A SURJECTIVE SUMMATION OPERATOR WITH NO LIPSCHITZ RIGHT INVERSE

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Abstract. We show that there exists a Banach space X which contains closed subspaces Y and Z with $Y + Z = X$ such that the associated surjective summation operator $\Sigma\colon Y \times Z \to X$ defined by $\Sigma(y, z) = y + z$ for $y \in Y$ and $z \in Z$ has no Lipschitz right inverse.

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1. Introduction and statement of the main result

Given closed subspaces Y and Z of a Banach space X, we consider the *summation operator* $\Sigma\colon Y \times Z \to X$ defined by

$$\Sigma(y, z) = y + z \qquad (y \in Y,\, z \in Z). \tag{1.1}$$

This operator is linear and bounded, with norm at most 1 provided that we equip $Y \times Z$ with the norm $\|(y, z)\| = \|y\| + \|z\|$ for $y \in Y$ and $z \in Z$. (As usual, we consider $Y \times Z$ as a vector space with respect to the coordinatewise defined operations.) Moreover, Σ is surjective if and only if $Y + Z = X$.

The purpose of this paper is to prove the following result.

Theorem 1.1. *There exists a Banach space X which contains closed subspaces Y and Z with $Y + Z = X$ such that the associated surjective summation operator $\Sigma\colon Y \times Z \to X$ defined by* (1.1) *has no Lipschitz right inverse.*

Outline of the proof of Theorem 1.1. The proof consists of three steps which we state in Propositions 1.2, 1.3 and 1.5 below. These results involve two technical notions, namely that two Banach spaces are "essentially incomparable" and that a Banach space is "hereditarily indecomposable". We refer to Definitions 3.1 and 4.1, respectively, for the formal definitions of these notions.

Proposition 1.2. *Let X be a Banach space which is reflexive or separable. Suppose that X contains closed subspaces Y and Z such that $Y + Z = X$ and*

(A) *the subspace $Y \cap Z$ is infinite-dimensional and essentially incomparable with both Y and Z.*

Then $\Sigma\colon Y \times Z \to X$ defined by (1.1) *has no Lipschitz right inverse.*

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Proposition 1.3. *Let X be a hereditarily indecomposable Banach space containing closed subspaces Y and Z such that $Y + Z = X$ and*

(B) *the subspace $Y \cap Z$ has infinite codimension in both Y and Z.*

Then condition (A) *in Proposition* 1.2 *is satisfied.*

Remark 1.4. Condition (B) evidently implies that the subspaces Y and Z are infinite-dimensional. In fact, the first part of condition (A) follows from this hypothesis alone: $Y \cap Z$ is infinite-dimensional whenever Y and Z are closed, infinite-dimensional subspaces of a hereditarily indecomposable Banach space X such that $Y + Z = X$; see Lemma 4.6 for details. However, we require the full strength of condition (B) in order to deduce the second part of condition (A).

Proposition 1.5. *There exists a reflexive (and hence separable) hereditarily indecomposable Banach space X containing closed subspaces Y and Z such that $Y + Z = X$ and condition* (B) *in Proposition* 1.3 *is satisfied.*

The proof of Theorem 1.1 is now immediate. Indeed, take X, Y and Z as in Proposition 1.5. Then Proposition 1.3 shows that condition (A) in Proposition 1.2 is satisfied. Since X is reflexive, we conclude that Σ has no Lipschitz right inverse. □

Organization. In the remainder of this section, we explain our motivation for proving Theorem 1.1. Next, in Section 2, we set up a general framework for studying whether the summation operator Σ has a right Lipschitz (or bounded linear) inverse (assuming that Σ is surjective, that is, $Y + Z = X$), before we apply it to prove Propositions 1.2, 1.3 and 1.5 in Sections 3–4. Finally, Section 5 contains a discussion of the main technical ingredient in the proof of Theorem 1.1.

Motivation. Let Y and Z be subsets of a Banach space X such that $Y + Z = X$. By the axiom of choice, there exist maps $a\colon X \to Y$ and $b\colon X \to Z$ which satisfy $x = a(x) + b(x)$ for every $x \in X$. We refer to pairs of maps (a, b) with this property as a *decomposition* of X into Y and Z, and say that a decomposition is *continuous* (respectively, *Lipschitz*) if the maps a and b are continuous (respectively, Lipschitz).

