On Cut Polytopes and Graph Minors

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Abstract

The *max-cut problem* is a fundamental and much-studied $\mathcal{NP}$-hard combinatorial optimisation problem, with a wide range of applications. Several authors have shown that the max-cut problem can be solved in polynomial time if the underlying graph is free of certain *minors*. We give a polyhedral counterpart of these results. In particular, we show that, if a family of valid inequalities for the cut polytope satisfies certain conditions, then there is an associated minor-closed family of graphs on which the max-cut problem can be solved efficiently.

Keywords. Max-cut, polytope, graph minor.

AMS subject classification. 90C27, 90C57, 05C83.

1 Introduction

Given an undirected graph G, with rational weights on the edges, the *max-cut problem* (MCP) calls for a partition of the node set into two subsets, such that the total weight of the edges crossing between the subsets is maximised. The MCP is a fundamental combinatorial optimisation problem, with a wide range of applications (e.g., [9, 24]). It is also $\mathcal{NP}$-hard in the strong sense [14]. Current exact algorithms struggle on graphs with more than 150 nodes or so (e.g., [19, 25]).

The convex hull of the incidence vectors of cuts in G is called the *cut polytope* of G, and denoted by CUT(G). Cut polytopes have been studied in great depth, and many families of valid and facet-defining inequalities are known (e.g., [3, 9]).

A graph G' is called a *minor* of G if it can be obtained from G via a series of edge deletions, edge contractions and/or deletions of isolated nodes. A celebrated theorem of Robertson and Seymour [26]

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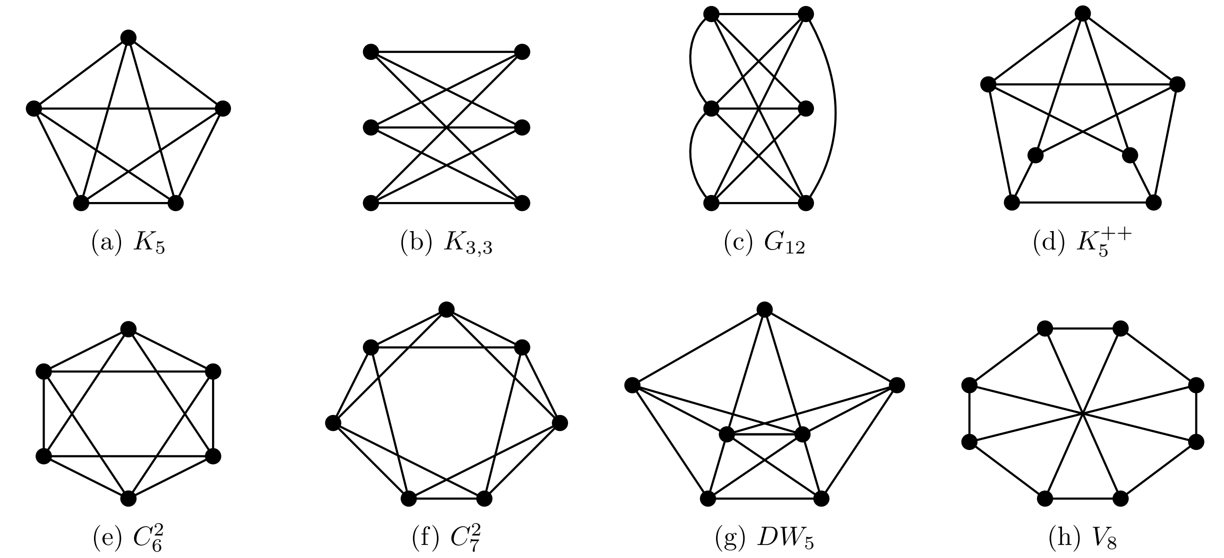


(a) K_5 (b) $K_{3,3}$ (c) G_{12} (d) K_5^{++}

(e) C_6^2 (f) C_7^2 (g) DW_5 (h) V_8

Figure 1: Eight fundamental graphs

states that any minor-closed family of graphs can be characterised by a finite set of minimal excluded minors (sometimes called “obstructions”). For algorithmic implications of this result, see, e.g., [4, 21].

Fig. 1 shows eight graphs that will be of importance in this paper. One can check that K_5 and $K_{3,3}$ are minors of G_{12} and K_5^{++}, which in turn are minors of C_7^2 and DW_5. One can also check that $K_{3,3}$ is a minor of V_8, and C_6^2 is a minor of DW_5. We remark that C_6^2 is sometimes called the *octahedron* (e.g., [10]).

Barahona [1] established a link between the MCP and graph minor theory, by showing that the MCP can be solved in polynomial time when G does not have K_5 as a minor. Barahona and Mahjoub [3] gave a polyhedral counterpart to this result, by giving a complete linear description of CUT(G) for K_5-minor-free graphs. Later on, Barahona [2] showed that the cut polytope of any K_5-minor-free graph is the projection of a polytope with a polynomial number of facets.

Truemper [27] generalised the result in [1], by showing that the MCP is solvable in polynomial time when G does not have G_{12} as a minor. Kamiński [17] proved an even more general result, by showing that the MCP is solvable in polynomial time on H-minor-free graphs, for any graph H that can be drawn in the plane with at most one edge crossing. Until now, however, no polyhedral counterparts of these results had been derived.

