C-IDEALS OF LIE ALGEBRAS

DAVID A. TOWERS

Department of Mathematics, Lancaster University
Lancaster LA1 4YF, England
Email: d.towers@lancaster.ac.uk

Abstract

A subalgebra B of a Lie algebra L is called a *c-ideal* of L if there is an ideal C of L such that $L = B + C$ and $B \cap C \leq B_L$, where B_L is the largest ideal of L contained in B. This is analogous to the concept of c-normal subgroup, which has been studied by a number of authors. We obtain some properties of c-ideals and use them to give some characterisations of solvable and supersolvable Lie algebras. We also classify those Lie algebras in which every one-dimensional subalgebra is a c-ideal.
Mathematics Subject Classification 2000: 17B05, 17B20, 17B30, 17B50.
Key Words and Phrases: Lie algebras, c-ideal, nilpotent, solvable, supersolvable, Frattini ideal.

1 Introduction

Throughout L will denote a finite-dimensional Lie algebra over a field F. If B is a subalgebra of L we define B_L, the *core* (with respect to L) of B to be the largest ideal of L contained in B. We say that a subalgebra B of L is a *c-ideal* of L if there is an ideal C of L such that $L = B + C$ and $B \cap C \leq B_L$. This is analogous to the concept of c-normal subgroup as introduced by Wang in [10]; this concept has since been further studied by a number of authors, including Li and Guo ([5] and [6]), Jehad ([4]), Wang ([11]), Wei ([12]) and Skiba ([7]).

The maximal subalgebras of a Lie algebra L and their relationship to the structure of L have been studied extensively. It is known that L is nilpotent if and only if every maximal subalgebra of L is an ideal of L. A further result is that every maximal subalgebra of L has codimension one in L if and only if L is supersolvable. In this paper we obtain some similar characterisations of solvable and supersolvable Lie algebras in terms of c-ideals.

A subalgebra B of L is a *retract* of L if there is an endomorphism $\theta : L \to L$ such that $\theta(b) = b$ for all $b \in B$ and $\theta(x) \in B$ for all $x \in L$. Such a map θ is called a *retraction*. Then it is easy to see that ideals of L and retracts of L are c-ideals of L; in the case of retracts the kernel of the retraction is an ideal that complements B. If F has characteristic zero then every Levi factor of L is a c-ideal of L.

In section one we give some basic properties of c-ideals; in particular, it is shown that c-ideals inside the Frattini subalgebra of a Lie algebra L are necessarily ideals of L. In section two we first show that all maximal subalgebras of L are c-ideals of L if and only if L is solvable. It is further shown that, over a field of characteristic zero or over an algebraically closed field of characteristic $p > 5$, L has a solvable maximal subalgebra that is a c-ideal if and only if L is solvable. Finally we have that if all maximal nilpotent subalgebras of L are c-ideals, or if all Cartan subalgebras of L are c-ideals and F has characteristic zero, then L is solvable.

In section three we show that if every maximal subalgebra of each maximal nilpotent subalgebra of L is a c-ideal of L then L is supersolvable. If each of the maximal nilpotent subalgebras of L has dimension at least two then the assumption of solvability can be removed. Similarly if the field has characteristic zero and L is not three-dimensional simple then this restriction can be removed. In the final section we classify those Lie algebras in which every one-dimensional subalgebra is a c-ideal.

If A and B are subalgebras of L for which $L = A + B$ and $A \cap B = 0$ we will write $L = A \oplus B$. The ideals $L^{(k)}$ and L^k are defined inductively by $L^{(1)} = L^1 = L$, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$, $L^{k+1} = [L, L^k]$ for $k \geq 1$. If A is a subalgebra of L, the *centralizer* of A in L is $C_L(A) = \{x \in L : [x, A] = 0\}$.

2 Preliminary results

First we give some basic properties of c-ideals.

Lemma 2.1 *(i) If B is a c-ideal of L and $B \leq K \leq L$ then B is a c-ideal of K.*

(ii) If I is an ideal of L and $I \leq B$ then B is a c-ideal of L if and only if B/I is a c-ideal of L/I.

Proof.

