$ implies that $f_N(U^p) \sim f_N(V^p)$ for all $n \trianglelefteq_o G$ and so we have that in particular, $f_N(U_p) \sim f_N(V_p)$ since it is unique up to G conjugation, thus we have that $U_p = V_p$, namely that $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$ since they share the same defining element. ∎

Proposition 4.21. *If $|U : K_G(U)| = p^k$ for some k, $p \neq 0$, and there exist $\{N_i | i \in I\}$, I infinite, such that U/N_i is a p-group, then we have that for $\mathcal{P} = \mathcal{P}_{U,p}$,*

$T_{\mathcal{P}} = G/U_p$ where U_p is the preimage of the p-Sylow of $G/K_G(U)$ in G.

Proof. Consider the following diagram.

$$\begin{array}{ccc} G & \longrightarrow & G/K_G(U) \\ | & & | \\ U_p & \longrightarrow & S \\ | & & | \\ K_G(U) & \longrightarrow & 1 \end{array}$$

By the previously stated lemma we have

$$\mathcal{P}_{U_p,p} = \mathcal{P}_{K_G(U),p} = \mathcal{P}_{U,p}$$

since the core is normal in G. Now consider $\varphi_{U_p}(G/U_p)$.

$$\begin{aligned} \varphi_{U_p}(G/U_p) &= |\{gU_p \mid h.gU_p = gU_p \quad \forall h \in U_p\}| \\ &= |\{gU_p \mid h^g U_p = U_p \quad \forall h \in U_p\}| \\ &= |\{gU_p \mid U_p^g \leq U_p\}| \\ &= |\{gU_p \mid U_p^g = U_p\}| \\ &= |N_G(U_p)/U_p| \\ &= |N_G(U_p) : U_p|. \end{aligned}$$

Now we have that $N_G(U_p) \leq N_G(K_G(U)) = G$

$$\Rightarrow p \nmid |N_G(U_p) : U_p|$$

since U_p is the preimage of a p-Sylow.

$$\Rightarrow G/U_p \notin \mathcal{P}$$

$$\Rightarrow T_{\mathcal{P}} = G/U_p.$$

$\square$

We are searching for a good way of classifying groups that induce these prime ideals. In the case of a minimal open normal group we have already shown that it's sufficient to check $U^p \sim V^p$.

Remark 4.22. If $|U : K_G(U)| = p^k p'$ then we necessarily have that $\nexists N_i \leq K_G(U)$ since otherwise $|U : N_i| = p^k p' q$ for some q and this is not a p-power. In particular we have that every N_i is normal in U but not normal in G.

Proposition 4.23. *Let $\mathcal{P} = \mathcal{P}_{U,p}$ and assume that there exists $\{N_i \trianglelefteq U | i \in I\}$, I infinite, such that U/N_i is a p-group.*
Then we have that $T_{\mathcal{P}} = G/U_p$ where U_p is the preimage of the p-Sylow subgroup of $N_G(U^p)/U^p$ for U^p the minimal subgroup $U \leq U^p \leq K_G(U)$ such that U/U^p is a p-group.

Proof. This is illustrated with the following diagram.

$$\begin{tikzcd}
 & N_G(U^p) \arrow[r] \arrow[d, dash] & G/K_G(U) \arrow[d, dash] \\
 & U_p \arrow[r] \arrow[d, dash] & S \arrow[d, dash] \\
U \arrow[r, dash] & U^p \arrow[r] \arrow[d, dash] & 1 \\
 & K_G(U) &
\end{tikzcd}$$

Applying the lemma we once again have

$$\mathcal{P}_{U,p} = \mathcal{P}_{U^p,p} = \mathcal{P}_{U_p,p}.$$

Additionally

$$\varphi_{U_p}(G/U_p) = |N_G(U_p) : U_p|,$$

$N_G(U_p) \leq N_G(U^p)$ since U^p characteristic in U_p,

$$\Rightarrow p \nmid |N_G(U_p) : U_p|$$

$$\Rightarrow G/U_p \notin \mathcal{P}$$

$$\Rightarrow T_{\mathcal{P}} = G/U_p.$$

$\square$

Proposition 4.24. *Assume $p \neq 0$ and that $U, V \leq_o G$ with respective infinite sets of open normal subgroups $\{N_i \trianglelefteq_o U \mid U/N_i \text{ is a } p\text{-group}\}, \{M_j \trianglelefteq_o V \mid V/M_j \text{ is a } p\text{-group}\}$. If there exists some N_i, M_j such that $N_i \sim M_j$, then we have that $\mathcal{P}_{U,p} = \mathcal{P}_{V,p}$.*

Proof. Assume that $\exists N_i \trianglelefteq U, M_j \trianglelefteq V$ such that $N_i \sim M_j$.

$$\varphi_U(X) \equiv \varphi_{N_i}(X) = \varphi_{M_j}(X) \equiv \varphi_V(X) \bmod p,$$

$$\Rightarrow \mathcal{P}_{U,p} = \mathcal{P}_{V,p}.$$

$\square$

Therefore we have that the prime ideals defined by the number of H-fixed points for $H \leq_o G$ are precisely the open prime ideals of $\widehat{B}(G)$. This is to say the prime ideal defined in definition 4.9 by $\mathcal{P}_{U,p}$ for p a prime or 0 and $U \leq_o G$.

4.3 Prime ideals in $\widehat{B}(S)$

Let S be a pro-p group. We seek to specifically examine the open prime ideals of $\widehat{B}(S)$ in order to classify the open prime ideals of the Burnside ring of a pro-p group. As for any profinite group, we have that either a prime ideal $\mathcal{P}$ has a minimal element in $I(S) \backslash I(S) \cap \mathcal{P}$ or it does not. The case where a minimal element exists, we have that the prime ideal is open, and the case where it does not, the prime ideal is closed but not open. Both of these can occur with pro-p groups as shown in the previous section with the prime ideals of $\mathbb{Z}_p$ and of $\mathbb{Z}_p \times \mathbb{Z}_p$ respectively.

Recall that if $I(S)\backslash I(S)\cap\mathcal{P}$ has a minimal element, then $\mathcal{P} = \mathcal{P}_{U,q}$ for $U \leq S, q$ a prime or 0 with the definition $\mathcal{P}_{U,q} = \{X \in \widehat{B}(S) | \varphi_U(X) \equiv 0 \bmod q\}$. Since for any $U \leq_o S$ we have $|S : U| = p^r$ for some $r \in \mathbb{N}_0$, we have that every subgroup is of p power index.

Lemma 4.25. *For S a pro-p group and any $U \leq_o S,\ K \leq_o S$ we have the following result*

$$\varphi_U(S/K) = |N_S(K) : K||\{g \in S/N_S(K) \mid U^g \leq K\}|.$$

Proof. Let $U, K \leq_o S$, then we have $\varphi_U(S/K) = |\{g \in S/K \mid U.gK = gK\}| = |\{g \in S/K \mid U^g K = K\}| = |\{g \in S/K \mid U^g \leq K\}| = |\{g \in S/K \mid U \leq {}^g K\}| = |N_S(K) : K||\{g \in S/N_S(K) \mid U^g \leq K\}|.$ □

Lemma 4.26. *If we suppose $K \trianglelefteq_o S$, then that $q \nmid \varphi_U(G/K), q \neq p$*

$$G/K \in \mathcal{P}_{U,q} \Leftrightarrow \varphi_U(G/K) \equiv 0 \bmod q \Leftrightarrow q \mid \ |\{\bar{g} \in G/N_G(K)|U^g \leq K\}|.$$

Corollary 4.27. *If $q = p$ then*

$$\mathrm{span}_{\mathbb{Z}}\{G/K|N_G(K) \neq K\} \subseteq \mathcal{P}_{U,p}.$$

Proof. As we noted before

$$\varphi_U(G/K) = |N_G(K) : K||\{\bar{g} \in G/N_G(K)|U^g \subseteq K\}|,$$

and if $|N_G(K) : K| \neq 1 \Rightarrow p\big||N_G(K) : K| \Rightarrow G/K \in \mathcal{P}_{U,p}$. □

It remains to observe what happens in the case that K is self normalizing.

Corollary 4.28. *$K \subseteq G$ is self normalizing $\Leftrightarrow \varphi_K(G/K) = 1$.*

Proof.

$$\varphi_K(G/K) = |N_G(K) : K| = 1 \Leftrightarrow N_G(K) = K$$

□

Combining the fact that $\mathcal{P}_{U,0} \subseteq \mathcal{P}_{U,p}$, we get that the only self normalizing subgroups we need to consider are the subgroups K such that $U \lesssim K, K = N_G(K)$.

Corollary 4.29.

$$I(S)\backslash I(S)\cap\mathcal{P}_{U,p}\subseteq\{S/K|K=N_S(K),U\lesssim K\}.$$

Corollary 4.30.

$$U\leq K_S(K)\Rightarrow U\subseteq{}^sK\ \forall s\in S\Rightarrow\varphi_U(K)=|S:K|.$$

Proposition 4.31. *Suppose that S is an abelian pro-p group, $U\leq_o S$ then the prime ideal $\mathcal{P}_{U,p}=\operatorname{span}_{\mathbb{Z}}\{I(S)\backslash\{S/S\}\}+p\mathbb{Z}[S/S]$ and $I(S)\backslash I(S)\cap\mathcal{P}_{U,0}=\{S/K|U\leq K\}$.*

Proof. First note that $\varphi_U(S/H)=|\{sH\mid U^s\leq K\}|$ for $U,H\leq_o S$ as a consequence of 2.23, and therefore we have that $\varphi_U(S/H)=0$ if $U\not\lesssim H$. Taking the number of U-fixed points of $X=\sum'_{H\leq_o S}x_H\cdot S/H\in\widehat{B}(S)$ with the series over the conjugacy class representatives, we use that $\varphi_U(S/H)=0$ for $U\not\lesssim H$ to restrict the series to a finite sum.

$$\begin{aligned}\varphi_U(X)&=\sum_{H\leq_o S}{}'\varphi_U(x_H\cdot S/H)\\&=\sum_{H\leq_o S}{}'x_H\cdot\varphi_U(S/H)\\&=\sum_{U\leq H\leq_o S}{}'x_H\cdot|S:H|.\end{aligned}$$

Since S is a pro-p group, $p\big||S:H|\ \forall H\neq S$.

$$\varphi_U(X)\equiv 0\bmod p\Leftrightarrow x_S\equiv 0\bmod p$$

$$\Rightarrow\mathcal{P}_{U,p}=\operatorname{span}_{\mathbb{Z}}\{I(S)\backslash\{S/S\}\}+p\mathbb{Z}[S/S]$$

and

$$\varphi_U(S/K)=0\Leftrightarrow U\not\leq K.$$

$\square$

Proposition 4.32. *Suppose that S is an abelian pro-p group, $q \neq p$,*

$$\mathcal{P}_{U,q} = \left\{ \sum_{H \leq_o S} a_H \cdot S/H \middle| \sum_{U \leq H \leq_o S} a_H \cdot |S : H| \equiv 0 \bmod q \right\}.$$

Proof. The prime ideal $\mathcal{P}_{U,q} = \{X \in \widehat{B}(S) \mid \varphi_U(X) \equiv 0 \bmod q\}$ and so we classify the elements with this property. The spanning set of isomorphism classes of transitive S-spaces is clear since if S is abelian, we can take S/H with $U \not\leq H$, and therefore $\varphi_U(S/H) = |\{sH \mid U^s \leq H\}| = |\{sH \mid U \leq H\} = |S : H|$, but $|S : H|$ is a power of p and so cannot be congruent to 0 mod q for $q \neq p$, therefore $S/H \notin \mathcal{P}_{U,q}$. Now suppose that $U \not\leq H \leq_o S$, then we have $\varphi_U(S/H) = 0$ and so $S/H \in \mathcal{P}_{U,q}$. Now suppose that $X = \sum_{H \leq_o S} a_H \; S/H \in \widehat{B}(S)$, $a_H \in \mathbb{Z}$, then $\varphi_U(X) = \varphi_U(\sum_{H \leq_o S} a_H \; S/H) = \sum_{U \leq H \leq_o S} a_H \; |S : H| \in \widehat{B}(S)$ and so we have the required result. $\square$

Theorem 4.33. *If S is a pro-p group then there is exactly one open prime ideal containing the element $p \cdot S/S \in \widehat{B}(S)$, namely*

$$\mathcal{P}_{S,p} = \mathrm{span}_{\mathbb{Z}}\{S/H | H <_o S\} + p\mathbb{Z}S/S.$$

Proof. Since $S = \varprojlim_{N \trianglelefteq_o S} S/N$ and we have that the $\cap_{N \trianglelefteq_o S} N = 1$, and any subgroup $U \leq_o S$ can be expressed as $U = \varprojlim_{N \trianglelefteq_o S} U/U \cap N$ such that $\cap_{N \trianglelefteq_o S} U \cap N = 1$ and we note that each $U/U \cap N$ is a p-group, it follows then that we have $U^p = 1$ for all $U \leq_o S$ by the previous chapter. $\square$

A natural question arises of if all closed ideals are open and we show using the following result that this is not always the case.

4.4 Existence of closed and not open prime ideals in $\widehat{B}(G)$

We return to the case when we have G a profinite group. We generalise a result in the previous section on pro-p groups to show some cases in which we can have closed and not open prime ideals, in particular we show that they can exist. This section is not claimed to be exhaustive of the ways that we can achieve a closed and not open prime ideal in $\widehat{B}(G)$.