The motivation behind this paper comes from the following general question: Among all possible decompositions of X into Y and Z, can we always find one that possesses a particular form of topological or algebraic regularity? It is a standard consequence of the Closed Graph Theorem that if Y and Z are closed, complementary subspaces of X (that is, they satisfy $Y \cap Z = \{0\}$ in addition to our standing hypothesis that $Y + Z = X$), then a bounded and linear decomposition exists. Naturally, this raises the question: If we drop the requirement that $Y \cap Z = \{0\}$, how much regularity of the decomposition can we retain? The following theorem implies that we can always find a continuous decomposition in this case.

Theorem 1.6 (Bartle–Graves [2, Corollary 17.67])**.** *Let $T\colon E \to F$ be a bounded linear surjection between Banach spaces E and F. Then T has a continuous (but not necessarily linear) right inverse.*

Indeed, assuming that Y and Z are closed subspaces of X such that $Y + Z = X$, we may apply Theorem 1.6 to the bounded linear surjection $\Sigma\colon Y \times Z \to X$ defined by (1.1) to obtain a continuous decomposition of X into Y and Z. With some extra work, one may strengthen this conclusion to produce a continuous decomposition that is pointwise Lipschitz on a dense subset of X, as shown in [20, Theorem 5.2].

In view of this result, and since the condition that $Y \cap Z = \{0\}$ seems fairly innocuous in this context, one may hope that, whenever Y and Z are closed subspaces of X such that $Y + Z = X$, it is always possible to obtain some higher level of regularity in the decomposition than the above, perhaps a Lipschitz decomposition, or maybe even one that is bounded and linear? However, Theorem 1.1 shows that this is not true in general: There exists a Banach space X containing closed subspaces Y and Z satisfying $Y + Z = X$ such that the bounded linear surjection $\Sigma\colon Y \times Z \to X$ defined by (1.1) has no Lipschitz right inverse, and therefore X has no Lipschitz decomposition into Y and Z.

Remark 1.7. Examples are known where a Banach space E contains a closed subspace F for which the quotient map from E onto E/F does not admit any Lipschitz, or even uniformly continuous, right inverse; see [1], [4, Example 1.20] and [14, Theorem 4.2]. The flavour of Theorem 1.1 is somewhat different from these examples. However, taking $E = Y \times Z$ and $F = \ker \Sigma$, where Y and Z are chosen as in Theorem 1.1 and $\Sigma\colon E = Y \times Z \to X$ is the bounded linear surjection defined by (1.1), we obtain another example where the quotient map from E onto E/F does not admit any Lipschitz right inverse.

A variant for ordered Banach spaces. The problem of how regular a decomposition one can find has a natural counterpart in the field of "positivity". Throughout this subsection, we consider real scalars only. A subset C of a Banach space X is a *wedge* if $C \neq \emptyset$, $C + C \subseteq C$ and $\lambda C \subseteq C$ for every $\lambda \in [0, \infty)$. A wedge C is *generating* if $C - C = X$, and it is a *cone* if $C \cap (-C) = \{0\}$.

It is a well-known, easy consequence of the axioms that wedges correspond to translation-invariant, positively homogeneous preorders on the underlying Banach space X in the following precise sense: Given a wedge C in a Banach space X, the relation $\geqslant$ on X defined by

$$x \geqslant y \iff x - y \in C$$

is a translation-invariant, positively homogeneous preorder, and conversely, given a translation-invariant, positively homogeneous preorder $\geqslant$ on X, the set $C = \{x \in X : x \geqslant 0\}$ is a wedge in X. Moreover, a wedge C is a cone if and only if the corresponding preorder $\geqslant$ is anti-symmetric (and hence a partial order). Due to this correspondence, the elements of a wedge C are called *positive.*

Let C be a norm-closed, generating wedge in a Banach space X. Then, using a generalization of Theorem 1.6, it was shown in [12, 19] that the continuous surjection $C \times C \to X$ defined by $(x, y) \mapsto x - y$ has a continuous right inverse. In other words, in a Banach space X preordered by a wedge, one can always decompose any element into the difference between two positive elements in a continuous manner. In the case where X is a Banach lattice, or an order unit space, the decomposition of an element into the difference between two positive elements can even be achieved in a Lipschitz manner, as shown in [24, Proposition 5.2] and [21, Proposition 3.2], respectively.

Closed subspaces are wedges, so Theorem 1.1 implies that, for closed wedges C and D in a Banach space X such that $C-D=X$, no Lipschitz decomposition of X into a difference between two elements of C and D exists in general. However, to the authors' knowledge, the following question remains open.