(i) Suppose that B is a c-ideal of L and $B \leq K \leq L$. Then there is an ideal C of L with $L = B + C$ and $B \cap C \leq B_L$. It follows that $K = (B + C) \cap K = B + C \cap K$, where $C \cap K$ is an ideal of K and $B \cap C \cap K \leq B_L \cap K \leq B_K$, and so B is a c-ideal K.

(ii) Suppose first that B/I is a c-ideal of L/I. Then there is an ideal C/I of L/I such that $L/I = B/I + C/I$ and $(B/I) \cap (C/I) \leq (B/I)_{L/I} = B_L/I$. It follows that $L = B + C$, where C is an ideal of L and $B \cap C \leq B_L$, whence B is a c-ideal of L.

Suppose conversely that I is an ideal of L with $I \leq B$ such that B is a c-ideal of L. Then there is an ideal C of L such that $L = B + C$ and $B \cap C \leq B_L$. Now $L/I = B/I + (C + I)/I$, where $(C + I)/I$ is an ideal of L/I and $(B/I) \cap (C + I)/I = (B \cap (C + I))/I = (I + B \cap C)/I \leq B_L/I = (B/I)_{L/I}$, so B/I is a c-ideal of L/I.

The *Frattini subalgebra* of L, $F(L)$, is the intersection of all of the maximal subalgebras of L. The *Frattini ideal*, $\phi(L)$, of L is $F(L)_L$. The next result shows, in particular, that c-ideals inside the Frattini subalgebra of a Lie algebra L are necessarily ideals of L.

Proposition 2.2 *Let B, C be subalgebras of L with $B \leq F(C)$. If B is a c-ideal of L then B is an ideal of L and $B \leq \phi(L)$.*

Proof. Suppose that $L = B + K$ and $B \cap K \leq B_L$. Then $C = C \cap L = C \cap (B + K) = B + C \cap K = C \cap K$ since $B \leq F(C)$. Hence $B \leq C \leq K$, giving $B = B \cap K \leq B_L$ and B is an ideal of L. It then follows from [8, Lemma 4.1] that $B \leq \phi(L)$.

3 Some characterisations of solvable algebras

Theorem 3.1 *Let L be a Lie algebra over any field F. Then all maximal subalgebras of L are c-ideals of L if and only if L is solvable.*

Proof. Let L be a non-solvable Lie algebra of smallest dimension in which maximal subalgebras are c-ideals of L. Then all proper factor algebras of L are solvable, by Lemma 2.1 (ii). Suppose first that L is simple. Let M be a maximal subalgebra of L. Then M is a c-ideal so there is an ideal C of L such that $L = M + C$ and $M \cap C \leq M_L = 0$, as L is simple. This yields that C is a non-trivial proper ideal of L, a contradiction. If L has two minimal ideals B_1 and B_2, then L/B_1 and L/B_2 are solvable and $B_1 \cap B_2 = 0$, so L is solvable. Hence L has a unique minimal ideal B and L/B is solvable.

Suppose there is an element $b \in B$ such that $\mathrm{ad}_L b$ is not nilpotent. Let $L = L_0 \oplus L_1$ be the Fitting decomposition of L relative to $\mathrm{ad}_L b$. Then $L \neq L_0$ so let M be a maximal subalgebra of L containing L_0. As M is a c-ideal there is an ideal C of L such that $L = M + C$ and $M \cap C \leq M_L$. Now $L_1 \leq B$ so $B \not\leq M_L$. It follows that $M_L = 0$ whence $M = L_0$ and $B = C = L_1$. But $b \in M \cap B = 0$. Hence every element of B is ad-nilpotent, yielding that B is nilpotent and so L is solvable, a contradiction.

Now suppose that L is solvable and let M be a maximal subalgebra of L. Then there is a $k \geq 2$ such that $L^{(k)} \leq M$, but $L^{(k-1)} \not\leq M$. We have that $L^{(k-1)}$ is an ideal of L, $L = M + L^{(k-1)}$ and $M \cap L^{(k-1)} \leq M_L$, so M is a c-ideal of L.