This paper constitutes a first step in filling this gap in the literature. Our approach is as follows. Given a (possibly infinite) family F of valid inequalities for cut polytopes of complete graphs, we define a (possibly infinite) family of graphs, called “F-friendly” graphs. Under certain conditions on F, we can show that (a) the family of F-friendly graphs is minor-closed, and (b) the max-cut problem can be solved in polynomial time on F-friendly graphs.

As a concrete application of our approach, we study the case in which F consists of all valid inequalities involving up to five nodes. We show that, for this choice of F, the set of F-friendly graphs is minor-closed and is a proper generalisation of the G_{12}-minor-free graphs. This implies Truemper's result.

The structure of the paper is as follows. Subsection 1.1 introduces some notation and terminology. Section 2 is a brief literature review. Section 3 defines F-friendly graphs and studies their properties. Section 4 gives the application to inequalities involving no more than five nodes. Conclusions and ideas for future research are given in Section 5.

1.1 Notation and terminology

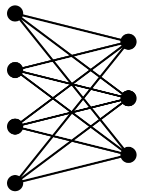
Throughout the paper, all graphs are undirected and simple. The complete graph on n nodes is denoted by K_n, and the complete bipartite graph with n nodes on one shore and m on the other is denoted by $K_{n,m}$. Given a graph $G = (V, E)$ and a set $S \subseteq V$, the set of edges in E that have exactly one end-node in S is called a *cut*, and is denoted by $\delta(S)$. A set $C \subseteq E$ is called a *circuit* if it induces a connected subgraph of G in which every node has degree 2.

For $n \geq 6$, we let C_n^2 denote the square of the circuit on n nodes (see Figs. 1e and 1f). For $n \geq 3$, we let DW_n denote the *double-wheel* graph, formed by taking a circuit C on n nodes, and connecting each node in C with two adjacent nodes not in C (see Fig. 1g). We also let $\mathcal{DW}$ denote the set of all double-wheel graphs.

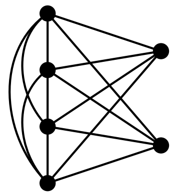
Contracting an edge of G means identifying the end-nodes, deleting the resulting loop, and replacing each pair of parallel edges (if any) with a single edge. Given a graph G and an edge e, we let $G \setminus e$ and $G \,/\, e$ denote the graphs obtained by deleting e and contracting e, respectively.

We say that G is *k-connected* if $|V| > k$ and G remains connected whenever any set of $k-1$ nodes is removed. The graph G is called a *k-sum* of G_1 and G_2 if there are cliques $S_i \subseteq V(G_i)$, $i = 1, 2$, each of cardinality k, such that G is obtained through the identification of S_1 and S_2, possibly followed by the removal of some edges inside the clique. A *k-separation* of G is a pair of induced subgraphs $\{G_1, G_2\}$ such that $V(G_1) \cup V(G_2) = V(G)$, $E(G_1) \cup E(G_2) = E(G)$, $V(G_1) \cap V(G_2) = k$ and $V(G_1) \setminus V(G_2) \neq \emptyset \neq V(G_2) \setminus V(G_1)$.

Finally, we say that a graph is *4-vertex-coverable* if there exists a set of four nodes such that every edge is incident on at least one node in the set (see Fig. 2). We also let $\mathcal{K}$ denote the class of all 4-connected 4-vertex-coverable graphs.



(a) $K_{4,3}$



(b) $K_6 \setminus e$

Figure 2: Two 4-vertex-coverable graphs

2 Literature Review

We now review the relevant literature, covering graph minors in Subsection 2.1, cut polytopes of complete graphs in Subsection 2.2, and cut polytopes of general graphs in Subsection 2.3.

2.1 Graph minors

Wagner [28] proved that a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as minors. He also proved that a graph has no K_5 minor if and only if it can be decomposed, via k-sums with $k \leq 3$, into planar graphs and copies of V_8. Hall [16] showed that a graph has no $K_{3,3}$ minor if and only if it can be decomposed, via k-sums with $k \leq 2$, into planar graphs and copies of K_5.

Results for other forbidden minors are given in, e.g., [10–13, 22, 23, 27]. Four of those results will be of relevance to us:

1. Truemper [27] showed that any G_{12}-minor-free graph can be constructed by applying k-sums, with $k \leq 3$, to planar graphs and/or copies of the seven non-planar graphs shown in Fig. 3.

2. Ding [10] showed that a graph has no C_6^2 minor if and only if and only if can be constructed by k-sums, with $k \leq 3$, of graphs in $\{K_1, K_2, K_3, K_4\} \cup \{C_{2n-1}^2 : n \geq 3\} \cup \{L_4', L_5, L_5', L_5'', P_{10}\}$, the last five graphs appearing in Figure 4.

3. Ferguson [13] showed that a graph has no K_5^{++} minor if and only if it can be constructed by applying k-sums, with $k \leq 3$, to planar graphs, V_8 and/or copies of non-planar graphs with no more than six nodes.

4. Ding *et al.* [12] showed that a 4-connected graph is C_7^2-minor-free if and only if it is either planar or belongs to $\mathcal{DW} \cup \mathcal{K} \cup \big\{K_6, L\big(K_{3,3}\big), \Gamma_1, \Gamma_2, \ldots, \Gamma_5\big\}$. Here, $L\big(K_{3,3}\big)$ denotes the line graph of $K_{3,3}$, and $\Gamma_1, \ldots, \Gamma_5$ are five specific graphs with no more than eight nodes. We omit details for brevity.