Question 1.8. Let C be a norm-closed, generating wedge (or cone) in a Banach space X. Is it always possible to find Lipschitz maps $a, b\colon X \to C$ such that $x = a(x) - b(x)$ for every $x \in X$?

2. A general framework

General conventions. All Banach spaces are over the same scalar field, either $\mathbb{R}$ or $\mathbb{C}$. The term "operator" means a bounded linear map between Banach spaces. Given a closed subspace W of a Banach space X, $Q_W\colon X \to X/W$ denotes the quotient map.

We begin with an elementary result which shows that writing a Banach space X as the sum of two closed subspaces corresponds to decomposing a certain quotient of X into two closed, complementary subspaces. This result is surely known to specialists, but as we have not been able to locate a proof of it in the literature, we include one.

Lemma 2.1. *Let X be a Banach space.*

(i) *Suppose that Y and Z are closed subspaces of X such that $Y+Z=X$, and set $W = Y\cap Z$. Then $Q_W[Y]$ and $Q_W[Z]$ are closed, complementary subspaces of the quotient space X/W.*

(ii) *Let W be a closed subspace of X, and suppose that F and G are closed, complementary subspaces of the quotient space X/W. Then $Y = Q_W^{-1}[F]$ and $Z = Q_W^{-1}[G]$ are closed subspaces of X such that $Y+Z=X$, $Y\cap Z = W$, $Y/W = F$ and $Z/W = G$.*

Proof. (i). We must show that the subspaces $Q_W[Y]$ and $Q_W[Z]$ of X/W are closed and satisfy: (1) $Q_W[Y]\cap Q_W[Z] = \{0\}$ and (2) $Q_W[Y]+Q_W[Z] = X/W$.

To verify that $Q_W[Y]$ is closed, take a sequence (f_n) in $Q_W[Y]$ such that the series $\sum_{n=1}^\infty \|f_n\|$ converges. For each $n\in\mathbb{N}$, choose $y_n \in Y$ such that $Q_W(y_n) = f_n$ and $\|y_n\| \leqslant \|f_n\| + 1/2^n$. Then $\sum_{n=1}^\infty y_n$ converges absolutely, and therefore, by completeness, it converges in norm to some $y\in Y$. Since Q_W is linear and continuous, it follows that $\sum_{n=1}^\infty f_n$ converges to $Q_W(y)\in Q_W[Y]$. This shows that the subspace $Q_W[Y]$ is complete and hence closed. A similar argument proves that $Q_W[Z]$ is closed.

(1). Suppose that $e\in Q_W[Y]\cap Q_W[Z]$, and write $e = Q_W(y) = Q_W(z)$ for some $y\in Y$ and $z\in Z$. Then $w = y-z$ belongs to $\ker Q_W = Y\cap Z$, and therefore $y = w+z\in Z$. This implies that $y \in Y\cap Z = W$, so $e = Q_W(y) = 0$. Conversely, it is clear that $0\in Q_W[Y]\cap Q_W[Z]$, and consequently $Q_W[Y]\cap Q_W[Z] = \{0\}$.

(2). Given $e\in X/W$, take $x\in X$ such that $Q_W(x) = e$, and write $x = y+z$ for some $y\in Y$ and $z\in Z$. Then $e = Q_W(y)+Q_W(z)\in Q_W[Y]+Q_W[Z]$. Conversely, the inclusion $Q_W[Y]+Q_W[Z]\subseteq X/W$ is clear, and (2) follows.

(ii). First, we note that $Y = Q_W^{-1}[F]$ and $Z = Q_W^{-1}[G]$ are closed subspaces of X because Q_W is continuous and linear.

Second, we show that $Y + Z = X$. Only the inclusion $X \subseteq Y + Z$ is non-trivial. Given $x \in X$, write $Q_W(x) = f + g$ for some $f \in F$ and $g \in G$, and choose $y, z \in X$ such that $Q_W(y) = f$ and $Q_W(z) = g$. The definitions of Y and Z imply that $y \in Y$ and $z \in Z$. Since $Q_W(x - y - z) = Q_W(x) - f - g = 0$, the element $w = x - y - z$ belongs to W. Consequently $y + w \in Y$, and therefore $x = (y + w) + z \in Y + Z$.

