Probabilistic Approach to Risk Processes with Level-Dependent Premium Rate

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Abstract

We study risk processes with level dependent premium rate. Assuming that the premium rate converges, as the risk reserve increases, to the critical value in the net-profit condition, we obtain upper and lower bounds for the ruin probability; our proving technique is purely probabilistic and based on the analysis of Markov chains with asymptotically zero drift.

We show that such risk processes give rise to heavy-tailed ruin probabilities whatever the distribution of the claim size, even if it is a bounded random variable. So, the risk processes with near critical premium rate provide an important example of a stochastic model where light-tailed input produces heavy-tailed output.

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1 Introduction

In context of the collective theory of risk, the classical *Cramér–Lundberg (Sparre Andersen) model* is defined as follows. An insurance company receives the constant inflow of premium at rate v, that is, the premium income is assumed to be linear in time with rate v. It is also assumed that the claims incurred by the insurance company arrive according to a homogeneous renewal process $N(t)$ with intensity λ and the sizes (amounts) $\zeta_n \ge 0$ of the claims are independent copies of a random variable ζ with finite mean. The ζ_n's are assumed independent of the process $N(t)$. The company has an initial risk reserve $x = R(0) \ge 0$. Then the risk reserve $R(t)$

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at time t is equal to

$$R(t) \quad = \quad x + vt - \sum_{i=1}^{N(t)} \zeta_i.$$

The probability

$$\begin{aligned} \psi(x) \quad &:= \quad \mathbb{P}\{R(t) < 0 \text{ for some } t \geq 0\} \\ &= \quad \mathbb{P}\Big\{\min_{t\geq 0} R(t) < 0\Big\} \end{aligned}$$

is the probability of ruin. We have

$$\psi(x) \quad = \quad \mathbb{P}\Big\{\sum_{i=1}^{N(t)} \zeta_i - vt > x \text{ for some } t \geq 0\Big\}.$$

Since $v > 0$, the ruin can only occur at a claim epoch. Therefore,

$$\psi(x) \quad = \quad \mathbb{P}\Big\{\sum_{i=1}^{n} \zeta_i - vT_n > x \text{ for some } n \geq 1\Big\},$$

where T_n is the nth claim epoch, so that $T_n = \tau_1 + \ldots + \tau_n$ where the τ_k's are independent copies of a random variable τ with finite mean $1/\lambda$, so that $N(t) := \max\{n \geq 1 : T_n \leq t\}$. Denote $X_i := \zeta_i - v\tau_i$ and $S_n := X_1 + \ldots + X_n$, then

$$\psi(x) \quad = \quad \mathbb{P}\Big\{\sup_{n\geq 1} S_n > x\Big\}.$$

This relation represents the ruin probability problem as the tail probability problem for the maximum of the associated random walk $\{S_n\}$. Let the *net-profit condition*

$$v \quad > \quad v_c \;:=\; \mathbb{E}\zeta/\mathbb{E}\tau \;=\; \lambda\mathbb{E}\zeta \tag{1}$$

hold, thus $\{S_n\}$ has a negative drift: $\mathbb{E}S_1 = \mathbb{E}\zeta_1 - v\mathbb{E}\tau < 0$. Hence by the strong law of large numbers $S_n \to -\infty$ a.s., so $\psi(x) \downarrow 0$ as $x \to \infty$.

If $v \leq v_c$ then $\psi(x) = 1$ for all x.

The most classical case is when the distribution of X_1 satisfies the following well-known Cramér condition: there exists a $\beta > 0$ such that

$$\mathbb{E}e^{\beta X_1} \quad = \quad 1. \tag{2}$$

Under this condition, the sequence $e^{\beta S_n}$ is a martingale and, by the Doob maximal inequality, the following Lundberg's inequality holds true

$$\psi(x) = \mathbb{P}\left\{\sup_{n\geq 1} e^{\beta S_n} > e^{\beta x}\right\} \leq e^{-\beta x}, \quad x > 0. \tag{3}$$

If we additionally assume that $\mathbb{E}X_1 e^{\beta X_1} < \infty$ and the distribution of X_1 is non-lattice, then the Cramér–Lundberg approximation holds, that is, there exists a constant $c_0 \in (0,1)$ such that

$$\psi(x) \sim c_0 e^{-\beta x} \quad \text{as } x \to \infty, \tag{4}$$

see e.g. Theorem VI.3.2 in Asmussen and Albrecher [1]; in the lattice case x must be taken as a multiple of the lattice step. The most important feature of these results is the fact that the upper bound (3) depends on the distribution of X_1 only via the parameter β. If the moment condition (2) on the distribution of X_1 does not hold then the tail asymptotics for $\psi(x)$ are typically determined by the tail of the claim size ζ. The most prominent situation is when the distribution of ζ is of subexponential type, see e.g. Embrechts and Veraverbeke [9]. We discuss this case in more detail later.

The risk models with non-constant premium rates have also become rather popular in the collective risk literature. There are two main approaches, one of them leads to a Markovian model when the premium rate is a function of the current level of the risk reserve $R(t)$, see e.g. Asmussen and Albrecher [1, Chapter VIII], Albrecher et al. [2], Boxma and Mandjes [5], Czarna et al. [6], Marciniak and Palmowski [16]; in the context of queueing theory when both the service and arrival rates depend on the current workload see e.g. Bekker et al. [3]. The second approach considers the premium rate that depends on the whole claims history, see e.g. Li et al. [15].

In this paper we follow the first approach and consider a risk process where the premium rate $v(y)$ only depends on the current level of risk reserve $R(t) = y$, so $R(t)$ satisfies the equality

$$R(t) \quad = \quad x + \int_0^t v(R(s))ds - \sum_{j=1}^{N(t)} \zeta_j; \tag{5}$$

hereinafter $v(y)$ is assumed to be a bounded càdlàg function bounded away from zero on each interval; as v is a càdlàd function, there are countably many at the most discontinuity points of v, which together with boundedness of $1/v$ on any interval implies by the Lebesgue–Vitali theorem that the function $1/v$ is Riemann integrable. The probability of ruin given initial risk reserve x is again denoted by $\psi(x)$, it is a decreasing function of x as it is in the classical case.

The ruin probability for such processes with level dependent premium rate is much less studied in the literature than with constant premium rate, and all known results are exact expressions for some particular distributions of τ, ζ and/or for particular choices of the rate function $v(y)$. The first example of the risk process where $\psi(x)$ is explicitly calculable is the case of exponentially distributed τ and ζ, say with parameters λ and μ respectively, so hence $v_c = \lambda/\mu$. In this case, for some $c_0 \in (0,1)$,

$$\begin{aligned} \psi(x) \quad &= \quad c_0 \int_x^\infty \frac{1}{v(y)} \exp\Big\{-\mu y + \lambda \int_0^y \frac{dz}{v(z)}\Big\} dy \\ &= \quad c_0 \int_x^\infty \frac{1}{v(y)} \exp\Big\{\lambda \int_0^y \Big(\frac{1}{v(z)} - \frac{1}{v_c}\Big) dz\Big\} dy, \end{aligned} \tag{6}$$

provided the outer integral is convergent from 0 to infinity, see, e.g. Corollary 1.9 in Albrecher and Asmussen [1, Ch. VIII]. Some further examples of solutions in closed form can be found in Albrecher et al. [2]. The authors of that paper use a purely analytical approach, which works however only in the situations where the Laplace transforms of ζ and τ are rational functions.

The main goal of our paper is to develop a probabilistic method of the asymptotic analysis of risk processes with level-dependent premium rate, which is not based on exact calculations and uses only moment and tail conditions on ζ and τ. The following two qualitatively different cases can be identified:

$$v(y) \quad \to \quad v_\infty \;>\; v_c \quad \text{as } y \to \infty; \tag{7}$$

$$v(y) \quad \to \quad v_c \quad \text{as } y \to \infty. \tag{8}$$

In the first case (7) one could expect that the ruin probability $\psi(x)$ decays similarly to the classical collective risk model with constant premium rate v_∞. In this paper we concentrate on a more difficult critical case (8) where the ruin is more likely due to the approaching the critical premium rate. Notice that it is quite different from the heavy traffic modelling where we consider dependence of a small parameter and study phenomena arising when we send it to zero. This approach goes back to Kingman [11] where the convergence to exponential limit was studied; see also Borovkov et al. [4] where in the framework of Markov chains the convergence to other laws has been discussed, say to a Γ-distribution.

We start by discretising the time; this procedure is standard for risk processes with constant rate. Since the ruin can only occur at a claim epoch, the ruin probability may be reduced to that for the embedded Markov chain $R_n := R(T_n)$, $n \ge 1$, $R_0 := x$, that is,

$$\psi(x) \quad = \quad \mathbb{P}\{R_n < 0 \text{ for some } n \ge 0\}.$$

So, our main goal is to analyse the down-crossing probabilities for the chain $\{R_n\}$. In contrast to the constant premium case, we deal with a Markov chain instead of a random walk with independent increments.

As mentioned above, we shall restrict our attention to the case (8) where $v(y)$ approaches the critical value v_c at infinity. Then the Markov chain $\{R_n\}$ has asymptotically zero drift, that is,

$$\mathbb{E}\{R_1 - R_0 \mid R_0 = x\} \quad \to \quad 0 \quad \text{as } x \to \infty, \tag{9}$$

see Theorem 1 below. The study of Markov chains with vanishing drift was initiated by Lamperti in a series of papers [12, 13, 14]. For further development in Lamperti's problem, see Menshikov et al. [17, 18]. We also show in Theorem 1 that under (8) the ruin probability decays slower than any exponential function, that is, for any $\lambda > 0$,

$$e^{\lambda x}\psi(x) \quad \to \quad \infty \quad \text{as } x \to \infty. \tag{10}$$

Our motivation to consider the critical case (8) is twofold. Firstly, this setting can model the following situations.

(i) Decreasing the premium rate makes an insurance company more attractive to new customers. Thus one has to analyse the impact of decreasing the premium rate on ruin probabilities.

(ii) One can also imagine the following strategy of an insurance company. The premium rate stays constant, but the company redirects (or invests) a portion of the risk reserve to other activities of the company. In that case $v(y)$ describes the portion of the remaining risk reserve.

Secondly, as well-known, in the classical Cramér–Lundberg model under the net-profit condition (1), the ruin probability decays slower than any exponential function if and only if the claim size tail distribution is so, see e.g. Embrechts and Veraverbeke [9]. As just mentioned, the risk processes under the critical premium rate (8) give rise to heavy-tailed ruin probabilities whatever the distribution of the claim size, even if it is a bounded random variable. So, the risk processes with near critical premium rate provide an important example of a stochastic model where light-tailed input produces heavy-tailed output. To the best of our knowledge, this effect was not observed before.

We want to investigate how the rate of convergence in (8) is reflected in how slowly the ruin probability $\psi(x)$ decreases. In Section 2 we get some intuition on what kind of phenomena we could expect here by considering the case of exponentially distributed ζ and τ. In Section 3 we show that the convergence $v(y) \to v_c$ implies that the ruin probability $\psi(x)$ decays slower than any exponential function, see (10). Section 4 is devoted to sufficient conditions on the rate function $v(y)$ which ensure that the embedded Markov chain R_n is transient and hence the ruin probability is strictly less than one. Notice that the ruin probability as a function of the initial risk reserve is a harmonic function for the chain R_n. Our approach to the study of asymptotic behaviour of the ruin probability $\psi(x)$ is based on constructing nearly harmonic functions for R_n. This approach allows us to investigate the whole spectrum of the rate of convergence of $v(y)$ to v_c. Specifically, in Section 5 we study the case where the premium rate approaches the critical value at rate of θ/x as $x \to \infty$ and show polynomial decay of the ruin probability,

$$\psi(x) \quad \asymp \quad \frac{1}{x^\rho} \quad \text{where } \rho = \frac{2\theta \mathbb{E}\tau}{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau} - 1.$$

Then in Section 5 we discuss the case of slower approach to the critical value, namely θ/x^α as $x \to \infty$, for some $\theta > 0$ and $0 < \alpha < 1$, and show Weibullian decay of the ruin probability which particularly implies the following logarithmic asymptotics

$$\log \psi(x) \quad \sim \quad -\frac{r_1}{1-\alpha} x^{1-\alpha} \quad \text{where } r_1 = \frac{2\theta \mathbb{E}\tau}{\mathbb{E}(v_c\tau - \zeta)^2}.$$

Finally, in Section 7 we study the case where the distribution of the claim size is so heavy that the moment conditions in the theorems proved in Section 5 are not met.

2 Intuition based on exponential case

Let us get some intuition on what kind of phenomena we could expect if $v(y) \to v_c$ by considering the case of exponentially distributed ζ and τ. As we have mentioned above, the ruin probability $\psi(x)$ is given in this case by (6). Combining (6) and (8), we obtain

$$\psi(x) \quad \sim \quad \frac{c_0}{v_c} \int_x^\infty \exp\Big\{\lambda \int_0^y \Big(\frac{1}{v(z)} - \frac{1}{v_c}\Big) dz\Big\} dy \quad \text{as } x \to \infty.$$

If the premium rate $v(z) \geq v_c$ approaches v_c at the rate of θ/z, $\theta > 0$, more precisely, if

$$\Big| v(z) - v_c - \frac{\theta}{z} \Big| \quad \leq \quad p(z) \quad \text{for all } z > 1, \tag{11}$$

where $p(z) > 0$ is an integrable at infinity decreasing function, then we get

$$\frac{1}{v(z)} \quad = \quad \frac{1}{v_c} - \frac{\theta}{v_c^2 z} + O(p(z) + 1/z^2)$$

and consequently

$$\lambda \int_0^y \Big(\frac{1}{v(z)} - \frac{1}{v_c}\Big) dz \quad = \quad -\frac{\theta\mu^2}{\lambda} \log y + c_1 + o(1) \quad \text{as } y \to \infty,$$

where c_1 is a finite real. Let $\theta > \lambda/\mu^2$. Then, for $C := c_0 e^{c_1}/(\theta\mu - \lambda/\mu) > 0$,

$$\psi(x) \quad \sim \quad \frac{C}{x^{\theta\mu^2/\lambda - 1}} \quad \text{as } x \to \infty. \tag{12}$$

A similar asymptotic equivalence can be obtained also in the case where the Laplace transforms of variables ζ_1 and τ_1 are rational functions, see Albrecher et al. [2].

