Two families of Dirac-like operators for Drinfeld's Hecke algebra

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Abstract

In this paper we define two generalisations of Dirac operators for Drinfeld's Hecke algebra. One generalisation, Parthasarathy operators inherit the notion of the Dirac inequality. The second generalisation, Vogan operators, inherit Dirac cohomology; if an operator has non-zero cohomology then it relates the infinitesimal character with a character of the group $\tilde{G}$. We prove properties about these operators and give a family of operators in each class.

1 Introduction

Dirac operators for real Lie groups, [1, 18, 20] have been significantly utilised in the study of $(\mathfrak{g}, K)$ representations. These ideas have been generalised to graded affine Hecke algebra [3] and Drinfeld's Hecke algebras [11, 7]. There are many other algebras which have has associated Dirac operators defined [10, 8, 15].

The main results for Dirac operators are the Dirac inequality and Dirac cohomology. The Dirac inequality [20, 12] notes that the square of the Dirac operator on a unitary module is non-negative. The square is equal to a sum of two elements, both central in different algebras. In the case of Lie algebras, these algebras are $\mathfrak{g}$ and the diagonal embedding of $\mathfrak{k}$. In the case of graded affine Hecke algebras $\mathbb{H}$, the first element is central in $\mathbb{H}$ and the second central in the diagonal embedding of $\tilde{G}$. We define a generalisation of Dirac operators for Drinfeld's Hecke algebras, Parthasarathy operators (Definition 3.1). These operators are such that each Parthasarathy operator leads to an associated inequality (Corollary 3.11). For graded affine Hecke algebras associated to the symmetric group $\mathbb{H}(S_n)$ we define a family of Parthasarathy operators (Corollary 3.10) . Unfortunately, every inequality given by this family is strictly weaker than the original Dirac inequality [2, 12] (Remark 3.12).

The second major result for Dirac operators is 'Vogan's conjecture' and Dirac cohomology. This states that if there exists non-zero Dirac cohomology [24] for an irreducible representation (X, π_X) then the Dirac operator relates the infinitesimal character of X with a character of a diagonal algebra. In the Lie algebra setting this diagonal algebra is $U(\mathfrak{k})$ and for graded affine Hecke algebras the diagonal algebra is $\mathbb{C}\tilde{G}$. Inspired by an infinite family of Dirac operators for the Dunkl angular momentum algebra [8] and the total Dunkl angular momentum algebra [9], we define another generalisation of the Dirac operator, Vogan operators (Definition 4.1). We give an infinite family of Vogan operators for Drinfeld's Hecke algebras (Definition 4.4 and Theorem 4.15). We then prove an equivalent Vogan conjecture for the cohomology of these operators (Theorem 5.14). If there is non-zero cohomology for a Vogan operator $\mathcal{D}_\omega$ (Definition 4.4) then there is a relation between the infinitesimal character and a character for $\widetilde{G}$ (Theorem 5.17). Furthermore, we prove a simple condition for an irreducible representation (X, π_X) to have non-zero cohomology for at least one Vogan operator $\mathcal{D}_\omega$ (Proposition 5.19). We show that this condition is satisfied for every irreducible representation of $\mathbb{H}(S_n)$.

It would be interesting to apply these ideas to Lie algebras [16] and operators as defined by Flake [15], Chan [10] and the local/global theory of for Cherednik algebras introduced by Ciubotaru and De Martino [13]. Application of these ideas to Lie algebras could lead to a generalisation of Dirac cohomology [16] and an enlargement of the Dirac series for real Lie groups [4].

2 Preliminaries

2.1 Drinfeld's Hecke algebra

In this section we define Drinfeld algebras as introduced by Drinfeld. Given a finite group G, anti-symmetric bilinear forms b_g for $g \in G$ and a representation (V, π_V) of G, then we construct an algebra

$$\mathbb{H} = \mathbb{C}G \rtimes T(V)/R.$$

Here R is the two sided ideal of $\mathbb{C}G \rtimes T(V)$ generated by the relations,

$$g^{-1}vg = \pi_V(g)(v) \text{ for all } g \in G \text{ and } v \in V,$$

and

$$[u, v] = \sum_{g \in G} b_g(u, v)g \text{ for all } v, u \in V.$$

We define a filtration on the algebra $\mathbb{C}G \rtimes T(V)/R$, a vector v has degree 1 and a group element $g \in G$ has degree 0.

Definition 2.1. *[14] An algebra of the form $\mathbb{H} = \mathbb{C}G \rtimes T(V)/R$ is a Drinfeld algebra if it satisfies a PBW criterion. That is the associated graded algebra is naturally isomorphic to*

$$\mathbb{C}G \rtimes S(V).$$

We state the conditions on the bilinear forms b_g such that $\mathbb{H}$ is a Drinfeld algebra. Define $G(b) = \{g \in G : b_g \neq 0\}$.

Theorem 2.2. *[14][21, Theorem 1.9] The algebra $\mathbb{H}$ is a Drinfeld algebra if and only if for every $g, h \in G$ and $u, v \in V$, $h' \in Z_G(g)$, $g' \in G(b) \setminus \ker \pi_V$,*

- $b_{g^{-1}hg}(u,v) = b_h(\pi_V(g)(u), \pi_V(g)(v))$
- $\operatorname{Ker} b_{g'} = V^{\pi_V(g')}$ *and* $\dim(V^{\pi_V(g')}) = \dim V - 2$,
- $\det(h'|_{V^{\pi_V(g')^\perp}}) = 1$,

where $V^{\pi_V(g)^\perp} = \{v - \pi_V(g)(v) : v \in V\}$.

Example 2.3. *Let G be the symmetric group S_n, acting by on the $n-1$ dimensional root space V. The root space V has a S_n invariant form $\langle\, , \rangle$. Consider roots $\Phi \subset V$ for S_n and let $\boldsymbol{c}$ be a parameter in $\mathbb{C}$. Define b_g for $g \in S_n$*

$$b_g(u,v) = \begin{cases} 0 & \text{if } g \text{ is not a three cycle,} \\ \boldsymbol{c}\left(\langle\alpha,u\rangle\langle\beta,v\rangle - \langle\alpha,v\rangle\langle\beta,u\rangle\right) & \text{for } g = s_\alpha s_\beta \text{ and } \alpha\beta \in \Phi^+ \text{ such that } \langle\alpha,\beta\rangle_{V^*} \neq 0. \end{cases}$$

This defines the graded affine Hecke algebra associated to S_n with representation $\mathbb{C}^{n-1}$.