Third, it follows directly from the definitions that

$$Y \cap Z = Q_W^{-1}[F] \cap Q_W^{-1}[G] = Q_W^{-1}[F \cap G] = Q_W^{-1}[\{0\}] = \ker Q_W = W$$

and

$$Y/W = Q_W[Y] = Q_W[[Q_W^{-1}[F]] = F,$$

with a similar argument showing that $Z/W = G$. □

Remark 2.2. Let Y and Z be closed subspaces of a Banach space X. An essential feature of the problem we consider is that $Y + Z = X$; otherwise the operator Σ is not surjective, so it cannot have a right inverse of any kind. This issue can of course be overcome by replacing X with $Y + Z$, but this approach will work only if the sum $Y + Z$ is closed in X; otherwise it is not complete, so $Y + Z$ fails to be a Banach space in its own right.

The question of when the sum of two closed subspaces Y and Z of a Banach space X is closed is well studied in the literature. It is folklore that $Y + Z$ is closed if Y or Z is finite-dimensional. On the other hand, concrete examples where $Y + Z$ is not closed are easy to find; see for instance [17, Exercise 1.84]. If $Y \cap Z = \{0\}$ (and Y and Z are both non-zero), then a standard result states that $Y + Z$ is closed if and only if

$$\inf\big\{\|y - z\| : y \in Y,\, z \in Z,\, \|y\| = \|z\| = 1\big\} > 0,$$

as mentioned in [10, p. 852], for instance. In the general case where $W = Y \cap Z$ is non-zero, and excluding the trivial case where $Y \subseteq Z$, one can use the above result in the quotient space X/W to show that $Y + Z$ is closed if and only if

$$\inf\left\{\frac{\|Q_Z(y)\|}{\|Q_W(y)\|} : y \in Y \setminus Z\right\} > 0;$$

see [15, Theorem 4.2 (page 219)].

The problem whether a surjective operator such as Σ has a bounded linear (or Lipschitz) right inverse has a well-known reformulation in the language of short exact sequences. Let us recall the basics of this terminology. We refer to [6] for a comprehensive modern account of this theory in the context of Banach spaces.

Definition 2.3. A *short exact sequence* of Banach spaces is a sequence of the form

$$0 \longrightarrow W \xrightarrow{\ S\ } E \xrightarrow{\ T\ } X \longrightarrow 0, \tag{2.1}$$

where W, E and X are Banach spaces, $S\colon W \to E$ is an injective operator, $T\colon E \to X$ is a surjective operator and $S[W] = \ker T$.

In particular, S has closed range, so it is an isomorphic embedding. In the context of Banach spaces, the Splitting Lemma from homological algebra reads as follows (see for instance [6, Lemma 2.1.5]).

Lemma 2.4. *The following three conditions are equivalent for a short exact sequence of the form* (2.1):

(a) *the operator S has a bounded linear left inverse;*
(b) *the operator T has a bounded linear right inverse;*
(c) *the subspace* $\ker T$ $(= S[W])$ *is complemented in E.*

Definition 2.5. When one, and hence all three, of the conditions in Lemma 2.4 are satisfied, we say that the short exact sequence (2.1) *splits*.

Lindenstrauss [16] and Godefroy and Kalton [9] showed that in certain cases one can pass from Lipschitz splittings to splittings in the above sense. We require the following result from their work.

Theorem 2.6 (Lindenstrauss; Godefroy–Kalton)**.** *Consider a short exact sequence of the form* (2.1)*, and suppose that the Banach space E is reflexive or that the Banach space X is separable. Then* (2.1) *splits if and only if it "Lipschitz splits" in the sense that the operator T has a Lipschitz right inverse (which need not be linear).*

Proof. This follows from [16, Corollary 2] for E reflexive, and from [9, Proposition 2.8 and Theorem 3.1], in the same way as in the proof of [9, Corollary 3.2], for X separable. □

The following lemma makes the connection between short exact sequences and Theorem 1.1 explicit.

Lemma 2.7. *Let Y and Z be closed subspaces of a Banach space X such that $Y + Z = X$, and set $W = Y \cap Z$. Then we have a short exact sequence*

$$0 \longrightarrow W \xrightarrow{\ \Delta\ } Y \times Z \xrightarrow{\ \Sigma\ } X \longrightarrow 0, \tag{2.2}$$

where the operator $\Delta\colon W \to Y \times Z$ is defined by $\Delta(w) = (w, -w)$ for $w \in W$, and $\Sigma\colon Y \times Z \to X$ is the summation operator defined by (1.1).