Proposition 4.34. *Suppose that $G \cong H \times K$ is an infinite profinite group such that H, K are both abelian, then we have that there exists a closed prime ideal of $\widehat{B}(G)$ which is not open if K is infinite.*

Proof. By Krull's separation lemma, if we find an ideal I and a multiplicative set X such that $I \cap X = \emptyset$, then we have that I is a prime ideal. Take $\mathcal{C} = \{G/(H \times K_j) \mid K_j \leq_o K\}$ and let $X = \{x \in X \mid x = \prod_{i \in S} g_i,\ g_i \in \mathcal{C},\ J \text{ finite}\}$, then we have that X is a multiplicative set. It therefore suffices to show that there is an ideal in $\widehat{B}(G)$ which does not contain any element of X.

Take $I = \mathrm{span}_{\mathbb{Z}}\{G/(H_i \times K_j) \mid H_i <_o H,\ K_j \leq_o K\}$ and we show that I is a well defined ideal. The additive group structure on I is inherent in taking the $\mathbb{Z}$-span. It remains to show that I is closed under multiplication by any element of $\widehat{B}(G)$. Since $\widehat{B}(G)$ has a basis in the form of $\{G/L \mid L \leq_o G\}$ as a free $\mathbb{Z}$-module, if we show that I is closed under multiplication by G/L for any $L \leq_o G$, by distributivity we have that I is closed under multiplication by any element of $\widehat{B}(G)$.

Take G/R with $R \leq_o G$ and take $G/J \in I$, then we have $G/J \times G/R = \sum_{g \in [J \backslash G/R]} G/(J \cap {}^gR)$. Since G is abelian, we have that ${}^gR = R$ for all $g \in G$ and $[J \backslash G/R] = [G/JR]$ since all elements commute and therefore we have that the double coset $JgR = gJR$. It follows that $G/J \times G/R = \sum_{g \in [J \backslash G/R]}(G/J \cap {}^gR) = \sum_{g \in [g/JR]} G/(J \cap R) = |G/JR| \cdot G/(J \cap R)$.

Define a map $\pi_1 : H \times K \to H$ by $\pi_1(h, k) = h$ to be the projection map into the first coordinate. Clearly, we must have $\pi_1(J \cap R) \leq \pi_1(J) \cap \pi_1(R)$. Since $G/J \in I$, we have that $\pi_1(J) <_o H$ and therefore $\pi_1(J \cap R) <_o H$. It follows that $G/(J \cap R) \in I$ by definition and it follows that I is an ideal.

Consider the subgroups $H \times K_i,\ H \times K_j$ for some $K_i,\ K_j \leq K$, then since we have a direct product, the coordinates entries are independent and so $(H \times$

$K_i) \cap (H \times K_j) = H \times (K_i \cap K_j)$. It follows that each element of X is a linear combination of transitive G-spaces of the form G/G_i where $\pi_1(G_i) = H$. By the definition of I, we must have that these two sets are disjoint and so $I \cap X = \emptyset$. It follows that I is a prime ideal which is closed but not open since there is an infinite set of transitive G-spaces which are not in I, namely $\mathcal{C}$. I is closed since we can take $I = \bigcap_{N \trianglelefteq_o G} I + \ker(\pi_N^G)$.

□

This property is quite restrictive since we require both H and K to be abelian groups, we show that it suffices for there to be only one infinite abelian group in the product, which to say that $G = H \times K$ where K is infinite abelian.

Proposition 4.35. *Suppose that G is an infinite profinite group that is virtually abelian, which is to say that $Z(G) \leq_o G$ and therefore of finite index, then $G \cong H \times Z(G)$ where H is a finite group. In this case there is a closed, non open ideal of $\widehat{B}(G)$.*

Proof. Firstly we consider the case when $Z(G) \neq G$. Suppose that G is virtually abelian, then by the definition in this case, we have that $Z(G) <_o G$. $Z(G)$ is clearly normal in G since every element of $Z(G)$ commutes with every element of G and so $Z(G)^g = Z(G)$. Therefore, the quotient group $G/Z(G)$ is well defined. It follows that for any $g \in G$, we can express $g = g' \cdot z$ where $z \in Z(G)$ and $g \in G/Z(G)$ in a unique way. We can form an isomorphism $\theta : G \to G/Z(G) \times Z(G)$.

Let $H = G/Z(G)$, and take the collection $\mathcal{C} = \{G/H \times Z_i | Z_i \leq_o Z(G)\}$ and set $X = \{x \in X \mid x = \prod_{i \in S} g_i,\ g_i \in \mathcal{C},\ J \text{ finite}\}$, then by similar reasoning as in proposition 4.34, we have that the ideal $I := \mathrm{span}_{\mathbb{Z}}\{G/H_i \times Z \mid H_i <_o H\}$ has $I \cap X = \emptyset$ and so is prime. In the case when $Z(G) = G$, the proof proceeds in the same way but instead we can consider the isomorphism $\theta : G \to G/H \times H$ for H any proper open subgroup of G. □

Proposition 4.36. *For $G = H \times K$ where H, K are infinite profinite groups which are closed under conjugation induced by elements of G, then there is a closed, not* *open prime ideal of* $\widehat{B}(G)$.

Proof. Let $G_i = H_i \times K_i$ and consider ${}^gG_i$ is still a direct product since if we take $(a,b) \in G_i$ and $g = (g_1, g_2) \in G$, then we have $(a,b)^g = (a^{g_1}, b^{g_2})$ and so $G_i^g = \{(a^{g_1}, b^{g_2}) \mid a \in H_i,\ b \in K_i\} = H^{g_1} \times K^{g_2}$. As before, we have that 1-generated subgroups are not open, 2-generated subgroups are open and direct products and 3-generated subgroups are isomorphic to a 2-generated subgroup. Therefore, we need only consider the 2-generated subgroups.

$$G/G_i \times G/G_j = \sum_{g \in [G_i \backslash G/G_j]} G/G_i \cap^g G_j$$
$$H \times K/H_i \times K_i \times H \times K/H_j \times K_j = \sum_g H \times K/(H_i \times K_i) \cap^g (H_j \times K_j).$$

However

$$\begin{aligned}(H_i \times K_i) \cap (H_j \times K_j) &= \{(a,b)|a \in H_i, H_j, b \in K_i, K_j\}\\ &= \{(a,b)|a \in H_i \cap H_j, b \in K_i \cap K_j\}\\ &= (H_i \cap H_j) \times (K_i \cap K_j)\end{aligned}$$

$\Rightarrow$ each summand is of the form $H \times K/H_r \times K_r$ such that

$$H_r \leq H_i \cap H_j, K_r \leq K_i \cap K_j.$$

So now taking

$$\mathcal{C} = \{H \times K/H_i \times K | H_i \leq H\}, X = \langle \mathcal{C} \rangle,$$

$$I = \langle H \times K/H_i \times H_j | H_i \leq_o H, K_j <_o K \rangle.$$

By the previous argument we get that $\exists P \supseteq I$ such that P is closed, not open, prime ideal. □

In fact, we can strengthen this argument to consider when the action of K on H can be non trivial. In this case we take the subgroups H, K to be normal such that they generate the whole group. This is to say that the elements of H do not necessarily commute with all elements of K, but that K-action stabilizes the subgroup H by conjugation.

Proposition 4.37. *Suppose that G is a profinite group, and $G = HK$ such that $H, K \leq G$ are infinite groups closed under conjugation in G, with $H \cap K \neq H, K$, then we have that there is a closed but not open prime ideal in $\widehat{B}(G)$.*

Proof. Suppose that H, K are closed under conjugation $\Rightarrow H, K \trianglelefteq G$. We want to find an infinite multiplicative set of elements. Take the collection

$$\mathcal{C} = \{HK/H_iK \mid H_i \leq_o H\}$$

and consider the set X generated by $\mathcal{C}$ multiplicatively.

$$HK/H_iK \times HK/H_jK = \sum_{g \in [H_iK \backslash HK/H_jK]} HK/H_iK \cap {}^gH_jK$$

$${}^g(H_jK) = ({}^gH_j)K = H_lK \text{ with } H_l \leq_o H$$

$$\Rightarrow H_iK \cap {}^gH_jK = (H_i \cap^g H_j)K$$

and so we have that each product is a linear combination of transitive G-spaces in $\mathcal{C}$. Considering

$$I = \mathrm{span}_{\mathbb{Z}}\{HK/H_iK_j \mid H_i \leq_o H,\ K_j <_o K\}$$

we have that $I \cap X = \emptyset$. Since X is multiplicative there exists a prime ideal P such that $P \cap X = \emptyset$ and $I \subseteq P$ and so we have that P is a (non empty) closed but not open prime ideal. $\square$

5 Units and idempotents of Burnside rings

5.1 Units of $\widehat{B}(G)$

Let G be a profinite group, if $u \in \widehat{B}^{\times}(G)$, then clearly the image in the ghost ring under the ring homomorphism $\varphi : \widehat{B}(G) \to \mathrm{Gh}(G)$ defined by $\varphi = (\varphi_U)_{U \leq_o G}$ where $\varphi_U : \widehat{B}(G) \to \mathbb{Z}$ must map to a unit in the ghost ring. Since the ghost ring is composed by a copy of $\mathbb{Z}$ for each conjugacy class of open subgroups, we see that $\mathrm{Gh}(G)^{\times} = \prod'_{H \leq_o G}\{\pm 1\}$ where we take the restricted product to take one representative from each G-conjugacy class of subgroups and so any element that maps to this subset must also be a unit of $\widehat{B}(G)$. In particular, we have that $\varphi(\widehat{B}(G)^{\times}) \subseteq \prod'_{H \leq_o G}\{\pm 1\}$.

Take $u \in \widehat{B}(G)^{\times}$, then since $\varphi_H(u) = \pm 1$ for each $H \leq_o G$, we have in particular that $\varphi_G(u) = \pm 1$. Take $u \in \widehat{B}(G)$ such that $u = \sum_{H \leq_o G} u_H \cdot G/H$ and $u_H \in \mathbb{Z}$, then

$$\varphi_G(u) = \varphi_G\left(\sum_{H \leq_o G} u_H \cdot G/H\right) = \sum_{H \leq_o G} u_H \cdot \varphi_G(G/H)$$

but $\varphi_G(G/H) = 0$ if $G \not\lesssim H$ and so $\varphi_G(u) = u_G \cdot \varphi_G(G/G) = u_G$, but since u is a unit, we have that $u_G = \pm 1$. Take $H \leq_o G$ a maximal subgroup of G, and recall from 2.23 that $\varphi_K(G/K) = |N_G(K) : K|$ for any $K \leq_o G$, then the number of H-fixed points is given by $\varphi_H(u) = u_G + u_H \cdot |N_G(H) : H|$. If $\varphi_H(u) = \varphi_G(u)$, then we have that $u_H = 0$ since $|N_G(H) : H| \neq 0$, otherwise we have $\varphi_H(u) \neq \varphi_G(u)$ and so $u_H \neq 0$. In the latter case, we see that $|N_G(H) : H| = 2$, $u_H = -u_G$ or $|N_G(H) : H| = 1$, $u_H = -2u_G$.

For any $K \leq_o H$, consider that $\varphi_K(G/H) = |\{gH \mid K \leq {}^gH\}$, by definition for any $g \in N_G(H)$ then we have ${}^gH = H$ and so it follows that $|N_G(H) : H|$ divides $\varphi_K(G/H)$. If H is a subgroup of G of index 2, then it must be normal, subsequently we have that if H is a subgroup of index 2, $\varphi_H(G/H) = |N_G(H) : H| = |G : H|$. Note that for any $K \leq_o G$, we have that $\varphi_K(G/H) \leq |G : H|$

since it corresponds to the number of elements in the G-orbit G/H which are fixed by K-action. Therefore for $K \leq_o H$ for H a subgroup of index 2, we have that $|G : H|n = \varphi_K(G/H) \leq |G : H|$ for some $n \in \mathbb{Z}$. It follows that $n = 0$ or $n = 1$, but for $K \leq H$, we have that it must be contained in some conjugate of H and so the fixed points cannot be 0, consequently $\varphi_K(G/H) = 2$.

Take $u := G/G - G/H$ for $|G : H| = 2$, then for any $K \not\lesssim H$, we have that $\varphi_K(u) = \varphi_K(G/G) - \varphi_K(G/H) = 1 - 0 = 1$. If $K \lesssim_o H$, we have $\varphi_K(u) = \varphi_K(G/G) - \varphi_K(G/H) = 1 - 2 = -1$. Therefore for any $K \leq_o G$, we have $\varphi_K(u) = \pm 1$ and so $u \in \widehat{B}(G)$ is a unit. Therefore if G is a profinite group with a subgroup of index 2, there is a non trivial unit of $\widehat{B}(G)$ of the form $G/G - G/H$.

If, on the other hand, we have that $|N_G(H) : H| = 1$, then we must necessarily have that $N_G(H) = H$, which is to say that it is self normalizing. By Wilson[26] Proposition 2.4.3, we have that the existence of a self normalizing subgroup implies that G is not pro-nilpotent. In this case, however, we do not immediately get units since $H \leq_o G$ implies that there is an open normal subgroup N of G such that $N \leq_o H$. Consider the fixed point map $\varphi_N(u_G\ G/G + u_H\ G/H) = u_G + u_H\varphi_N(G/H)$, but $\varphi_N(G/H) = |\{gH \in G/H \mid N^g \leq H\}| = |G : H|$ since $N \trianglelefteq_o G$, $N \leq H$. We have seen that $|G : H| \neq 2$ since otherwise it would be normal. It follows then that $u \neq \pm(G/G - 2G/H)$, and so we must include a correction term for such a u to be a unit.