2.2 Clifford Algebra

We define some basics on Clifford algebras, for more details see [19]. We assume that V has a G-invariant non-degenerate bilinear form $\langle\, , \rangle_V : V \times V \to \mathbb{C}$. We consider the Clifford algebra $\mathcal{C} := \mathcal{C}(V, \langle\, , \rangle_V)$ with canonical map $\gamma \colon V \to \mathcal{C}$. Let $\{u_i\}$ be a $\langle\, , \rangle_V$-orthonormal basis of V, then $\mathcal{C}$ is generated by $\{e_i = \gamma(u_i)\}$ satisfying

$$\{e_j, e_k\} = e_j e_k + e_k e_j = 2\langle e_j, e_k\rangle_V = 2\delta_{jk}\,.$$

The Clifford algebra is naturally $\mathbb{Z}_2$-graded with $\gamma(V)$ having degree $\bar{1}$. We extend this $\mathbb{Z}_2$ grading to $\mathbb{H} \otimes \mathcal{C}$ by giving every element in $\mathbb{H}$ degree $\bar{0}$.

Definition 2.4. *For a homogeneous element a in $\mathbb{H} \otimes \mathcal{C}$ with $\mathbb{Z}_2$-degree $|a| \in \mathbb{Z}_2$, we define $\epsilon(a) = (-1)^{|a|}$.*

In the Clifford algebra, there is a realisation of the group $\mathsf{Pin} := \mathsf{Pin}(V, B)$, which is a double covering of the orthogonal group $p \colon \mathrm{Pin} \to \mathsf{O}$. We define a double cover $\widetilde{\pi_V G} = p^{-1}(\pi_V G)$. Note that $\widetilde{\pi_V(G)}$ is not a double cover of G but it is a double cover of $\pi_V(G)$. We construct a cover of G. We define $\widetilde{G}$ to be the semi direct product $\operatorname{Ker} \pi_V \rtimes \widetilde{\pi_V(G)}$. Given $\widetilde{G}$ we can embed the group in $\mathbb{H} \otimes \mathcal{C}$ via

$$\rho : \widetilde{G} \to \mathbb{H} \otimes \mathcal{C},$$

$$\rho(\tilde{g}, h) = h p(\tilde{g}) \otimes \tilde{g}, \qquad \tilde{g} \in \widetilde{\pi_V(G)}, h \in \operatorname{Ker} \pi_V.$$

For a reflection $s \in G$ let $\tilde{s}$ denote a preimage in $\tilde{G}$, so $p(\tilde{s}) = \pi_V s$. Let θ be the nontrivial preimage of 1 in $\widetilde{G}$. The element θ is central in $\widetilde{G}$ and has order two: $\theta^2 = 1$.

Definition 2.5. *We define a character* $\operatorname{sgn} : \mathbb{C}\widetilde{G} \to \mathbb{C}$, *such that* $\operatorname{sgn}(\tilde{g}) = \det_{\pi_V}(p(\tilde{g}))$.

Example 2.6. *A Weyl group W (with simple roots Δ and positive roots Φ^+) has presentations*

$$W = \langle s_\alpha, \alpha \in \Phi^+ \mid s_\alpha^2 = 1, s_\alpha s_\beta s_\alpha = s_\gamma, \gamma = s_\alpha(\beta) \rangle,$$

$$W = \langle s_\alpha, \alpha \in \Delta \mid (s_\alpha s_\beta)^{m_{\alpha,\beta}} = 1 \rangle$$

while the double-cover has presentations

$$\widetilde{W} = \langle \theta, \tilde{s}_\alpha, \alpha \in \Phi^+ \mid \tilde{s}_\alpha^2 = 1 = \theta^2, \tilde{s}_\alpha \tilde{s}_\beta \tilde{s}_\alpha = \theta \tilde{s}_\gamma, \gamma = s_\alpha(\beta), \theta \text{ central} \rangle,$$

$$\widetilde{W} = \langle \theta, \tilde{s}_\alpha, \alpha \in \Delta \mid (\tilde{s}_\alpha \tilde{s}_\beta)^{m_{\alpha,\beta}} = (\theta)^{m_{\alpha,\beta}-1}, \theta \text{ central} \rangle.$$

The group algebra $\mathbb{C}\widetilde{G}$ splits into two subalgebras

$$\mathbb{C}\widetilde{G} = \frac{1}{2}(1+\theta)\mathbb{C}\widetilde{G} \oplus \frac{1}{2}(1-\theta)\mathbb{C}\widetilde{G}.$$

The algebra $\frac{1}{2}(1+\theta)\mathbb{C}\widetilde{G}$ is isomorphic to $\mathbb{C}G$. We shall denote the algebra $\frac{1}{2}(1 \pm z)\mathbb{C}\widetilde{G}$ by, respectively $\mathbb{C}\widetilde{G}_\pm$. The algebra $\mathbb{C}G_+$ is isomorphic to $\mathbb{C}G$. Furthermore, ρ is an embedding of $\mathbb{C}\widetilde{G}_-$ into $H \otimes \mathcal{C}$.

2.3 Hermitian forms

Let $*$ denote the anti-automorphism $\eta^* = \varepsilon(\eta^t)$, for all $\eta \in \mathcal{C}_\mathbb{R}$. Let also $\bullet$ be anti-linear form on $\mathbb{H}$ by $v^\bullet = -v$ for $v \in V$, and $w^\bullet = w^{-1}$, for all $w \in G$. We then define an anti-linear anti-involution $\star$ on $\mathbb{H} \otimes \mathcal{C}$ by taking the tensor product of these two anti-involutions.

Definition 2.7. *A Hermitian form $\langle\, , \rangle_X : X \times X \to \mathbb{C}$ is $\mathbb{H}$ invariant if*

$$\langle hx_1, x_2 \rangle_X = \langle x_1, h^\bullet x_2 \rangle_X \text{ for all } x_1, x_2 \in X \text{ and } h \in \mathbb{H}.$$

Definition 2.8. *A $\mathbb{H}$-module X is unitary if there exists a positive definition $\mathbb{H}$-invariant Hermitian form on X.*

Now fix, once and for all, (σ, S) an irreducible module (spinor module) for $\mathcal{C}$.

Definition 2.9. *There exists a positive definite form $\langle\, , \rangle_S$ on S. such that $\langle \gamma(v)s_1, s_2 \rangle = \langle s_1, v^* s_2 \rangle$ for all $v \in V$ and $s_1, s_2 \in S$. This endows S with a $*$-unitary $\mathcal{C}$-structure.*

For any $\bullet$-Hermitian module (π, X) of $\mathbb{H}$ we endow $X \otimes S$ with a $\star$-Hermitian structure $\langle x \otimes s, x' \otimes s' \rangle_{X \otimes S} = \langle x, x' \rangle_X \langle s, s' \rangle_S$ for all $x, x' \in X$ and $s, s' \in S$. If X is $\bullet$-unitary then the $\star$-Hermitian form on $X \otimes S$ is also positive definite and hence $\star$-unitary.