If the premium rate $v(z)$ approaches v_c at a slower rate of θ/z^α, $\theta > 0$ and $\alpha \in (0, 1)$, more precisely, if

$$\Big| v(z) - v_c - \frac{\theta}{z^\alpha} \Big| \quad \leq \quad p(z) \quad \text{for all } z > 1, \tag{13}$$

where $p(z) > 0$ is an integrable at infinity decreasing function, then we get

$$\frac{1}{v(z)} \quad = \quad \frac{1}{v_c} \sum_{j=0}^\infty \Big(-\frac{\theta}{v_c}\Big)^j \frac{1}{z^{\alpha j}} + O(p(z)).$$

Let $\gamma := \min\{k \in \mathbb{N} : k\alpha > 1\}$. Then

$$\frac{1}{v(z)} \quad = \quad \frac{1}{v_c} \sum_{j=0}^{\gamma-1} \Big(-\frac{\theta}{v_c}\Big)^j \frac{1}{z^{\alpha j}} + O(p_1(z)),$$

where $p_1(z) = p(z) + z^{-\gamma\alpha}$ is integrable at infinity. Consequently, if $1/\alpha$ is not integer, then

$$\begin{aligned}\lambda\int_0^y\Big(\frac{1}{v(z)}-\frac{1}{v_c}\Big)dz &= \frac{\lambda}{v_c}\int_1^y\sum_{j=1}^{\gamma-1}\Big(-\frac{\theta}{v_c}\Big)^j\frac{1}{z^{\alpha j}}dz + c_2 + o(1)\\ &= \frac{\lambda}{v_c}\sum_{j=1}^{\gamma-1}\Big(-\frac{\theta}{v_c}\Big)^j\frac{y^{1-\alpha j}}{1-\alpha j} + c_3 + o(1)\quad \text{as } y\to\infty,\end{aligned}$$

where c_3 is a finite real because $p_1(x)$ is integrable. In the case of integer $1/\alpha$ one has

$$\begin{aligned}\lambda\int_0^y\Big(\frac{1}{v(z)}-\frac{1}{v_c}\Big)dz &= \frac{\lambda}{v_c}\sum_{j=1}^{\gamma-2}\Big(-\frac{\theta}{v_c}\Big)^j\frac{y^{1-\alpha j}}{1-\alpha j}\\ &\qquad +\frac{\lambda}{v_c}\Big(-\frac{\theta}{v_c}\Big)^{\gamma-1}\log y + c_4 + o(1)\quad \text{as } y\to\infty.\end{aligned}$$

Let, for example, $\alpha\in(1/2,1)$. Then

$$\lambda\int_0^y\Big(\frac{1}{v(z)}-\frac{1}{v_c}\Big)dz = -\frac{\theta\mu^2}{\lambda(1-\alpha)}y^{1-\alpha} + c_3 + o(1)\quad \text{as } y\to\infty.$$

Therefore, for $C_1 := c_0e^{c_3}/\theta\mu > 0$ and $C_2 := \theta\mu^2/\lambda(1-\alpha) > 0$,

$$\psi(x) \sim C_1x^\alpha e^{-C_2x^{1-\alpha}}\quad \text{as } x\to\infty. \tag{14}$$

We are going to extend these results to not necessarily exponential distributions where there are no closed form expressions like (6) available for $\psi(x)$. In that case we can only derive lower and upper bounds for $\psi(x)$ which have the same decay rate at infinity.

3 Heavy-tailedness of the ruin probability in the critical case

Denote the jumps of the chain $\{R_n = R(T_n)\}$ by $\xi(x)$, that is,

$$\mathbb{P}\{\xi(x)\in B\} = \mathbb{P}\{R_1 - R_0 \in B \mid R_0 = x\}$$

for all Borel sets B. The dynamics of the risk reserve between two consequent claims is governed by the differential equation $R'(t) = v(R(t))$ where by R' we mean the right derivative of V. This equation is solvable in R because the function $1/v$ is Riemann integrable, due to the boundedness of $1/v$ and its right continuity. Let $V_x(t)$ denote its solution with initial value x, so then

$$V_x(t) = x + \int_0^t v(V_x(s))ds.$$

Therefore,

$$\xi(x) \quad =_{\rm st} \quad V_x(\tau) - x - \zeta \;\; = \;\; \int_0^\tau v(V_x(s))ds - \zeta,$$

where $=_{\rm st}$ stands for the equality in distribution.

To avoid trivial case where $\psi(x) = 0$ for all sufficiently large x, we assume that

$$\psi(x) \quad > \quad 0 \quad \text{for all } x. \tag{15}$$

A sufficient condition for that is that, for all $x_0 > 0$ there exists an $\varepsilon = \varepsilon(x_0) > 0$ such that

$$\mathbb{P}\{\xi(x) \le -\varepsilon\} \quad > \quad \varepsilon \quad \text{for all } x \in [0, x_0].$$

In its turn, for that it suffices to assume that the random variable ζ is unbounded, due to the inequality $\xi(y) \le \overline{v}\tau - \zeta$ valid for all y, where $\overline{v} := \sup_{z>0} v(z)$.

Theorem 1. *Let $v_c = \mathbb{E}\zeta/\mathbb{E}\tau$ and let $v(x) \to v_c$ as $x \to \infty$. Then the chain $\{R_n\}$ has asymptotically zero drift, that is,* (9) *holds true.*

If, in addition, (15) *holds true, then, for all $\lambda > 0$, $e^{\lambda x}\psi(x) \to \infty$ as $x \to \infty$.*

Proof. Since $v(y) \to v_c$, for all $t > 0$,

$$\int_0^t v(V_x(s))ds \quad \to \quad v_c t \quad \text{as } x \to \infty.$$

This implies the following convergence in distribution:

$$\xi(x) \quad \Rightarrow \quad v_c\tau - \zeta \quad \text{as } x \to \infty,$$

which implies the first statement by the dominated convergence theorem, due to the upper bound $|\xi(x)| \le_{st} \zeta + \overline{v}\tau$. It also implies that, for all $\lambda > 0$,

$$(e^{-\lambda\xi(x)} - 1)\mathbb{I}\{\xi(x) > -x\} \quad \Rightarrow \quad e^{\lambda(\zeta - v_c\tau)} - 1 \quad \text{as } x \to \infty.$$

Hence, as follows from Fatou's Lemma,

$$\begin{aligned}
\liminf_{x\to\infty} \mathbb{E}(e^{-\lambda\xi(x)} - 1)\mathbb{I}\{\xi(x) > -x\} \quad &\ge \quad \mathbb{E}e^{\lambda(\zeta - v_c\tau)} - 1 \\
&> \quad e^{\lambda\mathbb{E}(\zeta - v_c\tau)} - 1.
\end{aligned}$$

Recalling that $v_c = \mathbb{E}\zeta/\mathbb{E}\tau$, we get $\mathbb{E}(\zeta - v_c\tau) = 0$. Therefore, for all $\lambda > 0$ there exists an $\varepsilon = \varepsilon(\lambda) > 0$ such that

$$\mathbb{E}(e^{-\lambda\xi(x)} - 1)\mathbb{I}\{\xi(x) > -x\} \quad \ge \quad \varepsilon \quad \text{for all sufficiently large } x. \tag{16}$$

Let $\lambda > 0$. Consider a bounded decreasing function $U_\lambda(x) := \min(e^{-\lambda x}, 1)$. For all $x > 0$,

$$\begin{aligned}
\mathbb{E}(U_\lambda(x + \xi(x)) - U_\lambda(x)) \quad &\ge \quad \mathbb{E}\{e^{-\lambda(x+\xi(x))} - e^{-\lambda x};\; x + \xi(x) > 0\} \\
&= \quad e^{-\lambda x}\mathbb{E}\{e^{-\lambda\xi(x)} - 1;\; \xi(x) > -x\}.
\end{aligned}$$

Due to (16), there exists a sufficiently large $x_\lambda > 0$ such that

$$\mathbb{E}(U_\lambda(x+\xi(x)) - U_\lambda(x)) \quad \geq \quad 0 \quad \text{for all } x > x_\lambda.$$

Therefore, the process $\{U_\lambda(R_{n\wedge\tau_{B_\lambda}})\}$ is a bounded submartingale, where $B_\lambda := (-\infty, x_\lambda]$ and $\tau_B := \min\{n : R_n \in B\}$. Hence by the optional stopping theorem, for $z > x_\lambda$ and $x \in (x_\lambda, z)$,

$$\mathbb{E}_x U_\lambda(R_{\tau_{B_\lambda}\wedge\tau_{(z,\infty)}}) \;\geq\; \mathbb{E}_x U_\lambda(X_0) \;=\; U_\lambda(x).$$

Letting $z \to \infty$ we conclude that

$$\begin{aligned}
\mathbb{E}_x\{U_\lambda(R_{\tau_{B_\lambda}});\ \tau_{B_\lambda} < \infty\} &= \lim_{z\to\infty} \mathbb{E}_x\{U_\lambda(R_{\tau_{B_\lambda}});\ \tau_{B_\lambda} < \tau_{(z,\infty)}\} \\
&= \lim_{z\to\infty} \mathbb{E}_x U_\lambda(R_{\tau_{B_\lambda}\wedge\tau_{(z,\infty)}}) - \lim_{z\to\infty} \mathbb{E}_x\{U_\lambda(R_{\tau_{(z,\infty)}});\ \tau_{B_\lambda} > \tau_{(z,\infty)}\} \\
&\geq U_\lambda(x) - 0 \;=\; U_\lambda(x).
\end{aligned}$$

On the other hand, since U_λ is bounded by 1,

$$\mathbb{E}_x\{U_\lambda(R_{\tau_{B_\lambda}});\ \tau_{B_\lambda} < \infty\} \quad \leq \quad \mathbb{P}_x\{\tau_{B_\lambda} < \infty\}.$$

This allows us to deduce the lower bound

$$\mathbb{P}_x\{\tau_{B_\lambda} < \infty\} \quad \geq \quad U_\lambda(x) \;=\; e^{-\lambda x} \quad \text{for all } x > x_\lambda.$$

Hence the conclusion (ii) follows, because by the strong Markov property, for all $\lambda > 0$ and $x > 0$,

$$\begin{aligned}
\psi(x) \;=\; \mathbb{P}_x\{\tau_{(-\infty,0)} < \infty\} &\geq \mathbb{P}_x\{\tau_{B_\lambda} < \infty\} \inf_{y\in[0,x_\lambda]} \mathbb{P}_y\{\tau_{(-\infty,0)} < \infty\} \\
&\geq \mathbb{P}_x\{\tau_{B_\lambda} < \infty\}\psi(x_\lambda),
\end{aligned} \tag{17}$$

since $\psi(x)$ is decreasing; here $\psi(x_\lambda) > 0$ owing to the condition (15). $\square$

4 Transience of the embedded Markov chain

In this section we find conditions on the rate function $v(z)$ which ensure that the ruin probability is strictly less than one.

Theorem 2. *Let, for some $\theta > 0$,*

$$v(z) \quad \geq \quad v_c + \theta/z \quad \textit{for all sufficiently large } z. \tag{18}$$

Let both $\mathbb{E}\tau_1^2$ and $\mathbb{E}\zeta_1^2$ be finite. If

$$\theta \quad > \quad \frac{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau}{2\mathbb{E}\tau}, \tag{19}$$

then the underlying Markov chain $\{R_n = R(T_n)\}$ is transient in the sense that $\psi(x) < 1$ for all sufficiently large x.

If, in addition, the chain $\{R_n\}$ is transient in the common sense that $\psi(x) < 1$ for all $x > 0$ and

$$v(z) - v_c \quad \sim \quad \theta/z \quad \textit{as } z \to \infty, \tag{20}$$

then R_n^2/n weakly converges to a Γ-distribution with mean $2\mu + b$ and variance $(2\mu + b)2b$ where $\mu := \theta\mathbb{E}\tau$ and $b := \mathbb{V}\mathrm{ar}\,\zeta + v_c^2\mathbb{V}\mathrm{ar}\,\tau$.

As we see from the convergence to a Γ-distribution, in the case (19) the chain R_n escapes to infinity in probability at rate $\sqrt{n}$ in quite specific way as there is no law of large numbers. In the case where $v(z) - v_c \sim c/z^\alpha$ with $\alpha \in (0, 1)$, the chain R_n is transient too, however as follows from Lamperti [13, Theorem 7.1], it follows a law of large numbers, $R_n^{1+\alpha}/n \to c(1 + \alpha)$ as $n \to \infty$.

Below we prove Theorem 2 via Lyapunov (test) functions approach, so we start with moment computations for the jumps of $\{R_n\}$. Denote by $m_k(x)$ the kth moment of the jump $\xi(x)$ of the chain $\{R_n\}$ from state x, that is, $m_k(x) = \mathbb{E}\xi^k(x)$.

Lemma 3. *If both $\mathbb{E}\tau^2$ and $\mathbb{E}\zeta^2$ are finite, then, under the rate of convergence* (11), *as $x \to \infty$,*

$$\begin{aligned} m_1(x) &= \frac{\theta\mathbb{E}\tau}{x} + O(p(x) + 1/x^2), &(21)\\ m_2(x) &= \mathbb{V}\mathrm{ar}\,\zeta + v_c^2\mathbb{V}\mathrm{ar}\,\tau + O(1/x). &(22) \end{aligned}$$

If $\mathbb{E}\zeta^{\gamma_0+2} < \infty$ for some $\gamma_0 \geq 0$ then, for all $\delta > 0$,

$$\mathbb{P}\{\xi(x) < -\delta x\} \quad = \quad o(p_1(x)/x^{\gamma_0+1}) \quad \textit{as } x \to \infty, \tag{23}$$

for some decreasing integrable at infinity function $p_1(x)$.