Proposition 5.1. *For $u \in \widehat{B}(G)^\times$, u can be written as either X or $-X$ where $\varphi_G(X) = 1$.*

Proof. Let $u \in \widehat{B}(G)^\times$ be a unit, then as we have shown, we must have $\Rightarrow \varphi_G(u) = \pm 1$. In the case that $\varphi_G(u) = 1$, we are done with $u = X$.. Now suppose that $\varphi_G(u) = -1$, it follows that $\varphi_G(-u) = -\varphi_G(u) = 1$ and so we set $X = -u$. $\square$

Hence we have a clear bijection

$$\{u \in \widehat{B}(G)^\times | \varphi_G(u) = 1\} \longleftrightarrow \{u \in \widehat{B}(G)^\times | \varphi_G(u) = -1\}$$

so without loss of generality we can consider just one of these subsets in order to recover all possible units by multiplication by ± 1. We can therefore form an equivalence relation where each equivalence class $[u] = \{u, -u\}$. We choose the former subset as a set of representatives of this equivalence class. We shall generally use the notation of u for the representative of the equivalence class $[u]$ with $\varphi_G(u) = 1$.

Note that for $X \in \widehat{B}(G)$ with $X = \sum_{H \leq_o G} x_H \cdot G/H$, we have that $\varphi_K(X) = \sum'_{K \leq_o H} x_H \cdot \varphi_K(G/H)$ with the sum taken over conjugacy class representatives since φ_K is a ring homomorphism. Since $\varphi_K(G/H) = 0$ if $K \not\lesssim H$, we have that $\varphi_K(X) = \sum'_{K \leq_o H} x_H \cdot \varphi_K(G/H)$. In particular, $\varphi_G(X) = x_G$ and if $\varphi_G(X) = 1$ then $x_G = 1$. We therefore have that all of these representative units are of the form $u = G/G + \sum_{H <_o G} u_H \cdot G/H$. We use the notation $c_H(u) = u_H$ to denote the coefficient of G/H in the canonical expression of any $u \in \widehat{B}(G)$.

Proposition 5.2. *If G has no self normalizing subgroups, and $u \in \widehat{B}(G)^\times$ the canonical representative of the equivalence class $[u]$, then we have that $c_H(u) \neq 0$ with $c_H(u) = 0$ for each $H \lnsim_o K <_o G$ implies that $c_H(u) = -1$ and $|N_G(H) : H| = 2$.*

Proof. Suppose that $u \in \widehat{B}(G)^\times$ with $c_H(u) \neq 0$ and for each $H \lnsim_o K <_o G$, we have that $c_H(u) = 0$. Then by the previous observation, we have that for $u = \sum_{K \leq_o G} u_K \cdot G/K$, then $\varphi_H(u) = \sum'_{H \leq K \leq_o G} u_K \cdot \varphi_H(G/K)$. It follows that $\varphi_H(u) = u_G \cdot \varphi_H(G/G) + u_H \cdot \varphi_H(G/H)$ since all other u_K in this sum are 0 by our assumption.

Recalling from 2.23 that $\varphi_H(G/H) = \{gH \mid H^g \leq H\}$, we have that $\varphi_H(G/H) = |N_G(H) : H|$ and so $\varphi_H(u) = u_G + u_H \cdot |N_G(H) : H|$. Since u is the canonical representative, we have $u_G = \varphi_G(u) = 1$. It follows that $\varphi_H(u) = 1 + u_H \cdot |N_G(H) : H|$. Since u is a unit, we must have that $\varphi)H(u) = \pm 1$, and since $u_H \neq 0$ and $|N_G(H) : H| \neq 0$, then we must have $\varphi_H(u) = -1$ and subsequently $u_H \cdot |N_G(H) : H| = -2$. Since we have no self normalizing subgroups, then we have $|N_G(H) : H| = 2$ and $u_H = -1$.

$\square$

Consequently this holds for any nilpotent group G.

Remark 5.3. Suppose that $c_G(u) = 1$,

$$\Rightarrow u = 1 - X \text{ and } (1 - X)^2 = 1$$

$$\Rightarrow 1 - 2X + X^2 = 1$$

$$\Rightarrow X^2 = 2X.$$

Proposition 5.4. *Suppose that $u \in \widehat{B}(G)^\times$ and $u \neq 1$ with $u_G = 1$, then we have that there exists $K <_o G$ such that $\varphi_K(u) = -1$.*

Proof. This is clear since if we suppose that $\varphi_K(u) = 1$ for all $K \leq_o G$, then we have that $u = 1$ up to equivalence. Therefore we must have that there exists $K <_o G$ such that $\varphi_K(u) \neq 1$. However, since $\varphi_H(u) = \pm 1$ for all $H \leq_o G$, we have that $\varphi_K(u) = -1$. $\square$

Proposition 5.5. *Suppose that G has no self normalizing subgroups. Let $u \in \widehat{B}(G)^\times$ and*

$$\mathcal{H}_u = \{H \leq G | H \text{ maximal with } c_H(u) \neq 0\}.$$

Let K be maximal such that $K \notin \mathcal{H}_u$ and $c_H(u) \neq 0$ then $c_H(u) > 0$.

Proof.

$$\begin{aligned}
\varphi_K(u) &= \varphi_K(G/G + \sideset{}{'}\sum_{K \lesssim H, K \not\sim H} u_H G/H + u_K G/K + X' \\
&= \varphi_K(G/G) + \sideset{}{'}\sum_{K \lesssim H, K \not\sim H} u_H \varphi_K(G/H) + u_K \varphi_K(G/K) \\
&= 1 + u_K |N_G(K) : K| + \sideset{}{'}\sum_{K \lesssim H, K \not\sim H} u_H \varphi_K(G/H) \\
&= 1 - 2 \sideset{}{'}\sum_{K \lesssim H, K \not\sim H} |\{g \in G/N_G(H) | K^g \leq H\}| + u_K |N_G(K) : K|.
\end{aligned}$$

Case 1.

Suppose that $|\mathcal{H}_u \cap \{H \leq G | K \lesssim H\}| \geq 2$

$$\begin{aligned}\Rightarrow \varphi_K(u) = -1 \Rightarrow u_K \geq 0\\ \varphi_K(u) = 1 \Rightarrow u_K \geq 0\\ \Rightarrow u_K \geq 0 \Rightarrow u_K > 0.\end{aligned}$$

Case 2.

Suppose that $|\mathcal{H}_u \cap \{H \leq G | K \lesssim H\} = 1$ then

$$|\{g \in G/N_G(H) | K^g \leq H\}| = 1$$

$$\Rightarrow \varphi_K(u) = -1 + u_K |N_G(K) : K|$$

$$\Rightarrow u_K \geq 0$$

$$\Rightarrow u_K > 0.$$

$\square$

Corollary 5.6. *If we have that G is a group with no self-normalizing subgroups,*

$$|\mathcal{H}_u \cap \{H \leq G | K \lesssim H\}| = 1$$

$$\Rightarrow \varphi_K(u) = 1$$

and $u_K = 1, |N_G(K) : K| = 2.$

5.2 Units of $\widehat{B}(D_{2^\infty})$

We show the process of finding the units recursively by taking the pro-dihedral group that is discussed in Miller[15]. Take the presentation of the group given by

$G = \mathbb{Z}_2 \rtimes \{\pm 1\} \cong \varprojlim D_{2^n} = \left\{ \left(\begin{smallmatrix} \pm 1 & a \\ 0 & 1 \end{smallmatrix} \right) \;\middle|\; a \in \mathbb{Z}_2 \right\}$. Let $r = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, $s = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. We define $H_n = \langle r^{2^{n-1}} \rangle$, $K_n = \langle s, r^{2^n} \rangle$, $K'_n = \langle rs, r^{2^n} \rangle$. By [15]Miller §3, we have that every open subgroup of G is conjugate to exactly one of these for some $n \in \mathbb{N}$. This is to say that the set of these subgroups is a complete set of representatives of conjugacy classes of open subgroups of G.

Proposition 5.7. *Let $X \in \widehat{B}(G)^\times$ such that $c_{H_1} \neq 0$ and $c_{K_n}(X) = c_{K'_n}(X) = 0$ for all $n \in \mathbb{N}$, then we have that $c_{H_n}(X) = 0$.*

Proof. Assume $c_{H_k}(X) \neq 0$, H_k maximal with this property, $k \geq 2$, and consider $X = G/G - G/H_1 + x_{H_k} \cdot G/H_k + X'$ for some $X' \in \widehat{B}(G)$ such that $c_G(X') = c_{H_1}(X') = c_{K_n}(X') = c_{K'_n}(X') = 0$. It follows that by taking the H_k-fixed points of X, we have that $\varphi_{H_k}(X') = 0$. It therefore follows that

$$\begin{aligned} \varphi_{H_k}(X) &= \varphi_{H_k}(G/G - G/H_1 + X_{H_k} G/H_k) \\ &= 1 - |G : H_1| + X_{H_k} |G : H_k| \\ &= 1 - 2 + X_{H_k} 2^k \\ &\neq \pm 1 \text{ for } k \geq 2. \end{aligned}$$

□

Proposition 5.8. *Let $X \in \widehat{B}(G)^\times$ such that $c_{H_1}(X) \neq 0$, $c_{K_i}(X) \neq 0$ for some maximal K_i, and $c_{K'_j}(X) = 0$ for all $j \leq i$, then we have that $c_{H_{i+1}}(X) = 0$.*

Proof.

$$\begin{aligned} \varphi_{H_{i+1}}(X) &= \varphi_{H_{i+1}}(G/G - G/H_1 - G/K_i + X_{H_{i+1}} G/H_{i+1}) \\ &= 1 - 2 - 2^i + X_{H_{i+1}} 2^{i+1} \\ &= -1 - 2^i + X_{H_{i+1}} 2^{i+1} \\ \Rightarrow X_{H_{i+1}} &= 0 \end{aligned}$$

□

The same reasoning shows that if we interchange all occurances of K_m and K'_m for some $m \in \mathbb{N}$ in the above proposition then the same result holds.

5.3 Idempotents

In order to establish the idempotents within the Burnside ring of profinite groups. we first prove some background results around establishing stating the structure of idempotents. To do this we extend a result by Solomon [20]Solomon, Theorem 1 which was proved for finite posets, and prove an analogue to the Möbius inclusion function of subgroups in the case where we have a poset with a maximal element. In order to do this, we need a version of the Möbius inversion formula for a class of infinite posets. We establish one such μ for this thesis with the following definition.

Definition 5.9. Let G be a profinite group, we define the poset $P = \{H \mid H \leq_o G\}$ ordered by inclusion and we define a function $\mu : P \times P \to \mathbb{Z}$. For each $H, K \leq_o G$, we have a well defined function $\mu(H, K) \to \mathbb{Z}$ which is defined by $\mu(H, K) = 0$ if $H \not\leq K$, $\mu(H, H) = 1$ and $\sum_{K \leq J \leq H} \mu(J, H) = 0$.

Since $K, H \leq_o G$, we have that there are finitely many subgroups between H and K, we have that for each pair of open subgroups, the defining sum of the function has finitely many non zero summands. We abstract this result in order to prove in greater generality a method for finding idempotents for $\mathbb{Z}$-modules based by elements of posets. The Möbius function on a finite set is precisely the one given in definition 5.9, however this is not currently defined for an infinite set. With the following definition, we define what we mean by the Möbius function on an infinite set which is motivated by it adhering to the properties of the Möbius function in the finite case.

Definition 5.10. Suppose that P is a poset such that for each $H \in P$, there are finitely many $K \in P$ such that $H \leq K$. We define a Möbius function μ on P to be the map $\mu : P \times P \to \mathbb{Z}$ such that $\mu(H, K)$=0 if $H \not\leq K$, $\mu(H, H) = 1$ and the relation that $\sum_{K \leq J \leq H} \mu(J, H) = 0$.

For P a poset, we define the Möbius module of P, $M[P]$, to be the free $\mathbb{Z}$-module with basis elements of the poset. Note that since $\widehat{B}(G)$ has a basis as a free $\mathbb{Z}$-module of the form $\{G/H \mid H \leq_o G\}$ coincides with this description with the ordering on this basis given by subconjugation of the stabilisers, although we have yet to define products. We show in the following lemma a formulation for the product of elements in the Möbius module, this is a generalisation of the [20]Solomon, Theorem 1.

Lemma 5.11. *Let P be a poset such that for each $H \in P$, there are finitely many $K \in P$ such that $H \leq K$. Let μ be the Möbius function of P as given in definition 5.10. For each $(a,b) \in P \times P$ we define a function by $\varphi_{(a,b)} : P \to \mathbb{Z}$*

$$\varphi_{(a,b)}(p) = \sum_{q \in P_{(a,b)}} \mu(p,q), \quad p \in P$$

where $P_{(a,b)} = \{q \in P \mid q \leq a,\ q \leq b\}$. Define the product of elements in P by

$$ab = \sum_{p \in P} \varphi_{(a,b)}(p)p$$

and extend to $\mathcal{M}[P]$ by linearity, then $\mathcal{M}[P] \cong \prod \mathbb{Z}$. If K is a field then the Mobius algebra defined by

$$\mathcal{M}_K[P] = \mathcal{M}[P] \otimes_{\mathbb{Z}} K$$

is a semisimple algebra over K and its primitive idempotents are $e_a \otimes 1$ where

$$e_a = \sum_{b \in P} \mu(b,a)b,\ \ a \in P.$$

Proof. We first prove that since $p \in P$, the function $\varphi_{(a,b)}$ is well defined as a function into $\mathbb{Z}$. For $p \in P$, we have that there are finitely many $q \in p$ such that $p \leq q$ by the condition imposed on our poset P, and by definition of the Möbius function, we have that $\mu(p,q) = 0$ for any $q \ngeq p$. It follows that

$$\sum_{\substack{q \in P_{a,b} \\ p \leq q}} \mu(p,q) = \sum_{q \in P_{a,b}} \mu(p,q)$$

since any other summand is 0. Finally, we have that the condition on the summation is equivalent to $\{q \mid q \leq a,\ q \leq b,\ p \leq q\} \subseteq \{q \mid p \leq q\}$ which is finite for fixed $p \in P$. Therefore since this is a finite sum of finite values, we have that this certainly maps into $\mathbb{Z}$.

