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Intermittent phase dynamics of non-autonomous oscillators through time-varying phase

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Abstract

Oscillatory dynamics pervades the universe, appearing in systems of all scales. Whilst autonomous oscillatory dynamics has been extensively studied and is well understood, the very important problem of non-autonomous oscillatory dynamics is less well understood. Here, we provide a framework for non-autonomous oscillatory dynamics, within which we can define intermittent phenomena such as intermittent phase synchronisation. Moreover, we demonstrate this framework with a coupled pair of non-autonomous phase oscillators as well as a higher-dimensional system comprising of two interacting phase-oscillator networks.

Keywords: Oscillatory dynamics, Phase of non-autonomous oscillators, Instantaneous phase, Intermittent synchronization, Coupled non-autonomous oscillators
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1. Introduction

Oscillators are ubiquitous in our universe, with oscillatory dynamics appearing in systems ranging across scales and disciplines [1–14]. Mathematically, one way to formulate the concept of an oscillator is as an autonomous dynamical system possessing an attracting limit cycle [15, 16]; another formulation is simply as a differential equation on a one-dimensional circular state space with non-zero constant right-hand side, where the states in this state space are called *phases*. One can transform from the former to the latter by *phase-reduction*, and one can study oscillators or networks of oscillators in terms of the phase dynamics [3, 17]. The dynamics of autonomous phase oscillators and phase-oscillator networks has seen useful application to a wide variety of systems, such as arrays of Josephson junctions [15, 16], the swaying of footbridges by large crowds [18], and collections of fireflies [15], to name but a few.

However, this autonomous-dynamics description assumes that the oscillator or oscillator-network is not subject to time-dependent external influences, i.e. that it is effectively a "thermodynamically isolated" system (or, more precisely, that the oscillating system together with its power source and its receptor of dissipated energy can, as a whole, be treated as a thermodynamically isolated system). There are many systems exhibiting oscillatory dynamics, particularly living systems such as cells, that are *thermodynamically open*—i.e. continually exchanging matter and energy with their environment. For such systems, a description in terms of autonomous dynamical systems is inappropriate [19–21]. One common way to represent the effect of external influences upon a system is in terms of *noise*, where the net forcing from external influences is not described deterministically but is regarded as statistically approximable by a stochastic process [22–25]. Accordingly, theory of *stochastic oscillators* has been developed in [26–28]. However, modelling external influences as noise is simply not realistic for many systems, again notably including living systems. An alternative framework for studying open systems is that of *non-autonomous* dynamical systems [10, 20, 29, 30]. Important dynamical phenomena can occur in non-autonomous models, that

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are directly connected with the model's being non-autonomous, such as temporally intermittent synchrony between the phases of two oscillators [31], and temporally intermittent order among the phases of a large network of oscillators [32].

Some work on the important topic of deterministic non-autonomous oscillatory dynamics has been carried out, but relatively little compared to the amount that exists on autonomous oscillatory dynamics. In [33], the concept of *chronotaxic systems* was introduced to represent non-autonomous systems exhibiting oscillatory behaviour with a phase-stability that is not possible for autonomous systems. This was defined in terms of infinite-time dynamics, particularly the tracking of a pullback-attracting trajectory. In [31], through the example of non-autonomous Adler equations, the idea was expanded to allow for phase-stability induced by intermittent synchronisation, and did not rely on any infinite time-horizon. The setup was then generalised, and the nature of this stabilisation explored in more depth from a mathematically rigorous perspective, in [34]. The intermittent phase-synchronisation of [31] was extended to networks in [35].

Still, while definitions of an oscillator exist in the autonomous context and the stochastic context, there does not seem yet to exist a broadly-encompassing definition of a non-autonomous oscillator. A common approach to non-autonomous dynamics is to take an autonomous system and consider the effect that adding a non-autonomous perturbation has on the dynamics. However, we will suggest (in Sec. 2.2) that defining a non-autonous oscillator as the result of adding a non-autonomous perturbation to an autonomous limit-cycle oscillator is not an appropriate approach for systems such as biological oscillatory systems.

In this paper, we propose, with appropriate physical justification, a definition of a non-autonomous oscillator intended particularly for oscillatory systems whose external influences do not vary too rapidly in time; namely, we propose the general setup of [36] (except without the "fast term" $\boldsymbol{p}(t)$) applied to explicitly finite time-intervals as the appropriate definition. We similarly propose a definition of a *non-autonomous phase oscillator*. We propose a definition of a non-autonomous oscillator's *non-autonomous phase*, by which a link is provided from non-autonomous oscillators to non-autonomous phase-oscillators.

Within our framework of non-autonomous oscillatory dynamics, we define *synchronisation* of non-autonomous phase oscillators, and consider the phenomenon of *intermittent synchronisation* of a pair of oscillators observed and discussed in [31]; for this intermittency it is required that the system be non-autonomous. We then consider a type of system involving interaction between two networks of non-autonomous phase oscillators, and define synchronisation between the *macroscopic phases* of these two networks and consider intermittency of this synchronisation.

The structure of the paper is as follows: In Section 2, we review limit-cycle oscillators and phase oscillators in the classical autonomous setting, and define non-autonomous oscillators and non-autonomous phase oscillators. In Section 3, we define synchronisation of non-autonomous phase oscillators. In Section 4, we apply our non-autonomous framework to a coupled pair of oscillators, with particular focus on the phenomenon of intermittent synchronisation. In Section 5, we consider a coupled pair of non-autonomous phase-oscillator networks; we define synchronisation of the macroscopic phases, and numerically consider the phenomenon of intermittency of such synchronisation. Finally, Section 6 provides a summary of the results and perspective for further development.

2. Oscillators and phase

An oscillatory process can be modelled at varying levels of complexity—depending on the level of physical quantitative resolution deemed appropriate for theoretical and/or numerical investigation of the dynamical behaviours of interest—such as:

- one-dimensional models (*phase oscillators*), where the state variable is simply an angle (called the *phase*) describing "how far along the cycle" the oscillator is;
- finite-dimensional models, such as ordinary differential equations whose state variables represent a finite number of instantaneous positions and velocities;
- infinite-dimensional models, such as partial differential equations and delay differential equations.

The extraction of phases in a higher-than-one-dimensional model can sometimes allow one to reduce the model to a one-dimensional approximation [17]; and in the converse direction, for some oscillatory processes, especially *biological* oscillators, it may be appropriate to understand the physics in terms of the oscillator's being designed according to the optimal phase dynamics and then being built in a higher-dimensional space so as to approximately reflect that phase dynamics. For example, in the case of a system of interacting biological oscillatory processes [37–39], each coordinate of the state space could be an angle that directly represents the physiologically functional state of a biological oscillator, as opposed to being extracted by phase-reduction from a higher-dimensional model built directly from approximation of the material mechanics of the system. (This point is particularly relevant for non-autonomous oscillators, since biological and other oscillators or oscillator-networks that interact with their environment are more appropriately modelled as non-autonomous than as autonomous.)

We will now review the basic concepts of oscillatory dynamics in the autonomous setting, and then consider extension to the non-autonomous setting.

2.1. The classical autonomous case

2.1.1. Limit-cycle oscillators

For simplicity, we just consider finite-dimensional smooth differential equations. An *oscillator*, or *limit-cycle oscillator*, is a smooth ordinary differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{1}$$

periodically forced, almost-periodically forced [29], stationary-noise-forced [41], or asymptotically autonomous [42] systems—oscillatory processes such as biological oscillators will typically be subject to more freely time-dependent external influences that do not follow any particular infinite-time deterministic or statistical pattern. Thus, for such systems, dynamics is better described in terms of explicitly finite-time behaviour [43, 44]. Hence, long-time-asymptotic non-autonomous concepts such as forward-attractors and pullback-attractors [29] do not comprise the appropriate framework within which to describe non-autonomous oscillation for such systems.