Theorem 3.2 *Let L be a Lie algebra over a field F of characteristic zero. Then L has a solvable maximal subalgebra that is a c-ideal of L if and only if L is solvable.*

Proof. Suppose first that L has a solvable maximal subalgebra M that is a c-ideal of L. We show that L is solvable. Let L be a minimal counter-example. Then there is an ideal K of L such that $L = M + K$ and $M \cap K \leq M_L$. Now $M_L = 0$, since otherwise, L/M_L is solvable and M_L is solvable, whence L is solvable, a contradiction. It follows that $L = M \oplus K$. If R is the solvable radical of L then $R \leq M_L = 0$, so L is semisimple and $L^2 = L$. But $L^2 \leq M^2 + K \neq L$, a contradiction. The result follows.

The converse follows from Theorem 3.1.

For fields of characteristic $p > 0$ we have the following result.

Theorem 3.3 *Let L be a Lie algebra over an algebraically closed field F of characteristic greater than 5. Then L has a solvable maximal subalgebra that is a c-ideal of L if and only if L is solvable.*

Proof. Suppose first that L has a solvable maximal subalgebra M that is a c-ideal of L. We show that L is solvable. Let L be a minimal counter-example. Then, as above, $L = M \oplus K$ and K is a minimal ideal of L. We follow the contents of [13]: M defines a filtration in which $L_0 = M$, $L_{i+1} = \{x \in L_i : [x, L] = L_i\}$. When $L_1 = 0$ this filtration is called *short*; otherwise it is *long*. Suppose first that it is short. Then, as in the first two paragraphs of the proof of [13, Theorem 2.2], $L = \bigoplus_{i \in \mathbb{Z}_p} L_i$, $M = L_0$ and L_i is an irreducible M-submodule of L for each $i \neq 0$. Moreover, since K is an ideal of L, $K = \bigoplus_{0 \neq i \in \mathbb{Z}_p} L_i$. Let S be the subalgebra generated by L_1. Then S is spanned by commutators $c(x_1, \dots, x_n) = [x_1, [x_2, \dots [x_{n-1}, x_n]]$ with $x_i \in L_1$ and $n \geq 1$. Now $c(x_1, \dots, x_p) \in M \cap K = 0$ for all $x_1, \dots, x_p \in L_1$, so S is nilpotent. Also M idealizes S, so $M + S$ is a subalgebra of L, whence $L = M + S$ and S is an ideal of L. It follows that $K = S$ is nilpotent and L is solvable, a contradiction.

Now suppose that the filtration is long. Then the nilradical, N, of M acts nilpotently on K, by [13, Proposition 2.5]. Let $C = C_K(N)$. Then $C \neq 0$ and $M + C$ is a subalgebra of L. It follows that $L = M + C$, whence $K = C$.. But this means that N is an ideal of L, so that $N \subseteq M_L = 0$. We conclude that $M = 0$, a contradiction.

The converse follows from Theorem 3.1 as before.

Theorem 3.4 *Let L be a Lie algebra over any field F, such that all maximal nilpotent subalgebras of L are c-ideals of L. Then L is solvable.*

Proof. Let N be the nilradical of L and let $x \notin N$. Then $x \in B$ for some maximal nilpotent subalgebra B of L, and there is an ideal C of L such that $L = B + C$ and $B \cap C \leq B_L$. Clearly $x \notin B_L \leq N$, so $x \notin C$. Moreover, $L/C \cong B/(B \cap C)$ is nilpotent. So if $x \notin N$, there is an ideal C of L such that $x \notin C$ and L/C is nilpotent.

So let $x_1 \notin N$ and let C_1 be such an ideal with $x_1 \notin C_1$ and L/C_1 nilpotent. If $C_1 \leq N$ we have finished. If not, then choose $x_2 \in C_1 \setminus N$ and let C_2 be such an ideal with $x_2 \notin C_2$ and L/C_2 nilpotent. Clearly dim $(C_1 \cap C_2) <$ dim C_1. If $C_1 \cap C_2 \not\leq N$, choose $x_3 \in (C_1 \cap C_2) \setminus N$. Continuing in this way we find ideals $C_1, \dots, C_n$ of L such that $C_1 \cap \dots \cap C_n \leq N$ and

L/C_i is nilpotent for each $1 \leq i \leq n$. Since $L/(C_1 \cap \ldots \cap C_n)$ is solvable, the result follows.

