Physics Department

Geometrical Interpretation of Multipoles and Moments on Differential Manifolds

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Abstract

In this thesis, the construction of a specific family of linear functionals with support on a closed embedding $c : \mathbb{R} \hookrightarrow M$ upon a manifold is discussed. The construction is performed in a purely coordinate free fashion, based on the De Rham push-forward approach and generalised to define "tensorial currents" called "multipoles". Several geometrical and algebraic properties are investigated and two main useful classes of non-trivial coordinate representations are compared and related to the choices of some extra structures on the manifold (i.e. affine connection, foliation, adapted atlas, adapted frames). It is shown that in general, the transformation rules are not given by the action of the linear group, unless some information upon the "transverse" directions with respect to the closed embedding is provided. It is shown how the multipoles are the geometrical objects naturally arising when some specific one parameter families of compact support tensor fields are expanded asymptotically around the closed embedding. In case a one parameter family satisfies also an extra condition (i.e. self similarity) it is shown how to recover the well known standard definition of "moments", opening the door to a new completely covariant and coordinate free meaning of the concept of "multipole expansion" of functions and tensor field upon the differential manifolds. It is shown how these linear functionals admit a coordinates representation coinciding with the moments commonly defined to perform the Pole-Dipole approximation of an Energy-Momentum Tensor field in General Relativity, and when a Levi Civita connection is assumed on a pseudo-Riemmanian manifold, the first two multipoles related to an Energy Momentum tensor field expansion can easily satisfy the well known Mathisson-Papapetrou-Dixon equation. Since the proposed method of construction of the multipoles does not rely on a specific metric or a specific affine connection, a generalisation of the Pole-Dipole approximation for a non metric connection is easily achieved, casting the Mathisson-Papapetrou-Dixon equation in presence of a non null torsion. Because of this, there is hence the possibility to interpret the test particles and test charges within the Relativistic Theories (possibly beyond General Relativity) just as the multipole approximation of the regular sources of the interaction fields, with a new clear geometrical background.

Introduction

It has been several decades since Schwartz's theory of Distributions began to play a fundamental role in Science, thank to the ability to formalise rigorously some intuitions subtending fundamental mathematical concepts like Green functions, Laplace and Fourier transforms or integration and derivation of non-regular functions.

Concerning the Physics, in Continuum Classical Mechanics, Statistical Mechanics and Classical Field Theory as well as in Quantum Mechanics and Quantum Field theory, the distributions are essential mathematical tools with which, sooner or later, it is required to deal with. In Classical Electrodynamics, the only way to obtain the correct solution to the Maxwell's Equations for an electromagnetic field generated by point-like charges (the poles) without excluding the region where the charges are located, is to model the sources using non regular Schwartz distributions (i.e. the Dirac delta) and since the Maxwell's Equations are linear, it is possible to prove how the Electromagnetic Field of an arbitrary regular source admits a weak approximation in terms of a (possibly infinite) superposition of elementary Electromagnetic Fields generated by several poles spread appropriately in the space (multipole expansion).

Therefore in Classical Electromagnetism, the distributions provide a formal tool to describe the point-like sources and interpret them just as an ideal approximation of realistic regular sources when their size is beyond the scale fixed for the theory. The very same conclusion can be achieved within the Classical theory of Gravitation, where the point-like masses generating the Gravitational Field are interpreted as an ideal approximation of massive extended objects.

So, because of these aspects, the distributions seem to be the natural way to express classical point-like particles within the Classical Field Theories.

However, despite the many successful applications of the Schwartz's distributions in several branches of the Physics, their use in Relativistic Theories like General Relativity (GR) is still quite problematic.

In fact, even if the technology of the multipole expansion is massively and successfully used within perturbative approaches to find extraordinary important relativistic predictions such as weak lensing or gravitational waves radiation, the distributions are still basically used just as tools in order to find an approximation for the local coordinate expression of the geometrical objects encoding the physical information expressed in some fixed local charts, perhaps even after performing a non-covariant post-Newtonian approximation procedure or a non-general covariant linearisation.

In contrast with the Classical Theories, since one of the most stringent paradigm characterising the Relativistic Theories prescribes the mathematical objects carrying physical information must not depend at all on the choices made by the observers to map the events of the spacetime, it is clear that, in this perspective, usually the distributions are no more than a very powerful analysis tool, unable to carry true physical information and describe relativistic physical objects accordingly to a true full relativistic framework.

However, a very specific exception is provided by the attempt to approximate the motion of a free falling "spinning" extended object in Standard General Relativity. This approach, commonly known as the "Pole-Dipole approximation", despite its complicated intricacies, is able to provide a covariant method to characterise the motion of an extended "test" body with two tensor fields called Spin and Momentum, defined upon an appropriate worldline commonly interpreted as the trajectory of the object [1][2][3].

During the steps required to define the pole-dipole it is clear how the distributions definitely play a role defining the coordinate local expression of the Spin and the Momentum, but once again it is quite obscure if they are just mathematical tools to analyse local expressions for the Energy-Momentum tensor fields or are themselves the local expression of some more fundamental hidden geometrical object, maybe with a physical relevance [1][2][3][4][5][6][7].

So usually the distributions are just relegated to a shady gray zone, and it is preferred to focusing mainly on the terms that at fixed local coordinate system determine the action of these functionals, fixing step by step several constraints on them, in order to obtain something that can be possibly manipulated and interpreted a posteriori with physical considerations.[6][8][9][10][11]

This approach hides definitely non negligible potential risks, because it is almost impossible to distinguish clearly which assumptions made on the distributions are depending on the chosen physical model and which constraints, independently from the physics, must be fixed to obtain a well defined mathematical objects (linear functionals) with no caveats. Furthermore it is very obscure to understand, even at fixed physical model, which assumptions are just mere choices induced by usual customs and which constraints are essential to preserve the self-consistency of the theory.[7] [12] [9] [10]

Probably this is one of the main reasons that still prevents a massive methodical asymptotic approach (possibly to higher orders) to the dynamics within the relativistic theory beyond General Relativity or even when in General Relativity, other field of interactions are added to the model.[10]

Aimed by the purpose of mitigating this problem, in this work a new intrinsic geometrical covariant definition of a specific family of global continuous linear functionals acting on the smooth compact support tensor fields is given. These geometrical objects, very closely related to the De Rham currents [13] [14], provide a generalisation of the Schwartz distribution on the differential manifolds, formalising the intuitive concept of "distributional tensor fields". We will see how these linear functionals are able to approximate some specific one parameter families of regular compact support tensor fields and how one of their possible local coordinate representation coincides exactly with the usual definition of the moments of the local scalar fields characterising the coordinate expression of a regular compact support tensor field. Because of this strong correspondence, these specific kind of functionals are then simply called "multipoles".

Within this framework, we will see how the well known "Pole-Dipole" approximation of the Energy Momentum tensor related to an extended body in General Relativity admits a completely covariant and coordinate free geometrical interpretation in terms of this kind of linear functional. Thence in this perspective it is potentially allowed to interpret a point-like spinning free falling test particle as a first order multipole with support on a worldline which is packing, in a purely intrinsic coordinate free fashion, some relevant physical information concerning the first order approximation of the dynamics of an extended free falling object.

Since the proposed approach to the multipoles does not assume any specific metric or affine connection, we are able in principle to cast the pole and pole-dipole approximation even for a non fixed metric background and without assuming any a-priori constraints on the affine connection (i.e. Levi Civita connection).

In this perpective, the role played by the "transverse Dixon vector field" in order to fix the uniqueness [1][2][3] of the moments is shown to be one of the infinite possible method to establish a one to one relationship between the considered linear functionals and a specific set of tensor fields, so it is just a matter of choice to split the geometrical information carried by the multipoles. On the other hand the symmetry conditions usually imposed on the covariant derivatives has a much more deep nature, linked directly with the coordinate free definition of the multipoles [1][2][3].

To show the mathematical generality of these approach, the dynamical equation constraining the multipoles related to the first order asymptotic expansion of a compactly supported Energy Momentum tensor field for a non negligible torsion contribution is given, showing that in principle the torsion affects just the dynamics beyond the trivial order finding very similar result as shown in [9] [10] [14] [15].

However, even if the generalised Mathisson-Papapetrou-Dixon equation for the pole-dipole approximation for non torsion-less spaces is achieved, we would like to stress that one should consider this work just as a methodological introduction to the problem of interpreting geometrically the multipoles and the moments within relativistic models, rather than a concrete physical proposal, since no consideration about the physical interpretation and the physical aspects are made. Furthermore the relevant problem of back-reactions and divergences is completely ignored at this stage, and it must be deeply analysed separately in case one decides to promote the multipoles to be true physical sources for the interaction fields, rather than just an asymptotic intrinsic approximation of them.

Outline

In the beginning of the first chapter a brief review of the standard $\mathbb{R}$-linear operations upon tensor fields on differential manifolds is given, offering to the reader the possibility to become confident with the slightly different tensorial notation needed to play smoothly with the multipoles. The following section fixes some fundamental lemmas characterising the properties of higher order derivations upon tensor fields strongly needed to investigate multipoles coordinate expressions later.

The second chapter introduces the concept of $\mathbb{R}$-linear functionals acting on the class of smooth test tensor fields upon a differential manifold. The given definition is very general and despite the nice algebraic structure inherited naturally from the operations defined upon the tensor fields, the set of these linear functionals is extremely wide and it contains also a lot of pathological objects. This is not a concern, since the purpose of this chapter is to show the reader that, in principle, it is possible to translate all the operations upon tensor fields directly on the functionals, just relying on the definition of the action of such functionals on the test tensor fields.

In the third chapter, putting together the statements achieved previously, we will provide a general coordinate-free definition for two very specific subset of $\mathbb{R}$-linear functionals acting on the test tensor fields, named in order "Ellis set of multipoles" and "Dixon set of multipoles". These definitions are both strongly founded on the closed embedding concept as well as the De Rham push-forward concept, representing a natural generalisation of it. The general properties of rank, support and order of both this kind of multipole are discussed as well as how they are affected by the standard operations on functionals. It is shown how, the two sets coming from the two very different definitions given at the beginning, in practice each set is contained in the other set, so it is possible to conclude that the Ellis multipoles definition is completely equivalent to the Dixon multipole definition and the set of the multipoles is unique. Considering this, the two main local coordinates representation induced by the two equivalent definition are provided. Closing the chapter some considerations on the algebraic structure of $C^{\infty}(\mathbb{R})$-module of the set of multipoles are discussed.

The fourth chapter and fifth chapter are completely dedicated to the investigation of the Ellis local representation of multipoles and the Dixon local representation of multipoles, respectively. It is shown how both local representations are affected by severe issues when pursuing the attempt to associate uniquely a multipole to its local coordinate expression called "moments". In general there are infinite ways to work around this problem and all of them provides pros and cons. In these chapters several approaches to the uniqueness problem of the moments are investigated and their transformation rules are analysed when a change in the atlas of the manifold is performed.

In chapter four, using the Ellis local representation we are able to interpret the multipoles as the coefficients of an asymptotic expansion approximating a specific one parameter family of compactly supported tensor fields when the one parameter family tends to zero. In the very special case the one parameter family satisfies also the self-similarity condition, the moments induced by the Ellis representation coincide exactly with the standard usual definition of multipole moments of the coordinates expression for a specific tensor field belonging to the given family.

In chapter five using the Dixon local representation, it is proven how the "moments" coincides with the moments definition given by Dixon and widely use in General Relativity to perform the Pole-Dipole approximation. Despite the Ellis case, it is shown that the Dixon moments can be associated to a n-tuple of tensor fields with support on the image of the embedding, opening the door to possible interpretations about the physical information encoded inside the Dixon moments.

In the sixth chapter, we will show a specific application of the multipoles in Relativistic Theories reproducing the free falling particle dynamics directly imposing on the multipoles the same condition that must be satisfied by the Energy Momentum tensor (i.e. divergenceless and symmetry) inspired by the correspondence between multipoles and regular fields previously discussed. Assuming the Levi Civita connection it is shown how an order 0 multipole (a monopole) is able to reproduce the dynamics of a free falling particle and an order 1 multipole (a dipole) satisfies the Mathisson-Papapetrou-Dixon equations. Since no assumption upon the relation between metric and connection is needed, in the second part of the sixth chapter the generalised Mathisson-Papapetrou-Dixon equations are provided in case of non Levi Civita connection. It is interesting to notice that the contribution of the torsion affects the dynamics of the multipoles starting just from the first order, without any influence on the dynamics of the monopoles.

In the last chapter all the fundamental statements are recollected and some final comments concerning the nature of the multipoles and possible applications in Relativistic Theories are discussed. Some aspects of this work have been published in [20]

Contents

1 $\mathbb{R}$-linear Operations on Smooth Tensor Fields **1**

1.1 $C^\infty(M)$-linear operations on tensor fields 1

1.1.1 Introductory comments concerning the notation related to the lists of objects 1

1.1.2 Intrinsic definition for $C^\infty(M)$-linear operations on tensor fields . 2

1.1.3 Pull-back and push-forward of tensor fields 8

1.1.4 Local expressions for $C^\infty(M)$-linear operations on tensor fields . . 11

1.2 Differential operators acting on tensor fields and their properties 15

1.2.1 Definitions 15

1.2.2 Fundamental properties 23

1.2.3 Local expression of derivations on tensor fields 33

1.2.4 More properties 38

1.2.5 Torsion and Curvature 52

2 Functionals on Test Tensors Fields **69**

2.1 Definitions and standard linear operations 69

3 Multipoles on Differential Manifolds **81**

3.1 Coordinate-free definitions and basic properties 82

3.1.1 Two multipoles coordinate free definitions 82

3.1.2 Rank, support and order of a multipole 85

3.1.3 Multipoles as continuous maps 90

3.2 Local representation of the Ellis multipoles 92

3.2.1 Local expression of the action induced by a trivialization of TM and the Ellis definition 93

3.2.2 Ellis local expression of the multipoles 107

3.3 Local representation of the Dixon multipoles 111

3.3.1 Local expression of the action induced by a trivialisation of TM and the Dixon definition 112

3.3.2 Dixon local expression of the multipoles 128

3.4 $C^\infty(\mathbb{R})$ module structure of the multipoles up to order k 134

3.4.1 The multipoles set as a $C^\infty(\mathbb{R})$-module. 134

4 Concerning the Ellis Local Representation **137**

4.1 Problems arising from the general Ellis representation 138

4.1.1 A specific trivial example 138

4.1.2 Considerations 144
4.2 Isomorphism between multipoles and the Ellis representation induced by a the choice of an adapted coordinate system. 151
4.2.1 Coordinate system adapted to the closed embedding and transverse space 152
4.2.2 Ellis representation fixed by an adapted atlas and adapted Ellis moments of a multipole 154
4.2.3 The $C^\infty(\mathbb{R})$ free module structure of $\Upsilon^q_p(c)$ 169
4.3 Transverse basis for low order multipoles fixed by a transverse frame . . . 173
4.3.1 Transverse basis for the multipoles up to order 2 173
4.3.2 Degree of freedom in the choice of the general local charts 194
4.3.3 Degree of freedom in the choice of the adapted local frame defining the transverse directions 196
4.4 Brief discussion on the Ellis top order local representation 207
4.4.1 The Ellis top order local representation 208
4.4.2 Issues concerning the Ellis top order local representation 210
4.5 Squeezed Tensor fields, Weak Asymptotic Expansions and Adapted Ellis Moments 214
4.5.1 A specific realisation 215
4.5.2 Weak asymptotic expansions of "squeezed tensor fields" 217
4.5.3 Transverse self-similar squeezing of a compact support tensor field 234
4.6 Final considerations on the Ellis local representations 237

5 Concerning the Dixon Local Representation 239
5.1 Problems arising from the Dixon representation 239
5.1.1 A specific trivial example 240
5.1.2 Considerations 246
5.2 Isomorphism between multipoles and the Dixon representation induced by a choice of an adapted coordinate system 248
5.2.1 Dixon representation fixed by an adapted atlas and adapted Dixon moments of a multipole 248
5.3 Intrinsic interpretation of the Dixon parameters 260
5.3.1 The Dixon Generators 260
5.3.2 Adapted Dixon basis for the multipoles 266
5.3.3 Dixon parameters of the multipoles as tensor field restricted on a worldline 269
5.3.4 Covariant choice of the Dixon moments induced by a covector field on the worldline 273
5.4 Final considerations on the Dixon local representations 282

6 Specific applications of the multipoles to the Relativistic Dynamics 283
6.1 Dynamics of a free falling test particle modelled as a Dixon monopole . . 285
6.2 Dynamics of a free falling spinning test particles modelled as a Dixon dipole 288
6.3 Dynamics of a divergenceless symmetric Dixon dipole when Torsion occur 299

7 Conclusions 305

A Conventions and Notation **311**
A.1 Indices and Lists and Multi-indexed Lists 311
A.1.1 Introduction 311
A.1.2 Notation about the lists indexed by positive natural numbers . . . 312
A.1.3 Local coordinate expressions of points on the manifold 313
A.1.4 Local coordinate expression for tangent vectors 313
A.1.5 Local coordinate expression for covectors. 314
A.1.6 Local expression for small rank tensors 315
A.1.7 Recalling the standard Einstein notation 317
A.1.8 Generalisation for arbitrary rank tensors and condensed Einstein notation 318
A.1.9 Split notation for multipoles 321
A.1.10 Recalling the "Split Condensed Einstein Convention" 322

B Fiber Bundles, Tangent Tensors and Tensors Fields **325**
B.1 Tangent and cotangent bundles 325
B.1.1 Elements of fiber bundles and fields 325
B.1.2 Tangent bundle of a smooth manifold 333
B.1.3 Cotangent bundle 340
B.2 Tangent tensors at a point 346
B.2.1 Introduction to tangent tensors 346
B.2.2 Coordinate expressions induced by the choice of basis 350
B.2.3 Pull-back and Push-forward of tensors 352
B.2.4 Coordinate expression for standard operations on tensors 358
B.2.5 Coordinate transformations and basis changes. 361
B.3 Tensor fields 366
B.3.1 The tangent tensor bundle 366
B.3.2 The $c(\mathbb{R})$-constrained tangent tensor bundle $T^p_{q\,c(\mathbb{R})}M$ 370

C K-forms, operations and integration **375**
C.1 K-forms as antisymmetric tensor fields 375
C.1.1 Definitions 375
C.1.2 Specific operations on forms 379
C.1.3 Elements concerning orientation and integration of differential forms 385
C.1.4 Densities 395

Chapter 1

$\mathbb{R}$-linear Operations on Smooth Tensor Fields

1.1 $C^\infty(M)$-linear operations on tensor fields

In this chapter we are going to analyse several operations that can be performed upon the space of the smooth tensor fields. Some of them are standard, others are not so popular but fundamental to achieving the definition of the linear functionals we are interested in. All of them are needed to guarantee a correct approach to the multipoles as we will see in the next chapters. Because of this we decide to start from scratch, making explicit all we know about operations on tensor fields taking account of the slightly different notation defined in the appendix, required to face easily the multipoles. A lot of interesting properties for these operations are made explicit, some of them are very well known, others are quite specific and not so commonly used. However even if the approach can seem pedantic or even sterile, all the relevant properties of the multipoles are inherited directly from the behaviour of the many operations defined on the tensor fields on which they can act. We will work in a purely coordinate-free fashion as much as possible, in order to understand the intrinsic nature of the objects we are dealing with, independently with respect to the local coordinate charts chosen on the manifold. For any problem concerning the notation or the fundamental concepts related to the theory of bundles and tangent tensors, the reader is suggested to check the appendices.

1.1.1 Introductory comments concerning the notation related to the lists of objects

Considering we are going to massively manipulate indices and lists related to the coordinate expression of tensors and multipoles, a clear multi-index notation is required. A full explanation of it is given in the appendix. However we will briefly summarise some aspects of it. Let us consider $\mathbb{N}^+$ the set of all non-null natural numbers. Given $a, b \in \mathbb{N} \,|\, a < b$ we denote with $[a, b] = \{x \in \mathbb{N} | a \leq x \leq b\}$ a generic interval. Given a set U and $l \in \mathbb{N}^+$, a list I of elements in U with length l is a function $I : [a, a+l-1] \to U$. Hence a list is isomorphic to an indexed n-tuple of elements in U. We can use the standard round bracket notation $(u_a, ...u_{a+l-1})$ to denote the indexed n-tuples $I = \{(u(\mu), \mu) \,|\, u(\mu) \in U, \forall \mu \in [a, a+l-1]\}$. The letter μ is called index and it points uniquely to an element inside the list, therefore given a list I we can denote uniquely an element of it by just specifying the name of the list and the corresponding index. Given a list I we can define a sub-list J a subset of I such that it is a list. For instance given the list $(t_3, t_4, t_5, t_6, t_7, t_8)$ a good sub-list is given by (t_4, t_5, t_6). A natural generalisation of a list is the multi-indexed list. Given a set U and $l \in \mathbb{N}^+$, a multi-indexed list I of elements in U is a function

$$I : [a_1, a_1 + l_1 - 1] \times [a_2, a_2 + l_2 - 1] \times ... \times [a_n, a_n + l_n - 1] \to U \tag{1.1.1}$$

Hence a generic multi-indexed list can be written as:

$$I = \{(u(\mu_1, \mu_2, ..., \mu_n), (\mu_1, \mu_2, ..., \mu_n)) \mid u(\mu_1, \mu_2, ..., \mu_n) \in U, \forall \mu_i \in [a_i, a_i + l_i - 1]\}$$

There are several different lists and multi-indexed list in our work, all of them are used for different purposes. Often it is mandatory to face lists of lists, lists of indices or lists with unfixed length. There is no way to single out just one specific notation for the lists adapted to all the needs in a satisfactory way. We decided to be pragmatic, prioritising the attempt to make the operations and manipulations on tensors and multipoles as easy as possible. A list of natural numbers starting from 1 and ending at $a \in \mathbb{N}^+$ is denoted by $\overline{a}$. Hence accordingly to this notation $\overline{a} = (1, ..., a)$. A list of natural numbers starting from $a \in \mathbb{N}^+$ and ending in $b \in \mathbb{N}^+, b > a$ is denoted by $\overline{b} \setminus \overline{a}$. The length of the list is just $b - a$. By convention the empty list can be denoted both by $\overline{0}$. This is compatible with the previous notation in fact $\overline{a} \setminus \overline{a} = \overline{0} = \varnothing$. Accordingly to this, when it is convenient, a list $(u_1, ..., u_a)$ of elements in U, can be denoted just with $u_{\overline{a}}$. For list starting not from 1 for instance $(u_{a+1}, u_{a+2}, ..., u_{b-1}, u_b)$ we use the following notation $u_{\overline{b} \setminus \overline{a}}$. By convention a list as $u_{\overline{0}}$ is the empty list as well as $u_{\overline{a} \setminus \overline{a}}$. In our work we are going to use just specific multi-indexed lists. As stated before a multi-indexed list can be interpreted just as a list in which each element is singled out by a list of indices rather than just one index. Therefore a compact notation for the list of indices is required. The lists of indices by convention start from a number greater that 0 making the counting of the of indices more intuitive, hence it could be something like $(\mu_{a+1}, \mu_{a+2} ..., \mu_{b-1}, \mu_b)$. To express it in a compact way also for a list of indices of unfixed length we decided to use this notation $(\mu_{a+1}, ..., \mu_b) = \mu_{\overline{b} \setminus \overline{a}}$ accordingly with the previous notation set up for the list of natural numbers. The most common lists of indices start from 1 and end in a, making the notation very easy: $(\mu_1, ..., \mu_a) = \mu_{\overline{a}}$. By convention the list $\mu_{\overline{0}}$ is the empty list as well as $\mu_{\overline{a} \setminus \overline{a}}$.

1.1.2 Intrinsic definition for $C^\infty(M)$-linear operations on tensor fields

Considering the fact in this work we are going to use mainly local and global fields on manifolds, we decided to denote them using the very same notation indicating the tangent tensor. If talking about tangent tensors at a point, expressions as T^μ_ν mean a multi-indexed list of numbers, here T^μ_ν means a multi-indexed list of fields. Concerning the tangent geometrical object at a point (i.e tangent tensors and tangent vectors), we can interpret them just as a restriction of a field to a point on the manifold, using the standard notation accordingly, therefore no ambiguity raises from this change. In case the reader is not familiar with this, one can consult the section concerning the notations in the appendix. As it is showed in the appendix, rank p, q tensor fields are defined to be the sections of the bundle $T^p_q M$, but they can be naturally interpreted as the module of $\mathcal{F}(M)$-linear maps sending n-tuples of vector and covectors fields into $\mathcal{F}(M)$. Let us forget temporarily the tensor bundle structure existing on a manifold and let us investigate the tensor fields just from the algebraic perspective.

Definition 1: Given $U \subseteq M$, the set of all scalar fields $f : U \to \mathbb{R}$ is denoted by $\mathcal{F}(U)$

Definition 2: Given $U \subseteq M$, a ***local tensor field*** T on U with rank $p, q \in \mathbb{N}$ is a map:

$$T : (\times^p \Gamma_U T^\star M) \times (\times^q \Gamma_U TM) \to \mathcal{F}(U) \tag{1.1.2}$$

such that it is multilinear: $\forall i \in [1, p]$, $\forall j \in [1, q]$, $\forall f_1, f_2, g_1, g_2 \in \mathcal{F}(U)$, $\forall \alpha, \beta \in \Gamma_U T^\star M, \forall w, u \in \Gamma_U TM$

$$T(\omega^{\overline{i-1}}, f_1\alpha + g_1\beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j-1}}, f_2 w + g_2 u, v_{\overline{q}\backslash\overline{j}}) = \tag{1.1.3}$$
$$= f_1 f_2 T(\omega^{\overline{i-1}}, \alpha, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j-1}}, w, v_{\overline{q}\backslash\overline{j}}) + f_1 g_2 T(\omega^{\overline{i-1}}, \alpha, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j-1}}, u, v_{\overline{q}\backslash\overline{j}}) + \tag{1.1.4}$$
$$+ g_1 f_2 T(\omega^{\overline{i-1}}, \beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j-1}}, w, v_{\overline{q}\backslash\overline{j}}) + g_1 g_2 T(\omega^{\overline{i-1}}, \beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j-1}}, u, v_{\overline{q}\backslash\overline{j}}) \tag{1.1.5}$$

If $U = M$ then T is called ***global tensor field***.

Definition 3: The ***set of all local tensor fields*** with rank p, q defined on the open set U is denoted by $\Gamma_U T^p_q M$. The ***set of all global tensor fields*** with rank p, q is denoted by $\Gamma_M T^p_q M$.

All the discussions here concerning the tensor fields are done without specifying the domain U unless some specific constraints on the domains are required. However when a tensor is applied to vector fields and covector fields, we implicitly assume that they share the same domain U (or at least the domain of the tensor field is included in the domain of the arguments) unless the action is not defined. All the $C^\infty(M)$-linear operations defined for the smooth tensor fields can be defined in the same way in case the considered fields are not smooth, but this cannot be done anymore for the differential operations defined in the following sections, therefore from here, unless explicitly specified, we are going to consider just global smooth tensor fields acting on smooth vector and covector fields. In this particular case we have that the smooth tensor fields are:

$$T : (\times^p \Gamma_U T^\star M) \times (\times^q \Gamma_U TM) \to \mathcal{C}^\infty(U) \tag{1.1.6}$$

Inspired by the approach to the tangent tensors at a point, we can try to replicate on $\Gamma T^p_q M$ the same algebraic structures and operations defined for the tangent tensors:

Definition 4: Given two tensor fields $T, S \in \Gamma T^p_{qx} M$ we define a ***sum of tensor fields*** the map $+ : \Gamma T^p_q M \times \Gamma T^p_q M \to \Gamma T^p_q M$ such that:

$$[T+S](\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha^{\overline{p}}, v_{\overline{q}}) + S(\alpha^{\overline{p}}, v_{\overline{q}}) \quad , \quad \forall \alpha^{\overline{p}} \in \times^p \Gamma T^\star M \ , \ \forall v_{\overline{q}} \in \times^q \Gamma TM \tag{1.1.7}$$

Definition 5: Given a tensor field $T \in \Gamma T^p_q M$ and a scalar field $f \in C^\infty(M)$ we define a ***multiplication by a scalar field*** the map $\cdot : C^\infty(M) \times \Gamma T^p_q M \to \Gamma T^p_q M$ such that:

$$[fT](\alpha^{\overline{p}}, v_{\overline{q}}) = f[T(\alpha^{\overline{p}}, v_{\overline{q}})] \quad , \quad \forall \alpha^{\overline{p}} \in \times^p \Gamma T^\star M \ , \ \forall v_{\overline{q}} \in \times^q \Gamma TM \ , \ \forall f \in C^\infty(M) \tag{1.1.8}$$

Property 1: The algebraic structure $(\Gamma T^p_q M, +, \cdot)$ satisfies all the requirements to be a module on the ring $(C^\infty(M), +, \cdot)$. Furthermore it is a finitely generated $C^\infty(M)$ module. The null tensor field is by definition identified with the null map $0 \in T^p_q M$ such that

$$0(\alpha^{\overline{p}}, v_{\overline{q}}) = 0 \quad , \quad \forall \alpha^{\overline{p}} \in \times^p T^\star_x M \ , \ \forall v_{\overline{q}} \in \times^q T_x M \tag{1.1.9}$$

In general there is no way to build a global basis for the whole module. This is deeply related with the topology of the manifold on which the fields are defined (i.e. parallelizable manifolds) and must be investigated with the bundle technology.

The algebraic operations defined above are enough to endow $(T^p_{q\,x} M)$ with a linear structure that characterises it as a module, but these are not the only useful operations we are able to define on tensor fields. The standard multiplication of functions defined on the ring $C^\infty(M)$, induces another very important binary operation called tensor product. The definition of tensor product of sections on vector bundles is very abstract, it can be given in a very general way and it is deeply rooted in the bundles theory, but it is beyond our purposes to analyse in detail how it is possible to establish general canonical isomorphism and correspondences between algebraic structures on bundles. Again we settle here to give a simplistic definition of tensor product that is very effective for achieve our purposes.

Definition 6: Given two tensor fields $T \in \Gamma T^p_q M$ and $S \in \Gamma T^{p'}_{q'} M$ we define a ***tensor product of tensor fields*** the map $\otimes : \Gamma T^p_q M \times \Gamma T^{p'}_{q'} M \to \Gamma T^{p+p'}_{q+q'} M$ such that:

$$[T \otimes S](\alpha^{\overline{p}}, \beta^{\overline{p}'}, v_{\overline{q}}, u_{\overline{q}'}) = T(\alpha^{\overline{p}}, v_{\overline{q}}) S(\beta^{\overline{p}'}, u_{\overline{q}'}) \,, \tag{1.1.10}$$

$$\forall \alpha^{\overline{p}} \in \times^p \Gamma T^\star M \,,\ \forall v_{\overline{q}} \in \times^q \Gamma TM \quad , \quad \forall \beta^{\overline{p}'} \in \times^p \Gamma T^\star M \,,\ \forall u_{\overline{q}'} \in \times^q \Gamma TM \tag{1.1.11}$$

Considering the tensor fields are multi-linear maps, they must act on n-tuples of vector and covector fields. Therefore we have a natural action of the group of permutations on $\Gamma T^p_q M$ induced by the permutations on the n-tuples of vector and covector fields.

Definition 7: Let I and J be two of permutations of p and q elements respectively. Let P_I and P_J be their representations, acting respectively on the n-tuples $\alpha^{\overline{p}} \in \times^p \Gamma T^\star M$ and $v_{\overline{q}} \in \times^q \Gamma T_x M$ as following:

$$I(\alpha^{\overline{p}}) = (\alpha^{P_I(\overline{p})}) \tag{1.1.12}$$

$$J(\alpha^{\overline{p}}) = (v_{P_J(\overline{q})}) \tag{1.1.13}$$

Given a tensor field $T \in \Gamma T^p_q M$ we define a ***braiding map*** the map $\sigma^I_J : \Gamma T^p_q M \to \Gamma T^p_q M$ such that

$$[\sigma^I_J T](\alpha^{\overline{p}}, v_{\overline{q}}) = T(I(\alpha^{\overline{p}}), J(v_{\overline{q}})) = T(\alpha^{P_I(\overline{p})}, v_{P_J(\overline{q})}) \tag{1.1.14}$$

Of course anyone is free to choose its own notation to express the permutation I and J, however we decided to use the standard cycle decomposition because it offers a direct representation of the action upon the list of indices related to the coordinate representation of the tensors. It is very interesting to notice how the action of tensor fields upon vector fields and covector fields induces canonically an action of vector fields and covectors fields on the tensors fields:

Definition 8: Given a tensor field $T \in \Gamma T^p_q M$, with $q \geq 1$ we define a ***contraction with a vector field*** the map $\lrcorner : \Gamma TM \times \Gamma T^p_q M \to \Gamma T^p_{q-1}$ such that

$$[u \lrcorner T](\alpha^{\overline{p}}, v_{\overline{q-1}}) = T(\alpha^{\overline{p}}, u, v_{\overline{q-1}}) \tag{1.1.15}$$

Given a tensor $T \in \Gamma T^p_{q\,x} M$, with $p \geq 1$ we define a ***contraction with a covector field*** the map $\neg : \Gamma T^\star M \times \Gamma T^p_q M \to \Gamma T^{p-1}_q M$ such that:

$$[\beta \neg T](\overline{\alpha^{p-1}}, \overline{v_{q-1}}) = T(\beta, \overline{\alpha^{p-1}}, \overline{v_{q-1}}) \tag{1.1.16}$$

The definition of internal contraction can still be provided but, since in general TM and $T^\star M$ does not admit any global frame, it is much more tricky. Luckily we can account on the existence of a smooth partition of unity.

Definition 9: Let be TM and $T^\star M$ respectively, the tangent and cotangent bundles of a differential manifold M. Let be $\mathcal{A} = (U_j, \phi_j)$ an atlas on M, we know that it induces a local trivialisation of TM denoted by $(U_j \times \mathbb{R}^m)$ via the existence of a bunch of m $C^\infty(M)$-linearly independent smooth local sections $(e_{(j)\mu})$ such that $e_{(j)\mu} : U \to \tau_M^{-1}(U_i)$. Let (ψ_j) be a smooth partition of the unity subordinate to (U_j). Given a tensor $T \in T^p_{q\,x}M$, with $p, q \geq 1$ we define an ***internal contraction*** the map $i : T^p_{q\,x}M \to T^{p-1}_{q-1\,x}M$ such that:

$$[iT](\overline{\alpha^{p-1}}, \overline{v_{q-1}}) = \sum_{U_j \in \mathcal{A}} \psi_j T(e^\mu_{(j)}, \overline{\alpha^{p-1}}, e_{(j)\mu}, \overline{v_{q-1}}) \tag{1.1.17}$$

Property 2: One can prove that, even if the definition is given for a fixed local charts and basis, the internal contraction of a tensor field is a tensor field, furthermore this operation preserves globally the sections and it does not depend on the choice of local basis and local coordinates on the manifold.

Proof. Given $T \in \Gamma T^p_q M$, Let us suppose to have two different trivialisation of TM denoted by $(U_i, t_{(i)})$ and $(U_j, t_{(j)})$ induced by the local frames $(e_{\mu(i)})$ and $(e_{\mu(j)})$ respectively. For each $U_j \in \mathcal{A}$ let us define a set of local sections $e_{(j)\mu} \neg (e^\mu_{(j)} \lrcorner T) : U_j \to \tilde{\tau}_M^{-1}(U_j) \subset T^{p-1}_{q-1} M$ Let $U_{ij} = U_i \cap U_j$ be the overlaps, then $\forall x \in U_{ij}$:

$$e_{(j)\mu} \neg (e^\mu_{(j)} \lrcorner T) = \tag{1.1.18}$$

$$= T(e^\mu_{(j)}, \overline{\alpha^{p-1}}, e_{(j)\mu}, \overline{v_{q-1}})_{|x} = \Lambda^\mu_{\alpha|x} \overline{\Lambda}^\beta_{\mu|x} T(e^\alpha_{(i)}, \overline{\alpha^{p-1}}, e_\beta(i), \overline{v_{q-1}})_{|x} = \tag{1.1.19}$$

$$= \delta^\beta_\alpha T(e^\alpha_{(i)}, \overline{\alpha^{p-1}}, e_\beta(i), \overline{v_{q-1}})_{|x} = T(e^\mu_{(i)}, \overline{\alpha^{p-1}}, e_{(i)\mu}, \overline{v_{q-1}})_{|x} = e_{(i)\mu} \neg (e^\mu_{(i)} \lrcorner T) \tag{1.1.20}$$

Therfore the local sections $e_{(j)\mu} \neg (e^\mu_{(j)} \lrcorner T) : U_j \to \tilde{\tau}_M^{-1}(U_j) \subset T^{p-1}_{q-1} M$ satisfy the compatibility conditions and they can be glued toghether to define a global section. Furthermore we have that $\forall x \in U_i, \forall U_i \in \mathcal{A}$:

$$[iT](\alpha^{\overline{p-1}}, v_{\overline{q-1}})_{|x} = \sum_{U_j \in \mathcal{A}} \psi_{j|x} T(e^{\mu}_{(j)}, \alpha^{\overline{p-1}}, e_{(j)\mu}, v_{\overline{q-1}})_{|x} = \tag{1.1.21}$$

$$= \sum_{j \mid x \in supp(\psi_j)} \psi_{j|x} T(e^{\mu}_{(j)}, \alpha^{\overline{p-1}}, e_{(j)\mu}, v_{\overline{q-1}})_{|x} = \tag{1.1.22}$$

$$= \sum_{j \mid x \in supp(\psi_j)} \psi_{j|x} T(e^{\mu}_{(i)}, \alpha^{\overline{p-1}}, e_{(i)\mu}, v_{\overline{q-1}})_{|x} = \tag{1.1.23}$$

$$= T(e^{\mu}_{(i)}, \alpha^{\overline{p-1}}, e_{(i)\mu}, v_{\overline{q-1}})_{|x} \sum_{j \mid x \in supp(\psi_j)} \psi_{j|x} = T(e^{\mu}_{(i)}, \alpha^{\overline{p-1}}, e_{(i)\mu}, v_{\overline{q-1}})_{|x} = \tag{1.1.24}$$

$$= e_{(i)\mu} \urcorner (e^{\mu}_{(i)} \lrcorner T) \tag{1.1.25}$$

□

So we must conclude that $[iT]$ is a true global section that does not depend on the choice of frame.

Since $\Gamma T^p_q M$ is a $C^\infty(M)$ module, it is natural to define the $C^\infty(M)$-linear maps acting on them.

Definition 10: Given two tensor fields $T \in \Gamma T^p_q M$ and $S \in \Gamma T^{p'}_{q'} M$ we define a $C^\infty(M)$-linear map $\mathcal{L} : \Gamma T^p_q M \to \Gamma T^{p'}_{q'} M$ such that:

$$\mathcal{L}(fT + gS) = f\mathcal{L}(T) + g\mathcal{L}(S) \quad , \quad \forall f, g \in C^\infty(M) \tag{1.1.26}$$

It is very interesting and useful to notice that there is an isomorphism between the linear maps on tensors fields and the tensor fields themselves. In general the existence of this isomorphism strongly depends on the facts that the two tensor spaces admits the same trivialisation. In general there is no way to interpret the linear maps as tensors when two different trivialisation are chosen. This aspect is crucial during the study of the totally antisymmetric tensors, leading to the concept of "tensor density". Let $T^p_q M$ and $T^{p'}_{q'} M$ be two bundles, and $Lin(p, q, p', q') = \{\mathcal{L} : \Gamma T^p_q M \to \Gamma T^{p'}_{q'} M\}$ be the space of the linear maps, there always exists a unique tensor field $L \in \Gamma T^{q+p'}_{p+q'} M$ such that:

$$\mathcal{L}(T) = [i]^{p+q} [\sigma^{\overline{(p+q)}}]^p (T \otimes L) \tag{1.1.27}$$

therefore $Lin(p, q, p', q')$ is isomorphic to $\Gamma T^{q+p'}_{p+q'} M$ as a vector space and we can perform on them all the operation defined on tensors. The proof will be provided in the following section because extra structures are needed.

Property 3: The sum, multiplication by a scalar, tensor product, braiding maps, contractions and internal contractions are all $C^\infty(M)$-linear maps. Furthermore all of them are smooth.