So then, there is a natural opening for the provision of notions of "non-autonomous oscillators" that are not formulated with reference to an "unperturbed" time-independent oscillatory state, and that are explicitly defined with reference to consideration of finite-time behaviour.

In this paper, we will specifically concern ourselves with the scenario that external forces do not change rapidly in time. As in the autonomous case, we will only consider smooth dynamical systems. Given a smooth non-autonomous ordinary differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), t), \tag{7}$$

for any given time τ we can define the corresponding τ-*fibre* [29] to be the autonomous ordinary differential equation

$$\frac{d}{ds}\mathbf{x}(s) = \mathbf{F}(\mathbf{x}(s), \tau), \tag{8}$$

where, to avoid confusion, we will use the symbol s for the time-variable with respect to which the autonomous differential equation (8) is defined for fixed τ.

2.2.1. Non-autonomous oscillators

We will regard Eq. (7) as a *non-autonomous oscillator* on a time-interval $(\alpha, \beta) \subset \mathbb{R}$ if (cf. [36])

(a) for each $\tau \in (\alpha, \beta)$, the τ-fibre (8) is an autonomous limit-cycle oscillator, with a stable limit cycle $\mathcal{X}_\tau$ that depends continuously on τ (and hence the τ-fibre's set of isochrons of $\mathcal{X}_\tau$ also varies continuously with τ [45]);

(b) the variation of $\mathbf{F}(\,\cdot\,, t)$ with respect to t takes place on a slower timescale than the dynamics of the individual τ-fibres for $\tau \in (\alpha, \beta)$.

So the time-indexed family of autonomous limit cycles forms a "tube" $\mathcal{X}_{(\alpha,\beta)} := \bigcup_{\tau\in(\alpha,\beta)}(\{\tau\} \times \mathcal{X}_\tau)$ (depicted in Fig. 1), and the slow t-dependence of $\mathbf{F}(\,\cdot\,, t)$ implies that, intuitively speaking, solutions of (7) over the time-interval (α, β) that start near this tube will track it over time. More precisely, if a solution $\mathbf{x}(t)$ of (7) has that at some time $t_0 \in (\alpha, \beta)$, $\mathbf{x}(t_0)$ is in the basin of attraction of $\mathcal{X}_{t_0}$ under the t_0-fibre, then at every time $t < \beta$ subsequent to some initial transient time-period following the initial time t_0, the solution $\mathbf{x}(t)$ will lie very close to $\mathcal{X}_t$.

As in the autonomous case, we can also consider networks of non-autonomous oscillators, i.e. dynamical systems of the form

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}_i(\mathbf{x}_i(t), t) + \mathbf{G}_i(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t), t), \quad i = 1, \ldots, N, \tag{9}$$

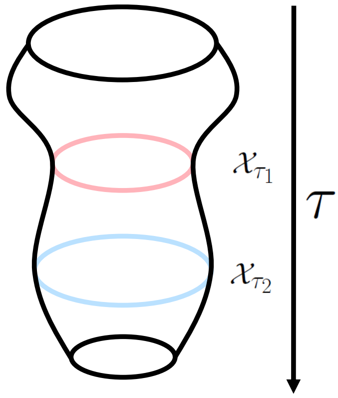

Figure 1: An illustration of the "tube" of autonomous limit cycles $\mathcal{X}_{(\alpha,\beta)}$, with two such limit cycles $\mathcal{X}_{\tau_1}$ and $\mathcal{X}_{\tau_2}$ highlighted in red and blue, respectively.

defined on a time-interval (α, β), where the "uncoupled" non-autonomous dynamical systems $\dot{\mathbf{x}}_i(t) = \mathbf{F}_i(\mathbf{x}_i(t), t)$ each fulfil the definition of a non-autonomous oscillator, and the coupling functions $\mathbf{G}_i$ also have slow-timescale dependence on their t-input.

2.2.2. Non-autonomous phase oscillators

Non-autonomous phase oscillators are the slowly time-dependent generalisation of the time-independent definition of a phase oscillator given in Sec. 2.1.2. In essence, a non-autonomous phase oscillator is a slowly time-dependent differential equation on the circle (again, understood as the set of all angles) whose solutions move round the circle with a progression of phase that is approximately proportional to the progression of time within timescales comparable to those of the autonomous fibres.

Namely, we define a *non-autonomous phase oscillator* on a time-interval $(\alpha, \beta) \subset \mathbb{R}$ as a smooth non-autonomous differential equation of the form

$$\dot{\theta}(t) = \omega(t) + \xi(\theta(t), t) \tag{10}$$

defined over $t \in (\alpha, \beta)$, where

(a) for each $\tau \in (\alpha, \beta)$, $\omega(\tau) > 0$ and $|\xi(\theta, \tau)|$ takes only small values compared to $\omega(\tau)$;

(b) the variation of $\omega(t)$ and $\xi(\,\cdot\,, t)$ with respect to t takes place on a slower timescale than the approximate periodicities $T(\tau) = \frac{2\pi}{\omega(\tau)}$ of the individual τ-fibres for $\tau \in (\alpha, \beta)$.

Here, $\omega(\tau)$ represents the approximate time-localised angular frequency of the oscillatory process locally around time τ.

Let us make some further comments on the above definition of a non-autonomous phase oscillator. The requirements that $\omega(t)$ varies slowly with t and that the spatially dependent term $\xi(\,\cdot\,, t)$ takes only small-magnitude values compared to $\omega(t)$ imply that the progression of phase is time-locally approximately proportional to the progression of time. One could consider generalising the definition of a non-autonomous phase oscillator by relaxing the requirement that the spatial dependence in the right-hand side is small. In other words, under this generalisation, a non-autonomous phase oscillator would be any system of the form

$$\dot{\theta}(t) = \zeta(\theta(t), t) \tag{11}$$

defined over $t \in (\alpha,\beta)$, where ζ is a smooth function that takes only positive (or only negative) values and has $\zeta(\theta,\tau)$ depending slowly on τ. Analogously to our discussion of spatial dependence in Sec. 2.1.2, we now address the question of whether the form (11) bears greater generality than our definition (10).

If—besides the basic modelling assumption that one excursion of the solution round the circle approximately represents one complete cycle of the physical oscillatory process being modelled—one does not seek to place any further constraints on how the phase-calibration of the circular state space depends on time, then one is free to perform any *slowly time-dependent* smooth phase-recalibration of the state space. In this case, the variable-transformation described for the autonomous case in Sec. 2.1.2 can be applied to each of the autonomous fibres of (11); for each fibre, this transformation is unique up to the specification of how one point is recalibrated. By choosing this specification in a time-independent manner, one then has a time-dependent variable-transformation that transforms (11) to a system of the form (10), where

- the time-dependence in this variable-transformation is slow due to the slow time-dependence of ζ;
- for each $\tau \in (\alpha,\beta)$, $\omega(\tau)$ is the angular frequency of the τ-fibre of (11);
- $\xi(\theta, t)$ is a small-magnitude term arising in the transformation-of-variables formula from the slow time-dependence of the variable-transformation.

So then, our requirement that the progression of phase is approximately proportional to the progression of time within autonomous-fibre timescales does not come with any loss of generality, if one does not place further constraints on the time-dependence of the phase-calibration of the state space. On the other hand, if one is dealing with an oscillatory process that progresses through a series of states according to a clearly identifiable repeating cycle but is subject to temporal modulation of the proportion of time spent around different parts of that cycle, one might prefer a calibration of the state space which time-independently identifies phases in the state space with points along the cycle. In such a case, the more general formulation in (11) would be necessary, i.e., one would not be able to assume that the progression of phase is approximately proportional to the progression of time on the fast timescale. But we will not focus on such scenarios in this paper.