Theorem 3.5 *Let L be a Lie algebra, over a field F of characteristic zero, in which every Cartan subalgebra of L is a c-ideal of L. Then L is solvable.*

Proof. Suppose that every Cartan subalgebra of L is a c-ideal of L, and that L has a non-zero Levi factor S. Let H be a Cartan subalgebra of S and let B be a Cartan subalgebra of its centralizer in the solvable radical of L. Then $C = H + B$ is a Cartan subalgebra of L (see [3]) and there is an ideal K of L such that $L = C + K$ and $C \cap K \leq C_L$. Now there is an $r \geq 2$ such that $L^{(r)} \leq K$. But $S \leq L^{(r)} \leq K$, so $C \cap S \leq C \cap K \leq C_L$ giving $C \cap S \leq C_L \cap S = 0$, a contradiction. It follows that $S = 0$ and hence that L is solvable.

Note: If $L^\infty = \cap_{i=1}^\infty L^i$ is abelian then the converse to the above theorem holds, by Theorem 4.4.1.1 of [14].

4 Some characterisations of supersolvable algebras

First we need some preliminary results concerning maximal nilpotent subalgebras of Lie algebras.

Lemma 4.1 *Let L be a Lie algebra over any field F, let A be an ideal of L and let U/A be a maximal nilpotent subalgebra of L/A. Then $U = C + A$, where C is a maximal nilpotent subalgebra of L.*

Proof. If $A \leq \phi(U)$ then $U/\phi(U)$ is nilpotent, whence U is nilpotent, by Theorem 6.1 of [8] and the result is clear. So suppose that $A \not\leq \phi(U)$. Then $U = A + M$ for some maximal subalgebra M of U. If we choose B to be minimal with respect to $U = A + B$, then $A \cap B \leq \phi(B)$, by Lemma 7.1 of [8]. Also $U/A \cong B/(A \cap B)$ is nilpotent, which yields that B is nilpotent. If we now choose C to be the biggest nilpotent subalgebra of U such that $U = A + C$, it is easy to see that C is a maximal nilpotent subalgebra of L.

Lemma 4.2 *Let L be a Lie algebra, over any field F, in which every maximal subalgebra of each maximal nilpotent subalgebra of L is a c-ideal of L, and let A be a minimal abelian ideal of L. Then every maximal subalgebra of each maximal nilpotent subalgebra of L/A is a c-ideal of L/A.*

Proof. Suppose that U/A is a maximal nilpotent subalgebra of L/A. Then $U = C + A$ where C is a maximal nilpotent subalgebra of L by Lemma 4.1. Let B/A be a maximal subalgebra of U/A. Then $B = B \cap (C + A) = B \cap C + A = D + A$ where D is a maximal subalgebra of C with $B \cap C \leq D$. Now D is a c-ideal of L so there is an ideal K of L with $L = D + K$ and $D \cap K \leq D_L$.

If $A \leq K$ we have $L/A = (D + K)/A = ((D + A)/A) + (K/A) = (B/A) + (K/A)$, and $(B/A) \cap (K/A) = (B \cap K)/A = ((D + A) \cap K)/A = (D \cap K + A)/A \leq (D_L + A)/A \leq (B/A)_{L/A}$.

If $A \not\leq K$, we have $A \cap K = 0$. Then $(A + K)/K$ is a minimal ideal of L/K, which is nilpotent, so $\dim A = 1$ and $LA \leq A \cap K = 0$. It follows that $A \leq C$ and $B = D$. We have $L = B + K$ and $B \cap K \leq B_L$, so $L/A = (B/A) + ((K+A)/A)$ and $(B/A) \cap ((K+A)/A) = (B \cap (K+A))/A = (B \cap K + A)/A \leq (B_L + A)/A \leq (B/A)_{L/A}$.