We automatically have that $P_{a,b} = \{q \mid q \leq a,\ q \leq b\} = P_{b,a}$. Since $\varphi_{a,b}(p)$ is finite for each $a, b, p \in P$, we have that $ab = \sum_{p \in P} \varphi_{a,b}(p)p$ is well defined as an element of $\mathcal{M}[P]$ since it is a $\mathbb{Z}$-linear combination of elements of P. We want to show that the product ab defined in the lemma is commutative, this follows naturally by $ab = \sum_{p \in P} \varphi_{a,b}(p)p = \sum_{p \in P} \sum_{q \in P_{a,b}} \mu(p,q)p = \sum_{p \in P} \sum_{q \in P_{b,a}} \mu(p,q)p = ba$.

We define a function $\zeta : P \times P \to \mathbb{Z}$ by

$$\zeta(a,b) = 1 \text{ if } a \leq b$$

$$\zeta(a,b) = 0 \text{ otherwise.}$$

Then for each $c \in P$ we make a $\mathbb{Z}$-linear map $\zeta_c : \mathcal{M}[P] \to \mathbb{Z}$ by defining $\zeta_c(a) = \zeta(c,a)$ for each $a \in P$ and taking the linear extension of this since P is a $\mathbb{Z}$-basis for $\mathcal{M}[P]$. This map is well defined since once again, there are only finitely many $a \in P$ such that $c \leq a$ by assumption and so any element $r \in \mathcal{M}[P]$ has finite coefficients for each $a \in P$. Then for $a, b \in P$

$$\begin{aligned}\zeta_c(ab) &= \sum_{p \in P} \varphi_{a,b}(p)\zeta_c(p) \\ &= \sum_{q \in P_{a,b}} \sum_{p \in P} \zeta(c,p)\mu(p,q) \\ &= \sum_{q \in P_{a,b}} \delta_{c,q} \\ &= \zeta_c(a)\zeta_c(b),\end{aligned}$$

therefore ζ_c is a homomorphism $\mathcal{M}[P] \to \mathbb{Z}$ since it is $\mathbb{Z}$-linear by definition.

Suppose that $x \in \mathcal{M}[P]$ such that $\zeta_c(x) = 0$ for each $c \in P$, and write $x = \sum_{a \in P} x(a)a$ with each $x(a) \in \mathbb{Z}$ and so $0 = \sum_{c \leq a} x(a)$ for each $c \in P$. By Zorn's lemma, we can take c maximal in P, we see that $x(c) = 0$, repeating the process inductively for $P' = P \backslash \{c\}$ removing a maximal element each time we have that $x(a) = 0$ for every $a \in P$ and therefore $x = 0$.

Take $x \in \mathcal{M}[P]$, $y \in \mathcal{M}[P]$ and suppose that $\zeta_c(x) = \zeta_c(y)$. Since ζ_c is a linear map, we can reformulate this to say $0 = \zeta_c(x) - \zeta_c(y) = \zeta_c(x - y)$ for every $c \in P$. Therefore we have $x - y = 0$ since we have that 0 is the unique element with $\zeta_c(0) = 0$ for all $c \in P$ and so we have $x = y$. Let $e_a = \sum_{b \in P} \mu(b,a)b$ for $a \in P$, we note again that this is a well defined element of $\mathcal{M}[P]$ since each $\mu(a,b)$ is finite. Take $x \in \mathcal{M}[P]$, $c \in P$ and consider $\zeta(xe_a) = \zeta_c(x)\zeta_c(e_a) = \zeta_c(x)\delta_{a,c}$ since if $c < a$, we have that $\sum_{c \leq b \leq a} \mu(b,a) = 0$ and if $c > a$, we have each $b \leq a$ has $\zeta_c(b) = 0$. We have then that $a = c$ and $\zeta_c(x)\delta_{a,c} = \zeta_a(x)\delta_{a,c}$. Applying $\zeta_c(e_a) = \delta_{a,c}$ and noting that $\zeta_a(x) = \zeta_c(\zeta_a(x)c)$ we have $\zeta_c(\zeta_a(x)e_a) = \zeta_c(xe_a)$. It follows that $xe_a = \zeta_a(x)e_a$.

For $b \neq a$, $b \in P$, we have that $e_b e_a = \zeta_a(e_b)e_a = \delta_{a,b}e_a$ and so in particular $e_a e_a = e_a$ and $e_a e_b = 0$ if $a \neq b$, therefore these are pairwise orthogonal idempotents in $\mathcal{M}[P]$. Now let $e = \sum_{a \in P} e_a$ and consider for $b, c \in P$ we have that $\zeta_c(be) = \zeta_c(b)\zeta_c(e) = \zeta_c(b)$, this holds for every $c \in P$ and so it follows that $be = b$ for every $b \in P$. It follows that e is an identity on $\mathcal{M}[P]$.

$x \in \mathcal{M}[P]$

$$\Rightarrow x = xe = \sum_{a \in P} \zeta_a(x) e_a.$$

Orthogonality shows associativity

$$\begin{aligned}(xy)z &= \sum_{a \in P} \zeta_a(x)\zeta_a(y)\zeta_a(z)e_a \\ &= x(yz).\end{aligned}$$

If K is a field then

$$\mathcal{M}_K[P] = \sum_{a \in P} K(e_a \otimes 1).$$

□

By definition, we have shown that this lemma can be applied to the poset of open subgroups of G since it satisfies the assumption that for each $H \in P$, there are finitely many groups $K \in P$ such that $H \leq K$. We claim that there is in fact an isomorphism from $\mathcal{M}_{\mathbb{Q}}[P]$ to $\mathbb{Q}\widehat{B}(G)$ and so we have that the idempotents of $\mathbb{Q}\widehat{B}(G)$ are given by the image of $e_a \otimes 1$ for each $a \in P$. Take the elements in the basis of $\mathbb{Q}\widehat{B}(G)$ given by $v_H = \frac{G/H}{|G:H|}$, then define the map $\theta : \mathbb{Q}\widehat{B}(G) \to \mathcal{M}_{\mathbb{Q}}[P]$ to be the linear extension of the map which has $\theta(v_H) = H \otimes 1$.

Addition clearly holds and so we check multiplication. It follows that the product of the basis elements$G/H \times G/K = \sum_{H\backslash G/K} G/H \cap {}^gK$, in order to see that this is in fact an isomorphism we have that this must coincide with the image in $\mathcal{M}_{\mathbb{Q}}[P]$. Taking $\cap_{g\in G} H^g = N$ and $\cap_{g \in G} K^g = M$ which are both open, we have that $N \cap M$ is an open subgroup contained in all conjugates of H and K. In Solomon's paper[20], it has been proved that there is an isomorphism for any Burnside algebra ($\mathbb{Q}B(G)$) of a finite group G to the Möbius algebra and so in particular we have that that the multiplication of elements can be shown to be equal since there are minimal open subgroup as a stabilizer of a summand.

This is to say that $\mathbb{Q}\widehat{B}(G) \cong \varprojlim_N \mathbb{Q}B(G/N) \cong \varprojlim_N \mathcal{M}_{\mathbb{Q}}[P/N] \cong \mathcal{M}_{\mathbb{Q}}[P]$. Equipped with this result, we can follow a similar process to Yoshida[27][4] in the finite case to show that we can define idempotents in the Burnside ring of profinite groups. In order to do so, we cite the Möbius inversion formula as proved in Rota[19] to approach equivalences of summations. Let $f(x)$ be a function into a commutative ring defined for x in a locally finite poset, then we have $g(x) := \sum_{y \leq x} f(y)$ if and only if $f(x) = \sum_{y \leq x} \mu(y, x) g(y)$.

We note that a poset P is defined to be locally finite if the interval $[H, K]$ is finite for each $H, K \in P$. As we have shown, each interval of this form in the poset of conjugacy classes of open subgroups must be finite as there are finitely many elements above H for each $H \leq_o G$ and so we can apply the theorem, in particular, let

$$f(H) = \frac{e_H}{|G : N_G(H)|} := \sum_{K \leq_o H} \frac{\mu(K, H)}{|G : K|} G/K$$

for each $H \in P$ and observe that the theorem then implies that for $g(K) = \frac{G/K}{|G:K|}$ we have

$$\frac{G/H}{|G : H|} = \sum_{K \leq_o H} \frac{e_K}{|G : N_G(K)|}$$

and so by rearrangement we have

$$G/H = \sum_{K \leq_o H} \frac{|G : H|}{|G : N_G(K)|} e_K.$$

Define $u_H = G/H$ for $H \in P$, the poset of open subgroups of G. Since e_H is defined in such a way since by the previous lemma, and the defined isomorphism θ, we see that the idempotents of $\mathcal{M}_{\mathbb{Q}}[P]$ where P denotes the poset of subgroups ordered by $\leq$ are given by $t_H = \sum_{K \leq_o H} \mu(K, H) K \otimes 1$, for ease we write this as $t_H = \sum_{K \leq_o H} \mu(H, K) K$, then taking the image under the inverse of the isomorphism θ given above, we have

$$\theta^{-1}\left(\sum_{K \leq_o H} \mu(K, H) K\right) = \sum_{K \leq_o H} \mu(K, H) \frac{G/K}{|G : K|}.$$

However, this sum is taken over subgroups and not conjugacy classes of subgroups and so we have that there are $|G : N_G(H)|$ isomorphic copies of each conjugacy class, it follows that

$$e_H = |G : N_G(H)| \sum_{K \leq_o H}{}' \frac{G/K}{|G : K|},$$

with the sum taken over the poset of conjugacy classes of open subgroups. Since each idempotent of $\mathcal{M}_{\mathbb{Q}}[P]$ can be written as a sum of idempotents of the form, e_a, we have that the idempotents of $\mathbb{Q}\widehat{B}(G)$ are all linear combinations of the e_H. Therefore we have proved the following theorem.

Theorem 5.12. *Let G be a profinite group, $H \leq_o G$ then*

$$e_H^G := |G : N_G(H)| \sum_{K \leq H}{}' \frac{\mu(K, H)}{|G : K|} u_K$$

is an idempotent in the Burnside algebra. Any idempotent of the Burnside algebra is a linear combination of these elements, with each e_H having coefficient $0, 1$ in the series.

We have a method to construct idempotents of $\mathbb{Q}\widehat{B}(G)$ which are orthogonal since e_a in the Möbius algebra are orthogonal. Consider $\varphi_H(e_H^G)$ for $e_H^G \in \mathbb{Q}\widehat{B}(G)$, then we have that $\varphi_H(e_H^G) = |G : N_G(H)|\mu(H, H)\frac{|N_G(H):H|}{|G:H|} = 1$. Proceeding with downwards induction, we note that it has been shown in the lemma that the unit in $\mathcal{M}_{\mathbb{Q}}[P]$ is expressed as the sum of all idempotents, by isomorphism this is also true in $\mathbb{Q}\widehat{B}(G)$. Let $G/G = \sum_{H \leq_o G} e_H^G$, then we have for each $H \leq_o G$, $\varphi_H(G/G) = \sum_{H \leq_o K} \varphi_H(e_K^G) = 1$ since for $H \not\leq K$, we have $\varphi_H(e_K^G) = 0$. Note that for any idempotent $e \in \mathbb{Q}\widehat{B}(G)$, we must have $\varphi_H(e) = 0, 1$ for each $H \in P$. It follows that $\varphi_H(e_K) = 0$ if $H \not\sim K$ since otherwise we would have $\varphi_H(G/G) \neq 1$ which is a contradiction.

$$\varphi_H(e_K) = \begin{cases} 1 \text{ if } H \sim_G K \\ \\ 0 \text{ otherwise.} \end{cases}$$

Recall that an inverse limit of groups is called pro-$\mathcal{C}$ if it is an inverse limit of groups of class $\mathcal{F}$. In particular, we have that if $G = \varprojlim_{N \trianglelefteq_o G}(G/N)$ is an inverse limit, then we have that G is pro-solvable if each G/N is solvable. Solvability has a key role in the Burnside ring of finite groups, as a consequence of [8]Dress, Proposition 2 shows that the only idempotents of $B(G)$ for G a finite group are given by 0 and 1.

Proposition 5.13. *Suppose that G is a profinite group such that $H \leq_o G$ is pro-solvable. Then we have that every idempotent $e \in \widehat{B}(G)$ is an inflation of an idempotent from $B(G/K)$ to $\widehat{B}(G)$ for $K = K_G(H) \leq_o G$.*

Proof. Let $e \in \widehat{B}(G)$ be an idempotent, then clearly since $\varphi_J(e_H) = \varphi_J((e_H)^2) \in \mathbb{Z}$, it follows that $\varphi_J(e) \in \{0, 1\}$ for each $J \leq_o G$. Since this holds, we also have that $\varphi_J(e) \in \{0, 1\}$ for each $J \leq_o H \leq_o G$. Recalling that $\varphi_J(X) = \varphi_J(\mathrm{res}^G_S(X))$ for all $X \in \widehat{B}(G)$, $J \leq_o S$. In particular, we have $\varphi_J(\mathrm{res}^G_H(e)) \in \{0, 1\}$ for all $J \leq_o H$. Since the ghost map is injective, we have that this uniquely determines an element in $\widehat{B}(H)$ which is itself an idempotent.

Since H is pro-solvable, we have that the only idempotents in $\widehat{B}(H)$ are given by $1 = H/H$ and 0 since we have that the image in the Burnside ring of each finite quotient must be solvable and so we must have that the idempotents of $\widehat{B}(H)$ are an inverse limit of idempotent elements. It follows that since H acts trivially on e, we have that that $K_G(H)$ acts trivially on e. By the quotient map $\pi^G_N : \widehat{B}(G) \to B(G/N)$, there is an element $\pi^G_N(e) \in B(G/N)$ with the required fixed points for $N = K_G(H)$. Therefore e can be expressed as an inflation. Moreover, it is the inflation of an idempotent since the fixed points must be $0, 1$ for each $J \leq_o G$. $\square$

Since $B(G/N)$ contains finitely many idempotents, we have that for a virtually pro-solvable group, that is a group which contains an open pro-solvable group, then there are as many idempotents as in the Burnside ring of the quotient of the group by the core of the open pro-solvable group. In particular, we have that there are at most finitely many. This gives a method for expressing the idempotents in $\widehat{B}(G)$ for G virtually pro-solvable, but we wish to find the idempotents for any profinite group G.