If $\mathbb{E}\tau^2\log(1 + \tau) < \infty$ and $\mathbb{E}\zeta^2\log(1 + \zeta) < \infty$, then, as $x \to \infty$,

$$\begin{aligned} \mathbb{E}\big\{|\xi(x)|^3;\ |\xi(x)| \leq \delta x\big\} &= o(x^2 p_1(x)), &(24)\\ \mathbb{E}\{\xi^2(x);\ |\xi(x)| > \delta x\} &= o(xp_1(x)). &(25) \end{aligned}$$

Proof. By (11),

$$\begin{aligned} v(y) &\leq v_c + \theta/y + p(y)\\ &\leq v_c + \theta/x + p(x) \quad \text{for all } y \geq x, \end{aligned}$$

therefore

$$\begin{aligned} V_x(t) - x &= \int_0^t v(V_x(s))ds\\ &\leq v_c t + \theta t/x + p(x)t, \quad t > 0. &(26) \end{aligned}$$

On the other hand, again by (11),

$$\begin{aligned} v(y) &\geq v_c + \theta/y - p(y)\\ &\geq v_c + \theta/y - p(x) \quad \text{for all } y \geq x. \end{aligned}$$

Hence,

$$\begin{aligned} V_x(t) - x &\geq v_c t + \theta \int_0^t \frac{ds}{V_x(s)} - p(x)t \\ &\geq v_c t + \theta \int_0^t \frac{ds}{x + (v_c + \theta/x + p(x))s} - p(x)t \\ &= v_c t + \frac{\theta}{v_c + \theta/x + p(x)} \log\bigl(1 + (v_c + \theta/x + p(x))t/x\bigr) - p(x)t, \end{aligned}$$

where the second inequality follows from the upper bound (26). Therefore,

$$V_x(t) - x \geq v_c t + \frac{\theta}{v_c + \theta/x + p(x)} \log\bigl(1 + v_c t/x\bigr) - p(x)t. \tag{27}$$

Since $\xi(x) = V_x(\tau) - x - \zeta$, it follows from (26) and (27) that

$$\begin{aligned} &v_c\tau - \zeta + \frac{\theta}{v_c + \theta/x + p(x)} \log\Bigl(1 + \frac{v_c\tau}{x}\Bigr) - p(x)\tau \\ &\qquad\qquad \leq \xi(x) \leq v_c\tau - \zeta + \frac{\theta\tau}{x} + p(x)\tau. \end{aligned} \tag{28}$$

Recalling that $v_c = \mathbb{E}\zeta/\mathbb{E}\tau$, we get

$$\frac{\theta}{v_c + \theta/x + p(x)} \mathbb{E} \log\Bigl(1 + \frac{v_c\tau}{x}\Bigr) - p(x)\mathbb{E}\tau \leq m_1(x) \leq \frac{\theta}{x}\mathbb{E}\tau + p(x)\mathbb{E}\tau.$$

By the inequality $\log(1+z) \geq z - z^2/2$ for $z \geq 0$,

$$\mathbb{E} \log\Bigl(1 + \frac{v_c\tau}{x}\Bigr) \geq \frac{v_c\mathbb{E}\tau}{x} - \frac{v_c^2\mathbb{E}\tau^2}{2x^2}.$$

Therefore, the relation (21) follows. From that expression we have

$$\begin{aligned} m_2(x) &= \mathbb{V}\mathrm{ar}\,\xi(x) + m_1^2(x) \\ &= \mathbb{V}\mathrm{ar}\,(V_x(\tau) - x - \zeta) + O(p^2(x) + 1/x^2) \\ &= \mathbb{V}\mathrm{ar}\,(V_x(\tau) - x) + \mathbb{V}\mathrm{ar}\,\zeta + O(p^2(x) + 1/x^2) \quad \text{as } x \to \infty. \end{aligned}$$

Recalling that

$$v_c t - p(x)t \leq V_x(t) - x \leq v_c t + \frac{\theta}{x} t + p(x)t,$$

we get

$$(v_c - p(x))\mathbb{E}\tau \leq \mathbb{E}(V_x(\tau) - x) \leq (v_c + \theta/x + p(x))\mathbb{E}\tau$$

and

$$(v_c - p(x))^2\mathbb{E}\tau^2 \leq \mathbb{E}(V_x(\tau) - x)^2 \leq (v_c + \theta/x + p(x))^2\mathbb{E}\tau^2.$$

Hence,

$$\mathbb{V}\mathrm{ar}\,(V_x(\tau)-x) \;=\; v_c^2\mathbb{V}\mathrm{ar}\,\tau+O(1/x) \quad \text{as } x\to\infty,$$

which in its turn implies (22).

Next, since $V_x(\tau)-x\ge 0$ and $\zeta\ge 0$, we have

$$\begin{aligned}\xi^2(x)\mathbb{I}\{\xi(x)>\delta x\} &= (V_x(\tau)-x-\zeta)^2\mathbb{I}\{V_x(\tau)-x-\zeta>\delta x\}\\ &\le (V_x(\tau)-x)^2\mathbb{I}\{V_x(\tau)-x>\delta x\}\\ &\le \overline{v}^2\tau^2\mathbb{I}\{\tau>\delta x/\overline{v}\},\end{aligned} \tag{29}$$

where $\overline{v}=\sup_z v(z)$, owing to the inequality $V_x(t)-x\le\overline{v}t$, which follows from (26). Similarly,

$$\begin{aligned}\xi^2(x)\mathbb{I}\{\xi(x)<-\delta x\} &= (V_x(\tau)-x-\zeta)^2\mathbb{I}\{V_x(\tau)-x-\zeta<-\delta x\}\\ &\le \zeta^2\mathbb{I}\{\zeta>\delta x\}.\end{aligned} \tag{30}$$

Then it follows from the finiteness of $\mathbb{E}\zeta^2\log(1+\zeta)$ and $\mathbb{E}\tau^2\log(1+\tau)$ that both tail expectations $\mathbb{E}\{\tau^2;\ \tau>\delta x/\overline{v}\}$ and $\mathbb{E}\{\zeta^2;\ \zeta>\delta x\}$ are of order $o(xp_1(x))$ for some decreasing integrable at infinity function $p_1(x)$, see Lemma 23. Hence the upper bound (25).

Further, the upper bound (24) follows from Lemma 18 with $\gamma=2$ and $\alpha=1$.

Finally,

$$\mathbb{P}\{\xi(x)<-\delta x\} \;\le\; \mathbb{P}\{\zeta>\delta x\} \;=\; o(p_1(x)/x^{\gamma_0+1}),$$

for some decreasing integrable at infinity function $p_1(x)$, due to Lemma 22 with $\gamma=\gamma_0+2$, $\beta=0$, and $\alpha=1$. Hence the upper bound (23). $\square$

Proof of Theorem 2. Let us consider the function $v_\theta(z):=\min(v(z),v_c+\theta/z)$. The dynamics of the risk reserve between two consequent claims with premium rate $v_\theta(z)$ is governed by the differential equation $R'(t)=v_\theta(R(t))$. Let $V_{\theta,x}(t)$ denote its solution with the initial value x, so then

$$V_{\theta,x}(t) \;=\; x+\int_0^t v_\theta(V_{\theta,x}(s))ds.$$

Since $v_\theta(z)\le v(z)$,

$$V_x(t) \;\ge\; V_{\theta,x}(t) \quad \text{for all } t>0. \tag{31}$$

For $\xi_\theta(x):=V_{\theta,x}(\tau)-x-\zeta$, denote $m_{\theta,k}(x):=\mathbb{E}\xi_\theta^k(x)$. Since $v_\theta(z)=v_c+\theta/z$ for all sufficiently large z, Lemma 3 applies. As a result we have

$$m_{\theta,1}(x) \;=\; \frac{\theta\mathbb{E}\tau}{x}+O(1/x^2) \quad \text{as } x\to\infty,$$

and

$$m_{\theta,2}(x) \;=\; \mathbb{V}\mathrm{ar}\,\zeta+v_c^2\mathbb{V}\mathrm{ar}\,\tau+O(1/x) \quad \text{as } x\to\infty.$$

Therefore,

$$\frac{2m_{\theta,1}(x)}{m_{\theta,2}(x)} \quad = \quad \frac{2\theta\mathbb{E}\tau}{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2\mathbb{V}\mathrm{ar}\,\tau} \cdot \frac{1}{x} + O(1/x^2) \quad \text{as } x \to \infty.$$

By the condition on θ, there exists an $\varepsilon > 0$ such that

$$\frac{2m_{\theta,1}(x)}{m_{\theta,2}(x)} \quad \geq \quad \frac{1+\varepsilon}{x} \quad \text{for all sufficiently large } x. \tag{32}$$

Further, again by Lemma 3 with $\gamma_0 = 0$, for any fixed $\delta > 0$,

$$\mathbb{P}\{\xi_\theta(x) \leq -\delta x\} \quad = \quad O(p(x)/x) \quad \text{as } x \to \infty, \tag{33}$$

for some decreasing integrable at infinity function $p(x)$, due to $\mathbb{E}\zeta^2 < \infty$.

The bounds (32) and (33) show that the conditions (11) and (13) from Theorem 3 in [8] hold true. In addition, there exists a sufficiently large x_0 such that the Markov chain $\{R_{\theta,n}\}$—the embedded Markov chain for the ruin process with premium rate $v_\theta(z)$—dominates above the level x_0 a similar Markov chain generated by a risk process with constant premium rate v_c. The latter represents a zero-drift random walk which is null-recurrent and hence satisfying the condition (12) from Theorem 3 in [8], thus

$$\mathbb{P}_x\Big\{\limsup_{n\to\infty} R_{\theta,n} = \infty\Big\} \;\to\; 1 \quad \text{as } x \to \infty.$$

Therefore, Theorem 3 from [8] applies and we conclude that

$$\mathbb{P}_x\{R_{\theta,n} \to \infty \text{ as } n \to \infty\} \;\to\; 1 \quad \text{as } x \to \infty,$$

which in its turn yields that

$$\mathbb{P}_x\{R_{\theta,n} > x_0 \text{ for all } n\} \;\to\; 1 \quad \text{as } x \to \infty.$$

Then the same property holds true for the original chain $\{R_n\}$, due to the domination property (31).

The convergence to a Γ-distribution follows from Theorem 4 in [8]. □

Remark 4. *It is worth mentioning that the condition* (19) *is close to be minimal one for* $\psi(x) < 1$. *More precisely, one can show that if*

$$v(z) \leq v_c + \theta/z \quad \text{for all sufficiently large } z$$

with some

$$\theta < \frac{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2\mathbb{V}\mathrm{ar}\,\tau}{2\mathbb{E}\tau}$$

then the chain $\{R_n\}$ *is recurrent or, equivalently,* $\psi(x) = 1$ *for all* $x > 0$.
This statement follows by similar arguments applied to a dominating Markov chain with premium rate $v_\theta(z) := \max(v(z), v_c + \theta/z)$ *that satisfies, for some* $\varepsilon > 0$,

$$2zm_{\theta,1}(z) \quad \leq \quad (1-\varepsilon)m_{\theta,2}(z) \quad \text{for all sufficiently large } z,$$

and hence the classical Lamperti criterion (see, e.g. Lamperti [12]) for recurrence of Markov chains applies.

5 Approaching critical premium rate at rate of θ/x

In this section we generalise the polynomial asymptotics (12) shown in Section 2 under exponential assumptions, to τ and ζ having general distributions. Then the result only depends on the first two moments of their distributions.

Theorem 5. *Assume* (15) *and the rate of convergence* (11) *with some* θ *satisfying* (19)*, that is,*

$$\theta \quad > \quad \frac{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau}{2\mathbb{E}\tau}.$$

Set

$$\rho \quad := \quad \frac{2\theta\mathbb{E}\tau}{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau} - 1.$$

If both $\mathbb{E}\tau^2 \log(1+\tau)$ *and* $\mathbb{E}\zeta^{\rho+2}$ *are finite then there exist positive constants* c_1 *and* c_2 *such that*

$$\frac{c_1}{(1+x)^\rho} \quad \le \quad \psi(x) \quad \le \quad \frac{c_2}{(1+x)^\rho} \quad \text{for all } x > 0.$$

These bounds are quite similar to the classical estimates (3) and (4). Indeed, they are universal and only depend on a single parameter ρ of the distribution of (ζ, τ). In contrast to the classical Cramer case, the crucial parameter ρ only depends on τ and ζ via the first two moments. A further advantage of the bounds in Theorem 5 is the fact that they are applicable to a wide class of claim size distributions: the only restriction is that the moment of order $\rho + 2$ should be finite; otherwise, the probability of ruin is higher, see Section 7.

By the condition on θ, $\rho > 0$. Define

$$q(x) := (\rho + 1)\min(1, 1/x)$$

and

$$Q(x) := \int_0^x q(y)dy \quad \to \quad \infty \quad \text{as } x \to \infty; \tag{34}$$

hereinafter we define $Q(x) = 0$ for $x < 0$. The increasing function $Q(x)$ is concave on the positive half line because $q(x)$ is decreasing. We have, for $c = \rho + 1$,

$$Q(x) \;=\; \int_0^x q(y)dy \;=\; (\rho+1)\log x + c \quad \text{for all } x \ge 1,$$

so the function $e^{-Q(x)}$ is integrable at infinity, due to $\rho > 0$. It allows us to define the following bounded decreasing function which plays the most important rôle in our analysis of the ruin probabilities:

$$U(x) \quad := \quad \int_x^\infty e^{-Q(y)}dy \quad \text{for } x \ge 0; \tag{35}$$

and $U(x) = U(0)$ for $x \le 0$. For all $x \ge 1$ we have

$$e^{-Q(x)} = e^{-c}/x^{\rho+1} \quad \text{and} \quad U(x) = e^{-c}/\rho x^{\rho}. \tag{36}$$

Let us also define the following auxiliary decreasing functions needed for our analysis. Without loss of generality we assume that $p_1(x) \le p(x) \le q(x)$ for all x, where $p_1(x)$ is given by Lemma 3; otherwise we can always consider the function $\max(p_1(x), p(x))$ instead of $p(x)$. Consider the functions $q_+(x) := q(x) + p(x)$ and $q_-(x) := q(x) - p(x)$ and let

$$\begin{aligned} Q_\pm(x) &:= \int_0^x q_\pm(y)dy, \\ U_\pm(x) &:= \int_x^\infty e^{-Q_\pm(y)}dy, \quad x \ge 0, \end{aligned} \tag{37}$$

and $U_\pm(x) = U_\pm(0)$ for $x \le 0$. We have $0 \le q_-(x) \le q(x) \le q_+(x)$, $0 \le Q_-(x) \le Q(x) \le Q_+(x)$ and $U_-(x) \ge U(x) \ge U_+(x) > 0$. Since

$$C_p := \int_0^\infty p(y)dy \quad \text{is finite,}$$

we have

$$Q_\pm(x) = Q(x) \pm C_p + o(1) \quad \text{as } x \to \infty. \tag{38}$$

Therefore,

$$U_\pm(x) \sim e^{\mp C_p} U(x) \sim \frac{e^{\mp C_p}}{\rho} x e^{-Q(x)} \quad \text{as } x \to \infty. \tag{39}$$

Since $p(x)$ is decreasing and integrable, $p(x)x \to 0$ as $x \to \infty$. We also assume that

$$p'(x) = O(1/x^2). \tag{40}$$

It follows from Lemma 24 that the condition on $p'(x)$ is always satisfied for a properly chosen function p.