Proof. It is clear that Δ is a bounded linear injection and Σ a bounded linear surjection, so the following chain of equations completes the proof:

$$\ker \Sigma = \{(y, z) \in Y \times Z : y + z = 0\} = \{(y, -y) : y \in Y \cap Z\} = \Delta[W]. \qquad \square$$

We conclude this collection of general material with a result which shows that to construct examples where the short exact sequence (2.2) does not split, one must work with closed subspaces which are not complemented.

Lemma 2.8. *Let X be a Banach space containing closed subspaces Y and Z such that $Y + Z = X$, and suppose that the subspace $W = Y \cap Z$ is complemented in Y or in Z. Then the short exact sequence* (2.2) *splits.*

Proof. Suppose that W is complemented in Y, so that there is an operator $P\colon Y \to W$ such that $Pw = w$ for every $w \in W$. Define $L\colon Y \times Z \to W$ by $L(y, z) = Py$. Then L is an operator which satisfies

$$L\Delta w = L(w, -w) = Pw = w \qquad (w \in W),$$

so condition (a) in Lemma 2.4 is satisfied. The case where W is complemented in Z is similar (except that we need to define $L(y, z) = -Pz$). $\square$

Remark 2.9. Let X be a Banach space which contains closed subspaces Y and Z such that $Y + Z = X$, and set $W = Y \cap Z$. We can represent this information in the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & W \arrow[r, "J_W^Y"] \arrow[d, "J_W^Z"'] & Y \arrow[r, "Q_Z\restriction_Y"] \arrow[d, "J_Y"] & X/Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "J_Z"] \arrow[d, "Q_Y\restriction_Z"'] & X \arrow[r, "Q_Z"] \arrow[d, "Q_Y"] & X/Z \arrow[r] & 0 \\
 & X/Y \arrow[r, equal] \arrow[d] & X/Y \arrow[d] & & \\
 & 0 & 0, & &
\end{tikzcd} \tag{2.3}$$

where J_W^Y, J_W^Z, J_Y and J_Z denote the inclusion maps and Q_Y and Q_Z the quotient maps. Comparing this diagram with [6, Diagram (2.34)], we see that the Banach space X can be viewed as the pushout of the diagram

$$\begin{tikzcd}
W \arrow[r, "J_W^Y"] \arrow[d, "J_W^Z"'] & Y \\
Z, &
\end{tikzcd} \tag{2.4}$$

and the sequence (2.2) arises as the "diagonal pushout sequence" of (2.3) in the sense of [6, page 92]. Alternatively, combining Lemma 2.7 with the explicit definition given in [6, page 73] of the pushout of (2.4) as the quotient space $(Y \times Z)/\{(w, -w) : w \in W\}$, we obtain a direct, non-diagrammatic proof that this pushout is X.

3. Essential incomparability and the proof of Proposition 1.2

Definition 3.1. An operator $S\colon E \to F$ between Banach spaces E and F is:

- *Fredholm* if the kernel of S is finite-dimensional and the range of S has finite codimension in F. (It is a standard result that the latter condition implies that S has closed range.)
- *inessential* if $I_E - TS$ is a Fredholm operator for every operator $T\colon F \to E$. (Here, and elsewhere, I_E denotes the identity operator on E.)

We say that a Banach space E is *essentially incomparable* with another Banach space F if every operator from E to F is inessential.

Remark 3.2. (i) The definitions of inessential operator and essential incomparability appear to lack symmetry. However, it is an elementary fact that $I_E - TS$ is a Fredholm operator if and only if $I_F - ST$ is a Fredholm operator (see for instance [18, Sätze 2.4-A–2.5-A]). This implies in particular that E is essentially incomparable with F if and only if every operator from F to E is inessential, and so we can make statements like "the Banach spaces E and F are essentially incomparable" without ambiguity.

(ii) The class of inessential operators forms a proper operator ideal in the sense of Pietsch [23, Section 3.4]. Specifically, this means that every finite-rank operator is inessential, the sum of two inessential operators is inessential, the composition of an inessential operator with an arbitrary operator from the left or the right is inessential, and the identity operator on an infinite-dimensional Banach space is never inessential.