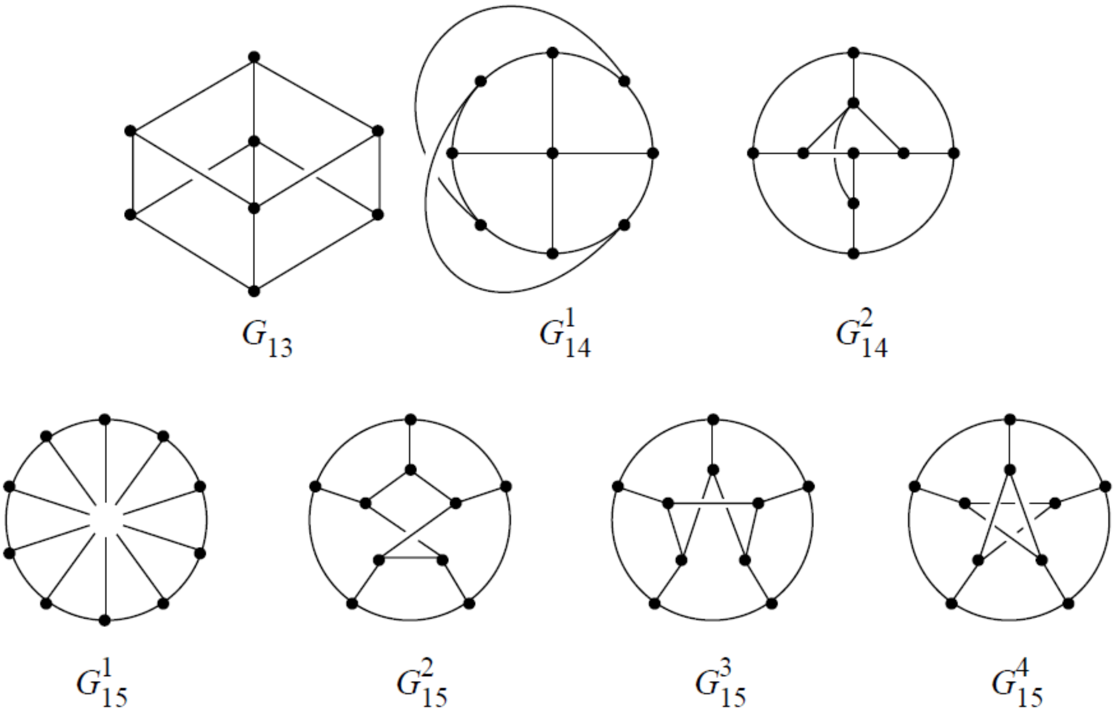


Figure 3: Seven non-planar G_{12}-minor-free graphs, as listed in [27, Theorem 10.5.21]

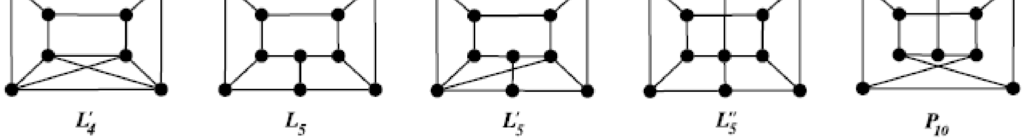


Figure 4: Five basic octahedron-minor-free graphs, as listed in [10, Theorem 1.2]

2.2 Cut polytopes of complete graphs

The cut polytope of K_n, denoted by CUT_n, is the convex hull of vectors $x \in \{0,1\}^{\binom{n}{2}}$ satisfying the following *triangle* inequalities [3]:

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \quad (1 \leq i < j < k \leq n), \tag{1}$$

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \quad (1 \leq i < j \leq n; k \neq i, j). \tag{2}$$

Many facets of CUT_n are known; see for instance [3, 9, 24]. We will need two results from [3]:

- Given any disjoint sets $S, T \subset V$ with $|S| > |T|$ and $|S| + |T| \geq 3$ and odd, the inequality

$$\sum_{\{i,j\} \subseteq S} x_{ij} + \sum_{\{i,j\} \subseteq T} x_{ij} - \sum_{i \in S} \sum_{j \in T} x_{ij} \leq \left\lfloor (|S| - |T|)^2 / 4 \right\rfloor \tag{3}$$

  defines a facet of CUT_n. (The set T may be empty.)

- Given any vector $v \in \mathbb{Z}^n$ with $\sum_{i=1}^n v_i$ odd, the inequality

$$\sum_{1 \leq i < j \leq n} v_i v_j x_{ij} \leq \left\lfloor \left(\sum_{i=1}^n v_i \right)^2 / 4 \right\rfloor \tag{4}$$

  is valid (though not necessarily facet-inducing) for CUT_n.

Note that the triangle inequalities are a special case of (3), which in turn are a special case of (4).

We will also need a definition from [7]. Let $\alpha^T x \leq \beta$ be any valid inequality for CUT_n. *Collapsing* the edge $\{i,j\}$ means constructing a valid inequality for CUT_{n-1} as follows. The edge $\{i,j\}$ is contracted, by identifying j with i. For any $k \in \{1, \ldots, n\} \setminus \{i,j\}$, the coefficient of x_{ik} in the new inequality is set to $\alpha_{ik} + \alpha_{jk}$. The coefficients for the edges that were not incident on i and j remain unchanged. (See Subsection 3.3 for examples.)