Lemma 4.3 *Let L be a Lie algebra, over any field F, in which every maximal subalgebra of each maximal nilpotent subalgebra of L is a c-ideal of L, and suppose that A is a minimal abelian ideal of L and M is a core-free maximal subalgebra of L. Then A is one dimensional.*

Proof. We have that $L = A \oplus M$ and A is the unique minimal ideal of L, by Lemma 1.4 of [9]. Let C be a maximal nilpotent subalgebra of L with $A \leq C$. If $C = A$, choose B to be a maximal subalgebra of A, so that $A = B + Fa$ and $B_L = 0$. Then B is a c-ideal of L so there is an ideal K of L with $L = B + K$ and $B \cap K \leq B_L = 0$. But now $L = A + K = K$, giving $B = 0$ and $\dim A = 1$.

So suppose that $C \neq A$. Then $C = A + M \cap C$. Let B be a maximal subalgebra of C containing $M \cap C$. Then B is a c-ideal of L, so there is an ideal K of L with $L = B + K$ and $B \cap K \leq B_L$. If $A \leq B_L \leq B$ we have $C = A + M \cap C \leq B$, a contradiction. Hence $B_L = 0$ and $L = B \oplus K$. Now $C = B + C \cap K$ and $B \cap C \cap K = B \cap K = 0$. As C is nilpotent this means that $\dim(C \cap K) = 1$. But $A \leq C \cap K$, so $\dim A = 1$, as required.

We can now prove our main result.

Theorem 4.4 *Let L be a solvable Lie algebra, over any field F, in which every maximal subalgebra of each maximal nilpotent subalgebra of L is a c-ideal of L. Then L is supersolvable.*

Proof. Let L be a minimal counter-example and let A be a minimal abelian ideal of L. Then L/A satisfies the same hypotheses, by Lemma 4.2. We thus have that L/A is supersolvable, and it remains to show that $\dim A = 1$.

If there is another minimal ideal I of L, then $A \cong (A+I)/I \leq L/I$, which is supersolvable and so $\dim A = 1$. So we can assume that A is the unique minimal ideal of L. Also, if $A \leq \phi(L)$ we have that $L/\phi(L)$ is supersolvable, whence L is supersolvable, by Theorem 7 of [2]. We therefore further assume that $A \not\leq \phi(L)$. It follows that $L = A \oplus M$, where M is a core-free maximal subalgebra of L.The result now follows from Lemma 4.3.

If L has no one-dimensional maximal nilpotent subalgebras, we can remove the solvability assumption from the above result.

Corollary 4.5 *Let L be a Lie algebra, over any field F, in which every maximal nilpotent subalgebra has dimension at least two. If every maximal subalgebra of each maximal nilpotent subalgebra of L is a c-ideal of L, then L is supersolvable.*

Proof. Let N be the nilradical of L and let $x \notin N$. Then $x \in C$ for some maximal nilpotent subalgebra C of L. Since $\dim C > 1$, there is a maximal subalgebra B of C with $x \in B$. Now there is an ideal K of L with $L = B + K$ and $B \cap K \leq B_L \leq C_L \leq N$. Clearly $x \notin K$, since otherwise $x \in B \cap K \leq N$. Moreover, L/K is nilpotent. We have shown that if $x \notin N$ there is an ideal K of L with $x \notin K$ and L/K nilpotent. Proceeding as in Theorem 3.4 we see that L is solvable. The result then follows from Theorem 4.4.

If L has a one-dimensional maximal nilpotent subalgebra then we can also remove the solvability assumption from Theorem 4.4 provided that the underlying field F has characteristic zero and L is not three-dimensional simple.

Corollary 4.6 *Let L be a Lie algebra over a field F of characteristic zero. If every maximal subalgebra of each maximal nilpotent subalgebra of L is a c-ideal of L, then L is supersolvable or three-dimensional simple.*

Proof. If every maximal nilpotent subalgebra of L has dimension at least two then L is supersolvable, by Corollary 4.5. So we need only consider the case where L has a one-dimensional maximal nilpotent subalgebra, Fx say.