Now if, as above, one is free to perform any slowly time-dependent smooth phase-recalibration of the state space, then by approximation (over timescales comparable to the slow timescale of the time-dependence of the system), it is actually possible to remove the spatial dependence altogether and work with the simpler model

$$\dot{\theta}(t) = \omega(t). \tag{12}$$

To be precise (cf. [46, Sec. 3.1]): Firstly, the term $\xi(\theta, t)$ arising from the above-described variable-transformation of (11) is exactly reciprocally proportional to the time-duration over which the form of t-dependence of $\zeta(\,\cdot\,, t)$ is stretched out, and hence we may assume without loss of generality that in our definition (10), not only ξ itself but also all the partial derivatives of ξ up to any given order are small. Consequently (e.g. by Eq. (42) of [34]), over timescales comparable to the timescale of the time-dependence of (10), the solution flow of (10) is approximately a (non-autonomous) flow of rigid rotations; in other words, over such timescales, if we fix one solution $\theta_*(t)$ of (10), the system (10) as a whole can be approximated by the differential equation

$$\dot{\theta}(t) = \omega(t) + \xi(\theta_*(t), t). \tag{13}$$

The t-dependent variable-transformation $\theta \mapsto \theta + \int_0^t \xi(\theta_*(\tau), \tau)\, d\tau$ then transforms (13) to (12); the t-dependence of this variable-transformation is slow due to the integrand $\xi(\theta_*(\tau), \tau)$ being small in magnitude.

Nevertheless, we will not take Eq. (12) as our definition of a non-autonomous phase oscillator, as this would place huge extra constraint on the set of admissible phase-calibrations beyond the basic requirement that the progression of phase is time-locally approximately proportional to the progression of time, since the integral $\int_0^t \xi(\theta_*(\tau), \tau)\, d\tau$ is typically not small for t comparable to the timescale of the time-dependence of (10).

Now we can define a *network of non-autonomous phase oscillators* by introducing coupling terms into a set of non-autonomous phase oscillators, where these coupling terms are also allowed to have slow time-dependence. Namely, a network of non-autonomous phase oscillators is a system of the form

$$\dot{\theta}_i(t) = \omega_i(t) + g_i(\theta_1(t), \ldots, \theta_N(t), t), \quad i = 1, \ldots, N, \tag{14}$$

where $\omega_1(t), \ldots, \omega_N(t)$ are the slowly time-dependent natural angular frequencies of the oscillatory processes, and the functions g_i have slow-timescale dependence on their t-input. The function g_i represents the combination of the forcing on θ_i from the rest of the network and the small-magnitude spatially dependent term ξ_i in the internal dynamics of θ_i.

2.2.3. Non-autonomous phases and phase-reduction

Suppose we have a non-autonomous oscillator as in Sec. 2.2.1. For each $\tau \in (\alpha,\beta)$, let U_τ be the basin of attraction of $\mathcal{X}_\tau$ under the τ-fibre (8). So one can define an assignment of a phase $\theta_{\mathbf{x}}^{[\tau]}$ to each point $\mathbf{x} \in U_\tau$ according to the description in Sec. 2.2.1 applied to the τ-fibre (8), and this assignment is unique up to specification of the phase assigned to one point in U_τ. It then remains to specify an assignment of a phase to one point in U_τ for each $\tau \in (\alpha,\beta)$.

Given such a specification yielding a smooth calibration of phases on $\mathcal{X}_{(\alpha,\beta)}$, [36] gives a phase-reduction whereby the phase of the non-autonomous oscillator may be approximated by a non-autonomous phase oscillator (10), where $\omega(t)$ is the angular frequency of the limit cycle of the t-fibre. We now present what is probably the simplest example of such a specification of phases.

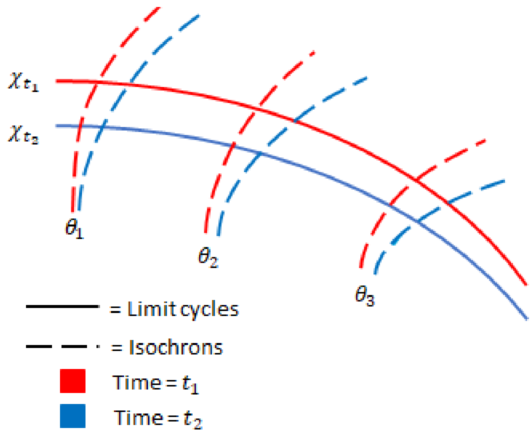

Figure 2: An illustration of the continuous time-dependence of the assignment of phases to points in the state space of a non-autonomous oscillator. The solid lines show the limit cycle of the t-fibre of the non-autonomous oscillator at a time $t = t_1$ (red) and a slightly later time $t = t_2$ (blue). The three red dashed lines indicate the set of points in the state space assigned certain phase-values $\theta_1, \theta_2, \theta_3$ respectively at the earlier time t_1. The three blue dashed lines indicate the set of points in the state space assigned the same three phase-values $\theta_1, \theta_2, \theta_3$ respectively at the slightly later time t_2.

Although in our non-autonomous setting the limit cycle $\mathcal{X}_\tau$ of the τ-fibre depends on τ, nonetheless for many oscillatory processes subject to temporal modulation of their oscillatory parameters, this modulation will not dramatically alter the range of quantitative states that the oscillator occupies during its cycles; mathematically, this would correspond to modelling the breadth of variation of $\mathcal{X}_\tau$ over $\tau \in (\alpha, \beta)$ as being not very great. In particular, if attraction to the limit cycles $\mathcal{X}_\tau$ is sufficiently robust against perturbation to the state variable $\mathbf{x}$, compared with how widely the limit cycle $\mathcal{X}_\tau$ itself varies with τ, then the intersection of the basins of attraction, $\bigcap_{\tau\in(\alpha,\beta)} U_\tau$, will be non-empty. So, under these assumptions, we can choose an arbitrary point $\mathbf{p} \in \bigcap_{\tau\in(\alpha,\beta)} U_\tau$ and then, for each time $\tau \in (\alpha, \beta)$, we take the assignment of phases $(\theta^{[\tau]}_{\mathbf{x}})_{\mathbf{x}\in U_\tau}$ for the autonomous τ-fibre such that $\theta^{[\tau]}_{\mathbf{p}} = 0$. This is illustrated in Figure 2. By [45], this yields a smooth calibration of phases on $\mathcal{X}_{(\alpha,\beta)}$.

For a non-autonomous oscillator network (9), it may be possible, as in [32, 47], to obtain a phase-reduction approximation of the network to a network of non-autonomous phase oscillators (14), where θ_i represents the phase of $\mathbf{x}_i$ and $\omega_i(t)$ is the angular frequency of the limit cycle of the unperturbed t-fibre

$$\frac{d}{ds}\mathbf{x}_i(s) = \mathbf{F}_i(\mathbf{x}_i(s), t).$$

3. Synchronisation

A fundamental phenomenon in physics is *synchronisation* of oscillatory processes. This phenomenon is not restricted to synchronisation of the states of different copies of the same oscillatory process, where only the initial state varies between the oscillators. Different oscillatory processes with different state spaces and with different natural frequencies can interact with each other, and these interactions can cause synchronisation phenomena where this synchronisation is defined in terms of the *phases* of the oscillatory processes. In other words, what we consider is *phase-synchronisation* [48]. Accordingly, an appropriate mathematical framework for studying synchronisation is coupled phase-oscillator dynamics.

In the autonomous case represented by Eq. (4) with $N \geq 2$, fixing two of the phase oscillators θ_i and θ_j, we will regard a solution $(\theta_1(t), \ldots, \theta_N(t))$ of (4) as being θ_i-θ_j *synchronous* if there exists a constant angle C such that

$$\theta_i(t) - \theta_j(t) \approx C \quad \forall\, t \geq 0. \tag{15}$$

If the phase-coupling functions in (4) fulfil the property (5), then there may exist solutions that fulfil Eq. (15) with exact equality rather than just approximate equality. (We will see an example of this for $N = 2$.)