2.3 Cut polytopes of general graphs

The cut polytope $\mathrm{CUT}(G)$ of a graph $G = (V, E)$ is the *projection* of CUT_n onto $\mathbb{R}^E$. Barahona [1] showed that, if the pair $\{G_1, G_2\}$ forms a k-separation of G, with $k \leq 3$, then a linear description of $\mathrm{CUT}(G)$ is obtained simply by juxtaposing the linear descriptions of $\mathrm{CUT}(G_1)$ and $\mathrm{CUT}(G_2)$. He used this to show that the MCP is polynomially solvable on graphs with no K_5 minor.

As mentioned in the introduction, Truemper [27] and Kamiński [17] proved that the MCP is polynomially solvable on some more general families of graphs. Their proofs did not involve polytopes, however.

The following results from [3] will also be useful:

- CUT(G) is the convex hull of the vectors $x \in \{0,1\}^E$ satisfying the *co-circuit* inequalities

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \le |F| - 1, \tag{5}$$

  for all chordless circuits $C \subseteq E$ and all $F \subseteq C$ with $|F|$ odd.

- CUT(G) is described by co-circuit inequalities and the trivial bounds $0 \le x_e \le 1$ if and only if G has no K_5 minor.

- Let $\alpha^T x \le \beta$ be valid for CUT(G). For any $S \subset V$, the "switched" inequality

$$\sum_{e \in E \setminus \delta(S)} \alpha_e x_e - \sum_{e \in \delta(S)} \alpha_e x_e \le \beta - \sum_{e \in \delta(S)} \alpha_e$$

  is also valid for CUT(G).

Finally, we mention that Barahona [2] showed that projecting the polytope defined by the triangle inequalities onto $\mathbb{R}^{|E|}$ yields the polytope defined by co-circuit inequalities and trivial bounds.

3 F-Friendly Graphs

In this section, we introduce the idea of F*-friendly* graphs. Subsection 3.1 gives a formal definition. Subsection 3.2 gives some procedures for checking whether a given graph is F-friendly. Subsection 3.3 gives a sufficient condition for F-friendly graphs to be minor-closed.

3.1 Definition

Let F be a (possibly infinite) set of valid inequalities for cut polytopes of complete graphs. We will make the following three assumptions:

- F is closed under permutations of nodes. That is, if the inequality

$$\sum_{1 \le i < j \le n} \alpha_{ij} x_{ij} \le \beta \tag{6}$$

  belongs to F, and ϕ is any permutation of $1, \ldots, n$, then the inequality

$$\sum_{1 \le i < j \le n} \alpha_{\phi(i),\phi(j)} x_{\phi(i),\phi(j)} \le \beta$$

belongs to F.

- F is closed under switching (see Subsection 2.3).

- F is closed under "zero-lifting". That is, if the inequality (6) is one of the inequalities for CUT_n that belongs to F, then we obtain an inequality for CUT_{n+1} that lies in F simply by assigning the coefficient zero to all edges incident on node $n + 1$ [7].

So, for example, F could consist of all triangle inequalities, all inequalities of the form (3), or all inequalities of the form (4).

Now, for a given integer $n \geq 3$, let $P_n(F)$ denote the polyhedron that is defined by all inequalities in F that are valid for CUT_n. Note that, by definition, $\text{CUT}_n \subseteq P_n(F)$ for all F and n. Also, given a graph $G = (V, E)$, with $|V| = n$, let $\pi_G(F)$ denote the projection of $P_n(F)$ into the subspace defined by the edges in E. By definition, we have $\text{CUT}(G) \subseteq \pi_G(F)$ for all F and G. This leads naturally to the following definition.

Definition 1 *A graph is F-*friendly *if $CUT(G) = \pi_G(F)$.*

To make this concept more concrete, we give a couple of examples.

Example 1 *Let B denote the set of all trivial bounds of the form $0 \leq x_e \leq 1$. A graph is B-friendly if and only if it is a forest. (This follows from the fact that a graph is a forest if and only if it contains no circuit, and the fact that, whenever G contains a circuit, $CUT(G)$ has at least one facet-defining co-circuit inequality.)*

Example 2 *Let K3 denote the set of all triangle inequalities. A graph is K3-friendly if and only if it does not contain K_5 as a minor. (This follows from the fact that $CUT(G)$ is described by co-circuit inequalities and trivial bounds if and only if G is K_5-minor free [3], and the fact that the triangle inequalities imply the trivial bounds [2].)*

3.2 Checking F-friendliness

A natural question is, given a set F of inequalities and a graph G, how can we check whether G is F-friendly? Here is a general procedure:

1. Enumerate all extreme points of $\text{CUT}(G)$.

2. Use a software package such as PORTA [6] or PANDA [20] to compute all facets of $\text{CUT}(G)$.

3. Let I be the set of all facet-defining inequalities. Partition the members of I into equivalence classes, based on permutation and switching.

4. For each equivalence class, select one inequality as a representative, and solve a linear program (LP), in which (i) there is a variable for each edge in K_n, (ii) the objective is to maximise the left-hand side of the inequality, and (iii) the constraints are the inequalities in F.

5. If the optimal profit of the LP solution exceeds the right-hand side, for any of the inequalities tested, the graph G is not F-friendly; otherwise, declare G as F-friendly.

For example, suppose we wish to check whether the graph K_5 is K3-friendly. There are two equivalence classes. One contains the triangle inequalities, and the other contains the inequalities (3) with $|S| + |T| = 5$. The inequality $\sum_{1 \leq i < j \leq 5} x_{ij} \leq 6$ is a representative of the latter class. Maximising the left-hand side of this inequality subject to all triangle inequalities yields a fractional point with $x_e = 2/3$ for all e. The profit of this point is $2/3 \times 10 = 20/3 > 6$. Thus, K_5 is not K3-friendly.