Proof. The sum and the multiplication by scalar are trivially $C^\infty(M)$-linear by definition. The tensor product is $C^\infty(M)$-linear due to its definition and by the distributivity of the multiplication with respect to the sum. The braiding map is $C^\infty(M)$-linear because of the commutativity of the sum and the multiplication, the contraction are linear by definition as well as the internal contraction that is a sum of contractions. The smoothness can be easily checked by fixing local smooth frames and analysing the local expression of the operations provided in the following section. □

1.1.3 Pull-back and push-forward of tensor fields

When one has two manifolds and a smooth map between them there is a canonical natural way to transport back and forward tangent structures between them called pull-back and push-forward. In the appendix the way to transport back and forward tangent structures at a single point of the manifold is defined. Here we will see how it is possible to transport also whole sections of the tangent, cotangent and tangent tensor bundles. There are some different way to interpret the pull-back and push-forward, some of them are very sophisticated involving functor and categories theory. Once again we prefer being pragmatic and we settle here to provide an operational definition of pull-back and push-forward of tensor fields without investigating in detail all the properties in terms of maps between categories. Let us recall the definition of pull-back and push-forward of functions. Let M and N be two manifolds and $\phi : U \subseteq M \to V \subseteq N$ is a local generic map between them. We define the pull-back of functions as the map $\phi^\star : \mathcal{F}(V) \to \mathcal{F}(U)$ such that:

$$\phi^\star(f) = f \circ \phi \quad , \quad \forall f \in \mathcal{F}(V) \tag{1.1.28}$$

Let us stress again that any function can be pulled back along any map. On the contrary, this is not the case for the push-forward. Given a function on M there is no general way to define a function on N. For pushing forward functions, one has to either restrict functions or restrict maps. If $\phi : U \subseteq M \to V \subseteq N$ is a local invertible map then we can define the push-forward of functions as the map $\phi_\star : \mathcal{F}(U) \to \mathcal{F}(V)$ such that:

$$\phi_\star(f) = f \circ \phi^{-1} \quad , \quad \forall f \in \mathcal{F}(U) \tag{1.1.29}$$

It is trivial to check that for an invertible map ϕ the push-forward is just the inverse map of the pull-back, furthermore $\phi^\star = (\phi^{-1})_\star$

Definition 11: Let be M and N two manifolds with the respective tangent bundles $(TM, M, \tau_M, \mathbb{R}^m)$ and $(TN, N, \tau_N, \mathbb{R}^n)$. Let $\phi : U \subseteq M \to V \subseteq N$ such that $V = \phi(U)$ be a local smooth map between them. We define the ***push-forward of local vector fields*** the map $\phi_\star : \Gamma_U TM \to \Gamma_V TN$ such that:

$$[\phi_\star(v)](f) = v(\phi^\star(f)) = v(f \circ \phi) \quad , \quad \forall v \in \Gamma_U TM \, , \, \forall f \in C^\infty(V) \tag{1.1.30}$$

Since ϕ is smooth then $\phi_\star(v)$ is well defined and if v is smooth obviously $\phi_\star(v)$ must be smooth as well.

Let us stress that any vector field can be pushed forward along any smooth map. On the contrary, this is not the case for the pull-back.

Definition 12: Let M and N be two manifolds with the respective tangent bundles $(TM, M, \tau_M, \mathbb{R}^m)$ and $(TN, N, \tau_N, \mathbb{R}^n)$. Let $\phi : U \subseteq M \to V \subseteq N$ such that $V = \phi(U)$ be a diffeomorphism between them. We define the ***pull-back of local vector fields*** the map $\phi^\star : \Gamma_V TN \to \Gamma_U TM$ such that:

$$[\phi^\star(v)](f) = v(\phi_\star(f)) = v(f \circ \phi^{-1}) \quad , \quad \forall v \in \Gamma_V TN \, , \, \forall f \in C^\infty(U) \tag{1.1.31}$$

Since ϕ is a diffeomorphism then $\phi^\star(v)$ is well defined and if v is smooth obviously $\phi^\star(v)$ must be smooth as well.

Again for an invertible map ϕ the push-forward is just the inverse map of the pull-back, furthermore $\phi_\star = (\phi^{-1})^\star$.

Definition 13: Let M and N be two manifolds with the respective cotangent bundles $(T^\star M, M, \tau_M, \mathbb{R}^m)$ and $(T^\star N, N, \tau_N, \mathbb{R}^m)$. Let $\phi : U \subseteq M \to V \subseteq N \mid V = \phi(U)$ be a local smooth map between them. We define the ***pull-back of local covector fields*** the map $\phi^\star : \Gamma_V TN \to \Gamma_U TM$ such that:

$$[\phi^\star(\alpha)](v) = \alpha(\phi_\star(v)) \quad , \quad \forall v \in \Gamma_U TM \, , \, \forall \alpha \in \Gamma_V T^\star N \tag{1.1.32}$$

Since ϕ is smooth then $\phi^\star(\alpha)$ is well defined and if α is smooth obviously $\phi^\star(\alpha)$ must be smooth as well.

Let us stress once again that any covector field can be pulled back along any smooth map but this is not the case for the push-forward.

Definition 14: Let be M and N two manifolds with the respective cotangent bundles $(T^\star M, M, \tau_M, \mathbb{R}^m)$ and $(T^\star N, N, \tau_N, \mathbb{R}^m)$. Let $\phi : U \subseteq M \to V \subseteq N \,|\, V = \phi(U)$ be a local diffeomorphism between them. We define the ***push-forward of local covector fields*** the map $\phi_\star : \Gamma_U TM \to \Gamma_U T^\star M$ such that:

$$[\phi_\star(\alpha)](v) = \alpha(\phi^\star(v)) = \alpha({\phi^{-1}}_\star(v)) \quad , \quad \forall v \in \Gamma_V TN \ , \ \forall \alpha \in \Gamma_U T^\star M \tag{1.1.33}$$

Since ϕ is smooth then $\phi_\star(\alpha)$ is well defined and if α is smooth obviously $\phi_\star(\alpha)$ must be smooth as well.

Again for an invertible map ϕ the push-forward is just the inverse map of the pull-back, and the relation $\phi^\star = (\phi^{-1})_\star$ can be easily checked.

Definition 15: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local smooth map between them. Let $x \in U$ a point, then we define the ***push-forward of covariant tensor fields*** the map $\phi_\star : \Gamma_U T^p M \to \Gamma_V T^p N$ such that:

$$[\phi_\star(T)](\alpha^{\overline{p}}) = T([\phi^\star(\alpha)]^{\overline{p}}) \quad , \quad \forall \alpha^{\overline{p}} \in \times^p \Gamma_V T^\star N \ , \ \forall T \in \Gamma_U T^p M \tag{1.1.34}$$

In the same way we define a ***pull-back of contravariant tensors fields*** the map $\phi^\star : \Gamma_V T_q N \to \Gamma_U T_q M$ such that:

$$[\phi^\star(T)](v_{\overline{q}}) = T([\phi_\star(v)]_{\overline{q}}) \quad , \quad \forall v_{\overline{q}} \in \times^p \Gamma_V TN \ , \ \forall T \in \Gamma_V T_q N \tag{1.1.35}$$

Definition 16: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local diffeomorphism between them. Let $x \in U$ a point, then we define the ***pull-back of covariant tensor fields*** the map $\phi_\star : \Gamma_V T^p N \to \Gamma_U T^p M$ such that:

$$[\phi^\star(T)](\alpha^{\overline{p}}) = T([\phi_\star(\alpha)]^{\overline{p}}) = T([{\phi^{-1}}^\star(\alpha)]^{\overline{p}}) \quad , \quad \forall \alpha^{\overline{p}} \in \times^p \Gamma_U T^\star M \ , \ \forall T \in \Gamma_V T^p N \tag{1.1.36}$$

In the same way we define a ***push-forward of contravariant tensor fields at*** x the map $\phi^\star : \Gamma_U T_q M \to \Gamma_V T_q N$ such that:

$$[\phi_\star(T)](v_{\overline{q}}) = T([\phi^\star(v)]_{\overline{q}}) = T([\phi^{-1}{}_\star(v)]^{\overline{p}}) \quad , \quad \forall v_{\overline{q}} \in \times^p \Gamma_V TN \ , \ \forall T \in \Gamma_U T_q M \tag{1.1.37}$$

Definition 17: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local diffeomorphism between them. Let $x \in U$ a point, then we define the ***push-forward of tensor fields*** the map $\phi_\star : \Gamma_U T^p_q M \to \Gamma_V T^p_q N$ such that $\forall \alpha^{\overline{p}} \in \times^p \Gamma_V T^\star N$, $\forall v^{\overline{q}} \in \times^q \Gamma_V TN$, $\forall T \in \Gamma_U T^p_q M$:

$$[\phi_\star(T)](\alpha^{\overline{p}}, v_{\overline{q}}) = T([\phi^\star(\alpha)]^{\overline{p}}, ([\phi^\star(v)]_{\overline{q}}) = T([\phi^\star(\alpha)]^{\overline{p}}, [\phi^{-1}{}_\star(v)]_{\overline{q}}) \tag{1.1.38}$$

In the same way we define a ***pull-back of tensor fields*** the map $\phi^\star : \Gamma_V T^p_q N \to \Gamma_U T^p_q M$ such that $\forall \alpha^{\overline{p}} \in \times^p \Gamma_U T^\star M$, $\forall v^{\overline{q}} \in \times^q \Gamma_U TM$, $\forall T \in \Gamma_V T^p_q N$:

$$[\phi^\star(T)](\alpha^{\overline{p}}, v_{\overline{q}}) = T([\phi_\star(\alpha)]^{\overline{p}}, [\phi_\star(v)]_{\overline{q}}) = T([\phi^{-1\star}(\alpha)]^{\overline{p}}, [\phi_\star(v)]_{\overline{q}}) \tag{1.1.39}$$

Property 4: One can easily check from the given definition of pull-back and push-forward of vectors, covectors and tensors are all $\mathbb{R}$-linear, hence the linear structures on the vector spaces are preserved. Furthermore since the pull-back and push-forwards are $\mathbb{R}$-linear maps between finite dimensional vector spaces, due to theorems of standard linear algebra, we know that fixing two basis on $T_x M$ and $T_y N$ they can be expressed by matrices. Since $\phi : U \subseteq M \to V \subseteq N$ is a diffeomorphism it is trivial to check from the definition that

$$\phi^\star \circ \phi_\star = \phi_\star \circ \phi^\star = id \tag{1.1.40}$$

therefore the pull-back is the inverse of the push-forward and vice-versa.

1.1.4 Local expressions for $C^\infty(M)$-linear operations on tensor fields

Since at fixed frame there is a one to one smooth relation between a tensor and its local expression, we can ask ourselves how the operations defined in the previous section affect the local expression of a tensors field. This is very useful because it allows us to single out for each operation defined above, the rules to manipulate directly the local expressions and to check local properties i.e. the smoothness. Finding the local expression manipulation rules can be performed easily given a smooth local frame and proceeding as it was done before in the case of tangent tensors. We have to remark that, since $\Gamma_U T^p_q M$ is a $\mathcal{F}(U)$-linear module, the multi-indexed list of coefficients are not just real number but belongs to $\mathcal{F}(U)$. If the tensor field is smooth then its local expression is a multi-indexed list of $C^\infty(U)$ functions. Let us suppose we have an open set U on which is defined a local smooth frame $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}) \in \Gamma_U T^p_q M$ that fixes a local trivialisation of the vector bundle $T^p_q M$.

1. ***Sum***:

$$(T+S)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = [T+S](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) + S(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + S^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.1.41}$$

   Example: $(g+h)_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{\mu\nu}(x)$

2. ***Multiplication by a scalar field***:

$$(fT)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = [fT](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = f \cdot [T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}})] = fT^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.1.42}$$

   Example: $(fg)_{\mu\nu}(x) = f(x)g_{\mu\nu}(x)$

3. ***Tensor product***:

$$(T \otimes S)^{\mu_{\overline{p}}\alpha_{\overline{p}'}}_{\nu_{\overline{q}}\beta_{\overline{q}'}} = [T \otimes S](e^{\mu_{\overline{p}}}, e^{\alpha_{\overline{p}'}}, e_{\nu_{\overline{q}}}, e_{\beta_{\overline{q}'}}) = T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) S(e^{\alpha_{\overline{p}'}}, e_{\beta_{\overline{q}'}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} S^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}} \tag{1.1.43}$$

   Example: $(g \otimes h)_{\mu\nu\alpha\beta}(x) = g_{\mu\nu}(x) h_{\alpha\beta}(x)$

4. ***Braiding***:

$$(\sigma^I_J T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}(x) = [\sigma^I_J T](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}})(x) = T(e^{\mu_{P_I(\overline{p})}}, e_{\nu_{P_J(\overline{q})}})(x) = T^{\mu_{P_I(\overline{p})}}_{\nu_{P_J(\overline{q})}}(x) \tag{1.1.44}$$

   Example: $(\sigma_{(12)} g)_{\mu\nu}(x) = g_{\nu\mu}(x)$

5. ***Contractions***:

$$(v \lrcorner T)^{\mu_{\overline{p}}}_{\nu_{\overline{q-1}}} = [v \lrcorner T](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q-1}}}) = T(e^{\mu_{\overline{p}}}, v, e_{\nu_{\overline{q-1}}}) = v^\alpha T(e^{\mu_{\overline{p}}}, e_\alpha, e_{\nu_{\overline{q-1}}}) = v^\alpha T^{\mu_{\overline{p}}}_{\alpha\nu_{\overline{q-1}}} \tag{1.1.45}$$

$$(\alpha \urcorner T)^{\mu_{\overline{p-1}}}_{\nu_{\overline{q}}} = [\alpha \urcorner T](e^{\mu_{\overline{p-1}}}, e_{\nu_{\overline{q}}}) = T(\alpha, e^{\mu_{\overline{p-1}}}, e_{\nu_{\overline{q}}}) = \alpha^\alpha T(e^\alpha, e^{\mu_{\overline{p-1}}}, e_\alpha, e_{\nu_{\overline{q}}}) = \alpha^\alpha T^{\alpha\mu_{\overline{p-1}}}_{\nu_{\overline{q}}} \tag{1.1.46}$$

Example: $(v \lrcorner g)_\nu(x) = v^\mu g_{\mu\nu}(x)$

6. ***Internal contraction***:

$$(iT)^{\mu_{\overline{p-1}}}_{\nu_{\overline{q-1}}} = iT(e^{\mu_{\overline{p-1}}}, e_{\nu_{\overline{q-1}}}) = iT(e^\alpha, e^{\mu_{\overline{p-1}}}, e_\alpha, e_{\nu_{\overline{q-1}}}) = T^{\alpha\mu_{\overline{p-1}}}_{\alpha\nu_{\overline{q-1}}} \tag{1.1.47}$$

Example: $(i\, Tor)_\mu(x) = Tor^\lambda_{\lambda\mu}(x)$

Lemma 1: Let $T^p_q M$ and $T^{p'}_{q'} M$ be two tangent tensor bundles, and $Lin(p,q,p',q') = \{\mathcal{L} : T^p_{qx} M \to T^{p'}_{q'x} M\}$ be the space of the linear map between them. There always exists an unique tensor $L \in \Gamma T^{q+p'}_{p+q'} M$ such that:

$$\mathcal{L}(T) = [i]^{p+q}[\sigma^{(\overline{p+q})}]^p (T \otimes L) \tag{1.1.48}$$

therefore $Lin(p,q,p',q')$ is isomorphic to $\Gamma T^{q+p'}_{p+q'} M$ as a module and we can perform on them all the operation defined on tensor fields.

Proof. Let us fix a trivialisation $(U_i, t_{(i)})$ on TM fixing the local smooth frames $e_{\mu(i)}$, and let us induce from it the local trivialisation on $T^p_q M$ and on $T^{p'}_{q'} M$. Then for each U_i we can write :

$$\mathcal{L}(T) = \mathcal{L}(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} \mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)}) \tag{1.1.49}$$

Considering the definition of $\mathcal{L}$ and since at fixed indices μ and ν we have a tensor, we can write $\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)}) = [\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)}$ where $[\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)}))]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}$ is a multi-indexed list of functions in $\mathcal{F}(U_i)$. Hence defining the functions:

$$L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}(i)}(x) = [\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)}]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}(x) \quad , \quad \forall x \in U_i \tag{1.1.50}$$

we can recast the expression as follow:

$$\mathcal{L}(T) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)}\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)}[\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} = \tag{1.1.51}$$

$$= T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}(i)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}(i)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} \tag{1.1.52}$$

On the other hand we have that on U_i the following holds:

$$[i]^{p+q}[\sigma^{(\overline{p+q+q'})}\sigma_{(\overline{p+q+p'})}]^{(p+q)}(L \otimes T) = \tag{1.1.53}$$

$$= \{[i]^{p+q}[\sigma^{(\overline{p+q+q'})}\sigma_{(\overline{p+q+p'})}]^{(p+q)}(L \otimes T)\}^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}(i)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} = \tag{1.1.54}$$

$$= \{[i]^{p+q}[\sigma^{(\overline{p+q})}]^{p}(T \otimes L)\}(e^{\alpha_{\overline{p}'}}_{(i)}, e_{\beta_{\overline{q}'}(i)}) \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} = L^{\alpha_{\overline{p}'}\nu_{\overline{q}}}_{\beta_{\overline{q}'}\mu_{\overline{p}}(i)} T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} \tag{1.1.55}$$

So, via a linear combination, we can define a local section of $T^{q+p'}_{p+q'}M$ for each U_i such that:

$$L^{\alpha_{\overline{p}'}\nu_{\overline{q}}}_{\beta_{\overline{q}'}\mu_{\overline{p}}(i)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e_{\nu_{\overline{q}}(i)} \otimes e^{\mu_{\overline{p}}}_{(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} := [\mathcal{L}(e^{\nu_{\overline{q}}}_{(i)} \otimes e_{\mu_{\overline{p}}(i)})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}} e_{\alpha_{\overline{p}'}(i)} \otimes e_{\nu_{\overline{q}}(i)} \otimes e^{\mu_{\overline{p}}}_{(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} . \tag{1.1.56}$$

Due to the linearity of the map it is very easy to check that given another trivialisation $(U_j, t_{(j)})$ of TM, on the overlap $U_{ij} = U_i \cap U_j$ the following holds:

$$[i]^{p+q}[\sigma^{(\overline{p+q})}]^{p}(T \otimes L) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}(i)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} = \tag{1.1.57}$$

$$= T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(j)} L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}(j)} \; e_{\alpha_{\overline{p}'}(j)} \otimes e^{\beta_{\overline{q}'}}_{(j)} = \Lambda^{\rho_{\overline{p}}}_{\mu_{\overline{p}}(ij)} \Lambda^{\delta_{\overline{q}'}(ij)}_{\beta_{\overline{q}'}(ij)} \overline{\Lambda}^{\nu_{\overline{q}}}_{\lambda_{\overline{q}}(ij)} \overline{\Lambda}^{\alpha_{\overline{p}'}}_{\gamma_{\overline{p}'}(ij)} T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} L^{\lambda_{\overline{q}}\gamma_{\overline{p}'}}_{\rho_{\overline{p}}\delta_{\overline{q}'}(j)} \; e_{\alpha_{\overline{p}'}(i)} \otimes e^{\beta_{\overline{q}'}}_{(i)} \tag{1.1.58}$$

Therefore we can conclude that the transition functions are:

$$L^{\alpha_{\overline{p}'}\nu_{\overline{q}}}_{\beta_{\overline{q}'}\mu_{\overline{p}}(i)} = \Lambda^{\rho_{\overline{p}}}_{\mu_{\overline{p}}(ij)}\Lambda^{\delta_{\overline{q}'}}_{\beta_{\overline{q}'}(ij)}\overline{\Lambda}^{\nu_{\overline{q}}}_{\lambda_{\overline{q}}(ij)}\overline{\Lambda}^{\alpha_{\overline{p}'}}_{\gamma_{\overline{p}'}(ij)}L^{\lambda_{\overline{q}}\gamma_{\overline{p}'}}_{\rho_{\overline{p}}\delta_{\overline{q}'}(j)} \tag{1.1.59}$$

and we can state that this can be interpreted as the local expression of a global section $L \in \Gamma T^{q+p'}_{p+q'}M$. To prove the relation is an isomorphism of modules one should prove that the sum and multiplication by scalar are preserved. This can be trivially verified from the definition of L, $\mathcal{L}$ and the trivialisation induced by the choice of the smooth frame. Let us remark that the existence of this isomorphism strongly depends on the facts that the two tensor spaces admits the same trivialisation. In general there is no way to interpret the linear maps as tensors when another trivialisation is chosen. We will see that this aspect can is crucial when we will define the basis of the totally anti-symmetric tensor. As we previously said, concerning the pull-back and the push-forward it is extremely easy to show that the action of the pull-back and the push-forward upon the local coordinate expression can be expressed by a matrix called the "Jacobian" matrix related to the transformation. □

1.2 Differential operators acting on tensor fields and their properties

As we will see later the multipoles are founded on the concept of derivations upon the test tensor fields. Considering that, a relevant section of the introductory chapter must be dedicated to their investigation. Once again since we want to achieve an analysis of the multipoles, we are not interested here in a complete review of the differential operators on tensor fields. We are going to focus ourselves mainly on their algebraic properties and on their coordinate expressions.

1.2.1 Definitions

Given a manifold M endowed with an atlas $(U_i, \varphi_{(i)})$, let us consider the tangent tensor bundle T^p_qM and a section $T \in \Gamma T^p_qM$. We know that at fixed trivialisation $(U_i, \tilde{t}_{(i)})$ we can induce local representatives of the tensor field denoted by $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} : U_i \to \mathbb{R}^{m^{p+q}}$ such that:

$$T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)}(x) = T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}(i)})_{|x} \tag{1.2.1}$$

If T is a smooth section, by definition the local expression $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)}(x)$ must be a multi-index list of smooth functions. Therefore using the coordinate $\varphi_{(i)}$ it is possible to define a new list of functions $\hat{T}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} : \mathbb{R}^m \to \mathbb{R}^{m^{p+q}}$ usually called the ***the coordinates of the local***

expression, such that:

$$\hat{T}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)}(x) = [T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)} \circ \varphi^{-1}_{(i)}](x^{\mu}_{(i)}) \tag{1.2.2}$$

and $\hat{T}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(i)}$ must be a multi-indexed list of smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^{m^{p+q}}$ so they can be differentiated in a standard way. Considering this it is natural to ask ourselves if it is possible to define intrinsic global differential operators on tensor fields such that coordinate expressions of the local expression are closely related to the standard differential operators defined on maps $\mathbb{R}^m$ to $\mathbb{R}^{m^{p+q}}$. If this was possible one can say that the information about the differentiable property of the sections in $\Gamma T^p_q M$ does not depend on trivialisation or coordinates but it is purely geometrical information encoded eventually, at fixed trivialisation, inside the derivatives of the local expression of tensor fields. Several different approach can be pursued to reach this goal. The most common consist in fixing the trivialisation in defining some differential operations that map the coordinates of local expressions into coordinates of local expressions, then checking if the compatibility condition is satisfied. Another very elegant approach focused on the very geometrical perspective, defines specific flows of diffeomorphisms on $T^p_q M$ (i.e parallel transport, pull-back and push-forward) such that they act naturally on the sections as derivations. In this work we are not going to follow one of these two paths. Considering our purposes we prefer to focus directly on the operative aspects of the the differential operators, defining them by imposing intrinsically the constraints concerning how they can act on the smooth sections in $\Gamma T^p_q M$. Then using the definitions and the properties, we will provide the local expression of these operators. Let us just remark that at this stage all the considered fields must be smooth.

Definition 18: Given a manifold M, let TM and $T^p_q M$ be the tangent and tangent tensor bundles respectively. We define the ***Lie derivative of tensor field*** the map

$$L : \Gamma TM \times \Gamma T^p_q M :\rightarrow \Gamma T^p_q M \tag{1.2.3}$$

such that:

1. it satisfies the $\mathbb{R}$-linearity in the first and second arguments, $\forall \lambda, \mu \in \mathbb{R}$, $\forall T, S \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$:

$$L_v(\lambda T + \mu S) = \lambda L_v(T) + \mu L_v(S) \tag{1.2.4}$$

$$L_{\lambda v+\mu u}(T) = \lambda L_v(T) + \mu L_u(T) \tag{1.2.5}$$

2. it satisfies the Leibniz rule with respect to the tensor product, $\forall T, S \in \Gamma T^p_q M, \forall v \in \Gamma TM$:

$$L_v(T \otimes S) = L_v(T) \otimes S + T \otimes L_v(S) \tag{1.2.6}$$

3. it satisfies the Leibniz rule with respect to the multiplication by a scalar, $\forall T \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$, $\forall f \in C^\infty(M)$

$$L_v(fT) = L_v(f)T + fL_v(T) \tag{1.2.7}$$

4. it satisfies the Leibniz rule with respect to both contractions, $\forall T \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$, $\forall \alpha \in \Gamma T^\star M$

$$L_v(u\lrcorner\, T) = L_v(u)\lrcorner\, T + u\lrcorner\, L_v(T) \tag{1.2.8}$$

$$L_v(\alpha\urcorner\, T) = L_v(\alpha)\urcorner\, T + \alpha\urcorner\, L_v(T) \tag{1.2.9}$$

5. it satisfies the Jacobi Identity $\forall T \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$

$$[L_u, L_v](T) = L_{[u,v]}(T) \tag{1.2.10}$$

6. it commutes with the internal contraction $\forall T \in \Gamma T^p_q M, \forall u \in \Gamma TM$

$$iL_u(T) = L_u(iT) \tag{1.2.11}$$

7. it commutes with the braiding maps $\forall T \in \Gamma T^p_q M, \forall u \in \Gamma TM$

$$\sigma^I_J L_u(T) = L_u(\sigma^I_J T) \tag{1.2.12}$$

8. it satisfies the rule for scalar fields $\forall T \in \Gamma T^p_q M, \forall u \in \Gamma TM$, $\forall f \in C^\infty(M)$:

$$L_v(f) = v(f) = d[f](v) \tag{1.2.13}$$

9. it satisfies the rule for vector fields $L_v(u) = [v, u]$, $\forall u \in \Gamma TM$

Property 5: Let us notice that the Lie Derivative satisfies all the properties needed to be a well defined derivation on the C^∞-module of the smooth tensor fields.

The Lie derivative is a very nice and useful operator, it has a very important geometrical meaning deeply rooted in the concept of the flow of diffeomorphisms of the manifold M in itself. Although this perspective is very interesting and powerful, especially to model and express how the flows of transformations on the basis act naturally on the sections of $\Gamma T^p_q M$ via the pull-back and push-forward functor of bundles, this definition is all we need for the purpose of this work. The Lie derivatives play an essential role in the definition of the symmetries of geometrical and physical objects, but unfortunately an exhaustive show of these very important concepts cannot be provided properly in this work, because it would be out of the main topic. We suggest the casual reader to have a look to the book [21], to get familiar with the concepts of smooth bundle morphisms, pull-back and push-forward of structures and symmetries. The Lie Derivative is not the only derivation which can be defined. In contrast with the Lie derivative which does not require any structure other than the differentiable structure of M, introducing an extra structure called "affine connection", it is possible to define another derivation on the tensor fields.

Definition 19: Given a manifold M let TM and $T^p_q M$ be the tangent and the tangent tensor bundles respectively. We define the ***Covariant derivative of tensor field*** the map

$$\nabla : \Gamma TM \times \Gamma T^p_q M \to \Gamma T^p_q M \tag{1.2.14}$$

such that:

1. it satisfies the $\mathbb{R}$-linearity in the second argument, $\forall \lambda, \mu \in \mathbb{R}$, $\forall T, S \in \Gamma T^p_q M, \forall v \in$

ΓTM :

$$\nabla_v(\lambda T + \mu S) = \lambda \nabla_v(T) + \mu \nabla_v(S) \tag{1.2.15}$$

2. it satisfies the $C^\infty(M)$-linearity in the first argument, $\forall f, g \in C^\infty(M)$, $\forall T \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$:

$$\nabla_{fv+gu}(T) = f\nabla_v(T) + g\nabla_u(S) \tag{1.2.16}$$

3. it satisfies the Leibniz rule with respect to the tensor product, $\forall T, S \in \Gamma T^p_q M, \forall v \in \Gamma TM$:

$$\nabla_v(T \otimes S) = \nabla_v(T) \otimes S + T \otimes \nabla_v(S) \tag{1.2.17}$$

4. it satisfies the Leibniz rule with respect to multiplication by a scalar, $\forall T \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$, $\forall f \in C^\infty(M)$

$$\nabla_v(fT) = \nabla_v(f)T + f\nabla_v(T) \tag{1.2.18}$$

5. it satisfies the Leibniz rule with respect to both contractions, $\forall T \in \Gamma T^p_q M, \forall v, u \in \Gamma TM$, $\forall \alpha \in \Gamma T^\star M$

$$\nabla_v(u \lrcorner T) = \nabla_v(u) \lrcorner T + u \lrcorner \nabla_v(T) \tag{1.2.19}$$

$$\nabla_v(\alpha \urcorner T) = \nabla_v(\alpha) \urcorner T + \alpha \urcorner \nabla_v(T) \tag{1.2.20}$$

6. it commutes with the internal contraction $\forall T \in \Gamma T^p_q M, \forall u \in \Gamma TM$

$$i\nabla_u(T) = \nabla_u(iT) \tag{1.2.21}$$

7. it commutes with the braiding maps $\forall T \in \Gamma T^p_q M, \forall u \in \Gamma TM$

$$\sigma^I_J \nabla_u(T) = \nabla_u(\sigma^I_J T) \tag{1.2.22}$$

8. it satisfies the rule for scalar fields $\forall T \in \Gamma T^p_q M, \forall u \in \Gamma TM$, $\forall f \in C^\infty(M)$:

$$\nabla_v(f) = v(f) = d[f](v) \tag{1.2.23}$$

In contrast with the Lie Derivative, because of the $C^\infty(M)$-linearity in the first term, we can interpret ∇ in different way with respect to the previous definition. The new interpretation will be very useful to define the multipoles. Given a tensor field T, let us consider $\nabla(T) : \Gamma TM \to \Gamma T^p_q M$ defined as:

$$[\nabla(T)](\alpha^{\overline{p}}, u, v_{\overline{q}}) = [\nabla_v(T)](\alpha^{\overline{p}}, v_{\overline{q}}) \tag{1.2.24}$$

Since $\nabla_v(T)$ is $C^\infty(M)$-linear in the first argument, we must conclude that $\nabla(T)$ is a C^∞-multilinear map in all its arguments therefore $\nabla(T) \in \Gamma T^p_{q+1} M$. Furthermore from the properties of the covariant derivative one can prove that it must be a smooth tensor field since T is smooth. Considering this we induce another definition of ∇:

Definition 20: Given a manifold M let TM and $T^p_q M$ be the tangent and the tangent tensor bundles respectively. We define the ***covariant differential*** as the map

$$\nabla : \Gamma T^p_q M \to \Gamma T^p_{q+1} M \tag{1.2.25}$$

such that:

$$u \lrcorner \nabla(T) = \nabla_u(T) \quad , \qquad \forall u \in \Gamma TM \tag{1.2.26}$$

where $\nabla_u(T)$ satisfies all the properties defining the covariant derivative.

Definition 21: Given a manifold M let $T^p_q M$ (with $p \geq 1$) be the tangent tensor bundle.

We define the ***divergence*** as the map

$$div : \Gamma T^p_q M \to \Gamma T^{p-1}_q M \tag{1.2.27}$$

such that:

$$div(T) = i[\nabla(T)] \tag{1.2.28}$$

Definition 22: Given a manifold M let TM and $T^p_q M$ be the tangent and the tangent tensor bundles respectively. We define recursively the ***k-th covariant differential*** as the map

$$\nabla^k : \Gamma T^p_q M \to \Gamma T^p_{q+k} M \tag{1.2.29}$$

such that:

$$\nabla^0(T) = T \quad , \quad \forall T \in T^p_q M \tag{1.2.30}$$

$$\tag{1.2.31}$$

$$\nabla^1(T) := \nabla(T) \quad , \quad \forall T \in T^p_q M \tag{1.2.32}$$

and

$$\nabla^k(T) = \nabla(\nabla^{k-1}(T)) \quad , \quad \forall T \in T^p_q M \quad , \quad \forall k \in \mathbb{N}^+ \tag{1.2.33}$$

Definition 23: Given a manifold M let TM and $T^p_q M$ be the tangent and the tangent tensor bundles respectively. We define recursively the ***k-th covariant derivative*** the map

$$\nabla^k : \times^k \Gamma TM \times \Gamma T^p_q M \to \Gamma T^p_q M \tag{1.2.34}$$

such that:

$$\nabla^k(u^{\overline{k}}, T) = \nabla^k_{u^{\overline{k}}}(T) = u_k \lrcorner \dots \lrcorner u_1 \lrcorner \nabla^k(T) \quad , \quad \forall T \in T^p_q M, \forall k \in \mathbb{N}^+ \tag{1.2.35}$$

Definition 24: Given a manifold M let TM and $T^p_q M$ (with $p \geq k$) be the tangent and tangent tensor bundles respectively. We define recursively the ***k-th divergence*** the map

$$div : \Gamma T^p_q M \to \Gamma T^{p-k}_q M \tag{1.2.36}$$

such that:

$$div^1(T) = div(T) \tag{1.2.37}$$

and

$$div^k(T) = div(div^{k-1}(T)) \tag{1.2.38}$$

By convention $div^0 = id$

Definition 25: From the higher order covariant derivatives we can extract two useful differential operators. Let be $\{K_i\}$ with $i \in [1, k!]$ the set of all the possible permutations upon k elements:

$$\nabla^k_{()} : \Gamma T^p_q M \to \Gamma T^p_{(k)+q} M \quad \text{such that} \quad \nabla^k_{()}(T) = \frac{1}{k!} \sum_{i=1}^{k!} \sigma_{K_i}\{\nabla^k(T)\} \tag{1.2.39}$$

$$\nabla^k_{[]} : \Gamma T^p_q M \to \Gamma T^p_{[k]+q} M \quad \text{such that} \quad \nabla^k_{[]}(T) = \frac{1}{k!} \sum_{i=1}^{k!} (-1)^{\sharp(K_i)} \sigma_{K_i}\{\nabla^k(T)\} \tag{1.2.40}$$

where $\sharp(K_i)$ is the sign of the permutation K_i

These two operators will be extremely useful later, when casting one of the two definition of the multipoles.

1.2.2 Fundamental properties

In this subsection we are going to list and prove a lot of useful properties concerning the derivations defined above. These will be widely used in the analysis of the multipoles.

Property 6: The covariant differential ∇ commutes with respect to the braiding map σ^I but it does not commute with all the other braiding maps as σ_J, in particular it commutes with just braiding map of permutation such that $J(1) = 1$. In other words it commutes just if the braiding map acts without touching the first argument.

Proof. It is enough to notice that the covariant derivative commutes with both braiding maps to write

$$[\sigma^I \nabla T](\alpha^{\overline{p}}, u, v_{\overline{q}}) = [\nabla T](\alpha^{P_I(\overline{p})}, u, v_{\overline{q}}) = \nabla_u T(\alpha^{P_I(\overline{p})}, v_{\overline{q}}) = \tag{1.2.41}$$
$$=\sigma^I[\nabla_u T](\alpha^{\overline{p}}, v_{\overline{q}}) = [\nabla_u(\sigma^I T)](\alpha^{\overline{p}}, v_{\overline{q}}) = \nabla(\sigma^I T)(\alpha^{\overline{p}}, u, v_{\overline{q}}) \tag{1.2.42}$$

$$[\sigma_J \nabla T](\alpha^{\overline{p}}, v_{\overline{q}+1}) = [\sigma_J \nabla T](\alpha^{\overline{p}}, v_1, v_{\overline{q}+1\setminus\overline{1}}) = [\nabla T](\alpha^{\overline{p}}, v_{P_J(1)}, v_{P_J(\overline{q}+1\setminus\overline{1})}) = \tag{1.2.43}$$
$$= \nabla_{v_{P_J(1)}} T(\alpha^{P_I(\overline{p})}, v_{P_J(\overline{q}+1\setminus\overline{1})})) \tag{1.2.44}$$

If $P_J(1) = 1$ then $v_{P_J(1)} = v_1$ then we have that:

$$[\sigma_J \nabla T](\alpha^{\overline{p}}, v_{\overline{q+1}}) = \nabla_{v_{P_J(1)}} T(\alpha^{P_I(\overline{p})}, v_{P_J(\overline{q+1}\setminus\overline{1})})) = \nabla_{v_1} T(\alpha^{P_I(\overline{p})}, v_{P_J(\overline{q+1}\setminus\overline{1})})) = \tag{1.2.45}$$
$$= \sigma_J[\nabla_{v_1}(T)](\alpha^{\overline{p}}, v_{\overline{q+1}\setminus\overline{1}}) = [\nabla_{v_1}(\sigma_J T)](\alpha^{\overline{p}}, v_{\overline{q+1}\setminus\overline{1}}) = [\nabla(\sigma_J T)](\alpha^{\overline{p}}, v_{\overline{q+1}}) \tag{1.2.46}$$

□

Property 7: The covariant differential satisfies:

1. the "generalised Leibniz" rule with respect to the product $\otimes$:

$$\nabla(T \otimes S) = \nabla(T) \otimes S + \sigma_{\overline{q+1}}(T \otimes \nabla(S)) \tag{1.2.47}$$

2. a "generalised Leibniz" rule with respect to the contraction with covector fields $\neg$:

$$\nabla(\alpha \neg T) = i(\sigma_{(12)}[\nabla(\alpha) \otimes T]) + \alpha \neg \nabla(T) \tag{1.2.48}$$

3. a "generalised Leibniz" rule with respect to the contraction with vector fields $\lrcorner$:

$$\nabla(v \lrcorner T) = i(\sigma_{(12)}[\nabla(v) \otimes T]) + v \lrcorner (\sigma_{(12)} \nabla(T)) \tag{1.2.49}$$

Proof. Given T and S arbitrary tensor fields, let v and α be a vector and a covector field respectively, therefore

1. using the definitions and the properties of the covariant derivatives:

$$u \lrcorner \nabla(T \otimes S) = \nabla_u(T \otimes S) = \nabla_u(T) \otimes S + T \otimes \nabla_u(S) = \tag{1.2.50}$$
$$= [u \lrcorner \nabla(T)] \otimes S + T \otimes [u \lrcorner \nabla(S)] = \tag{1.2.51}$$
$$= u \lrcorner [\nabla(T) \otimes S] + u \lrcorner [\sigma_{\overline{q+1}}(T \otimes \nabla(S))] = \tag{1.2.52}$$
$$= u \lrcorner [\nabla(T) \otimes S + \sigma_{\overline{q+1}}(T \otimes \nabla(S))] \quad , \quad \forall u \in \Gamma TM \tag{1.2.53}$$

2. in the same way we can prove:

$$u \lrcorner \nabla(\alpha \neg T) = \nabla_u(\alpha \neg T) = \nabla_u(\alpha) \neg (T) + \alpha \neg \nabla_u(T) = \tag{1.2.54}$$
$$= u \lrcorner \{i(\sigma_{(12)}[\nabla(\alpha) \otimes T])\} + u \lrcorner \{\alpha \neg \nabla(T)\} = \tag{1.2.55}$$
$$= u \lrcorner [i(\sigma_{(12)}[\nabla(\alpha) \otimes T]) + \alpha \neg \nabla(T)] \quad , \quad \forall u \in \Gamma TM \tag{1.2.56}$$

3. as well as:

$$u \lrcorner \nabla(v \lrcorner T) = \nabla_u(v \lrcorner T) = \nabla_u(v) \lrcorner (T) + v \lrcorner \nabla_u(T) = \tag{1.2.57}$$
$$= u \lrcorner \{i(\sigma_{(12)}[\nabla(v) \otimes T])\} + u \lrcorner \{v \lrcorner \sigma_{(12)} \nabla(T)\} = \tag{1.2.58}$$
$$= u \lrcorner [i(\sigma_{(12)}[\nabla(v) \otimes T]) + v \lrcorner \sigma_{(12)} \nabla(T)] \quad , \quad \forall u \in \Gamma TM \tag{1.2.59}$$

□

Property 8: The covariant differential ∇ does not commute with the internal contraction. In particular we have that given $T \in \Gamma T^p_q M$.

$$\nabla(iT) = i[\sigma_{(12)}\nabla(T)] \tag{1.2.60}$$

On the other hand using the definition of the covariant derivative we can state that:

$$i(u\lrcorner\nabla(T)) = (u\lrcorner\nabla(iT)) \tag{1.2.61}$$

Proof. Considering the properties of the covariant derivatives:

$$i(u\lrcorner\nabla(T)) = i(\nabla_u(T)) = \nabla_u(iT) = (u\lrcorner\nabla(iT)) \tag{1.2.62}$$

Combining the properties above with the definition of i we can prove trivially the thesis. □

Property 9: The covariant divergence satisfies the following rule

$$div(v \otimes T) = div(v) \otimes T + \nabla_v(T) \tag{1.2.63}$$

Proof.

$$div(v \otimes T) = i(\nabla(v \otimes T)) = i(\nabla(v) \otimes T + v \otimes \nabla(T)) = \tag{1.2.64}$$
$$= i(\nabla(v)) \otimes T + i(v \otimes \nabla(T)) = div(v) \otimes T + \nabla_v(T) \tag{1.2.65}$$

□

In contrast to the higher order differentials and divergences, the higher order covariant derivatives are not merely the composition of lower order covariant derivatives.

$$\nabla^k(T)(\alpha^{\overline{p}}, u_{\overline{k}}, v_{\overline{q}}) \neq \nabla_{u_1}(...\nabla_{u_k}(T))(\alpha^{\overline{p}}, v_{\overline{q}}) \tag{1.2.66}$$

In fact the second term is not a $C^\infty(M)$-multi-linear in the u_i.