In order for synchronisation between θ_i and θ_j to be physically realisable, it is necessary that there be a sufficiently attractive invariant subset A of the state space in which solutions are θ_i-θ_j synchronous, so that it will be possible for some solutions where $\theta_i(t)$ and $\theta_j(t)$ initially progress at different rates to eventually become θ_i-θ_j synchronous after some time. The size of the basin of attraction of A would determine how realisable and robust this synchronisation is.

We now consider how to define synchronisation in the setting of non-autonomous networks of phase oscillators represented by (14). Due to the explicit time-dependence in (14), two points (generalised from [31]) are worth noting:

(1) To define synchronisation in the non-autonomous context as requiring that the phase difference remain approximately the same across time-durations comparable to the timescale of the time-dependent driving is inappropriately strong. In the example studied in [31], an important synchronisation phenomenon does take place, that would be disregarded if this were required to qualify as synchronisation.

(2) The dynamics of some of the autonomous fibres of a non-autonomous system can exhibit synchronising dynamics while others do not, and so synchronisation in the non-autonomous context should not be defined as a time-independent property of the system (14) as a whole, but rather should be defined with reference to subintervals of the time-interval on which (14) is defined. As exemplified in [31], it is very readily possible for two oscillators to be synchronous for some long time-duration and then to become asynchronous, and indeed any number of alternations between time-intervals of synchrony and time-intervals of asynchrony is possible. In other words, *temporally intermittent synchronisation* of oscillators is readily possible in this non-autonomous setting.

Let us take (14) to be defined on a time-interval (α, β). We will regard a solution $(\theta_1(t), \ldots, \theta_N(t))$ of (14) as being θ_i-θ_j synchronous (for some i and j) on a time-interval $(a, b) \subset (\alpha, \beta)$ if there exists a slowly t-dependent angle $C(t)$ defined over $t \in (a, b)$ such that

$$\theta_i(t) - \theta_j(t) \approx C(t) \quad \forall\, t \in (a, b). \tag{16}$$

We regard this synchrony as being *stable*, and hence physically realisable, if for each $\tau \in (a, b)$, $(\theta_1(\tau), \ldots, \theta_N(\tau))$ lies close to an attractive invariant set A_τ of the τ-fibre of (14) in which solutions of the τ-fibre of (14) are θ_i-θ_j synchronous, where A_τ has continuous dependence on τ over $\tau \in (a, b)$.

4. Intermittent phase-synchronisation between two coupled oscillators

We now illustrate the concepts in Secs. 2 and 3—particularly non-autonomous phase-synchronisation and its intermittency—with a simple example of a two-oscillator system. Although our ultimate focus will be on a coupled-phase-oscillator system, to study synchronisation between the phase oscillators, nevertheless we will start with a higher-dimensional model built according to the phase dynamics that we ultimately will be focusing on. We do this in order to illustrate how, using the concepts in Sec. 2, one may set up the question of phase-synchronisation of two oscillatory processes if one is starting with a higher-dimensional model of the processes.

4.1. General model

A very simple dynamical system that has been used extensively as a theoretical model of biological oscillations is the *Poincaré oscillator* [49, 50]. This is a two-dimensional system which, in its classical autonomous form, is defined in polar coordinates as

$$\begin{aligned} \dot{r}(t) &= kr(t)(1 - r(t)) \\ \dot{\theta}(t) &= \omega \end{aligned} \tag{17}$$

where $k > 0$ is a parameter representing the stability of oscillation, and $\omega > 0$ is a parameter representing the angular frequency of the oscillation. The unit circle is a stable limit cycle that attracts all solutions except the repelling equilibrium at the origin. This system is smooth everywhere except at the origin, and so if we either exclude the origin from the state space or else apply a smoothing modification of the right-hand side of the radial equation locally around $r = 0$, this system fulfils the definition of an autonomous limit-cycle oscillator in Sec. 2.1.1. This model is designed so as to have an immediate phase-reduction: simply take the angular component of the polar-coordinate expression (17).

Since living systems are continually interacting with their environment, the parameters of biological oscillations are continually being modulated over time; in particular, the modulation of *frequency* over time is a very important aspect of biological oscillations [51, 52]. This makes non-autonomous modelling of biological oscillatory systems appropriate. In the case of the Poincaré oscillator, we can generate a non-autonomous version of the Poincaré oscillator by allowing the parameters k and ω to modulate over time; but also, whereas the autonomous system (17) has the amplitude of oscillation normalised to 1 without loss of generality, for a non-autonomous model it may also be appropriate to allow the amplitude to modulate over time. So then, we define the non-autonomous version of the Poincaré oscillator to be a system of the form

$$\begin{aligned} \dot{r}(t) &= k(t)r(t)(1 - \tfrac{r(t)}{a(t)}) \\ \dot{\theta}(t) &= \omega(t) \end{aligned} \tag{18}$$

defined over a time-interval (α,β), where $\omega(t) > 0$ is the time-localised angular frequency, $a(t) > 0$ is the time-localised amplitude, and $k(t) > 0$ represents the time-localised stability, and these three parameters are all assumed to have smooth, slow time-dependence. For each $\tau \in (\alpha,\beta)$, the stable limit cycle $\mathcal{X}_\tau$ of the autonomous τ-fibre is the origin-centred circle of radius $a(\tau)$, and the corresponding basin of attraction U_τ is the whole of $\mathbb{R}^2 \setminus \{(0, 0)\}$. Again, the system (18) is smooth everywhere except at the time-independent fixed point at the origin, and so if we exclude the origin from the state space, the system (18) fulfils the definition of a non-autonomous oscillator in Sec. 2.2.1. By picking a point $\mathbf{p} = (p, 0) \in \mathbb{R}^2$ with $p > 0$, the phase-calibration described in Sec. 2.2.3 corresponds simply to taking the angular component of the polar-coordinate representation of the state space, and so we have an exact phase-reduction to the non-autonomous phase oscillator

$$\dot{\theta}(t) = \omega(t).$$

We now consider introducing a coupling between two non-autonomous Poincaré oscillators. For simplicity, this coupling will not itself be time-dependent. It will act purely on the phase variables, and will depend only on the phase difference. Namely, we will consider a system of the form

$$\begin{aligned} \dot{r}_1(t) &= k_1(t)r_1(t)(1 - \tfrac{r_1(t)}{a_1(t)}) \\ \dot{\theta}_1(t) &= \omega_1(t) + \tilde{g}_1(\theta_1(t) - \theta_2(t)) \\ \dot{r}_2(t) &= k_2(t)r_2(t)(1 - \tfrac{r_2(t)}{a_2(t)}) \\ \dot{\theta}_2(t) &= \omega_2(t) + \tilde{g}_2(\theta_2(t) - \theta_1(t)) \end{aligned} \tag{19}$$

defined over a time-interval (α,β), where once again, the six time-dependent parameters are assumed to have smooth, slow time-dependence, and the coupling functions $\tilde{g}_1$ and $\tilde{g}_2$ are assumed to be smooth. If we exclude the repelling invariant set $(\mathbb{R}^2 \times \{(0, 0)\}) \cup (\{(0, 0)\} \times \mathbb{R}^2)$ from the state space (corresponding to the exclusion of $(0, 0)$ from the set of states of each of the two oscillators), then this is a smooth system, and so it fulfils the definition of a network of non-autonomous oscillators in Sec. 2.2.1 with $N = 2$. Applying the above-described phase calibration of non-autonomous Poincaré oscillators to each of the two oscillators in the system (19) then trivially gives an exact phase-reduction of (19) to a network of non-autonomous phase oscillators

$$\begin{aligned} \dot{\theta}_1(t) &= \omega_1(t) + \tilde{g}_1(\theta_1(t) - \theta_2(t)) \\ \dot{\theta}_2(t) &= \omega_2(t) + \tilde{g}_2(\theta_2(t) - \theta_1(t)). \end{aligned} \tag{20}$$

The system (20) is a generalisation of the setup considered in [31, 53]. Note that the coupling in (20) fulfils property (5); as a consequence, we will see that those autonomous fibres of (20) that exhibit synchronisation between θ_1 and θ_2 do so with exact equality in Eq. (15).