Unfortunately, our experience is that this procedure is useful only for graphs with up to 25 edges or so. Fortunately, for some graphs (namely, graphs that are not 4-connected), we can use *k-sums* to speed up the process further.

Proposition 1 *The k-sum of F-friendly graphs, with $k \leq 3$, is F-friendly.*

Proof. As mentioned in Subsection 2.3, Barahona [1] showed that, if the pair $\{G_1, G_2\}$ forms a k-separation of G, with $k \leq 3$, then $\text{CUT}(G)$ has no further facet-defining inequalities other than the ones appearing in the description of $\text{CUT}(G_1)$ and $\text{CUT}(G_2)$. Now let $n1$ and $n2$ be the number of nodes in G_1 and G_2, respectively. If G_1 is F-friendly, then every inequality that defines a facet of $\text{CUT}(G_1)$ is implied by inequalities in F that are valid for CUT_{n1}. By zero-lifting, such an inequality is also implied by inequalities in F that are valid for CUT_{n1+n2}. A similar argument applies to G_2. Thus, every valid inequality for G is implied by inequalities in F that are valid for CUT_{n1+n2}. That is, G is F-friendly. □

3.3 When are F-friendly graphs minor-closed?

Recall (Example 1) that a graph is B-friendly if and only if it is a forest. This is equivalent to saying that the graph does not contain K_3 as a minor. Thus, the set of B-friendly graphs is closed under taking minors. Recall also (Example 2) that a graph is K3-friendly if and only if it does not contain K_5 as a minor. Thus, the K3-friendly graphs are minor-closed as well.

This leads immediately to the question: what conditions need to be imposed on the set F to ensure that the set of F-friendly graphs is minor-closed? We do not have a full answer, but we do have a sufficient condition. In order to present it, we will need the following lemma and definition.

Lemma 1 *Regardless of the set F, the set of F-friendly graphs is closed under edge deletion.*

Proof. Let $G = (V, E)$ and $e \in E$. By definition, $\mathrm{CUT}(G \setminus e)$ is the projection of $\mathrm{CUT}(G)$ onto $\mathbb{R}^{E \setminus \{e\}}$, and $\pi_{G\setminus e}(F)$ is the projection of $\pi_G(F)$ onto the same subspace. Thus, if $\pi_G(F) = \mathrm{CUT}(G)$, then $\pi_{G\setminus e}(F) = \mathrm{CUT}(G \setminus e)$. □

Definition 2 *We say that a set F of inequalities is "closed under collapsing" if, given any inequality in F and any edge e, the inequality obtained by collapsing e satisfies one of the following three conditions:*

- *It belongs to F.*
- *It is implied by two or more inequalities in F.*
- *It is vacuous (being equivalent to $0 \le 0$).*

We illustrate this definition with a few examples.

Example 3 *The set K3 is closed under collapsing. For example, if we take the triangle inequality $x_{12} - x_{13} - x_{23} \le 0$ and collapse the edge $\{1, 2\}$, we obtain the trivial inequality $-2x_{13} \le 0$, which is implied by triangle inequalities. If we collapse on $\{1, 3\}$ or $\{2, 3\}$ instead, we obtain the vacuous inequality $0 \le 0$. If we collapse any other edge, the triangle inequality remains unchanged.*

Example 4 *The set of inequalities of the form (4) is closed under collapsing. For example, if we collapse the edge $\{n - 1, n\}$, the resulting inequality can be written as*

$$\sum_{1 \le i < j \le n-1} v'_i v'_j x_{ij} \le \left\lfloor \left(\sum_{i=1}^{n-1} v'_i \right)^2 / 4 \right\rfloor ,$$

where v' is a new vector of length $n - 1$, obtained from v by replacing the last two components, v_{n-1} and v_n, with a single component equal to $v_{n-1} + v_n$.

Example 5 *The set of inequalities of the form (3) is not closed under collapsing. For example, if we take the inequality $\sum_{1 \le i < j \le 7} x_{ij} \le 12$ and collapse the edge $\{6, 7\}$, we obtain the inequality*

$$\sum_{1 \le i < j \le 5} x_{ij} + 2 \sum_{1 \le i \le 5} x_{i6} \le 12.$$

This latter inequality is of the form (4), and it defines a facet of CUT_6 [3, Theorem 2.4], but it is not of the form (3).

We are now ready for the final result in this section.

Theorem 1 *If F is closed under collapsing, then the family of F-friendly graphs is minor-closed.*

Proof. By Lemma 1, it suffices to prove closure under edge contraction. For a given graph $G = (V, E)$ and edge $e = \{i, j\} \in E$, let A be the set of nodes in G that are adjacent to both i and j, and let P be the face of $\text{CUT}(G)$ induced by the trivial bound $x_e \geq 0$. Note that, if x is an extreme point of P, then it satisfies not only the equation $x_e = 0$, but also the equation $x_{ik} = x_{jk}$ for all $k \in A$. Thus, we can construct $\text{CUT}(G/e)$ as follows: take P, eliminate the variable x_e, and identify the variables x_{ik} and x_{jk} for all $k \in A$. This implies that every valid inequality for $\text{CUT}(G/e)$ can be obtained by collapsing a valid inequality for $\text{CUT}(G)$. Thus, if $\pi_G(F) = \text{CUT}(G)$, then $\pi_{G/e}(F) = \text{CUT}(G/e)$. □

4 K5-Friendly Graphs

Let K5 be the set of all inequalities of the form (3) such that $|S| + |T| \leq 5$. (Note that K5 includes all triangle inequalities.) Given that K5 contains $O(n^5)$ inequalities, it is possible to optimise a linear function over $P_n(K5)$ in polynomial time. Accordingly, one can solve the MCP on K5-friendly graphs in polynomial time. In this section, we explore these graphs in detail.