Lemma 6. *As $x \to \infty$,*

$$\mathbb{E}U_+(x + \xi(x)) - U_+(x) = p(x)(1 + o(1))e^{-Q_+(x)} \tag{41}$$

and

$$\mathbb{E}U_-(x + \xi(x)) - U_-(x) = -p(x)(1 + o(1))e^{-Q_-(x)}. \tag{42}$$

Proof. We start with the following decomposition:

$$\begin{aligned} \mathbb{E}U_\pm(x + \xi(x)) - U_\pm(x) = \; & \mathbb{E}\{U_\pm(x + \xi(x)) - U_\pm(x);\ \xi(x) < -x/2\} \\ & + \mathbb{E}\{U_\pm(x + \xi(x)) - U_\pm(x);\ |\xi(x)| \le x/2\} \\ & + \mathbb{E}\{U_\pm(x + \xi(x)) - U_\pm(x);\ \xi(x) > x/2\}. \end{aligned} \tag{43}$$

The third term on the right hand side is negative because $U_\pm$ decreases and it may be bounded below as follows:

$$\begin{aligned}\mathbb{E}\{U_\pm(x+\xi(x))-U_\pm(x);\ \xi(x)>x/2\} &\ge -U_\pm(x)\mathbb{P}\{\xi(x)>x/2\}\\ &= o\big(p_1(x)e^{-Q_\pm(x)}\big),\end{aligned}\tag{44}$$

due to the upper bound (25) which implies $\mathbb{P}\{\xi(x)>x/2\}=o(p_1(x)/x)$, and due to the relations (38) and (39). Further, the first term on the right hand side of (43) is positive and possesses the following upper bound:

$$\begin{aligned}\mathbb{E}\{U_\pm(x+\xi(x))-U_\pm(x);\ \xi(x)<-x/2\} &\le \mathbb{E}\{U_\pm(x+\xi(x));\ \xi(x)<-x/2\}\\ &= o\big(p_1(x)e^{-Q_\pm(x)}\big),\end{aligned}\tag{45}$$

due to the upper bound (23) with $\gamma_0=\rho$ and due to the relations (36) and (38).

To estimate the second term on the right hand side of (43), we make use of Taylor's expansion:

$$\begin{aligned}&\mathbb{E}\{U_\pm(x+\xi(x))-U_\pm(x);\ |\xi(x)|\le x/2\}\\ &\quad= U'_\pm(x)\mathbb{E}\{\xi(x);|\xi(x)|\le x/2\}+\frac{1}{2}U''_\pm(x)\mathbb{E}\{\xi^2(x);|\xi(x)|\le x/2\}\\ &\qquad\qquad+\frac{1}{6}\mathbb{E}\big\{U'''_\pm(x+\theta\xi(x))\xi^3(x);|\xi(x)|\le x/2\big\}\\ &\quad= U'_\pm(x)m_1(x)+\frac{1}{2}U''_\pm(x)m_2(x)\\ &\qquad -U'_\pm(x)\mathbb{E}\{\xi(x);|\xi(x)|>x/2\}-\frac{1}{2}U''_\pm(x)\mathbb{E}\{\xi^2(x);|\xi(x)|>x/2\}\\ &\qquad\qquad+\frac{1}{6}\mathbb{E}\big\{U'''_\pm(x+\theta\xi(x))\xi^3(x);|\xi(x)|\le x/2\big\},\end{aligned}\tag{46}$$

where $0\le\theta=\theta(x,\xi(x))\le 1$. By the construction of $U_\pm$,

$$U'_\pm(x)=-e^{-Q_\pm(x)},\qquad U''_\pm(x)=q_\pm(x)e^{-Q_\pm(x)}=(q(x)\pm p(x))e^{-Q_\pm(x)}.\tag{47}$$

Then it follows that

$$\begin{aligned}U'_\pm(x)m_1(x)+\frac{1}{2}U''_\pm(x)m_2(x) &= e^{-Q_\pm(x)}\Big(-m_1(x)+(q(x)\pm p(x))\frac{m_2(x)}{2}\Big)\\ &= \frac{m_2(x)}{2}e^{-Q_\pm(x)}\bigg(-\frac{2m_1(x)}{m_2(x)}+q(x)\pm p(x)\bigg)\\ &= \pm\frac{m_2(x)}{2}e^{-Q_\pm(x)}p(x)(1+o(1)),\end{aligned}\tag{48}$$

by Lemma 3 which yields

$$\frac{2m_1(x)}{m_2(x)} = q(x)+o(p(x)+1/x^2))\quad\text{as } x\to\infty.$$

It follows from (25) and (47) that

$$U'_\pm(x)\mathbb{E}\{\xi(x);|\xi(x)|>x/2\}+\frac{1}{2}U''_\pm(x)\mathbb{E}\{\xi^2(x);|\xi(x)|>x/2\} = o(p(x)e^{-Q_\pm(x)}).\tag{49}$$

Finally, let us estimate the last term in (46). Notice that by the condition (40) on the derivative of $p(x)$,

$$\begin{aligned} U'''_\pm(x) &= \big(q'(x) \pm p'(x) - (q(x) \pm p(x))^2\big) e^{-Q_\pm(x)} \\ &= O(1/x^2) e^{-Q_\pm(x)}, \end{aligned}$$

hence,

$$U'''_\pm(x+y) = O(1/x^2) e^{-Q_\pm(x)}$$

as $x \to \infty$ uniformly for $|y| \le x/2$ which implies

$$\big|\mathbb{E}\big\{U'''_\pm(x+\theta\xi(x))\xi^3(x); |\xi(x)| \le x/2\big\}\big| \le \frac{c_1}{x^2}\mathbb{E}\big\{|\xi^3(x)|;\ |\xi(x)| \le x/2\big\} e^{-Q_\pm(x)}.$$

Then, in view of (24),

$$\big|\mathbb{E}\big\{U'''_\pm(x+\theta\xi(x))\xi^3(x);\ |\xi(x)| \le x/2\big\}\big| = o\big(p_1(x) e^{-Q_\pm(x)}\big). \tag{50}$$

Substituting (48)–(50) into (46), we obtain that

$$\mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ |\xi(x)| \le x/2\} = \pm m_2(x) p(x)(1+o(1)) e^{-Q_\pm(x)}. \tag{51}$$

Substituting (44)—or (45)—and (51) into (43) and recalling that $p_1(x) \le p(x)$, we finally come to the desired conclusions. □

Lemma 6 implies the following result.

Corollary 7. *There exists an $\widehat{x}$ such that, for all $x > \widehat{x}$,*

$$\begin{aligned} \mathbb{E}U_-(x+\xi(x)) &\le U_-(x), \\ \mathbb{E}U_+(x+\xi(x)) &\ge U_+(x). \end{aligned}$$

Proof of Theorem 5. The process $U_-(R_n)$ is bounded above by $U_-(0)$. Let $\widehat{x}$ be any level guaranteed by the last corollary, $B = (-\infty, \widehat{x}]$ and $\tau_B = \min\{n \ge 1 : X_n \in B\}$.

By Corollary 7, $U_-(R_{n\wedge\tau_B})$ is a bounded supermartingale. Hence by the optional stopping theorem, for $z > \widehat{x}$ and $x \in (\widehat{x}, z)$,

$$\mathbb{E}_x U_-(R_{\tau_B \wedge \tau_{(z,\infty)}}) \le \mathbb{E}_x U_-(R_0) = U_-(x).$$

Letting $z \to \infty$ we conclude that

$$\begin{aligned} \mathbb{E}_x\{U_-(R_{\tau_B});\ \tau_B < \infty\} &= \lim_{z\to\infty} \mathbb{E}_x\{U_-(R_{\tau_B});\ \tau_B < \tau_{(z,\infty)}\} \\ &= \lim_{z\to\infty} \mathbb{E}_x U_-(R_{\tau_B\wedge\tau_{(z,\infty)}}) - \lim_{z\to\infty} \mathbb{E}_x\{U_-(R_{\tau_{(z,\infty)}});\ \tau_B > \tau_{(z,\infty)}\} \\ &\le U_-(x) - 0 = U_-(x). \end{aligned}$$

On the other hand, since U_- is decreasing,

$$\mathbb{E}_x\{U_-(R_{\tau_B});\ \tau_B < \infty\} \ge U_-(\widehat{x})\mathbb{P}_x\{\tau_B < \infty\}.$$

Therefore,

$$\mathbb{P}_x\{\tau_B < \infty\} \;\le\; \frac{U_-(x)}{U_-(\widehat{x})}, \tag{52}$$

which implies, by (39), that, for some constant $c_2 < \infty$,

$$\mathbb{P}_x\{R_n \le \widehat{x} \text{ for some } n\} \;\le\; c_2 U(x) \quad \text{for all } x > \widehat{x}.$$

Thus,

$$\begin{aligned} \mathbb{P}_x\{R_n \le 0 \text{ for some } n\} &\le \mathbb{P}_x\{R_n \le \widehat{x} \text{ for some } n\} \\ &\le c_2 U(x) \quad \text{for all } x > \widehat{x}. \end{aligned}$$

This gives the desired upper bound.

On the other hand, the process $\{U_+(R_{n\wedge\tau_B})\}$ is a bounded submartingale due to the lower bound provided by Corollary 7. Hence again by the optional stopping theorem, for $x > x_0$,

$$\mathbb{E}_x\{U_+(R_{\tau_B});\ \tau_B < \infty\} \;\ge\; \mathbb{E}_x U_+(R_0) \;=\; U_+(x).$$

On the other hand, since U_+ is bounded by $U_+(0)$,

$$\mathbb{E}_x\{U_+(R_{\tau_B});\ \tau_B < \infty\} \;\le\; U_+(0)\mathbb{P}_x\{\tau_B < \infty\}.$$

This allows us to deduce a lower bound

$$\mathbb{P}_x\{\tau_B < \infty\} \;\ge\; \frac{U_+(x)}{U_+(0)},$$

which completes the proof of the lower bound, for some constant $c_1 > 0$,

$$\mathbb{P}_x\{R_n \le \widehat{x} \text{ for some } n\} \;\ge\; c_1 U(x) \quad \text{for all } x > \widehat{x},$$

due to (39). To complete the proof of the lower bound it remains to refer to the arguments in (17). $\square$

6 Approaching critical premium rate at rate of θ/x^α

In this section we consider the case (13) with some $\alpha \in (0,1)$. The asymptotic behaviour of the ruin probability under this rate of approaching the critical value v_c is described in the next theorem. Define

$$\gamma := \min\{k \ge 1 : \alpha k > 1\}.$$

Theorem 8. *Assume* (15) *and the rate of convergence* (13). *Let* $\mathbb{E}\tau^{\gamma+1} < \infty$ *and* $\mathbb{E}e^{r\zeta^{1-\alpha}} < \infty$ *for some*

$$r \;>\frac{r_1}{1-\alpha},$$

where

$$r_1 \quad := \quad \frac{2\theta\mathbb{E}\tau}{\mathbb{E}(v_c\tau-\zeta)^2} \quad = \quad \frac{2\theta\mathbb{E}\tau}{\mathbb{V}\mathrm{ar}\,\zeta+v_c^2\mathbb{V}\mathrm{ar}\,\tau}. \tag{53}$$

Then there exist constants r_2, r_3, ..., $r_{\gamma-1}\in\mathbb{R}$, defined recursively in the proof below, and $0<C_1<C_2<\infty$ such that

(i) if $\alpha=1/(\gamma-1)$ for an integer $\gamma\ge 2$, then, for all $x>1$,

$$\frac{C_1x^\alpha}{x^{r_{\gamma-1}}}\exp\biggl\{-\sum_{j=1}^{\gamma-2}\frac{r_j}{1-\alpha j}x^{1-\alpha j}\biggr\} \ \le\ \psi(x) \ \le\ \frac{C_2x^\alpha}{x^{r_{\gamma-1}}}\exp\biggl\{-\sum_{j=1}^{\gamma-2}\frac{r_j}{1-\alpha j}x^{1-\alpha j}\biggr\}, \tag{54}$$

(ii) if $\alpha<1/(\gamma-1)$ then

$$C_1x^\alpha\exp\biggl\{-\sum_{j=1}^{\gamma-1}\frac{r_j}{1-\alpha j}x^{1-\alpha j}\biggr\} \ \le\ \psi(x) \ \le\ C_2x^\alpha\exp\biggl\{-\sum_{j=1}^{\gamma-1}\frac{r_j}{1-\alpha j}x^{1-\alpha j}\biggr\}. \tag{55}$$

As seen from these bounds, the ruin probability is decaying, roughly speaking, as a Weibullian distribution with shape parameter $1-\alpha$. However further terms in the exponent are needed to make lower and upper bounds precise up to a constant multiplier.

In order to prove the last theorem, we firstly derive asymptotic estimates for the moments of $V_x(\tau)-x$.

Lemma 9. *Let $\mathbb{E}\tau^\gamma<\infty$ and there exists an $x_0\ge 0$ such that*

$$v_-(x) \quad\le\quad v(x) \ \le\ v_+(x) \quad \text{for all } x\ge x_0, \tag{56}$$

where both $v_-(x)$ and $v_+(x)$ are decreasing functions on $[x_0,\infty)$. Then, for all $k\le\gamma$,

$$\mathbb{E}\tau^k\left(v_-(x+\tau v_+(x))\right)^k \ \le\ \mathbb{E}(V_x(\tau)-x)^k \quad\le\quad v_+^k(x)\mathbb{E}\tau^k, \quad x\ge x_0. \tag{57}$$

If, in addition, $\mathbb{E}\tau^{\gamma+1-\alpha}<\infty$ and (13) holds true, then there exists an integrable decreasing function $p_1(x)$ such that, for all $k\le\gamma$,

$$\mathbb{E}(V_x(\tau)-x)^k \quad=\quad (v_c+\theta/x^\alpha)^k\mathbb{E}\tau^k+O(p_1(x)) \quad \text{as } x\to\infty. \tag{58}$$

Proof. Fix some $x\ge x_0$. Due to (56), $v(z)\le v_+(x)$ for all $z\ge x$. Hence,

$$\begin{aligned} V_x(t) \quad &= \quad x+\int_0^t v(V_x(s))ds \\ &\le \quad x+\int_0^t v_+(x)ds \ =\ x+tv_+(x), \end{aligned} \tag{59}$$

and the inequality on the right hand side of (57) follows. It follows from the left hand side inequality in (56) and from the last upper bound for $V_x(t)$ that

$$V_x(t) - x \;\geq\; \int_0^t v_-(V_x(t))ds \;\geq\; tv_-(x + tv_+(x)), \tag{60}$$

and the left hand side bound in (57) is proven.