Corollary 5.14. *If G virtually pro-solvable then there are finitely many idempotents in $\widehat{B}(G)$.*

Proposition 5.15. *Let G be a profinite group, then for $N \trianglelefteq_o H \leq_o G$ then we have that there is an open subgroup H_N such that H/H_N is solvable where*

$$H_N = \bigcap_{\substack{K \trianglelefteq H \\ N \leq K \\ H/K \, solvable}} K.$$

Proof. Since we have that N is open in G and that each $K \geq N$, then we have that each subgroup K is itself open. Additionally, since N is open, there are finitely many subgroups between N and G and therefore this intersection is an intersection of finitely many subgroups, and therefore an open subgroup. It remains to show that this is a subgroup with the property that H/H_N is solvable. This is equivalent to showing that the intersection of any two subgroups M, K such that G/M solvable and G/K is solvable, then $G/M \cap K$ is solvable. Clearly we have $M \cap K$ is a normal subgroup, but this follows since $G/M \cap K$ can be seen as a subgroup of $G/M \times G/K$ which is itself solvable. $\square$

Proposition 5.16. *Let $G \geq_o H$ be a profinite group defined by the inverse limits*

$$\varprojlim_{N \trianglelefteq_o G} G/N \geq \varprojlim_{N \trianglelefteq_o G} H/H \cap N.$$

The set $\{H/H \cap N \mid N \trianglelefteq_o G\}$ admits an inverse system of subgroups with the notation of the previous Proposition $\{H/H_{H\cap N_i}\}_{N_i}$ with compatible maps $\{\varphi_{ji} : H/H_{H\cap N_i} \to H/H_{H\cap N_j}\}_{i \leq j}$.

Proof. Suppose that $j \leq i$ then define $H_i := H_{H\cap N_i}$, $H_j := H_{H\cap N_j}$, it follows from the definition that the H_i and H_j can be expressed respectively as

$$H_i = \bigcap_{\substack{H\cap N_i \leq K \\ H/K \text{ solvable}}} K, \quad H_j = \bigcap_{\substack{H\cap N_j \leq K \\ H/K \text{ solvable}}} .K$$

Since we know that H/H_i is solvable and H/H_j is pro-solvable, we have that $H_i \leq H_j$ since H_j appears in the defining intersection of H_i, therefore we have that there is a natural homomorphism given by the projections between the quotient maps.

$$\Rightarrow \varphi_{ji} : H/H_i \to H/H_j$$
$$hH_i \to hH_j$$

and so these do form an inverse system since we have chosen the subgroups to be contained appropriately. □

Proposition 5.17. *Let G be a profinite group, then for each $H \leq_o G$ we have that there exists a minimal closed subgroup $K \leq H$ such that H/K is pro-solvable. We call such an element $H^{(\infty)}$.*

Proof. Take $H \leq_o G$, then we have that there is an inverse limit on the subgroup as a profinite group $H \cong \varprojlim_{N \trianglelefteq_o G}(H/(H \cap N))$ induced from $G \cong \varprojlim_{N \trianglelefteq_o G}(G/N)$. Then by the notation in the previous theorem we have that $H_{H\cap N}$ is the minimal subgroup of H above N such that $H/H_{H\cap N}$ is solvable. $H \cap N \leq H_{H\cap N} \leq H$. By the previous Proposition, these subgroups form an inverse system and so we can take the inverse limit $K := \varprojlim(H/H_{H\cap N})$. There is a group homomorphism $\theta : H \to \varprojlim(H/H_{H\cap N})$ defined by the relation $\theta(gH \cap N) = gH_{H\cap N}$. We get that

$$ker(\theta) = \{h \in H \mid h \in H_{H\cap N} \forall N \trianglelefteq_o G\}$$

$$\Rightarrow ker(\theta) = \bigcap_{N \trianglelefteq_o G} H_{H\cap N}$$

$\Rightarrow$ There is a unique minimal closed subgroup of H such that $H/ker(\theta)$ is a pro-solvable group. This is also evident since $H^{(\infty)}$ is the intersection of infinitely many open (and therefore closed) subgroups of G. □

Theorem 5.18. *Let G be a profinite group and P the poset on the open subgroups of G with inclusion ordering. If $\mu(H,G) \neq 0$ for μ the Möbius function as in* *definition 5.10, then we have that $H = G$ or H is the intersection of finitely many open subgroups of G, which is to say $H = \bigcap_{i\in I} M_i$ such that each $M_i \leq_o G$ is a maximal subgroup.*

We prove the generalisation of [13]Hall, Theorem 2.3 in order to derive a simplification of the representation of idempotents in the Burnside ring of a profinite group. In essence, this result states when the Möbius function $\mu(H, K)$ can be non zero. Utilising this with the expression given in Theorem 5.12 allows us to discount the 0 entries of the idempotent e_H^G.

Proof. Let G be a profinite group, then we have that there are at most finitely many subgroups above any $H \leq_o G$, $H \in P$. Clearly we have that $G \in P$. Let $f(H)$ be any function defined $\forall H \in P$ and let $g(H)$ be the function $g(H) = \sum_{K\leq H} f(K)$. P is locally finite since each $H \in P$ is contained in finitely many open subgroups since it is of finite index, subsequently we have by the Möbius inversion formula

$$f(G) = \sum_{H\in S} \mu(H, G) g(H)$$

$$c_S(G) = 1$$

$$\sum_{H\leq K\leq G} \mu(K, G) = 0$$

For $H \leq_o G$, take the finite poset $S = \{K \mid H \leq_o K \leq_o G\} \subseteq P$ so that we can apply this. Suppose that $H \leq G$ and that H is not the meet of maximal subgroups.

Assume the result is true for $K > H$ so that $\mu(K, G) = 0$ for each $K > H$. Let M be the meet of all maximal members of S which contain H, then we have that

$$-\mu(H, G) = \sum_{K>M} \mu(K, G),$$

but all the terms on the right vanish and so we can conclude that $\mu(H, G) = 0$. □

Definition 5.19. Let G be a profinite group, then define

$$\mathcal{L}_M(G) = \{K \leq G \mid K = \cap M_i\} \cup \{G\}$$

where each M_i a maximal open subgroup of G and

$$\mathcal{L}_M^O(G) = \{H \in \mathcal{L}_M(G) \mid H \leq_o G\}$$

be the meet lattice of the maximal subgroups and the meet lattice of finitely many maximal open subgroups respectively. This is to say the poset ordered by inclusion.

Definition 5.20. [26](2.5) For G a profinite group, define the Frattini subgroup $\Phi(G)$ to be the intersection of all open maximal subgroups of G. The Frattini is normal in G.

Proposition 5.21. *For G a profinite group then we have that*

$$\mathcal{L}_M^O(G) \cong \mathcal{L}_M^O(G/\Phi(G))$$

as lattices where $\Phi(G)$ is the Frattini subgroup of G.

Proof. Take $H \in \mathcal{L}_M^O(G)$, then H is either equal to G or is the intersection of finitely many maximal open subgroups. It follows that $\Phi(G) \leq H$ since $\Phi(G)$ is the intersection of all open maximal subgroups. Therefore we have $H/\Phi(G)$ is a well determined unique element of the lattice $\mathcal{L}_M^O(G/\Phi(G))$. □

Applying Theorem 5.12 and proposition 5.21 to the fact that $\mu(K, H) \neq 0$ then it follows that K is an intersection of maximal subgroups of H, then we have that $\mu(H, G) \neq 0$ if and only if $H \in \mathcal{L}_M^O(G)$. Consequently, we get an equivalent expression for the idempotent of $\mathbb{Q}\widehat{B}(G)$ as follows.

Proposition 5.22. *Let G be a profinite group, $H \leq_o G$ then we have that*

$$e_H^G = \sum_{K \in \mathcal{L}_M^O(G)} \mu(K, H) \frac{|G : N_G(H)|}{|G : K|} G/K.$$

The following is motivated by tom Dieck[23](Proposition 6), we have shown that we can form an inverse limit of minimal normal open subgroups such that their quotient group is solvable. By the notation of proposition 5.17, we have that for H the normal subgroup which is the kernel of the quotient is denoted by H^{∞}. These will allow us to make a connection between the closed perfect subgroups of G and the idempotents of $\widehat{B}(G)$. Note that a closed perfect subgroup always exists since $1 \leq G$ is a closed subgroup for all profinite groups G. We prove a result that is the analogue to a result in Bouc's survey paper on the Burnside ring of finite groups[4](Proposition 3.3.4).

Proposition 5.23. *Let G be a profinite group and π a set of primes. Suppose that $\mathcal{F}$ is a family of open subgroups closed under conjugation. Let $[\mathcal{F}]$ the set of G-conjugacy classes of these groups, then the following are equivalent.*

1.
$$\sum_{H \in [\mathcal{F}]} e_H^G \in \mathbb{Z}_{(\pi)} \otimes \widehat{B}(G) \subseteq \mathbb{Q}\widehat{B}(G)$$

2. *Let $H, K \leq_o G$ be subgroups such that K/H is (cyclic) of prime order $p \in \pi$ then*
$$H \in \mathcal{F} \Leftrightarrow K \in \mathcal{F}.$$

Proof. Suppose that $e \in \mathbb{Q}\widehat{B}(G)$ is an idempotent such that $e = \sum_{H \leq_o G} y_H \cdot G/H$, with $y_H \in \mathbb{Z}_{(\pi)}$. Clearly then, we have that $e \in \mathbb{Z}_{(\pi)}\widehat{B}(G)$. Take $H \leq_o K$ such that K/H is a cyclic group of prime order, then by lemma 4.15 we have that $\varphi_K(X) \equiv \varphi_H(X)$ for each $X \in \widehat{B}(G)$, in particular we can take the $\mathbb{Q}$-linear extension of the ring homomorphism φ_H and the result still holds. As we have shown following theorem 5.14, $\varphi_H(e_K^G) = 1$ if $H \sim K$ and 0 otherwise. Consider

$$\varphi_H(e) = \varphi_H\left(\sum_{L \in \mathcal{F}}' e_L^G\right) = \sum_{L \in \mathcal{F}}' \varphi_H(e_L^G),$$

then since we are taking the restricted sum over the conjugacy classes of subgroups in $\mathcal{F}$, we have that $\varphi_H(e) = \varphi_H(e_H) = 1$ if $[H] \in [\mathcal{F}]$ and $\varphi_H(e) = 0$ otherwise.

However, since $\varphi_K(e) \equiv \varphi_H(e) \bmod p$, and if $p \in \pi$, we have that $[H] \in [\mathcal{F}]$ if and only if $[K] \in [\mathcal{F}]$.

Conversely, suppose that we have $H \leq_o G$, then there is an open normal subgroup contained in H. Consider the map $\pi_N^G : \mathbb{Q}\widehat{B}(G) \to \mathbb{Q}B(G/N)$ to be the $\mathbb{Q}$-linear extension of the projection map, and take $e \in \mathbb{Q}\widehat{B}(G)$, then we have that $\varphi_{K/N}(X) = \varphi_K(X)$ for each $K \geq N$, it follows that in particular, $\varphi_{K/N}(X) = 0$ or 1 for each $K/N \leq G/N$. Therefore, since all of the fixed points under subgroups of G/N are either 0 or 1, we have that $\pi_N^G(e)$ is an idempotent in $\mathbb{Q}B(G/N)$. By Bouc[4](theorem 3.3.4), we have that for each pair of open subgroups such that K/H a cyclic group of prime order, we have that this idempotent lies within $\mathbb{Z}_{(\pi)}$ and so all coefficients y_H in the expression $e = \sum_{H \leq_o G} y_H \cdot G/H$ are in $\mathbb{Z}_{(\pi)}$. Repeating for all $H \leq_o G$ achieves the required result that $y_H \in \mathbb{Z}_{(\pi)}$ for every $H \leq_o G$ and so $e \in \mathbb{Z}_{(\pi)}\widehat{B}(G)$.

□

If we apply this result with the set π to be the set of all primes, we have the idempotents of $\mathbb{Q}\widehat{B}(G)$ since all primes are coprime to the denominator of the rational coefficients y_H such that $e \in \mathbb{Q}\widehat{B}(G)$ is expressed in the form $e = \sum_{H \leq_o G} y_H \cdot G/H$. We therefore look for irreducible families of subgroups which can define idempotents. Recall that a subgroup $P \leq G$ is said to be perfect if $[P, P] = P = P^{(\infty)}$, where the sum is taken over each conjugacy class of subgroups.

Proposition 5.24. *Let G be a profinite group, P a closed perfect subgroup of G, then the G-space defined by*

$$f_J^G = \sum_{\substack{H \leq_o G, \\ H^{(\infty)} = J}}' e_H^G$$

is a virtual almost finite G-space.

Proof. In order to show that this is a virtual almost finite G-space, we are required to show that there are finitely many fixed points under the action of each open subgroups $K \leq_o G$ and that each element in f_J^G is in a finite orbit by definition 2.19.

This is to say that each element in f_J^G has an open stabilizer subgroup, by Dress-Siebeneicher[10](2.2.2) we then have that this can be regarded with the discrete topology. Virtual, in this case, merely states that we can have both negative and non negative coefficients for each G orbit in the formal expression.