2.4 The original Dirac element

If V has a G-invariant symmetric bilinear form then one can define a Dirac operator $\mathcal{D}$. In [11] (resp. [7]) Dirac cohomology is defined for faithful (resp. non-faithful Drinfeld algebra). In this section we recall definitions and a formula for $\mathcal{D}^2$ from [7]. Given any basis $\{v_i\}$ of V and dual basis $\{v^i\}$ with respect to $\langle\, ,\, \rangle_V$ we define the Dirac element

$$\mathcal{D} = \sum_i v_i \otimes v^i \in \mathbb{H} \otimes \mathcal{C}.$$

For every $g \in G(b)$ set,

$$\mathbf{k}_g = \sum_{i,j} b_g(v_i, v^j) v^i v_j \in \mathcal{C},$$

and

$$\mathbf{h} = \sum_i v_i v^i \in \mathbb{H}.$$

Define the set $G(b) = \{g \in G : b_g \neq 0\}$, write $\widetilde{G}(b)$ for the cover of this subset. For every coset representative $g \in G(b)/\operatorname{Ker}(\pi_V)$ define

$$\tilde{g} = \alpha\beta \in \mathcal{C}, \qquad c_{\tilde{g}} = \frac{b_g(\alpha,\beta)}{1 - \langle \alpha, \beta \rangle^2} \in \mathbb{C}, \qquad e_g = \frac{b_g(\alpha,\beta)\langle \alpha\beta \rangle}{1 - \langle \alpha, \beta \rangle^2} \in \mathbb{C}.$$

Every $w \in \widetilde{g}(b)$ can be written as $h^{-1}gh$ where g is a coset representative of $\widetilde{G}(b)/\operatorname{Ker}\pi_V$ and $h \in \operatorname{Ker}\pi_V$. Lemma [7, Lemma prod of alpha] gives $g = s_\alpha s_\beta$ and $\tilde{g} = \alpha\beta \in \mathcal{C}$. We define, for $w = h^{-1}gh \in \widetilde{G}$, define

$$\tilde{g} = \alpha\beta \in \mathcal{C}, \qquad c_{\tilde{g}} = c_{\tilde{g}}, \qquad e_w = h e_g h^{-1}.$$

Let us define the Casimir elements, $\Omega_{\mathbb{H}}$ in $\mathbb{H}$ and $\Omega_{\widetilde{G}}$ in $\widetilde{G}$.

$$\Omega_{\mathbb{H}} = \mathbf{h} - \sum_{g \in G(b)/\operatorname{Ker}\pi_V} e_g g \in \mathbb{H}^G,$$

$$\Omega_{\widetilde{G}} = \sum_{\substack{h \in \operatorname{Ker}\pi_V \\ g \in G(b)/\operatorname{Ker}\pi_V}} h^{-1}\tilde{g}hc_{\tilde{g}} \in \mathbb{C}[\widetilde{G}]^{\widetilde{G}}.$$

We give a formula for $\mathcal{D}^2$. This is equivalent to [11, Theorem 2.7],. The only variation being that $\ker\pi_V$ replaces 1.

Theorem 2.10. *[7, Theorem 2.4][11, c.f. Theorem 2.7] The square of the Dirac element can be expressed as a sum of the two Casimir elements plus even terms from the Clifford algebra:*

$$\mathcal{D}^2 = -\Omega_{\mathbb{H}} \otimes 1 + \pi_V(\Omega_{\widetilde{G}}) + \frac{1}{2} \otimes \sum_{w \in \ker\pi_V} \boldsymbol{k}_w.$$

Lemma 2.11. *[11, Lemma 2.4]The operator $\mathcal{D}$ sgn-commutes with $\widetilde{G}$,*

$$\rho(\tilde{g})\mathcal{D}\rho(\tilde{g})^{-1} = \operatorname{sgn}(\tilde{g})\mathcal{D}$$

for every $\tilde{g} \in \widetilde{G}$.

3 Parthasarathy Operators

Definition 3.1. *Let $\mathbb{H}$ be a Drinfeld algebra with group algebra $\mathbb{C}G$. We say an operator $\mathcal{P} \in \mathbb{H} \otimes \mathcal{C}$ is a Parthasarathy operator if the following holds:*

1. *$\mathcal{P}^* = \mathcal{P}$*
2. *$\mathcal{P}^2 = z_1 + z_2$, where $z_1 \in Z(\mathbb{H}) \otimes 1$ and $z_2 \in (Z(\rho(\mathbb{C}\widetilde{G}))$.*

3.1 A family of Parthasarathy operators for $\mathbb{H}$

Let $\mathbb{H}$ be a Drinfeld algebra, with a Dirac operator $\mathcal{D} \in \mathbb{H} \otimes \mathcal{C}$.

Definition 3.2. *For $\Xi \in \mathbb{C}\widetilde{G}_-$, we say that $\Xi = \sum \lambda_{\tilde{g}}\tilde{g}$ is P-admissible if*

1. *$\Xi^\bullet = \Xi$,*
2. *$\operatorname{sgn}(\tilde{g}) = -1$ for all $\lambda_{\tilde{g}} \neq 0$,*
3. *$\Xi^2 \in Z(\mathbb{C}\widetilde{G}_-)$.*

Remark 3.3. *The original Dirac operator (Section 2.4) is a Parthasarathy Dirac operator if and only if $\kappa_g = 0$ for every $g \in \operatorname{Ker} \pi_V$.*

For the remainder of this section let us assume that $\kappa_g = 0$ for all $g \in \ker \pi_V$.

Theorem 3.4. *For all P-admissible elements $\Xi \in \mathbb{C}\widetilde{G}_-$, the operator*

$$\mathcal{D}_\Xi = \mathcal{D} + \rho\Xi,$$

is a Parthasarathy operator.

Proof. Clearly, $\mathcal{D}^*_\Xi = \mathcal{D}_\Xi$. Furthermore, $\mathcal{D}^2_\Xi = (\mathcal{D} + \rho\Xi)^2 = \mathcal{D}^2 + \rho\Xi^2 + \mathcal{D}\rho\Xi + \rho\Xi\mathcal{D}$. Note that $\rho\Xi \in \rho\mathbb{C}\widetilde{G}$, hence $\rho\Xi\mathcal{D} = \sum \lambda_{\tilde{g}}\tilde{g}\mathcal{D} = \operatorname{sgn}(\tilde{g})\mathcal{D}\sum \lambda_{\tilde{g}}\tilde{g} = -\mathcal{D}\rho\Xi$. We conclude that,

$$\mathcal{D}^2_\Xi = \mathcal{D}^2 + \rho\Xi^2 = z_1 + z_2$$

where $z_1 \in Z(\mathbb{H})$ and $z_2 = \Omega_{\widetilde{G}} + \Xi^2 \in Z\rho(\widetilde{G})$.