Owing to (13), $v(z)$ is sandwiched between the two eventually decreasing functions $v_\pm(z) := v_c + \theta/z^\alpha \pm p(z)$. Therefore, applying the right hand side bound in (57) we get

$$\begin{aligned} \mathbb{E}(V_x(\tau) - x)^k &\leq (v_c + \theta/x^\alpha + p(x))^k \mathbb{E}\tau^k \\ &= (v_c + \theta/x^\alpha)^k \mathbb{E}\tau^k + O(p(x)) \quad \text{as } x \to \infty. \end{aligned} \tag{61}$$

From the lower bound in (57) we deduce that, for all $k \leq \gamma$,

$$\begin{aligned} \mathbb{E}(V_x(\tau) - x)^k &\geq \mathbb{E}\tau^k \Big(v_c + \frac{\theta}{(x + \tau v_+(x))^\alpha} - p(x)\Big)^k \\ &\geq \mathbb{E}\tau^k \Big(v_c + \frac{\theta}{(x + \overline{v}\tau)^\alpha}\Big)^k + O(p(x)), \quad \overline{v} = \sup_z v(z). \end{aligned}$$

By the inequality $1/(1+y)^\alpha \geq 1 - \alpha y \wedge 1$, we infer that, for $c_2 = \alpha\overline{v}$,

$$\frac{1}{(x + \overline{v}t)^\alpha} \;\geq\; \frac{1}{x^\alpha}\Big(1 - \frac{c_2 t}{x} \wedge 1\Big).$$

Therefore, for all $k \leq \gamma$,

$$\begin{aligned} \mathbb{E}(V_x(\tau) - x)^k &\geq \mathbb{E}\tau^k \Big(v_c + \frac{\theta}{x^\alpha} - \frac{c_2\theta\tau}{x^{\alpha+1}}\mathbb{I}\{\tau \leq x/c_2\} - \frac{\theta}{x^\alpha}\mathbb{I}\{\tau > x/c_2\}\Big)^k + O(p(x)) \\ &\geq \Big(v_c + \frac{\theta}{x^\alpha}\Big)^k \mathbb{E}\tau^k - \frac{c_3}{x^\alpha}\mathbb{E}\{\tau^k;\; \tau > x/c_2\} \\ &\qquad -c_3 \sum_{j=1}^k \frac{1}{x^{j(\alpha+1)}}\mathbb{E}\{\tau^{k+j};\; \tau \leq x/c_2\} - c_3 p(x), \end{aligned} \tag{62}$$

for some $c_3 < \infty$. Then, due to the integrability of $p(x)$, in order to prove that

$$\mathbb{E}(V_x(\tau) - x)^k \;\geq\; (v_c + \theta/x^\alpha)^k \mathbb{E}\tau^k - p_1(x) \tag{63}$$

for some decreasing integrable function $p_1(x)$, it suffices to show that

$$x^{-\alpha}\mathbb{E}\{\tau^\gamma;\; \tau > x\}$$

and

$$x^{-j(\alpha+1)}\mathbb{E}\{\tau^{\gamma+j};\; \tau \leq x\}$$

are bounded by decreasing integrable at infinity functions. Indeed, the integral of the first function—which decreases itself—is finite due to the finiteness of the $(\gamma+1-\alpha)$ moment of τ. Concerning the second function, first notice that

$$x^{-j(\alpha+1)}\mathbb{E}\{\tau^{\gamma+j};\; \tau \leq x\} \;\leq\; \frac{\mathbb{E}\{\tau^{\gamma+1};\; \tau \leq x\}}{x^{1+\alpha}}, \quad j \geq 1.$$

The right hand side is bounded by a decreasing integrable at infinity function due to the moment condition on τ and Lemma 18. So, (63) is proven which together with (61) completes the proof. $\square$

Proposition 10. *Assume the rate of convergence* (13)*. If both* $\mathbb{E}\tau^{1+\gamma}$ *and* $\mathbb{E}\zeta^{1+\gamma}$ *are finite, then, for all* $k \le \gamma$,

$$m_k(x) \;=\; \sum_{j=0}^{k} \frac{a_{k,j}}{x^{\alpha j}} + O(x^{\alpha(k-1)} p_2(x)) \quad \text{as } x \to \infty,$$

where $p_2(x)$ *is a decreasing integrable at infinity function and*

$$a_{k,j} \;:=\; \binom{k}{j} \theta^j \mathbb{E}\tau^j (v_c\tau - \zeta)^{k-j}, \quad j \le k \le \gamma.$$

In addition,

$$\mathbb{E}\{|\xi^k(x)|;\ |\xi(x)| > x^\alpha\} \;=\; o(x^{\alpha(k-1)} p_2(x)) \quad \text{as } x \to \infty. \tag{64}$$

Proof. It follows from the definition of $\xi(x)$ that

$$\mathbb{E}\xi^k(x) \;=\; \mathbb{E}(V_x(\tau) - x - \zeta)^k \;=\; \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}(V_x(\tau) - x)^i \mathbb{E}(-\zeta)^{k-i}.$$

Applying Lemma 9, we then obtain

$$\begin{aligned}
m_k(x) \;:=\; \mathbb{E}\xi^k(x) \;&=\; \sum_{i=0}^{k} \binom{k}{i} \Big(v_c + \frac{\theta}{x^\alpha}\Big)^i \mathbb{E}\tau^i \mathbb{E}(-\zeta)^{k-i} + O(p_1(x)) \\
&=\; \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}\tau^i \mathbb{E}(-\zeta)^{k-i} \sum_{j=0}^{i} \binom{i}{j} v_c^{i-j} \Big(\frac{\theta}{x^\alpha}\Big)^j + O(p_1(x)) \\
&=:\; \sum_{j=0}^{k} \frac{a_{k,j}}{x^{\alpha j}} + O(p_1(x)) \quad \text{as } x \to \infty,
\end{aligned}$$

where

$$\begin{aligned}
a_{k,j} \;&:=\; \binom{k}{j} \theta^j \sum_{i=j}^{k} \binom{k-j}{i-j} \mathbb{E}\tau^i \mathbb{E}(-\zeta)^{k-i} v_c^{i-j} \\
&=\; \binom{k}{j} \theta^j \mathbb{E} \sum_{i=0}^{k-j} \binom{k-j}{i} \tau^{i+j} (-\zeta)^{k-j-i} v_c^i \\
&=\; \binom{k}{j} \theta^j \mathbb{E}\tau^j (v_c\tau - \zeta)^{k-j}.
\end{aligned}$$

It is immediate from (59) that $V_x(\tau) - x \le \overline{v}\tau$ where $\overline{v} = \sup_z v(z)$. Then

$$\begin{aligned}
\mathbb{E}\{|\xi^k(x)|;\ |\xi(x)| > x^\alpha\} \;&\le\; \mathbb{E}\{(V_x(\tau) - x)^k;\ V_x(\tau) - x > x^\alpha\} + \mathbb{E}\{\zeta^k; \zeta > x^\alpha\} \\
&\le\; \overline{v}^k \mathbb{E}\{\tau^k;\ \tau > x^\alpha/\overline{v}\} + \mathbb{E}\{\zeta^k;\ \zeta > x^\alpha\}.
\end{aligned}$$

Since $\mathbb{E}\tau^{\gamma+1} < \infty$, for all $k \le \gamma$,

$$x^{-\alpha(k-1)}\mathbb{E}\{\tau^k;\ \tau > x^\alpha/\overline{v}\} = o\left(\frac{1}{x^{\alpha(k-1)}x^{\alpha(\gamma+1-k)}}\right) = o\left(\frac{1}{x^{\alpha\gamma}}\right) \quad \text{as } x \to \infty.$$

By the definition of the γ, $\alpha\gamma > 1$. The function $1/x^{\alpha\gamma}$ is integrable at infinity. The same arguments work for ζ, so the value of $x^{-\alpha(k-1)}\mathbb{E}\{|\xi^k(x)|;\ |\xi(x)| > x^\alpha\}$ is bounded by a decreasing integrable at infinity function, and the proof is complete. □

Proof of Theorem 8. We first show that there exist constants $r_1, r_2, \dots, r_{\gamma-1}$ such that

$$q(x) := \sum_{j=1}^{\gamma-1} \frac{r_j}{(b+x)^{\alpha j}}$$

satisfies

$$-m_1(x) + \sum_{j=2}^{\gamma}(-1)^j \frac{m_j(x)}{j!} q^{j-1}(x) = o(p_3(x)), \tag{65}$$

where p_3 is a decreasing integrable function and b is a positive number.

We can determine all these numbers recursively. Indeed, as proven in Proposition 10,

$$m_1(x) = \frac{\theta\mathbb{E}\tau}{x^\alpha} + o(p_2(x)) \quad \text{as } x \to \infty$$

and

$$m_2(x) = \mathbb{V}\mathrm{ar}\,\zeta + v_c^2\mathbb{V}\mathrm{ar}\,\tau + O(x^{-\alpha} + x^\alpha p_2(x)) \quad \text{as } x \to \infty.$$

For r_1 defined defined in (53),

$$-m_1(x) + \sum_{j=2}^{\gamma}(-1)^j \frac{m_j(x)}{j!} q^{j-1}(x) = O(x^{-2\alpha} + p_2(x)) \quad \text{as } x \to \infty,$$

for any choice of $r_2, r_3, \dots, r_{\gamma-1}$. Then we can choose r_2 such that the coefficient of $x^{-2\alpha}$ is also zero,

$$r_2 = \frac{\mathbb{E}(v_c\tau - \zeta)^3 r_1^2/3 - 2\theta\mathbb{E}\tau(v_c\tau - \zeta)r_1}{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2\mathbb{V}\mathrm{ar}\,\tau},$$

and so on. It is clear that the numbers $r_1, r_2, \dots, r_{\gamma-1}$ do not depend on the parameter b. Therefore, we can take b so large that the function $q(x)$ is decreasing on $[0, \infty)$.

As in the previous section, we define

$$Q(x) = \int_0^x q(y)dy \quad \text{and} \quad U(x) = \int_x^\infty e^{-Q(y)}dy,\ x \ge 0.$$

For $x<0$ we set $U(x)=U(0)$. It is immediate from the definition of $q(x)$ that

$$Q(x) \quad = \quad \int_0^x \sum_{j=1}^{\gamma-1} \frac{r_j}{(b+z)^{\alpha j}} dz, \quad x \ge 0.$$

and

$$U(x) \quad = \quad \int_x^\infty \exp\Big\{-\int_0^y \sum_{j=1}^{\gamma-1} \frac{r_j}{(b+z)^{\alpha j}} dz\Big\} dy, \quad x \ge 0.$$

We define also, for $p(x) \ge \max(p_1(x), p_2(x))$,

$$q_\pm(x) = q(x) \pm p(x), \quad Q_\pm(x) = \int_0^x q_\pm(y) dy \quad \text{and} \quad U_\pm(x) = \int_x^\infty e^{-Q(y)} dy.$$

We further assume that, for all $1 \le k \le \gamma - 1$,

$$q^{(k)}(x) = o(q^\gamma(x)), \qquad p^{(k)}(x) = o(q^\gamma(x)) \quad \text{as } x \to \infty \tag{66}$$

and

$$q^\gamma(x) \quad = \quad o(p(x)) \quad \text{as } x \to \infty. \tag{67}$$

If $q(x) \sim c/x^\alpha$ where $\gamma\alpha < 2$, then it follows from Lemma 24 that the condition on the derivatives of $p(x)$ is always satisfied for a properly chosen function p, so the condition (66) on the derivatives of p does not restrict generality under this specific choice of $r(x)$.

It is clear that

$$U_\pm(x) \sim e^{\mp C_p} U(x) \quad \text{as } x \to \infty.$$

Noting that

$$\frac{U'(x)}{\Big(\frac{1}{q(x)} e^{-Q(x)}\Big)'} = \frac{-e^{-Q(x)}}{(-q'(x)/q^2(x) - 1) e^{-Q(x)}} \to 1 \quad \text{as } x \to \infty$$

and applying the L'Hôpital rule, we conclude that, as $x \to \infty$,

$$U(x) \sim \frac{e^{-Q(x)}}{q(x)} \quad \text{and} \quad U_\pm(x) \sim e^{\mp C_p} \frac{e^{-Q(x)}}{q(x)} \sim \frac{e^{-Q_\pm(x)}}{q(x)}. \tag{68}$$

Lemma 11. *As $x \to \infty$, we have the following estimates:*

$$\mathbb{E}U_+(x+\xi(x)) - U_+(x) \quad = \quad \frac{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau + o(1)}{2} p(x) e^{-R_+(x)}, \tag{69}$$

$$\mathbb{E}U_-(x+\xi(x)) - U_-(x) \quad = \quad -\frac{\mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau + o(1)}{2} p(x) e^{-R_-(x)}. \tag{70}$$

Proof. We start with the following decomposition:

$$\begin{aligned}
\mathbb{E}U_\pm(x+\xi(x)) - U_\pm(x) &= \mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ \xi(x) < -x^\alpha\} \\
&\quad + \mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ |\xi(x)| \le x^\alpha\} \\
&\quad + \mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ \xi(x) > x^\alpha\}. \qquad (71)
\end{aligned}$$

The third term on the right hand side is negative because $U_\pm$ decreases and it may be bounded below as follows:

$$\begin{aligned}
\mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ \xi(x) > x^\alpha\} &\ge -U_\pm(x)\mathbb{P}\{\xi(x) > x^\alpha\} \\
&= o\big(p_2(x)e^{-Q_\pm(x)}\big), \qquad (72)
\end{aligned}$$

due to the upper bound (64) which implies $\mathbb{P}\{\xi(x) > x^\alpha\} = o(p_2(x)/x^\alpha)$, and due to the relation (68).