Property 10: From the definition one can express recursively the action of $\nabla^k_{u_{\overline{k}}}$ on T as composition of lower order covariant derivatives:

$$[\nabla^k_{u_{\overline{k}}}(T)] = \nabla_{u_1}\{\nabla^{k-1}_{u_{\overline{k}\setminus\overline{1}}}(T)\} - \sum_{i=2}^{i=k}\{\nabla^{k-1}_{u_{\overline{i-1}\setminus\overline{1}}\ \nabla_{u_1}(u_i)\ u_{\overline{k}\setminus\overline{i}}}(T)\} \tag{1.2.67}$$

Proof. Let us start computing explicitly $\nabla_{u_1}\{\nabla^{k-1}_{u_{\overline{k}\setminus\overline{1}}}(T)\}(\alpha^{\overline{p}}, v_{\overline{q}})$:

$$\nabla_{u_1}\{\nabla^{k-1}_{u_{\overline{k}\setminus\overline{1}}}(T)\} = \nabla_{u_1}\{u_k\lrcorner \ldots \lrcorner u_2\lrcorner\nabla^{k-1}(T)\} = \tag{1.2.68}$$

$$= u_k\lrcorner \ldots \lrcorner u_2\lrcorner\nabla_{u_1}(\nabla^{k-1}(T)) + \nabla_{u_1}(u_k)\lrcorner \ldots \lrcorner u_2(\nabla^{k-1}(T)) + \cdots \tag{1.2.69}$$

$$= u_k\lrcorner \ldots \lrcorner u_2\lrcorner\nabla_{u_1}(\nabla^{k-1}(T)) + \sum_{i=2}^{i=k}\{\nabla^{k-1}_{u_{\overline{i-1}\setminus\overline{1}}\ \nabla_{u_1}(u_i)\ u_{\overline{k}\setminus\overline{i}}}(T)\} \tag{1.2.70}$$

□

Example: Let us consider a remarkable instance:

$$\nabla_u(\nabla_v(T)) = \nabla_u(v\lrcorner\nabla(T)) = \nabla_u(v)\lrcorner\nabla(T) + v\lrcorner\nabla_u(\nabla(T)) = \tag{1.2.71}$$

$$= \nabla_u(v)\lrcorner[\nabla(T)] + v\lrcorner u\lrcorner[\nabla(\nabla(T))] = \nabla_u(v)\lrcorner[\nabla(T)] + v\lrcorner u\lrcorner\nabla^2(T) \tag{1.2.72}$$

Therefore we can conclude that:

$$\nabla^2_{uv}(T) = \nabla_u\nabla_v(T) - \nabla_{\nabla_u(v)}(T) \tag{1.2.73}$$

Lemma 2: Given $T \in \Gamma T^p_q M$ and a connection ∇ there always exists a set of maps:

$$Z_{(n,k)} : \times^n(\Gamma TM) \to \Gamma T^k_0 M \tag{1.2.74}$$

not linear in all the arguments such that:

$$\nabla^{n-1}_{u_{\overline{n-1}}}(\nabla_{u_n}(T)) = \sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{n}})] \otimes \nabla^k(T)) \tag{1.2.75}$$

Proof. We can prove the statement by induction:

1. For $n = 1$ it is trivial:

$$\nabla_u(\nabla_v(T)) = \nabla^2_{u,v}(T) + \nabla_{\nabla_u(v)}(T) = i^2((u \otimes v) \otimes \nabla^2(T)) + i(\nabla_u(v) \otimes \nabla(T)) \tag{1.2.76}$$

2. Let us suppose that the statement is true for the step n.

$$\nabla^{n-1}_{u_{\overline{n-1}}}(\nabla_{u_n}(T)) = \sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{n}})] \otimes \nabla^k(T)) \tag{1.2.77}$$

3. If we are able to prove the statement for the $n + 1$ using 1) and 2) then the thesis is true $\forall n \in \mathbb{N}$

$$\nabla^{n}_{u_{\overline{n}}}(\nabla_{u_{n+1}}(T)) = \nabla_{u_1}(\nabla^{n-1}_{u_{\overline{n}\setminus\overline{1}}}(\nabla_{u_{n+1}}(T))) - \tag{1.2.78}$$

$$-\sum_{i=2}^{n}\{\nabla^{n-1}_{u_{\overline{i-1}\setminus\overline{1}}\ \nabla_{u_1}(u_i)\ u_{\overline{n}\setminus\overline{i}}}(\nabla_{u_{n+1}}(T))\} \tag{1.2.79}$$

Now we can use the inductive step:

$$\nabla^{n}_{u_{\overline{n}}}(\nabla_{u_{n+1}}(T)) = \nabla_{u_1}(\sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^k(T))) - \tag{1.2.80}$$

$$-\sum_{i=2}^{i=n}\{\sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{i-1}\setminus\overline{1}}, \nabla_{u_1}(u_i), u_{\overline{n+1}\setminus\overline{i}})] \otimes \nabla^k(T))\} \tag{1.2.81}$$

and considering that the covariant derivative ∇_u commutes with the contraction i:

$$\nabla^n_{u_{\overline{n}}}(\nabla_{u_{n+1}}(T)) = \sum_{k=1}^{n} i^k(\nabla_{u_1}[Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^k(T)) - \tag{1.2.82}$$

$$-\sum_{i=2}^{i=n}\{\sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{i-1}\setminus\overline{1}}, \nabla_{u_1}(u_i), u_{\overline{n+1}\setminus\overline{i}})] \otimes \nabla^k(T))\} = \tag{1.2.83}$$

$$= \sum_{k=1}^{n} i^k(\nabla_{u_1}[Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^k(T)) + \tag{1.2.84}$$

$$+ \sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla_{u_1}\nabla^k(T)) - \tag{1.2.85}$$

$$-\sum_{i=2}^{n}\{\sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{i-1}\setminus\overline{1}}, \nabla_{u_1}(u_i), u_{\overline{n+1}\setminus\overline{i}})] \otimes \nabla^k(T))\} = \tag{1.2.86}$$

$$= \sum_{k=1}^{n} i^k(\nabla_{u_1}[Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^k(T)) + \tag{1.2.87}$$

$$+ \sum_{k=1}^{n} i^{k+1}(u_1 \otimes [Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^{k+1}(T)) - \tag{1.2.88}$$

$$-\sum_{i=2}^{n}\{\sum_{k=1}^{n} i^k([Z_{(n,k)}(u_{\overline{i-1}\setminus\overline{1}}, \nabla_{u_1}(u_i), u_{\overline{n+1}\setminus\overline{i}})] \otimes \nabla^k(T))\} = \tag{1.2.89}$$

$$= \sum_{k=1}^{n} i^k(\nabla_{u_1}[Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^k(T)) + \tag{1.2.90}$$

$$+ \sum_{k=2}^{n+1} i^k(u_1 \otimes [Z_{(n,k)}(u_{\overline{n+1}\setminus\overline{1}})] \otimes \nabla^k(T)) - \tag{1.2.91}$$

$$-\sum_{k=1}^{n}\sum_{i=2}^{n}\{i^k([Z_{(n,k)}(u_{\overline{i-1}\setminus\overline{1}}, \nabla_{u_1}(u_i), u_{\overline{n+1}\setminus\overline{i}})] \otimes \nabla^k(T))\} = \tag{1.2.92}$$

$$\tag{1.2.93}$$

Now we can re-sum order by order in k defining some new non linear maps $Z_{(n+1,k)} : \times^{n+1}TM \to \Gamma T^k_0 M$ as linear combinations, composition with covariant derivatives and tensor products of maps $Z_{(n,j)}$, therefore we have:

$$\nabla^n_{u_{\overline{n}}}(\nabla_v(T)) = \sum_{k=1}^{(n+1)} i^k([Z_{(n+1,k)}(u_{\overline{n+1}})] \otimes \nabla^k(T)) \tag{1.2.94}$$

$\square$

Lemma 3: Given $T \in \Gamma T^p_q M$, the following two hold:

1. The higher order differential commutes with the upper braiding maps σ^I

$$\nabla^k(\sigma^I T) = \sigma^I \nabla^k(T) \tag{1.2.95}$$

2. The higher order differential does not commute with the lower braiding maps σ_J. In general given a permutation J of q vector fields one can state that:

$$\nabla^k(\sigma_J T) = \sigma_{J'} \nabla^k(T) \tag{1.2.96}$$

where J' is a new permutation of $q+k$ vector fields such that

$$\begin{cases} v_{P_{J'}(m)} = v_m \quad , \quad \forall m \in [1,k] \\ v_{P_{J'}(l+k)} = v_{P_J(l)} \quad , \quad \forall l \in [1,r] \end{cases} \tag{1.2.97}$$

Proof.

1. The first is trivial because the first differential commutes with σ^I as proven before. therefore:

$$\nabla^k(\sigma^I(T)) = \nabla^{k-1}(\nabla(\sigma^I(T))) = \nabla^{k-1}(\sigma^I(\nabla T)) \tag{1.2.98}$$

Iterating the process k times we have the thesis.

2. The second is a bit more tricky and it must be proved by induction. For $k = 1$ we have

$$[\nabla(\sigma_J T)](\alpha^{\overline{p}}, v_{\overline{q+1}}) = [\nabla_{v_1}(\sigma_J(T))](\alpha^{\overline{p}}, v_{\overline{q+1}\backslash\overline{1}}) = \sigma_J[\nabla T](\alpha^{\overline{p}}, v_{\overline{q}+1\backslash\overline{1}}) = \tag{1.2.99}$$

$$= [\nabla_{v_1} T](\alpha^{\overline{p}}, v_{P_J(\overline{q+1}\backslash\overline{1})}) = \nabla_{v_1} T(\alpha^{\overline{p}}, v_{P_J(\overline{q+1}\backslash\overline{1})})) = \sigma_{J'}[\nabla(T)](\alpha^p, v_{\overline{q+1}}) \tag{1.2.100}$$

Now let us suppose this is true for the step k therefore we can prove the relation still holds for $k+1$

$$[\nabla^{k+1}(\sigma_J T)](\alpha^{\overline{p}}, v_{\overline{q+k+1}}) = [\nabla^{k+1}_{v_{\overline{k+1}}}(\sigma_J(T))](\alpha^{\overline{p}}, v_{\overline{q+k+1}\setminus\overline{k+1}}) = \tag{1.2.101}$$

$$=\{\nabla_{v_1}[\nabla^k_{v_{\overline{k+1}\setminus\overline{1}}}(\sigma_J(T)) - \sum_{i=2}^{k+1}\nabla^k_{v_{\overline{i-1}\setminus\overline{1}}\ \nabla_{v_1}(v_i)\ v_{\overline{k+1}\setminus\overline{i}}}(\sigma_J(T))\}(\alpha^{\overline{p}}, v_{\overline{q+k+1}\setminus\overline{k+1}}) \tag{1.2.102}$$

Now using the inductive steps

$$[\nabla^{k+1}(\sigma_J T)](\alpha^{\overline{p}}, v_{\overline{q+k+1}}) = \tag{1.2.103}$$
$$=\nabla_{v_1}[\sigma_J(\nabla^k_{v_{\overline{k+1}\setminus\overline{1}}}(T))](\alpha^{\overline{p}}, v_{\overline{q+k+1}\setminus\overline{k+1}}) - \tag{1.2.104}$$
$$-\sum_{i=2}^{k+1}\sigma_J(\nabla^k_{v_{\overline{i-1}\setminus\overline{1}}\ \nabla_{v_1}(v_i)\ v_{\overline{k+1}\setminus\overline{i}}}(T))(\alpha^{\overline{p}}, v_{\overline{q+k+1}\setminus\overline{k+1}}) = \tag{1.2.105}$$
$$=\sigma_J\big\{\nabla_{v_1}[\nabla^k_{v_{\overline{k+1}\setminus\overline{1}}}(T)] - \sum_{i=2}^{k+1}\nabla^k_{v_{\overline{i-1}\setminus\overline{1}}\ \nabla_{v_1}(v_i)\ v_{\overline{k+1}\setminus\overline{i}}}(T)\big\}(\alpha^{\overline{p}}, v_{\overline{q+k+1}\setminus\overline{k+1}}) = \tag{1.2.106}$$
$$=\big\{\nabla_{v_1}[\nabla^k_{v_{\overline{k+1}\setminus\overline{1}}}(T)] - \sum_{i=2}^{k+1}\nabla^k_{v_{\overline{i-1}\setminus\overline{1}}\ \nabla_{v_1}(v_i)\ v_{\overline{k+1}\setminus\overline{i}}}(T)\big\}(\alpha^{\overline{p}}, v_{P_J(\overline{q+k+1}\setminus\overline{k+1})}) = \tag{1.2.107}$$
$$=\big\{\nabla^{k+1}_{v_{\overline{k+1}}}(T)\big\}(\alpha^{\overline{p}}, v_{P_J(\overline{q+k+1}\setminus\overline{k+1})}) = \big\{\nabla^{k+1}(T)\big\}(\alpha^{\overline{p}}, v_{\overline{k+1}}, v_{P_J(\overline{q+k+1}\setminus\overline{k+1})}) = \tag{1.2.108}$$
$$=\sigma_{J'}[\nabla^{k+1}(T)](\alpha^{\overline{p}}, v_{\overline{q+k+1}}) \tag{1.2.109}$$

□

Lemma 4: Let $T \in \Gamma T^p_q M$ a tensor field and let $L : \Gamma T^p_q M \to \Gamma T^{p'}_{q'} M$ be a smooth $C^\infty(M)$-linear map. We know that $L \in \Gamma T^{q+p'}_{p+q'} M$ due to the isomorphism. The following relation holds:

$$\nabla(L(T)) = \{\nabla L\}(T) + \{\mathbb{I} \otimes L\}(\nabla T) \tag{1.2.110}$$

Proof. The proof is easily performed fixing an arbitrary local frame (e_μ) on TM and the canonic dual (e^ν) on $T^\star M$ and using the standard properties of the covariant differential:

$$\nabla(L(T)) = \nabla(L(T)^{\mu_{\overline{p}'}}_{\nu_{\overline{q}'}} e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}}) = \tag{1.2.111}$$

$$=\nabla(L(T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} e_{\lambda_{\overline{p}}} \otimes e^{\rho_{\overline{q}}})^{\mu_{\overline{p}'}}_{\nu_{\overline{q}'}} e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}}) = \nabla(T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} L(e_{\lambda_{\overline{p}}} \otimes e^{\rho_{\overline{q}}}))^{\mu_{\overline{p}'}}_{\nu_{\overline{q}'}} e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}} = \tag{1.2.112}$$

$$=\nabla(T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} L^{\rho_{\overline{q}}\mu_{\overline{p}'}}_{\lambda_{\overline{p}} b_{\overline{q}'}} e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}}) = \nabla(T)^{\lambda_{\overline{p}}}_{rho_{\overline{q}}} L^{\lambda_{\overline{q}}\mu_{\overline{p}'}}_{\lambda_{\overline{p}}\nu_{\overline{q}'}} e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}} + T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \nabla(L)^{\rho_{\overline{q}}\mu_{\overline{p}'}}_{\lambda_{\overline{p}}\nu_{\overline{q}'}} e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}} = \tag{1.2.113}$$

$$= = \nabla_\sigma(T)^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \delta^\sigma_\eta L^{\rho_{\overline{q}}\mu_{\overline{p}'}}_{\rho_{\overline{p}}\nu_{\overline{q}'}} e^\eta \otimes e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}} + T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \nabla_\eta(L)^{\rho_{\overline{q}}\mu_{\overline{p}'}}_{\lambda_{\overline{p}}\nu_{\overline{q}'}} e^\eta \otimes e_{\mu_{\overline{p}'}} \otimes e^{\nu_{\overline{q}'}} = \tag{1.2.114}$$

$$= = \{\nabla L\}(T) + \{\mathbb{I} \otimes L\}(\nabla T) \tag{1.2.115}$$

□

Lemma 5: Let be $L : \Gamma T^p_q M \to \Gamma T^{p'}_{q'} M$ a smooth C^∞ linear application on the tensor fields. We know that $L \in \Gamma T^{p+p'}_{q+q'} M$ due to the isomorphism between linear applications on tensors fields and tensor fields. The following holds:

$$\nabla^n(L(T)) = \sum_{k=0}^n A_{(n,k)}(\nabla^k T) \tag{1.2.116}$$

where $A_{(n,k)}$ is a bunch of smooth C^∞-linear applications $\Gamma T^p_{k+q} M \to \Gamma T^{p'}_{n+q'} M$

In particular one has that:

$$\nabla^n(L(T)) = \sum_{k=0}^n \binom{n}{k} [\sigma^{(\overline{1+p'})} \ \mathbb{I}\otimes]^k \nabla^{n-k}(L)(\nabla^k T) \tag{1.2.117}$$

Proof. The proof is easily performed by induction: For n=1 the proof is given by the previous lemma, then let us suppose that the formula holds for the step n, then we have that at the step n+1:

$$\nabla^{n+1}(L(T)) = \nabla(\nabla^n L(T)) = \nabla \sum_{k=0}^n \binom{n}{k} [\sigma^{(\overline{1+p'})} \ \mathbb{I}\otimes]^k \nabla^{n-k}(L)(\nabla^k T)) = \tag{1.2.118}$$

$$= \sum_{k=0}^n \binom{n}{k} \nabla \big\{ [\sigma^{(\overline{1+p'})} \ \mathbb{I}\otimes]^k \nabla^{n-k}(L)(\nabla^k T) \big\} = \tag{1.2.119}$$

$$= \sum_{k=0}^n \binom{n}{k} \nabla \big\{ [\sigma^{(\overline{1+p'})} \ \mathbb{I}\otimes]^k \nabla^{n-k}(L) \big\} (\nabla^k T) + \tag{1.2.120}$$

$$+ \sum_{k=0}^n \binom{n}{k} [\sigma^{(\overline{1+p'})} \ \mathbb{I}\otimes]^{k+1} \nabla^{n-k}(L) \big(\nabla^{k+1} T) = \tag{1.2.121}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big\}(\nabla^{k}T)+ \tag{1.2.122}$$

$$+\sum_{k=1}^{n+1}\binom{n}{k-1}[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big(\nabla^{k}T\big)= \tag{1.2.123}$$

$$=\nabla^{n+1}(L)\big(T\big)+\sum_{k=1}^{n}\binom{n}{k}\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big\}(\nabla^{k}T)+ \tag{1.2.124}$$

$$+\sum_{k=1}^{n}\binom{n}{k-1}[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big(\nabla^{k}T\big)+\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{n}L\big\}(\nabla^{n}T)= \tag{1.2.125}$$

$$=\nabla^{n+1}(L)\big(T\big)+\sum_{k=1}^{n}\Big[\binom{n}{k}+\binom{n}{k-1}\Big]\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big\}(\nabla^{k}T)+ \tag{1.2.126}$$

$$+\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{n}L\big\}(\nabla^{n}T)= \tag{1.2.127}$$

$$=\nabla^{n+1}(L)\big(T\big)+\sum_{k=1}^{n}\Big[\binom{n+1}{k}\Big]\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big\}(\nabla^{k}T)+\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{n}L\big\}(\nabla^{n}T)= \tag{1.2.128}$$

$$=\sum_{k=0}^{n+1}\Big[\binom{n+1}{k}\Big]\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big\}(\nabla^{k}T) \tag{1.2.129}$$

and now it's enough to define the new linear application

$$A_{(n+1,k)}=\Big[\binom{n+1}{k}\Big]\big\{[\sigma^{(\overline{1+p'})}\ \mathbb{I}\otimes]^{k}\nabla^{n-k+1}(L)\big\}$$

to end up with

$$\nabla^{n+1}(L(T))=\sum_{k=0}^{n+1}A_{(n+1,k)}(\nabla^{k}T) \tag{1.2.130}$$

$\square$

Lemma 6: Let $\nabla^{n}:\Gamma T^{p}_{q}M\to\Gamma T^{p}_{n+q}M$ be the n-th covariant differential. Given an arbitrary tensor field $T\in\Gamma T^{p}_{q}M$, there always exists a set of $C^{\infty}(M)-linear$ maps

$H_{(k,n)} : \Gamma T^{p+p'}_{k+q+q'}M \to \Gamma T^{p+p'}_{n+q+q'}M, (\forall s \in [0,n])$ such that:

$$\nabla^n(S \otimes T) = \sum_{k=0}^{n} H_{(k,n)}(\nabla^{n-k}(T) \otimes \nabla^k(S)) \tag{1.2.131}$$

Proof. The proof can be performed by induction. The first case:

$$\nabla(S \otimes T) = \nabla(T) \otimes S + \sigma_{\overline{1+q}}(T \otimes \nabla(S)) \tag{1.2.132}$$

has already been proved. Now let us suppose the thesis holds for the case $\nabla^n(S \otimes T)$ and let us prove it for the next step:

$$\nabla^{n+1}(S \otimes T) = \nabla[\sum_{k=0}^{n} H_{(k,n)}(\nabla^{n-k}(T) \otimes \nabla^k(S))] = \tag{1.2.133}$$

$$= \sum_{k=0}^{n} \{\nabla[H_{(k,n)}](\nabla^{n-k}(T) \otimes \nabla^k(S)) + [\sigma^{1+p+q}(\ \mathbb{I} \otimes H_{(k,n)})](\nabla(\nabla^{n-k}(T) \otimes \nabla^k(S)))) = \tag{1.2.134}$$

$$= \sum_{k=0}^{n} \nabla[H_{(k,n)}](\nabla^{n-k}(T) \otimes \nabla^k(S)) + \tag{1.2.135}$$

$$+ [\sigma^{1+p+q}(\ \mathbb{I} \otimes H_{(k,n)})](\nabla^{n-k+1}(T) \otimes \nabla^k(S) + \sigma_{n-k+q+1}\nabla^{n-k}(T) \otimes \nabla^{k+1}(S))))\Big\} \tag{1.2.136}$$

Now it is enough to re-sum order by order in k redefining appropriate new $C^\infty(M)$-linear maps $H_{(k,n+1)}$ to end up with :

$$\nabla^{n+1}(S \otimes T) = \sum_{k=0}^{n} H_{(k,n+1)}(\nabla^{n+1-k}(T) \otimes \nabla^k(S)) \tag{1.2.137}$$

□

1.2.3 Local expression of derivations on tensor fields

Let us suppose to fix a local frame (e_μ) on TM, we know that this choice induces a local trivialisation of the bundles TM , $T^\star M$ and $T^p_q M$. Let us consider the Lie derivative first. The action on the functions is given by:

$$L_v(f) = v(f) \quad , \quad \forall f \in C^\infty(M) \tag{1.2.138}$$

therefore the coordinate expression induced by the local frame:

$$L_v(f) = v^\mu e_\mu(f) \quad , \quad \forall f \in C^\infty(M) \tag{1.2.139}$$

If a natural frame ∂_μ is chosen then:

$$L_v(f) = v^\mu \partial_\mu(f) \quad , \quad \forall f \in C^\infty(M) \tag{1.2.140}$$

Concerning the vector fields we have that:

$$\{L_u(v)\}(f) = [u, v](f) = u^\mu e_\mu(v^\nu e_\nu(f)) - v^\mu e_\mu(u^\nu e_\nu(f)) = \tag{1.2.141}$$
$$= u^\mu e_\mu(v^\nu)e_\nu(f) + u^\mu v^\nu e_\mu(e_\nu(f)) - u^\mu v^\nu e_\nu(e_\mu(f)) - v^\mu e_\mu(u^\nu)e_\nu(f) = \tag{1.2.142}$$
$$= u^\mu e_\mu(v^\nu)e_\nu(f) - v^\mu e_\mu(u^\nu)e_\nu(f) + v^\mu u^\mu [e_\mu, e_\nu]^\lambda e_\lambda(f) = \tag{1.2.143}$$
$$= u^\mu e_\mu(v^\nu)e_\nu(f) - v^\mu e_\mu(u^\nu)e_\nu(f) + v^\mu u^\mu C^\lambda_{\mu\nu} e_\lambda(f) \tag{1.2.144}$$

Let us remark that $C^\lambda_{\mu\nu}$ cannot be considered the components of a tensor field because it does not change with the usual linear rule when the frame transforms. If a natural frame ∂_μ is chosen then:

$$\{L_u(v)\}(f) = [u, v](f) = u^\mu \partial_\mu(v^\nu)\partial_\nu(f) - v^\mu \partial_\mu(u^\nu)\partial_\nu(f) + v^\mu u^\mu [\partial_\mu, \partial_\nu](f) = \tag{1.2.145}$$
$$= u^\mu \partial_\mu(v^\nu)\partial_\nu(f) - v^\mu \partial_\mu(u^\nu)\partial_\nu(f) \quad , \quad \forall f \in C^\infty(M) \tag{1.2.146}$$

For the covector fields

$$\{L_u(\alpha(v))\} = u(\alpha(v)) = \{L_u(\alpha)\}(v) + \alpha(L_u(v)) = \{L_u(\alpha)\}(v) + \alpha(L_u(v)) \tag{1.2.147}$$

therefore we have:

$$
\begin{aligned}
&\{L_u(\alpha)\}(v) = u(\alpha(v)) - \alpha(L_u(v)) = && (1.2.148)\\
=&u^\mu e_\mu(\alpha_\nu v^\nu) - \alpha_\nu u^\mu e_\mu(v^\nu)\alpha_\nu + v^\mu e_\mu(u^\nu)\alpha_\nu - v^\mu u^\mu C^\lambda_{\mu\nu}\alpha_\lambda = && (1.2.149)\\
=&u^\mu e_\mu(\alpha_\nu)v^\nu + u^\mu \alpha_\nu e_\mu(v^\nu) - \alpha_\nu u^\mu e_\mu(v^\nu)\alpha_\nu + v^\mu e_\mu(u^\nu)\alpha_\nu - v^\mu u^\mu C^\lambda_{\mu\nu}\alpha_\lambda = && (1.2.150)\\
=&u^\mu e_\mu(\alpha_\nu)v^\nu + v^\mu e_\mu(u^\nu)\alpha_\nu - v^\mu u^\mu C^\lambda_{\mu\nu}\alpha_\lambda && (1.2.151)
\end{aligned}
$$

If a natural frame ∂_μ is chosen then $C^\lambda_{\mu\nu} = 0$:

$$
\{L_u(\alpha)\}(v) = v^\nu\{u^\mu \partial_\mu(\alpha_\nu) + \partial_\nu(u^\mu)\alpha_\mu\} \tag{1.2.152}
$$

For the tensor fields we can follow the same procedure to state:

$$
\begin{aligned}
&L_u[T(\alpha^{\overline{p}}, v_{\overline{q}})] = L_u[v_q\lrcorner ... v_1\lrcorner\alpha^p\neg ... \alpha^1\neg T] = && (1.2.153)\\
&= L_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) + \sum_{i=1}^{p} T(\alpha^{\overline{i-1}}, L_u(\alpha^i), \alpha^{\overline{p}\setminus\overline{i}}, v_{\overline{q}}) + \sum_{j=1}^{q} T(\alpha^{\overline{p}}, v_{\overline{j-1}}, L_u(v_j), v_{\overline{q}\setminus\overline{j}}) && (1.2.154)
\end{aligned}
$$

Therefore we can write for each $\alpha^{\overline{p}} \in \times^p\Gamma T^\star M$ and for each $v_{\overline{q}} \in \times^q \Gamma TM$:

$$
L_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) = L_u[T(\alpha^{\overline{p}}, v_{\overline{q}})] - \sum_{i=1}^{p} T(\alpha^{\overline{i-1}}, L_u(\alpha^i), \alpha^{\overline{p}\setminus\overline{i}}, v_{\overline{q}}) - \sum_{j=1}^{q} T(\alpha^{\overline{p}}, v_{\overline{j-1}}, L_u(v_j), v_{\overline{q}\setminus\overline{j}}) \tag{1.2.155}
$$

Substituting the coordinate expression of $L_u(v)$ and $L_u(\alpha)$, expanding the coordinate expression of T and using the Leibniz rules we can conclude easily that:

$$
\begin{aligned}
&L_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) = \Big\{u^\lambda e_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) - \sum_{i=1}^{p} T_{\nu_{\overline{q}}}^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}[e_\alpha(u^{\mu_i}) - C^{\mu_i}_{\alpha\lambda}u^\lambda] + && (1.2.156)\\
&+\sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}[e_{\nu_j}(u^\beta) - C^\beta_{\nu_j\lambda}u^\lambda]\Big\}(\alpha_{\mu_{\overline{p}}}, v^{\nu_{\overline{q}}}) && (1.2.157)
\end{aligned}
$$

If a natural frame ∂_μ is chosen then $C^\lambda_{\mu\nu} = 0$ and we can conclude that:

$$L_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) = \Big\{u^\lambda \partial_\lambda (T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) - \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \partial_\alpha (u^{\mu_i}) + \tag{1.2.158}$$

$$+ \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \partial_{\nu_j}(u^\beta)\Big\}(\alpha_{\mu_{\overline{p}}}, v^{\nu_{\overline{q}}}) \tag{1.2.159}$$

Concerning the covariant derivatives the approach is identical:

$$\nabla_v(f) = v(f) = v^\mu e_\mu(f) \quad , \quad \forall f \in C^\infty(M) \tag{1.2.160}$$

let us remark that $\Gamma^\mu_{\nu\lambda}$ cannot be interpreted as the coordinate expression of a tensor field because it does not change with the right transformation rules when the trivialisation is changed. If a natural frame is chosen then:

$$\nabla_v(f) = v(f) = v^\mu \partial_\mu(f) \quad , \quad \forall f \in C^\infty(M) \tag{1.2.161}$$

Concerning vector fields one has:

$$\{\nabla_u(v)\}(f) = \{\nabla_u(v^\nu e_\nu)\}(f) = u(v^\nu) e_\nu(f) + v^\nu \{\nabla_u(e_\nu)\}(f) = \tag{1.2.162}$$

$$= u^\mu e_\mu(v^\nu) e_\nu(f) + v^\nu u^\mu \{\nabla_{e_\mu}(e_\nu)\}(f) = \tag{1.2.163}$$

$$= u^\mu e_\mu(v^\nu) e_\nu(f) + v^\nu u^\mu \{\nabla_{e_\mu}(e_\nu)\}^\lambda e_\lambda(f) = \tag{1.2.164}$$

$$= u^\mu e_\mu(v^\nu) e_\nu(f) + v^\nu u^\mu \Gamma^\lambda_{\mu\nu} e_\lambda(f) \tag{1.2.165}$$

If a natural trivialization is chosen then we have:

$$\{\nabla_u(v)\}(f) = u^\mu \partial_\mu(v^\lambda) \partial_\lambda(f) + v^\nu u^\mu \Gamma^\lambda_{\mu\nu} \partial_\lambda(f) \tag{1.2.166}$$

For the covector fields it is enough to calculate:

$$\nabla_u(\alpha(v)) = u(\alpha(v)) = \{\nabla_u(\alpha)\}(v) + \alpha(\{\nabla_u(v)\}) \tag{1.2.167}$$

Therefore for a generic vector field $v \in \Gamma TM$ we can state:

$$\{\nabla_u(\alpha)\}(v) = u(\alpha(v)) - \alpha(\{\nabla_u(v)\}) = \tag{1.2.168}$$
$$= u^\mu e_\mu(\alpha_\nu v^\nu) - \alpha_\lambda u^\mu e_\mu(v^\lambda) - \alpha_\lambda v^\nu u^\mu \Gamma^\lambda_{\mu\nu} = \tag{1.2.169}$$
$$= u^\mu e_\mu(\alpha_\nu) v^\nu + u^\mu e_\mu(v^\nu)\alpha_\nu - \alpha_\lambda u^\mu e_\mu(v^\lambda) - \alpha_\lambda v^\nu u^\mu \Gamma^\lambda_{\mu\nu} = \tag{1.2.170}$$
$$= u^\mu e_\mu(\alpha_\nu) v^\nu - \alpha_\lambda v^\nu u^\mu \Gamma^\lambda_{\mu\nu} \tag{1.2.171}$$

In case of natural frame:

$$\{\nabla_u(\alpha)\}(v) = u^\mu \partial_\mu(\alpha_\nu) v^\nu - \alpha_\lambda v^\nu u^\mu \Gamma^\lambda_{\mu\nu} \tag{1.2.172}$$

For the tensor field the calculation can be performed in the same way:

$$\nabla_u[T(\alpha^{\overline{p}}, v_{\overline{q}})] = \nabla_u[v_q \lrcorner ... v_1 \lrcorner \alpha^p \neg ... \alpha^1 \neg T] = \tag{1.2.173}$$
$$= \nabla_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) + \sum_{i=1}^{p} T(\alpha^{\overline{i-1}}, \nabla_u(\alpha^i), \alpha^{\overline{p}\setminus\overline{i}}, v_{\overline{q}}) + \sum_{j=1}^{q} T(\alpha^{\overline{p}}, v_{\overline{j-1}}, \nabla_u(v_j), v_{\overline{q}\setminus\overline{j}}) \tag{1.2.174}$$

Therefore we can write for each $\alpha^{\overline{p}} \in \times^p \Gamma T^\star M$ and for each $v_{\overline{q}} \in \times^q \Gamma TM$:

$$\nabla_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) = \nabla_u[T(\alpha^{\overline{p}}, v_{\overline{q}})] - \sum_{i=1}^{p} T(\alpha^{\overline{i-1}}, \nabla_u(\alpha^i), \alpha^{\overline{p}\setminus\overline{i}}, v_{\overline{q}}) - \sum_{j=1}^{q} T(\alpha^{\overline{p}}, v_{\overline{j-1}}, \nabla_u(v_j), v_{\overline{q}\setminus\overline{j}}) \tag{1.2.175}$$

Substituting the coordinate expression of $L_u(v)$ and $L_u(\alpha)$, expanding the coordinate expression of T and using the Leibniz rules we can conclude easily that:

$$\nabla_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) = \Big\{ u^\lambda e_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) + \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \Gamma^{\mu_i}_{\alpha\lambda} u^\lambda + \tag{1.2.176}$$
$$- \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \Gamma^{\beta}_{\nu_j\lambda} u^\lambda \Big\} (\alpha_{\mu_{\overline{p}}}, v^{\nu_{\overline{q}}}) \tag{1.2.177}$$

If a natural frame ∂_μ is chosen then we can conclude that:

$$\nabla_u\{T\}(\alpha^{\overline{p}}, v_{\overline{q}}) = \Big\{u^\lambda \partial_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) + \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \Gamma^{\mu_i}_{\alpha\lambda} u^\lambda + \tag{1.2.178}$$

$$-\sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \Gamma^{\beta}_{\nu_j\lambda} u^\lambda\Big\}(\alpha_{\mu_{\overline{p}}}, v^{\nu_{\overline{q}}}) \tag{1.2.179}$$

Considering this we can find the local coordinate expression related to the covariant differential and divergence:

$$\nabla(T)^{\mu_{\overline{p}}}_{\lambda\nu_{\overline{q}}} = e_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) + \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \Gamma^{\mu_i}_{\alpha\lambda} - \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \Gamma^{\beta}_{\nu_j\lambda} \tag{1.2.180}$$

$$\nabla(T)^{\mu_{\overline{p}}}_{\mu_1\nu_{\overline{q}}} = e_{\mu_1}(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) + \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \Gamma^{\mu_i}_{\alpha\mu_1} - \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \Gamma^{\beta}_{\nu_j\mu_1} \tag{1.2.181}$$

and if the natural frame is chosen:

$$\nabla(T)^{\mu_{\overline{p}}}_{\lambda\nu_{\overline{q}}} = \partial_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) + \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \Gamma^{\mu_i}_{\alpha\lambda} - \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \Gamma^{\beta}_{\nu_j\lambda} \tag{1.2.182}$$

$$\nabla(T)^{\mu_{\overline{p}}}_{\mu_1\nu_{\overline{q}}} = \partial_{\mu_1}(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) + \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \Gamma^{\mu_i}_{\alpha\mu_1} - \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \Gamma^{\beta}_{\nu_j\mu_1} \tag{1.2.183}$$

1.2.4 More properties

Now that we have made explicit the coordinate expression of the actions of the derivations upon the local coordinate expression of a tensor field, we are able to prove more properties concerning these differential operators.

Property 11: Let $T \in \Gamma T^p_q M$ be a tensor field , $f \in C^\infty(M)$ be a scalar field and $u \in \Gamma TM$ a vector field . The Lie derivative behaves as follow:

$$\{L_{fu}(T)\} = \tag{1.2.184}$$

$$= f L_u\{T\} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{u \otimes [df \neg \sigma^{\overline{s}}(T)]\Big\} + \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{df \otimes [u \neg \sigma_{\overline{r}}(T)]\Big\} \tag{1.2.185}$$

where $\overline{\sigma^{\overline{s}}}$ and $\overline{\sigma_{\overline{r}}}$ are respectively the inverse briding maps of $\sigma^{\overline{s}}$ and $\sigma_{\overline{r}}$

Proof. The proof can be performed locally and then we can glue together the local sections as proven in the first chapter. Let us fix the natural trivialisation choosing the natural local frame (∂_μ).

$$L_{fu}\{T\}(e^{\mu_{\overline{p}}}, e_{\mu_{\overline{q}}}) = f u^\lambda \partial_\lambda (T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) - \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} \partial_\alpha (f u^{\mu_i}) + \tag{1.2.186}$$

$$+ \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} \partial_{\nu_j}(f u^\beta) = \tag{1.2.187}$$

$$= f L_u\{T\}(e^{\mu_{\overline{p}}}, e_{\mu_{\overline{q}}}) - \sum_{i=1}^{p} T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}} u^{\mu_i} \partial_\alpha(f) + \sum_{j=1}^{q} T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}} u^\beta \partial_{\nu_j}(f) = \tag{1.2.188}$$

$$= \Big[f L_u\{T\} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{u \otimes [df \neg \sigma^{\overline{s}}(T)]\Big\} + \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{df \otimes [u \neg \sigma_{\overline{r}}(T)]\Big\}\Big](e^{\mu_{\overline{p}}}, e_{\mu_{\overline{q}}}) \tag{1.2.189}$$

□

Corollary 1: From the property we can state trivially that:

$$L_{fu}\{T\} = f L_u(T) - \sum_{s=1}^{p} \Big\{ i \Big[\sigma^{(1,s+1)}(df \otimes u \otimes T)\Big]\Big\} + \sum_{r=1}^{q} \Big\{ i \Big[\sigma_{(1,r+1)}(df \otimes u \otimes T)\Big]\Big\} \tag{1.2.190}$$

Proof. The required result follows trivially from the coordinate expression found above. □

Corollary 2: From the previous property we can cast the following statement:

$$L_v T^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{v^{\alpha}_{(i)} e_{(i)\alpha}} T^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{1.2.191}$$

$$= v^{\alpha}_{(i)} L_{\alpha} T^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}[e_{(i)\alpha} \otimes d(\ v^{\alpha}_{(i)}) \neg (\sigma^{\overline{s}} T)]^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}[d(v^{\alpha}_{(i)}) \otimes (e_{(i)\alpha} \lrcorner \sigma_{\overline{r}} T)]^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \tag{1.2.192}$$

or equivalently

$$L_v T^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{v^{\alpha}_{(i)} e_{(i)\alpha}} T^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{1.2.193}$$

$$= v^{\alpha}_{(i)} L_{\alpha} T^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} - \sum_{s=1}^{p} \left\{ i \left[\sigma^{(1,s+1)} (d(v^{\alpha}_{(i)}) \otimes e_{(i)\alpha} \otimes T) \right] \right\}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \tag{1.2.194}$$

$$+ \sum_{r=1}^{q} \left\{ i \left[\sigma_{(1,r+1)} (d(v^{\alpha}_{(i)}) \otimes e_{(i)\alpha} \otimes T) \right] \right\}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \tag{1.2.195}$$

where we used the notation $L_{\alpha} T_{(i)} = L_{e_{(i)\alpha}} T$.