Define a map c_K that for any G-space X, $X = \sum_{H \leq_o G} x_H \cdot G/H$, each x_H a coefficient, then we define $c_H(X) = x_H$. Then, applying this to e_H^G for some $H \leq_o G$, we have that we sum over all conjugates of K in order to establish the number of isomorphic transitive G-spaces. It follows that

$$c_K(e_H^G) = \sum_{K' \sim K} \frac{|G : N_G(H)|}{|G : K'|} \mu(K', H).$$

Note that since $|G : N_G(K)| \leq |G : K|$ is finite, there are finitely many conjugates of K in G and therefore we have that this is a finite sum. Clearly, since $H, K \leq_o G$ we have that $|G : N_G(H)|$, $|G : K|$ and $\mu(K', H)$ are finite and so $c_K(e_H^G)$ is finite.

We now consider f_J^G,

$$\begin{aligned} c_K(f_J^G) &= c_K \left(\sideset{}{'}\sum_{\substack{H \leq_o G, \\ H^{(\infty)} = J}} e_H^G \right) \\ &= \sideset{}{'}\sum_{\substack{H \leq_o G \\ H^{(\infty)} = J}} c_K(e_H^G) \\ &= \sideset{}{'}\sum_{\substack{K \leq H \leq_o G, \\ H^{(\infty)} = J}} c_K(e_H^G). \end{aligned}$$

However, since $|G : K|$ is finite, we have that there are finitely many groups containing K for each $K \leq_o G$, therefore once again this is a finite sum of finite summands and therefore finite itself. Since $c_K(f_J^G)$ is finite for each $K \leq_o G$, we have that for $L \leq_o G$,

$$\varphi_L(f_J^G) = \sideset{}{'}\sum_{K \leq_o G} \varphi_L(c_K(f_J^G) G/K) = \sideset{}{'}\sum_{L \leq_o K \leq_o G} c_K(f_J^G) \varphi_L(G/K).$$

Since $K \leq_o G$ we have $\varphi_L(G/K)$ is finite and there are finitely many summands, we have that $\varphi_L(f_J^G)$ is finite for each $L \leq_o G$. We therefore conclude that this is a virtual essentially finite G-space, since each orbit is finite we can regard it with the discrete topology and so it is a virtual almost finite G-space.

□

Therefore the isomorphism class of f_J^G as a G-space is an idempotent in the Burnside ring. Since the $H^{(\infty)}$ we have defined are perfect subgroups, each open subgroup $H \leq_o G$ contains precisely one subgroup of this form since if it were to contain two, we would contradict the minimality of the subgroup $H^{(\infty)}$. Therefore, the closed perfect subgroups partition the sets since they are disjoint and the union of the sets $\{H \leq_o G \mid H^{(\infty)} = P\}$ is all the open subgroups of G. Therefore, we have that since each conjugacy class of open subgroup appears exactly once across these sets, we can make the following corollary.

Corollary 5.25. *Let G be a profinite group, P a closed perfect subgroup of G, then*

$$G/G = \sum_{P \leq_c G} f_P^G$$

with the sum taken over all conjugacy classes of closed perfect subgroups of G.

Proof. Let G be a profinite group, then we have that in $\mathbb{Q}\widehat{B}(G)$, then the sum of all idempotents is of the following form

$$G/G = \sum_{H \leq_o G} e_H^G.$$

Since $H^{(\infty)}$ is a closed perfect subgroup $\forall H \leq_o G$, and is unique, we have that each conjugacy class of subgroups appears exactly one in the sum. It follows that by summing over the closed perfect subgroups P and then over open subgroups $H \leq_o G$ such that $H^{(\infty)}$, we have a series which includes e_H^G exactly once for each conjugacy class of $H \leq_o G$.

$$G/G = \sum_{P \leq_c G} \sum_{H(\infty)=P} e_H^G = \sum_{P \leq_c G} f_P^G.$$

$\square$

Corollary 5.26. *Let G be a profinite group and π a set of primes. For $H \leq_o G$, define H^π to be the minimal normal subgroup of H such that H/H^π is a pro-solvable pro-π group. A group with the property $H = H^\pi$ is called π-perfect. Define an element r_J^G by*

$$r_J^G = \sum_{\substack{H \leq_o G \\ H^\pi \sim_G J}} e_H^G$$

Then the set $\{r_J^G \mid J \text{ a closed } \pi\text{-perfect subgroup } \ \exists H \leq_o G,\ H^\pi \sim_G J\}$ is a complete set of orthogonal idempotents of $\mathbb{Z}_{(\pi)}\widehat{B}(G)$.

Proof. Firstly, we show that H^π is a well defined element that exists. Suppose that H has two subgroups $K_1,\ K_2 \trianglelefteq H$ such that H/K_1 and H/K_2 are pro-solvable pro-π groups, then we have that $K_1 \cap K_2 \trianglelefteq H$ and we have that $K_1K_2 \leq H$ implies that K_1K_2/K_1 is a π-group and $K_1/K_1 \cap K_2$ is a π-group by the tower theorem for subgroups. It follows that $H/K_1 \cap K_2$ must be a π-group by definition. Therefore, we can take the intersection of all such subgroups to find H^π. We always have $H/H = 1$ is a pro-solvable pro-π group and therefore there always exists at least one subgroup K such that H/K is a pro-solvable pro-π group.

By Proposition 5.25, we have that for $\mathcal{F}$ a family of subgroups closed under conjugation, then $f = \sum'_{H \in \mathcal{F}} e_H^G$, where the sum is over a representative of each conjugacy class of open subgroups, is an idempotent in $\mathbb{Z}_{(\pi)}\widehat{B}(G)$ if and only if for all $K, H \leq_o G$ such that H is normal in K such that K/H is a group of order $p \in \pi$, then $K \in \mathcal{F} \Leftrightarrow H \in \mathcal{F}$. We first show that each r_J^G as defined above is an idempotent in $\mathbb{Z}_{(\pi)}$ using this equivalence.

Suppose that we have $H, K \leq_o G$ such that K/H is a cyclic group of order p. Then by definition, we have that $K^\pi \trianglelefteq K$, since $H \leq K$ and K/H is both solvable and a p-group, then we have that $H \cap K^\pi$ since K^π is the intersection of all normal subgroups such that the quotient is a solvable p-group. This implies that $K^\pi \trianglelefteq H$ but $H^\pi \trianglelefteq K^\pi$ and so we have that $K^\pi = H^\pi$, therefore for a closed π-perfect subgroup $J \leq G$, we have a family $\mathcal{F} = \{H \leq_o G \mid H^\pi \sim_G J\}$ such that $H \in \mathcal{F} \Leftrightarrow K \in \mathcal{F}$ and so defines an idempotent in $\mathbb{Z}_{(\pi)}\widehat{B}(G)$ of the form r_J^G since this family being closed under conjugation is clear.

In order to show orthogonality, assume that we have closed π-perfect subgroups $J, K \leq G$ and consider $x = r_J^G \cdot r_K^G$. Take $H \leq_o G$, then we compare the fixed points under H-action, $\varphi_H(x) = \varphi_H(r_J^G) \cdot \varphi_H(r_K^G)$. Recall that by definition of the e_H^G, we have that $\varphi_H(r_J^G) = \delta_{H^\pi, J}$ and $\varphi_H(r_K^G) = \delta_{H^\pi, K}$. We have then $\varphi_H(x) = \varphi_H(r_K^G) \cdot \varphi_H(r_J^G) = \delta_{H^\pi, K} \cdot \delta_{H^\pi, J}$. If this is 0 everywhere then we have that they are orthogonal. Assume they are not 0 everywhere, then by definition of delta we have that $H^\pi = J = K$ for some $H \leq_o G$. But then we have $r_J^G = r_K^G$, therefore they are mutually orthogonal.

Finally we show that there is no proper sub family of $\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathcal{F}'$ defines an idempotent. Suppose that there is such a family and take $H \in \mathcal{F}'$ and $K \in \mathcal{F}$. Since $\mathcal{F}'$ is a subfamily, we have that $H \in \mathcal{F}$ and so $H^\pi = J$. Since H, K are open, we can take an open normal subgroup $N \leq H \cap K$, then $H^\pi N$ is a normal subgroup of H and K such that the quotient $H/H^\pi N = K/K^\pi N$, both of which are solvable. Since the composition groups of both of these are formed of cyclic groups of prime order, we have that $H \in \mathcal{F}$ and so the families must be equal.

$\square$

If we apply this for $\pi = \{\text{all primes}\}$ then consequently we get that the perfect subgroups are the ones that form the orthogonal idempotents for $\widehat{B}(G)$. In this case, the closed π-perfect subgroups become closed perfect subgroups.

5.4 Idempotents of $\widehat{B}(A_5 \times \mathbb{Z}_p)$

Example 5.27. Let $G = A_5 \times \mathbb{Z}_p$ for $p \geq 7$ where $\mathbb{Z}_p$ is the p-adic integers and take the profinite completion. The closed perfect subgroups of G are given by $\{A_5 \times e, 1 \times e\}$ where e denotes the trivial group of $\mathbb{Z}_p$ since the projection into each group in the direct product must be perfect and we know that $\mathbb{Z}_p$ is Abelian. We know that the order of A_5 does not divide p and so we have no other perfect subgroups since there does not exist a subgroup of order $|A_5| = 120$ in $\mathbb{Z}_p$.

As we have proved, it is sufficient to look at the closed perfect subgroups of G in order to determine a complete set of orthogonal idempotents of $\widehat{B}(G)$. It is clear then that the orthogonal idempotent admitted by these are $\{f^G_{A_5 \times e}, f^G_{1 \times e}\}$. These are both not $0, 1$ in the Burnside ring since we have that there exist open subgroups such that these perfect groups are given by the commutator subgroup. Explicitly, we have $[G, G] = A_5 \times e$ and $[1 \times \mathbb{Z}_p, 1 \times \mathbb{Z}_p] = 1 \times e$.

Both of these subgroups are open and so in both cases we have an open subgroup such that $\varphi_G(f^G_{A_5 \times e}) = 1$, $\varphi_{1 \times \mathbb{Z}_p}(f^G_{1 \times e}) = 1$ and conversely $\varphi_G(f^G_{A_5 \times e}) = 0$, $\varphi_{1 \times \mathbb{Z}_p}(f^G_{A_5 \times e}) = 0$ since the idempotents are orthogonal. It follows that neither can be $0, G/G \in \widehat{B}(G)$ and therefore are non trivial examples of idempotents. Substituting in the expression shown in Proposition 5.24 for e^G_H, we see an explicit expression for the idempotents.

$$\begin{aligned} f_{A_5 \times 1} &= \sum_{\substack{H \leq_o G, \\ H^{(\infty)} \sim A_5 \times e}} e^G_H \\ &= \sum_{\substack{H \leq_o G, \\ H^{(\infty)} \sim A_5 \times e}} \sum_{K \in \mathcal{L}^O_M(H)} \mu(K, H) \frac{|G : N_G(H)|}{|G : K|} G/K. \end{aligned}$$

In order to evaluate the summands, we first select $H \leq_o G$ such that $H^{(\infty)} \sim A_5 \times e$. Then we use the notation that $\pi_1(H) = H_1$, $\pi_2(H) = H_2$ are the projections into either coordinates respectively. Any H satisfying this condition must be of the form $A_5 \times H_2$ where $H_2 \leq_o \mathbb{Z}_p$. Using this, we know that the Möbius function on the element $K \in \mathcal{L}_M^O(H)$ has $\mu(K,H)=0$ unless $K=H$ or is the intersection of maximal subgroups of H. Since every subgroup of A_5 can be expressed as the intersection of maximal subgroups of A_5, we see that the poset is equivalent to the poset $\mathcal{L}_M^O(A_5) \times \{H_2, pH_2\}$ with the product ordering.

In order to further simplify the coefficients, we consider the value of the terms $|G : N_G(A_5 \times H_2)|$ for each $H_2 \leq_o \mathbb{Z}_p$. For any $H_2 \leq_o \mathbb{Z}_p$, we have that $H_2 = p^i\mathbb{Z}_p$ for some $i \in \mathbb{N}_0$. We combine these to get an expression for $f^G_{A_5 \times e}$.

$$
\begin{aligned}
f^G_{A_5\times 1} &= \sum_{H_2 \leq_o \mathbb{Z}_p} \; \sum_{K \in \mathcal{L}_M^O(A_5 \times H_1)} \mu(K, A_5 \times H_2) \frac{|G : N_G(A_5 \times H_2)|}{|G : K|} G/K \\
&= \sum_{i \in \mathbb{N}_0} \; \sum_{K \in \mathcal{L}_M^O(A_5 \times p^i\mathbb{Z}_p)} \frac{\mu(K,H)}{|A_5 : \pi_1(K)||\mathbb{Z}_p : \pi_2(K)|} G/K \\
&= \sum_{i \in \mathbb{N}_0} \; \sum_{K \in \mathcal{L}_M^O(A_5)} \frac{\mu(K \times p^i\mathbb{Z}_p, A_5 \times p^i\mathbb{Z}_p)}{|A_5 : K||\mathbb{Z}_p : p^i\mathbb{Z}_p|} G/(K \times p^i\mathbb{Z}_p) + \\
&\qquad \frac{\mu(K \times p^{i+1}\mathbb{Z}_p, A_5 \times p^i\mathbb{Z}_p)}{|A_5 : K||\mathbb{Z}_p : p^{i+1}\mathbb{Z}_p|} G/(K \times p^{i+1}\mathbb{Z}_p) \\
&= \sum_{i \in \mathbb{N}_0} \sum_{K \in \mathcal{L}_M^O(A_5)} \frac{\mu(K, A_5)}{|A_5 : K|p^i} G/(K \times p^i\mathbb{Z}_p) + \frac{-\mu(K, A_5)}{|A_5 : K|p^{i+1}} G/(K \times p^{i+1}\mathbb{Z}_p).
\end{aligned}
$$

Let $K_i = K \times p^i\mathbb{Z}_p$ then for $i \geq 1$

$$
\begin{aligned}
c_{K_i}(f^G_{A_5\times 1}) &= \sum_{K' \sim K} \frac{\mu(K', A_5)}{p^i|A_5 : K|} - \frac{\mu(K', A_5)}{p^i|A_5 : K|} \\
&= 0
\end{aligned}
$$

$$
\Rightarrow f^G_{A_5\times 1} = \sum_{K \in \mathcal{L}_M^O(A_5)} \frac{\mu(K, A_5)}{|A_5 : K|} G/(K \times \mathbb{Z}_p)
$$

$$
\begin{aligned}
\Rightarrow f^G_{1\times 1} &= G/G - f^G_{A_5\times 1} \\
&= \sum_{K \in \mathcal{L}_M^O(A_5) \setminus A_5} \frac{\mu(K, A_5)}{|A_5 : K|} G/(K \times \mathbb{Z}_p).
\end{aligned}
$$

Since $A_5 \times \mathbb{Z}_p$ is virtually pro-solvable since it contains an open subgroup of the form $1 \times \mathbb{Z}_p$, we see that these idempotents can be expressed as the inflation of an idempotent from $B(G/1 \times \mathbb{Z}_p)$, and that we have as many idempotents as we have in $B(A_5)$. This raises a question of whether it is possible to find idempotents which are not the inflation of idempotents for some $B(G/N)$ and shall be the motivation behind the next section.