Suppose first that L is semisimple, so $L = S_1 \oplus \ldots \oplus S_n$, where S_i is a simple ideal of L for $1 \leq i \leq n$. Let $n > 1$. If $x \in S_i$ then choosing $s \in S_j$ with $j \neq i$ we have that $Fx + Fs$ is a two-dimensional abelian subalgebra, which contradicts the maximality of Fx. If $x \notin S_i$ for every $1 \leq i \leq n$, then x has non-zero projections in at least two of the S_k's, say $s_i \in S_i$ and $s_j \in S_j$. But then $Fx + Fs_i$ is a two-dimensional abelian subalgebra, a contradiction again. It follows that L is simple. But then Fx is a Cartan subalgebra of L, which yields that L has rank one and thus is three dimensional.

So now let L be a minimal counter-example. We have seen that L is not semisimple, so it has a minimal abelian ideal A. By Lemma 4.2, L/A is supersolvable or three-dimensional simple. In the former case, L is solvable and so supersolvable, by Theorem 4.4. In the latter case, $L = A \oplus S$ where S is three-dimensional simple, and so a core-free maximal subalgebra of L. It follows from Lemma 4.3 that $\dim A = 1$. But now $C_L(A) = A$ or L. In the former case $S \cong L/A = L/C_L(A) \cong Inn(A)$, a subalgebra of $Der(A)$, which is impossible. Hence $L = A \oplus S$, where A and S are both ideals of L, and again L has no one-dimensional maximal nilpotent subalgebras.

5 One-dimensional c-ideals

First we note that one-dimensional c-ideals are easy to classify.

Proposition 5.1 *Let L be a Lie algebra over any field F. Then the one-dimensional subalgebra Fx of L is a c-ideal of L if and only if*

(i) Fx is an ideal of L; or

(ii) $x \notin L^2$.

Proof. Let Fx be a c-ideal of L. Then there is an ideal K of L such that $L = Fx + K$ and $Fx \cap K \leq (Fx)_L$. But $Fx \cap K = Fx$ or 0. The former implies that Fx is an ideal of L, and the latter implies that $x \notin L^2 \leq K$.

Conversely, suppose that $x \notin L^2$. Then there is a subspace K of L of codimension one in L such that $L^2 \leq K$ and $x \notin K$. Clearly $L = Fx \oplus K$ and K is an ideal of L, whence Fx is a c-ideal of L.

We shall denote by $Z(L)$ the *centre* of L; that is $Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$. The *abelian socle* of L, Asoc$L$, is the sum of the minimal abelian ideals of L. We say that L is *almost abelian* if $L = L^2 \oplus Fx$, where L^2 is abelian and $[x, y] = y$ for all $y \in L^2$.

Theorem 5.2 *Let L be a Lie algebra over any field F. Then all one-dimensional subalgebras of L are c-ideals of L if and only if*

(i) $L^3 = 0$*; or*

(ii) $L = A \oplus B$*, where A is an abelian ideal of L and B is an almost abelian ideal of L.*

Proof. Suppose that all one-dimensional subalgebras of L are c-ideals of L. First note that the one-dimensional ideals are inside Asoc$L$. If Fx is not an ideal of L then there is an ideal M of L such that $L = Fx + M$ and $Fx \cap M \leq (Fx)_L = 0$, so Fx is complemented by an ideal of codimension one in L.

Now let A be a minimal ideal of L and let $a \in A$. If $A \neq Fa$ then there is an ideal M of codimension one in L which complements Fa. But this implies that $M \cap A = 0$, whence $A = Fa$, a contradiction. It follows that all minimal ideals are one dimensional. Put Asoc$L = Fa_1 \oplus \ldots \oplus Fa_r$. Suppose that $[x, a_i] = \lambda a_i$, $[x, a_j] = \mu a_j$ and $\lambda \neq \mu$. Then $F(a_i + a_j)$ is not an ideal of L, and so there is an ideal M of L with $L = F(a_i + a_j) \oplus M$. Clearly one of a_i, a_j does not belong to M: suppose $a_i \notin M$. Then $L = Fa_i \oplus M$ and $a_i \in Z(L)$. Hence Asoc$L = Z \oplus D$, where $Z = Z(L)$ and $[x, a] = \lambda_x a$ for all $a \in D$ and $\lambda_x \neq 0$.