Lemma 7: Let $T \in \Gamma T^p_q M$ be a tensor field, $f \in C^{\infty}(M)$ an arbitrary scalar field and (e_{λ}) an arbitrary local frame defined on the open $U \subseteq M$. There always exists a bunch of local smooth scalar fields $f^{\lambda_{\overline{k}} \rho_{\overline{q}} \mu_{\overline{p}}}_{\sigma_{\overline{p}} \nu_{\overline{q}}} : U \to M$ such that the following relation about the local expression is satisfied:

$$L_{\lambda_{\overline{k}}} (fT)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k} L_{\alpha_{\overline{j}}} (T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}} \sigma_{\overline{q}} \mu_{\overline{p}}}_{\lambda_{\overline{k}} \rho_{\overline{p}} \nu_{\overline{q}}} \tag{1.2.196}$$

Proof. We can prove it easily via induction. The step 1 is trivial because of the Leibniz rule:

$$L_{\lambda} (fT)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_{\lambda}(f) T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + f L_{\lambda}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.197}$$

If we suppose the thesis holds for the generic step k we can use again the Leibniz rule to prove it for the step $k+1$. Now we can use the inductive step. Let us suppose that the thesis holds for the case k and let us prove that it still holds for $k+1$.

$$[L_{\lambda_{\overline{k+1}}}(fT)]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_{\lambda_1}[L_{\lambda_{\overline{k+1}\backslash\overline{1}}}(fT)]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.198}$$

$$= L_{\lambda_1}\Big[\sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}} e_{\beta_{\overline{p}}} \otimes e^{\gamma_{\overline{q}}}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.199}$$

$$= \sum_{j=0}^{k+1}\Big[L_{\lambda_1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e_{\beta_{\overline{p}}} \otimes e^{\gamma_{\overline{q}}} + L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}\Big[L_{\lambda_1}(e_{\beta_{\overline{p}}} \otimes e^{\gamma_{\overline{q}}})\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.200}$$

A this point it is enough to re-sum order by order, defining a new bunch of local smooth scalar fields $g^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ as appropriate linear combinations of $f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ and Lie derivatives of the frame to obtain:

$$[L_{\lambda_{\overline{k+1}}}(L_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.201}$$

□

Corollary 3: In the very same way it is possible to prove that:

$$L_{m_{\overline{k}}}(fT)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k} L_{n_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{n_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{m_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.202}$$

where $m_{\overline{k}}$ and $n_{\overline{j}}$ are bunches of indices running in $[1, dim(M)-1] \subset \mathbb{N}$

Lemma 8: Given a tensor field $T \in \Gamma T^p_q M$, and an arbitrary vector field $v \in \Gamma TM$ and an arbitrary local frame (e_λ) defined on the open $U \subseteq M$. There always exists a bunch of local smooth scalar fields $f^{\lambda_{\overline{k}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\sigma_{\overline{p}}\nu_{\overline{q}}} : U \to M$ such that the following relation about the local expression is satified:

$$[L_{\lambda_{\overline{k}}}(L_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.203}$$

Proof. We can prove it via induction. Let us start with $k = 1$. Therefore:

$$L_\lambda(L_v(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_v(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - L_{[e_\lambda,v]^\sigma e_\sigma}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.204}$$

$$= v^\alpha L_\alpha(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d(v^\alpha) \neg \sigma^{\overline{s}}(L_\lambda T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.205}$$

$$+ \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{d(v^\alpha) \otimes [e_\alpha \neg \sigma_{\overline{r}}(L_\lambda T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + L_{[e_\lambda,v]^\alpha e_\alpha}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.206}$$

$$= v^\alpha L_\alpha(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d(v^\alpha) \neg \sigma^{\overline{s}}(L_\lambda T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.207}$$

$$+ \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda,v]^\alpha) \otimes [e_\alpha \neg \sigma_{\overline{r}}(L_\lambda T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d([e_\lambda,v]^\alpha) \neg \sigma^{\overline{s}}(T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.208}$$

$$+ \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{d(v^\alpha) \otimes [e_\alpha \neg \sigma_{\overline{r}}(T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + [e_\lambda,v]^\alpha L_{e_\alpha}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.209}$$

Now one can notice that the operations inside the sums are $C^\infty(M)$-linear in the terms $L_\lambda T$ and T therefore, we can decompose them using the local frame obtaining:

$$L_\lambda(L_v(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.210}$$

$$= v^\alpha L_\alpha(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d(v^\alpha) \neg \sigma^{\overline{s}}(L_\lambda T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.211}$$

$$+ \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda,v]^\alpha) \otimes [e_\alpha \neg \sigma_{\overline{r}}(L_\lambda T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d([e_\lambda,v]^\alpha) \neg \sigma^{\overline{s}}(T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.212}$$

$$+ \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{d(v^\alpha) \otimes [e_\alpha \neg \sigma_{\overline{r}}(T)]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + [e_\lambda,v]^\alpha L_{e_\alpha}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.213}$$

$$= v^\alpha L_\alpha(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + [e_\lambda,v]^\alpha L_{e_\alpha}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.214}$$

$$- \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d(v^\alpha) \neg \sigma^{\overline{s}}(L_\lambda T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} e_{\sigma_{\overline{p}}} \otimes e^{\rho_{\overline{q}}}))]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.215}$$

$$+ \sum_{r=1}^{q} \overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda,v]^\alpha) \otimes [e_\alpha \neg \sigma_{\overline{r}}(L_\lambda T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} e_{\sigma_{\overline{p}}} \otimes e^{\rho_{\overline{q}}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.216}$$

$$- \sum_{s=1}^{p} \overline{\sigma^{\overline{s}}}\Big\{e_\alpha \otimes [d([e_\lambda,v]^\alpha) \neg \sigma^{\overline{s}}(T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} e_{\sigma_{\overline{p}}} \otimes e^{\rho_{\overline{q}}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.217}$$

$$+\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d(v^\alpha)\otimes[e_\alpha\neg\sigma_{\overline{r}}(T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.218}$$

$$=v^\alpha L_\alpha(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+[e_\lambda,v]^\alpha L_{e_\alpha}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.219}$$

$$-L_\lambda T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\alpha\otimes[d(v^\alpha)\neg\sigma^{\overline{s}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}}))]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.220}$$

$$+L_\lambda T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda,v]^\alpha)\otimes[e_\alpha\neg\sigma_{\overline{r}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.221}$$

$$-T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\alpha\otimes[d([e_\lambda,v]^\alpha)\neg\sigma^{\overline{s}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.222}$$

$$+T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d(v^\alpha)\otimes[e_\alpha\neg\sigma_{\overline{r}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho\overline{q}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.223}$$

$$=v^\alpha L_\alpha(L_\lambda(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+[e_\lambda,v]^\alpha L_{e_\alpha}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.224}$$

$$+L_\lambda T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big[-\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\alpha\otimes[d(v^\alpha)\neg\sigma^{\overline{s}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho\overline{q}}))]\Big\}+ \tag{1.2.225}$$

$$+\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda,v]^\alpha)\otimes[e_\alpha\neg\sigma_{\overline{r}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho\overline{q}})]\Big\}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.226}$$

$$+T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big[-\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\alpha\otimes[d([e_\lambda,v]^\alpha)\neg\sigma^{\overline{s}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.227}$$

$$+\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d(v^\alpha)\otimes[e_\alpha\neg\sigma_{\overline{r}}(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})]\Big\}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.228}$$

It is enough to consider the action of the maps upon the local tensor frame $(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})$ and re-sum order by order in the Lie derivatives to define the local smooth scalar fields:

$$L_\lambda(L_v(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=L_{\alpha_1}(L_{\alpha_2}(T))^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_1\alpha_2\,\mu_{\overline{p}}\rho_{\overline{q}}}_{\lambda\,\sigma_{\overline{p}}\nu_{\overline{q}}}+L_{\alpha_1}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_1\,\mu_{\overline{p}}\rho_{\overline{q}}}_{\lambda\,\sigma_{\overline{p}}\nu_{\overline{q}}}+(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\mu_{\overline{p}}\rho_{\overline{q}}}_{\lambda\,\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.229}$$

Now we can use the inductive step. Let us suppose that the thesis holds for the case k and let us prove that it still holds for $k+1$.

$$[L_{\lambda_{\overline{k+1}}}(L_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=L_{\lambda_1}[L_{\lambda_{\overline{k+1}\setminus\overline{1}}}(L_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.230}$$

$$=L_{\lambda_1}\Big[\sum_{j=0}^{k+1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.231}$$

$$=\sum_{j=0}^{k+1}\left[L_{\lambda_1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}\right]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}}+L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}\left[L_{\lambda_1}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})\right]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.232}$$

A this point re-summing order by order, defining a new bunch of local smooth scalar fields $g^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ as an appropriate linear combinations of $f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ and Lie derivatives of the frame we can obtain:

$$[L_{\lambda_{\overline{k+1}}}(L_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\sum_{j=0}^{(k+1)+1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.233}$$

□

Property 12: Given a tensor field $T\in\Gamma T^p_qM$ and a vector field $u\in\Gamma TM$ the lie derivative can always be written in terms of covariant derivatives as follows:

$$\{L_u(T)\}(\alpha^{\overline{p}},v_{\overline{q}})= \tag{1.2.234}$$

$$=\{\nabla_u(T)\}(\alpha^{\overline{p}},v_{\overline{q}})-\sum_{i=1}^{q}T(\alpha^{\overline{i-1}},\alpha^i\lrcorner[\nabla(u)],\alpha^{\overline{p}\setminus\overline{i}},v_{\overline{q}})+\sum_{j=1}^{p}T(\alpha^{\overline{p}},v_{\overline{j-1}},v_j\lrcorner[\nabla(u)],v_{\overline{q}\setminus\overline{j}}) \tag{1.2.235}$$

Proof. The proof can be performed locally and then we can glue together the local sections as described in the appendix. Let us fix a local natural trivialisation by choosing the local frame ∂_μ.

$$L_u\{T\}(e^{\mu_{\overline{p}}},e_{\mu_{\overline{q}}})=\Big\{u^\lambda\partial_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})-\sum_{i=1}^{p}T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}}\partial_\alpha(u^{\mu_i})+\sum_{j=1}^{q}T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}\partial_{\nu_j}(u^\beta)\Big\}= \tag{1.2.236}$$

$$=\Big\{u^\lambda\partial_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})-\sum_{i=1}^{p}T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}}\partial_\alpha(u^{\mu_i})+\sum_{j=1}^{q}T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}\partial_{\nu_j}(u^\beta)\Big\}+ \tag{1.2.237}$$

$$\mp\sum_{i=1}^{p}T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}}\Gamma^{\mu_i}_{\alpha\lambda}(u^\lambda)\pm\sum_{j=1}^{q}T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}\Gamma^{\beta}_{\nu_j\lambda}(u^\lambda)= \tag{1.2.238}$$

$$=\Big\{u^\lambda\partial_\lambda(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})+\sum_{i=1}^{p}T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}}\Gamma^{\mu_i}_{\alpha\lambda}(u^\lambda)-\sum_{j=1}^{q}T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}\Gamma^{\beta}_{\nu_j\lambda}(u^\lambda)\Big\}+ \tag{1.2.239}$$

$$-\sum_{i=1}^{p}\Big\{T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}}[\partial_\alpha(u^{\mu_i})+\Gamma^{\mu_i}_{\alpha\lambda}(u^\lambda)]\Big\}+\sum_{j=1}^{q}\Big\{T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}[\partial_{\nu_j}(u^\beta)+\Gamma^{\beta}_{\nu_j\lambda}(u^\lambda)]\Big\}= \tag{1.2.240}$$

$$=\nabla_u(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}-\sum_{i=1}^{p}\Big\{T^{\mu_{\overline{i-1}}\alpha\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{q}}}\nabla_\alpha(u)^{\mu_i}\Big\}+\sum_{j=1}^{q}\Big\{T^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}}\beta\nu_{\overline{q}\setminus\overline{j}}}\nabla_{\nu_j}(u)^\beta\Big\} \tag{1.2.241}$$

□

Corollary 4: From the property we can state trivially that:

$$L_u\{T\}=\nabla_u(T)-\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}+\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\} \tag{1.2.242}$$

Proof. the required result follows trivially from the coordinate expression found above □

Lemma 9: Given a tensor field $T\in\Gamma T^p_qM$, and an arbitrary vector field $u\in\Gamma TM$ and an arbitrary local frame (e_λ) defined on the open $U\subseteq M$. There always exists a bunch of local smooth scalar fields $f^{\lambda_{\overline{k}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\sigma_{\overline{p}}\nu_{\overline{q}}}:U\to M$ such that the following relation about the local expression is satified:

$$[L_{\lambda_{\overline{k}}}(\nabla_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\sum_{j=0}^{k+1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.243}$$

Proof. We can prove it via induction. Let us start with $k=1$. Therefore:

$$L_\lambda(\nabla_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=L_\lambda\Big(L_u(T)+\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}+ \tag{1.2.244}$$

$$-\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\}\Big)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.245}$$

$$=L_\lambda\Big(L_u(T)\Big)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+L_\lambda\Big(\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}+ \tag{1.2.246}$$

$$-\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\}\Big)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.247}$$

By previous lemma we know that there must exists a bunch of local smooth scalar fields denoted by $\hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}$ allowing us to rewrite the first term as:

$$[L_{\lambda_{\overline{k}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.248}$$

and recast the expression as follow:

$$L_\lambda(\nabla_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.249}$$

$$= \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} + \sum_{s=1}^{p} L_\lambda \Big\{ i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.250}$$

$$- \sum_{r=1}^{q} L_\lambda \Big\{ i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.251}$$

$$= \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} + \tag{1.2.252}$$

$$+ \sum_{s=1}^{p} \Big\{ i\Big[\sigma^{(1,s+1)}(L_\lambda[\nabla(u)]\otimes T + \nabla(u)\otimes L_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.253}$$

$$- \sum_{r=1}^{q} \Big\{ i\Big[\sigma_{(1,r+1)}(L_\lambda[\nabla(u)]\otimes T + \nabla(u)\otimes L_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.254}$$

Now one can notice that the operations inside the sums are $C^\infty(M)$-linear in the terms $L_\lambda T$ and T therefore, we can decompose them using the local frame obtaining:

$$L_\lambda(\nabla_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}() = \tag{1.2.255}$$

$$= \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} + \tag{1.2.256}$$

$$+ \sum_{s=1}^{p} \Big\{ i\Big[\sigma^{(1,s+1)}(L_\lambda[\nabla(u)]\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \sum_{s=1}^{p} \Big\{ i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes L_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.257}$$

$$- \sum_{r=1}^{q} \Big\{ i\Big[\sigma_{(1,r+1)}(L_\lambda[\nabla(u)]\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{r=1}^{q} \Big\{ i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes L_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.258}$$

$$= \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} + \tag{1.2.259}$$

$$+\sum_{s=1}^{p}\left\{i\Big[\sigma^{(1,s+1)}(L_\lambda[\nabla(u)]\otimes T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.260}$$

$$+\sum_{s=1}^{p}\left\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes L_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.261}$$

$$-\sum_{r=1}^{q}\left\{i\Big[\sigma_{(1,r+1)}(L_\lambda[\nabla(u)]\otimes T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.262}$$

$$-\sum_{r=1}^{q}\left\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes L_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.263}$$

$$=\sum_{j=0}^{k+1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}+ \tag{1.2.264}$$

$$+T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{s=1}^{p}\left\{i\Big[\sigma^{(1,s+1)}(L_\lambda[\nabla(u)]\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.265}$$

$$+L_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{s=1}^{p}\left\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.266}$$

$$-T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{r=1}^{q}\left\{i\Big[\sigma_{(1,r+1)}(L_\lambda[\nabla(u)]\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.267}$$

$$-L_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{r=1}^{q}\left\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\right\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.268}$$

Now it is enough to consider the action of the maps upon the local tensor frame $(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})$ and re-sum order by order in the Lie derivatives to define the local smooth scalar fields

$$L_\lambda(\nabla_v(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.269}$$

$$=\sum_{j=0}^{k+1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\hat{f}^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}+L_{\alpha_1}(L_{\alpha_2}(T))^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}g^{\alpha_1\alpha_2\;\mu_{\overline{p}}\rho_{\overline{q}}}_{\lambda\;\sigma_{\overline{p}}\nu_{\overline{q}}}+ \tag{1.2.270}$$

$$+L_{\alpha_1}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}g^{\alpha_1\;\mu_{\overline{p}}\rho_{\overline{q}}}_{\lambda\;\sigma_{\overline{p}}\nu_{\overline{q}}}+(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}g^{\mu_{\overline{p}}\rho_{\overline{q}}}_{\lambda\;\sigma_{\overline{p}}\nu_{\overline{q}}}= \tag{1.2.271}$$

$$=\sum_{j=0}^{k+1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.272}$$

Now we can use the inductive step. Let us suppose that the thesis holds for the case k and let us prove that it still holds for $k+1$.

$$[L_{\lambda_{\overline{k+1}}}(\nabla_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=L_{\lambda_1}[L_{\lambda_{\overline{k+1}\setminus\overline{1}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.273}$$

$$=L_{\lambda_1}\Big[\sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}} e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.274}$$

$$=\sum_{j=0}^{k+1}\Big[L_{\lambda_1}L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}} + L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}\Big[L_{\lambda_1}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})\Big]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.275}$$

At this point it is enough to re-sum order by order, defining a new bunch of local smooth scalar fields $h^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ as appropriate linear combinations of $f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ and Lie derivatives to obtain:

$$[L_{\lambda_{\overline{k+1}}}(\nabla_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\sum_{j=0}^{(k+1)+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.276}$$

□

Lemma 10: Given a tensor field $T\in\Gamma T^p_q M$, and an arbitrary vector field $u\in\Gamma TM$ and an arbitrary local frame (e_λ) defined on the open $U\subseteq M$. There always exists a bunch of local smooth scalar fields $f^{\lambda_{\overline{k}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\sigma_{\overline{p}}\nu_{\overline{q}}}: U\to M$ such that the following relation about the local expression is satisfied:

$$[\nabla^k_{\lambda_{\overline{k}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.277}$$

Proof. We can prove it via induction. Let us start with $k=1$. Therefore:

$$\nabla_\lambda(L_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\nabla_\lambda\Big(\nabla_u(T)-\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}+ \tag{1.2.278}$$

$$+\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\}\Big)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.279}$$

$$=\nabla_\lambda\Big(\nabla_u(T)\Big)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+\nabla_\lambda\Big(\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}+ \tag{1.2.280}$$

$$-\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\}\Big)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.281}$$

By the previous lemma we know we can write

$$[\nabla_{\lambda_{\overline{k}}}(\nabla_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k+1} \nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,k)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.282}$$

where $Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,k)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}}$ are the local expression of $C^\infty(M)$ linear maps acting on the tensor fields $\nabla^j(T)$. Then recasting the expression as follow:

$$\nabla_\lambda(L_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.283}$$
$$= \sum_{j=0}^{1+1} \nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,1)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}} - \sum_{s=1}^{p} \nabla_\lambda \Big\{ i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.284}$$
$$+ \sum_{r=1}^{q} \nabla_\lambda \Big\{ i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.285}$$
$$= \sum_{j=0}^{1+1} \nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,1)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}} + \tag{1.2.286}$$
$$- \sum_{s=1}^{p} \Big\{ i\Big[\sigma^{(1,s+1)}(\nabla_\lambda[\nabla(u)]\otimes T + \nabla(u)\otimes\nabla_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.287}$$
$$+ \sum_{r=1}^{q} \Big\{ i\Big[\sigma_{(1,r+1)}(\nabla_\lambda[\nabla(u)]\otimes T + \nabla(u)\otimes\nabla_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.288}$$

Now one can notice that the operations inside the sums are $C^\infty(M)$-linear in the terms $\nabla_\lambda T$ and T therefore, we can decompose them using the local frame obtaining:

$$\nabla_\lambda(L_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}() = \tag{1.2.289}$$
$$= \sum_{j=0}^{1+1} \nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,1)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}} + \tag{1.2.290}$$
$$- \sum_{s=1}^{p} \Big\{ i\Big[\sigma^{(1,s+1)}(\nabla_\lambda[\nabla(u)]\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p} \Big\{ i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes\nabla_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \tag{1.2.291}$$
$$+ \sum_{r=1}^{q} \Big\{ i\Big[\sigma_{(1,r+1)}(\nabla_\lambda[\nabla(u)]\otimes T)\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \sum_{r=1}^{q} \Big\{ i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes\nabla_\lambda[T])\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{1.2.292}$$
$$= \sum_{j=0}^{1+1} \nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,1)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}} + \tag{1.2.293}$$

$$-\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla_\lambda[\nabla(u)]\otimes T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.294}$$

$$-\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes\nabla_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.295}$$

$$+\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla_\lambda[\nabla(u)]\otimes T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.296}$$

$$+\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes\nabla_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.297}$$

$$=\sum_{j=0}^{1+1}\nabla^{j}_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}Z(u)^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(j,1)\lambda\sigma_{\overline{p}}\nu_{\overline{q}}}+ \tag{1.2.298}$$

$$-T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla_\lambda[\nabla(u)]\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.299}$$

$$-\nabla_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{s=1}^{p}\Big\{i\Big[\sigma^{(1,s+1)}(\nabla(u)\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.300}$$

$$+T^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla_\lambda[\nabla(u)]\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.301}$$

$$+\nabla_\lambda[T]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\sum_{r=1}^{q}\Big\{i\Big[\sigma_{(1,r+1)}(\nabla(u)\otimes e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})\Big]\Big\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.302}$$

Now it is enough to consider the action of the maps upon the local tensor frame $(e_{\sigma_{\overline{p}}}\otimes e^{\rho_{\overline{q}}})$ and re-sum order by order in the covariant derivatives to define the local smooth scalar fields

$$\nabla_\lambda(L_u(T))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\sum_{j=0}^{1+1}\nabla_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.303}$$

Now we can use the inductive step. Let us prove that it still holds for $k+1$.

$$[\nabla^{k+1}_{\lambda_{\overline{k+1}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}=\nabla_{\lambda_1}[\nabla^{k}_{\lambda_{\overline{k+1}\setminus\overline{1}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}-\sum_{l=2}^{l=k+1}\{\nabla^{k}_{\lambda_{\overline{l-1}\setminus\overline{1}}\nabla_{\lambda_1}(e_{\lambda_l})\lambda_{\overline{k+1}\setminus l}}(L_u(T))\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.304}$$

$$=\nabla_{\lambda_1}[\nabla^{k}_{\lambda_{\overline{k+1}\setminus\overline{1}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}-\sum_{l=2}^{l=k+1}\nabla_{\lambda_1}(e_{\lambda_l})^{\beta}\{\nabla^{k}_{\lambda_{\overline{l-1}\setminus\overline{1}}\beta\lambda_{\overline{k+1}\setminus l}}(L_u(T))\}^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}= \tag{1.2.305}$$

$$=\nabla_{\lambda_1}[\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.306}$$

$$-\sum_{l=2}^{l=k+1}\nabla_{\lambda_1}(e_{\lambda_l})^{\beta}\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{l-1}\setminus\overline{1}}\beta\lambda_{\overline{k+1}\setminus l}\sigma_{\overline{p}}\nu_{\overline{q}}}= \tag{1.2.307}$$

$$=\sum_{j=0}^{k+1}\nabla_{\lambda_1}[\nabla^j_{\alpha_{\overline{j}}}(T)]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.308}$$

$$+\nabla_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\nabla_{\lambda_1}[f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.309}$$

$$-\sum_{l=2}^{l=k+1}\nabla_{\lambda_1}(e_{\lambda_l})^{\beta}\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{l-1}\setminus\overline{1}}\beta\lambda_{\overline{k+1}\setminus l}\sigma_{\overline{p}}\nu_{\overline{q}}}= \tag{1.2.310}$$

$$=\sum_{j=0}^{k+1}\nabla_{\lambda_1}[\nabla^j_{\alpha_{\overline{j}}}(T)]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.311}$$

$$+\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\nabla_{\lambda_1}[f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.312}$$

$$-\sum_{l=2}^{l=k+1}\nabla_{\lambda_1}(e_{\lambda_l})^{\beta}\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{l-1}\setminus\overline{1}}\beta\lambda_{\overline{k+1}\setminus l}\sigma_{\overline{p}}\nu_{\overline{q}}}= \tag{1.2.313}$$

$$=\sum_{j=0}^{k+1}\left[\nabla^{j+1}_{\lambda_1\alpha_{\overline{j}}}(T)+\sum_{l=2}^{j=1}\nabla_{\lambda_1}(e_{\alpha_l})^{\beta}\nabla^j_{\alpha_{\overline{l}\setminus 1}\beta\lambda_{\overline{j+1}\setminus\overline{l}}}(T)\right]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.314}$$

$$+\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\nabla_{\lambda_1}[f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.315}$$

$$-\sum_{l=2}^{l=k+1}\nabla_{\lambda_1}(e_{\lambda_l})^{\beta}\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{l-1}\setminus\overline{1}}\beta\lambda_{\overline{k+1}\setminus l}\sigma_{\overline{p}}\nu_{\overline{q}}}= \tag{1.2.316}$$

$$=\sum_{j=0}^{k+1}\nabla^{j+1}_{\lambda_1\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}+ \tag{1.2.317}$$

$$+\sum_{j=0}^{k+1}\left[\sum_{l=2}^{j=1}\nabla_{\lambda_1}(e_{\alpha_l})^{\beta}\nabla^j_{\alpha_{\overline{l}\setminus 1}\beta\lambda_{\overline{j+1}\setminus\overline{l}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\right]f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.318}$$

$$+\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\nabla_{\lambda_1}[f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}(e_{\beta_{\overline{p}}}\otimes e^{\gamma_{\overline{q}}})]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}+ \tag{1.2.319}$$

$$-\sum_{l=2}^{l=k+1}\nabla_{\lambda_1}(e_{\lambda_l})^{\beta}\sum_{j=0}^{k+1}\nabla^j_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{l-1}\setminus\overline{1}}\beta\lambda_{\overline{k+1}\setminus l}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.320}$$

The reader can easily notice that the expression is completely $C^\infty(M)$- linear in all the terms like $\nabla^j(T)$, therefore it is possible to resum order by order, defining a new bunch of local smooth scalar fields $h^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ as appropriate linear combinations of $f^{\alpha_{\overline{j}}\rho_{\overline{q}}\beta_{\overline{p}}}_{\lambda_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\gamma_{\overline{q}}}$ and covariant derivatives to obtain:

$$[\nabla^{k+1}_{\lambda_{\overline{k+1}}}(L_u(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k+1}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{1.2.321}$$

Therefore we have the thesis. □

1.2.5 Torsion and Curvature

Given the definition of higher order covariant differential, it is possible to recast the concepts of Torsion tensor field and the Riemann Curvature tensor field, accordingly to our intrinsic operatorial language. This will be very useful to split the action of the higher order covariant differential, taking account the symmetric and anti-symmetric parts.

Definition 26: Given the covariant differential operator ∇ we can define the ***torsion of*** ∇ the $C^\infty(M)$-multilinear map $Tor : \Gamma TM \times \Gamma TM \to \Gamma TM$ such that $\forall v, u \in \Gamma TM$:

$$Tor(u, v) = v \lrcorner u \lrcorner Tor = \nabla_u(v) - \nabla_v(u) - L_u(v) = \nabla_u(v) - \nabla_v(u) - [u, v] \tag{1.2.322}$$

Property 13: The local coordinate expression for the torsion is just:

$$Tor^\mu(e_\nu, e_\lambda)e_\mu = \nabla_{e_\nu}(e_\lambda) - \nabla_{e_\lambda}(e_\nu) - [e_\lambda, e_\nu]^\mu = \Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\lambda\nu} - [e_\lambda, e_\nu]^\mu \tag{1.2.323}$$

If a natural trivialization is chosen then:

$$Tor^\mu(e_\nu, e_\lambda)e_\mu = \Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\lambda\nu} \tag{1.2.324}$$

Definition 27: Given the covariant differential operator ∇ we can define the ***curvature of*** ∇ or equivalently ***the Riemann tensor field*** the $C^\infty(M)$-multilinear map $R : \Gamma TM \times \Gamma TM \times \Gamma TM \to \Gamma TM$ such that $\forall v, u, w \in \Gamma TM$:

$$R(u, v, w) = \nabla_u(\nabla_v(w)) - \nabla_v(\nabla_u(w)) - \nabla_{L_u(v)}(w) = \tag{1.2.325}$$

$$= \nabla_u(\nabla_v(w)) - \nabla_v(\nabla_u(w)) - \nabla_{[u,v]}(w) \tag{1.2.326}$$

Property 14: Given the covariant differential operator ∇, the curvature operator can be defined equivalently by:

$$R(u, v, w) = 2\nabla^2_{[uv]}(w) + Tor(u, v)\lrcorner\nabla(w) \quad , \quad \forall u, v, w \in \Gamma TM \tag{1.2.327}$$

Proof.

$$\begin{aligned}
R(u, v, w) &= \nabla_u(\nabla_v(w)) - \nabla_v(\nabla_u(w)) - \nabla_{[u,v]}(w) = && (1.2.328)\\
=&\nabla^2_{uv}(w) + \nabla_{\nabla_u(v)}(w) - \nabla^2_{vu}(w) - \nabla_{\nabla_v(u)}(w) - \nabla_{[u,v]}(w) = && (1.2.329)\\
=&\nabla^2_{uv}(w) - \nabla^2_{uv}(w) + \nabla_{\nabla_u(v)-\nabla_v(u)-[u,v]}(w) = && (1.2.330)\\
=&R(u, v, w) = 2\nabla^2_{[uv]}(w) + Tor(u, v)\lrcorner\nabla(w) && (1.2.331)
\end{aligned}$$

$\square$

Property 15: The local coordinate expression for the curvature is just:

$$\begin{aligned}
R^\mu(e_\nu, e_\lambda, e_\rho)e_\mu &= \nabla_{e_\nu}(\nabla_{e_\lambda}(e_\rho)) - \nabla_{e_\lambda}(\nabla_{e_\nu}(e_\rho)) - \nabla_{[e_\lambda,e_\nu]}(e_\rho) = && (1.2.332)\\
=&\nabla_{e_\nu}(\Gamma^\alpha_{\lambda\rho}e_\alpha) - \nabla_{e_\lambda}(\Gamma^\alpha_{\nu\rho}e_\alpha) - \nabla_{[e_\lambda,e_\nu]}(e_\rho) = && (1.2.333)\\
=&e_\nu(\Gamma^\mu_{\lambda\rho})e_\mu + \Gamma^\alpha_{\lambda\rho}\nabla_{e_\nu}(e_\alpha) - e_\lambda(\Gamma^\mu_{\nu\rho})e_\mu - \Gamma^\alpha_{\nu\rho}\nabla_{e_\lambda}(e_\alpha) - \nabla_{[e_\lambda,e_\nu]}(e_\rho) = && (1.2.334)\\
=&e_\nu(\Gamma^\mu_{\lambda\rho})e_\mu + \Gamma^\alpha_{\lambda\rho}\Gamma^\mu_{\nu\alpha}e_\mu - e_\lambda(\Gamma^\mu_{\nu\rho})e_\mu - \Gamma^\alpha_{\nu\rho}\Gamma^\mu_{\lambda\alpha}e_\mu - \nabla_{[e_\lambda,e_\nu]}(e_\rho) && (1.2.335)
\end{aligned}$$

If a natural trivialization is chosen then:

$$R^\mu(e_\nu, e_\lambda, e_\rho) = \partial_\nu(\Gamma^\mu_{\lambda\rho}) - \partial_\lambda(\Gamma^\mu_{\nu\rho}) - \Gamma^\alpha_{\nu\rho}\Gamma^\mu_{\lambda\alpha} + \Gamma^\alpha_{\lambda\rho}\Gamma^\mu_{\nu\alpha} \tag{1.2.336}$$

Property 16: Given $\alpha \in \Gamma T^\star M$, the following holds:

$$\nabla^2_{[uv]}(f) = -\frac{1}{2}Tor(u, v)\lrcorner\nabla(f) \tag{1.2.337}$$

Proof.

$$0 = \nabla_u(\nabla_v(f)) - \nabla_v(\nabla_u(f)) - \nabla_{[u,v]}(f) = \tag{1.2.338}$$

$$
\begin{aligned}
=&\nabla_u(v\lrcorner\nabla(f)) - \nabla_v(u\lrcorner\nabla(f)) - [u,v]\lrcorner\nabla(f) = && (1.2.339)\\
=&\nabla_u(v)\lrcorner\nabla(f) + v\lrcorner\nabla_u(\nabla(f)) - \nabla_v(u)\lrcorner\nabla(f) - u\lrcorner\nabla_v(\nabla(f)) - [u,v]\lrcorner\nabla(f) = && (1.2.340)\\
=&Tor(u,v)\lrcorner\nabla(f) + u\lrcorner v\lrcorner\nabla(\nabla(f)) - v\lrcorner u\lrcorner\nabla(\nabla(f)) = && (1.2.341)\\
=&Tor(u,v)\lrcorner\nabla(f) + 2\nabla^2_{[uv]}(f) && (1.2.342)
\end{aligned}
$$

□

Property 17: Given $\alpha \in \Gamma T^\star M$, the following holds:

$$
-\alpha(R(u,v,w)) = \nabla_u(\nabla_v(\alpha))(w) - \nabla_v(\nabla_u(\alpha))(w) - \nabla_{[u,v]}(\alpha)(w) \tag{1.2.343}
$$

Proof. Considering that the covariant derivative must satisfy the Leibniz rule with respect to the contraction we can state:

$$
\begin{aligned}
0 =& \nabla_u(\nabla_v(\alpha(w))) - \nabla_v(\nabla_u(\alpha(w))) - \nabla_{[u,v]}(\alpha(w)) = && (1.2.344)\\
=&\nabla_u(\nabla_v(\alpha)(w) + \alpha(\nabla_v(w))) - \nabla_v(\nabla_u(\alpha)(w) + \alpha(\nabla_u(w))) - \nabla_{[u,v]}(\alpha(w)) = && (1.2.345)\\
=&[\nabla_u\nabla_v(\alpha)](w) + [\nabla_v(\alpha)](\nabla_u(w)) + [\nabla_u(\alpha)](\nabla_v(w))) + [\nabla_v\nabla_u(\alpha)](w) + && (1.2.346)\\
-&[\nabla_v\nabla_u(\alpha)](w) - [\nabla_u(\alpha)](\nabla_v(w)) - [\nabla_v(\alpha)](\nabla_u(w))) - [\nabla_u\nabla_v(\alpha)](w) + && (1.2.347)\\
-&[\nabla_{[u,v]}(\alpha)](w)) - \alpha(\nabla_{[u,v]}(w)) = -\alpha(\nabla_u\nabla_v(w) - \nabla_v\nabla_u(w) - \nabla_{[u,v]}(w)) + && (1.2.348)\\
+&[\nabla_u\nabla_v(\alpha)](w) - [\nabla_v\nabla_u(\alpha)](w) - [\nabla_{[u,v]}(\alpha)](w) = && (1.2.349)\\
=&\alpha(R(u,v,w)) + [\nabla_u\nabla_v(\alpha)](w) - [\nabla_v\nabla_u(\alpha)](w) - [\nabla_{[u,v]}(\alpha)](w) && (1.2.350)
\end{aligned}
$$

□

Definition 28: Given the covariant differential operator ∇ we can define a $C^\infty(M)$-linear map $P : \Gamma T^p_q M \to \Gamma T^p_{[2]+q} M$ anti-symmetric in the first two arguments: $u\lrcorner v\lrcorner P(T) = -v\lrcorner u\lrcorner P(T)$ such that the local coordinate expression is :

$$
[v\lrcorner u\lrcorner P(T)]^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} = u^\gamma v^\delta P(T)_{\gamma\,\delta\,\beta_{\overline{q}}}^{\alpha_{\overline{p}}} = u^\gamma v^\delta \Big\{ \sum_{i=1}^{p} R^{\alpha_i}_{\gamma\,\delta\,\mu} T^{\alpha_{\overline{i-1}}\,\mu\,\alpha_{\overline{p}\setminus\overline{i}}}_{\beta_{\overline{q}}} - \sum_{i=1}^{q} R^{\mu}_{\gamma\,\delta\,\beta_i} T^{\alpha_{\overline{p}}}_{\beta_{\overline{i-1}}\,\mu\,\beta_{\overline{q}\setminus\overline{i}}} \Big\} \tag{1.2.351}
$$

Definition 29: Given Tor the torsion of ∇, we can define a $C^\infty(M)$ linear map $Q : \Gamma T^p_{q+1} M \to \Gamma T^p_{[2]+q-1} M$ antisymmetric in the first two arguments: $u\lrcorner v\lrcorner Q(T) = -v\lrcorner u\lrcorner Q(T)$ such that the local coordinate expression is:

$$v \lrcorner u \lrcorner Q(T)^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} = u^{\gamma} v^{\delta} Q(T)_{\gamma\,\delta\,\beta_{\overline{q}}}{}^{\alpha_{\overline{p}}} = u^{\delta} v^{\varepsilon} Tor^{\gamma}_{\delta\varepsilon} T^{\alpha_{\overline{p}}}_{\gamma\,\beta_{\overline{q}}} \tag{1.2.352}$$

Lemma 11: The linear map P satisfies a generalised Leibniz rule with respect to the tensor product of tensor fields:

$$P(T \otimes S) = P(T) \otimes S + (\sigma_{\overline{q+2}})^2 [T \otimes P(S)] \tag{1.2.353}$$

Proof. It is quite trivial if we fix a local trivialisation:

$$P(T \otimes S)_{\gamma\,\delta\,\beta_{\overline{q+q'}}}{}^{a_{\overline{p+p'}}} = \tag{1.2.354}$$

$$= \sum_{i=1}^{p+p'} R^{\alpha_i}_{\gamma\,\delta\,\mu} (T \otimes S)^{\alpha_{\overline{i-1}}\,\mu\,\alpha_{\overline{p+p'}\setminus\overline{i}}}_{\beta_{\overline{q+q'}}} - \sum_{i=1}^{q+q'} R^{\mu}_{\gamma\,\delta\,\beta_i} (T \otimes S)^{\alpha_{\overline{p+p'}}}_{\beta_{\overline{i-1}}\,\mu\,\beta_{\overline{q+q'}\setminus\overline{i}}} = \tag{1.2.355}$$

$$= \sum_{i=1}^{p} R^{\alpha_i}_{\gamma\,\delta\,\mu} T^{\alpha_{\overline{i-1}}\,\mu\,\alpha_{\overline{p}\setminus\overline{i}}}_{\beta_{\overline{q}}} S^{\alpha_{\overline{p'}\setminus\overline{p}}}_{\beta_{\overline{q'}\setminus\overline{q}}} - \sum_{i=1}^{q} R^{\mu}_{\gamma\,\delta\,\beta_i} T^{\alpha_{\overline{p}}}_{\beta_{\overline{i-1}}\,\mu\,\beta_{\overline{q}\setminus\overline{i}}} S^{\alpha_{\overline{p'}\setminus\overline{p}}}_{\beta_{\overline{q'}\setminus\overline{q}}} + \tag{1.2.356}$$

$$+ \sum_{i=p+1}^{p'} R^{\alpha_i}_{\gamma\,\delta\,\mu} S^{\alpha_{\overline{i-1}\setminus\overline{p}}\,\mu\,\alpha_{\overline{p'}\setminus\overline{i}}}_{\alpha_{\overline{q'}\setminus\overline{q}}} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} - \sum_{i=q+1}^{q'} R^{\mu}_{\gamma\,\delta\,\beta_i} S^{\alpha_{\overline{p'}\setminus\overline{p}}}_{\beta_{\overline{i-1}\setminus\overline{p}}\,\mu\,\beta_{\overline{q'}\setminus\overline{i}}} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} = \tag{1.2.357}$$

$$= P(T)_{\gamma\,\delta\,\beta_{\overline{q}}}{}^{\alpha_{\overline{p}}} S^{\alpha_{\overline{p'}\setminus\overline{p}}}_{\beta_{\overline{q'}\setminus\overline{q}}} + T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} P(S)_{\gamma\,\delta\,\beta_{\overline{q'}\setminus\overline{q}}}{}^{\alpha_{\overline{p'}\setminus\overline{p}}} = \tag{1.2.358}$$

$$= \big\{ P(T) \otimes S + (\sigma_{\overline{q+2}})^2 [T \otimes P(S)] \big\}_{\gamma\,\delta\,\beta_{\overline{q+q'}}}{}^{\alpha_{\overline{p+p'}}} \tag{1.2.359}$$

□

Property 18: By fixing a local trivialisation, one can easily check that given $u, v, w \in \Gamma TM$ and $\alpha \in \Gamma T^{\star}M$ the following holds:

$$v \lrcorner u \lrcorner P(w) = R(v, u, w) \tag{1.2.360}$$

$$\{ v \lrcorner u \lrcorner P(\alpha) \}(w) = -\alpha(R(v, u, w)) \tag{1.2.361}$$

Since the scalar fields can be interpreted as rank $0,0$ tensor fields it is natural to investigate what happens to P when applied to the tensor fields.