□

3.2 P-admissible elements

Throughout this section we only consider the symmetric group S_n. Recall S_n and $\tilde{S_n}$ have presentations:

$$S_n = \langle s_{ij}, 1 \leq i < j \leq n \mid (s_{ij}s_{kl})^{m_{ijkl}} = 1 \rangle$$

where

$$m_{ijkl} = \begin{cases} 2 & |\{i,j,k,l\}| = 4, \\ 3 & |\{i,j,k,l\}| = 3, \\ 1 & |\{i,j,k,l\}| = 2. \end{cases}$$

Similary for $\widetilde{S}_n$

$$\tilde{S_n} = \langle \theta, \tilde{s}_{ij}, 1 \leq i < j \leq n \mid (\tilde{s}_{ij}\tilde{s}_{kl})^{m_{ijkl}} = (\theta)^{m_{ijkl}-1}, \theta \text{ central} \rangle.$$

Definition 3.5. *[5, 3.1] The Jucys-Murphy elements in $\mathbb{C}S_n^-$ for $i = 1, \ldots, n$ are,*

$$M_j = \sum_{i=1}^{j-1} \tilde{s}_{ij}.$$

Remark 3.6. *[5, 3.1] The Jucys-Murphy elements anti-commute , that is*

$$M_i M_j = -M_j M_i \text{ if } i \neq j.$$

Lemma 3.7. *[5, 3.2] The even centre $Z(\tilde{S}_{n-})_0$ is spanned by the set of symmetric polynomials of the Jucys-Murphy elements.*

Proposition 3.8. *Every odd symmetric power-polynomial in the squares of the Jucys-Murphy elements is a square of an odd symmetric polynomial in the Jucys-Murphy elements.*

Proof. Suppose that k is odd. Let $P_k = \sum_{i=1}^{n} (M_i^2)^k$ and let $Q_k = \sum_{i=1}^{n} (M_i)^k$. We claim that $P_k = Q_k^2$.

$$Q_k^2 = (\sum_{i=1}^{n} (M_i)^k)^2 = \sum_{i=1}^{n} (M_i)^k \sum_{j=1}^{n} (M_j)^k = \sum_{i=1}^{n} (M_i)^{2k} + \sum_{i \neq j} (M_i)^k (M_j)^k + (M_j)^k (M_i)^k$$

Now since k is odd and if $i \neq j$ then $(M_i)^k (M_j)^k = -(M_j)^k (M_i)^k$. Hence we have shown that $Q_k^2 = P_k$.

□

Theorem 3.9. *Let $G = S_n$, let Q_j be an odd power polynomial in the Jucys-Murphy elements Q_j. Then $\sqrt{-1}Q_j$ is P-admissible.*

Proof. The element, Q_j, consists of sums of odd polynomials in $\tilde{s}_{kl}$, all of which have odd *sgn* group elements. Each pseudo reflection $\tilde{s}_{kl}$ is such that $\rho \tilde{s}_{kl}^{\bullet} = -\rho \tilde{s}_{kl}$, hence Q_j is skew-adjoint and $\sqrt{-1}Q_j$ is self-adjoint. To complete the proof we note that Q_j^2 is central in $\mathbb{C}\widetilde{G}_-$. □

Corollary 3.10. *Let $\mathbb{H}(S_n)$ be a graded affine Hecke algebra with the symmetric group. For every odd j, then the operator*

$$\mathcal{D} + \sqrt{-1}Q_j$$

is a Parthasarathy operator.

3.3 A family of Dirac inequalities

Suppose that X is unitary (Definition 2.7), then any Parthasarathy operator $\mathcal{P}$ is self-adjoint. Furthermore, $X \otimes S$ has a $\mathbb{H} \otimes \mathcal{C}$-invariant positive definite form. Hence, $\mathcal{P}^2$ is a positive operator.

Corollary 3.11 (Generalised Parthasarathy inequality)**.** *For every Parthasarathy operator $\mathcal{P}$ and unitary module (X, π_X), then $\mathcal{P}^2$ is positive operator on $X \otimes S$ and*

$$\pi_X(z_1) + \pi_X(z_2) \geq 0.$$

Here, using Definition 3.1, $\mathcal{P}^2 = z_1 + z_2$, with $z_1 \in Z(\mathbb{H})$, $z_2 \in Z(\rho(\mathbb{C}\widetilde{G}))$.

Remark 3.12. *Unfortunately, every Parthasarathy operator defined in Definition 3.2 gives an inequality weaker than the original Dirac inequality. Let $\mathcal{D}_\Xi$ be defined as in Definition 3.2, then Ξ is self-adjoint. Therefore, $\pi_X(\Xi)^2$ is a positive operator. The inequality associated to $\mathcal{D}_\Xi$ is*

$$-\pi_X(\Omega_{\mathbb{H}}) + \pi_X(\Omega_{\widetilde{G}}) + \pi_X(\Xi)^2 \geq 0.$$

Since $\pi_X(\Xi)^2$ is positive this is less restrictive than the original Dirac inequality,

$$-\pi_X(\Omega_{\mathbb{H}}) + \pi_X(\Omega_{\widetilde{G}}) \geq 0.$$

An interesting question, which the author intends to study, is whether there are any Parthasarathy operators in $\mathbb{H} \otimes \mathcal{C}$ which lead to new relations between the centre of $\mathbb{H}$ and the centre of $Z(\widetilde{G})$.

4 Vogan operators

Definition 4.1. *Let $\mathbb{H}$ be a Drinfeld algebra, we say an operator $\mathcal{V} \in \mathbb{H} \otimes \mathcal{C}$ is a Vogan operator if the following holds:*

1. *$\mathcal{V}^* = \mathcal{V}$*
2. *The operator $\mathcal{V}$ is* sgn*-invariant under the action of $\rho\widetilde{G}$.*

Definition 4.2. *Let X be a $\mathbb{H}$-module and S a spinor for $\mathcal{C}$, the operator $\mathcal{V}$ acts on $X \otimes S$ as $\mathcal{V}_X$. Define $H(X, \mathcal{V})$ as*

$$\operatorname{Ker} \mathcal{V}_{X\otimes S} / \operatorname{Ker} \mathcal{V}_{X\otimes S} \cap \operatorname{im} \mathcal{V}_{X\otimes S}.$$

Proposition 4.3. *The cohomology of $\mathcal{V}$ is a $\widetilde{G}$ module.*

Proof. This follows from the fact that $\mathcal{V}$ is $\operatorname{sgn} -\widetilde{G}$ invariant. □

4.1 A family of Vogan operators for $\mathbb{H}$

Definition 4.4. *We define the* sgn-*centre of* $\mathbb{C}\widetilde{G}_-$ *to be:*

$$Z_{\text{sgn}}(\widetilde{G}_-) = \{g \in \mathbb{C}\widetilde{G}_- : gh = \text{sgn}(h)hg \quad \textit{for all } h \in \mathbb{C}\widetilde{G}\}.$$

Furthermore, we say an element is sgn-*central if it is contained in the* sgn-*centre.*

The ungraded centre of $Z^{ug}\mathbb{C}\widetilde{G}_-$ is equal to $\text{Hom}_{\widetilde{G}}(\text{triv}, \mathbb{C}\widetilde{G}_-)$ and the sgn-centre of $\mathbb{C}\widetilde{G}_-$ is equal to $\text{Hom}_{\widetilde{G}}(\text{sgn}, \mathbb{C}\widetilde{G}_-)$.