Further, the first term on the right hand side of (71) is positive. To obtain an upper bound for that expectation we first notice that, due to the fact that $Q(z)$ is monotone increasing,

$$U_\pm(x-y) - U_\pm(x) = \int_{x-y}^{x} e^{-Q_\pm(z)}dz \le e^{2C_p}\int_{x-y}^{x} e^{-Q(z)}dz \le e^{2C_p}ye^{-Q(x-y)}.$$

Since $q(x)$ is chosen to be decreasing, $Q(z)$ is concave and, consequently,

$$Q(x-y) \ge Q(x) - Q(y).$$

Using this inequality we obtain

$$U_\pm(x-y) - U_\pm(x) \le e^{2C_p}ye^{-Q(x)}e^{Q(y)}$$

and

$$\begin{aligned}
&\mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ \xi(x) < -x^\alpha\} \\
&\quad \le e^{2C_p}e^{-Q(x)}\mathbb{E}\{-\xi(x)e^{Q(\xi(x))}; \xi(x) < -x^\alpha\} \\
&\quad \le e^{2C_p}e^{-Q(x)}\mathbb{E}\{\zeta e^{Q(\zeta)}; \zeta > x^\alpha\}.
\end{aligned}$$

The moment assumption on ζ implies that the decreasing function $\mathbb{E}\{\zeta e^{Q(\zeta)}; \zeta > x^\alpha\}$ is integrable at infinity. As a result we have

$$\mathbb{E}\{U_\pm(x+\xi(x)) - U_\pm(x);\ \xi(x) < -x^\alpha\} = o\big(p_1(x)e^{-Q_\pm(x)}\big). \qquad (73)$$

To estimate the second term on the right hand side of (71), we make use of

Taylor's expansion with $\gamma+1$ terms:

$$\begin{aligned}
&\mathbb{E}\{U_\pm(x+\xi(x))-U_\pm(x);\ |\xi(x)|\le x^\alpha\}\\
&= \sum_{k=1}^{\gamma}\frac{U_\pm^{(k)}(x)}{k!}\mathbb{E}\{\xi^k(x);|\xi(x)|\le x^\alpha\}\\
&\qquad\qquad +\mathbb{E}\Big\{\frac{U_\pm^{(\gamma+1)}(x+\theta\xi(x))}{(\gamma+1)!}\xi^{\gamma+1}(x);\ |\xi(x)|\le x^\alpha\Big\},\\
&= \sum_{k=1}^{\gamma}\frac{U_\pm^{(k)}(x)}{k!}m_k(x)-\sum_{k=1}^{\gamma}\frac{U_\pm^{(k)}(x)}{k!}\mathbb{E}\{\xi^k(x);|\xi(x)|>x^\alpha\}\\
&\qquad\qquad +\mathbb{E}\Big\{\frac{U_\pm^{(\gamma+1)}(x+\theta\xi(x))}{(\gamma+1)!}\xi^{\gamma+1}(x);\ |\xi(x)|\le x^\alpha\Big\},
\end{aligned}\tag{74}$$

where $0\le\theta=\theta(x,\xi(x))\le 1$. By the construction of $U_\pm$,

$$U'_\pm(x)=-e^{-Q_\pm(x)},\qquad U''_\pm(x)=q_\pm(x)e^{-Q_\pm(x)}=(q(x)\pm p(x))e^{-Q_\pm(x)},\tag{75}$$

and, for $k=3,\ \dots,\ \gamma+1$,

$$U_\pm^{(k)}(x)=-(e^{-Q_\pm(x)})^{(k-1)}\ =\ (-1)^k\big(q_\pm^{k-1}(x)+o(p(x))\big)e^{-Q_\pm(x)}\quad\text{as } x\to\infty,$$

where the remainder terms in the parentheses on the right are of order $o(p(x))$ by the conditions (66) and (67). By the definition of $q_\pm(x)$,

$$q_\pm^{k-1}(x)\ =\ (q(x)\pm p(x))^{k-1}=q^{k-1}(x)+o(p(x))\quad\text{for all } k\ge 3,$$

which implies the relation

$$U_\pm^{(k)}(x)\ =\ (-1)^k\big(q^{k-1}(x)+o(p(x))\big)e^{-Q_\pm(x)}\quad\text{as } x\to\infty.\tag{76}$$

From these equalities we get $|U_\pm^{(k)}(x)|\le Cx^{-\alpha(k-1)}e^{-Q_\pm(x)}$, Combining this with (64), we obtain

$$\sum_{k=1}^{\gamma}\frac{U_\pm^{(k)}(x)}{k!}\mathbb{E}\{\xi^k(x);|\xi(x)|>x^\alpha\}=o\left(p_2(x)e^{-Q_\pm(x)}\right)\quad\text{as } x\to\infty.\tag{77}$$

It follows from the equalities (75) and (76) that

$$\begin{aligned}
&\sum_{k=1}^{\gamma}\frac{U_\pm^{(k)}(x)}{k!}m_k(x)\\
&= e^{-Q_\pm(x)}\left(\sum_{k=1}^{\gamma}(-1)^k\frac{r^{k-1}(x)}{k!}m_k(x)+o(p(x))\pm p(x)\frac{m_2(x)}{2}\right)\\
&= e^{-Q_\pm(x)}\left(o(p(x))\pm p(x)\frac{m_2(x)}{2}\right)\quad\text{as } x\to\infty,
\end{aligned}\tag{78}$$

by the equality (65). Owing to the condition (66) on the derivatives of $r(x)$ and (67),

$$U_\pm^{(\gamma+1)}(x) \quad = \quad (-1)^{\gamma+1}(q^\gamma(x)+o(q^\gamma(x)))e^{-Q_\pm(x)} \quad \text{as } x\to\infty.$$

Then, the last term in (74) possesses the following bound:

$$\begin{aligned}
\Big|\mathbb{E}\Big\{\frac{U_\pm^{(\gamma+1)}(x+\theta\xi(x))}{(\gamma+1)!}\xi^{\gamma+1}(x);\ |\xi(x)|\le x^\alpha\Big\}\Big| \\
&= O\big(q^\gamma(x)e^{-Q_\pm(x)}\big)\mathbb{E}\big\{|\xi(x)|^{\gamma+1};\ |\xi(x)|\le x^\alpha\big\} \\
&= o\big(p(x)e^{-Q_\pm(x)}\big) \quad \text{as } x\to\infty,
\end{aligned}$$

by the condition (67). Therefore, it follows from (74), (77) and (78) that

$$\begin{aligned}
\mathbb{E}\{U_\pm(x+\xi(x))-U_\pm(x);\ |\xi(x)|\le x^\alpha\} \\
&= \pm p(x)\frac{m_2^{[s(x)]}(x)}{2}e^{-Q_\pm(x)}+o\big(p(x)e^{-Q_\pm(x)}\big) \quad \text{as } x\to\infty.
\end{aligned}$$

Together with (72), (73), and (71) this completes the proof. □

The remaining part of the proof repeats literally the final part of the proof of Theorem 5 and we omit it. □

7 Heavy-tailed claim sizes

In this section we study the case where the distribution of the claim size is so heavy that the moment conditions in Theorem 5 are not met.

We assume that $v(x)$ converges towards v_c at rate θ/x and that the distribution of ζ is regularly varying at infinity with index $-(\beta+2)$ for some $\beta\in(0,\rho)$. Then $\mathbb{E}\zeta^{\rho+2}$ is infinite and, consequently, Theorem 5 does not apply.

Theorem 12. *Assume the rate of convergence* (11) *with some* θ *satisfying* (19). *Assume also that* $\mathbb{E}\tau^2\log(1+\tau)<\infty$ *and that*

$$\mathbb{P}\{\zeta>x\}=x^{-2-\beta}L(x) \tag{79}$$

for some slowly varying at infinity function $L(x)$ *and* $\beta\in(0,\rho)$. *Then there exist constants* C_1 *and* C_2 *such that*

$$C_1\int_x^\infty y\mathbb{P}\{\zeta>y\}dy\le\psi(x)\le C_2\int_x^\infty y\mathbb{P}\{\zeta>y\}dy \quad \textit{for all } x>0.$$

Under the condition (79), by Karamata's theorem,

$$\int_x^\infty y\mathbb{P}\{\zeta>y\}dy\sim\frac{1}{\beta}x^2\mathbb{P}\{\zeta>x\} \quad \text{as } x\to\infty.$$

Therefore, the claim of Theorem 12 can be reformulated in the following way:

$$\widehat{C}_1 x^{-\beta} L(x) \le \psi(x) \le \widehat{C}_2 x^{-\beta} L(x).$$

Notice that, for the classical ruin process with constant premium rate and with claim size of subexponential type, $\psi(x)$ is asymptotically equivalent to the integral $\int_x^\infty \mathbb{P}\{\zeta > y\}dy$ (see e.g. [10, Section 5.11]). So, the main difference between our case and the classical one is that the probability of ruin is higher in our case owing to the additional weight y in the integral, which is not surprising and reflects the fact that our system is close to a critical one, $v(y) \to v_c$ as $y \to \infty$.

Notice that the condition (15) follows by (79).

The proof of Theorem 12 is split into two parts, where we derive the upper and lower bounds. For both, we need the following result on the left tail distribution of the jumps of the chain $\{R_n\}$.

Lemma 13. *If the distribution of ζ is long-tailed, that is, if*

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\zeta > x+u\}}{\mathbb{P}\{\zeta > x\}} = 1 \quad \textit{for any fixed } u,$$

then, uniformly for all $x \ge 0$,

$$\mathbb{P}\{\xi(x) < -y\} \quad \sim \quad \mathbb{P}\{\zeta > y\} \quad \textit{as } y \to \infty.$$

Proof. Using the equality $\xi(x) = V_x(\tau) - x - \zeta$, we get the following upper bound

$$\begin{aligned} \mathbb{P}\{\xi(x) < -y\} &= \mathbb{P}\{\zeta - (V_x(\tau) - x) > y\} \\ &\le \mathbb{P}\{\zeta > y\}. \end{aligned}$$

For a lower bound, let us notice that, for any fixed u,

$$\begin{aligned} \mathbb{P}\{\xi(x) \le -y\} &\ge \mathbb{P}\{\zeta > y+u\}\mathbb{P}\{V_x(\tau) - x < u\} \\ &\sim \mathbb{P}\{\zeta > y\}\mathbb{P}\{V_x(\tau) - x < u\} \quad \text{as } y \to \infty, \end{aligned}$$

due to the long-tailedness of the distribution of ζ. Also, by the stochastic boundedness of the family of random variables $\{V_x(\tau) - x,\ x \ge 0\}$,

$$\inf_x \mathbb{P}\{V_x(\tau) - x < u\} \quad \to \quad 1 \quad \text{as } u \to \infty,$$

which implies the following lower bound, uniformly for all $x \ge 0$,

$$\mathbb{P}\{\xi(x) \le -y\} \quad \ge \quad \mathbb{P}\{\zeta > y\}(1 + o(1)) \quad \text{as } y \to \infty,$$

hence the desired result. $\square$

7.1 Proof of the upper bound

As in the previous sections, we analyse the behaviour of the chain $R_n = R(T_n)$, $n \geq 0$. In order to understand the impact of large claim sizes on ruin probabilities from the point of view of an upper bound, we introduce an auxiliary chain with jumps truncated below. For every $x \geq 0$ we define jump $\widetilde{\xi}(x)$ as follows:

$$\mathbb{P}\{\widetilde{\xi}(x) \in B\} := \mathbb{P}\{\xi(x) \in B \mid \xi(x) \geq -x/2\}, \quad B \in \mathcal{B}(\mathbb{R}).$$

Let $\{\widetilde{R}_n\}$ be a Markov chain with jumps $\widetilde{\xi}(x)$. The connection between $\{\widetilde{R}_n\}$ and $\{R_n\}$ is described in the next lemma.

Lemma 14. *Set $A_n := \{\xi(R_k) \geq -R_k/2$ for all $k < n\}$. Then, for all Borel sets $B_1, B_2, \ldots, B_n$ we have*

$$\mathbb{P}_x\{R_1 \in B_1, \ldots, R_n \in B_n; A_n\} \quad = \quad \mathbb{E}_x\left\{\prod_{k=0}^{n-1} g(\widetilde{R}_k);\ \widetilde{R}_1 \in B_1, \ldots, \widetilde{R}_n \in B_n\right\},$$

where

$$g(x) := \mathbb{P}\{\xi(x) \geq -x/2\} \in (0,1).$$

Proof. We use the induction in n. If $n = 1$, then

$$\begin{aligned}
\mathbb{P}_x\{R_1 \in B_1; A_1\} &= \mathbb{P}\{x + \xi(x) \in B_1, \xi(x) > -x/2\} \\
&= g(x)\mathbb{P}\{x + \widetilde{\xi}(x) \in B_1\} \\
&= \mathbb{E}_x\{g(\widetilde{R}_0); \widetilde{R}_1 \in B_1\}.
\end{aligned}$$

For the induction step $n - 1 \to n$ it suffices to apply the Markov property:

$$\begin{aligned}
&\mathbb{P}_x\{R_1 \in B_1, ..., R_n \in B_n; A_n\} \\
&= \int_{B_{n-1}} \mathbb{P}_x\{R_1 \in B_1, \ldots, R_{n-1} \in dy; A_{n-1}\}\mathbb{P}\{y + \xi(y) \in B_n, \xi(y) \geq -y/2\} \\
&= \int_{B_{n-1}} \mathbb{E}_x\left[\prod_{k=0}^{n-2} g(\widetilde{R}_k);\ \widetilde{R}_1 \in B_1, ..., \widetilde{R}_{n-1} \in dy\right] g(y)\mathbb{P}\{y + \widetilde{\xi}(y) \in B_n\} \\
&= \mathbb{E}_x\left\{\prod_{k=0}^{n-1} g(\widetilde{R}_k);\ \widetilde{R}_1 \in B_1, ..., \widetilde{R}_n \in B_n\right\},
\end{aligned}$$

which completes the proof. □

Let $\widetilde{\psi}(x)$ denote the ruin probability for the chain $\{\widetilde{R}_n\}$, that is,

$$\widetilde{\psi}(x) = \mathbb{P}_x\{\widetilde{R}_n < 0 \text{ for some } n \geq 1\}.$$