Let $\Lambda : L \to F$ be given by $\Lambda(x) = \lambda_x$. This is a one-dimensional representation of L. Hence, either Im $\Lambda = 0$, in which case $D = 0$, or else $L = \text{Ker } \Lambda \oplus Fx$ and $\lambda_x = 1$. Put $L = Z \oplus D \oplus C \oplus Fx$, where $C \subseteq \text{Ker } \Lambda$.

If $y \notin$ Asoc$L$ then Fy is complemented by an ideal M and $L^2 \leq M$, so $y \notin L^2$. This yields that $L^2 \leq$ Asoc$L$. Clearly $D \leq L^2$. If $L^2 \leq Z$ then $L^3 = 0$ and we have case (i). So suppose that $D \neq 0$ and let $a \in D$. If there is an element $z \in Z \cap L^2$, then $F(z + a)$ is not an ideal of L and so $z + a \notin L^2$, a contradiction. Hence $L^2 = D$.

Let $c \in C$. If $[x, c] = 0$ then Fc is an ideal of L and $c \in C \cap$ Asoc$L = 0$. So suppose $[x, c] \neq 0$. Then $[x, c] \in D$ so $[x, c - [x, c]] = 0$. This implies that $F(c - [x, c])$ is an ideal of L, whence $c - [x, c] \in$ Asoc$L$. But now $c \in C \cap$ Asoc$L = 0$. Hence $C = 0$ and L is as described in (ii).

Now suppose that $L^3 = 0$. If $x \in L^2$ then Fx is an ideal of L. If $x \notin L^2$ there is a subspace M of codimension one in L containing L^2 such that $x \notin M$. This implies that Fx is a c-ideal of L.

Finally suppose that L is as in (ii): say $L = Z \oplus A \oplus Fx$ where $Z = Z(L)$, A is abelian and $[x, a] = a$ for all $a \in A$. Let $z + a + \alpha x \in L$. If $z \neq 0$ then choosing $M = Z_1 \oplus A \oplus Fx$ where $Z = Z_1 \oplus Fz$ shows that $F(z + a + \alpha x)$ is a c-ideal of L. So suppose $z = 0$. If $\alpha = 0$ then Fa is an ideal of L. If $\alpha \neq 0$ then choosing $M = Z \oplus A$ shows that $F(a + \alpha x)$ is a c-ideal of L.

References

[1] Barnes, D. W.: Nilpotency of Lie algebras. *Math. Z.* **79**, 237–238 (1962).

[2] Barnes, D. W.: On the cohomology of soluble Lie algebras. *Math. Z.* **101**, 343–349 (1967).

[3] Dixmier, J.: Sous-algèbres de Cartan et décompositions de Levi dans les algèbres de Lie (French). *Trans. Roy. Soc. Canada Sect. III* (3) **50**, 17–21 (1956).

[4] Jehad,J.: Some conditions for solubility. *Math. J. Okayama Univ.* **42**, 1–5 (2000).

[5] Li, D., Guo, X.: The influence of c-normality on the structure of finite groups. *J. Pure and Applied Algebra* **150**, 53–60 (2000).

[6] Li, D., Guo, X.: The influence of minimal subgroups on the structure of finite groups, II. *Comm. Algebra* **26**, 1913–1922 (1998).

[7] Skiba, A. N.: A note on c-normal subgroups of finite groups. *Algebra and Discrete Math.* **3**, 85–95 (2005).

[8] Towers, D. A.: A Frattini theory for algebras. *Proc. London Math. Soc.* (3) **27**, 440–462 (1973).

[9] Towers, D. A.: On Lie algebras in which modular pairs of subalgebras are permutable. *J. Algebra* **68**, 369–377 (1981).

[10] Wang, Y.: C-normality of groups and its properties. *J. Algebra* **180**, 954–965 (1996).

[11] Wang, Y.: The influence of minimal subgroups on the structure of finite groups. *Acta Math. Sinica* **16**, 63–70 (2000).

[12] Wei, H.: On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups. *Comm. Algebra* **29**, 2193–2200 (2001).

[13] Weisfeiler, B.: On subalgebras of simple Lie algebras of characteristic $p > 0$'. *Trans. Amer. Math. Soc.* **286**, 471–503 (1984).

[14] Winter, D. J.: Abstract Lie algebras. M.I.T. Press, Cambridge, Mass. (1972).