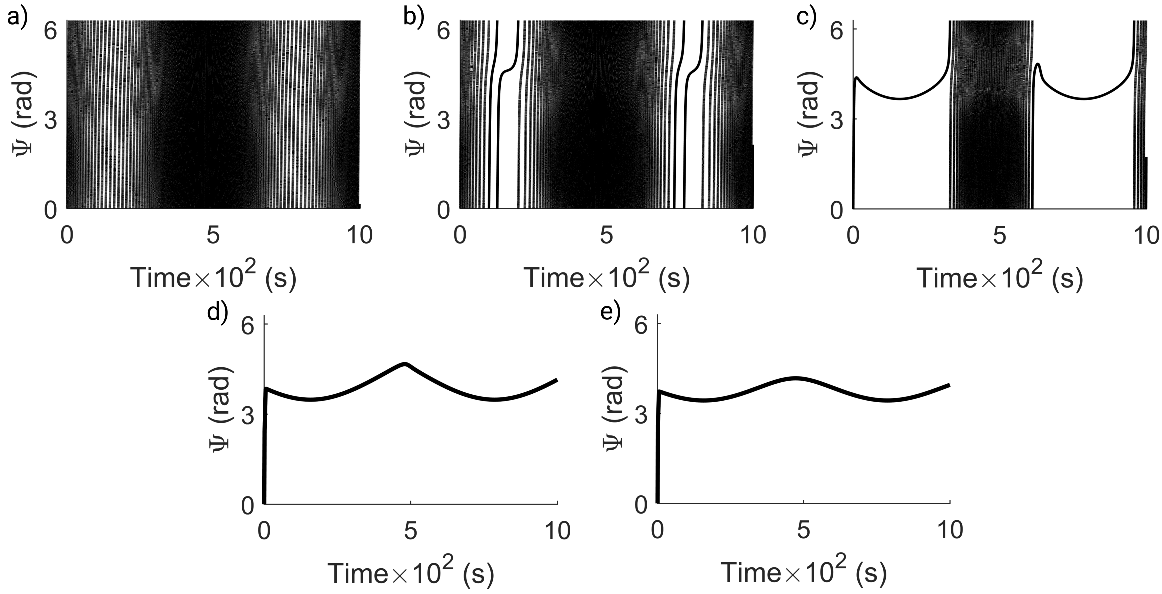

Figure 3: Plots of the phase difference $\psi(t)$ against time t for the system $\dot{\psi}(t) = \Delta\omega_0 - K\sin(\omega_m t) + \gamma\sin(\psi(t))$ with $\Delta\omega_0 = 1$ rad/s, $K = 0.5$ rad/s, $\omega_m = 0.01$ rad/s, and various values of γ to show the three synchronisation scenarios. (a) $\gamma = 0.25$ rad/s and the oscillators do not synchronise. (b) $\gamma = 0.5$ rad/s, lying at the boundary between no synchronisation and intermittent synchronisation. (c) $\gamma = 1$ rad/s, and the oscillators synchronise intermittently. (d) $\gamma = 1.5$ rad/s, lying at the boundary between intermittent synchronisation and synchronisation at all times. (e) $\gamma = 1.75$ rad/s and the two oscillators are synchronised at all times. Conditions for the three regimes are given by Eqs. (24)–(26).

4.2. Synchronisation

The phase difference $\psi(t) = \theta_2(t) - \theta_1(t)$ obeys the one-dimensional non-autonomous differential equation

$$\dot{\psi}(t) = \Delta\omega(t) + g(\psi(t)) \tag{21}$$

with $\Delta\omega(t) := \omega_2(t) - \omega_1(t)$ and $g(\psi) := \tilde{g}_2(\psi) - \tilde{g}_1(-\psi)$. Let us assume that g is unimodal, in the sense that there is a unique ψ-value at which g has a local maximum and a unique ψ-value at which g has a local minimum (and hence this local maximum and local minimum are indeed the global maximum and global minimum).

Write $V \subset \mathbb{R}$ for the range of the function $g(\cdot)$, and write $U \subset \mathbb{R}$ for the range of $-\Delta\omega(t)$ over $t \in (\alpha, \beta)$. Then we can split into three main possibilities regarding synchronisation between θ_1 and θ_2:

(I) Synchronisation over the whole time $t \in (\alpha, \beta)$; this corresponds to the case that U is contained in the interior of V. In this case, for every $\tau \in (\alpha, \beta)$ the τ-fibre of (21) given by

$$\frac{d}{ds}\psi(s) = \Delta\omega(\tau) + g(\psi(s)) \tag{22}$$

has a stable equilibrium point $C(\tau)$ that attracts all solutions of (22) except for a single unstable equilibrium point of (22), and the stable equilibrium $C(\tau)$ point has slow, continuous dependence on τ.

(II) No synchronisation; this corresponds to the case that the closure of U has empty intersection with V. In this case, Eq. (22) has no equilibrium points for every $\tau \in (\alpha, \beta)$, and the phase difference $\psi(t)$ drifts monotonically throughout the time-interval (α, β).

(III) Intermittent synchronisation; this corresponds to the case that the interior of U intersects both V and $\mathbb{R} \setminus V$. In this case, there are alternations between time-intervals during which Eq. (22) has a stable equilibrium as described in case (I) and time-intervals where the phase drifts monotonically as described in case (II); these correspond respectively to time-intervals in which θ_1 and θ_2 are synchronised and time-intervals in which θ_1 and θ_2 are not synchronised.

The stability of solutions of Eq. (21) can be described and quantified as in [34].

In the case that g is not unimodal, one can make a similar division into cases (I), (II) and (III), except that in cases (I) and (III), during "times of synchronisation" the phase-difference $\psi(t)$ may undergo a *bifurcation-induced tipping* [34, 54] where the oscillators briefly desynchronise and then resynchronise to a new phase-difference value significantly different from the phase-difference value just before the desynchronisation.

4.3. Example

Let us now illustrate the results of Sec. 4.2 with one of the main systems considered in [31]. For some constant parameters $\gamma > 0$, $k > 0$, $\omega_{1,0} > 0$, $\omega_{2,0} > 0$ and $\omega_m > 0$ with ω_m very small compared to the other parameters, we take the system (20) with

$$\begin{aligned}
\omega_1(t) &= \omega_{1,0}[1 + k\sin(\omega_m t)] \\
\tilde{g}_1(\psi) &= 0 \\
\omega_2(t) &= \omega_{2,0} \\
\tilde{g}_2(\psi) &= \gamma\sin(\psi).
\end{aligned}$$

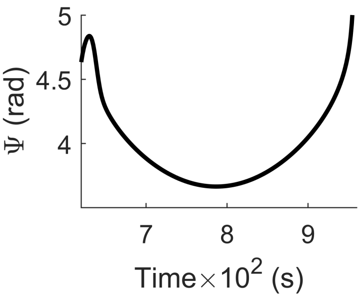

Figure 4: $\dot{\psi}(t) = \Delta\omega_0 - K\sin(\omega_m t) + \gamma\sin(\psi(t))$ with $\Delta\omega_0 = 1$ rad/s, $K = 0.5$ rad/s, $\omega_m = 0.01$ rad/s and $\gamma = 1$ rad/s, exhibiting intermittent synchronisation.