Definition 4.5. *A homogeneous element* $\omega \in \mathbb{C}\widetilde{G}_-$ *is called* V**-admissible** *if* ω *is* sgn-*central and* $\omega^\bullet = \omega$. *For any* V*-admissible* $\omega \in Z_{\text{sgn}}\widetilde{G}_-$, *define*

$$\mathcal{D}_\omega := \mathcal{D} + \rho\omega \in \mathbb{H} \otimes \mathcal{C}.$$

Theorem 4.6. *The elements* $\mathcal{D}_\omega$ *are all Vogan operators. Furthermore, they are precisely the modification of* $\mathcal{D}$ *by elements in* $\rho\mathbb{C}\widetilde{G}$ *which are Vogan operators.*

4.2 A formula for $\mathcal{D}_\omega^2$

Lemma 4.7. *The square of* $\mathcal{D}_\omega$ *is equal to a central element in* $\mathbb{H}$ *plus a central element in* $\widetilde{G}$, *a linear term in* $\mathcal{D}_\omega$ *and a correction quadratic term in* $\mathcal{C}$;

$$(\mathcal{D}_\omega)^2 = -\Omega_{\mathbb{H}} \otimes 1 + \rho(\Omega_{\widetilde{G}} + \omega^2) + +(1 + \text{sgn}(\rho(\omega)))\rho(\omega)\mathcal{D} + \frac{1}{2} \otimes \sum_{w \in \ker \pi_V} \boldsymbol{k}_w.$$

Proof. The following calculation is simple algebra,

$$\begin{aligned}
(\mathcal{D}_\omega)^2 &= (\mathcal{D} + \rho(\omega))^2 \\
&= (\mathcal{D})^2 + \rho(\omega)^2 + \mathcal{D}\rho(\omega) + \rho(\omega)\mathcal{D} \\
&= (\mathcal{D})^2 + \rho(\omega)^2 + \text{sgn}(\rho(\omega))\rho(\omega)\mathcal{D} + \rho(\omega)\mathcal{D}.
\end{aligned}$$

Finishing the proof with application of Theorem 2.10. □

Lemma 4.8 (sgn invariance of D_ω)**.** *For every* $\tilde{g} \in \widetilde{G}$, *we have the invariance property:*

$$\rho(\tilde{g})\mathcal{D}_\omega\rho(\tilde{g})^{-1} = \text{sgn}(p(\tilde{g}))\mathcal{D}_\omega.$$

Proof. Both $\mathcal{D}$ and ω are sgn invariant, hence so it their sum $\mathcal{D}_\omega = \mathcal{D} + \rho(\omega)$ □

4.3 V-Admissible elements

Definition 4.9. *[9, Definition 6.3] Let us the define the* θ*-centre of* $\mathbb{C}\widetilde{G}$,

$$Z^\theta(\mathbb{C}\widetilde{G}) = \{a \in \mathbb{C}\widetilde{G} | a\tilde{g} = \theta^{\text{sgn}(p(\tilde{g}))}\tilde{g}a \textit{ for all } \tilde{g} \in \widetilde{G}\}.$$

Proposition 4.10. *[9, Proposition 6.4] The θ-centre of $\mathbb{C}\widetilde{G}$ is spanned by elements of the form*

$$C^{\theta}_{\tilde{g}} = \sum_{\tilde{h}\in\widetilde{G}} \theta^{|l(\tilde{h})|}\tilde{h}^{-1}\tilde{g}\tilde{h} = \sum_{\tilde{h}\in\widetilde{G}_{\overline{0}}} \tilde{h}^{-1}\tilde{g}\tilde{h} + \theta \sum_{\tilde{h}\in\widetilde{G}_{\overline{1}}} \tilde{h}^{-1}\tilde{g}\tilde{h}$$

for any choice of $\tilde{g} \in \widetilde{G}$.

Proof. Given any $\tilde{g}$, then $C^{\theta}_{\tilde{g}}$ is in $Z^{\theta}\mathbb{C}\widetilde{G}$. Furthermore, any $a \in Z^{\theta}\mathbb{C}\widetilde{G}$ that has a non-zero coefficient of $\tilde{g}$, then there exists a non-zero scalar t such that $a - tC^{\theta}_{\tilde{g}}$ is θ-central with no coefficient of $\tilde{g}$. Continuing the process shows that a is in the space spanned by $C^{\theta}_{\tilde{g}}$. □

Theorem 4.11. *[9, Theorem 6.5]*
The sgn*-centre of $\mathbb{C}\widetilde{G}_-$ is the projection of the θ-centre of $\mathbb{C}\widetilde{G}$*

$$Z^{\mathrm{sgn}}(\mathbb{C}\widetilde{G}_-) = \frac{1-\theta}{2} Z^{\theta}\mathbb{C}\widetilde{G}.$$

In particular, the sgn-centre of $\mathbb{C}\widetilde{G}_-$ is spanned by elements of the form

$$C^{\mathrm{sgn}}_{\tilde{g}} = \sum_{\tilde{h}\in\widetilde{G}} (-1)^{|l(\tilde{h})|}\tilde{h}^{-1}\tilde{g}\tilde{h} \in \mathbb{C}\widetilde{G}_-.$$

Denote by $\widetilde{G}_{\bar{0}}$ the even subgroup of $\widetilde{G}$.

Lemma 4.12. *[9, Lemma 6.7] Suppose that $\tilde{g} \in \widetilde{G}$ is even, define the $\tilde{g}$ conjugacy class, $C(\tilde{g}) = \{\tilde{w} \in \widetilde{G} : \tilde{g} = \tilde{h}^{-1}\tilde{g}\tilde{h}, \tilde{h} \in \widetilde{G}\}$, then the element $Z^{\epsilon}_{\tilde{g}}$ is non zero if and only if the conjugacy class $C(\tilde{g})$ splits into two conjugacy classes in $\widetilde{G}_{\overline{0}}$.*

Theorem 4.13. *[22] [23, Theorem 2.7] Let λ be an even partition of n. The $\tilde{S}_n$ conjugacy classes C_λ (or $C^{\pm}_{\lambda}$) if already split) split into two $\tilde{A}_n$ conjugacy classes if and only if $\lambda \in DP^+_n$. Here DP^+_n is the set of distinct partitions of n which are even.*

Corollary 4.14. *The sgn-centre of $\mathbb{C}\widetilde{G}_-$ has basis*

$$\{C^{\mathrm{sgn}}_{\tilde{g}} : C(\tilde{g}) \textit{ splits into two conjugacy classes in } \widetilde{G}_{\overline{0}}\}.$$

Due to Corollary 4.14 we may assume that any V-admissible element in $\mathbb{C}\widetilde{G}_-$ is even.