Throughout this section, we let K_6^- denote the graph obtained by deleting one edge from K_6.

4.1 Four useful lemmas

We start with the following lemma.

Lemma 2 *K5 is closed under collapsing.*

Proof. Suppose first that $|S| = 5$ and $T = \emptyset$. Assume w.l.o.g. that $S = \{1, \ldots, 5\}$ and we collapse the edge $\{4, 5\}$. The resulting inequality is $x_{12} + x_{13} + x_{23} + 2(x_{14} + x_{24} + x_{34}) \leq 6$. This is the sum of the triangle inequalities $x_{12} + x_{14} + x_{24} \leq 2$, $x_{13} + x_{14} + x_{34} \leq 2$ and $x_{23} + x_{24} + x_{34} \leq 2$. A similar argument can be used when $|S| \in \{3, 4\}$. □

Together with Theorem 1, this implies that the set of K5-friendly graphs is minor-closed. It also implies that there exists a finite set of minimal K5-unfriendly graphs, by the Robertson–Seymour Theorem [26]. The following proposition gives three of them.

Lemma 3 *K_6, C_7^2 and DW_5 are minimal K5-unfriendly graphs.*

Proof. It is shown in [3] that K_6 and DW_5 give rise to inequalities that are facet-defining for CUT_n, but are not of the form (3). The same is shown for C_7^2 in [15]. Hence, all three graphs are K5-unfriendly.

We have enumerated all non-isomorphic graphs that can be obtained from one of the three graphs mentioned by either deleting or contracting an edge. Then, using the method described in Subsection 3.2, we have verified that all such graphs are K5-friendly. □

Now recall that $\mathcal{K}$ denotes the class of all 4-connected 4-vertex-coverable graphs. We will find the two following lemmas useful later on.

Lemma 4 *If a graph G on n nodes is in $\mathcal{K}$, and $n \leq 8$, then G is K5-friendly.*

Proof. The unique maximal member of $\mathcal{K}$ with $n \leq 8$ is obtained by taking 8 nodes, selecting 4 of those nodes, and then connecting each of the selected nodes to every other node. This graph has 8 nodes and 22 edges. Using the method described in Subsection 3.2, we have verified that this graph is K5-friendly. □

Lemma 5 *If a graph G on n nodes is in $\mathcal{K}$, and $n \geq 9$, then G contains K_6^- as a minor.*

Proof. Let the nodes forming the cover of all other nodes be $1, 2, 3$ and 4, i.e., each of the nodes $5, \ldots, n$ is connected to each of $1, \ldots, 4$. If we contract the edges $(1, n), (2, n-1)$ and $(3, n-2)$ of G, we obtain a graph G' in which nodes $1, \ldots, 4$ form a clique, hence nodes $1, \ldots, 6$ induce K_6^- as a subgraph. Therefore, the original graph G contains K_6^- as a minor. □

4.2 Some families of K5-friendly graphs

In this subsection, we show that various families of graphs are K5-friendly. Our first result is fairly straightforward.

Proposition 2 *Let H be any 3-connected graph with no more than ten edges. All H-minor-free graphs are K5-friendly.*

Proof. Ding and Liu [11] characterised the H-minor-free graphs for each such graph H. In particular, Theorems 3.2-3.9 in [11] show that, for any such H, the H-minor-free graphs can be constructed via k-sums, with $k \leq 2$, from 3-connected planar graphs, K_5, V_8, and/or 3-vertex-covered graphs. Barahona and Mahjoub [3] showed that planar graphs and V_8 are K3-friendly, which implies that they are also K5-friendly. K_5 is K5-friendly by definition. Finally, the edge-wise maximal 3-vertex-covered graphs can be obtained by 3-summing copies of K_4 over a common triangle. Therefore, by [1, Theorem 3.1], their cut polytopes are described by triangle inequalities, since K_4 is planar. □

In particular, graphs with no $K_{3,3}$ minor are K5-friendly. Let us now provide a different sufficient condition for K5-friendliness.

Proposition 3 *If a graph contains neither K_5^{++} nor K_6 as a minor, it is K5-friendly.*

Proof. [13, Theorem 4.1] states that any non-planar graph without a K_5^{++} minor can be composed by k-summing planar graphs (with $k \leq 3$), and then, if necessary, k-summing the resulting graph (with $k \leq 2$) with V_8 and/or non-planar graphs with at most six nodes. We know that V_8 is K5-friendly, as mentioned in the proof of Proposition 2. By Lemma 3, any graph with at most six nodes, other than K_6 itself, is K5-friendly. □

Our next result is quite more involved.

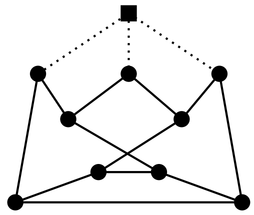
Theorem 2 *A graph without a G_{12} minor is K5-friendly.*

Proof. [27, Theorem 10.5.21] states that every graph without G_{12} minors is a k-sum ($k = 0, 1, 2, 3$) or is planar, or is isomorphic to K_5, $K_{3,3}$, G_8, G_{13}, G_{14}^1, G_{14}^2, G_{15}^1, G_{15}^2, G_{15}^3, or G_{15}^4 (see Figure 3). All these graphs, except G_{15}^4 (the Petersen graph), can be composed through repetitive 3-sums of K_4's and subgraphs of K_6^-, which is K5-friendly by Lemma 3.