So there is unidirectional coupling, with θ_1 driving θ_2. Equation (21) becomes

$$\dot{\psi}(t) = \omega_{2,0} - \omega_{1,0}[1 + k\sin(\omega_m t)] + \gamma\sin(\psi(t)). \tag{23}$$

Here, $g(\psi) = \tilde{g}_2(\psi) = \gamma\sin(\psi)$, so g is unimodal. Let us define the constants $\Delta\omega_0 := \omega_{2,0} - \omega_{1,0}$ and $K := \omega_{1,0}k$. Then, assuming that the time-duration $\beta - \alpha$ over which the system is considered is greater than $\frac{\pi}{\omega_m}$, the three synchronisation scenarios described above are as given in [31], namely:

(I) Synchronisation over all time:

$$\gamma > |\Delta\omega_0| + K. \tag{24}$$

(II) No synchronisation:

$$\gamma < |\Delta\omega_0| - K. \tag{25}$$

(III) Intermittent synchronisation:

$$|\Delta\omega_0| - K < \gamma < |\Delta\omega_0| + K. \tag{26}$$

These scenarios are illustrated numerically in Fig. 3. Here, Eq. (23) is simulated using a Runge-Kutta 4-step algorithm, with a fixed step size of 0.001 s; the initial condition is taken as $\psi(0) = 0$.

The non-constancy of the phase-difference during synchronisation over time-durations comparable to the timescale ω_m of the variation of $\Delta\omega(t)$ with t can be seen in plots (c–e) of Fig. 3. For the intermittent synchronisation case shown in plot (c), we zoom in on one of the time-intervals of synchronisation in Fig. 4.

5. Intermittent macroscopic-phase synchronisation between two mean-field-coupled networks

Oscillator networks comprising of a large number of oscillators are an important kind of model appearing across a range of disciplines [37, 55]. In such applications, it is common to construct the network's interactions with its surroundings and other model components not directly in terms of the network's individual oscillators, but rather in terms of its *collective* or *macroscopic* behaviour, as represented by observables such as its *macroscopic phase* (which is meaningful only at times when there is a significant level of collective order among the oscillators in the network).

Accordingly, just as in Sec. 4.3 we have considered synchronisation between two unidirectionally sinusoid-coupled phase oscillators, so we will now illustrate our non-autonomous framework through considering analogous synchronisation between two unidirectionally coupled processes, where

- each of the two processes is described by the macroscopic behaviour of a phase-oscillator network;
- the coupling is a sinusoidal phase coupling in which the driving phase is the macroscopic phase of the driving process (as in [56]).

This is illustrated in Fig. 5. As in the case of two individual phase oscillators, synchronisation will be defined in terms of a macroscopic-phase difference that stays approximately constant on timescales comparable to those of the autonomous fibres but is allowed to drift over longer timescales.

5.1. General model

In our model, we suppose that we have two networks of non-autonomous phase-oscillators and we then introduce a unidirectional coupling between them.

Let us label the driving and driven network respectively as network A and network B, and let N_A and N_B denote the number of oscillators in the respective networks; then, using the shorthands

$$\boldsymbol{\theta}_A(t) := (\theta_{A1}(t), \ldots, \theta_{AN_A}(t))$$
$$\boldsymbol{\theta}_B(t) := (\theta_{B1}(t), \ldots, \theta_{BN_B}(t)),$$

we consider the system

$$\begin{aligned} \dot{\theta}_{Ai}(t) &= \omega_{Ai}(t) + g_{Ai}(\boldsymbol{\theta}_A(t), t), \quad i = 1, \ldots, N_A \\ \dot{\theta}_{Bi}(t) &= \omega_{Bi}(t) + g_{Bi}(\boldsymbol{\theta}_A(t), t) \\ &\quad + \gamma(t) R_{N_A}(\boldsymbol{\theta}_A(t)) \sin(\Phi_{N_A}(\boldsymbol{\theta}_A(t)) - \theta_{Bi}(t)), \\ &\quad i = 1, \ldots, N_B. \end{aligned} \tag{27}$$

Here, we use the notations,

$$R_n(\theta_1, \ldots, \theta_n) := \frac{1}{n}\left|\sum_{j=1}^{n} e^{i\theta_j}\right|,$$

$$\Phi_n(\theta_1, \ldots, \theta_n) := \operatorname{Arg}\left(\sum_{j=1}^{n} e^{i\theta_j}\right) \quad \text{provided } R_n(\theta_1, \ldots, \theta_n) \neq 0.$$

In Eq. (27), $\theta_{A1}, \ldots, \theta_{AN_A}$ are the phase-oscillators in the driving network A, and $\theta_{B1}, \ldots, \theta_{BN_B}$ are the phase-oscillators in the driven network B. $\omega_{A1}(t), \ldots, \omega_{AN_A}(t) > 0$ are the natural angular frequencies of the oscillators in A, and $\omega_{B1}(t), \ldots, \omega_{BN_B}(t) > 0$ are the natural angular frequencies of the oscillators in B.

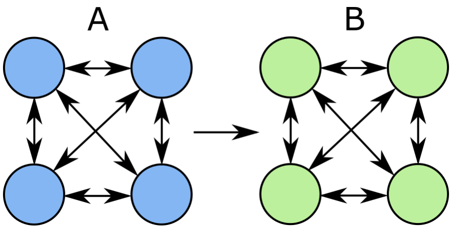

Figure 5: Diagram of the model (27) analysed in Section 5, for simplicity's sake illustrated for $N_A = N_B = 4$ nodes per network. Network nodes are represented by circles, and edges by the connecting lines, representing bidirectional coupling between the oscillators within each network. The line connecting network A to network B represents the unidirectional coupling from the collective dynamics of A to the oscillators in B.

$g_{A1}, \dots, g_{AN_A}$ are the internal phase-coupling functions for network A, and $g_{B1}, \dots, g_{BN_B}$ are the internal phase-coupling functions for network B. $\gamma(t)$ is the strength of coupling from A to B; $R_{N_A}(\boldsymbol{\theta}_A(t))$ is called the *Kuramoto order parameter* of network A, and represents how closely aligned the phases of the oscillators in A are at each time t. $\Phi_{N_A}(\boldsymbol{\theta}_A(t))$ quantifies the *macroscopic phase* of A—it is physically meaningful only when $R_{N_A}(\boldsymbol{\theta}_A(t))$ is not too small, but this is not a problem as $R_{N_A}(\boldsymbol{\theta}_A(t))$ itself appears as a pre-factor in front of the coupling term containing $\Phi_{N_A}(\boldsymbol{\theta}_A(t))$. Once again, it is assumed that the dependencies of $\omega_{Ai}(t)$, $\omega_{Bi}(t)$, $g_{Ai}(\cdot, t)$, $g_{Bi}(\cdot, t)$ and $\gamma(t)$ on t are all slow compared to the timescales of the autonomous fibres of (27). The order and the macroscopic phase of A have a combined representation by a single complex number

$$\frac{1}{N_A} \sum_{j=1}^{N_A} e^{i\theta_{Aj}} = R_{N_A}(\boldsymbol{\theta}_A(t)) e^{i\Phi_{N_A}(\boldsymbol{\theta}_A(t))},$$

called the *mean field* or *complex Kuramoto order parameter* of A.

5.2. Synchronisation

We say that a solution $(\boldsymbol{\theta}_A(t), \boldsymbol{\theta}_B(t))$ of (27) *exhibits synchrony between A and B* on a time-interval (a, b) if, defining

$$\begin{aligned} \Phi_A(t) &= \Phi_{N_A}(\boldsymbol{\theta}_A(t)) \\ \Phi_B(t) &= \Phi_{N_B}(\boldsymbol{\theta}_B(t)) \\ \Psi(t) &= \Phi_B(t) - \Phi_A(t), \end{aligned}$$

there exists a slowly t-dependent angle $C(t)$ defined over $t \in (a, b)$ such that

$$\Psi(t) \approx C(t) \quad \forall\, t \in (a, b),$$

analogously to Eq. (16) for individual oscillators. We regard this synchrony as *stable* if at each $\tau \in (a, b)$, $(\boldsymbol{\theta}_A(\tau), \boldsymbol{\theta}_B(\tau))$ lies close to an attractive invariant set S_τ of the τ-fibre of (27) (with S_τ depending continuously on τ) in which solutions of the τ-fibre of (27) have that $\Psi(s)$ is approximately constant over all time $s \geq 0$.