5.5 Finite and infinite idempotents

We wish to answer the question of if it is possible to find idempotents in the Burnside ring of a profinite group that are the isomorphism classes of virtual almost finite G-spaces which are not finite G-sets. We begin by proving a basic result on the the idempotents of $\widehat{B}(G)$.

Proposition 5.28. *Let G be a profinite group, every finite G-set can be represented as the inflation from $B(G/N)$ to $\widehat{B}(G)$ for some $N \trianglelefteq_o G$.*

Proof. Suppose that $X \in \widehat{B}(G)$ is a finite G-set, then X can expressed by the element $X = \sum_{H \leq_o G} x_H \cdot G/H$ where at most finitely many x_H are non zero. Take $K = \bigcap_{X_H \neq 0} H$, then we must have that $K \leq_o G$ since it is the intersection of finitely many open subgroups. In particular, be the definition of the topology, we must have that there is an open normal subgroup of G contained within K. That is to say that there exists $N \trianglelefteq_o G$ such that $N \leq K$. Consequently, we have $\sum_{H \leq_o G} x_H \cdot G/H = \mathrm{Inf}_{G/N}^{G} \left(\sum_{N \leq H \leq_o G} x_H \cdot (G/N)/(H/N) \right)$. □

Corollary 5.29. *Every idempotent in $\widehat{B}(G)$ which is a finite G-set is an inflation from some $N \trianglelefteq_o G$. That is to say, the inflation from the Burnside ring of some finite group as defined in 2.24.*

We note that $G = A_5 \times \mathbb{Z}_p$ is an example of a virtually pro-solvable group and so these idempotents can be expressed as the inflation from $B(A_5 \times \mathbb{Z}_p/1 \times \mathbb{Z}_p) \cong B(A_5)$.

Definition 5.30. Let G be a profinite group, $N \trianglelefteq_o G$, then for $e \in B(G/N)$ an idempotent, we let $\hat{e} := \operatorname{Inf}{}^{G}_{G/N} e \in \widehat{B}(G)$ be the idempotent corresponding to the inflation in the respective ring.

Proposition 5.31. *For $e \in B(G/N)$, $N \trianglelefteq_o G$ and $K \leq_o N$, we have that*

$$\varphi_K(\hat{e}) = \varphi_N(\hat{e}) = \varphi_{N/N}(e).$$

Proof.

$$\varphi_N(\hat{e}) = \sum_{H \leq_o G} |x_H| \cdot |\{x \in G/H | N.x = x\}|$$

As a virtual G-space but $K \leq N$ and N acts trivially on $\hat{e}$ since $\hat{e} \in \operatorname{Im}(\operatorname{Inf}{}^{G}_{G/N})$

$$\Rightarrow \; N.x = x \; \forall x \in \hat{e}$$

$$\Rightarrow \; K.x = x \; \forall x \in \hat{e}$$

$$\begin{aligned}
\varphi_K(\hat{e}) &= \sum_{H \leq_o G} |x_H| \cdot |\{x \in G/H | K.x = x\}| \\
&= \sum_{H \leq_o G} |x_H| \cdot |\{x \in G/H | N.x = x\}| \\
&= \varphi_N(\hat{e})
\end{aligned}$$

But N acting trivially on $\hat{e}$

$$\Rightarrow \varphi_N(\hat{e}) = \varphi_{N/N}(\hat{e}),$$

□

This covers the case when we have that the idempotents are virtual finite G-sets, but can we have virtual almost finite G-spaces? And if so, when do they occur? We break this question down into two cases.

1. We have a perfect subgroup P such that f_P^G is isomorphism class of an (infinite) almost finite G-space,

2. We have infinitely many orthogonal idempotents f_P^G but all of which are finite G-sets and it is possible to take an infinite series of these which is infinite.

We shall show that the first of these can occur using an example, thus proving that idempotents that are infinite almost finite G-spaces can exist.

Example 5.32. Let $G = \prod_{\mathbb{N}} A_5$, we then construct the profinite completion of G. To create the profinite completion, we require a base for the topology of open normal subgroups, which is to say normal subgroups of finite index. Take any normal subgroup of G, then it must be normal in each coordinate (although it is not necessarily a direct product). However, since this subgroup must be of finite index, we have that for each open normal subgroup N, there is a maximal natural number j such that $\pi_j(H) = A_5$ for all $k \geq j$ and we are free to take any element in any coordinate for $k \geq j$.

Note that this is possible since even a diagonal subgroup $H \leq_o G$ must have finitely many $k \in \mathbb{N}$ such that $\pi_k(H) \neq A_5$. It is therefore sufficient to take a sub base for the topology defined by $\prod_{i<j} 1 \times \prod_{i \geq j} A_5$. Any intersection of subgroups of this form is again a subgroup of this form and so is a well defined filter base.

We take the compatible maps, for $i < j$,

$$\phi_{j,i} : \prod_{k=1}^{j} A_5 \to \prod_{k=1}^{i} A_5$$

to be the projection maps into the first i coordinates. These are compatible in the usual way since $\phi_{k,j}\phi_{j,i} = \phi_{k,i}$ for $i \leq j \leq k$. Then we have a sub collection of open normal subgroups of $\widehat{G}$ are given by the kernels of the maps

$$\phi_i : \widehat{G} \to \prod_{k=1}^{i} A_5.$$

It is important to note that these are not all of the open normal subgroups, since we can also have diagonals in the sense that we could have a group H such that $\pi_i(x) = \pi_j(x) \quad \forall x \in H$ where π_i, π_j are the projection into the i, j coordinate respectively, however each of these must contain the kernel of some ϕ_k and so it is sufficient to consider the groups which contain the kernels of the maps. We give a base for the topology in terms of the kernels,

$$N_j = \ker(\phi_j) = \prod_{k=1}^{j} 1 \times \prod_{\mathbb{N}\setminus\{1,\ldots,j\}} A_5$$

The closed perfect subgroups of $\widehat{G}$ are given by the Cartesian product of any combination of diagonals, trivial groups and copies of A_1. It is worth noting that all perfect subgroups are isomorphic to $\widehat{G} \cong \prod A_5$.

$$H \leq_o \widehat{G} \Rightarrow \exists j : N_j \leq H.$$

Taking $S = \{\prod_I 1 \times \prod_J A_5 \mid I \cup J = \mathbb{N}, I \cap J = \emptyset\}$ a subset of the set of closed perfect subgroups of G, $P_{\widehat{G}} = \{P \leq_c \widehat{G} \mid [P,P] = P\}$, we see that there are infinitely many orthogonal idempotents. For ease of notation, from now on we shall use G to refer to the inverse limit $\widehat{G}$. Consider in particular the idempotent given by f_G^G since we have that G is perfect.

$$\begin{aligned} f_G^G &= \sum_{H^{(\infty)}=G} e_H^G \\ &= e_G^G \\ &= \sum_{\mathcal{L}_M^O(G)} \frac{|G : N_G(G)|}{|G : K|} \mu(K,G) G/K \\ &= \sum_{\mathcal{L}_M^O(G)} \frac{\mu(K,G)}{|G : K|} G/K. \end{aligned}$$

We can set $H_{j,M} = \prod_{i \neq j} A_5 \times M$ where A_5 is maximal in A_5.

$$\Rightarrow \mu(H_{j,M}, G) = -1$$

Since we know that A_5 is simple, we have that the maximal subgroups are self normalizing, and therefore the number of conjugates that appear in the expression is equal to the index $|G : H_{j,M}|$

$$\Rightarrow c_{H_{j,M}}(f_G^G) = -|G : H_{j,M}|.$$

$$\Rightarrow c_{H_{j,M}}(f_G^G) = \begin{cases} -6 \text{ if } M \cong D_{10} \\ -5 \text{ if } M \cong A_4 \\ -10 \text{ if } M \cong S_3. \end{cases}$$

Therefore we have infinitely many summands which are non zero and non cancelling.

$$\Rightarrow f_G^G \text{ is an almost finite } G\text{-space which is not finite.}$$

We can generalise this result to the following using similar arguments.

Lemma 5.33. *If G is a profinite group and $G \cong \prod_{i\in\mathbb{N}} G_i$ with each $G_i \neq 1$ a finite perfect group, then we have $\widehat{B}(G)$ contains an idempotent which is an infinite G-space.*

Proof. Consider the idempotent f_G^G. Clearly we have that $f_G^G = e_G^G$, in order to show that this is not equal to 1, we find a proper open normal subgroup, namely $P = \prod_{i\neq j} G_i$, since this is open, we have $\varphi_P(f_G^G) = \varphi_P(e_G^G) = 0$ and so $\varphi_H(f_G^G) = 0 \quad \forall H <_o G$. The following series is well defined since for each open subgroup K, there are at most finitely many conjugates of $K \in G$ and so despite being an uncountable set, there are only finitely many summands for each G/K.

$$\begin{aligned} f_G^G = e_G^G &= \sum_{K\in\mathcal{L}_M^O(G)} \mu(K,G)\frac{|G : N_G(G)|}{|G : K|}G/K \\ &= \sum_{K\in\mathcal{L}_M^O(G)} \frac{\mu(K,G)}{|G : K|}G/K \end{aligned}$$

But in particular we have that for maximal subgroups $M \leq_o G$, $\mu(M,G) \neq 0$. Since the series is given over the open subgroups and not the conjugacy classes of open subgroups, we check that these do not cancel termwise. Given that we have uncountably many maximal subgroups we get that there are infinitely many non cancelling terms and so we shall be done.

$$G/M_1 = G/M_2 \Leftrightarrow M_1 \sim_G M_2.$$

Take M maximal and suppose that we have $c_g : M \to M^g$ to be the conjugation map by $g \in G$. Then suppose we have that M^g is not maximal, then $M^g \leq H$ implies $M \leq {}^gH$, but M is maximal in G and so ${}^gH = M$ or ${}^gH = G$. However, since $|G : M| = |G :^g M|$ we have a contradiction.

$$\Rightarrow \mu(M, G) = \mu(M^g, G) = -1$$

for M maximal, in particular $sgn(\mu(M, G)) = sgn(\mu(M^g, G))$

$$\Rightarrow c_M(f_G^G) \neq 0 \quad \forall M \leq_o G \text{ maximal.}$$

□

We can use the same principles as defined in this example to prove the following result.

Corollary 5.34. *Let G be a perfect profinite group with infinitely many open maximal subgroups, then we have that there is an idempotent which is not a finite G-set in $\widehat{B}(G)$ and f_G^G will be an isomorphism class of an almost finite G-space.*

Theorem 5.35. *Let G be a perfect profinite group with $\Phi(G)$ not open, then we have an infinite idempotent in $\widehat{B}(G)$.*

To assist with our investigation of the second case, when we have infinitely many idempotents all of which are finite G-sets, we prove the following result.

Proposition 5.36. *Let G be an infinite profinite group with infinitely many idempotents of $\widehat{B}(G)$, then we have that $\mathcal{K} = \{K \mid c_K(f_P^G) \neq 0$ for some $P \in P_G\}$ is an infinite set.*

Proof. Suppose that $\mathcal{K}$ is a finite set, then we have that the intersection of all elements of $\mathcal{K}$ must be an open subgroup since it is the intersection of finitely many open subgroups. It follows there exists an open normal subgroup $N \trianglelefteq_o G$ such that $N \leq \bigcap_{\mathcal{K}} K$. Consequently, we have that all idempotents of $\widehat{B}(G)$ are expressible as an inflation $\hat{e} = \mathrm{Inf}_{G/N}^{G}(e)$ for e some idempotent of $B(G/N)$. There are only finitely many idempotents in $B(G/N)$ and so we have a contradiction, therefore we must have that $\mathcal{K}$ is an infinite set. □

We note that G must not be virtually pro-solvable since otherwise we would have only finitely many idempotents. If we choose a perfect group, then it cannot have infinitely many maximal subgroups since otherwise we will have that f_G^G is an infinite almost finite G-space. Since we require all of the othogonal idempotents to be finite G-sets, we have that they can each be given by an inflation, and we need infinitely many perfect subgroups. We begin by proving a result that shows one way in which we can find finite G-sets as idempotents.