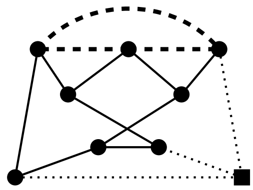
To show this, we apply an inverse procedure: we select in the given graph G a node v of degree 3, add edges (if missing) between any two of its neighbors and then delete v to obtain graph G'. Practically we apply what is know in the literature as $Y - \Delta$ operation, in order to show that G is a 3-sum of G' and K_4. If this procedure can be repeated $n - 6$ times and the last graph obtained is a subgraph of K_6^-, the initial graph is K5-friendly.

For G_{13}, if we apply this procedure first for the upper-most node and then for the lowest one, we obtain a graph with 6 nodes that is a subgraph of K_6^- because the upper-left and the lower-right nodes in that graph are not connected with an edge. For G_{14}^1, the nodes to be selected for applying this procedure are the upper-most, the right-most and the lowest. For G_{14}^2, the nodes to be selected are the left-most, the right-most and the 'central' one. Regarding G_{15}^1, if we number the nodes from 0 to 9, the procedure is applied to nodes $0, 2, 4$ and 6. For G_{15}^2 and G_{15}^3, we present the procedure in Figures 5 and 6 respectively. (In these figures, we use squares to represent the selected nodes with degree three. The edges that are removed in the process are dotted, while the ones that are introduced are dashed.)

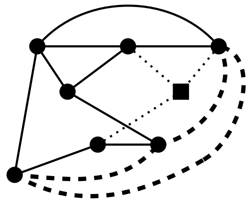
To show that the Petersen graph is K5-friendly, we use the fact that $\mathrm{CUT}(G_{15}^4)$ is defined by four families of inequalities [8]. Three of them are the bounds inequalities and cycle inequalities for cycles of length 5 or 6 that are known to be implied by triangle inequalities in CUT_n. To show a representative inequality of the fourth family, let $0, \ldots, 4$ be the nodes of the inner 5-cycle in the Petersen graph, $5, \ldots, 9$ be the ones in the outer 5-cycle and node i in the inner cycle be a neighbor of node $i + 5$ in



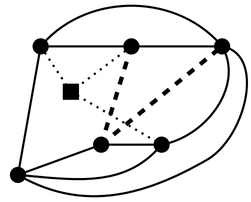
(a) $G_{15}^{2} = K_4 \oplus_3 G_{15,1}^{2}$



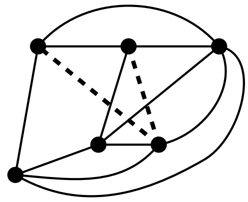
(b) $G_{15,1}^{2} = K_4 \oplus_3 G_{15,2}^{2}$



(c) $G_{15,2}^{2} = K_4 \oplus_3 G_{15,3}^{2}$

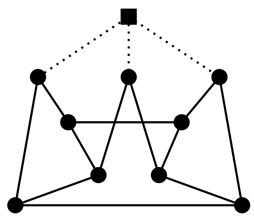


(d) $G_{15,3}^{2} = K_4 \oplus_3 G_{15,4}^{2}$

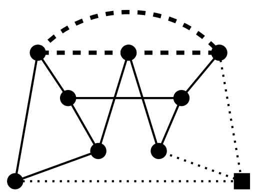


(e) $G_{15,4}^{2} \simeq K_6^{-}$

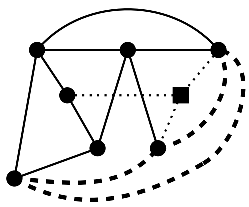
Figure 5: Iterative application of the $Y - \Delta$ operator on G_{15}^{2} that shows its K_5-friendliness.



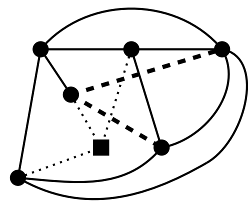
(a) $G_{15}^{3} = K_4 \oplus_3 G_{15,1}^{3}$



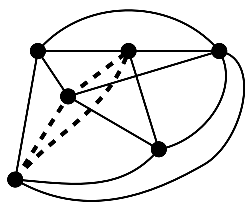
(b) $G_{15,1}^{3} = K_4 \oplus_3 G_{15,2}^{4}$



(c) $G_{15,2}^{3} = K_4 \oplus_3 G_{15,3}^{3}$



(d) $G_{15,3}^{3} = K_4 \oplus_3 G_{15,4}^{3}$



(e) $G_{15,4}^{3} \simeq K_6^{-}$

Figure 6: Iterative application of the $Y - \Delta$ operator on G_{15}^{3} that shows its K_5-friendliness.

the outer one. Then, the inequality has the form:

$$x_{01} + x_{12} + x_{23} + x_{34} + x_{04} + x_{57} + x_{79} + x_{69} + x_{68} + x_{58} + 2x_{05} + 2x_{16} + 2x_{27} + 2x_{38} + 2x_{49} \leq 16,$$

and is the sum of a K_5 inequality and certain triangle inequalities. □

Note that Proposition 3 is not implied by Theorem 2, since K_5^{++} is not a minor of G_{12}, and G_{12} is a minor of K_6.