Lemma 12: Let $f \in C^\infty(M)$ a smooth scalar field. We have that:

$$P(f) = 0 \tag{1.2.362}$$

Proof. P is $C^\infty(M)$-linear hence:

$$P(fu) = fP(u) \quad \forall u \in \Gamma TM \tag{1.2.363}$$

but to satifies Leibniz we need:

$$P(fu) = P(f \otimes u) = P(f) \otimes u + f \otimes P(u) = P(f) \otimes u + f(P(u)) \quad \forall u \in \Gamma TM \tag{1.2.364}$$

therefore we must conclude the thesis. □

Lemma 13: The action of the anti-symmetrised second covariant differential $\nabla^2_{[]}$ on a generic tensor field $T \in \Gamma T^p_q M$ can be written in terms of the maps P and Q defined above:

$$\nabla^2_{[]}(T) = \frac{1}{2}P(T) - \frac{1}{2}Q(\nabla T) \tag{1.2.365}$$

Proof. Let us consider the scalar field first.

$$\nabla^2_{[uv]}(f) = 0 - \frac{1}{2}Tor(u,v) \lrcorner \nabla(f) = v \lrcorner u \lrcorner \{\frac{1}{2}P(f) - \frac{1}{2}Q(\nabla(f))\} \tag{1.2.366}$$

The remaining part of the proof can be performed by induction on the rank of the tensor.

1. Let us suppose that $T \in TM$, therefore:

$$R(u,v,w) = \nabla_u(\nabla_v(T)) - \nabla_v(\nabla_u(T)) - \nabla_{[u,v]}(T) = \tag{1.2.367}$$

$$=\nabla^2_{uv}(T) + \nabla_{\nabla_u(v)}(T) - \nabla^2_{uv}(T) - \nabla_{\nabla_v(u)}(T) - \nabla_{[u,v]}(T) = \tag{1.2.368}$$
$$=\nabla^2_{uv}(T) - \nabla^2_{uv}(T) + \nabla_{\nabla_u(v)-\nabla_v(u)-[u,v]}(T) = \tag{1.2.369}$$
$$=R(u,v,T) = 2\nabla^2_{[uv]}(T) + Tor(u,v)\lrcorner\nabla(T) \tag{1.2.370}$$

and we can conclude that:

$$\nabla^2_{[uv]}(T) = \frac{1}{2}R(u,v,T) - \frac{1}{2}Tor(u,v)\lrcorner\nabla(T) = \tag{1.2.371}$$
$$=v\lrcorner u\lrcorner\{\frac{1}{2}P(T) - \frac{1}{2}Q(\nabla(T))\} \tag{1.2.372}$$

Let us suppose now that $T \in T^\star M$. By the previous lemma we know that:

$$u\lrcorner v\lrcorner P(T) = -T(R(u,v,w)) = \tag{1.2.373}$$
$$=\nabla_u(\nabla_v(T))(w) - \nabla_v(\nabla_u(T))(w) - \nabla_{[u,v]}(T)(w) \tag{1.2.374}$$

Using it in the explicit expression of $\nabla^2_{[]}$ we can state:

$$2\{\nabla^2_{[uv]}(T)\}(w) = \tag{1.2.375}$$
$$=\{\nabla_u(\nabla_v(T)) - \nabla_{\nabla_u(v)}(T) - \nabla_v(\nabla_u(T)) - \nabla_{\nabla_v(u)}(T)\}(w) = \tag{1.2.376}$$
$$=\{\nabla_u(\nabla_v(T)) - \nabla_{\nabla_u(v)}(T) - \nabla_v(\nabla_u(T)) - \nabla_{\nabla_v(u)}(T)\}(w) + \tag{1.2.377}$$
$$\pm\nabla_{[u,v]}(T)(w) = -T(R(u,v,w)) - Tor(u,w)\lrcorner\nabla(T) = \tag{1.2.378}$$
$$=v\lrcorner u\lrcorner\{P(T) - Q(\nabla(T))\} \tag{1.2.379}$$

2. Now let us assume the thesis holds for a generic tensor $S \in \Gamma T^p_q M$, we can prove that it must be true also for $T \in \Gamma T^{p+1}_q$ or equivalently for $T \in \Gamma T^p_{q+1}$. Let us consider $\hat{T} \in \Gamma T^{p+1}_q M$ and let us fix a local frame $\{e_a\}$:

$$2\nabla^2_{[u_1u_2]}(T) = 2\, u_2\lrcorner u_1\lrcorner\nabla^2_{[]}(T) = 2\nabla^2_{[u_1u_2]}(e_a \otimes T^a) \tag{1.2.380}$$

where T^a is an n-tuple of tensors belonging to $\Gamma T^p_q M$. So it's enough to use the definition of ∇^2 and the Leibniz rule:

$$\nabla_{uv}(e_a \otimes T^a) = \nabla_u(\nabla_v(e_a \otimes T^a)) - \nabla_{\nabla_u(v)}(e_a \otimes T^a) = \tag{1.2.381}$$

$$
\begin{aligned}
=&\nabla_u(\nabla_v(e_a)) \otimes T^a + e_a \otimes \nabla_u(\nabla_v(T^a)) + \nabla_u(e_a) \otimes \nabla_v(T^a) + && (1.2.382)\\
&+\nabla_v(e_a) \otimes \nabla_u(T^a) - \nabla_{\nabla_u(v)}(e_a) \otimes T^a - e_a \otimes \nabla_{\nabla_u(v)}(T^a) && (1.2.383)
\end{aligned}
$$

So the anti-symmetrized part is:

$$
\begin{aligned}
&\nabla_{[u}(\nabla_{v]}(e_a)) \otimes T^a + e_a \otimes \nabla_{[u}(\nabla_{v]}(T^a)) + && (1.2.384)\\
&+\nabla_{[u}(e_a) \otimes \nabla_{v]}(T^a) + \nabla_{[v}(e_a) \otimes \nabla_{u]}(T^a) + && (1.2.385)\\
&-\nabla_{\nabla_u(v)}(e_a) \otimes T^a - e_a \otimes \nabla_{\nabla_u(v)}(T^a) + && (1.2.386)\\
&+\nabla_{\nabla_u(v)}(e_a) \otimes T^a - e_a \otimes \nabla_{\nabla_u(v)}(T^a) = && (1.2.387)\\
&=\nabla_{[u}(\nabla_{v]}(e_a)) \otimes T^a - \nabla_{\nabla_u(v)}(e_a) \otimes T^a + && (1.2.388)\\
&+\nabla_{\nabla_u(v)}(e_a) \otimes T^a + e_a \otimes \nabla_{[u}(\nabla_{v]}(T^a)) = && (1.2.389)\\
&=\frac{1}{2} e_a \otimes [v \lrcorner u \lrcorner P(T^a)] + \frac{1}{2}[v \lrcorner u \lrcorner P(e_a)] \otimes T^a + && (1.2.390)\\
&-\frac{1}{2} v \lrcorner u \lrcorner Q(\nabla(e_a)) \otimes T^a - \frac{1}{2} e_a \otimes [v \lrcorner u \lrcorner Q(\nabla(T^a))] = && (1.2.391)\\
&=\frac{1}{2} v \lrcorner u \lrcorner [P(T) - Q(\nabla T)] && (1.2.392)
\end{aligned}
$$

The proof for $T \in \Gamma T^p_{q+1} M$ follows in the same way.

□

Lemma 14: The following holds:

$$
[\nabla^n(P)](f) = 0 \quad , \quad \forall f \in C^\infty(M), \forall n \in \mathbb{N}^+ \tag{1.2.393}
$$

Proof. The proof can be performed by induction. The step 1 is trivial because considering the properties of the covariant derivative, the covariant differential of the null tensor field is null. So if $P(f) = 0 \Rightarrow \nabla[P(f)] = 0$. This means that $\forall v \in \Gamma TM$ we have that

$$
0 = \nabla_v[P(f)] = \nabla_v(P)(f) + P(\nabla_v(f)) = \nabla_v(P)(f) + 0 \quad \forall v \in \Gamma TM \tag{1.2.394}
$$

So we end up with:

$$
\forall v \in \Gamma TM, \quad \nabla_v(P)(f) = 0 \Rightarrow \nabla(P)(f) = 0 \tag{1.2.395}
$$

If we suppose that the step n is satisfied, it is trivial to prove that also the n+1 step holds in the same way as we did in the step one , so the thesis follows. □

Lemma 15: Let $L : \Gamma T^p_q M \to \Gamma T^{p'}_{q'} M$ be a C^∞ linear map then $\forall k \in \mathbb{N}^+$ the following holds:

$$\nabla^k\{P\}[L(T)] = \{\nabla^k(P)[L]\}](T) - \{[\mathbb{I}\otimes]^{k+2}L\}[\nabla^k(P)[T]] \tag{1.2.396}$$

Proof. The proof can be easily performed locally by fixing a local frame. As usual as proven in the first chapter we can glue together the local sections to prove the relation still holds for global sections.

$$\{\nabla^k(P)\}[L(T)]_{\gamma_{\overline{k}}\,\delta\,\varepsilon\,\beta_{\overline{q}}}{}^{\alpha_{\overline{p}}} = \tag{1.2.397}$$

$$= \sum_{i=1}^{p} \nabla^k_{\gamma_{\overline{k}}}(R)^{\alpha_i}_{\delta\,\varepsilon\,\sigma}[L(T)]^{\alpha_{\overline{i-1}}\,\sigma\,\alpha_{\overline{p}\setminus\overline{i}}}_{\beta_{\overline{q}}} - \sum_{i=1}^{q} \nabla^k_{\gamma_{\overline{k}}}(R)^{\sigma}_{\delta\,\varepsilon\,\beta_i}[L(T)]^{\alpha_{\overline{p}}}_{\beta_{\overline{i-1}}\,\sigma\,\beta_{\overline{q}\setminus\overline{i}}} = \tag{1.2.398}$$

$$= \sum_{i=1}^{p} \nabla^k_{\gamma_{\overline{k}}}(R)^{\alpha_i}_{\delta\,\varepsilon\,\sigma}[L^{\alpha_{\overline{i-1}}\,\sigma\,\alpha_{\overline{p}\setminus\overline{i}}\tau_{\overline{m}}}_{\beta_{\overline{q}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}}] - \sum_{i=1}^{q} \nabla^k_{\gamma_{\overline{k}}}(R)^{\sigma}_{\delta\,\varepsilon\,\beta_i}[L^{\alpha_{\overline{p}}\,\tau_{\overline{m}}}_{\beta_{\overline{i-1}}\,\sigma\,\beta_{\overline{q}\setminus\overline{i}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}}] = \tag{1.2.399}$$

$$= \sum_{i=1}^{p} \nabla^k_{\gamma_{\overline{k}}}(R)^{\alpha_i}_{\delta\,\varepsilon\,\sigma} L^{\alpha_{\overline{i-1}}\,\sigma\,\alpha_{\overline{p}\setminus\overline{i}}\tau_{\overline{m}}}_{\beta_{\overline{q}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}} - \sum_{i=1}^{q} \nabla^k_{\gamma_{\overline{k}}}(R)^{\sigma}_{\delta\,\varepsilon\,\beta_i} L^{\alpha_{\overline{p}}\,\tau_{\overline{m}}}_{\beta_{\overline{i-1}}\,\sigma\,\beta_{\overline{q}\setminus\overline{i}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}} + \tag{1.2.400}$$

$$\pm \sum_{i=1}^{m} \nabla^k_{\gamma_{\overline{k}}}(R)^{\tau_i}_{\delta\,\varepsilon\,\sigma} L^{\alpha_{\overline{p}}\tau_{\overline{i-1}}\,\sigma\,\tau_{\overline{m}\setminus\overline{i}}}_{\beta_{\overline{q}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{t_{\overline{m}}} \mp \sum_{i=1}^{l} \nabla^k_{\gamma_{\overline{k}}}(R)^{\sigma}_{\delta\,\varepsilon\,\upsilon_i}[L]^{\alpha_{\overline{p}}\,\tau_{\overline{m}}}_{\beta_{\overline{q}}\upsilon_{\overline{i-1}}\,\sigma\,\upsilon_{\overline{l}\setminus\overline{i}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}} = \tag{1.2.401}$$

$$= [\nabla^k(P)(L)](T)_{\gamma_{\overline{k}}\,\delta\,\varepsilon\,\beta_{\overline{q}}}{}^{\alpha_{\overline{p}}} + \tag{1.2.402}$$

$$- \sum_{i=1}^{m} \nabla_{\gamma_{\overline{k}}}(R)^{\tau_i}_{\delta\,\varepsilon\,\sigma} L^{\alpha_{\overline{p}}\tau_{\overline{i-1}}\,\sigma\,\tau_{\overline{m}\setminus\overline{i}}}_{\beta_{\overline{q}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}} + \sum_{i=1}^{l} \nabla_{\gamma_{\overline{k}}}(R)^{\sigma}_{\delta\,\varepsilon\,\upsilon_i}[L]^{\alpha_{\overline{p}}\,\tau_{\overline{m}}}_{\beta_{\overline{q}}\upsilon_{\overline{i-1}}\,\sigma\,\upsilon_{\overline{l}\setminus\overline{i}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m}}} = \tag{1.2.403}$$

$$= [\nabla^k\{P\}(L)](T)_{\gamma_{\overline{k}}\,\delta\,\varepsilon\,\beta_{\overline{q}}}{}^{\alpha_{\overline{p}}} + \tag{1.2.404}$$

$$- \sum_{i=1}^{m} \nabla_{\gamma_{\overline{k}}}(R)^{\sigma}_{\delta\,\varepsilon\,\tau_i} L^{a_{\overline{p}}\tau_{\overline{m}}}_{\beta_{\overline{q}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l}}}_{\tau_{\overline{m-i}}\,\sigma\,\tau_{\overline{m}\setminus i}} + \sum_{i=1}^{l} \nabla_{\gamma_{\overline{k}}}(R)^{\upsilon_i}_{\delta\,\varepsilon\,\sigma}[L]^{\alpha_{\overline{p}}\,\tau_{\overline{m}}}_{\beta_{\overline{q}}\upsilon_{\overline{l}}} T^{\upsilon_{\overline{l-i}}\,\sigma\,\upsilon_{\overline{l}\setminus\overline{i}}}_{\tau_{\overline{m}}} = \tag{1.2.405}$$

$$= \{\nabla^k(P)[L]\}(T)_{\gamma_{\overline{k}}\,\delta\,\varepsilon\,\beta_{\overline{q}}}{}^{\alpha_{\overline{p}}} - [\mathbb{I}\otimes]^{k+2}L(\nabla^k(P)(T))^{\alpha_{\overline{p}}}_{\gamma_{\overline{k}}\,\delta\,\varepsilon\,\beta_{\overline{q}}} = \tag{1.2.406}$$

$$= \Big\{[\nabla^k(P)[L]](T) - \{[\mathbb{I}\otimes]^{k+2}L\}[\nabla^k(P)[T]]\Big\}^{a_{\overline{p}}}_{c_{\overline{k}}\,d\,e\,b_{\overline{q}}} \tag{1.2.407}$$

□

Lemma 16: Let $T \in \Gamma T_q M$ be a tensor field and let K be a permutation of q elements. Given $(i_k j_k)$ the list of transposition decomposition of K (with $k \in [1, a] \subset \mathbb{N}^+$), there always exists an n-tuple of linear maps $C^\infty(M)$ linear map $A_{(K)i_kj_k} : \Gamma T^p_q M \to \Gamma T^p_q M$ such that:

$$[\mathbb{I} - \sigma_K](T) = [T - \sigma_K(T)] = \sum_{k=1}^{a} A_{(K)i_kj_k}\big(\{\mathbb{I} - \sigma_{(i_kj_k)}\}(T)\big) \tag{1.2.408}$$

Proof. The proof follows immediately considering that all the permutations form a group with respect the composition and each permutation can be created by many compositions of transpositions. Let us suppose that the permutation K can be written (not uniquely) as a composition of transpositions (i_kj_k) with $k \in [1, a] \subset \mathbb{N}$ therefore:

$$\begin{aligned}
&[T - \sigma_J(T)] = && (1.2.409)\\
=&T \pm \sigma_{(i_1j_1)}(T) \pm \sigma_{(i_1j_1)}(\sigma_{(i_2j_2)}(T)) + && (1.2.410)\\
\pm& \cdots \pm \sigma_{(i_1j_1)}(\sigma_{(i_2j_2)}(...(\sigma_{(i_{a-1}j_{a-1})}(T)))) - \sigma_J(T) = && (1.2.411)\\
=&\{T - \sigma_{(i_1j_1)}(T)\} + \sigma_{(i_1j_1)}\{T - \sigma_{(i_2j_2)}(T)\} + && (1.2.412)\\
+&\sigma_{(i_1j_1)}\sigma_{(i_2j_2)}\{T - \sigma_{(i_3j_3)}(T)\} + \cdots \sigma_{(i_1j_1)}...\sigma_{(i_{a-1}j_{a-1})}\{T - \sigma_{(i_aj_a)}(T)\} && (1.2.413)
\end{aligned}$$

This is a sum of $C^\infty(M)$-linear maps acting on $\{T - \sigma_{(i_kj_k)}(T)\}$ for each $k \in [1, a]$, therefore it is a $C^\infty(M)$-linear combination $\{T - \sigma_{(i_kj_k)}(T)\}$ for each $k \in [1, a]$. If for each transposition in the list (i_kj_k) we define the linear maps $A_{(K)i_kj_k}$ as follow

$$A_{(K)i_kj_k} = \sigma_{i_1j_1} \circ ... \circ \sigma_{i_{k-1}j_{k-1}} \tag{1.2.414}$$

we can recast the previous expression:

$$[\mathbb{I} - \sigma_K](T) = [T - \sigma_K(T)] = \sum_{k=1}^{a} A_{(K)i_kj_k}\big(\{\mathbb{I} - \sigma_{(i_kj_k)}\}(T)\big) \tag{1.2.415}$$

□

Let us remark that since each permutation cannot always be decomposed uniquely as a composition of transposition, in general for the same permutation K we can admit different equivalent linear combination, one for each different decomposition.

Lemma 17: Let $T \in \Gamma T_q M$ be a tensor field an let (ij) (with $i, j \in [1, n] \subset \mathbb{N}^+$) a transposition. There always exists an n-tuple of linear maps $C^\infty(M)$ linear map $B_{(ij)k}$:

$\Gamma T^p_q M \to \Gamma T^p_q M$ such that:

$$\frac{1}{2}(\mathbb{I} - \sigma_{(ij)})\nabla^n T = \sum_{k=i+1}^{j} B_{(ij)k}\{\nabla^{k-1}\nabla^2_{[]}\nabla^{n-k-1}(T)\} \tag{1.2.416}$$

Proof. For each n-tuple of vector fields $u_{\overline{n}} \in \times^n \Gamma TM$ we can write:

$$u_n \lrcorner \ldots \lrcorner u_1 \lrcorner \frac{1}{2}(\mathbb{I} - \sigma_{(ij)})\nabla^n T = \frac{1}{2}\{\nabla^n_{u_{\overline{n}}}(T) - \nabla^n_{u_{\overline{i-1}}u_j u_{\overline{j-1}\setminus\overline{i}}u_i u_{\overline{n}\setminus\overline{j}}}(T)\} = \tag{1.2.417}$$

$$= \frac{1}{2}\Big\{\nabla^n_{u_{\overline{n}}}(T) - \nabla^n_{u_{\overline{i-1}}u_j u_{\overline{j-1}\setminus\overline{i}}u_i u_{\overline{n}\setminus\overline{j}}}(T) + \tag{1.2.418}$$

$$\pm \sum_{k=i+2}^{j-1} \nabla^n_{u_{\overline{i-1}}u_{\overline{k-1}\setminus\overline{i}}u_i u_k u_{\overline{j-1}\setminus\overline{k}}u_j u_{\overline{n}\setminus\overline{j}}}(T) + \tag{1.2.419}$$

$$\mp \sum_{k=i+2}^{j-1} \nabla^n_{u_{\overline{i-1}}u_{\overline{k-1}\setminus\overline{i}}u_j u_k u_{\overline{j-1}\setminus\overline{k}}u_i u_{\overline{n}\setminus\overline{j}}}(T) \pm \nabla^n_{u_{\overline{i-1}}u_{\overline{j-1}\setminus\overline{i}}u_j u_i u_{\overline{n}\setminus\overline{j}}}(T)\Big\} = \tag{1.2.420}$$

$$= \Big\{\sum_{k=i+1}^{j-1}\big[\nabla^n_{u_{\overline{i-1}}u_{\overline{k-1}\setminus\overline{i}}[u_i u_k]u_{\overline{j-1}\setminus\overline{k}}u_j u_{\overline{n}\setminus\overline{j}}}(T) - \nabla^n_{u_{\overline{i-1}}u_{\overline{k-1}\setminus\overline{i}}[u_j u_k]u_{\overline{j-1}\setminus\overline{k}}u_i u_{\overline{n}\setminus\overline{j}}}(T)\big] + \tag{1.2.421}$$

$$+ \nabla^n_{u_{\overline{i-1}}u_{\overline{j-1}\setminus\overline{i}}[u_i u_j]u_{\overline{n}\setminus\overline{j}}}(T)\Big\} = \tag{1.2.422}$$

$$= u_n \lrcorner \ldots \lrcorner u_1 \lrcorner \Big\{\sum_{k=i+1}^{j-1}\Big(\mathbb{I} - \sigma_{(ij)}\Big)\big[\sigma_{\overline{k}\setminus\overline{i}}\{\nabla^{k-1}\nabla^2_{[]}\nabla^{n-k-1}(T)\}\big] + \sigma_{\overline{j}\setminus\overline{i}}\{\nabla^{j-1}\nabla^2_{[]}\nabla^{n-j-1}(T)\}\Big\} \tag{1.2.423}$$

If we define the $C^\infty(M)$-linear maps $B_{(ij)k}$ such that:

$$\begin{cases} B_{(ij)k} = \Big(\mathbb{I} - \sigma_{(ij)}\Big) \circ \sigma_{\overline{k}\setminus\overline{i}} & \forall k \in [i+1, j-1] \\ B_{(ij)k} = \sigma_{\overline{k}\setminus\overline{i}} & k = j \end{cases} \tag{1.2.424}$$

then we can rewrite the expression as follow:

$$\frac{1}{2}(\mathbb{I} - \sigma_{(ij)})\nabla^n T = \sum_{k=i+1}^{j} B_{(ij)k}\{\nabla^{k-1}\nabla^2_{[]}\nabla^{n-k-1}(T)\} \tag{1.2.425}$$

$\square$

Theorem 1: Let $\nabla^n : \Gamma T^p_q M \to \Gamma T^p_{n+q} M$ be the n-th covariant differential. Given an arbitrary tensor field $T \in \Gamma T^p_q M$, there always exists a set of $C^\infty(M)$-linear maps $C_{(s,n)} : \Gamma T^p_{s+q} M \to \Gamma T^p_{n+q} M$ ($\forall s \in [0, n-1]$) such that the covariant differential can be always decomposed in a totally symmetric part plus a $C^\infty(M)$-linear combination of lower order covariant differentials as follow:

$$\nabla^n(T) = \nabla^n_{()}(T) + \sum_{s=0}^{n-1} C_{(r,n)}[\nabla^s(T)] \tag{1.2.426}$$

Proof. The proof can be performed using the previous lemmas. Let us start considering this:

$$\nabla^n(T) = \frac{1}{n!} n! \nabla^n(T) \pm \frac{1}{n!} \sum_{K \in \prod(n) \setminus id} \sigma_K[\nabla^n(T)] = \tag{1.2.427}$$

$$= \frac{1}{n!} \sum_{K \in \prod(n)} \sigma_K[\nabla^n(T)] + \frac{n! - 1}{n!} \nabla^n(T) - \frac{1}{n!} \sum_{K \in \prod(n) \setminus id} \sigma_K[\nabla^n(T)] = \tag{1.2.428}$$

$$= \nabla^n_{()}(T) + \frac{1}{n!} \sum_{K \in \prod(n) \setminus id} \{\mathbb{I} - \sigma_K\}[\nabla^n(T)] \tag{1.2.429}$$

where $\prod(n)$ denotes the set of all permutation of n elements. Now we can use the previous lemma to express $\{\mathbb{I} - \sigma_K\}[\nabla^n(T)]$ as a linear combination of transpositions:

$$\nabla^n(T) = \nabla^n_{()}(T) + \frac{1}{n!} \sum_{K \in \prod(n) \setminus id} \{\mathbb{I} - \sigma_K\}[\nabla^n(T)] = \tag{1.2.430}$$

$$= \nabla^n_{()}(T) + \frac{1}{n!} \sum_{K \in \prod(n) \setminus id} \sum_{m=1}^{a_K} A_{(K) i_m j_m} \{\mathbb{I} - \sigma_{(i^K_m j^K_m)}\}[\nabla^n(T)] \tag{1.2.431}$$

where $a_K \in \mathbb{N}$ is the length of the chosen transposition decomposition of K denoted by $(i^K_m j^K_m)$. Now again using the previous lemma we can rewrite the terms $\{\mathbb{I} - \sigma_{(i^K_m j^K_m)}\}[\nabla^n(T)]$ as follow:

$$\nabla^n(T) = \nabla^n_{()}(T) + \frac{1}{n!} \sum_{K \in \prod(n) \setminus id} \sum_{m=1}^{a_K} A_{(K) i_m j_m} \{\mathbb{I} - \sigma_{(i^K_m j^K_m)}\}[\nabla^n(T)] = \tag{1.2.432}$$

$$=\nabla^n_{()}(T)+\frac{2}{n!}\sum_{K\in\prod(n)\backslash id}\sum_{m=1}^{a_K}A_{(K)i_mj_m}\Big[\sum_{l=i^K_m+1}^{j^K_m}B_{(i^K_mj^K_m)l}\{\nabla^{l-1}\nabla^2_{[]}\nabla^{n-l-1}(T)\}\Big]= \tag{1.2.433}$$

$$=\nabla^n_{()}(T)+\frac{2}{n!}\sum_{K\in\prod(n)\backslash id}\sum_{m=1}^{a_K}\sum_{l=i^K_m+1}^{j^K_m}A_{(K)i_mj_m}\big(B_{(i^K_mj^K_m)l}\{\nabla^{l-1}(\nabla^2_{[]}(\nabla^{n-l-1}(T)))\}\big) \tag{1.2.434}$$

If we write the operator $\nabla^2_{[]}$ in terms of the $C^\infty(M)$ linear maps P and Q defined previously we obtain:

$$\nabla^n(T)=\nabla^n_{()}(T)+\frac{1}{n!}\sum_{K\in\prod(n)\backslash id}\sum_{m=1}^{a_K}\sum_{l=i^K_m+1}^{j^K_m}A_{(K)i_mj_m}\big(B_{(i^K_mj^K_m)l}\{\nabla^{l-1}(\nabla^2_{[]}(\nabla^{n-l-1}(T)))\}\big)= \tag{1.2.435}$$

$$=\nabla^n_{()}(T)+\frac{1}{n!}\sum_{K\in\prod(n)\backslash id}\sum_{m=1}^{a_K}\sum_{l=i^K_m+1}^{j^K_m}A_{(K)i_mj_m}\big(B_{(i^K_mj^K_m)l}\{\nabla^{l-1}(P(\nabla^{n-l-1}(T))-Q(\nabla^{n-l}(T)))\}\big) \tag{1.2.436}$$

At last we can use the rule found previously stating the action of ∇^n on $L(T)$ where L is an arbitrary $C^\infty(M)$-linear map. Let us just consider the terms $\nabla^{l-1}[P(\nabla^{n-l-1}(T)]$ and $\nabla^{l-1}[Q(\nabla^{n-l}(T)]\}$. They can be rewritten as:

$$\nabla^{l-1}[P(\nabla^{n-l-1}(T)]=\sum_{r=0}^{l-1}\Big[\binom{l-1}{r}\Big]\big\{[\sigma^{(\overline{1+p})}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(P)\big\}(\nabla^r(\nabla^{n-l-1}(T))) \tag{1.2.437}$$

$$\nabla^{l-1}[Q(\nabla^{n-l}(T)]=\sum_{r=0}^{l-1}\Big[\binom{l-1}{r}\Big]\big\{[\sigma^{(\overline{1+p})}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(Q)\big\}(\nabla^r(\nabla^{n-l}(T))) \tag{1.2.438}$$

Therefore:

$$\nabla^{l-1}\big(P(\nabla^{n-l-1}(T))-Q(\nabla^{n-l}(T)\big)= \tag{1.2.439}$$

$$=\sum_{r=0}^{l-1}\binom{l-1}{r}\Big[\big\{[\sigma^{\overline{(1+p)}}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(P)\big\}(\nabla^r(\nabla^{n-l-1}(T)))+ \tag{1.2.440}$$

$$-\big\{[\sigma^{\overline{(1+p)}}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(Q)\big\}(\nabla^r(\nabla^{n-l}(T)))\Big] \tag{1.2.441}$$

Substituting this expression into the previous one we have:

$$\nabla^n(T)= \tag{1.2.442}$$

$$=\nabla^n_{()}(T)+\frac{1}{n!}\sum_{K\in\prod(n)\backslash id}\sum_{m=1}^{a_K}\sum_{l=i^K_m+1}^{j^K_m}A_{(K)i_mj_m}\Big(B_{(i^K_mj^K_m)l}\sum_{r=0}^{l-1}\binom{l-1}{r}\Big\{ \tag{1.2.443}$$

$$\big\{[\sigma^{\overline{(1+p)}}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(P)\big\}(\nabla^r(\nabla^{n-l-1}(T)))-\big\{[\sigma^{\overline{(1+p)}}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(Q)\big\}(\nabla^r(\nabla^{n-l}(T)))\Big\}\Big)= \tag{1.2.444}$$

$$=\nabla^n_{()}(T)+\frac{1}{n!}\sum_{K\in\prod(n)\backslash id}\sum_{m=1}^{a_K}\sum_{l=i^K_m+1}^{j^K_m}A_{(K)i_mj_m}\Big(B_{(i^K_mj^K_m)l}\sum_{r=0}^{l-1}\binom{l-1}{r}\Big\{ \tag{1.2.445}$$

$$\big\{[\sigma^{\overline{(1+p)}}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(P)\big\}(\nabla^{r+n-l-1}(T))-\big\{[\sigma^{\overline{(1+p)}}\ \mathbb{I}\otimes]^r\nabla^{l-1-r}(Q)\big\}(\nabla^{r+n-l}(T))\Big\}\Big) \tag{1.2.446}$$

At this point it is enough to notice that the whole expression is just a very complicated $C^\infty(M)$-linear combination of $C^\infty(M)$-linear maps acting on the set

$$\{\nabla^s(T)\quad,\quad\forall s\in[0,n-1]\}$$

hence it is enough to re-sum together order by order all the covariant differentials defining a new set of linear maps $C_{(s,n)}:\Gamma T^p_{s+q}M\to T^p_{n+q}M$ as linear combinations and compositions of the linear maps $A_{(K)i_mj_m}$, $B_{(i^K_mj^K_m)l}$, $\nabla^{l-1-r}(P)$ and $\nabla^{l-1-r}(Q)$ to rewrite the previous expression as:

$$\nabla^n(T)=\nabla^n_{()}(T)+\sum_{s=0}^{n-1}C_{(r,n)}[\nabla^s(T)] \tag{1.2.447}$$

□

Corollary 5: Let $\nabla^n:\Gamma T^p_qM\to\Gamma T^p_{n+q}M$ be the n-th covariant differential. Given an arbitrary tensor field $T\in\Gamma T^p_qM$, there always exists a set of $C^\infty(M)$-linear maps

$D_{(s,n)} : \Gamma T^p_{(s)+q} M \to \Gamma T^p_{(n)+q} M$ ($\forall s \in [0, n-1]$)such that the covariant differential can be decomposed in a totally symmetric part plus a $C^\infty(M)$-linear combination of lower order symmetrized covariant differentials as follow:

$$\nabla^n(T) = \nabla^n_{(\,)}(T) + \sum_{s=0}^{n-1} D_{(s,n)}[\nabla^s_{(\,)}(T)] \tag{1.2.448}$$

Proof. The proof can be easily performed by induction: The case n=2 is trivial, it is enough to consider $\nabla^2(T) = \nabla^2_{(\,)}(T) + \nabla^2_{[\,]}(T) = \nabla^2_{(\,)}(T) - \frac{1}{2}Q(\nabla(T)) + \frac{1}{2}P(T)$ Let us suppose this is true for a generic n and let us prove it for the step $n+1$. Using the previous theorem we can state that:

$$\nabla^{n+1}(T) = \nabla^{n+1}_{(\,)}(T) + \sum_{s=0}^{n-1} C_{(s,n)}(\nabla^s(T)) \tag{1.2.449}$$

and using the inductive step we can conclude:

$$\nabla^{n+1}(T) = \nabla^{n+1}_{(\,)}(T) + \sum_{s=0}^{n} C_{(s,n+1)}\Big(\nabla^s_{(\,)}(T) + \sum_{r=0}^{s-1} D_{(r,s)}(\nabla^r_{(\,)}(T))\Big) \tag{1.2.450}$$

Resumming order by order the covariant differentials and defining $D_{(s,n+1)}$ the new set of $C^\infty(M)$-linear maps obtained summing and composing $D_{(r,s)}$ and $C_{(s,n+1)}$ we can conclude:

$$\nabla^{n+1}(T) = \nabla^{n+1}_{(\,)}(T) + \sum_{s=0}^{n} D_{(s,n+1)}[\nabla^s_{(\,)}(T)] \tag{1.2.451}$$

□

Property 19: Let us remark a very important property concerning the higher order covariant differential enlightened by the corollary. The action of an higher order covariant differential ∇^n upon a tensor field can be written as a $C^\infty(M)$-linear combination of lower order symmetrized covariant differentials.

Lemma 18: Let $\gamma : \mathbb{R} \to M$ be a geodesic curve, $\nabla_{\dot\gamma}(\dot\gamma) = 0, \quad \forall t \in \mathbb{R}$ and let $T \in T^p_q M$ be a tensor field. The following relation between the higher order covariant derivatives along the curves and the higher order covariant differentials holds:

$$\frac{D^n}{ds^n}(T) = (\dot{\gamma}\lrcorner)^n \nabla^n(T) = i^n[(\otimes^n \dot{\gamma}) \otimes \nabla^n T] \tag{1.2.452}$$

Proof. We can prove it by induction, let's start from $n = 1$

$$\frac{D}{ds}(T) = \nabla_{\dot{\gamma}}(T) = \dot{\gamma}\lrcorner\nabla(T) \tag{1.2.453}$$

Let us suppose this is true for the step n and lets prove it for the step $n + 1$

$$\frac{D^{n+1}}{ds^{n+1}}(T) = \frac{D}{ds}[\frac{D^n}{ds^n}(T)] = \frac{D}{ds}[(\dot{\gamma}\lrcorner)^n \nabla^n(T)] = (\dot{\gamma}\lrcorner)^n \frac{D}{ds}[\nabla^n(T)] \tag{1.2.454}$$

because $\frac{D}{ds}(\dot{\gamma}) = \nabla_{\dot{\gamma}}(\dot{\gamma}) = 0$. So one has:

$$\frac{D^{n+1}}{ds^{n+1}}(T) = (\dot{\gamma}\lrcorner)^n \frac{D}{ds}[\nabla^n(T)] = (\dot{\gamma}\lrcorner)^n \nabla_{\dot{\gamma}}[\nabla^n(T)] = (\dot{\gamma}\lrcorner)^n \dot{\gamma}\lrcorner\nabla[\nabla^n(T)] = (\dot{\gamma}\lrcorner)^{n+1}[\nabla^{n+1}(T)] \tag{1.2.455}$$

So we have the thesis. □

Lemma 19: Let $U \subset M$ be an open subset of the manifold and let $c : \mathbb{R} \hookrightarrow M$ be a closed embedding such that $c(\mathbb{R}) \cap U \neq \varnothing$. Let $(e_0 ... e_{m-1})$ be a local frame on TM with support on U inducing a local trivialization of $T^p_q M$ such that $e_{0|_{c(\mathbb{R})\cap U}} = \dot{c}$. Given a tensor field $T \in \Gamma_U T^p_q M$, then the following rules on the local expressions of T hold:

$$c^\star(L_{e_0}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) = \frac{d}{dt}(c^\star(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})) \tag{1.2.456}$$

$$c^\star(\nabla_{e_0}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) = \frac{D}{dt}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{1.2.457}$$

Proof. The proof follows immediately using the coordinate expressions of the given differential operations as well as the definition of pull-back of scalar fields. □

This result will be massively used later to perform the proofs related to the properties of the local representations of the multipoles.

Chapter 2

Functionals on Test Tensors Fields

In this chapter we are going to define the general concept of $\mathbb{R}$-linear functionals acting upon the $C^\infty(M)$-module of the compactly supported smooth tensor fields. At this stage no concept of topology is given on the set of the functionals since the definition is just algebraic. We are going to show then that some operation can be naturally defined due to the $\mathbb{R}^m$-linearity therefore they form an algebraic structure. In general the set of all $\mathbb{R}$-linear functionals can be very large, including an infinite number of very pathological objects. Furthermore since no topology is fixed, at this point there is still no concept of "neighborhood of a functionals", "continuity" and "convergence of the sequences" without mention concepts like metric, norm and distances. However to perform some kind of functional analysis a topology is required. Fixing the weak topology on the test tensor fields, it is possible to consider just the linear functionals preserving the topological structures. This aspect is fundamental expecially to perform standard analysis on manifolds and to cast theorems and properties, however again the topic would be too wide to be treated here in details. Once again we settle here to provide an essential but self-consistent approach to the mathematical objects we are interested in, then focusing ourselves strictly on the intrinsic geometrical and algebraic operative definition of the multipoles, that by definition inherits automatically the continuity with respect to the weak topology of the test tensor field set on which they act.

2.1 Definitions and standard linear operations

Definition 30: Given a tensor field $T \in \Gamma T^p_q M$ we define ***the support*** of T to be the closure of the set of points where T is not null :

$$supp(T) = \overline{\Big\{ x \in M \mid T(x) \neq 0 \Big\}} \qquad (2.1.1)$$

Definition 31: Given the tangent tensor bundle $T^p_q M$ we define the set of the ***smooth***

compact support tensor fields as:

$$\Gamma_0 T^q_p M = \{\phi \in \Gamma T^q_p M \mid \phi \text{ has a compact support on } M\} \tag{2.1.2}$$

By notation, the action of the functional on a test tensor field is written as $\mathcal{T}(\phi) = [\phi, \mathcal{T}]$ using the standard convention about the Schwartz distribution.

Definition 32: We define a $\mathbb{R}$-***linear functional*** $\mathcal{T}$ ***on*** $\Gamma_0 T^p_q(M)$ as a map

$$\mathcal{T} : \Gamma_0 T^p_q M \to \mathbb{R} \tag{2.1.3}$$

such that:

$$\forall \lambda, \mu \in \mathbb{R}\ ,\ \forall \psi, \phi \in \Gamma_0 T^p_q M \Rightarrow [\lambda\phi + \mu\psi, \mathcal{T}] = \lambda[\phi, \mathcal{T}] + \mu[\psi, \mathcal{T}] \tag{2.1.4}$$

We denote with $\mathcal{J}^q_p(M)$ the set of all these $\mathbb{R}$-linear functionals acting on $\Gamma_0 T^p_q M$

Definition 33: Given $\mathcal{T}, \mathcal{S} \in \mathcal{J}^q_p(M)$ define an equality of functionals as the following relation:

$$\mathcal{T} = \mathcal{S} \Leftrightarrow [\phi, \mathcal{T}] = [\phi, \mathcal{S}], \forall\, \phi \in \Gamma_0 T^p_q M \tag{2.1.5}$$

Property 20: It's easy to check that $=$ is symmetric, reflexive and transitive with cardinality of each equivalence class equal to 1.