Table 1: Parameters of the phase network system for the simulations shown in Fig. 6.

Parameter	Value
$\mu_{A\omega}$	π rad/s
$\sigma_{A\omega}$	0.5 rad/s
M_A	0.5
ω_{mA}	0.1 rad/s
$\mu_{B\omega}$	2π rad/s
$\sigma_{B\omega}$	0.5 rad/s
M_B	0.5
ω_{mB}	0.2 rad/s
K_A	8 rad/s
K_B	0.5 rad/s
N	100

5.3. Example

We now carry out a numerical investigation of synchronisation and intermittent synchronisation for the system (27) in the case that the coupling functions are time-independent and are of Kuramoto type, and the natural frequencies of the individual oscillators are sinusoidally modulated. Specifically, we take

$$\begin{aligned} N_A &= N_B =: N \\ g_{Ai}(\theta_1, \dots, \theta_N, t) &= \frac{K_A}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \\ g_{Bi}(\theta_1, \dots, \theta_N, t) &= \frac{K_B}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \\ \gamma(t) &= \gamma \\ \omega_{Ai}(t) &= \omega_{Ai,0} + M_A \omega_{Ai,0} \sin(\omega_{mA} t) \\ \omega_{Bi}(t) &= \omega_{Bi,0} + M_B \omega_{Bi,0} \sin(\omega_{mB} t). \end{aligned}$$

Here, the constants $K_A, K_B > 0$ are internal coupling strengths, and the coupling strength from A to B is a constant parameter $\gamma > 0$. The constants $\omega_{Ai,0}$ ($i = 1, \dots, N$) are the natural centre frequencies of the oscillators in network A, and likewise $\omega_{Bi,0}$ ($i = 1, \dots, N$), the natural centre frequencies of the oscillators in network B. The constant $\omega_{mA} > 0$ is the frequency of modulation of the natural frequencies of all the oscillators in network A; and likewise $\omega_{mB} > 0$ for network B. The constant $0 < M_A < 1$ is such that for each oscillator θ_{Ai} in network A, the amplitude of frequency modulation is $M_A \omega_{Ai,0}$; and M_B is analogous for network B.

In Fig. 6, we show numerical simulation results. We take $N = 100$. The values of the internal coupling constants K_A, K_B and the frequency modulation parameters $M_A, \omega_{mA}, M_B, \omega_{mB}$ are shown (along with other parameters) in Table 1. The values of

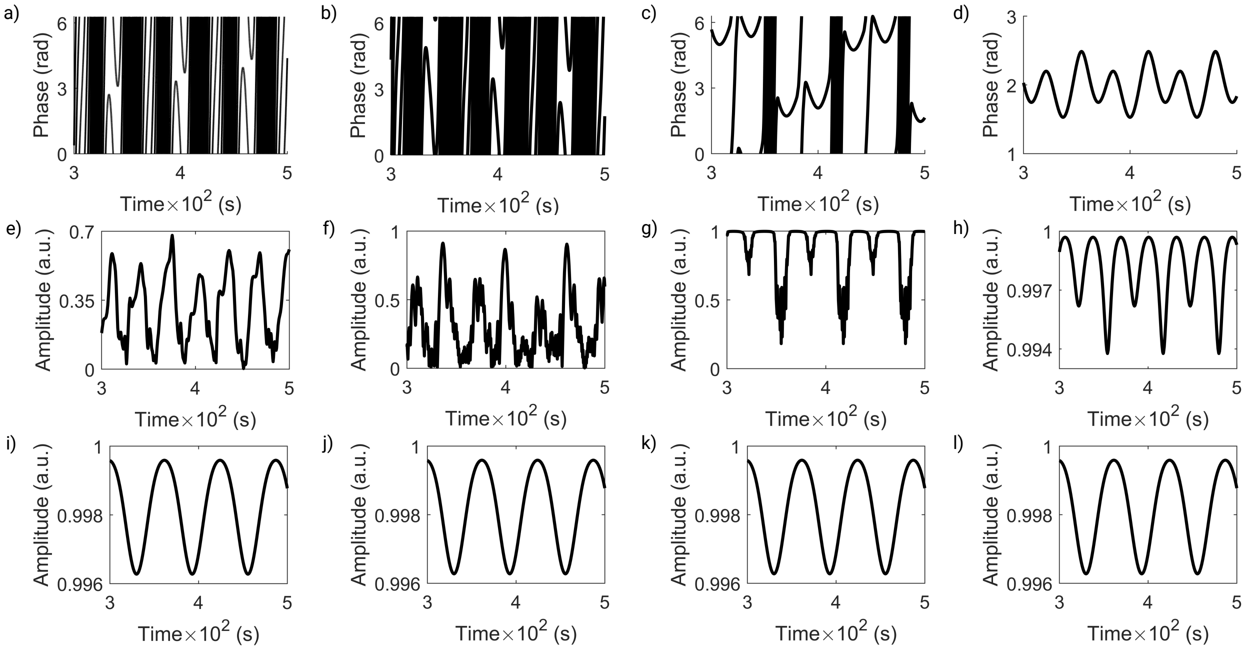

Figure 6: Synchronisation regimes for a unidirectionally coupled pair of oscillator networks. Results are for the system (27) with the coupling functions and natural frequencies of the form specified at the start of Sec. 5.3, with parameters (besides γ) given in Table 1. Plots (a,e,i) show results for $\gamma = 0$ rad/s, (b,f,j) for $\gamma = 0.5$ rad/s, (c,g,k) for $\gamma = 5$ rad/s, and (d,h,l) for $\gamma = 10$ rad/s: In (a–d) are shown the phase difference $\Psi(t)$ between the macroscopic phase of A and the macroscopic phase of B, as defined in Sec. 5.2; in (e–h) are shown the Kuramoto order parameter of network B; and in (i–l) are shown the Kuramoto order parameter of network A. For $\gamma = 0$ rad/s (i.e. no coupling) and 0.5 rad/s (weak coupling), the networks are not synchronised; for $\gamma = 5$ rad/s (intermediate coupling), they are intermittently synchronised; and for $\gamma = 10$ rad/s (strong coupling), they are synchronised at all time. The plot shows results over the time-interval from 300 s to 500 s after the start of the simulation; this 200 s time-interval corresponds to just over 3 periods of the modulation of the natural frequencies of the oscillators in network A.

$\omega_{Ai,0}$ ($i = 1, \ldots, 100$) were randomly sampled from a Gaussian distribution of mean $\mu_{A\omega} = \pi$ rad/s and standard deviation $\sigma_{A\omega} = 0.5$ rad/s. The values of $\omega_{Bi,0}$ ($i = 1, \ldots, 100$) were also randomly sampled from a Gaussian distribution, of a different mean $\mu_{B\omega} = 2\pi$ rad/s but the same standard deviation $\sigma_{B\omega} = 0.5$ rad/s. For both networks, the standard deviation is sufficiently small compared to the mean that the probability of obtaining a negative value for one of the values $\omega_{Ai,0}$ or $\omega_{Bi,0}$ from a sample size of 100 is immeasurably small. Note that for each i, having a positive value for $\omega_{Ai,0}$ (resp. $\omega_{Bi,0}$) ensures that the time-dependent natural frequency $\omega_{Ai}(t)$ (resp. $\omega_{Bi}(t)$) always remains positive, due to the fact that M_A and M_B are less than 1. The initial conditions $\theta_{A1}(0), \ldots, \theta_{A100}(0), \theta_{B1}(0), \ldots, \theta_{B100}(0)$ were randomly sampled from the uniform distribution on $[-\pi, \pi]$. Having fixed the natural centre frequencies and the initial phases, we simulated the system (27) for different values of the A-to-B coupling strength γ. Just as with Fig. 3, the numerical simulations were conducted using a Runge-Kutta 4-step algorithm, with a fixed step size of 0.001 s.