The following can be shown in a manner similar to Theorem 2.

Proposition 4 *If a graph contains neither the octahedron nor C_7^2 as a minor, it is K5-friendly.*

Proof. [10, Theorem 1.2] proves that a graph is octahedron-free if and only if it is constructed by k-sums ($k = 0, 1, 2, 3$) of graphs in $\{K_1, K_2, K_3, K_4\} \cup \{C^2_{2n-1} : n \geq 3\} \cup \{L'_4, L_5, L'_5, L''_5, P_{10}\}$, the last five graphs appearing in Figure 4. As all graphs $\{C^2_{2n-1} : n \geq 3\}$ have C_7^2 as a minor and C_5^2 is K_5, it follows that a graph containing neither the octahedron nor C_7^2 as a minor can be constructed by k-sums ($k = 0, 1, 2, 3$) of graphs in $\{K_1, K_2, K_3, K_4, K_5\} \cup \{L'_4, L_5, L'_5, L''_5, P_{10}\}$.

All complete graphs with less than 6 nodes are trivially K5-friendly as subgraphs of K_6^-, which is K5-friendly by Lemma 3. All graphs $\{L'_4, L_5, L'_5, L''_5, P_10\}$ can be composed through repetitive 3-sums of K_4's and subgraphs of K_6^-, thus shown to be K5-friendly by Proposition 1. To check this, observe that each among the graphs in Figure 4 has a sufficiently large number of nodes of degree 3 for the procedure of successive $Y - \Delta$ operations in Theorem 2 to be applied. Specifically, the upper 4 nodes of L'_4, L_5, and L'_5 have degree 3, the same holding for the outer 6 nodes of L''_5 and the outer 5 nodes of P_{10}. Repeating the procedure of successive $Y - \Delta$ operations per graph shown in Figure 4 and for the aforementioned nodes of degree 3 always leaves a subgraph of K_6^-. □

Let us now provide a similar result regarding 4-connected graphs.

Proposition 5 *If a 4-connected graph has none of K_6^-, C_7^2 and DW_5 as a minor, then it is K5-friendly.*

Proof. Let us observe that 4-connected graphs without any of K_6^-, C_7^2 or DW_5 as a minor are a subset of C_7^2-free graphs. By [12, Corollary 1.2], 4-connected C_7^2-free graphs are only planar graphs $\mathcal{P}$, double wheels $\mathcal{DW}$, 4-vertex covered graphs, $L(K_{3,3})$ and graphs $\Gamma_1, \ldots \Gamma_5$ (as listed in [12]). The last six individual graphs can be shown to be K5-friendly through the procedure of Subsection 3.2. Planar graphs are K5-friendly. Among the double wheels $\mathcal{DW}$, observe that DW_3 and DW_4 are the only ones not having DW_5 as a minor, and both of those are minors of K_6^-, hence being K5-friendly. Last, graphs in $\mathcal{K}$ are either K5-friendly or contain K_6^- as a minor, as implied by Lemmas 4 and 5. □

Interestingly, we can also show the following.

Lemma 6 *G_{12} is the unique maximal common minor of K_6, C_7^2 and DW_5.*

Proof. G_{12} is trivially a minor of K_6. It can be obtained from C_7^2, viewed as $K_7 \setminus C_7$, by contracting an edge connecting two nodes of distance 3 in the C_7. It can also be obtained from DW_5, viewed as $K_7 \setminus C_5$, by contracting an edge connecting two non-adjacent nodes in the C_5 and then deleting two (appropriately chosen) edges.

To show that G_{12} is a maximal common minor, let us consider C_7^2. To obtain a common minor with K_6, one must delete a node or contract an edge. Deleting any node and the adjacent edges from C_7^2 yields a subgraph of G_{12}. One can contract an edge connecting two nodes of distance 2 or 3 in C_7^2. One can check that, in either case, one obtains G_{12}. □

4.3 A conjecture

The results in the previous subsection, together with experiments that we have conducted with various graphs, lead us to make the following conjecture:

Conjecture 1 *A graph is K5-friendly if and only if it does not contain any of K_6, C_7^2 and DW_5 as a minor.*

We note that this conjecture is consistent with Propositions 3 and 4, since (a) K_5^{++} is a minor of both C_7^2 and DW_5, and (b) the octahedron is a minor of both K_6 and DW_5. Moreover, we have verified that the conjecture holds for $n \leq 8$.

5 Further Work

We conclude the paper with some suggestions for further work. The most obvious direction would be the complete characterisation of K5-friendly graphs, although this would seem to require a complete characterisation of C_7^2-minor-free graphs (or at least graphs with no K_6, C_7^2 or DW_5 minor).

We remark that polynomial-time separation algorithms are known for two exponentially-large families of facet-defining inequalities for the cut polytope: the *switched odd bicycle wheel* inequalities [5] and certain inequalities related to *circulants* [18]. It would be natural to study graphs that are friendly with respect to either or both of those families. First, however, one would have to check whether those families are closed under collapsing.

Relevant to that is whether the sufficient condition in Theorem 1, for the set of F-friendly graphs to be minor-closed, is also necessary. As it stands, we do not even know whether friendliness with respect to the inequalities (3) is minor-closed (see Example 5).

More generally, one could attempt to apply the concept of F-friendliness to other graph optimisation problems, besides max-cut. Note however that an operation analogous to collapsing may not exist for some problems.

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