With no effort it is possible to define some operation over the functional as well as we have done before with the tensor fields. Many of them are induced by the operations defined on tensor fields due to the $\mathbb{R}^m$-linearity.

Definition 34: Given $\mathcal{J}^p_q(M)$ we define the ***sum of functionals*** the map $+ : \mathcal{J}^q_p(M) \times \mathcal{J}^q_p(M) \to \mathcal{J}^q_p(M)$ such that:

$$[\phi, \mathcal{T} + \mathcal{S}] = [\phi, \mathcal{T}] + [\phi, \mathcal{S}] \tag{2.1.6}$$

Definition 35: Given $\mathcal{J}_q^p(M)$ we define the ***product with a smooth function*** the map $\cdot : C^\infty \times \mathcal{J}_p^q(M) \to \mathcal{J}_p^q(M)$ such that:

$$[\phi, f \cdot \mathcal{T}] = [f\phi, \mathcal{T}] \tag{2.1.7}$$

Definition 36: Given $\mathcal{J}_p^q(M)$ we define the ***a product with tensor field*** the map $\cdot : \Gamma T_{p'}^{q'}(M) \times \mathcal{J}_p^q(M) \to \mathcal{J}_{p'+p}^{q'+q}(M)$ such that:

$$[\psi \otimes \phi, S \cdot \mathcal{T}] = [\{i^{p'+q'}\}\{\sigma^{\overline{p'+q'}}\}^{p'}(S \otimes \psi)\phi, \mathcal{T}] \tag{2.1.8}$$

Property 21: In case $v \in \Gamma TM$ we have that $[\phi, v \cdot \mathcal{T}] = [v \lrcorner\, \phi, \mathcal{T}]$. In case $\alpha \in \Gamma TM$ we have that $[\phi, \alpha \cdot \mathcal{T}] = [\alpha \urcorner\, \phi, \mathcal{T}]$

Property 22: The definition of product with vector fields satisfies:

1. Associativity with respect to the tensor product, $\forall S \in \Gamma T_{p'}^{q'} M$, $\forall Q \in \Gamma_{p''}^{q''} M$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$(S \otimes Q) \cdot \mathcal{T} = S \cdot (Q \cdot \mathcal{T}) \tag{2.1.9}$$

2. Distributivity with respect to the sum of functionals and the sum of tensors, $\forall Q \in \Gamma T_{p'}^{q'} M$, $\forall S \in \Gamma_{p''}^{q''} M$, $\forall \mathcal{T}, \mathcal{S} \in \mathcal{J}_p^q(M)$

$$Q \cdot (\mathcal{T} + \mathcal{S}) = Q \cdot \mathcal{T} + Q \cdot \mathcal{S} \tag{2.1.10}$$

$$(Q + R) \cdot (\mathcal{T}) = Q \cdot \mathcal{T} + R \cdot \mathcal{T} \tag{2.1.11}$$

The same holds for the product with a scalar field identifying the tensor product with the standard product.

Proof. These properties follow trivially from the definition of "product with scalar fields" and "product with tensor fields" and from the properties of the operation on tensor fields. □

Definition 37: Let I and J be two of permutations of p and q elements respectively. Let $\sigma^I_J : \Gamma T^p_q M \to \Gamma T^p_q M$ be the braiding map of tensor fields. Given $\mathcal{J}^p_q(M)$ we define the ***braiding map of functionals*** the map $\sigma^J_I : \mathcal{J}^q_p(M) \to \mathcal{J}^q_p(M)$ such that:

$$[\phi, \sigma^J_I(\mathcal{T})] = [\sigma^I_J(\phi), \mathcal{T}] \tag{2.1.12}$$

Of course anyone is free to choose their own notation to express the permutation I and J, however we decided to use the standard cycle decomposition because it offers a direct representation of their action on the list of indices related to the coordinate representation of the tensors. It is very interesting to notice how the action of tensor fields upon vector fields and covector fields induces canonically an action of vector fields and covectors fields on the tensor fields:

Definition 38: Given $\mathcal{J}^q_p(M)$ (with $p \geq 1$) we define ***the contraction with a vector field*** the map $\lrcorner : \Gamma TM \times \mathcal{J}^q_p(M) \to \mathcal{J}^q_{p-1}(M)$ such that:

$$[\phi, v \lrcorner \mathcal{T}] = [v \otimes \phi, \mathcal{T}] \tag{2.1.13}$$

Given $\mathcal{J}^q_p(M)$ (with $q \geq 1$) we define ***the contraction with a covector field*** the map $\urcorner : \Gamma T^\star M \times \mathcal{J}^q_p(M) \to \mathcal{J}^{q-1}_p(M)$ such that:

$$[\phi, \alpha \urcorner \mathcal{T}] = [\alpha \otimes \phi, \mathcal{T}] \tag{2.1.14}$$

Definition 39: Let be $\mathcal{A} = (U_j, \phi_j)$ an atlas on M , we know that it induces a local trivialisation of TM and $T^\star M$ denoted by $(U_j \times R^m)$ via the existence of a bunch of m $C^\infty(M)$-linearly independent smooth local sections $(e_{(j)\mu})$ and $(e^\mu_{(j)})$. Let (ψ_j) be a smooth partition of the unity subordinate to (U_j). Given $\mathcal{J}^q_p(M)$ (with $q \geq 1$) , (with $p, q \geq 1$) we define an internal contraction the map $i : \mathcal{J}^q_p(M) \to \mathcal{J}^{q-1}_{p-1}(M)$ such that:

$$[\phi, iT] = [\phi, \sum_{U_j \in \mathcal{A}} \psi_j e_{(j)\mu} \lrcorner (e^\mu_{(j)} \urcorner \mathcal{T})] \tag{2.1.15}$$

Property 23: Even if the definition of internal contraction is given at fixed frame, the operation does not depend on the frame and it defines intrinsically a good $\mathbb{R}^m$ linear functional on M.

Proof. It follows immediately from the fact that, fixed another local trivialisation (U_i, ϕ_i)

and another local frame $e'_{(j)\mu}$ on the overlap with (U_i, ϕ_i), we have :

$$[\phi, \psi'_j \psi_i e'_{(j)\mu} \lrcorner (e'^{\mu}_{(j)} \neg \mathcal{T})] = [\phi, \psi'_j \psi_i e_{(i)\mu} \Lambda^{\mu}_{\nu} \lrcorner (e^{\nu}_{(i)} \overline{\Lambda}^{\nu}_{\sigma} \neg \mathcal{T}) = [\phi, \psi'_j \psi_i e_{(j)\mu} \lrcorner (e^{\mu}_{(i)} \neg \mathcal{T})] \tag{2.1.16}$$

Given two atlases $\mathcal{A} = (U_i, \phi_i)$ and $\mathcal{B} = (U'_j, \phi'_j)$ and two smooth partition of the unity (ψ_j) and (ψ'_i) respectively since (U_i) and (U'_j) both cover all M we have:

$$[\phi, \sum_{U_i \in \mathcal{A}} \psi_i e_{(i)\mu} \lrcorner (e^{\mu}_{(i)} \neg \mathcal{T})] = [\phi, \sum_{U'_j \in \mathcal{B}} \psi'_j \sum_{U_i \in \mathcal{A}} \psi_i e_{(i)\mu} \lrcorner (e^{\mu}_{(i)} \neg \mathcal{T})] \tag{2.1.17}$$

$$= [\phi, \sum_{U'_j \in \mathcal{B}} \sum_{U_i \in \mathcal{B}} \psi'_j \psi_i e_{(i)\mu} \lrcorner (e^{\mu}_{(i)} \neg \mathcal{T})] = [\phi, \sum_{U_i \in \mathcal{A}} \sum_{U'_j \in \mathcal{B}} \psi'_j \psi_i e'_{(j)\mu} \lrcorner (e'^{\mu}_{(j)} \neg \mathcal{T})] = \tag{2.1.18}$$

$$= [\phi, \sum_{U_i \in \mathcal{A}} \psi_i \sum_{U'_j \in \mathcal{B}} \psi'_j e'_{(j)\mu} \lrcorner (e'^{\mu}_{(j)} \neg \mathcal{T})] = [\phi, \sum_{U'_j \in \mathcal{B}} \psi'_j e'_{(j)\mu} \lrcorner (e'^{\mu}_{(j)} \neg \mathcal{T})] \tag{2.1.19}$$

□

Property 24: The braiding map, both the contractions and the internal contraction are $C^\infty(M)$ linear.

Proof. It follows trivially from the definitions and the properties of the operation on tensor fields □

We are going now to define the derivations upon the functionals. The definition can be weird in appearance but we will motivate it a-posteriori checking it is compatible with the standard definitions of derivations upon the tensor fields satisfying the same property and defining the same algebra.

Definition 40: Let $\mathcal{J}^q_p(M)$ be the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T^p_q M$ and define the ***Lie derivative of functionals*** the map $L : \Gamma TM \times \mathcal{J}^q_p(M) \to \mathcal{J}^q_p(M)$ defined as:

$$[\phi, L_v \mathcal{T}] = -[L_v \phi, \mathcal{T}] \quad , \quad \forall \phi \in \Gamma_0 T^p_q M \, , \, \forall v \in \Gamma TM \, , \, \forall \mathcal{T} \in \mathcal{J}^q_p(M) \tag{2.1.20}$$

Property 25: The definition of Lie derivative satisfies all the standard good properties:

1. It is $\mathbb{R}$-linear with respect to both the first and second argument, $\forall \lambda, \mu \in \mathbb{R}, \forall \mathcal{T}, \forall v, w \in$

ΓTM , $\forall \mathcal{T}, \mathcal{S} \in \mathcal{J}_p^q(M)$:

$$L_v(\lambda\mathcal{T} + \mu\mathcal{S}) = \lambda L_v(\mathcal{T}) + \mu L_v(\mathcal{S}) \tag{2.1.21}$$

$$L_{\lambda v+\mu w}(\mathcal{T}) = \lambda L_v(\mathcal{T}) + \mu L_w(\mathcal{T}) \tag{2.1.22}$$

2. It satisfies the Leibniz rule with respect to the product with a tensor field, $\forall v \in \Gamma TM$, $\forall S \in \Gamma T_{p'}^{q'} M$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$L_v(S \cdot \mathcal{T}) = L_v(S) \cdot \mathcal{T} + S \cdot L_v(\mathcal{T}) \tag{2.1.23}$$

3. It satisfies the Leibniz rule with respect to the product with a scalar field, $\forall v \in \Gamma TM$, $\forall f \in C^\infty(M)$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$L_v(S \cdot \mathcal{T}) = L_v(S) \cdot \mathcal{T} + S \cdot L_v(\mathcal{T}) \tag{2.1.24}$$

4. It satisfies the Leibniz rule with respect to both the contractions, $\forall v, u \in \Gamma TM, \forall \alpha \in \Gamma T^\star M$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$L_v(u \lrcorner \mathcal{T}) = L_v(u) \lrcorner \mathcal{T} + u \lrcorner L_v(\mathcal{T}) \tag{2.1.25}$$

$$L_v(\alpha \urcorner \mathcal{T}) = L_v(\alpha) \urcorner \mathcal{T} + \alpha \urcorner L_v(\mathcal{T}) \tag{2.1.26}$$

5. It satisfies the Jacobi identity, $\forall v, u \in \Gamma TM$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$[L_u, L_v](\mathcal{T}) = L_u(L_v(\mathcal{T})) - L_v(L_u(\mathcal{T})) = L_{[u,v]}\mathcal{T} \tag{2.1.27}$$

6. It commutes with the internal contraction $\forall v \in \Gamma TM$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$i(L_v(\mathcal{T})) = L_u(i(\mathcal{T})) \tag{2.1.28}$$

Proof. These properties follow trivially from the definition of Lie derivative of functionals and from the properties of the Lie derivative of tensor fields. $\square$

Definition 41: Let $\mathcal{J}_p^q(M)$ be the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T_q^p M$ and define the ***covariant derivative of functionals*** the map $\nabla : \Gamma TM \times \mathcal{J}_p^q(M) \to \mathcal{J}_p^q(M)$ defined as $\forall \phi \in \Gamma_0 T_q^p M$, $\forall v \in \Gamma TM$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$[\phi, \nabla_v \mathcal{T}] = -[\nabla_v \phi + div(v)\phi, \mathcal{T}] = -[div(v \otimes \phi), \mathcal{T}] \tag{2.1.29}$$

Property 26: The definition of covariant derivative satisfies all the standard good properties:

1. It is $C^\infty M$-linear with respect to the first argument and $\mathbb{R}$-linear with respect to the second argument, $\forall f, g \in C^\infty M$, $\forall \lambda, \mu \in \mathbb{R}$, $\forall \mathcal{T}$, $\forall v, w \in \Gamma TM$, $\forall \mathcal{T}, \mathcal{S} \in \mathcal{J}_p^q(M)$:

$$\nabla_v(\lambda \mathcal{T} + \mu \mathcal{S}) = \lambda \nabla_v(\mathcal{T}) + \mu \nabla_v(\mathcal{S}) \tag{2.1.30}$$

$$L_{fv+gw}(\mathcal{T}) = f L_v(\mathcal{T}) + g L_w(\mathcal{T}) \tag{2.1.31}$$

2. It satisfies the Leibniz rule with respect to the product with a tensor field, $\forall v \in \Gamma TM$, $\forall S \in \Gamma T_{p'}^{q'} M$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$\nabla_v(S \cdot \mathcal{T}) = \nabla_v(S) \cdot \mathcal{T} + S \cdot \nabla_v(\mathcal{T}) \tag{2.1.32}$$

3. It satisfies the Leibniz rule with respect to the product with a scalar field, $\forall v \in \Gamma TM$, $\forall f \in C^\infty(M)$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$\nabla_v(S \cdot \mathcal{T}) = \nabla_v(S) \cdot \mathcal{T} + S \cdot \nabla_v(\mathcal{T}) \tag{2.1.33}$$

4. It satisfies the Leibniz rule with respect to both the contractions, $\forall v, u \in \Gamma TM, \forall \alpha \in \Gamma T^\star M$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$\nabla_v(u \lrcorner \mathcal{T}) = \nabla_v(u) \lrcorner \mathcal{T} + u \lrcorner \nabla_v(\mathcal{T}) \tag{2.1.34}$$

$$\nabla_v(\alpha \urcorner \mathcal{T}) = \nabla_v(\alpha) \urcorner \mathcal{T} + \alpha \urcorner \nabla_v(\mathcal{T}) \tag{2.1.35}$$

5. It commutes with the internal contraction $\forall v \in \Gamma TM$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$i(\nabla_v(\mathcal{T})) = \nabla_u(i(\mathcal{T})) \tag{2.1.36}$$

Proof. These properties follows trivially from the definition of Lie derivative of functionals and from the properties of the Lie derivative of tensor fields. For the linearity in the first term:

$$\begin{aligned}
&[\phi, \nabla_{fv+gw}\mathcal{T}] = -[f\nabla_v(\phi)] - [g\nabla_w(\phi), \mathcal{T}] - [div(fv)\phi, \mathcal{T}] - [div(fv)\phi, \mathcal{T}] = && (2.1.37)\\
&= - [f\nabla_v(\phi), \mathcal{T}] - [\nabla_v(f)\phi, \mathcal{T}] - [div(v)f\phi, \mathcal{T}]- && (2.1.38)\\
&-g\nabla_w(\phi), \mathcal{T}] - [\nabla_w(g)\phi, \mathcal{T}] - [div(w)g\phi, \mathcal{T}] = && (2.1.39)\\
&= - [\nabla_v(f\phi), \mathcal{T}] - [div(v)f\phi, \mathcal{T}] - [\nabla_w(g\phi), \mathcal{T}] - [div(w)g\phi, \mathcal{T}] = && (2.1.40)\\
&=[\phi, f\nabla_v\mathcal{T}] + [\phi, g\nabla_w\mathcal{T}] && (2.1.41)
\end{aligned}$$

Concerning the Leibniz rule with respect to the product "$\cdot$":

$$\begin{aligned}
&[\psi \otimes \phi, \nabla_v(S \cdot \mathcal{T})] = -[\nabla_v(\psi \otimes \phi), (S \cdot \mathcal{T})] - [div(v)\psi \otimes \phi, (S \cdot \mathcal{T})] = && (2.1.42)\\
&= - [S(\nabla_v(\psi)) \otimes \phi, \mathcal{T})]- && (2.1.43)\\
&\quad - [S(\psi) \otimes \nabla_v(\phi), \mathcal{T})] - [div(v)S(\psi) \otimes \phi, (S \cdot \mathcal{T})] = && (2.1.44)\\
&= - [\nabla_v(S(\psi)) \otimes \phi - (\nabla_v S)(\psi) \otimes \phi, \mathcal{T})] - [S(\psi) \otimes \nabla_v(\phi), \mathcal{T})]+ && (2.1.45)\\
&\quad - [div(v)S(\psi) \otimes \phi, (S \cdot \mathcal{T})] = && (2.1.46)\\
&= - [\nabla_v(S(\psi) \otimes \phi), \mathcal{T})] + [\nabla_v(\phi), \nabla_v(S) \cdot \mathcal{T})] - [div(v)S(\psi) \otimes \phi, (S \cdot \mathcal{T})] = && (2.1.47)\\
&=[\psi \otimes \phi, S \cdot \nabla_v\mathcal{T}] + [\nabla_v(\phi), \nabla_v(S) \cdot \mathcal{T})] && (2.1.48)
\end{aligned}$$

In the same manner one can prove the Leibniz rule respect to the contraction and that the covariant derivative commutes with "i" $\square$

Because of the $C^\infty(M)$-linearity in the first term of the covariant derivative, we can define the ∇ operator as well as it has been done before with the tensor fields.

Definition 42: Let $\mathcal{J}_p^q(M)$ be the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T_q^p M$ and define the ***covariant differential of functionals*** the map $\nabla : \mathcal{J}_p^q(M) \to \mathcal{J}_{p+1}^q(M)$ defined as $\forall v \in \Gamma TM$, $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$v \lrcorner (\nabla(\mathcal{T})) = \nabla_v(\mathcal{T}) \tag{2.1.49}$$

Definition 43: Let $\mathcal{J}_p^q(M)$ (with $q \geq 1$) be the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T_q^p M$ and define the ***divergence of functionals*** the map div : $\mathcal{J}_p^q(M) \to \mathcal{J}_p^{q-1}(M)$ defined as $\forall \mathcal{T} \in \mathcal{J}_p^q(M)$:

$$div(\mathcal{T}) = i(\nabla(\mathcal{T})) \tag{2.1.50}$$

Lemma 20: Let $\mathcal{J}_p^q(M)$ (with $q \geq 1$) be the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T_q^p M$. The following relations between the divergence and the covariant differential hold

$$[\phi, \nabla \mathcal{T}] = -[div(\phi), \mathcal{T}] \tag{2.1.51}$$

$$[\phi, div(\mathcal{T})] = -[\nabla \phi, \mathcal{T}] \tag{2.1.52}$$

Proof. Let us fix an atlas $\mathcal{A} = (U_i, \varphi_i)$ on M inducing a trivialization on TM due to the local frame $(e_{(i)\mu})$ and let (ψ_i) be a smooth partition of the unity subordinate to the covering (U_i). Considering that all the sections $\phi \in \Gamma_0 T_p^{q+1} M$ can be generated locally from elements of $\Gamma_0 T_p^q M$ with a linear combination, we can write:

$$\phi = \sum_{U_i \in \mathcal{A}} \psi_i \; e_{(i)\mu} \otimes \phi'^{\mu}_{(i)} \tag{2.1.53}$$

where $\phi'^{\mu}_{(i)} \in \Gamma_0 T^q_p M$ a sum is implicitly assumed over the repeating indices as prescribed by the standard Einstein convention.

$$[\phi, \nabla\mathcal{T}] = \sum_{U_i\in\mathcal{A}} [\psi_i\ e_{(i)\mu} \otimes \phi'^{\mu}_{(i)}, \nabla\mathcal{T}] = \sum_{U_i\in\mathcal{A}} [\psi_i\ \phi'^{\mu}_{(i)}, e_{(i)\mu} \lrcorner \nabla\mathcal{T}] = \tag{2.1.54}$$

$$= \sum_{U_i\in\mathcal{A}} [\psi_i\ \phi'^{\mu}_{(i)}, \nabla_{e_{(i)\mu}}\mathcal{T}] = \sum_{U_i\in\mathcal{A}} -[\nabla_{e_{(i)\mu}}(\psi_i \phi'^{\mu}_{(i)}) + div(e_{(i)\mu})\psi_i\phi'^{\mu}_{(i)}, \mathcal{T}] = \tag{2.1.55}$$

$$= -\ [div(\sum_{U_i\in\mathcal{A}} \psi_i e_{(i)\mu} \otimes \phi'^{\mu}_{(i)}), \mathcal{T}] = -[div(\phi), \mathcal{T}] \tag{2.1.56}$$

In the same way we can say that:

$$[\phi, div(\mathcal{T})] = \sum_{U_i\in\mathcal{A}} [\phi, \psi_i e_{(i)\mu} \lrcorner e^{\mu}_{(i)} \urcorner \nabla(\mathcal{T})] = \sum_{U_i\in\mathcal{A}} [\phi, \psi_i e^{\mu}_{(i)} \urcorner e_{(i)\mu} \lrcorner \nabla(\mathcal{T})] = \tag{2.1.57}$$

$$= \sum_{U_i\in\mathcal{A}} [(e^{\mu}_{(i)} \otimes \psi_i\phi), \nabla_{e_{(i)\mu}}\mathcal{T}] = \tag{2.1.58}$$

$$= \sum_{U_i\in\mathcal{A}} -[\nabla_{e_{(i)\mu}}(\psi_i e^{\mu}_{(i)} \otimes \phi) + div(e_{(i)\mu}) e^{\mu}_{(i)} \otimes \psi_i\phi, \mathcal{T}] = \tag{2.1.59}$$

$$= \sum_{U_i\in\mathcal{A}} -[\nabla_{e_{(i)\mu}}(e^{\mu}_{(i)}\psi_i) \otimes \phi + \psi_i e^{\mu}_{(i)} \otimes \nabla_{e_{(i)\mu}}(\phi) + div(e_{(i)\mu}) e^{\mu}_{(i)} \otimes \psi_i\phi, \mathcal{T}] = \tag{2.1.60}$$

$$= \sum_{U_i\in\mathcal{A}} -[div(\psi_i e_{(i)\mu} \otimes e^{\mu}_{(i)}) \otimes \phi + \psi_i e^{\mu}_{(i)} \otimes \nabla_{e_{(i)\mu}}\phi, \mathcal{T}] = \tag{2.1.61}$$

$$= \sum_{U_i\in\mathcal{A}} -[\psi_i e^{\mu}_{(i)} \otimes \nabla_{e_{(i)\mu}}\phi, \mathcal{T}] = -[\nabla(\phi), \mathcal{T}] \tag{2.1.62}$$

where

$$\sum_{U_i\in\mathcal{A}} div(\psi_i e_{(i)\mu} \otimes e^{\mu}_{(i)}) = div(\sum_{U_i\in\mathcal{A}} \psi_i e_{(i)\mu} \otimes e^{\mu}_{(i)}) = \sum_{U_i\in\mathcal{A}} \psi_i div(e_{(i)\mu} \otimes e^{\mu}_{(i)}) = \tag{2.1.63}$$

$$= \sum_{U_i\in\mathcal{A}} \psi_i i(\nabla(e_{(i)\mu} \otimes e^{\mu}_{(i)})) = \sum_{U_i\in\mathcal{A}} \psi_i i(\nabla(e_{(i)\mu}) \otimes e^{\mu}_{(i)} + e_{(i)\mu} \otimes \nabla(e^{\mu})) = \tag{2.1.64}$$

$$\sum_{U_i\in\mathcal{A}} \psi_i(\Gamma^{\mu}_{\nu\lambda} e_{(i)\mu} \otimes e^{\lambda}_{(i)} \otimes e^{\nu}_{(i)} - e_{(i)\mu} \otimes \Gamma^{\mu}_{\nu\lambda} e^{\lambda}_{(i)} \otimes e^{\nu}_{(i)}) = 0 \tag{2.1.65}$$

and

$$v \neg \sum_{U_i \in \mathcal{A}} \psi_i (e^{\mu}_{(i)} \otimes \nabla_{e_{(i)\mu}}(\phi)) = \sum_{U_i \in \mathcal{A}} \psi_i v^{\mu}_{(i)} \nabla_{e_{(i)\mu}}(\phi) = \nabla_v(\phi) = v \neg \nabla(\phi) \tag{2.1.66}$$

holding $\forall v \in \Gamma TM$ has been used. Therefore:

$$\nabla(\phi) = \sum_{U_i \in \mathcal{A}} \psi_i e^{\mu}_{(i)} \otimes \nabla_{e_{(i)\mu}}(\phi)$$

□

Definition 44: Let $\mathcal{J}^q_p(M)$ the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T^p_q M$ we define recursively the ***k-th covariant differential of functionals*** the map $\nabla^k : \mathcal{J}^q_p(M) \to \mathcal{J}^q_{p+k}(M)$ defined as $\forall \mathcal{T} \in \mathcal{J}^q_p(M)$:

$$\begin{cases} \nabla^0(\mathcal{T}) = \mathcal{T} \\ \nabla^k(\mathcal{T})) = \nabla(\nabla^{k-1}(\mathcal{T})) \quad , \quad \forall k > 0 \end{cases} \tag{2.1.67}$$

Definition 45: Let $\mathcal{J}^q_p(M)$ (with $q > k$) the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T^p_q M$ we define recursively the ***k-th covariant differential of functionals*** the map $\nabla^k : \mathcal{J}^q_p(M) \to \mathcal{J}^{q-k}_p(M)$ defined as $\forall \mathcal{T} \in \mathcal{J}^q_p(M)$:

$$\begin{cases} div^0(\mathcal{T}) = \mathcal{T} \\ div^k(\mathcal{T})) = div(div^{k-1}(\mathcal{T})) \quad , \quad \forall k > 0 \end{cases} \tag{2.1.68}$$

Lemma 21: Let $\mathcal{J}^q_p(M)$ (with $q \geq 1$) the space of the $\mathbb{R}$-linear functionals on the test tensor fields $\Gamma_0 T^p_q M$. The following relations between the k-th divergence and the k-th covariant differential hold

$$[\phi, \nabla^k \mathcal{T}] = (-1)^k [div^k(\phi), \mathcal{T}] \tag{2.1.69}$$

$$[\phi, div^k(\mathcal{T})] = (-1)^k[\nabla^k\phi, \mathcal{T}] \tag{2.1.70}$$

Proof. We can prove it via induction. The step $k = 1$ was already proven, let us suppose that the thesis holds for k and let us prove it in case of $k + 1$.

$$[\phi, \nabla^{k+1}\mathcal{T}] = [\phi, \nabla(\nabla^k\mathcal{T}] = -[div(\phi), \nabla^k\mathcal{T}] = (-1)^{k+1}[div^k(div(\phi))] = \tag{2.1.71}$$
$$=(-1)^{k+1}[div(div^k(\phi))] = (-1)^{k+1}[div^{k+1}(\phi)] \tag{2.1.72}$$

$$[\phi, div^{k+1}(\mathcal{T})] = [\phi, div(div^k(\mathcal{T})] = -[\nabla(\phi), div^k(\mathcal{T})] = (-1)^{k+1}[\nabla^k(\nabla(\phi))] = \tag{2.1.73}$$
$$=(-1)^{k+1}[\nabla(\nabla^k(\phi))] = (-1)^{k+1}[\nabla^{k+1}(\phi)] \tag{2.1.74}$$

□

Property 27: It is trivial to check that $\mathcal{J}_p^q(M)$ together, with the sum of functionals and the product with scalar fields, forms a left module over the ring $(C^\infty(M), +, \cdot)$

Chapter 3

Multipoles on Differential Manifolds

As it has been already stated, the whole set of the $\mathbb{R}^m$-linear functionals on the test tensor fields is very wide and it includes a huge number of pathological objects. The majority of them are affected by a lot of mathematical "issues" and they do not satisfy even the most basical topological requirements (for instance the continuity) usually needed when attempting to build mathematical models of the world around us. We are going then to restrict our investigation to a subset of the linear functionals acting on the test tensor fields called the "multipole set". Loosely speaking, our definition of multipoles can be considered a sort of generalisation of the concept of De Rham currents to the tensor field algebra [13][14]. In [13] the De Rham push-forward of forms is defined in order to study homology, here the definition is recalled to interpret a 1-form on $\mathbb{R}$ as a linear functional and then to push it forward through a closed embedding $c : \mathbb{R} \hookrightarrow M$ to induce naturally some linear functionals on M. One can recall the standard definition of currents just restricting ourselves to the completely anti-symmetric test tensor fields on which the $\mathbb{R}$-linear functionals must act and defining the wedge product of forms and a distribution and the differential as particular cases of the operations we defined on generic $\mathbb{R}$-linear functionals on test tensor fields. Although a complete analysis of the set of the multipoles as a topological space with respect to several possible topologies should be provided, this would be too long to be shown here as a mere part of this work. To show the geometrical properties and some possible applications of the multipoles upon the differential manifolds, a topological analysis is not really fundamental, since the definition implies automatically the continuity of these kind of functionals with respect the standard weak topology of the test tensor fields. So looking forward to the purposes we would like to achieve, it can be omitted. Despite this, let us stress that a complete investigation of this aspect is however essential for a complete study of the multipoles, expecially when trying to formalize the concept of weak limits, squeezed tensor fields and the convergence of multipoles. Since we are focusing mainly on how to intrinsically build some geometrical objects which can be interpreted as a formalization of the standard concept of multipoles and moments, we are going to provide a reasonable definition of the multipoles and then check that this is able to single out the geometrical object related with the common already well known definition of "moments" for the coordinate expression of the compact support scalar fields and more generally of the compact support tensor fields. In analogy with the standard De Rham currents, we are going to show that this particular class of $\mathbb{R}$-linear functional is continuous with respect the weak topology defined on the test tensor fields space. As showed later, since the multipoles are the functionals playing an essential role in the considered scenarios, we are not investigating here if the given definition is able to cover all the possible tensor-valued distributions that can be built on a differential manifold. Although this is a crucial aspect, is beyond the purpose of this work to show that all the possible tensor-valued distributions are multipoles. So this is still an open question. However we will see later how the set of the multipoles is closed with respect to the operations involved to perform the asymptotic expansions.

3.1 Coordinate-free definitions and basic properties

We are going now to propose what we think can be considered a nice coordinate free definition of the multipoles. Despite in the beginning it could seem very obscure and abstract, we will prove how, from that one, it is possible to recall all the already existing usual definitions of "multipoles moments" in several different context (General Relativity, Special Relativity, Newtonian Mechanics, Standard Analysis on $\mathbb{R}$ and $\mathbb{R}^m$, Statistics). Without any extra work, the minimal algebraic environment defined above for generic $\mathbb{R}$-linear functionals can be entirely inherited, allowing us to perform standard operations on the multipoles as a specific case of what has been already defined in the previous chapter.

3.1.1 Two multipoles coordinate free definitions

Let $c : \mathbb{R} \hookrightarrow M$ be a closed embedding that is also called worldline. Let $\alpha \in \Gamma\Lambda^1\mathbb{R}$ be a global smooth one form. There is a natural way to induce from α a $\mathbb{R}$-linear functional on test tensor fields with support on the image of the embedding c.

Definition 46: Let $c : N \hookrightarrow M$ be a closed embedding and $\alpha \in \Gamma\Lambda^k N$ a smooth global k-form over N. We define the ***De Rham push-forward of*** α ***through the embedding*** c of the linear functionals acting on the set of the compact support smooth forms over M:

$$c_\zeta(\alpha) : \Gamma_0\Lambda M \to \mathbb{R} \tag{3.1.1}$$

such that:

$$\begin{cases} [\phi, c_\zeta(\alpha)] = \int_N c^\star(\phi) \wedge \alpha \,, \quad \forall\phi \in \Gamma_0\Lambda M \mid deg(\phi) = dim(N) - deg(\alpha) \\ \\ [\phi, c_\zeta(\alpha)] = 0 \,, \quad \forall\phi \in \Gamma_0\Lambda M \mid deg(\phi) \neq dim(N) - deg(\alpha) \end{cases} \tag{3.1.2}$$

where $c^\star$ is the usual pullback of differential forms along the embedding c.

Example: We would like to show here the particular case in which $N = \mathbb{R}$. This is very important since it will be the case we are interested in trying to define the multipoles along the worldlines. Let $c : \mathbb{R} \hookrightarrow M$ be a closed embedding, called also worldline and $\alpha \in \Gamma\Lambda\mathbb{R}$ a smooth form over $\mathbb{R}$. We define the ***De Rham push-forward of*** α ***through the worldline*** c the linear functionals acting on the set of the compact support smooth forms over M:

$$c_\zeta(\alpha) : \Gamma_0\Lambda M \to \mathbb{R} \tag{3.1.3}$$

such that:

$$\begin{cases} [\phi, c_\zeta(\alpha)] = \int_{\mathbb{R}} c^\star(\phi) \wedge \alpha \,, \quad \forall \phi \in \Gamma_0\Lambda M \mid deg(\phi) = 1 - deg(\alpha) \\ \\ [\phi, c_\zeta(\alpha)] = 0 \,, \quad \forall \phi \in \Gamma_0\Lambda M \mid deg(\phi) \neq 1 - deg(\alpha) \end{cases} \tag{3.1.4}$$

where $c^\star$ is the usual pullback of differential forms along worldline c.

Definition 47: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline. We define the ***De Rham top core of*** c the set :

$$Cor(c) = \{c_\zeta(\alpha) \,, \forall \alpha \in \Gamma\Lambda^1\mathbb{R}\} \tag{3.1.5}$$

the set of all the functionals induced by the smooth global top forms on $\mathbb{R}$ through the De Rham push-forward.

Property 28: Let us notice that, from the definition of De Rham push-forward and top core of c, if $\alpha \in \Gamma\Lambda^1\mathbb{R}$ therefore $c_\zeta(\alpha)$ can just acts with a non null result over $\Gamma_0\Lambda^0 M$. Since the space $\Gamma_0\Lambda^0 M = \Gamma_0 T^0_0 M$ of the compact support scalar field can be interpreted also as set of all the rank 0 smooth test tensor fields, we can say that $Cor(c) \subset \mathcal{J}^0_0(M)$ interpreting it as a set of $\mathbb{R}$-linear functionals acting on the rank 0 smooth test tensor fields.

This property is fundamental, allowing us to interpret the De Rham push-forward of the top forms also as $\mathbb{R}$-linear functionals on the test tensor fields. Once this identification has been done, it is enough to use the the standard operation on the $\mathbb{R}$-linear functionals on the test tensor fields restricted to the $Cor(c)$ to define new specific subset of $\mathbb{R}$-linear functionals on the test tensor fields.

Definition 48: We define the ***De Rham p-q core of*** c the set of $\mathbb{R}$-linear functionals $Cor^p_q(c) \subset \mathcal{J}^p_q(M)$ satisfying:

$$Cor^q_p(c) = \{T \cdot c_\zeta(\alpha), \forall \alpha \in \Gamma\Lambda^1\mathbb{R}, \forall T \in \Gamma T^q_p M\} \tag{3.1.6}$$

Property 29: Let us stress that the set $Cor^q_p(c)$ is not just the set of the couples like (T, α) , $\forall \alpha \in \Gamma\Lambda^1\mathbb{R}, \forall T \in \Gamma T^q_p M$, in fact the couples (T, α) and $(kT, k^{-1}\alpha)$ for each $k \in \mathbb{R}, k \neq 0$ define the same object belonging in $Cor^q_p(c)$.

Definition 49: We define the set of the ***Ellis multipoles*** $\Upsilon^p_q(c) \subset \mathcal{J}^p_q(M)$ the **closure of the set** $Cor^p_q(c)$ **with respect three operations**:

1. Sum of functionals:

$$\mathcal{T} + \mathcal{S} \in \Upsilon^p_q(c) \ , \ \forall \mathcal{T}, \mathcal{S} \in \Upsilon^p_q(c) \tag{3.1.7}$$

2. Product with scalar fields:

$$f \cdot \mathcal{T} \in \Upsilon^p_q(c) \ , \ \forall \mathcal{T} \in \Upsilon^p_q(c), \forall f \in \Gamma\Lambda^0 M \tag{3.1.8}$$

3. Lie Derivative:

$$L_v \mathcal{T} \in \Upsilon^p_q(c) \ , \ \forall \mathcal{T} \in \Upsilon^p_q(c), \forall v \in \Gamma TM \tag{3.1.9}$$

Definition 50: Given a connection ∇ on the manifold M, we define the set of the ***Dixon multipoles*** $\Delta^p_q(c) \subset \mathcal{J}^p_q(M)$ to be the **closure of** $Cor^q_p(c)$ **with respect three operations:**

1. Sum of functionals:

$$\mathcal{T} + \mathcal{S} \in \Upsilon^p_q(c) \ , \ \forall \mathcal{T}, \mathcal{S} \in \Upsilon^p_q(c) \tag{3.1.10}$$

2. Product with scalar fields:

$$f \cdot \mathcal{T} \in \Upsilon^p_q(c) \ , \ \forall \mathcal{T} \in \Upsilon^p_q(c), \forall f \in \Gamma\Lambda^0 M \tag{3.1.11}$$

3. Divergence of a contraction with a vector field:

$$div(v \cdot \mathcal{T}) \in \Delta^p_q(c) \ , \ \forall \mathcal{T} \in \Delta^p_q(c), \forall v \in \Gamma TM \tag{3.1.12}$$

Although $\Upsilon^p_q(c)$ and $\Delta^p_q(c)$ seem to denote two very different sets of $\mathbb{R}$-linear functionals, we will be able to prove that $\Upsilon^p_q(c) = \Delta^p_q(c)$. Therefore a-posteriori we can state that these two definitions are completely equivalent. Considering this, a careful reader could ask which role is played by the connection ∇ used in the definition of Dixon multipoles since the same set can be defined without any connection ∇ due to the Ellis definition. We will see that ∇ will give an essential contribution to define easily in a completely covariant way some particular classes of $C^\infty(\mathbb{R})$-linear independent multipoles. These can be used to generate via $C^\infty(\mathbb{R})$-linear combinations the whole set of multipoles and we will be able to link $\Delta^p_q(c)$ with the space of the tensor fields with support on the image of the worldline $\Gamma T^p_{q+k\ c} M$. This very important geometrical structure emerges just if we fix a connection on the manifold M, without which we would not be able to cast tensorial equations defining the generators. This would cause a lot of trouble when trying to separate the covariant mathematical and physical information from coordinate dependent properties occurring just when a particular coordinate system on the manifold is chosen. Because of this result we definitely prefer to talk about "Ellis and Dixon representation of the multipoles" rather than "Dixon or Ellis multipoles" in the next chapters.

3.1.2 Rank, support and order of a multipole

Even though it has not been proved yet that $\Upsilon^q_p(c) = \Delta^q_p(c)$ we are able to define three fundamental concept concerning the multipoles independently from which definition has been chosen.

Definition 51: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T} \in \Delta^q_p(c)$ or equivalently $\mathcal{T} \in \Upsilon^q_p(c)$ a multipole. The ***rank of a multipole*** is defined to be $rank(\mathcal{T}) = (q, p)$.