Theorem 4.15. *The V-admissible elements in $\mathbb{C}\widetilde{G}_-$ are equal to the real-span of the set*

$$\{C^{\mathrm{sgn}}_{\tilde{g}} + C^{\mathrm{sgn}}_{\tilde{g}^{-1}}, \sqrt{-1}(C^{\mathrm{sgn}}_{\tilde{g}} - C^{\mathrm{sgn}}_{\tilde{g}^{-1}}) : C(\tilde{g}) \textit{ splits into two conjugacy classes in } \widetilde{G}_{\overline{0}}\}.$$

Proof. The V-admissible elements are elements in the sgn-centre which are self adjoint. Since $\rho\tilde{g}^\bullet = \rho\tilde{g}^{-1}$, then taking a basis for the sgn-centre from Corollary 4.14 and adding or subtracting $C^{\mathrm{sgn}}_{\tilde{g}^{-1}})$ enforces this set to be self adjoint (or skew adjoint respectively). Multiplying by $\sqrt{(}-1)$ forces the skew adjoint operators to be self adjoint. This then spans all self-adjoint operators in the sgn-centre. □

Corollary 4.16. *Let $G = S_n$ and let $\{g\}$ be the set of elements in S_n associated to an even partition λ which has distinct cycles. Then $C(\tilde{g})$ splits in $\tilde{A}_n = (\tilde{S}_n)_{\overline{0}}$. Then from Lemma 4.12, $C^{\mathrm{sgn}}_{\tilde{g}} \neq 0$. The group $\widetilde{S}_n$ is ambivalent, that is, $\tilde{g}^{-1}$ is always conjugate to $\tilde{g}$. Therefore every nonzero element $C^{\mathrm{sgn}}_{\tilde{g}} = \frac{1}{2}(C^{\mathrm{sgn}}_{\tilde{g}} + C^{\mathrm{sgn}}_{\tilde{g}^{-1}})$ is an admissible element and $C^{\mathrm{sgn}}_{\tilde{g}} - C^{\mathrm{sgn}}_{\tilde{g}^{-1}} = 0$ for every $C^{\mathrm{sgn}}_{\tilde{g}}$.*

5 Vogan's Dirac morphism

In this section we prove that there exits a Vogan morphism for each Vogan operator we defined in Definition 4.4 and that Vogan's conjecture holds for each of these operators. Furthermore, we show that each operator $\mathcal{D}_\omega$ defines a map between $\operatorname{Irr}\widetilde{G}$ and $\operatorname{Spec}\mathcal{B}$ for a suitably chosen abelian algebra $\mathcal{B}$. We finish by showing that the defined family of Vogan operators gives rise to non-zero cohomology of (X, π_X) whenever there is an admissible element that is not in the kernel of π_X.

5.1 The linear map d_ω

Definition 5.1. *Let $\mathcal{D}_\omega$ be a Vogan operator as defined in Definition 4.4. We define a map from $\mathbb{H}\otimes\mathcal{C}$ to $\mathbb{H}\otimes\mathcal{C}$.*

$$d_\omega : \mathbb{H}\otimes\mathcal{C} \to \mathbb{H}\otimes\mathcal{C},$$

where $d_\omega(a) = \mathcal{D}_\omega a - \epsilon(a)\mathcal{D}_\omega$, for $a \in \mathbb{H}\otimes\mathcal{C}$.

Remark 5.2. *The map d_ω is an odd derivation, i.e.,*

$$d_\omega(ab) = d_\omega(a)b + \epsilon(a)d_\omega(b)$$

for all $a, b \in \mathbb{H}\otimes\mathcal{C}$.

Lemma 5.3. *The image under ρ of $\mathbb{C}\widetilde{G}$ is in the kernel of d_ω*

$$\rho(\mathbb{C}\widetilde{G}) \subset \ker d_\omega.$$

Proof. This follows from the fact that $\mathcal{D}_\omega$ is $\mathrm{sgn}-\widetilde{G}$ invariant. □

The operator $\mathcal{D}_\omega$ intertwines the sgn and triv $\widetilde{G}$ isotypic components $\mathbb{H}\otimes\mathcal{C}^{\mathrm{triv}}$ and $\mathbb{H}\otimes\mathcal{C}^{\mathrm{sgn}}$. We define d^{triv}_ω and d^{sgn}_ω to be d_ω restricted to the triv and sgn isotypic components respectively. Since the kernel of d_ω, contains $\rho(\mathbb{C}\widetilde{G})$, then the kernel of d^{triv}_ω, contains the triv-isotypic component $\rho(\mathbb{C}\widetilde{G}^{\widetilde{G}})$.

Theorem 5.4. *The kernel of d_ω^{triv} equals:*

$$\ker d_\omega^{\mathrm{triv}} = \operatorname{im} d_\omega^{\mathrm{sgn}} \oplus \rho(\mathbb{C}\widetilde{G}^{\widetilde{G}}).$$

We prove this theorem in the following section.

5.2 The map $\bar{d_\omega}$

In this section we proof Theorem 5.4 by reducing to the associated graded algebra. This method is identical method to [3, 11]. The algebra $\mathbb{H}$ is filtered with $g \in \mathbb{C}G$ degree zero and $v \in V$ degree one. Similarly $\mathcal{C}$ is filtered with $\gamma(V) \in \mathcal{C}$ of degree one. Hence $\mathbb{H} \otimes \mathcal{C}$ is a filtered algebra and has an associated graded algebra $\mathrm{gr}\mathbb{H} \otimes \mathrm{gr}\mathcal{C}$.