Lemma 3.3. *Let W, Y and Z be Banach spaces, and suppose that W is essentially incomparable with both Y and Z. Then W is essentially incomparable with $Y \times Z$.*

Proof. Define $J_Y\colon Y \to Y \times Z$ and $P_Y\colon Y \times Z \to Y$ by $J_Y y = (y, 0)$ and $P_Y(y, z) = y$ for $y \in Y$ and $z \in Z$, and define $J_Z\colon Z \to Y \times Z$ and $P_Z\colon Y \times Z \to Z$ analogously. Since $I_{Y\times Z} = J_Y P_Y + J_Z P_Z$, every operator $S\colon W \to Y \times Z$ can be written as $S = J_Y(P_Y S) + J_Z(P_Z S)$, where the operators $P_Y S\colon W \to Y$ and $P_Z S\colon W \to Z$ are inessential by hypothesis. This implies that S is inessential because the inessential operators form an operator ideal. □

Lemma 3.4. *Consider a short exact sequence of the form*

$$0 \longrightarrow W \xrightarrow{\ S\ } E \xrightarrow{\ T\ } X \longrightarrow 0,$$

and suppose that W is infinite-dimensional and essentially incomparable with E. Then the short exact sequence does not split.

Proof. The essential incomparability of W with E implies that the operator $S\colon W \to E$ is inessential; that is, $I_W - LS$ is a Fredholm operator for every operator $L\colon E \to W$. In particular $I_W - LS \neq 0$ because W is infinite-dimensional, so condition (a) in Lemma 2.4 fails. □

Combining Lemmas 3.3 and 3.4 with Lemma 2.7, we obtain the following conclusion.

Corollary 3.5. *Let X be a Banach space containing closed subspaces Y and Z such that $Y+Z=X$ and the subspace $W=Y\cap Z$ is infinite-dimensional and essentially incomparable with both Y and Z. Then the short exact sequence*

$$0 \longrightarrow W \xrightarrow{\ \Delta\ } Y\times Z \xrightarrow{\ \Sigma\ } X \longrightarrow 0$$

from (2.2) *does not split.*

Finally, Proposition 1.2 follows by combining Corollary 3.5 with Theorem 2.6, bearing in mind that if X is reflexive, then the subspaces Y and Z are also reflexive, so $Y\times Z$ is reflexive.

4. Hereditarily indecomposable Banach spaces and the proofs of Propositions 1.3 and 1.5

Definition 4.1. A Banach space E is:

- *decomposable* if E contains a pair of infinite-dimensional, closed, complementary subspaces; that is, $E=F+G$ for some infinite-dimensional, closed subspaces F and G with $F\cap G=\{0\}$.
- *indecomposable* if E is infinite-dimensional and fails to be decomposable.
- *hereditarily indecomposable,* abbreviated HI, if E is infinite-dimensional and every closed, infinite-dimensional subspace of E is indecomposable.

The fact that there exist Banach spaces which are HI (and hence indecomposable) is a highly non-trivial result due to Gowers and Maurey [10]. In line with common practice, we shall write *HI space* instead of "hereditarily indecomposable Banach space".

Remark 4.2. Every non-separable reflexive Banach space E is decomposable. This is a well-known consequence of the fact that E has the "separable complementation property" (this follows for instance from [11, Theorem 3.42 and Proposition 3.43]), which means that every separable subspace of E is contained in a separable complemented subspace. Indeed, since every infinite-dimensional Banach space contains basic sequences, E contains an infinite-dimensional, separable subspace, so it also contains an infinite-dimensional, separable complemented subspace F. Take a closed subspace G of E such that $F+G=E$ and $F\cap G=\{0\}$. Then G must be non-separable and therefore infinite-dimensional, so E is decomposable.

Definition 4.3. An operator $S\colon E\to F$ between Banach spaces E and F is *strictly singular* if the restriction of S to any closed, infinite-dimensional subspace of E is not an isomorphism onto its range.

Theorem 4.4 (Ferenczi)**.** *Every operator on an HI space is either Fredholm or strictly singular.*

Proof. A proof of this result is contained in the final paragraph of the proof of [7, Theorem 2]. We observe that while [7, Theorem 2] itself is stated for real scalars only, the proof we require applies in both the real and complex case. (However, for complex scalars, Theorem 4.4 is also an easy consequence of the famous result of Gowers and Maurey [10] that every operator on a complex HI space is the sum of a scalar multiple of the identity and a strictly singular operator.) □

Corollary 4.5. *Let W be a closed subspace of infinite codimension in an HI space E. Then every operator from E to W is strictly singular, and hence W and E are essentially incomparable.*

Proof. Let $S\colon E \to W$ be an operator, and denote the inclusion map by $J_W\colon W \to E$. Since the range of the operator $J_W S\colon E \to E$ is contained in W, it has infinite codimension in E, so $J_W S$ cannot be a Fredholm operator. It is therefore strictly singular by Theorem 4.4, and hence S is strictly singular.