Definition 52: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T} \in \Delta^q_p(c)$ or equivalently $\mathcal{T} \in \Upsilon^q_p(c)$ a multipole. We define ***the order of a multipole*** as:

$$ord(\mathcal{T}) = \mathbf{min}\{k \in \mathbb{N} \mid [\lambda^{k+n+1}\phi, \mathcal{T}] = 0 \ , \forall n \in \mathbb{N} \ , \ \forall \ \phi \in \Gamma_0 T^p_q M \ , \ \forall \lambda \in \Gamma\Lambda^0 M \mid c^\star(\lambda) = 0\} \tag{3.1.13}$$

Definition 53: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T} \in \Delta^q_p(c)$ or equivalently $\mathcal{T} \in \Upsilon^q_p(c)$ a multipole. We define ***the support of a multipole*** as:

$$supp(\mathcal{T}) = \cap\{M/U \mid U \subset M \text{ is open and } [\phi, \mathcal{T}] = 0, \tag{3.1.14}$$

$$\forall \phi \in \Gamma_0 T^p_q M \mid supp(\phi) \subset U\}$$

Lemma 22: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T} \in \Delta^q_p(c)$ or equivalently $\mathcal{T} \in \Upsilon^q_p(c)$ a multipole, then:

$$supp(\mathcal{T}) \subseteq c(N) \tag{3.1.15}$$

Proof. To prove it, it's enough to show that $\forall \mathcal{T} \in Cor^q_p(c)$ holds. In fact we know using the sum, the product with tensors the Covariant Derivative and the Lie Derivative, the support is still a subset of $c(N)$.
Let us start fixing $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M, ψ_i an arbitary smooth partition of the unity subordinate to the atlas and let $(e_{(i)\mu})$ be the local frames defined on U_i inducing a trivialisation of TM, $T^\star M$ and $T^p_q M$. In this way:

$$\phi = \sum_{U_i \in \mathcal{A}} \psi_i \phi = \sum_{U_i \in \mathcal{A}} \psi_i \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} e_{(i)\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}_{(i)} \tag{3.1.16}$$

Let us consider $U = M \setminus c(N)$ then since c is a closed embedding we have that U is open. $\forall\, \phi \in \Gamma_0 T^p_q M \mid supp(\phi) \subset U$. We have that $\forall \mathcal{T} \in Cor^q_p(c)$:

$$[\phi, \mathcal{T}] = [\phi, T \cdot c_\varsigma(\alpha)] = \tag{3.1.17}$$
$$= \int_N c^\star(T(\phi)) \wedge \alpha = \int_N c^\star(T(\sum_{U_i \in \mathcal{A}} \psi_i \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} e_{(i)\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}_{(i)} \phi)) \wedge \alpha = \tag{3.1.18}$$
$$= \int_N c^\star(\sum_{U_i \in \mathcal{A}} \psi_i \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T(e_{(i)\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}_{(i)} \phi)) \wedge \alpha = \tag{3.1.19}$$
$$= \sum_{U_i \in \mathcal{A}} \int_N c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}) \wedge \alpha = \sum_{U_i \in \mathcal{A}} \int_N 0 \cdot c^\star(\psi_i T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}) \wedge \alpha = 0 \tag{3.1.20}$$

Hence from the definition of support, $\forall \mathcal{T} \in Cor^q_p(c)$ we have that $supp(\mathcal{T}) \subseteq c(N)$. Now considering that $\forall v \in TM \Rightarrow supp(\nabla_v(\phi)) \subseteq supp(\phi) \subset U$, $\forall v \in TM \Rightarrow supp(L_v(\phi)) \subseteq$

$supp(\phi) \subset U$and $supp(\mathcal{S}+\mathcal{T}) \subset supp(\mathcal{S}) \cup supp(\mathcal{T})$ we can conclude the thesis. $\square$

Lemma 23: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T}, \mathcal{S} \in \Delta^q_p(c)$ or alternatively $\mathcal{T}, \mathcal{S} \in \Upsilon^q_p(c)$ a multipole. The following statements on the order of the multipoles hold:

1. $ord(\mathcal{T}+\mathcal{S}) \leq \mathbf{max}\{ord(\mathcal{T}), ord(\mathcal{S})\}$
2. $ord(\mathcal{T}) = 0\,,\ \forall \mathcal{T} \in Cor^q_p(c)$
3. $ord(f \cdot \mathcal{T}) = ord(\mathcal{T})$
4. If $\mathcal{T}, \mathcal{S} \in \Delta^q_p(c)$ then $ord(div(v \cdot \mathcal{T})) \leq ord(\mathcal{T}) + 1$.
5. If $\mathcal{T}, \mathcal{S} \in \Upsilon^q_p(c)$ then $ord(L_v(\mathcal{T})) \leq ord(\mathcal{T}) + 1$

Proof. Let us consider a non null $\mathcal{T} \in Cor^q_p(c)$. We know by definition that $\exists T \in \Gamma T^p_q M$ and $\exists \alpha \in \Gamma\Lambda^1\mathbb{R}$ such that $\mathcal{T} = T \cdot c_\varsigma(\alpha)$. We know that

$$ord(\mathcal{T}) = \mathbf{min}\{k \in \mathbb{N} | [\lambda^{k+n+1}\phi, \mathcal{T}] = 0, \quad \forall \phi \in \Gamma_0 T^p_q M, \quad \forall n \in \mathbb{N}, \quad \forall \lambda \in \Gamma\Lambda^0 M | c^\star(\lambda) = 0\}$$

So

$$0 = [\lambda^{k+n+1}\phi, \mathcal{T}] = \int_N c^\star(T(\lambda^{k+n+1}\phi)) \wedge \alpha = \tag{3.1.21}$$

$$= \int_N c^\star(T(\phi)) c^\star(\lambda^{k+n+1}) \wedge \alpha$$

The minimum integer value of k we can have to satisfy the equation is 0 so we conclude that $ord(\mathcal{T}) = 0$ For $f \cdot \mathcal{T}$ the prove is trivial:

$$[\lambda^{k+n+1}\phi, f \cdot T] = [\lambda^{k+n+1}(f\phi), \cdot T] = [\lambda^{k+n+1}(\phi'), \cdot T]. \tag{3.1.22}$$

So the order is preserved.

For $\mathcal{S}+\mathcal{T}$ we have two different cases $ord(\mathcal{S}) \leq ord(\mathcal{T})$ or $ord(\mathcal{S}) > ord(\mathcal{T})$. Let us suppose to have the first case, the other can be proved in the same manner:

$$[\lambda^{k+n+1}\phi, \mathcal{T}+\mathcal{S}] = [\lambda^{k+n+1}\phi, \mathcal{T}] + [\lambda^{k+n+1}\phi, \mathcal{S}] \tag{3.1.23}$$

If we set $k = ord(\mathcal{T})$ we have:

$$[\lambda^{ord(\mathcal{T})+n+1}\phi, \mathcal{T}+\mathcal{S}] = [\lambda^{ord(\mathcal{T})+n+1}\phi, \mathcal{T}] + [\lambda^{ord(\mathcal{T})+n+1}\phi, \mathcal{S}] = 0 + 0 = 0 \tag{3.1.24}$$

So by definition of order we have that $ord(\mathcal{T}+\mathcal{S}) \leq ord(\mathcal{T})$. For $div(v \cdot \mathcal{T})$ we have:

$$[\lambda^{k+n+1}\phi, div(v \cdot \mathcal{T})] = -[\lambda^{k+n}\nabla_v(\lambda)\phi, \mathcal{T}] - [\lambda^{k+1+n}\nabla_v(\phi), \mathcal{T}] \tag{3.1.25}$$

If we set $k = ord(\mathcal{T}) + 1$ we have:

$$[\lambda^{ord(\mathcal{T})+2+n}\phi, \nabla_v\mathcal{T}] = -[\lambda^{ord(\mathcal{T})+1+n}\nabla_v(\lambda)\phi, \mathcal{T}] - [\lambda^{ord(\mathcal{T})+2+n}\nabla_v(\phi), \mathcal{T}] = 0 + 0 = 0 \tag{3.1.26}$$

So by definition of order we have that $ord(div(v \cdot \mathcal{T})) \leq ord(\mathcal{T}) + 1$. The identical proof can be performed for the Lie derivative. $\square$

Lemma 24: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T} \in \Delta^p_q(c)$ or equivalently $\mathcal{T} \in \Upsilon^p_q(c)$ a multipole. The order of a multipole always exists and it is unique:

$$\exists! k \in \mathbb{N} \; : \; ord(\mathcal{T}) = k \tag{3.1.27}$$

therefore ord is a function mapping surjectively the multipoles into $\mathbb{N}$

Proof. The existence is trivial. Since $\forall \mathcal{T} \in Cor^q_p(c)$ the order is well defined and it is 0. We already proved that taken a multipole with order k the action of the sums, multiplications with scalars and derivatives affect the order as expressed by the previous lemma, thence the definition is compatible with respect to all the operations needed to define the multipoles, hence all the multipoles admits a well defined order. The uniqueness is also trivial because the order is defined as the minimum of a well defined subset of $\mathbb{N}$. $\square$

Property 30: Let us stress that even if $ord(\mathcal{T}) = ord(\mathcal{S}) = k$ we have that $ord(\mathcal{T}+\mathcal{S}) \leq k$. So the set of all the multipoles with fixed order k cannot be interpreted as a module (or a vector space) (e.g $div(v \cdot c_\zeta(\alpha)) + div(w \cdot c_\zeta(\alpha))$, $v_{|\dot{c}} = f\dot{c} + u$, $w_{|\dot{c}} = g\dot{c} - u$, $\forall u \in \Gamma T_c M$). By contrast the set

$$\overset{(k)}{\Delta}{}^q_p(c) = \{\mathcal{T} \in \Delta^q_p(c) \mid ord(\mathcal{T}) \leq k\} \tag{3.1.28}$$

is closed with respect to the sum and "product with scalar field". In the same way the set of all the multipoles with fixed order k cannot be interpreted as a module (or a vector space) (e.g $L_v(L_w((T))) - L_w(L_v((T))) = L_{[v,w]}\mathcal{T}$). By contrast the set

$$\overset{(k)}{\Upsilon}{}^q_p(c) = \{\mathcal{T} \in \Upsilon^q_p(c) \,|\, ord(\mathcal{T}) \leq k\} \tag{3.1.29}$$

is closed with respect to the sum and "product with scalar field".

So considering the properties of the product respect the sum one can say that $\overset{(k)}{\Delta}{}^p_q(c)$ and $\overset{(k)}{\Upsilon}{}^p_q(c)$ are modules over the ring $(C^\infty(M), +, \cdot)$.

Definition 54: Let $\Upsilon^q_p(c)$ be the set of the Ellis multipoles. We define the ***subset of the Ellis multipoles up to the order*** k the set $\overset{(k)}{\Upsilon}{}^q_p(c)$ satisfying:

$$\overset{(k)}{\Upsilon}{}^q_p(c) = \{\mathcal{T} \in \Upsilon^q_p(c) \,|\, ord(\mathcal{T}) \leq k\} \tag{3.1.30}$$

Let $\Delta^q_p(c)$ be the set of the Dixon multipoles. We define the ***subset of the Dixon multipoles up to the order*** k the set $\overset{(k)}{\Delta}{}^q_p(c)$ satisfying:

$$\overset{(k)}{\Delta}{}^q_p(c) = \{\mathcal{T} \in \Delta^q_p(c) \,|\, ord(\mathcal{T}) \leq k\} \tag{3.1.31}$$

Property 31: Trivially we have that:

$$\forall h, k \in \mathbb{N} \,|\, h < k \Rightarrow \overset{(h)}{\Delta}{}^q_p(c) \subset \overset{(k)}{\Delta}{}^q_p(c) \tag{3.1.32}$$

$\overset{(h)}{\Delta}{}^q_p(c)$ is then a submodule of $\overset{(k)}{\Delta}{}^q_p(c)$. In the same way:

$$\forall h, k \in \mathbb{N} \,|\, h < k \Rightarrow \overset{(h)}{\Upsilon}{}^q_p(c) \subset \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.1.33}$$

$\overset{(h)}{\Upsilon}{}^q_p(c)$ is then a submodule of $\overset{(k)}{\Upsilon}{}^q_p(c)$

These properties are very important. The fact that $\overset{(k)}{\Upsilon}{}^q_p(c)$ and $\overset{(k)}{\Delta}{}^q_p(c)$ can be considered as $C^\infty(M)$-modules allows us to express each multipole up to the k order as a linear combination of several generators induced by the choice of the atlas on the manifold so the trivialisation of the tangent bundle TM.

3.1.3 Multipoles as continuous maps

As stated previously the multipoles are continuous linear functionals. With the term "continuity" we refer to the common meaning made explicit in [13] slightly generalized for the test tensor fields. In this perspective the multipoles are continuous with respect the usual weak topology defined on the smooth test tensor fields.

Definition 55: Let M be a manifold,let $\mathcal{A} = (U_i, \varphi_{(i)})$ be an atlas and let $\mathcal{T}$ be a $\mathbb{R}$-linear functional. We define $\mathcal{T}$ to be continuous if given an arbitrary sequence of smooth test tensor field $\{\phi_n \mid n \in \mathbb{N}\}$ with supports all contained in a compact set $K \subset U_i$ and satisfying:

$$\forall k \in \mathbb{N} \quad \Rightarrow \quad \lim_{n\to\infty} L_{\lambda_{\overline{k}}}(\phi_n)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = 0$$

the following holds:

$$\lim_{n\to\infty} [\phi_n, \mathcal{T}] = 0 \tag{3.1.34}$$

Loosely speaking we say that a linear functional is "continuous" with respect to the weak topology defined on the smooth test tensor field if it preserves the uniform convergence of the sequences of smooth test tensor fields to 0. This notion of continuity can be considered an adaptation of the one introduced by Schwartz and by De Rham [13].

Given the definition of multipoles in terms of De Rham push-forward together with the definition of continuity, it is easy to check that all the multipoles must be continuous with respect to the weak topology.

Lemma 25: Let M be a manifold, let $\mathcal{A} = (U_i, \varphi_{(i)})$ be an atlas. Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{T} \in \Upsilon^q_p(c)$ a multipole of rank (q, p). For each arbitrary sequence of smooth test tensor fields $\{\phi_n \mid n \in \mathbb{N}\}$ with supports all contained in a compact set $K \subset U_i$ and satisfying:

$$\forall k \in \mathbb{N} \quad \Rightarrow \quad \lim_{n\to\infty} L_{\lambda_{\overline{k}}}(\phi_n)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = 0$$

the following holds:

$$\lim_{n\to\infty} [\phi_n, \mathcal{T}] = 0 \tag{3.1.35}$$

therefore all the multipoles are continuous functionals.

Proof. To show the thesis it is enough to check that $Cor^q_p(c)$ is a set of continuous functionals and that the operations with respect is required the algebraic closure of $Cor_p^q(c)$ are able to preserve the continuity of the functionals. Let us start considering the manifold M, a closed embedding $c : \mathbb{R} \hookrightarrow M$ and the set $Cor(c)$. This is the set of the standard De Rham $m-1$ currents (with m the dimension of the manifold) with support on the image of the embedding c. As showed by De Rham in [13] these functionals are continuous. Now let us consider the set

$$Cor_p^q(c) = \{T \cdot c_\zeta(\alpha), \forall \alpha \in \Gamma\Lambda^1\mathbb{R}, \forall T \in \Gamma T_p^q M\} \tag{3.1.36}$$

The action of these functionals on a sequence of test tensor fields is given by the definition of the product with a tensor field:

$$[\phi_n, T \cdot c_\zeta(\alpha)] = [\{i^{p+q}\}\{\sigma^{\overline{p+q}}\}^p(T \otimes \phi_n), c_\zeta(\alpha)] = [\phi'_n, \cdot c_\zeta(\alpha)] \tag{3.1.37}$$

where $\phi'_n = \{i^{p+q}\}\{\sigma^{\overline{p+q}}\}^p(T \otimes \phi_n)$ is a sequence of smooth test scalar fields and $c_\zeta(\alpha)$ a De Rham $m-1$ current belongings to $Cor(c)$. Because of the continuity of the tensor product, the braiding map and the internal contraction, if ϕ_n converges to 0 as required in the previous definition then ϕ'_n converges to 0 in the same way. Hence, for an arbitrary sequence ϕ_n converging to 0 as required when $n \to \infty$, we have:

$$\lim_{n\to\infty} [\phi_n, T \cdot c_\zeta(\alpha)] = \lim_{n\to\infty} [\phi'_n, \cdot c_\zeta(\alpha)] = 0 \tag{3.1.38}$$

So referring to [13] once again, we can conclude that also $T \cdot c_\zeta(\alpha)$ must be continuous. Hence we can state that all the functionals in $Cor_p^q(c)$ must be continuous. Now we must show that the operations with respect to which is required the algebraic closure of $Cor_p^q(c)$ are able to preserve the continuity of the $\mathbb{R}$-linear functionals.

1. Sum. Let $\mathcal{T}$ and $\mathcal{S}$ be two continuous $\mathbb{R}$-linear functionals. The action of the functional $\mathcal{T} + \mathcal{S}$ on a sequence of test tensor fields is given by the definition of sum:

   $$[\phi_n, \mathcal{T} + \mathcal{S}] = [\phi_n, \mathcal{T}] + [\phi_n, \mathcal{S}] \tag{3.1.39}$$

   So using the continuity and the linearity of the two functionals we have:

   $$\lim_{n\to 0}[\phi_n, \mathcal{T} + \mathcal{S}] = \lim_{n\to 0}\{[\phi_n, \mathcal{T}] + [\phi_n, \mathcal{S}]\} = \lim_{n\to 0}[\phi_n, \mathcal{T}] + \lim_{n\to 0}[\phi_n, \mathcal{S}] = 0 \tag{3.1.40}$$

2. Let $\mathcal{T}$ be a continuous $\mathbb{R}$-linear functionals and $f \in C^\infty(M)$. The action of the functional $f \cdot \mathcal{T}$ on a sequence of test tensor fields is given by:

$$[\phi_n, f \cdot \mathcal{T}] = [f\phi_n, \mathcal{T}] = [\phi'_n, \mathcal{T}] \tag{3.1.41}$$

If ϕ_n converges to 0 as required in the previous definition then $\phi'_n = f\phi_n$ converges to 0 in the same way. Hence, for an arbitrary sequence ϕ_n converging to 0 as required when $n \to \infty$, using the continuity of $\mathcal{T}$, we have:

$$\lim_{n\to 0}[\phi_n, f \cdot \mathcal{T}] = \lim_{n\to 0}[\phi'_n, \mathcal{T}] = 0 \tag{3.1.42}$$

3. Let $\mathcal{T}$ be a continuous $\mathbb{R}$-linear functionals and $v \in \Gamma TM$. The action of the functional $L_v\mathcal{T}$ on a sequence of test tensor fields is given by:

$$[\phi_n, L_v\mathcal{T}] = -[L_v(\phi_n), \mathcal{T}] = -[\phi'_n, \mathcal{T}] \tag{3.1.43}$$

If ϕ_n converges to 0 as required in the definition of continuity, then $\phi'_n = L_v(\phi_n)$ converges to 0 in the same way. Hence, for an arbitrary sequence ϕ_n converging to 0 as required when $n \to \infty$, using the continuity of $\mathcal{T}$, we have:

$$\lim_{n\to 0}[\phi_n, L_v\mathcal{T}] = \lim_{n\to 0}[\phi'_n, \mathcal{T}] = 0 \tag{3.1.44}$$

Therefore, considering this, we can state that the set $Cor^q_p(c)$ is a set of continuous linear functionals and all the operations used to define the closure automatically preserve the continuity of each element. So the whole set of multipoles $\Upsilon^q_p(c)$, must be continuous. □

3.2 Local representation of the Ellis multipoles

In the previous section the Ellis multipoles $\Upsilon^q_p(c)$ have been defined in a coordinate free manner, interpreting them as a specific family of $\mathbb{R}$-linear functionals acting on the test tensor fields. This is very useful because this definition show how the multipoles on the manifolds, as well as tensor fields, connection and curves, can be considered as primitive intrinsic geometric objects, not necessarily linked to any "regular" field. However at this stage we have to face two big problems. First of all, for now, these particular functionals are called multipoles by definition, and we have still to show if and how they can be linked and compared with the standard definition of multipoles (e.g the multipole of a probability distribution or a multipole related to an extended bodies in classical mechanics). What has been done until now does not provide any formal element or rigorous methods to relate them. The second problem arises from the fact that, in practice, we are not able to achieve all our purposes just working with intrinsic abstract objects on the manifolds. As well as we manipulate directly the local expressions of the fields (e.g tensor fields) to perform easily the operations defined on them, we need to express the multipoles directly in terms of their action on the local expressions of the test tensor fields. We are going to see how the Ellis definition of the multipoles leads us naturally to an explicit expression of the action of a multipole in terms of contraction, derivations and integration of the local expressions of the test tensor fields. This is what we are going to call the "Ellis local expression of the action of a multipole". Since the $\mathbb{R}$-linear functionals are entirely defined by their action on the test tensor fields, we can use the Ellis local expression of the action of the multipoles, to define a local expression of the multipole themselves. Considering that the $\mathbb{R}$-linear functionals are entirely defined by how they act on the test tensor fields, once the Ellis local expression of the action of the multipoles has been made explicit, we can use it to define "a local expression of the multipole" as a $C^\infty(\mathbb{R})$-linear combination with respect to an elementary set of multipoles with compact support generating the whole module.

3.2.1 Local expression of the action induced by a trivialization of TM and the Ellis definition

Let us remark that at this stage, the elements belonging to $\overset{(k)}{\Upsilon}{}^q_p(c)$ are intrinsic geometrical objects and their definition depends just on the existence of compactly supported forms over M and a closed embedding $c : \mathbb{R} \hookrightarrow M$, and not at all on other structures like coordinates, metric , Killing vectors, Levi Civita connection, ADM fibration, just to quote some. The obtained local expression coincides exactly with the generalisation of the functionals defined in [17][18] on the manifold and we will see that they are related to the usual concept of multipole expansion of a tensor field. Since c is assumed to be a closed embedding we know that there must exist a specific atlas such that the worldline can be covered by a single adapted local chart. However, since we do not want to restrict ourselves using a very specific local chart, in general no constraints are given on the choice of the atlas covering the manifold. In some specific cases the adapted local charts are used as a tool to perform the proofs.

Lemma 26: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$ and ψ_i a smooth partition of the unity subordinate to (U_i). For each test tensor $\phi \in \Gamma_0 T^p_q M$, there exists a $N \in \mathbb{N}$ and at least one appropriate bunch of local smooth scalar fields $\{\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c(N)\cap U_i}\Lambda^0\mathbb{R} \mid U_i \in \mathcal{A}\}$ defining a global smooth section of $\Gamma\Lambda^1\mathbb{R}$

via the linear combination:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.1}$$

Proof. We provide here a sketch of proof about the existence of at least one non trivial set of local scalar field defining a global smooth 1-form over $\mathbb{R}$. Let us start by considering the atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Since TM is a vector bundle and the chosen trivialisation is compatible with the vector structure then we know that on each overlap $U_i \cap U_j$ we have that $e_{(i)\mu} = e_{(j)\nu}\Lambda^{\nu}_{(ij)\mu}$. As it has been showed in the first chapter this induces a trivialisation on $T^\star M$ and $T^p_q M$ compatible with vector structures of these bundles. Knowing this, to show that $\forall\phi \in \Gamma_0 T^p_q M$ the expression defines a good global one form on $\Gamma\Lambda^0\mathbb{R}$ is enough to prove that we can build a bunch of smooth local scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ such that they define a good global smooth section of $\Gamma\Lambda^0\mathbb{R}$ (in other words a good smooth global scalar field) via the linear combination:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star(\psi_i)c^\star\Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \tag{3.2.2}$$

Let us fix the index i and let us consider an arbitrary bunch of smooth scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} : c(\mathbb{R}) \cap U_i \to M$, then the expression:

$$\sum_{k=0}^{N} c^\star\Big(L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \tag{3.2.3}$$

is still a good smooth local scalar field defined on $c(\mathbb{R}) \cap U_i$ because the components $L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}$ are smooth local scalar fields and we are performing just pullbacks and sums preserving the smoothness. From the bundle theory we know that it can be considered the local expression of a global section if and only if the local sections are compatible, in other words if given two arbitrary $c \cap U_i$ and $c \cap U_j$ with a non empty overlap, $\forall x \in$

$(c(\mathbb{R}) \cap U_i) \cap (c(\mathbb{R}) \cap U_j)$ we have that:

$$\Big[\sum_{k=0}^{N} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}\Big]_{|t} = \Big[\sum_{k=0}^{N} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(j)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}\Big]_{|t} \tag{3.2.4}$$

Since ϕ is a tensor and $L_{\lambda_{\overline{k}}} = L_{e_{(i)\lambda_{\overline{k}}}} = L_{e_{(i)\lambda_1}}...L_{e_{(i)\lambda_k}}$ is the composition of k Lie derivative, we can use the local trivialisation induced by $(e_{(i)\mu})$ calculating explicitly the transformation rules for the components $L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}$ in terms of (in general very complicated) linear combination of Lie derivatives of $\Lambda^{\mu}_{(ij)\nu}$ up to the N-th order and the new components $L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}$ as follows:

$$\Big[\sum_{k=0}^{N} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(j)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}\Big]_{|t} = \tag{3.2.5}$$

$$=\Big[\sum_{k=0}^{N} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}\Big]_{|t} = \Big[\sum_{k=0}^{N}\sum_{s=0}^{k} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\beta_{(ji)\eta_{\overline{s}}\mu_{\overline{p}}\sigma_{\overline{q}}}^{\ \ \ \ \lambda_{\overline{k}}\nu_{\overline{q}}\rho_{\overline{p}}}\alpha_{(i)\ \ \ \ \rho_{\overline{p}}}^{\eta_{\overline{s}}\ \sigma_{\overline{q}}}\Big]_{|t} \tag{3.2.6}$$

where $\beta_{(ji)\eta_{\overline{s}}\mu_{\overline{p}}\sigma_{\overline{q}}}^{\ \ \ \ \lambda_{\overline{k}}\nu_{\overline{q}}\rho_{\overline{p}}}$ is a bunch of scalar fields formed by linear combinations of pullbacks along c of Lie derivatives of $\Lambda^{\mu}_{(ij)_{\nu}}$ with respect the local frame $(e_{(i)\mu})$. The explicit expression of $\beta_{(ji)\eta_{\overline{s}}\mu_{\overline{p}}\sigma_{\overline{q}}}^{\ \ \ \ \lambda_{\overline{k}}\nu_{\overline{q}}\rho_{\overline{p}}}$is very complicated and there is no general rule to find it, unless computing it in each case for each different atlas. For brevity we are not going to consider them in detail here however in principle they can always be calculated at fixed order N for each k. Considering this, if we have a bunch of scalar fields $\alpha_{(i)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}$ such that:

$$\alpha_{(j)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} = \sum_{s=1}^{k}\beta_{(ji)\eta_{\overline{s}}\mu_{\overline{p}}\sigma_{\overline{q}}}^{\ \ \ \ \lambda_{\overline{k}}\nu_{\overline{q}}\rho_{\overline{p}}}\alpha_{(i)\ \ \ \ \rho_{\overline{p}}}^{\eta_{\overline{s}}\ \sigma_{\overline{q}}} \quad , \quad \forall k \in [0,N], \forall U_i, U_j \in \mathcal{A} \mid U_i \cap U_j \neq \varnothing \tag{3.2.7}$$

they satisfy the compatibility condition for each test tensor ϕ, therefore they are able to define a good global smooth scalar field on $\mathbb{R}$ so a global 1-form on $\mathbb{R}$. In general this is a very strong constraint, possibly depending on the properties of the manifold M, but we are able to find at least two examples showing that gluing in an appropriate way the local scalar fields $\alpha_{(i)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}$ is always possible. The first case is trivial. For each value of $k \in [0,N]$ and for each open set $U_i \in \mathcal{A}$ let us consider a set of null local scalar field

$\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = 0$. Therefore:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = 0 \tag{3.2.8}$$

is the global null top form over $\mathbb{R}$ formed by gluing together trivially the null local scalar fields. The second example is more relevant. Let us suppose $\mathcal{A}$ being a minimal atlas of M, therefore there must exists at least an $U_i \in \mathcal{A}$ such that $\forall U_j \in \mathcal{A} \Rightarrow U_i \neq \cup\{U_j \in \mathcal{A} \mid U_j \neq U_i\}$. Let $b_i : U_i \subset M \to \mathbb{R}$ be a smooth bump function such that $supp(b) \subset U_i \setminus \cup\{U_j \in \mathcal{A} \mid U_j \neq U_i\}$ therefore given an arbitrary bunch of smooth scalar fields $f^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} : U_i \to \mathbb{R}$ we can define:

$$\begin{cases} \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = b \cdot f^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \\ \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = 0 \quad , \quad \forall j \neq i \end{cases} \tag{3.2.9}$$

It is easy to check that it satisfies the compatibility conditions. □

Although we did not write explicitly each transition functions, this lemma is fundamental, in fact it shows how there exists some appropriate linear combinations of local sections of $\Lambda^1\mathbb{R}$ defined by

$$\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma_U \Lambda^1\mathbb{R} \tag{3.2.10}$$

that can be glued together to form a global top form over $\mathbb{R}$. The transition functions can be calculated and presented case by case by fixing N for each k. In general they are very complicated involving a large number of terms growing almost factorially with respect to N, however to provide the definition of Ellis local representation it is just enough to prove the existence.

Theorem 2: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. For each Ellis multipole $\mathcal{T} \in \Upsilon_p^q(c)$, there always exists at least an $N \in \mathbb{N} \mid N \geq ord(\mathcal{T})$ and bunch of local smooth scalar field $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ defining a global smooth top form:

$$c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.11}$$

such that, $\forall\phi \in \Gamma_0 T^p_q M$, $\mathcal{T}$ acts on the local expression of ϕ as follow:

$$[\phi,\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big) c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.12}$$

Proof. To prove the statement it's enough to show that all $Cor^q_p(c)$ can be written in that way and then prove that the given expression is closed under the operations we used to define $\Upsilon^q_p(c)$. Let us consider a functional $\mathcal{S} \in Cor^q_p(c)$, we know by definition that there exists at least a $T \in T^q_p$ and a $\alpha \in \Gamma\Lambda^1\mathbb{R}$ such that $\mathcal{S} = T\cdot c_\zeta(\alpha)$ So we can write the action of this functional on an arbitrary test tensor field ϕ as an action on its local coordinate expression:

$$[\phi,\mathcal{S}] = [\phi, T\cdot c_\zeta(\alpha)] = \int_{\mathbb{R}} c^\star(T(\phi))\wedge\alpha = \tag{3.2.13}$$

$$= \int_{\mathbb{R}} c^\star\Big(T\Big[\sum_{U_i\in\mathcal{A}} \psi_i\phi_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} e_{(i)}^{\nu_{\overline{q}}} \otimes e_{(i)\mu_{\overline{p}}}\Big]\Big)\wedge\alpha = \tag{3.2.14}$$

$$= \int_{\mathbb{R}} c^\star\Big(\sum_{U_i\in\mathcal{A}} \psi_i\phi_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} T\Big[e_{(i)}^{\nu_{\overline{q}}} \otimes e_{(i)\mu_{\overline{p}}}\Big]\Big)\wedge\alpha = \tag{3.2.15}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\psi_i\phi_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} T_{(i)\mu_{\overline{p}}}^{\nu_{\overline{q}}}\Big)\wedge\alpha = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\phi_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big) c^\star(\psi_i) c^\star(T_{(i)\mu_{\overline{p}}}^{\nu_{\overline{q}}})\cdot\tilde{\alpha} ds \tag{3.2.16}$$

We know that $ord(\mathcal{S}) = 0$,therefore fixed arbitrarily an $N \in \mathbb{N}$ we can define a bunch of local scalar fields:

$$\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = \begin{cases} \tilde{\alpha}_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \quad , \quad \forall j \leq ord(\mathcal{S}) \\ 0 \quad , \quad ord(\mathcal{S}) < j \leq N \end{cases} \tag{3.2.17}$$

and rewrite the expression as follow:

$$[\phi, \mathcal{S}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) c^\star(\psi_i) \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.18}$$

Since by construction, $\forall \phi \in \Gamma_0 T^p_q M$ the local scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ satisfies :

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = c^\star(T(\phi)) \wedge \alpha \tag{3.2.19}$$

we have immediately that:

$$\sum_{k=0}^{N} c^\star \Big(L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) \alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.20}$$

which can be interpreted as a local expression of the global smooth form $c^\star(T(\phi)) \wedge \alpha \in \Gamma\Lambda^1\mathbb{R}$. So we have the thesis for each element of $Cor^p_q(c)$.

Now let us assume that a generic $\mathcal{T}, \mathcal{S} \in \overset{(k)}{\Upsilon}{}^q_p(c)$ satisfy the thesis, then we would like to prove that $\forall \phi \in \Gamma_0 T^p_q M$:

1. there always exists $L \in \mathbb{N} \mid L \geq ord(\mathcal{S} + \mathcal{T})$ a bunch of local smooth scalar field $\beta_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c \cap U_i} \Lambda^0 \mathbb{R}$ defining a global smooth top form:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{L} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) \beta_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^0\mathbb{R} \tag{3.2.21}$$

   such that, $\mathcal{T} + \mathcal{S}$ acts on the local expressions of ϕ as follows:

$$[\phi, \mathcal{S}+\mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{L} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big) c^\star(\psi_i) \beta_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{3.2.22}$$

2. there always exists $L \in \mathbb{N} \mid L \geq ord(f \cdot \mathcal{T})$ a bunch of local smooth scalar field $\beta_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} \in \Gamma_{c \cap U_i} \Lambda^0 \mathbb{R}$ defining a global smooth top form:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{L} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big) \beta_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} ds \in \Gamma \Lambda^0 \mathbb{R} \tag{3.2.23}$$

such that, $f \cdot \mathcal{T}$ acts on the local expressions of ϕ as follows:

$$[\phi, f \cdot \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{L} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big) c^\star(\psi_i) \beta_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{3.2.24}$$

3. For each smooth global vector field $v \in \Gamma TM$ there always exists $M \in \mathbb{N} \mid M \geq ord(L_v \mathcal{T})$ a bunch of local smooth scalar field $\gamma_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} \in \Gamma_{c \cap U_i} \Lambda^0 \mathbb{R}$ defining a global smooth top form:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{M} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big) \gamma_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} ds \in \Gamma \Lambda^0 \mathbb{R} \tag{3.2.25}$$

such that, $L_v \mathcal{T}$ acts on the local expressions of ϕ as follows:

$$[\phi, L_v \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{M} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big) c^\star(\psi_i) \gamma_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{3.2.26}$$

Let us start with the first:

$$[\phi, \mathcal{S}+\mathcal{T}] = [\phi,\mathcal{S}] + [\phi,\mathcal{T}] = \tag{3.2.27}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{K} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}} ds + \tag{3.2.28}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{J} L_{\lambda_{\overline{j}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\hat{\alpha}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}} ds = \tag{3.2.29}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} \Big[c^\star\Big(\sum_{k=0}^{K} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}} + c^\star\Big(\sum_{j=0}^{J} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}}\Big] ds \tag{3.2.30}$$

Let us suppose to have $J \leq K$, the other case follows in the same manner. Since $J \leq K$ we can always define a $M > K$ and new bunch of smooth local scalar fields:

$$\beta^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\mu_{\overline{p}}} = \begin{cases} \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\mu_{\overline{p}}} + \hat{\alpha}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\mu_{\overline{p}}}, \forall k \leq J \\ \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\mu_{\overline{p}}}\ ,\ J < k \leq K \\ 0 \quad , \quad K < k \leq M \end{cases} \tag{3.2.31}$$

Therefore using it on the expression

$$[\phi, \mathcal{S}+\mathcal{T}] = \tag{3.2.32}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} \Big[c^\star\Big(\sum_{k=0}^{K} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}} + c^\star\Big(\sum_{j=0}^{J} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}}\Big] ds = \tag{3.2.33}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} \Big[c^\star\Big(\sum_{k=0}^{K} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}} + c^\star\Big(\sum_{j=0}^{J} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}}\Big] ds = \tag{3.2.34}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{M} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\beta^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\mu_{\overline{p}}} ds \tag{3.2.35}$$

By construction, the relation $ord(\mathcal{S}+\mathcal{T}) \leq \max\{\mathcal{S},\mathcal{T}\} \leq K \leq M$, furthermore

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\sum_{k=0}^{M} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\beta^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds = \tag{3.2.36}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\sum_{k=0}^{K} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds + c^\star\Big(\sum_{j=0}^{J} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds \tag{3.2.37}$$

Since we assumed both $\mathcal{S}$ and $\mathcal{T}$ satisfy the thesis, then both the second terms in the right hand side of the equation are well defined global smooth 1-forms on $\mathbb{R}$. Considering that the sum of two smooth global 1-forms is still a smooth one form we have immediately that

$$c^\star\Big(\sum_{k=0}^{M} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\beta^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds \tag{3.2.38}$$

can be interpreted as the local expression of a global smooth form defined on the whole $\mathbb{R}$. To prove the closure with respect the product with scalar let us suppose that $\mathcal{T}$ satisfy the thesis, then we can use the previous lemma:

$$L_{\lambda_{\overline{k}}}(f\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \sum_{j=0}^{k} L_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}} \tag{3.2.39}$$

to manipulate the expression:

$$[\phi, f\cdot\mathcal{T}] = [f\phi,\mathcal{T}] = \tag{3.2.40}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(f\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds = \tag{3.2.41}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{k} L_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds \tag{3.2.42}$$

Now it is enough to re-sum order by order all the terms defining a new bunch of scalar fields $\beta_{(i)}^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ as linear combinations of $c^\star(f_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}})$ and $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ to end up with:

$$[\phi, f\cdot\mathcal{T}] = \tag{3.2.43}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{k} L_{\alpha_{\overline{j}}}(\phi)_{(i)\rho_{\overline{q}}}^{\sigma_{\overline{p}}} f_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}\Big) c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{3.2.44}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N} L_{\alpha_{\overline{j}}}(\phi)_{(i)\rho_{\overline{q}}}^{\sigma_{\overline{p}}}\Big) c^\star(\psi_i)\beta_{(i)}^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.45}$$

Since by hypothesis, $\forall\phi\in\Gamma_0 T^p_q M$ the local scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ define a good global smooth form on $\mathbb{R}$ with

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.46}$$

and since $\forall\phi\in\Gamma_0 T^p_q M, \forall f\in C^\infty(M)\ :\ f\phi\in\Gamma_0 T^p_q M$, we have that by construction:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(f\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{j=0}^{N} L_{\alpha_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\beta_{(i)}^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.47}$$

is a global smooth 1- form over $\mathbb{R}$ and:

$$c^\star\Big(\sum_{j=0}^{N} L_{\alpha_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\beta_{(i)}^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.48}$$

can be interpreted as its local expression. To prove the closure with respect the Lie derivatives let us suppose that $\mathcal{T}$ satisfy the thesis, then we can use the previous lemma:

$$L_{\lambda_{\overline{k}}}(L_v(\phi))^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}} \tag{3.2.49}$$

to manipulate the expression:

$$[\phi, L_v\mathcal{T}] = -[L_v\phi, \mathcal{T}] = \tag{3.2.50}$$

$$= - \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(L_v\phi)^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}} ds = \tag{3.2.51}$$

$$= - \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}} ds \tag{3.2.52}$$

Now it is enough to re-sum order by order all the terms defining a new bunch of scalar fields $\gamma^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}}$ as linear combinations of $c^\star(f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}})$ and $\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}}$ to end up with:

$$[\phi, L_v\mathcal{T}] = \tag{3.2.53}$$

$$= - \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}} ds = \tag{3.2.54}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N+1} L_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}\Big) c^\star(\psi_i)\gamma^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}} ds \tag{3.2.55}$$

Since by hypothesis, $\forall\phi \in \Gamma_0 T^p_q M$ the local scalar fields $\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}}$ define a good global smooth form on $\mathbb{R}$ with

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\qquad\mu_{\overline{p}}} ds \tag{3.2.56}$$

and since $\forall\phi \in \Gamma_0 T^p_q M, \forall v \in \Gamma TM \Rightarrow L_v\phi \in \Gamma_0 T^p_q M$, we have that by construction:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(L_v\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{j=0}^{N+1} L_{\alpha_{\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\gamma^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{3.2.57}$$

is a global smooth 1- form over $\mathbb{R}$ and:

$$c^\star\Big(\sum_{j=0}^{N+1} L_{\alpha_{\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\gamma^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{3.2.58}$$

can be interpreted as its local expression. □

Theorem 3: Let $c : \mathbb{R} \hookrightarrow M$. For each Ellis multipole $\mathcal{T} \in \Upsilon^q_p(c)$, $\forall u \in \Gamma TM$ then $div(u\cdot\mathcal{T}) \in \Upsilon^q_p(c)$. In other words the set $\Upsilon^q_p(c)$ is closed with respect to the "divergence of a contraction with a vector field".