Definition 5.5. *The map d_ω defines a map on the associated graded algebra*

$$\bar{d_\omega} : \mathrm{gr}\mathbb{H} \otimes \mathrm{gr}\mathcal{C} \to \mathrm{gr}\mathbb{H} \otimes \mathrm{gr}\mathcal{C}.$$

Lemma 5.6. *The map $\bar{d_\omega}$ is equal to $\bar{d_0}$ for any choice of $\omega \in \rho\mathbb{C}\widetilde{G}$. Hence, the proof in [11] holds for the following statements.*

Proof. Because $\mathcal{D}_\omega = \mathcal{D}_0 + l.o.t.$ then $\mathrm{gr}\mathcal{D}_\omega = \mathrm{gr}\mathcal{D}_0$ and $\bar{d_\omega} = \bar{d_0}$ for any $\omega \in \mathbb{C}\widetilde{G}_-$. ☐

Proposition 5.7. *(cf. [3, Proposition 4.14]). The space $\ker \bar{d_0}^{\mathrm{triv}}$ decomposes as*

$$\ker \bar{d_0}^{\mathrm{triv}} = \operatorname{im} \bar{d_0}^{\mathrm{sgn}} \oplus \rho(\mathbb{C}\bar{\widetilde{G}}^{\widetilde{G}}).$$

Corollary 5.8. *We have the following decomposition of $\ker \bar{d_\omega}^{\mathrm{triv}}$*

$$\ker \bar{d_\omega}^{\mathrm{triv}} = \operatorname{im} \bar{d_\omega}^{\mathrm{sgn}} \oplus \rho(\mathbb{C}\bar{\bar{\widetilde{G}}}^{\widetilde{G}}).$$

5.2.1 Induction

We have proved that $\ker \bar{d_\omega} = \operatorname{im} \bar{d_\omega} \oplus \rho(\mathbb{C}\bar{\bar{\widetilde{G}}}^{\widetilde{G}})$. To prove the equivalent statement for d_ω (Theorem 5.4) an induction on the filtration of $\mathbb{H} \otimes \mathcal{C}$. The proof is verbatim to [11, 3.3] and we omit it here.

5.3 Vogan's Dirac homomorphism

Theorem 5.9. *The projection $\zeta_\omega : \ker d_\omega^{\mathrm{triv}} \to \mathbb{C}\widetilde{G}^{\widetilde{G}}$ defined by Theorem 3.5 is an algebra homomorphism.*

Proof. Let $z_1, z_2 \in \ker d_\omega^{\mathrm{triv}}$, then $z_i = \zeta(z_i) + d_\omega^{\mathrm{sgn}}(a_i)$. Multiplying z_1, with z_2, we find

$$z_1 z_2 = \zeta(z_1)\zeta(z_2) + d_\omega^{\mathrm{sgn}}(a_1)\zeta(z_2) + \zeta(z_1) d_\omega^{\mathrm{sgn}}(a_2) + d_\omega^{\mathrm{sgn}}(a_1) d_\omega^{\mathrm{sgn}}(a_2).$$

However, d_ω^{sgn} is a derivation and $d_\omega^{\mathrm{sgn}}(\rho\mathbb{C}\widetilde{G}) = 0$, hence

$$z_1 z_2 = \zeta(z_1)\zeta(z_2) + d_\omega^{\mathrm{sgn}}(a_1\zeta(z_2)\zeta(z_1)a_2) + d_\omega^{\mathrm{sgn}}(a_1)d_\omega^{\mathrm{sgn}}(a_2).$$

To finish the proof, we remark that $(d_\omega^{\mathrm{sgn}})^2 = 0$. Therefore $d_\omega^{\mathrm{sgn}}(a_1)d_\omega^{\mathrm{sgn}}(a_2) = d_\omega^{\mathrm{sgn}}(a_1 d_\omega^{\mathrm{sgn}}(a_2))$. We can conclude that z_1z_2 is equal to $\zeta(z_1)\zeta(z_2)$ plus an element in the image of d_ω^{sgn}. ◻

Definition 5.10. *Let $\mathcal{B}$ be an abelian subalgebra of $\mathbb{H} \otimes \mathcal{C}^{\widetilde{G}}$ containing $\Omega_{\tilde{\mathbb{H}}}$.*

Proposition 5.11. *The dual of ζ_ω is a map between $\operatorname{Irr}\widetilde{G}$ and $\operatorname{Spec}\mathcal{B}$,*

$$\zeta_\omega^* : \operatorname{Irr}\widetilde{G} \to \operatorname{Spec}\mathcal{B}.$$

Remark 5.12. *If $\kappa_1 = 0$, then we can take $\mathcal{B} = Z(\mathbb{H}) \otimes 1$. Then ζ_ω is a map between the characters of $\widetilde{G}$ and infinitesimal characters for $\mathbb{H}$.*

5.4 Vogan's conjecture

Definition 5.13. *Let $\mathcal{V}$ be an operator in $\mathbb{H}\otimes\mathcal{C}$. We say that Vogan's conjecture for $\mathcal{V}$ holds if, for all $z \in Z(\mathbb{H})$, there exists a unique $\zeta(z) \in Z(\rho\mathbb{C}\widetilde{G})$ and $a, b \in \mathbb{H} \otimes \mathcal{C}$ such that*

$$z \otimes 1 = \zeta(z) + \mathcal{V}a + a\mathcal{V}.$$

Theorem 5.14. *Vogan's conjecture holds for the family of Vogan operators defined in Definition 4.4.*

Proof. By Theorem 5.4, for every $z \in \ker d_\omega^{\mathrm{triv}}$, there exists an $a \in \mathbb{H} \otimes \mathcal{C}$ such that $z = d_\omega^{\mathrm{sgn}}(a) + \zeta_\omega(z)$. The claim follows by observing that $Z(\mathbb{H})$ is in the kernel of d_ω^{triv} and $d_\omega^{\mathrm{sgn}}(a) = \mathcal{D}_\omega a + a\mathcal{D}_\omega$. ◻

5.5 Dirac cohomology

Let (X, π_X) be a representation of $\mathbb{H}$, We say that X is an admissible module if the decomposition of X into $\Omega_{\mathbb{H}}$-generalized eigenspaces

$$X = \bigoplus_{\lambda\in\mathbb{C}} X_\lambda$$

is such that every X_λ is finite dimensional.

Definition 5.15. *For an irreducible $\mathbb{H}$-representation (X, π_X), let $\chi : Z(\mathbb{H}) \to \mathbb{C}$ be the infinitesimal character $\pi_X|_{Z(\mathbb{H})}$.*

Definition 5.16. *Let X be an admissible $\mathbb{H}$-module and let S be a spinor for $\mathcal{C}$, then $X \otimes S$ is a $\mathbb{H} \otimes \mathcal{C}$ module and $\mathcal{D}_\omega \in \mathbb{H} \otimes \mathcal{C}$ acts*

$$(D_\omega)_X : X \otimes S \to X \otimes S,$$

The Dirac ω-cohomology of X (and S) is defined as

$$H(X, \omega) = \ker(D_\omega)_X / \ker(D_\omega)_X \cap \operatorname{im}(D_\omega)_X.$$

Since $\mathcal{D}_\omega$ sgn-commutes with $\widetilde{G}$ the Dirac ω-cohomology of an admissible X is a finite dimension $\widetilde{G}$ module, or zero.