Let $\widetilde{H}_x$ be the renewal measure of $\{\widetilde{R}_n\}$ with starting point x:

$$\widetilde{H}_x(B) = \sum_{n=0}^{\infty} \mathbb{P}_x\{\widetilde{R}_n \in B\}, \quad B \in \mathcal{B}(\mathbb{R}).$$

Lemma 15. *The following inequality holds true*

$$\psi(x) \quad \leq \quad \widetilde{\psi}(x) + \int_0^\infty (1 - g(y))\widetilde{H}_x(dy). \tag{80}$$

Proof. Let

$$\tau_0 := \inf\{n \geq 1 : R_n < 0\}$$

and

$$A_{\tau_0} := \{\xi(R_k) \geq -R_k/2 \text{ for all } k < \tau_0\}.$$

Noting that

$$\{\tau_0 < \infty\} \subseteq (\{\tau_0 < \infty\} \cap A_{\tau_0}) \cup A^c_{\tau_0},$$

we get

$$\mathbb{P}_x\{\tau_0 < \infty\} \leq \mathbb{P}_x\{\tau_0 < \infty, A_{\tau_0}\} + \mathbb{P}_x\{A^c_{\tau_0}\}. \tag{81}$$

Using now Lemma 14 with $B_1 = ... = B_{n-1} = [0, \infty)$ and $B_n = (-\infty, 0)$ we obtain

$$\mathbb{P}_x\{\tau_0 = n, A_n\} = \mathbb{E}_x\left\{\prod_{k=0}^{n-1} g(\widetilde{R}_k);\ \widetilde{\tau}_0 = n\right\} \leq \mathbb{P}_x\{\widetilde{\tau}_0 = n\}, \quad n \geq 1,$$

where

$$\widetilde{\tau}_0 := \inf\{n \geq 1 :\ \widetilde{R}_n < 0\}.$$

This implies that

$$\mathbb{P}_x\{\tau_0 < \infty, A_{\tau_0}\} \leq \mathbb{P}_x\{\widetilde{\tau}_0 < \infty\} = \widetilde{\psi}(x). \tag{82}$$

To bound the second probability term on the right hand side of (81), we firstly apply the total probability law twice

$$\begin{aligned}
\mathbb{P}_x\{A^c_{\tau_0}\} &= \mathbb{P}_x\{\xi(R_k) < -R_k/2 \text{ for some } k < \tau_0\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}_x\{A_n, \tau_0 > n, \xi(R_n) < -R_n/2\} \\
&= \sum_{n=0}^{\infty} \int_0^\infty \mathbb{P}_x\{R_n \in dy, A_n, \tau_0 > n\}\mathbb{P}\{\xi(y) < -y/2\},
\end{aligned}$$

and then we apply again Lemma 14 to the probability on the right hand side:

$$\begin{aligned}
\mathbb{P}_x\{A^c_{\tau_0}\} &\leq \sum_{n=0}^{\infty} \int_0^\infty \mathbb{P}_x\{\widetilde{R}_n \in dy\}(1 - g(y)) \\
&= \int_0^\infty (1 - g(y))\widetilde{H}_x(dy).
\end{aligned}$$

Plugging now this bound and (82) into (81), we get the desired upper bound. $\square$

In order to get an upper bound for $\psi(x)$ we need upper bounds for both terms on the right hand side of (80). It turns out that $\widetilde{\psi}(x)$ can be estimated by the method used in the proof of Theorem 5.

Lemma 16. *Assume that $\mathbb{E}\tau^2\log(1+\tau)$ and $\mathbb{E}\zeta^2\log(1+\zeta)$ are finite. Then there exists a constant C such that*

$$\mathbb{P}_x\{\widetilde{R}_n \le y \text{ for some } n \ge 1\} \le C\frac{1+y^\rho}{x^\rho} \quad \text{for all } 0<y<x. \tag{83}$$

In particular,

$$\widetilde{\psi}(x) \le \frac{C}{x^\rho} \quad \text{for all } x>0.$$

Proof. Let $U_-(x)$ be the function defined in the proof of Theorem 5. By the definition of $\widetilde{\xi}(x)$,

$$\begin{aligned}
\mathbb{E}U_-(x+\widetilde{\xi}(x)) - U_-(x) &= \frac{1}{g(x)}\mathbb{E}\{U_-(x+\xi(x)) - U_-(x);\ \xi(x) \ge -x/2\} \\
&= \frac{1}{g(x)}\mathbb{E}\{U_-(x+\xi(x)) - U_-(x);\ |\xi(x)| \le x/2\} \\
&\quad + \frac{1}{g(x)}\mathbb{E}\{U_-(x+\xi(x)) - U_-(x);\ \xi(x) > x/2\}.
\end{aligned}$$

Since the estimates (24) and (25) are valid under the conditions of the present lemma, we may apply (44) and (51) to get

$$\mathbb{E}U_-(x+\widetilde{\xi}(x)) - U_-(x) = -\frac{1+o(1)}{g(x)}p(x)e^{-Q_-(x)}.$$

Therefore, there exists $\widehat{x}$ such that $U_-(\widetilde{R}_{n\wedge\tau_B})$ is a bounded supermartingale, where $B=(-\infty,\widehat{x}]$. Then, applying the optional stopping theorem, we get the desired upper bound. $\square$

We now turn to the second term in (80). Firstly we state the following upper bounds for the renewal function.

Lemma 17. *The following bounds hold true:*

$$\widetilde{H}_x(0,y] \le C(1+y^2) \quad \text{for all } x,y>0 \tag{84}$$

and

$$\widetilde{H}_x(0,y] \le C\frac{1+y^{2+\rho}}{x^\rho} \quad \text{for all } 0<y<x. \tag{85}$$

Proof. Firstly note that

$$\mathbb{P}\{\widetilde{\xi}(u) < -u/2\} = 0.$$

Next, by using Lemma 3, we conclude that

$$\mathbb{E}\widetilde{\xi}(u) \sim \frac{\theta \mathbb{E}\tau}{u}$$

and

$$\mathbb{E}\widetilde{\xi}^2(u) \to \mathbb{V}\mathrm{ar}\,\zeta + v_c^2 \mathbb{V}\mathrm{ar}\,\tau \quad \text{as } u \to \infty.$$

Using these estimates one can easily see that all conditions of Lemma 4 in [8] are met. This implies (84). To prove (85) it suffices to notice that

$$\widetilde{H}_x(0,y] \le \mathbb{P}_x\{\widetilde{R}_n \le y \text{ for some } n \ge 1\} \sup_{u \le y} \widetilde{H}_u(0,y]$$

and to apply (83) and (84). □

Now we are ready to bound the second term in (80). Since $\xi(x) \ge_{st} -\zeta$,

$$\int_0^\infty (1-g(y))\widetilde{H}_x(dy) \quad \le \quad \int_0^\infty \mathbb{P}\{\zeta > y/2\}\widetilde{H}_x(dy). \tag{86}$$

Fubini's theorem implies that

$$\begin{aligned}
\int_0^\infty \mathbb{P}\{\zeta > y/2\}\widetilde{H}_x(dy) &= \int_0^\infty \widetilde{H}_x(dy)\int_{y+0}^\infty \mathbb{P}\{2\zeta \in du\} \\
&= \int_0^\infty \mathbb{P}\{2\zeta \in du\}\int_0^{u-0} \widetilde{H}_x(dy) \\
&= \int_0^\infty \widetilde{H}_x(0,u)\mathbb{P}\{2\zeta \in du\}.
\end{aligned}$$

Next, by Lemma 17,

$$\begin{aligned}
\int_0^\infty \widetilde{H}_x(0,y]\mathbb{P}\{2\zeta \in dy\} &\le C\int_0^x \frac{1+y^{2+\rho}}{x^\rho}\mathbb{P}\{2\zeta \in dy\} + C\int_x^\infty (1+y^2)\mathbb{P}\{2\zeta \in dy\} \\
&\le C_1(1/x^\rho + x^2\mathbb{P}\{\zeta \ge x/2\}),
\end{aligned}$$

owing to the regular variation of the distribution of ζ and Karamata's theorem. Since $\beta < \rho$, we conclude that

$$\int_0^\infty \widetilde{H}_x(0,y]\mathbb{P}\{2\zeta \in dy\} \quad \le \quad C_2 x^2 \mathbb{P}\{\zeta > x\}.$$

Together with (86) it yields that

$$\int_0^\infty (1-g(y))\widetilde{H}_x(dy) \quad \le \quad C_2 x^2 \mathbb{P}\{\zeta > x\}.$$

This estimate and Lemma 16 imply the desired upper bound.

7.2 Proof of the lower bound

To start with, we notice that, for all $x > 0$ and $N \geq 1$,

$$\begin{aligned}
\psi(x) &= \sum_{n=0}^{\infty} \int_0^{\infty} \mathbb{P}_x\{R_n \in dy,\ R_k \geq 0 \text{ for all } k \leq n\} \mathbb{P}\{\xi(y) < -y\} \\
&\geq \inf_{y \in [x/2, 2x]} \mathbb{P}\{\xi(y) < -y\} \sum_{n=0}^{N} \mathbb{P}_x\{R_k \in [x/2, 2x] \text{ for all } k \leq n\} \\
&\geq \inf_{y \in [x/2, 2x]} \mathbb{P}\{\xi(y) < -y\} N \mathbb{P}_x\{R_k \in [x/2, 2x] \text{ for all } k \leq N\}.
\end{aligned}$$

Due to Lemma 13,

$$\begin{aligned}
\inf_{y \in [x/2, 2x]} \mathbb{P}\{\xi(y) < -y\} &\geq \inf_{y \in [x/2, 2x]} \mathbb{P}\{\xi(y) < -2x\} \\
&\sim \mathbb{P}\{\zeta > 2x\} \quad \text{as } x \to \infty.
\end{aligned}$$

Consequently, putting $N = \delta x^2$, we get the following lower bound

$$\psi(x) \geq c\mathbb{P}\{\zeta > x\} \delta x^2 \mathbb{P}_x\{R_k \in [x/2, 2x] \text{ for all } k \leq \delta x^2\},$$

for every $\delta > 0$. Thus, it only remains to show that we can choose a $\delta > 0$ so small that the probability on the right hand side is bounded away from zero.

We start by stating the following decomposition

$$\begin{aligned}
&\mathbb{P}_x\{R_k \notin [x/2, 2x] \text{ for some } k \leq \delta x^2\} \\
&\quad \leq \mathbb{P}_x\{R_k \leq x/2 \text{ for some } k \geq 1\} + \mathbb{P}_x\Big\{\max_{k \leq \delta x^2} R_k > 2x, R_n \geq x/2 \text{ for all } n \geq 1\Big\}.
\end{aligned} \tag{87}$$

It follows from Lemma 3 that for every $\varepsilon < \rho$ there exists an x_0 such that

$$\frac{2x m_1(x)}{m_2(x)} \geq 1 + \varepsilon \quad \text{for all } x \geq x_0.$$

Noting that

$$\mathbb{P}\{\xi(x) \leq -\gamma x\} \leq \mathbb{P}\{\zeta > \gamma x\} = o(1/x^{2+\beta_0})$$

for every $\beta_0 < \beta$, we infer that all the conditions of Lemma 1 in [8] hold true and, consequently, there exists an x_0 such that

$$\mathbb{P}_x\{R_n \leq z \text{ for some } n \geq 1\} \leq (z/x)^{\beta_0} \quad \text{for all } x > z > x_0.$$

In particular,

$$\mathbb{P}_x\{R_k \leq x/2 \text{ for some } k \geq 1\} \leq 1/2^{\beta_0} \quad \text{for all } x > 2x_0. \tag{88}$$

To bound the second probability on the right hand side of (87) we introduce a martingale

$$M_k := R_k - R_0 - \sum_{j=0}^{k-1} m_1(R_j), \quad k \geq 0.$$

Due to Lemma 3, we may assume that x_0 is so large that $ym_1(y) \le 2\theta\mathbb{E}\tau$ for all $y \ge x_0$. This implies that, for $R_0 = x$,

$$\max_{k\le\delta x^2} R_k \quad\le\quad x + \max_{k\le\delta x^2} M_k + 4\theta\delta x\mathbb{E}\tau$$

on the event $\{R_k \ge x/2$ for all $k \ge 1\}$. Consequently,

$$\mathbb{P}_x\Big\{\max_{k\le\delta x^2} R_k > 2x, R_k \ge x/2 \text{ for all } k \ge 1\Big\} \quad\le\quad \mathbb{P}_x\Big\{\max_{k\le\delta x^2} M_k > (1-c_1\delta)x\Big\},$$

where $c_1 := 4\theta\mathbb{E}\tau$. Applying the Doob inequality to the right hand side and noting that $\mathbb{E}_x M_k^2 \le c_2 k$ for all k and x, we obtain

$$\mathbb{P}_x\Big\{\max_{k\le\delta x^2} R_k > 2x, R_k \ge x/2 \text{ for all } k \ge 1\Big\} \quad\le\quad \frac{c_2\delta}{(1-c_1\delta)^2}.$$

Plugging this estimate and (88) into (87), we conclude that

$$\mathbb{P}_x\{R_k \notin [x/2, 2x] \text{ for some } k \le \delta x^2\} \quad\le\quad \frac{1}{2^{\beta_0}} + \frac{c_2\delta}{(1-c_1\delta)^2} \quad \text{for all } x \ge 2x_0.$$

Choosing $\delta > 0$ sufficiently small, we can make the right hand side less than 1, hence

$$\inf_{x\ge 2x_0} \mathbb{P}_x\{R_k \in [x/2, 2x] \text{ for all } k \le \delta x^2\} \quad>\quad 0.$$

This completes the proof of the lower bound.