Plot (a) of Fig. 6 shows the graph of $\Psi(t)$ against t when $\gamma = 0$, corresponding to the absence of A-to-B coupling. Due to the unrelatedness of the natural frequencies of network A and the natural frequencies of network B, we expect no synchronisation, and this agrees with what we see in plot (a). Plot (b) shows the graph of $\Psi(t)$ against t for a small coupling strength, $\gamma = 0.5$ rad/s; we see qualitatively similar results to plot (a), once again indicating an absence of synchronisation.

Plot (c) shows the graph of $\Psi(t)$ against t for an intermediate coupling strength, $\gamma = 5$ rad/s. Here, we see intermittency between: epochs of synchronisation; isolated phase slips of size $+2\pi$; and epochs of non-synchronisation reflected in an unconstrained growth of $\Psi(t)$. Plot (g) shows the Kuramoto order parameter $R_{N_B}(\boldsymbol{\theta}_B(t))$ for the driven network, namely network B, for the same simulation as in plot (c); this plot indicates the following:

- During the epochs of synchronisation seen in plot (c), the driving from A that is responsible for this synchronisation causes the phases in network B to be themselves largely aligned with each other—i.e. $R_{N_B}(\boldsymbol{\theta}_B(t)) \approx 1$, which is equivalent to saying that $\theta_{Bi}(t) \approx \Phi_B(t)$ for most or all $i \in \{1, \ldots, 100\}$. Hence in particular, during these times of synchrony, the phases of the individual oscillators in B are themselves largely synchronous with the driving phase Φ_A, in the sense that the time-locally approximately constant value $\Psi(t)$ plotted in plot (c) is, at each time t, approximately equal to the phase difference $\theta_{Bi}(t) - \Phi_A(t)$ for most or all $i \in \{1, \ldots, 100\}$.
- By contrast, network B's level of order $R_{N_B}(\boldsymbol{\theta}_B(t))$ dips to significantly less than 1 during the phase slips and during the epochs of non-synchronisation.

Plot (d) shows the graph of $\Psi(t)$ against t for a high coupling strength, $\gamma = 10$ rad/s; and plot (h) again shows the Kuramoto order parameter of network B for the same simulation. In plot (d), we see synchronisation at all times, and in plot (h) we see that $R_{N_B}(\boldsymbol{\theta}_B(t)) \approx 1$ at all times. Thus, the coinciding of Φ_A-Φ_B synchrony with high mutual alignment of phases within B that we discussed in the intermediate-coupling-strength case also holds in this high-coupling-strength case.

Let us note from plots (c) and (d) that, during synchrony, the macroscopic-phase difference does not remain approximately constant across the time-intervals of synchrony shown, but undergoes slow-timescale drift as a result of the time-dependence of the oscillators' natural frequencies in our non-autonomous model; this is in accord with the t-dependence of $C(t)$ in our definition of synchronisation in Sec. 5.2.

In conclusion, just as we saw in Sec. 4 how a non-autonomous system consisting of two phase oscillators can exhibit regimes of no synchronisation, of intermittent synchronisation, and of synchronisation at all times, so likewise we see how a non-autonomous system consisting of *two networks* of phase-oscillator can exhibit these three regimes. In the latter case, the three regimes are defined in terms of the macroscopic phases of the two networks. Both cases involved a unidirectional phase coupling, where the different regimes were produced by changing the coupling strength.

6. Summary and conclusions

In this paper, we have provided a framework for describing the time-dependent phase dynamics of oscillatory processes subject to temporal modulation of their parameters, such as their time-localised frequency. This has included a definition of non-autonomous oscillators and their non-autonomous phase, and of non-autonomous phase oscillators. Our definition of non-autonomous oscillators and their phases is adapted from the setup considered in [36]. We defined phase-synchronisation in the non-autonomous setting in terms of the time-localised behaviour of phase differences. We considered such synchronisation and its stability in terms of the autonomous fibres of the non-autonomous phase oscillator system. We applied this to consideration of *intermittency* [31, 32, 35] of synchronisation both between two interacting oscillators and between two interacting oscillator-networks.

Let us now make some final remarks. Firstly, we mentioned in Sec. 1 that the kind of intermittent synchronisation considered in [31] cannot occur in the classical autonomous setting, and we saw further in Secs. 3 and 4 that even *during* epochs of time-localised synchronisation, a natural notion of synchronisation for the autonomous setting cannot be directly applied to the analogous non-autonomous synchronisation phenomenon due to the slow-timescale phase drift arising from the slow time-dependence of the system. This intermittency and slow-timescale phase drift have been observed in [31] for an interacting pair of oscillators, and we have now observed a similar phenomenon in Sec. 5 for an interacting pair of oscillator-networks. Since the quantitative characteristics of various real-world oscillatory systems are modulated in time by external influences, synchronisation phenomena of the kind considered in this paper that are not described by classical autonomous synchronisation are highly physically relevant.

Secondly, having just emphasised the importance of a non-autonomous approach to the investigation of dynamics of externally influenced oscillatory processes, let us briefly discuss implications of this for analysis of experimental data. For signals recorded from systems that are regarded as autonomous limit-cycle oscillators, one approach towards extracting phase-valued time-series is to extract a protophase time-series using the Hilbert transform and then apply a protophase-to-phase conversion [57]. This conversion involves a time-averaging procedure to average out noise, if noise is considered to be present. The need for a protophase-to-phase conversion essentially arises by virtue of the possible nonlinearity of the measured oscillations. For systems involving oscillatory processes whose quantitative characteristics are modulated over time, time-frequency representation enables a time-evolving description of the frequency content of signals measured from the system. Highly noise-resilient ridge-extraction from linear time-frequency representations such as the continuous wavelet transform has been developed in [58], which can be applied to the time-dependent fundamental mode in the representation of an oscillatory component, to give a phase time-series free of the kind of nonlinearities that protophase-to-phase conversion is designed to deal with. Accordingly, the notions of synchronisation of non-autonomous phase oscillators and non-autonomous phase-oscillator networks discussed in this paper, as well as temporal intermittency thereof, can be experimentally investigated via time-dependent phase-extraction methods such as ridge-extraction.

A future direction for this work would be to consider more closely the link between theoretical non-autonomous phase dynamics and time-frequency-representation-based time-series analysis methods. This can lead to more advanced time-series analysis tools built from dynamical theory for inferring more about underlying phase dynamics from experimentally measured signals. As a starting point, it may be useful to establish whether—given a smooth non-autonomous oscillator, together with an observable function h on the state space that maps the solution $\mathbf{x}(t)$ onto a time-series $h(\mathbf{x}(t))$—there exists a smoothly time-dependent isochron foliation of the autonomous fibres under which, in the singular limit of timescale separation, the phase $\theta(t)$ of $\mathbf{x}(t)$ approximately matches the phase-valued time-series obtained from the above-described ridge-extraction applied to the time-series $h(\mathbf{x}(t))$.

In conclusion, an appropriate non-autonomous treatment of phase dynamics, such as this present paper contributes towards, has the potential to lead to considerable progress throughout all sciences where interacting oscillators are involved, including all those discussed in the references [1–14] mentioned at the start of Sec. 1.

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Highlights

1. A framework for defining phase-dynamics phenomena for non-autonomous oscillatory systems is introduced.
2. Interaction between pairs of non-autonomous oscillators and between networks of non-autonomous oscillators is considered.
3. Phenomena such as synchronization in oscillatory processes subject to time-dependent modulation from their environment are formulated mathematically.
4. Relevant for many real-world problems, including metabolic networks, cardio-respiratory interactions, brain dynamics, electrons moving on the surface of liquid helium or financial markets.