Theorem 5.17. *Let X be an admissible irreducible $\mathbb{H}$-module. Let $\mathcal{B}$ be an algebra as in Definition 5.10 Suppose $H(X,\omega) \neq 0$. If there is a nonzero $\sigma \otimes \chi$-isotypic $\mathbb{C}\widetilde{G} \otimes B$ component in $H(X,\omega)$ for $\sigma \in \operatorname{Irr}(\widetilde{G})$ and $\chi \in \operatorname{Spec} B$, then*

$$\chi_X = \zeta_\omega^*(\sigma).$$

Proof. Once one has proved a Vogan conjecture for $\mathcal{D}_\omega$ (that is Theorem 5.14) then the proof is identical to [11, Theorem 3.14] given [11, Theorem 3.8]. □

Recall Theorem 4.15 states that all V-admissible elements for $\mathbb{C}\tilde{G}_-$ are even. Suppose that ω is V-admissible, because every group element occurring in ω is even then ω and $\mathcal{D}$ commute.

Proposition 5.18. *If X is a $\bullet$-unitary $\mathbb{H}$-module, then $H(X,\omega) = \ker(\mathcal{D}_\omega)$.*

Proof. If X is $\bullet$-Hermitian then the image and kernel of $\mathcal{D}_\omega$ are orthogonal with respect to $\langle\, , \,\rangle_{X\otimes S}$ and hence $\ker(\pi\mathcal{D}_\omega) \cap \operatorname{im}(\pi\mathcal{D}_\omega) = 0$ □

Proposition 5.19. *Let (π_X, X) be a $\bullet$-unitary module for $\mathbb{H}$. Suppose that there exists an admissible element $\omega \in \mathbb{C}\widetilde{G}_-$ such that $\pi_X(\omega) \neq 0$. Then, there exists an admissible $\lambda\omega \in \mathbb{C}\widetilde{G}_-$ such that $H(X, \lambda\omega) \neq 0$.*

Proof. Since X is Hermitian then $H(X,\omega) = \ker(\mathcal{D}_C) = \ker(\mathcal{D}_C^2)$. We study the kernel of the operator $(\mathcal{D}_\omega - 2\rho(\omega))\mathcal{D}_\omega = \mathcal{D}^2 - \rho(\omega)^2$. The elements $\mathcal{D}^2$ and $\rho(\omega)$ commute, hence have simultaneous eigenvalues. Given $\pi(\rho(\omega))^2 \neq 0$ it has positive real eigenvalues. Let $\pi_{X\otimes S}\mathcal{D}^2$ and $\pi_{X\otimes S}(C)^2$ have positive simultaneous eigenvalues $d \geq 0$ and $c > 0$ respectively. One can modify $\mathcal{D}_\omega$ to $\mathcal{D}_{\lambda\omega}$, with $\lambda = \sqrt{\frac{c}{d}}$. This ensures that

$$\mathcal{D}_0^2 - \lambda^2\rho(\omega)^2$$

has a non-zero kernel on $X \otimes S$. Thus proving that there exists a non-zero kernel for the operator $(\mathcal{D}_\omega - 2\rho(\omega))\mathcal{D}_\omega$. Since the composition of injective functions is injective, we find that one of $(\mathcal{D}_\omega - 2\rho(\omega))$ or $\mathcal{D}_\omega$ has a non-zero kernel. □

Remark 5.20. *When $G = S_n$, we have shown that the set of V-admissible elements in $\mathbb{R}\tilde{W}_-$ is equal to conjugacy classes associated to even partitions. From [17], $Z_0(\mathbb{R}\tilde{W}_-)$ is equal to $\{\sigma(M_1^2, \ldots, M_n^2) \mid \sigma \text{ real symmetric polynomial}\}$. The element $\sigma(M_1^2, \ldots, M_n^2)$ acts on an irreducible representation by evaluating σ at specific real values [6, Corollary 6.3]. In particular, we can conclude that for every $\mathbb{C}\tilde{W}$-module there is an admissible element that does not act by zero.*

Theorem 5.21. *Let $\mathbb{H}$ be a degenerate affine Hecke algebra associated to the symmetric group S_n. Then for any unitary $\mathbb{H}$-module X, there exists a Vogan operator $\mathcal{D}_\omega$ such that*

$$H(X,\omega) \neq 0.$$

5.6 Vogan inequalities

Throughout this section we assume that $\kappa_g = 0$ for all $g \in \ker \pi_V$.
Recall Theorem 4.15 states that all V-admissible elements for $\mathbb{C}\tilde{G}_-$ are even. Let $\omega = \sum c_{\tilde{g}}\tilde{g}$, this then implies $\mathrm{sgn}(\tilde{g}) = 1$ for every $c_{\tilde{g}} \neq 0$. Suppose that ω is V-admissible, because every group element occurring in ω is even then ω and $\mathcal{D}$ commute. On a $\star$-unitary module $X \otimes S$, the operator, $\mathcal{D}_\omega^2$ is positive.

Proposition 5.22. *Let (X, π_X) be a $\bullet$-unitary module, then $(X \otimes S, \pi_{X\otimes S})$ is $\star$-unitary and $\mathcal{D}_\omega^2 = \mathcal{D}^2 + 2\mathcal{D}\rho(\omega) + \rho(\omega)^2 \geq 0$. Hence for every V-admissible ω the following inequality holds ,*

$$\pi_X(\Omega_{\mathbb{H}}) \leq \pi_{X\otimes S}\Omega_{\tilde{G}} + \pi_{X\otimes S}((\omega + 2\mathcal{D})\omega).$$

Specialising the above inequality to $\omega = 0$ gives the Dirac inequality. Proposition 5.19 states that if there exists an ω such that $\pi_{X\otimes S}(\omega) \neq 0$ then there exists a V-admissible element such that the above inequality becomes an equality. For the Hecke algebra associated to the symmetric group there is always non-zero ω-cohomology, this implies that for a particular choice of ω it is always possible to make the inequality above strict.

6 Parthasarathy and Vogan operators

Proposition 6.1. *Consider elements of the form $\mathcal{D} + \rho A_{\widetilde{G}}$, where $A_{\widetilde{G}} \in \mathbb{C}\widetilde{G}_-$. The only operator of this form that is both a Vogan and Parthasarathy operator is*

$$\mathcal{D} + 0 = \mathcal{D}.$$

Proof. Suppose that $\mathcal{D} + \rho A_{\widetilde{G}}$ is a Parthasarathy operator, by construction sgn of every group element in $A_{\tilde{g}}$ is -1. Now suppose that $\mathcal{D} + \rho A_{\widetilde{G}}$ is a Vogan operator, then by Theorem 4.15, sgn of every group element is 1. Therefore $A_{\widetilde{G}}$ has no nonzero coefficient of group elements and hence must be zero.

□

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