8 Appendix

Lemma 18. *Let $\alpha \in (0,1]$ and $\gamma \ge \alpha$. Let a family of positive random variables $\{\xi_\theta,\ \theta \in \Theta\}$ possess a majorant Ξ, $\xi_\theta \ \le_{st}\ \Xi$ for all $\theta \in \Theta$, such that $\mathbb{E}\Xi^{\gamma+1-\alpha} < \infty$. Then there exists a decreasing integrable at infinity function $p(x)$ such that*

$$\sup_{\theta\in\Theta} \mathbb{E}\{\xi_\theta^{\gamma+1};\ \xi_\theta \le x\} = o(x^{1+\alpha}p(x)) \quad \text{as } x \to \infty.$$

Proof. Integration by parts yields that

$$\begin{aligned}
\mathbb{E}\{\xi_\theta^{\gamma+1};\ \xi_\theta \le x\} &= -\int_0^x y^{\gamma+1} d\mathbb{P}\{\xi_\theta > y\} \\
&= -x^{\gamma+1}\mathbb{P}\{\xi_\theta > x\} + (\gamma+1)\int_0^x y^\gamma \mathbb{P}\{\xi_\theta > y\}dy \\
&\le (\gamma+1)\int_0^x y^\gamma \mathbb{P}\{\Xi > y\}dy,
\end{aligned}$$

by the majorisation condition. Therefore, by the Markov inequality,

$$\begin{aligned}
\mathbb{E}\{\xi_\theta^{\gamma+1};\ \xi_\theta \le x\} &\le (\gamma+1)\int_0^x y^\alpha \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\}dy \\
&= (\gamma+1)x^{1+\alpha}p(x),
\end{aligned}$$

where

$$p(x) \;:=\; \frac{1}{x^{1+\alpha}} \int_0^x y^\alpha \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\} dy.$$

The finiteness of $\mathbb{E}\Xi^{\gamma+1-\alpha}$ implies integrability at infinity of $p(x)$. Indeed,

$$\begin{aligned}
\int_0^\infty p(x)dx &= \int_0^\infty \frac{dx}{x^{1+\alpha}} \int_0^x y^\alpha \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\} dy \\
&= \int_0^\infty y^\alpha \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\} dy \int_y^\infty \frac{dx}{x^{1+\alpha}} \\
&= \frac{1}{\alpha} \int_0^\infty \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\} dy \\
&= \frac{\mathbb{E}\Xi^{\gamma+1-\alpha}}{\alpha} \;<\; \infty,
\end{aligned}$$

by the moment condition on Ξ. In addition, the function $p(x)$ is decreasing because

$$\begin{aligned}
&\frac{d}{dx} \frac{1}{x^{1+\alpha}} \int_0^x y^\alpha \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\} dy \\
&\quad = -\frac{1+\alpha}{x^{2+\alpha}} \int_0^x y^\alpha \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > y\} dy + \frac{1}{x} \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > x\} \\
&\quad \le -\frac{1+\alpha}{x^{2+\alpha}} \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > x\} \int_0^x y^\alpha dy + \frac{1}{x} \mathbb{E}\{\Xi^{\gamma-\alpha};\ \Xi > x\} \\
&\quad = 0.
\end{aligned}$$

The proof is complete due to the next Lemma 19. □

Lemma 19. *Let $p(x) > 0$ be a decreasing function which is integrable at infinity. Then there exists a decreasing integrable at infinity function $p_1(x) > 0$ such that $p_1(x)/p(x) \to \infty$ as $x \to \infty$.*

Proof. Without loss of generality we assume that p is a left-continuous function. Since $p(x)$ is integrable at infinity, there exists an increasing sequence $x_k \to \infty$, $k \ge 0$, such that $x_0 = 0$ and

$$\int_{x_k}^\infty p(y)dy \;\le\; 1/k^3 \quad \text{for all } k \ge 1.$$

Since $p(x)$ decreases, a sequence x_k may be chosen in such a way that

$$(k+2)p(x_{k+1}) \;<\; (k+1)p(x_k) \quad \text{for all } k \ge 1,$$

Due to this condition the following sequence y_k such that $x_k < y_k < x_{k+1}$ for all k is well-defined:

$$y_k \;:=\; \sup\{x \ge x_k : (k+1)p(x) \ge (k+2)p(x_{k+1})\}.$$

Define a function $p_1(x)$ as follows:

$$p_1(x) \;:=\; \begin{cases} (k+1)p(x) & \text{for } x \in [x_k, y_k], \\ (k+2)p(x_{k+2}) & \text{for } x \in (y_k, y_{k+1}], \end{cases}$$

which is decreasing by construction. Since

$$p_1(x) \quad \geq \quad (k+1)p(x) \quad \text{for all } x \in [x_k, x_{k+1}],$$

the function $p_1(x)$ satisfies the condition $p_1(x)/p(x) \to \infty$ as $x \to \infty$. Lastly, its integral may be bounded as follows:

$$\begin{aligned}
\int_{x_1}^{\infty} p_1(x)dx &= \sum_{k=1}^{\infty} \int_{x_k}^{x_{k+1}} p_1(x)dx \\
&= \sum_{k=1}^{\infty}(k+1)\int_{x_k}^{y_k} p(x)dx + \sum_{k=1}^{\infty}(k+2)p(x_{k+1})(x_{k+1}-y_k) \\
&\leq \sum_{k=1}^{\infty}(k+2)\int_{x_k}^{x_{k+1}} p(x)dx \\
&\leq \sum_{k=1}^{\infty}\frac{k+2}{k^3} \;<\; \infty,
\end{aligned}$$

where the last bound is due to the choice of x_k, which completes the proof. □

Lemma 20 (Denisov [7]). *Let $p(x) > 0$ be a decreasing function which is integrable at infinity. Then there exists a decreasing integrable at infinity function $p_1(x) > 0$ which dominates $p(x)$ and is regularly varying at infinity with index -1.*

Lemma 21. *Let $\xi \geq 0$ be a random variable and let $V(x) \geq 0$ be an increasing function such that $\mathbb{E}V(\xi) < \infty$. Let $U(x) \geq 0$ be a function such that the function $f(x) := V(x)/xU(x)$ increases and satisfies the condition*

$$\sup_{x>1}\frac{f(2x)}{f(x)} \quad < \quad \infty. \tag{89}$$

Then there exists an increasing function $s(x) \to \infty$ of order $o(x)$ such that

$$\mathbb{E}\{U(\xi);\ \xi > s(x)\} \quad = \quad o(p(x)xU(x)/V(x)) \quad \text{as } x \to \infty,$$

where $p(x)$ is a decreasing integrable at infinity function which is only determined by ξ and $V(x)$.

Proof. Since $\mathbb{E}V(\xi) < \infty$, the decreasing function

$$p_1(x) \;:=\; \mathbb{E}\{V(\xi)/\xi;\ \xi > x\}$$

is integrable at infinity. Then by Lemmas 19 and 20,

$$\mathbb{E}\{V(\xi)/\xi;\ \xi > x\} \;=\; o(p(x)) \quad \text{as } x \to \infty,$$

where a decreasing function $p(x)$ is integrable and regularly varying at infinity with index -1. Hence, due to the increase of $V(x)/xU(x)$,

$$\begin{aligned}
\mathbb{E}\{U(\xi);\ \xi > x\} &= \mathbb{E}\Big\{\frac{U(\xi)\xi}{V(\xi)}V(\xi)/\xi;\ \xi > x\Big\} \\
&\le \frac{\mathbb{E}\{V(\xi)/\xi;\ \xi > x\}}{V(x)/xU(x)} \\
&= o(p(x)xU(x)/V(x)) \quad \text{as } x \to \infty.
\end{aligned}$$

Therefore, for any $n \in \mathbb{N}$,

$$\mathbb{E}\{U(\xi);\ \xi > x/n\} = o(p(x)xU(x)/V(x)) \quad \text{as } x \to \infty$$

because the function $p(x)$ is regularly varying at infinity and owing to (89). Hence, there exists an increasing sequence $x_n \to \infty$ such that

$$\mathbb{E}\{U(\xi);\ \xi > x/n\} \le p(x)xU(x)/nV(x) \quad \text{for all } x \ge x_n.$$

Then the level function $s(x) = \frac{x}{n}\mathbb{I}\{x \in (x_n, x_{n+1}]\}$ is of order $o(x)$ and delivers the stated result. □

Lemma 22. *Let $\xi \ge 0$ be a random variable with finite γth moment for some $\gamma \in [1, \infty)$. Let $\alpha \in [1/\gamma, 1]$. Then, for all $\beta \in [0, \gamma - 1/\alpha]$, there exists an increasing function $s(x) \to \infty$ of order $o(x^\alpha)$ such that*

$$\mathbb{E}\{\xi^\beta;\ \xi > s(x)\} = o(p(x)/x^{\alpha(\gamma-\beta)-1}) \quad \text{as } x \to \infty,$$

where $p(x)$ is a decreasing integrable at infinity function which is only determined by ξ, γ, and α.

Proof. Put $\eta = \xi^{1/\alpha}$ and $V(x) = x^{\alpha\gamma}$. As follows from Lemma 21 with $U(x) = x^{\alpha\beta}$, since $\mathbb{E}\xi^\gamma = \mathbb{E}V(\eta) < \infty$, there exists a regularly varying at infinity with index -1 function $p(x)$ which is integrable at infinity and a function $s(x) = o(x)$ such that

$$\begin{aligned}
\mathbb{E}\{\eta^{\alpha\beta};\ \eta > s(x)\} &= o(p(x)xU(x)/V(x)) \\
&= o(p(x)/x^{\alpha(\gamma-\beta)-1}) \quad \text{as } x \to \infty,
\end{aligned}$$

which can be rewritten as

$$\mathbb{E}\{\xi^\beta;\ \xi > s^\alpha(x)\} = o(p(x)/x^{\alpha(\gamma-\beta)-1}) \quad \text{as } x \to \infty,$$

and the proof is complete. □

Lemma 23. *Let $\xi \ge 0$ be a random variable and let $V(x)$ be a non-negative function such that $\mathbb{E}V(\xi)\log(1+\xi) < \infty$. Then there exists an increasing function $s(x) \to \infty$ of order $o(x)$ such that,*

$$\mathbb{E}\{V(\xi);\ \xi > s(x)\} = o(p(x)x) \quad \text{as } x \to \infty,$$

where $p(x)$ is a decreasing integrable at infinity function.

Proof. It follows almost immediately because

$$\begin{aligned}\int_1^\infty \frac{\mathbb{E}\{V(\xi);\ \xi > x\}}{x} dx &= \int_1^\infty \frac{dx}{x} \int_x^\infty V(y)\mathbb{P}\{\xi \in dy\} \\ &= \int_1^\infty V(y)\mathbb{P}\{\xi \in dy\} \int_1^y \frac{dx}{x} \\ &= \int_1^\infty V(y)(\log y)\mathbb{P}\{\xi \in dy\} \ < \ \infty.\end{aligned}$$

Hence, by Lemmas 19 and 20,

$$\mathbb{E}\{V(\xi);\ \xi > x\} \ = \ o(p(x)x) \quad \text{as } x \to \infty,$$

where a decreasing function $p(x)$ is integrable and regularly varying at infinity with index -1. Then concluding arguments as in Lemma 21 complete the proof. $\square$

Lemma 24. *Let $p(x) > 0$ be a decreasing function which is integrable at infinity. Then, for any $k \ge 1$, there exists a decreasing integrable at infinity function $p_k(x) \ge p(x)$ such that it is k times differentiable and, for all $j \le k$,*

$$\frac{d^j}{dx^j} p_k(x) \ = \ O(1/x^{1+j}) \quad \textit{as } x \to \infty.$$

Proof. Consider a decreasing function $p_k(x)$ defined by the equality

$$p_k(x) \ := \ 2^k \int_{x/2}^\infty dy_k \int_{y_k/2}^\infty dy_{k-1} \ldots \int_{y_3/2}^\infty dy_2 \int_{y_2/2}^\infty \frac{p(y_1)}{y_1^k} dy_1.$$

Firstly, since the function $p(x)/x^k$ decreases,

$$\int_{y_2/2}^\infty \frac{p(y_1)}{y_1^k} dy_1 \ \ge \ \int_{y_2/2}^{y_2} \frac{p(y_1)}{y_1^k} dy_1 \ \ge \ \frac{y_2}{2} \frac{p(y_2)}{y_2^k} \ = \ \frac{1}{2} \frac{p(y_2)}{y_2^{k-1}},$$

so repetition of this lower bound eventually leads to the inequalities

$$p_k(x) \ \ge \ 2^k \int_{x/2}^x \frac{1}{2^{k-1}} \frac{p(y_k)}{y_k} dy_k \ \ge \ 2^k \frac{x}{2} \frac{1}{2^{k-1}} \frac{p(x)}{x} \ = \ p(x).$$

Secondly, $p_k(x)$ is integrable at infinity because

$$\int_{y_2/2}^\infty \frac{p(y_1)}{y_1^k} dy_1 \ \le \ p(y_2/2) \int_{y_2/2}^\infty \frac{1}{y_1^k} dy_1 \ = \ O\Big(\frac{p(y_2/2)}{y_2^{k-1}}\Big),$$

and hence after $k-1$ steps we arrive at upper bound

$$p_k(x) \ \le \ c \int_{x/2}^\infty \frac{p(y_k/2^{k-1})}{y_k} dy_k, \quad c < \infty,$$

where the integral on the right hand side is integrable with respect to x, since

$$\begin{aligned}\int_0^\infty dx \int_{x/2}^\infty \frac{p(y/2^{k-1})}{y} dy &= \int_0^\infty \frac{p(y/2^{k-1})}{y} dy \int_0^{2y} dx \\ &= 2\int_0^\infty p(y/2^{k-1}) dy \ < \ \infty.\end{aligned}$$

Thirdly,

$$\begin{aligned}\frac{d^k}{dx^k} p_k(x) &= -\frac{2^k}{2}\frac{d^{k-1}}{dx^{k-1}} \int_{x/4}^\infty dy_{k-1} \dots \int_{y_3/2}^\infty dy_2 \int_{y_2/2}^\infty \frac{p(y_1)}{y_1^k} dy_1 \\ \dots \\ &= (-1)^k \frac{2^k}{2\cdot 4 \cdot \ldots \cdot 2^k} \frac{p(x/2^k)}{(x/2^k)^k} \ = \ O(p(x/2^k)/x^k) \quad \text{as } x \to \infty.\end{aligned}$$

Since $p(x)$ is decreasing and integrable at infinity, $p(x) = O(1/x)$ as $x \to \infty$, so $p_k^{(k)}(x) = O(1/x^{1+k})$. Integrating the kth derivative $k-j$ times we get that the jth derivative of $p_k(x)$ is not greater than $(k-j)$th integral of c/x^{1+k} which is of order $O(1/x^{1+j})$. This completes the proof. □

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