Proposition 5.37. *Suppose that $P \leq_o G$, a perfect subgroup of G, such that $\Phi(P) \leq_o G$,*

$$\Rightarrow \ f_P^G \text{ is a finite } G\text{-set.}$$

Proof. Consider

$$\begin{aligned} f_P^G &= \sum_{\substack{H \leq_o G \\ H^{(\infty)} = P}} e_H^G \\ &= \sum_{\substack{H \leq_o G \\ H^{(\infty)} = P}} \sum_{K \in \mathcal{L}_M^O(H)} \frac{|G : N_G(H)|}{|G : K|} \mu(K, H) G/K. \end{aligned}$$

$$\begin{aligned} K \in \mathcal{L}_M^O(H) &\Rightarrow \Phi(H) \leq K \\ H^{(\infty)} = P &\Rightarrow P \leq H \\ &\Rightarrow \Phi(P) \leq \Phi(H) \\ &\Rightarrow \Phi(P) \leq K \end{aligned}$$

$\Phi(P) \leq_o G \Rightarrow$ there are finitely many K such that $\Phi(P) \leq K$.

$$\Rightarrow \; f_P^G \text{ is a finite } G\text{-set.}$$

□

Example 5.38. Let $G = SL_2(\mathbb{Z}_p)$, then we have that $[SL_2(\mathbb{Z}_p), SL_2(\mathbb{Z}_p)] = G$, which is to say it is equal to its derived subgroup. This must be perfect since in each of the finite quotients of its inverse limit, we have that G/N will be have $[G/N, G/N] = G/N$ and therefore is perfect in G/N. Note that $\Phi(G) \leq_o G$ since G is virtually pro-p, with an open normal subgroup N_1 which denotes the matrices congruent to the identity mod p. The Frattini of N_1 is open in G and so the Frattini of G is open in G since $\Phi(N_1) \leq \Phi(G)$.

Consequently, taking a group H such that $H = G \times C_p$ gives a group which has an open perfect subgroup with open Frattini, namely $1 \times G$ is perfect since it is an inverse limit of perfect groups. This again is virtually pro-p and so we can use $\Phi(K := N_1 \times C_p) = [K, \bar{K}]K^p = \Phi(N_1) \times 1 = N_2$ which is open in H

Since we know that $\Phi(G)$ is the intersection of all maximal open subgroups of G, we clearly see that it is open if we have finitely many open maximal subgroups. Therefore, in order to find a group which has infinitely many idempotents as the form listed above, we are looking for a group that has infinitely many perfect subgroups, each of which with finitely many maximal subgroups. We see that the example of $G_1 = \varprojlim_{i \in I} A_5^i$ does not adhere to these conditions since whilst it has infinitely many open perfect subgroups, we see that each of these subgroups has infinitely many maximal subgroups. In fact, we can also claim that in fact all idempotents of the form $f_P^{G_1}$ are infinite G_1-spaces since we have that each open perfect subgroup is isomorphic to G_1.

Consequently, we look to other such groups which are perfect, for example we can take the group $G_2 = SL_n(\mathbb{Z}_p)$ for $p \geq 7$, we have that this is a perfect virtually pro-p group which has an open normal subgroup which is a finitely generated pro-p group and therefore has open Frattini. Whilst this satisfies the condition that there are finitely many maximal subgroups, we see that we only have finitely many open perfect subgroups, in fact we get that G_2 itself is the only open perfect subgroup since all others have infinite index.

If we can find a perfect group, that similar to G_1 has infinitely many open perfect subgroups isomorphic to the whole group and that similar to G_2 has an open Frattini/finitely many open maximal subgroups then we will have found a group with infinitely many idempotents that are finite G-sets.

5.6 Units from idempotents

We now consider the link between idempotents and units within the Burnside ring. We can classify both these classes of elements by their image in the Ghost ring, the former taken to have an entry of $0, 1$ in each coordinate and the latter with $1, -1$ in each coordinate. Using our results for the Burnside algebra, $\mathbb{Q} \otimes \widehat{B}(G)$, we have a method of constructing every unit from every idempotent in the Burnside algebra.

Lemma 5.39. *If $u \in \widehat{B}^{\times}(G)$ then $u = \sum_{H \in \mathcal{H}} e_H^G - \sum_{H \in \mathcal{H}'} e_H^G$ where $\mathcal{H} \cup \mathcal{H}' = \{[H] | H \leq_o G\}$ and $\mathcal{H} \cap \mathcal{H}' = \emptyset$.*

Proof. Take $u \in \widehat{B}^{\times}(G)$

$$\Rightarrow \varphi_K(u) = \pm 1 \ \forall H \leq_o G$$

Then we also have that $\varphi_K(e_H^G) = \begin{cases} 1 \text{ if } H \sim K \\ 0 \text{ otherwise} \end{cases}$

$$\Rightarrow K \in \mathcal{H} \text{ if } \varphi_K(u) = 1, \quad K \in \mathcal{H}' \text{ if } \varphi_K(u) = -1$$

$$\Rightarrow u = \sum_{H \in \mathcal{H}} e_H^G - \sum_{H \in \mathcal{H}'} e_H^G.$$

□

Consequently we have that every unit in the Burnside ring is expressible as the difference of two different idempotents in the Burnside algebra. Since the sets $\mathcal{H}, \mathcal{H}'$ are disjoint sets whose union cover all conjugacy classes of open subgroups of G, we have that the unit can be determined solely by $\mathcal{H}$.

Definition 5.40. Let $u \in \widehat{B}^{\times}(G)$, $\mathcal{H} \subseteq [s_G]$ then we let $u = u_{\mathcal{H}}$ to be the unit such that $\varphi_H(u) = 1$ for all $H \in \mathcal{H}$.

Clearly we have the units which can be expressed as the difference of idempotents not just within the Burnside algebra but idempotents within the Burnside ring.

Theorem 5.41. *Let G be a profinite group,*

$$S \subseteq P(G) = \{[J] | J \leq_c G, [J, J] = J\}$$

then we have that

$$u = \sum_{J \in S} f_J^G - \sum_{J \in P(G) \backslash S} f_J^G \in \widehat{B}^{\times}(G).$$

Proof. Take $K \leq_o G$

$$\Rightarrow [K^{(\infty)}] \in P(G)$$

$$\Rightarrow [K^{(\infty)}] \in S \text{ or } [K^{(\infty)}] \in P(G) \backslash S$$

$$\Rightarrow \varphi_K(u) = \begin{cases} 1 \text{ if } [K^{(\infty)}] \in S \\ -1 \text{ if } [K^{(\infty)}] \in P(G) \backslash S \end{cases}$$

$$\Rightarrow u \in \widehat{B}^{\times}(G).$$

□

Corollary 5.42. *Every unit, u, that can be written as $u = 1 - 2e$ with $e \in \widehat{B}(G)$ an idempotent is of the above form.*

Proof.

$$u = 1 - 2e = \sum_{J \in P(G)} f_J^G - 2 \sum_{J \in S} f_J^G = \sum_{J \in P(G) \backslash S} f_J^G - \sum_{J \in S} f_J^G.$$

$\square$

The natural question arises whether these are all of the units or if there are units which cannot be expressed in the form $1 - 2e$ for $e \in \widehat{B}(G)$. It can be noted that it is always true that every unit in $\widehat{B}(G)$ is of the form $1 - 2e$ for $e \in \mathbb{Q}\widehat{B}(G)$. We give an example to show that this is not always an exhaustive list of units.

Example 5.43. Let $G = \mathbb{Z}_2$, then we have that there is a unit in the form $u = \mathbb{Z}_2/\mathbb{Z}_2 - \mathbb{Z}_2/2\mathbb{Z}_2$ since we have a subgroup of index 2. However, G is pro-solvable and so the only idempotents in $\widehat{B}(G)$ are 0,1 so it cannot be written in the form $1 - 2e$ for $e \in \widehat{B}(G)$.

6 Open Questions

Throughout this thesis, I have proved the generalisation of many results concerning the Burnside rings of finite groups to the Burnside rings of profinite groups. A key aspect has been comparing and contrasting when the classifying results for structures in the Burnside rings for profinite groups aligns similarly to that of the Burnside rings of finite groups and where it diverges. The difference is where interesting open questions arise.

I have proved the existence of profinite groups G for which the Burnside ring $\widehat{B}(G)$ has closed but not open prime ideals. For H a finite group, then $B(H) \cong \widehat{B}(G)$ only has prime ideals which are both closed and open. These lead to the natural question of what structure must G have to ensure that there are ideals which are closed but not open. Given the strong connection between the prime ideal spectrum and idempotents, and we have shown that the idempotents are defined by perfect subgroups of G, it is natural to assume that the prime ideal structure on $\widehat{B}(G)$ is connected to the existence of perfect subgroups of G in some way.

I have also shown that every unit in $\widehat{B}(G)$ can be expressed as the difference of two idempotents in $\mathbb{Q}\widehat{B}(G)$. In example 5.43 it is demonstrated that the idempotents do not themselves be in $\widehat{B}(G)$ since it is possible that the terms with non integer coefficients may cancel. Therefore, can we classify when the partitions $\mathcal{H}, \mathcal{H}'$ of the set of open subgroups of G as in lemma 5.39 admit a unit within $\widehat{B}(G)$? Equivalently, can we describe all idempotents $e \in \mathbb{Q}\widehat{B}(G)$ such that $1-2e$ is in $\widehat{B}(G)$?

Finally, we have that $\{\mathrm{res}_S^G(G/H) \mid H \leq_o S \text{ fully normalized}\}$ is a basis for $\widehat{B}(\mathcal{F})$ for $\mathcal{F}$ a pro-fusion system over S defined by $\mathcal{F} = \mathcal{F}_S(G)$ with $S \leq_o G$. Can we then derive a similar result for a pro-fusion system over S given by G where S is not open in G? Each finite quotient of the fusion system $\mathcal{F}_i$ over the p-group S_i must necessarily be induced by some finite G_i. It follows that for $\mathcal{F}_i \cong \mathcal{F}_{S_i}(G_i)$, we have that there is a basis of $B(\mathcal{F}_i)$ given by $\{\mathrm{res}_{S_i}^{G_i}(S_i/H) \mid H \leq S_i \text{ fully normalized}\}$. Can we show that these basis elements in the finite quotients form an inverse limit?

References

[1] Kasper K.S. Andersen, Bob Oliver, and Joana Ventura. Reduced, tame and exotic fusion systems. *Proceedings of the London Mathematical Society*, 105(1):87–152, February 2012.

[2] Bernhard Banaschewski and Marcel Erné. On krull's separation lemma. *Order*, 10(3):253–260, 1993.

[3] Jamison Barsotti and Robbie Carman. Burnside rings of fusion systems and their unit groups. *Journal of Group Theory*, 23:709 – 729, 2019.

[4] Serge Bouc. Burnside rings. In M. Hazewinkel, editor, *Handbook of Algebra*, volume 2, pages 739–804. North-Holland, 2000.

[5] Nicolas Bourbaki. *General Topology: Chapters 1–4*. Springer Berlin Heidelberg, 1995.

[6] William Burnside. *Theory of Groups of Finite Order*. Cambridge Library Collection - Mathematics. Cambridge University Press, 2012.

[7] J. D. Dixon, M. P. F. Du Sautoy, A. Mann, and D. Segal. *Analytic Pro-P Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 1999.

[8] Andreas Dress. A characterisation of solvable groups. *Mathematische Zeitschrift*, 110(3):213–217, 1969.

[9] Andreas WM Dress. *Notes on the theory of representations of finite groups*. Univ., 1971.

[10] Andreas W.M. Dress and Christian Siebeneicher. The burnside ring of profinite groups and the witt vector construction. *Advances in Mathematics*, 70(1):87–132, 1988.

[11] David Gluck. Idempotent formula for the burnside algebra with applications to the p-subgroup simplicial complex. *Illinois J. Math.*, 25(1):63–67, March 1981.

[12] R. L. Goodstein. Sets and transfinite numbers, by m. m. zuckerman. pp xvi, 423. £5·95. 1974. sbn 0 02 432110 9 (collier-macmillan). *The Mathematical Gazette*, 59(408):128–128, 1975.

[13] P. HALL. THE EULERIAN FUNCTIONS OF A GROUP. *The Quarterly Journal of Mathematics*, os-7(1):134–151, 01 1936.

[14] Markus Linckelmann. *Introduction to Fusion Systems*. Univ., 2007.

[15] Lance Edward Miller. A witt-burnside ring attached to a pro-dihedral group, 2011.

[16] Sejong Park. Realizing a fusion system by a single finite group. *Archiv der Mathematik*, 94(5):405–410, April 2010.

[17] Lluís Puig. *Frobenius Categories versus Brauer Blocks.* Birkhäuser Basel, 2009.

[18] Sune Reeh. The abelian monoid of fusion-stable finite sets is free. *Algebra & Number Theory*, 9(10):2303–2324, December 2015.

[19] Gian Carlo Rota. On the foundations of combinatorial theory i. theory of möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2(4):340–368, 1964.

[20] Louis Solomon. The burnside algebra of a finite group. *Journal of Combinatorial Theory*, 2(4):603–615, 1967.

[21] Radu Stancu and Peter Symonds. Fusion systems for profinite groups. *Journal of the London Mathematical Society*, 89(2):461–481, 01 2014.

[22] Michio Suzuki. *Group theory I.* Springer-Verlag, Berlin; New York, 1982.

[23] Tammo tom Dieck. Idempotent elements in the burnside ring. *Journal of Pure and Applied Algebra*, 10(3):239–247, 1977.

[24] Tammo tom Dieck. *Transformation Groups.* De Gruyter, Berlin, New York, 1987.

[25] Gareth Wilkes. *Part III Profinite Groups.* Univ., 2020.

[26] John S. Wilson. *Profinite groups / John S. Wilson.* London Mathematical Society monographs. new series ; 19. Clarendon Press ; Oxford University Press, Oxford : New York, 1998.

[27] Tomoyuki Yoshida. Idempotents of burnside rings and dress induction theorem. *Journal of Algebra*, 80(1):90–105, 1983.