The essential incomparability of W and E follows because every strictly singular operator is inessential (see for instance [23, §26.7.3], bearing in mind that in the notation of [23], $\mathfrak{S}$ and $\mathfrak{R}$ denote the ideals of strictly singular and inessential operators, respectively). □

We shall also require the following elementary, standard fact about intersections of subspaces of an indecomposable Banach space. We include a simple proof for completeness.

Lemma 4.6. *Let Y and Z be closed, infinite-dimensional subspaces of an indecomposable Banach space X, and suppose that $Y + Z = X$. Then $Y \cap Z$ is infinite-dimensional.*

Proof. To prove the contrapositive, let Y and Z be closed, infinite-dimensional subspaces of a Banach space X such that $Y+Z = X$, and suppose that the subspace $W = Y\cap Z$ is finite-dimensional. Then W is complemented in Y, so Y contains a closed subspace Y_0 such that $Y_0\cap W = \{0\}$ and $Y_0+W = Y$. Since $W \subseteq Z$, we have $Y_0+Z = Y_0+W+Z = Y+Z = X$, and $Y_0 \cap Z = Y_0 \cap Y \cap Z = Y_0 \cap W = \{0\}$, which shows that Y_0 and Z are closed, complementary subspaces of X. Moreover, Y_0 is infinite-dimensional because it has finite codimension in Y, so X is decomposable. □

Proof of Proposition 1.3. Suppose that Y and Z are closed subspaces of an HI space X such that $Y + Z = X$ and $W = Y\cap Z$ has infinite codimension in both Y and Z. Then Y and Z must both be infinite-dimensional, so Lemma 4.6 implies that W is infinite-dimensional. Applying Corollary 4.5 twice, first with $E = Y$ and then with $E = Z$, we see that W is essentially incomparable with both Y and Z. □

Theorem 4.7 (Ferenczi)**.** *There exists a reflexive HI space X which contains a closed subspace W such that the quotient space X/W is decomposable.*

Proof. In [8, Section 3], Ferenczi constructs an HI space E which is not “quotient hereditarily indecomposable”. The latter term is defined in [8, Definition 1]. Negating it, we see that E contains closed subspaces $W \subset X$ such that the quotient space X/W is decomposable. Moreover, E is reflexive, as stated in the second line of [8, Section 3]. Since reflexivity and being HI both pass to closed, infinite-dimensional subspaces, we conclude that X has the stated properties. □

Remark 4.8. More generally, Argyros and Felouzis [3, Section 8] have shown that many classical decomposable Banach spaces, including ℓ_p and $L_p[0,1]$ for $1 < p < \infty$, can be realized as quotient spaces of reflexive HI spaces.

Proof of Proposition 1.5. For ease of reference, let us recall that the statement we seek to prove is:

> *There exists a reflexive (and hence separable) HI space X containing closed subspaces Y and Z such that $Y + Z = X$ and the subspace $Y \cap Z$ has infinite codimension in both Y and Z.*

Theorem 4.7 shows that there exists a reflexive HI space X which contains a closed subspace W such that the quotient space X/W is decomposable. Consequently we can apply Lemma 2.1(ii) to obtain subspaces Y and Z of X with the specified properties. Remark 4.2 explains why X must be separable. ☐

Remark 4.9. Ferenczi's Banach space E that we used in the proof of Theorem 4.7 above is defined as the pushout of a diagram

$$\begin{tikzcd} V \arrow[r, "U_1"] \arrow[d, "U_2"'] & X_1 \\ X_2, & \end{tikzcd} \tag{4.1}$$

where X_1 and X_2 are suitably constructed HI spaces, the operators U_1 and U_2 are isometries, and the Banach space V is infinite-dimensional. This follows from [8, Proposition 23]. In fact, reading this result carefully, one can see that $X = E$ satisfies the conclusions of Theorem 4.7 (using our notation, not Ferenczi's, who writes $\widehat{X}$ for the space we call E); that is, we do not need to pass to a subspace of E to find an HI space which admits a decomposable quotient space.

The reason is that [8, Proposition 23] states that E is an HI space whose dual space E^* is not HI. Therefore E^* contains a closed, infinite-dimensional subspace F which is decomposable. Let $J_F\colon F \to E^*$ be the inclusion map. Its adjoint $J_F^*\colon E^{**} \to F^*$ is a quotient map. Since E is reflexive and decomposability passes from F to its dual, we conclude that E admits a decomposable quotient space, namely F^*.

Scrutinizing the proof of [8, Proposition 23], we see that $F = W^\perp$, where W denotes the canonical isometric image in E of the subspace V used in the pushout construction (4.1) above. This implies that $F^* \cong E/W$ by reflexivity and standard duality results.

In summary, Ferenczi's work produces a reflexive HI space E which is the pushout of the diagram (4.1) and such that the quotient space E/W is decomposable (where W is the canonical image in E of the subspace V, as above). One could use this result as the starting point of a proof of Theorem 1.1 instead of the final piece of the jigsaw, as we have done.

5. Reflections on the proof of Theorem 1.1

It is natural to wonder whether we really need the heavy machinery of Ferenczi's HI space with a decomposable quotient in order to apply Proposition 1.2 to prove Theorem 1.1. In other words, is it possible to find a more "classical", or less "exotic", Banach space X containing closed subspaces Y and Z such that $Y+Z=X$ and the intersection $Y\cap Z$ is infinite-dimensional and essentially incomparable with both Y and Z? Note that, by the definition of essential incomparability, this cannot happen if $Y\cap Z$ is complemented in Y or in Z.

While we cannot answer the above question in complete generality, we know that it is impossible in many cases. This relies on the following notion: A Banach space X is *subprojective* if every closed, infinite-dimensional subspace of X contains a closed, infinite-dimensional subspace which is complemented in X. Pfaffenberger [22] has shown that every inessential operator on a subprojective Banach space is strictly singular. It is not hard to generalize this result to operators defined only on subspaces. For good measure, we provide a proof.

Lemma 5.1. *Let W be a closed, infinite-dimensional subspace of a subprojective Banach space X. Then every inessential operator from W to X is strictly singular, and consequently W and X are not essentially incomparable.*

Proof. To prove the contrapositive, suppose that $S\colon W\to X$ is an operator which is not strictly singular. Then W contains a closed, infinite-dimensional subspace Y such that the restriction of S to Y is an isomorphism onto $S[Y]$.

The subprojectivity of X means that we can find a closed, infinite-dimensional subspace Z of $S[Y]$ and an operator $P\colon X\to Z$ such that $Pz=z$ for every $z\in Z$. Set $V=Y\cap S^{-1}[Z]$. We observe that the restriction $\widetilde{S}\colon V\to Z$ given by $\widetilde{S}v=Sv$ for $v\in V$ is an isomorphism, so we may consider the composite operator $T=J_V\widetilde{S}^{-1}P\colon X\to W$, where $J_V\colon V\to W$ denotes the inclusion map. It satisfies

$$TSv=J_V\widetilde{S}^{-1}P\widetilde{S}v=v\qquad(v\in V),$$

which shows that $V\subseteq\ker(I_W-TS)$. This implies that I_W-TS is not a Fredholm operator because V is isomorphic to Z and therefore infinite-dimensional. Hence S is not inessential.

It follows that W and X are not essentially incomparable because the inclusion map from W to X is evidently not strictly singular and therefore not inessential. □

Since subprojectivity passes to closed subspaces simply by restricting the relevant projection, Lemma 5.1 implies that the answer to the question posed above is "no" whenever X is subprojective. More precisely, in this case, given any pair of closed subspaces Y and Z of X whose intersection $Y\cap Z$ is infinite-dimensional, we see that $Y\cap Z$ is neither essentially incomparable with Y nor Z, so Corollary 3.5 and Proposition 1.2 do not apply.

Many "standard" Banach spaces are subprojective, including the following:

(i) The classical sequence spaces c_0 and ℓ_p for $1\leqslant p<\infty$.
(ii) The Lebesgue spaces $L_p[0,1]$ for $2\leqslant p<\infty$ [13, Corollaries 1–2]. (By contrast, we remark that $L_p[0,1]$ is **not** subprojective for $1\leqslant p<2$. Indeed, set $q=2$ if

$p = 1$ and choose $q \in (p, 2)$ otherwise. Then $L_p[0, 1]$ contains a subspace which is isomorphic to ℓ_q, but no complemented subspace of $L_p[0, 1]$ is isomorphic to ℓ_q.)

(iii) The quasi-reflexive James spaces J_p for $1 < p < \infty$.

(iv) The Tsirelson space T.

(v) The Schreier space $X[\mathcal{S}_\alpha]$ associated with the Schreier family $\mathcal{S}_\alpha$ for every countable ordinal α, and also the p^{th} variant of $X[\mathcal{S}_1]$ defined in [5].

(vi) The Baernstein spaces B_p for $1 < p < \infty$.

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