Proof. Given $\mathcal{T} \in \Upsilon^q_p(c)$ we have that:

$$[\phi, div(u\cdot\mathcal{T})] = -[\nabla_u(\phi), \mathcal{T}] = -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\nabla_u\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{3.2.59}$$

Using the result of a previous lemma we can state that, fixing a local frame, the local components of a tensor fields must satisfy:

$$[L_{\lambda_{\overline{k+1}}}(\nabla_v(T))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)+1} L_{\alpha_{\overline{j}}}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{3.2.60}$$

Therefore we can use it to manipulate the expression:

$$[\phi, div(v\cdot\mathcal{T})] = -[\nabla_v\phi, \mathcal{T}] = \tag{3.2.61}$$

$$= - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}} (\nabla_v \phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds = \tag{3.2.62}$$

$$= - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}} (\phi)^{\ \sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \tag{3.2.63}$$

Now it is enough to re-sum order by order all the terms defining a new bunch of scalar fields $\gamma^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}$ as linear combinations of $c^\star(f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}})$ and $\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}$ to end up with:

$$[\phi, L_v \mathcal{T}] = \tag{3.2.64}$$

$$= - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k+1} L_{\alpha_{\overline{j}}} (\phi)^{\ \sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{\alpha_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds = \tag{3.2.65}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{j=0}^{N+1} L_{\alpha_{\overline{j}}} (\phi)^{\ \sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} \Big) c^\star(\psi_i) \gamma^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \tag{3.2.66}$$

Since by hypothesis, $\forall \phi \in \Gamma_0 T^p_q M$ the local scalar fields $\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}$ defines a good global smooth form on $\mathbb{R}$ with

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \tag{3.2.67}$$

and since $\forall \phi \in \Gamma_0 T^p_q M, \forall v \in \Gamma TM \Rightarrow L_v \phi \in \Gamma_0 T^p_q M$, we have that by construction:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}} (\nabla_v \phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{j=0}^{N} L_{\alpha_{\overline{j}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \gamma^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \tag{3.2.68}$$

is a global smooth 1- form over $\mathbb{R}$ and:

$$c^\star\Big(\sum_{j=0}^{N+1} L_{\alpha_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\gamma_{(i)}^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.69}$$

can be interpreted as its local expression. □

Corollary 6: Since $\forall \mathcal{T} \in \Upsilon_p^q(c)$, $\forall u \in \Gamma TM$ we have that $div(u \cdot \mathcal{T}) \in \Upsilon_p^q(c)$, the set $\Upsilon_p^q(c)$ is closed with respect the "divergence of a contraction with a vector field". Therefore by definition $\forall \mathcal{T} \in \Upsilon_p^q(c) \Rightarrow \mathcal{T} \in \Delta_p^q(c)$. This means that $\Upsilon_p^q(c) \subseteq \Delta_p^q(c)$ in other words the Ellis multipoles are just a subset of the Dixon multipoles.

We will prove later that on the other hand the Dixon multipoles are just a subset of the Ellis multipoles, hence the Ellis and Dixon definition define the same set of geometrical objects called simply multipoles. For this reason we definitely prefer to talk about Ellis and Dixon local representation (or equivalently Ellis and Dixon local expression) of the multipoles rather than the Ellis and Dixon multipoles, to avoid any misleading redundancy talking about the intrinsic geometrical objects.

Definition 56: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given a multipole $\mathcal{T} \in \Upsilon_p^q(c)$, we define the ***local Ellis representation of the action of $\mathcal{T}$*** the integral:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \quad , \quad \forall \phi \in \Gamma_0 T_q^p M \tag{3.2.70}$$

satisfying the following condition:

1. the 1-form:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.71}$$

   is a global smooth form on $\mathbb{R}$.

2. $\forall \phi \in \Gamma_0 T^p_q M$, the action of $\mathcal{T}$ can be expressed with that integral:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.2.72}$$

Property 32: The existence of the Ellis representation of the action for each multipole $\mathcal{T} \in \Upsilon^q_p(c)$ is guaranteed by the previous theorem.

3.2.2 Ellis local expression of the multipoles

Since the $\mathbb{R}$-linear functionals are entirely defined by their action on the test tensor fields we can use the Ellis local expression of the action of the multipoles, to define a local expression of the multipole themselves.

Corollary 7: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. For each Ellis multipole $\mathcal{T} \in \Upsilon^q_p(c)$, there always exists at least an $N \in \mathbb{N} \mid N \geq ord(\mathcal{T})$ and bunch of local smooth scalar field $\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c \cap U_i} \Lambda^0 \mathbb{R}$ defining a global smooth top form:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.73}$$

such that, $\mathcal{T}$ can be written as:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k L_{\lambda_{\overline{k}}} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta \Big(\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \Big) \Big\} \tag{3.2.74}$$

Proof. It follows directly from the previous theorem and from the definition of action of a $\mathbb{R}$-linear functionals on the test tensor fields. $\forall \phi \in \Gamma_0 T^p_q M$, $\mathcal{T}$ acts on the local expression of ϕ as follow:

$$[\phi, \mathcal{T}] = \tag{3.2.75}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds = \tag{3.2.76}$$

$$= [\phi, \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1^k) L_{\lambda_{\overline{k}}} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta \Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds \Big) \Big\}] \tag{3.2.77}$$

□

Property 33: From the standard bundle theory we can state that if

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.78}$$

is good global smooth top form on $\mathbb{R}$, therefore it cannot depend on the choices of the smooth partition of the unity (ψ_i), therefore the multipole $\mathcal{T}$ must not depend on the chosen smooth partition of the unity.

Definition 57: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given a multipole $\mathcal{T} \in \Upsilon^q_p(c)$, let

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds \quad , \quad \forall \phi \in \Gamma_0 T^p_q M \tag{3.2.79}$$

be the local Ellis representation of the action of $\mathcal{T}$. Since

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k L_{\lambda_{\overline{k}}} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta \Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds \Big) \Big\} \tag{3.2.80}$$

we define the right hand term ***the Ellis local representation of the multipole*** $\mathcal{T}$.

Example: It is very easy to see how this expression represents a generalisation of the

Ellis multipoles given in [17][18]. Let us fix the manifold M to be $\mathbb{R}^4$ endowed with a metric $\eta = diag(-1,1,1,1)$. Let $c : \mathbb{R} \to \mathbb{R}^4$ a worldline parametrized such that $\eta(\dot{c}(\tau), \dot{c}(\tau)) = -1$. Let us fix a global coordinate system $\overline{x} = (x^0, x^1, x^2, x^3)$. An arbitrary multipole action on $\phi \in \Gamma_0 T^p_q M$ can be then expressed as:

$$\int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\;\mu_{\overline{p}}} d\tau = \tag{3.2.81}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} c^\star \Big(\partial_{\lambda_{\overline{k}}} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\;\mu_{\overline{p}}} d\tau = \tag{3.2.82}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} [\partial_{\lambda_{\overline{k}}} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}]_{|_{c(\tau)}} \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\;\mu_{\overline{p}}} d\tau = \tag{3.2.83}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}^4} \delta^4(\overline{x} - \overline{c}(\tau)) \partial_{\lambda_{\overline{k}}} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}(\overline{x}) d^4x \Big] \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\;\mu_{\overline{p}}} d\tau = \tag{3.2.84}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}^4} (-1)^k \partial_{\lambda_{\overline{k}}} \delta^4(\overline{x} - \overline{c}(\tau)) \phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}(\overline{x}) d^4x \Big] \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\;\mu_{\overline{p}}} d\tau = \tag{3.2.85}$$

$$= \sum_{k=0}^{N} (-1)^k \int_{\mathbb{R}^4} \Big[\int_{\mathbb{R}} \partial_{\lambda_{\overline{k}}} \delta^4(\overline{x} - \overline{c}(\tau)) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\quad\;\mu_{\overline{p}}} d\tau \Big] \phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}(\overline{x}) d^4x = \tag{3.2.86}$$

If $p = 0$ and $q = 1$ and α is renamed j (absorbing a sign factor), then the expression is reduced just to the usual one cast in [17]

$$\int_{\mathbb{R}} c^\star \Big(L_{\lambda_{\overline{k}}}(\phi)_\mu \Big) j^{\lambda_{\overline{k}}\,\mu} d\tau = \int_{\mathbb{R}^4} \Big[\sum_{k=0}^{N} \int_{\mathbb{R}} \partial_{\lambda_{\overline{k}}} \delta^4(\overline{x} - \overline{c}(\tau)) j^{\lambda_{\overline{k}}\mu} d\tau \Big] \phi_\mu(\overline{x}) d^4x = \tag{3.2.87}$$

$$= \Big\langle \phi, \sum_{k=0}^{N} \int_{\mathbb{R}} \partial_{\lambda_{\overline{k}}} \delta^4(\overline{x} - \overline{c}(\tau)) j^{\lambda_{\overline{k}}\mu} d\tau \Big\rangle = \Big\langle \phi, \overset{(N)}{\mathcal{J}}{}^\mu \Big\rangle \tag{3.2.88}$$

and one can interpret as usual:

$$\overset{(N)}{\mathcal{J}}{}^{\mu} = \sum_{k=0}^{N} \int_{\mathbb{R}} \partial_{\lambda_{\overline{k}}} \delta^4(\overline{x} - \overline{c}(\tau)) j^{\lambda_{\overline{k}}\mu} d\tau \tag{3.2.89}$$

a multipolar singular electromagnetic current [17]. For the monopole case $N = 0$ the current $\mathcal{J}^{\mu}$ can represent the electromagnetic current of a point-like charge moving on the space-time just if and only if it satisfy the conservation law

$$\partial_{\mu} \overset{(N)}{\mathcal{J}}{}^{\mu} = 0 \tag{3.2.90}$$

and this fix uniquely the form for the distributional electromagnetic monopole:

$$\mathcal{J}^{\mu} = q \int_{\mathbb{R}} \delta^4(\overline{x} - \overline{c}(\tau)) \frac{dc^{\mu}}{d\tau} d\tau \tag{3.2.91}$$

Solving the Maxwell equations for this distributional current leads us to the well know Liénard-Wiechert electromagnetic potential that according to the experiments, seems to be able to provide a satisfactory representation of the field generated by a moving point-like charge. For higher order distributional currents it is possible to find some generalised predictions as showed in [17][18][19]

Property 34: The existence of the Ellis representation for each multipole $\mathcal{T} \in \Upsilon^q_p(c)$ is guaranteed by the previous corollary.

As we can see, loosely speaking, the Ellis representation of a multipole consists in writing a multipole $\mathcal{T} \in \Upsilon^q_p(c)$ as a linear combination of a bunch of specific multipoles with support on the patches U_i of the atlas $\mathcal{A}$ and glued together such that, for each test tensor field, they can induce a a global smooth 1-form on $\mathbb{R}$ (depending on the test tensor field and its Lie derivatives) which, if integrated, gives exactly the action of the multipole upon the test tensor field. However, as we stated from the beginning the rules defining the gluing conditions are in general extremely non trivial, and they must be calculated case by case by asking the integrand to be a global smooth top form over $\mathbb{R}$.

Corollary 8: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. A bunch of scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} : U_i \subseteq M \to \mathbb{R}$ defining a global top form on $\mathbb{R}$ by:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.92}$$

always identify also a multipole due to the Ellis local representation:

$$\mathcal{T} := \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k L_{\lambda_{\overline{k}}}\Big\{\psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}}\Big] c_{\zeta}\Big(\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds\Big)\Big\} \tag{3.2.93}$$

Definition 58: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. A bunch of scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} : U_i \subseteq M \to \mathbb{R}$ defining a global top form on $\mathbb{R}$ by:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.2.94}$$

are called ***Ellis local parameters of the multipole*** $\mathcal{T}$:

$$\mathcal{T} := \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k L_{\lambda_{\overline{k}}}\Big\{\psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}}\Big] c_{\zeta}\Big(\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds\Big)\Big\} \tag{3.2.95}$$

3.3 Local representation of the Dixon multipoles

In the previous section the Ellis representation of the multipoles has been defined. This is very convenient because it allows us to write explicitly the result of the action of a multipole just in terms of $\mathbb{R}$-linear operations on the components of the test tensor fields. This local representation depends just on the choice of a trivialisation of TM, therefore it is very essential and no extra structure is needed. Although this can seem a very strong advantage, actually the absence of further structure,(i.e the affine structure) can lead us to some relevant practical problems. Analysing in detail the properties of the

Ellis representation in the further chapter, the reader can realise how the lack of any structure beyond just the differential structure of the manifold causes a lot of issues. In general the Ellis local representation of a multipole is not unique, there is no way to link covariantly the order of the multipole with the number of Lie derivatives forming the representation and the equations and constraints concerning the multipoles in general cannot be translated into equation and constraints on local representation in a unique and covariant way, creating a lot of trouble when trying to separate the relevant coordinate independent pieces of information from the properties that hold just fixing a specific trivialization of TM possibly induced by a particular coordinate system. An additional source of trouble is given also by the very tricky way the local Ellis parameters glue together to define a smooth global one form built from the Lie derivatives of the test tensor fields taken with respect to the local frames. As it will be discussed in the third part, for several reasons these issues are not something we can just ignore, if we want to work within a fully relativistic framework (i.e General Relativity, Covariant Electromagnetism, Extended Theories of Gravitation). Luckily, we will see how, adding an affine structure on the manifold M encoded by a global connection ∇ and defining the multipoles in the Dixon way, it is possible to define a much more convenient local representation of the multipoles in terms of covariant derivatives rather than Lie derivatives, without losing any generality. As we are going to show in the next chapters the Dixon local representation of the multipoles will satisfies many more nice properties at the price of introducing an affine structure on M. We will see how this local expression is in the very same form found by Dixon and Tulczyjew in [1][2][3][4].

3.3.1 Local expression of the action induced by a trivialisation of TM and the Dixon definition

Let us remark that at this stage, the elements belonging to $\Delta_p^q(c)$ are intrinsic geometrical objects and their definition depends just on the existence of compact support forms over M and a closed embedding $c : \mathbb{R} \hookrightarrow M$, and not at all on other structures like coordinates except a connection ∇. Let us specify here that no assumption are made on the connection apart the globality, no constraints concerning metric compatibility or null torsion are assumed.

Lemma 27: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialization of TM due the local frame $(e_{(i)\mu})$. Let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$ and ψ_i a smooth partition of the unity subordinate to (U_i). For each test tensor $\phi \in \Gamma_0 T^p_q M$, there exists at least one appropriate bunch of local smooth completely simmetric in the indices $\lambda_{\overline{k}}$ scalar fields $\{\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c(N)\cap U_i}\Lambda^0\mathbb{R} \mid U_i \in \mathcal{A}\}$ defining a global smooth section of $\Gamma\Lambda^1\mathbb{R}$ via the linear combination:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.3.1}$$

Proof. We provide here a sketch of proof about the existence of local scalar fields defining a global smooth 1-form over $\mathbb{R}$. Let start considering the atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Since TM is a vector bundle and the chosen trivialisation is compatible with the vector structure then we know that on each overlap $U_i \cap U_j$ we have that $e_{(i)\mu} = e_{(j)\nu}\Lambda^{\nu}_{(ij)\mu}$. As it has been showed in the first chapter this induces a trivialisation on $T^{\star}M$ and T^p_qM compatible with vector structures of these bundles. Known this, to show that $\forall\phi \in \Gamma_0 T^p_q M$ the expression defines a good global one form on $\Gamma\Lambda^0\mathbb{R}$ is enough to prove that we can build a bunch of smooth local scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ such that they define a good global smooth section of $\Gamma\Lambda^0\mathbb{R}$ (aka a good smooth global scalar field) via the linear combination:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^{\star}(\psi_i)c^{\star}\Big(\sum_{k=0}^{N} \nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \tag{3.3.2}$$

where just the terms $\nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}$ completely symmetric in $\lambda_{\overline{k}}$ can survive to the contraction with the completely symmetric bunch of scalars fields. Let us fix the index i and let us consider an arbitrary bunch of smooth scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} : c(\mathbb{R}) \cap U_i \to M$, then the expression:

$$\sum_{k=0}^{N} c^{\star}\Big(\nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \tag{3.3.3}$$

is still a good smooth local scalar field defined on $c(\mathbb{R}) \cap U_i$ because the components $\nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}$ are smooth local scalar fields and we are performing just pullback and sums preserving the smoothness. From the bundle theory we know that it can be considered the local expression of a global section if and only if the local sections are compatible, in other words if given two arbitrary $c \cap U_i$ and $c \cap U_j$ with a non empty overlap, $\forall x \in (c(\mathbb{R}) \cap U_i) \cap (c(\mathbb{R}) \cap U_j)$ we have that:

$$\Big[\sum_{k=0}^{N} c^{\star}\Big(\nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}\Big]_{|t} = \Big[\sum_{k=0}^{N} c^{\star}\Big(\nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(j)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(j)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}\Big]_{|t} \tag{3.3.4}$$

Since ϕ is a global test tensor field and $\nabla^k_{()}(\phi)$ is a global test tensor field as well, we can use the local trivialisation induced by $(e_{(i)\mu})$ and calculate explicitly the transformation rules for the components $\nabla^k_{(\lambda_{\overline{k}})}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}$ in terms of linear combination of products of $\Lambda^{\mu}_{(ij)\nu}$

and the new components $\nabla^k_{(\lambda_{\overline{k}})}(\phi)^{\ \mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}$ as follows:

$$\Big[\sum_{k=0}^{N} c^\star\Big(\nabla^k_{(\lambda_{\overline{k}})}(\phi)^{\ \mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}\Big)\alpha^{(\lambda_{\overline{k}})\ \nu_{\overline{q}}}_{(j)\qquad \mu_{\overline{p}}}\Big]_{|t} = \tag{3.3.5}$$

$$=\Big[\sum_{k=0}^{N} c^\star\Big(\nabla^k_{(\lambda_{\overline{k}})}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}\Big]_{|t} = \Big[\sum_{k=0}^{N} c^\star\Big(\nabla^k_{(\lambda_{\overline{k}})}(\phi)^{\ \mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}\Big)c^\star(\Lambda_{(ji)\eta_{\overline{k}}}^{\ \lambda_{\overline{k}}}\overline{\Lambda}^{\sigma_{\overline{p}}}_{(ji)\mu_{\overline{p}}}\Lambda^{\rho_{\overline{p}}}_{(ji)\nu_{\overline{q}}})\alpha^{\eta_{\overline{s}}\ \sigma_{\overline{q}}}_{(i)\qquad \rho_{\overline{p}}}\Big]_{|t} \tag{3.3.6}$$

where $\Lambda_{(ji)\eta_{\overline{k}}}^{\ \lambda_{\overline{k}}}\overline{\Lambda}^{\sigma_{\overline{p}}}_{(ji)\mu_{\overline{p}}}\Lambda^{\rho_{\overline{p}}}_{(ji)\nu_{\overline{q}}}$ is a bunch of scalar fields formed by linear combinations of pullbacks along c of products of $\Lambda^{\mu}_{(ij)_\nu}$ with respect the local frame. Considering this, knowing that the symmetric part of the differential operators $\nabla^k_{()}$ are completely $C^\infty(M)$ linear independent, we can state that the only way to satisfy the compatibility condition for each test tensor ϕ is to have a bunch of scalar fields $\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}$ completely symmetric in $\lambda_{\overline{k}}$ such that:

$$\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(j)\qquad \mu_{\overline{p}}} = c^\star(\Lambda_{(ji)\eta_{\overline{k}}}^{\ \lambda_{\overline{k}}}\overline{\Lambda}^{\sigma_{\overline{p}}}_{(ji)\mu_{\overline{p}}}\Lambda^{\rho_{\overline{p}}}_{(ji)\nu_{\overline{q}}})\alpha^{\eta_{\overline{k}}\ \sigma_{\overline{q}}}_{(i)\qquad \rho_{\overline{p}}} \quad , \quad \forall k \in [0,N], \forall U_i, U_j \in \mathcal{A} \mid U_i \cap U_j \neq \varnothing \tag{3.3.7}$$

Just in this case, they are able to define a good global smooth scalar field on $\mathbb{R}$ so a global 1-form on $\mathbb{R}$. It is very interesting to notice that the transformation rules this the bunch of scalar fields $\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}$ is completely tensorial and does not mix the different orders in k. □

It is interesting to notice that without requiring the complete symmetry in the indices contracted with the first k indices of the covariant differential, we are not able to split uniquely the different orders and when changing the local frame, the terms contracted with the higher order differentials are mixed together in a tensorial way. This time we are able to write explicitly, without any trouble, the transformation rules split for each order. They are very simple because we are just involving linear combinations of the components of the tensor fields and the transformation matrix ruling the change of frame, without any interference of terms related to different order covariant differentials. So in contrast with the Ellis case, this time the gluing condition we need in order to define a smooth global top form on $\mathbb{R}$ are very simple and can be easily expressed at each order without any trouble. So we can conclude that there always exists an appropriate linear combination of local sections of $\Lambda^1\mathbb{R}$ defined by

$$\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma_U \Lambda^1\mathbb{R} \tag{3.3.8}$$

that can be glued toghether to form a global top form over $\mathbb{R}$ and the bunch of local smooth scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ must transform like the components of a tensor field when a change of local frame is performed.

Lemma 28: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialization of TM due to the local frame $(e_{(i)\mu})$. Let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$ and ψ_i a smooth partition of the unity subordinate to (U_i). For each test tensor a bunch of local smooth scalar field $\{\alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c(N)\cap U_i}\Lambda^0\mathbb{R} | U_i \in \mathcal{A}\}$ completely symmetric in $\lambda_{\overline{k}}$ defines a global smooth section of $\Gamma\Lambda^1\mathbb{R}$ via the linear combination:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) \alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{3.3.9}$$

if and only if there exists a set of global sections $\alpha^{(k)} \in \Gamma T_p^{(k)+q}{}_{c(\mathbb{R})}M, \forall k \in [0,N] \subset \mathbb{N}$ completely symmetric in k such that:

$$\alpha^{(k)}(e_{(i)}^{\lambda_{\overline{k}}}, e_{(i)}^{\nu_{\overline{q}}}, e_{(i)\mu_{\overline{p}}}) = \alpha_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \tag{3.3.10}$$

Proof. Given $c : \mathbb{R} \hookrightarrow M$, the bundle $T_p^{k+q}{}_{c(\mathbb{R})}M$ is the bundle defined as:

$$\Big(\bigsqcup_{s\in\mathbb{R}} T_p^{k+q}{}_{c(s)}M, \mathbb{R}, \pi, \mathbb{R}^{m^{k+q+p}}\Big) \tag{3.3.11}$$

where the projection $\pi : T_p^{k+q}{}_{c(\mathbb{R})}M \to \mathbb{R}$ is defined as $\pi(\alpha_s^{(k)}) = s, \forall \alpha_{c(s)}^{(k)} \in T_p^{k+q}{}_{c(s)}M$. Given an open set $U \subset M$ and $I \subset \mathbb{R}$ such that $U \cap c(I) \neq \varnothing$ we can always induce local sections in $\Gamma_I T_p^{k+q}{}_{c(\mathbb{R})}M$ from local sections in $\Gamma_U T_p^{k+q}M$ due to the composition with the map $c : \mathbb{R} \hookrightarrow M$. Since c is a smooth closed embedding the induced smooth sections are smooth. Because of this reason, fixing $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $e_{(i)\mu} : U \to TM$, also induces automatically a trivialisation of $T_p^{k+q}{}_{c(\mathbb{R})}M$ just composing the local frame with the closed embedding c. This bundle is a vector bundle and this kind of trivialisation compatible with the vector structure. Given two open sets $I_i, I_j \subseteq \mathbb{R}$ such that $I_i \cap I_j \neq \varnothing$ using the definition of this particular trivialisation we have that:

$$\alpha_{(j)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} = c^{\star}(\Lambda_{(ji)\eta_{\overline{k}}}^{\lambda_{\overline{k}}}\overline{\Lambda}_{(ji)\mu_{\overline{p}}}^{\sigma_{\overline{p}}}\Lambda_{(ji)\nu_{\overline{q}}}^{\rho_{\overline{p}}})\alpha_{(i)}^{\eta_{\overline{k}}\,\sigma_{\overline{q}}}{}_{\rho_{\overline{p}}} \quad , \quad \forall I_i, I_j \in \mathcal{A} \mid U_i \cap U_j \neq \varnothing \tag{3.3.12}$$

therefore the same condition about the compatibility coincides exactly with the request to be a good smooth global section of the bundle $T_p^{k+q}{}_{c(\mathbb{R})}M$. In addition to this we are imposing the total symmetry in the $\lambda_{\overline{k}}$ indices. Since $\alpha_{(j)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ satisfies the transformation rules of a tensor fields restricted to $c(\mathbb{R})$ we know that the property of symmetry is preserved, this means that is an intrinsic property and the sections itself must belong to $\Gamma T_p^{(k)+q}{}_{c(\mathbb{R})}M$ the smooth tensor fields restricted to $c(\mathbb{R})$ and symmetric in the first k arguments. □

This lemma is very important because it fixes a one to one relation between the smooth tangent tensor field of M restricted to the sub-manifold $c(\mathbb{R})$ and the 1-form on $\mathbb{R}$ defined above. We will see how this form fixes the local Dixon representation of the action of a multipole, therefore this lemma will allow us to associate to the multipoles a bunch of smooth tensor field restricted on the worldline. This link is very useful and we will analyse it in detail in the next chapters. For now we just settle to make explicit the geometrical nature of the smooth local scalar fields $\alpha_{(i)}^{\eta_{\overline{k}}\,\sigma_{\overline{q}}}{}_{\rho_{\overline{p}}} : \mathbb{R} \to \mathbb{R}$ showing that they can be interpreted as the local expression of an appropriate bunch $\alpha^{(k)} \in \Gamma T_p^{(k)+q}{}_{c(\mathbb{R})}$of global smooth tensors fields restricted on the sub-manifold $c(\mathbb{R})$. Since this identification is done we gain automatically the good transformation rules that we need to define the 1- form

$$\sum_{k=0}^{N} c^{\star}\left(\nabla^k_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\right)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.13}$$

we are going to use that later to explicit the Dixon local representation of the multipoles.

Theorem 4: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialization of TM due to the local frame $(e_{(i)\mu})$. Let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$ and ψ_i a smooth partition of the unity subordinate to (U_i). For each Dixon multipole $\mathcal{T} \in \Delta_q^p(c)$, there always exists at least $N \in \mathbb{N} \mid N \geq ord(\mathcal{T})$ and a bunch of global smooth sections $\alpha^{(k)} \in \Gamma T_p^{(k)+q}{}_{c(\mathbb{R})}M$ completely symmetric in the first k arguments such that $\forall \phi \in \Gamma_0 T^p_q M$ the multipole $\mathcal{T}$ acts on the local expression of ϕ as follows:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^\star \Big(\nabla^k_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{3.3.14}$$

Proof. To prove the statement it's enough to show that all $Cor^q_p(c)$ can be written in that way and then prove that the given expression is closed under the operations we used to define $\Delta^q_p(c)$. Let us consider a functional $\mathcal{S} \in Cor^q_p(c)$, we know by definition that there exists at least a $T \in T^q_p$ and a $\alpha \in \Gamma\Lambda^1\mathbb{R}$ such that $\mathcal{S} = T \cdot c_\zeta(\alpha)$ So we can write the action of this functional on an arbitrary test tensor field ϕ as an action on its local coordinate expression:

$$[\phi, \mathcal{S}] = [\phi, T \cdot c_\zeta(\beta)] = \int_{\mathbb{R}} c^\star(T(\phi)) \wedge \beta = \tag{3.3.15}$$

$$= \int_{\mathbb{R}} c^\star \Big(T \Big[\sum_{U_i \in \mathcal{A}} \psi_i \phi^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} e^{\nu_{\overline{q}}}_{(i)} \otimes e_{(i)\mu_{\overline{p}}} \Big] \Big) \wedge \beta = \tag{3.3.16}$$

$$= \int_{\mathbb{R}} c^\star \Big(\sum_{U_i \in \mathcal{A}} \psi_i \phi^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T \Big[e^{\nu_{\overline{q}}}_{(i)} \otimes e_{(i)\mu_{\overline{p}}} \Big] \Big) \wedge \beta = \tag{3.3.17}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\psi_i \phi^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \Big) \wedge \beta = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\phi^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) c^\star(T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}) \cdot \tilde{\beta} ds \tag{3.3.18}$$

We know that $ord(\mathcal{S}) = 0$,therefore fixed an arbitrary $N \in \mathbb{N}$ we can define a bunch of local scalar fields:

$$\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} = \begin{cases} c^\star(T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}) \cdot \tilde{\beta} & , \quad \forall k = 0 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.3.19}$$

and rewrite the expression as follow:

$$[\phi, \mathcal{S}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\phi^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\ \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds \tag{3.3.20}$$

Since by construction, $\forall \phi \in \Gamma_0 T^p_q M$ the local scalar fields $\alpha_{(i)\mu_{\overline{p}}}^{\ \ \nu_{\overline{q}}}$ satisfy the transformation rules :

$$\alpha_{(j)\mu_{\overline{p}}}^{\ \ \nu_{\overline{q}}} = c^\star(T^{\nu_{\overline{q}}}_{(j)\mu_{\overline{p}}}) \cdot \tilde{\beta} = c^\star(\Lambda^{\nu_{\overline{q}}}_{(ji)\beta_{\overline{q}}} \overline{\Lambda}^{\beta_{\overline{p}}}_{(ji)\mu_{\overline{p}}} T^{\alpha_{\overline{q}}}_{(j)\beta_{\overline{p}}}) \cdot \tilde{\beta} = \tag{3.3.21}$$

$$= c^\star(\Lambda^{\nu_{\overline{q}}}_{(ji)\alpha_{\overline{q}}} \overline{\Lambda}^{\beta_{\overline{p}}}_{(ji)\mu_{\overline{p}}}) c^\star(T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}) \cdot \tilde{\beta} \tag{3.3.22}$$

which can be interpreted as the local expression induced by the atlas $\mathcal{A}$ and the trivialisation $(e_{(i)\mu})$ of a smooth global section $\alpha \in \Gamma T^q_p{}_{c(\mathbb{R})} M$. So we have the thesis for each elements of $Cor^p_q(c)$.

Now let us assume that a generic $\mathcal{T}, \mathcal{S} \in \Delta^q_p(c)$ satisfies the thesis, then we would like to prove that $\forall \phi \in \Gamma_0 T^p_q M$:

1. there always exists at least a $N \in \mathbb{N} | N \geq ord(\mathcal{S} + \mathcal{T})$ and a bunch of global smooth sections $\beta^{(k)} \in \Gamma T^{(k)+q}_p{}_{c(\mathbb{R})} M$ such that, $\mathcal{T} + \mathcal{S}$ acts on the local expressions of ϕ as follow:

$$[\phi, \mathcal{S} + \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^\star \left(\nabla^k_{\lambda_{\overline{k}}} (\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \right) c^\star(\psi_i) \beta_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.23}$$

2. there always exists at least a $N \in \mathbb{N} \mid N \geq ord(f \cdot \mathcal{T})$ a bunch of global smooth sections $\beta^{(k)} \in \Gamma T^{(k)+q}_p{}_{c(\mathbb{R})} M$ such that, $f \cdot \mathcal{T}$ acts on the local expressions of ϕ as follow:

$$[\phi, f \cdot \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^\star \left(\nabla^k_{\lambda_{\overline{k}}} (\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \right) c^\star(\psi_i) \beta_{(i)}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.24}$$

3. For each smooth global vector field $v \in \Gamma TM$ there always exists at least a $N \in \mathbb{N} \mid N \geq ord(div(v \cdot \mathcal{T}))$ a bunch of global smooth sections $\gamma^{(k)} \in \Gamma T^{(k)+q}_p{}_{c(\mathbb{R})} M$ such that $div(v \cdot \mathcal{T})$ acts on the local expressions of ϕ as follow:

$$[\phi, div(v\cdot\mathcal{T})] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{N}\int_{\mathbb{R}} c^{\star}\Big(\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{3.3.25}$$

Let us start with the first:

$$[\phi, \mathcal{S}+\mathcal{T}] = [\phi,\mathcal{S}] + [\phi,\mathcal{T}] = \tag{3.3.26}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{K}\int_{\mathbb{R}} c^{\star}\Big(\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds + \tag{3.3.27}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{J}\int_{\mathbb{R}} c^{\star}\Big(\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \tag{3.3.28}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} \Big[c^{\star}\Big(\sum_{k=0}^{K}\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} + c^{\star}\Big(\sum_{k=0}^{J}\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}\Big] ds \tag{3.3.29}$$

Let us suppose to have $ord(\mathcal{S}) \leq ord(\mathcal{T}) \leq K$, the other case follow in the same manner. Since $ord(\mathcal{S}) \leq ord(\mathcal{T})$ we can always define an $N \geq K \geq ord(\mathcal{T}) \geq ord(\mathcal{S}+\mathcal{T})$ and a new bunch of smooth local scalar fields:

$$\beta^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} = \begin{cases} \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} + \hat{\alpha}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}\ , \quad \forall k \leq J \\ \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}\ ,\ J < k \leq K \\ 0,\ \ K < k \leq N \end{cases} \tag{3.3.30}$$

Therefore using it on the expression

$$[\phi, \mathcal{S}+\mathcal{T}] = \tag{3.3.31}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} \Big[c^{\star}\Big(\sum_{k=0}^{K}\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} + c^{\star}\Big(\sum_{k=0}^{J}\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i)\hat{\alpha}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}\Big] ds = \tag{3.3.32}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} \Big[c^\star \Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) c^\star(\psi_i) \tilde{\alpha}_{(i)}^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.33}$$

Since by construction, $\forall \phi \in \Gamma_0 T^p_q M$ the local scalar fields $\alpha_{(i)\mu_{\overline{p}}}^{\;\;\nu_{\overline{q}}}$ and $\alpha_{(i)\mu_{\overline{p}}}^{\;\;\nu_{\overline{q}}}$ satisfies the right linear transformation rules therefore the sum of them still satisfies the same transformation rule, therefore it can be interpreted as the local expression induced by the atlas $\mathcal{A}$ and the trivialisation $(e_{(i)\mu})$ of a smooth global sections $\beta^{(k)} \in \Gamma T^q_p{}_{c(\mathbb{R})} M$. To prove the closure with respect the multiplication with scalar fields it is enough to use the previous lemma

$$\nabla^k(f\phi) = \sum_{j=0}^{k} H_{(j,k)}[\nabla^{k-j}(f)\nabla^j(\phi)] \tag{3.3.34}$$

and explicit the local coordinate expression:

$$\nabla^k_{\lambda_{\overline{k}}}(f\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k} H_{(j,k)\lambda_k\nu_{\overline{q}}\sigma_{\overline{p}}}^{\qquad\quad \mu_{\overline{p}}\rho_{\overline{q+k}}} \nabla^{k-j}_{\rho_{\overline{k-j}}}(f) \nabla^j_{\rho_{\overline{k}\setminus\overline{k-j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{k+q}\setminus\overline{k}}} \tag{3.3.35}$$

We can use it in the equation:

$$[\phi, f\cdot\mathcal{T}] = [f\phi, \mathcal{T}] = \tag{3.3.36}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^\star \Big(\nabla^k_{\lambda_{\overline{k}}}(f\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) c^\star(\psi_i) \alpha_{(i)}^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{3.3.37}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k} H_{(j,k)\lambda_k\nu_{\overline{q}}\sigma_{\overline{p}}}^{\qquad\quad \mu_{\overline{p}}\rho_{\overline{q+k}}} \nabla^{k-j}_{\rho_{\overline{k-j}}}(f) \nabla^j_{\rho_{\overline{k}\setminus\overline{k-j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{k+q}\setminus\overline{k}}} \Big) c^\star(\psi_i) \alpha_{(i)}^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.38}$$

At this stage it is enough to re-sum order by order in ∇^j defining a new bunch of scalar fields $\gamma_{(i)}^{\lambda_{\overline{j}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ as appropriate (a quite complicated) linear combination of $c^\star(H_{(j,k)\lambda_k\nu_{\overline{q}}\sigma_{\overline{p}}}^{\qquad\quad \mu_{\overline{p}}\rho_{\overline{q+k}}} \nabla^{k-j}_{\rho_{\overline{k-j}}}(f))$

and $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$. Therefore we have:

$$[\phi, f\cdot\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{j=0}^{N} \int_{\mathbb{R}} c^{\star}\Big(\nabla^{j}_{\lambda_{\overline{j}}}(\phi)\Big) c^{\star}(\psi_i)\gamma_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.39}$$

We still miss the symmetry in the $\lambda_{\overline{j}}$ indices but this is not an issue. In fact we know from the previous lemma that each term $\nabla^j(\phi)$ can be written in the following way:

$$\nabla^j(\phi) = \nabla^j_{(\,)}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + \sum_{s=0}^{j-1} D_{s,j}[\nabla_{(\,)}(\phi)] \tag{3.3.40}$$

so we have:

$$[\phi, f\cdot\mathcal{T}] = \tag{3.3.41}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{j=0}^{N}\sum_{s=0}^{j} D_{(s,j)\lambda_k\nu_{\overline{q}}\sigma_{\overline{p}}}{}^{\mu_{\overline{p}}\alpha_{\overline{s}}\rho_{\overline{q}}} \nabla^{s}_{(\alpha_{\overline{s}})}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big) c^{\star}(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.42}$$

Once again, one can re-sum order by order in ∇^s defining a new bunch of scalar fields $\beta_{(i)}^{\lambda_{\overline{s}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ as an appropriate linear combination of $c^{\star}(D_{(s,j)\lambda_k\nu_{\overline{q}}\sigma_{\overline{p}}}{}^{\mu_{\overline{p}}(\alpha_{\overline{s}})\rho_{\overline{q}}})$ and $\gamma_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ to end up with:

$$[\phi, f\cdot\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^{\star}\Big(\nabla^{j}_{(\lambda_{\overline{k}})}(\phi)\Big) c^{\star}(\psi_i)\beta_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{3.3.43}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^{\star}\Big(\nabla^{j}_{\lambda_{\overline{k}}}(\phi)\Big) c^{\star}(\psi_i)\beta_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.44}$$

Let us stress that by construction the bunch of local scalar fields $\beta_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ is completely symmetric in the $\lambda_{\overline{k}}$ indices.

The previous definition leads us to conclude also:

$$\sum_{k=0}^{N} c^\star\Big(\nabla^k_{\lambda_{\overline{k}}}(f\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\ \mu_{\overline{p}}}ds = \sum_{k=0}^{N} c^\star\Big(\nabla^k_{(\lambda_{\overline{k}})}(f\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\quad\ \mu_{\overline{p}}}ds = \tag{3.3.45}$$

$$= c^\star\Big(\sum_{j=0}^{N}\nabla^j_{\lambda_{\overline{j}}}(\phi)\Big)c^\star(\psi_i)\beta^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\quad\ \mu_{\overline{p}}}ds = c^\star\Big(\sum_{j=0}^{N}\nabla^j_{(\lambda_{\overline{j}})}(\phi)\Big)c^\star(\psi_i)\beta^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\quad\ \mu_{\overline{p}}}ds \tag{3.3.46}$$

then it is enough to use the previous lemma to state that $\beta^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\quad\ \mu_{\overline{p}}}$ are the local expression of a bunch of global sections $\beta^{(j)} \in \Gamma T^{(j)+q}_{p\quad c(\mathbb{R})}M$. Concerning the closure with respect the "divergence of the product with vector fields" it is enough to explicit the expression:

$$[\phi, div(u\cdot\mathcal{T})] = -[\nabla_u(\phi), \mathcal{T}] = \tag{3.3.47}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\nabla^j_{\lambda_{\overline{j}}}(\nabla_u\phi)\Big)c^\star(\psi_i)\alpha^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}_{(i)\quad\ \mu_{\overline{p}}}ds \tag{3.3.48}$$

Now using the lemma in the first section we can say that there always exists a set of smooth maps (in general they are not C^∞ -linear)

$$Z_{(j,k)} : \times^{j+1}(\Gamma TM) \to \Gamma T^k_0 M \tag{3.3.49}$$

such that for each bunch of smooth local frame $e_{\lambda_{\overline{j}}}$ and a vector field u:

$$\nabla^j_{\lambda_{\overline{j}}}(\nabla_u(\phi)) = \sum_{k=1}^{j+1} i^k([Z_{(j,k)}(e_{\lambda_{\overline{k}}, u})] \otimes \nabla^k(\phi)) \tag{3.3.50}$$

and using the other lemma about the symmetric decomposition of the covariant derivative we can write:

$$\nabla^j_{\lambda_{\overline{j}}}(\nabla_u(\phi)) = \sum_{k=1}^{j+1} i^k([Z_{(j,k)}(e_{\lambda_{\overline{k}}, u})] \otimes \nabla^k(\phi)) = \tag{3.3.51}$$

$$=\sum_{k=1}^{j+1} i^k([Z_{(j,k)}(e_{\lambda_{\bar{j}}},u)]\otimes\sum_{l=0}^{k} D_{(l,k)}[\nabla^l_{()}(\phi))] \tag{3.3.52}$$

where $D_{(l,k)}$ are smooth $C^\infty(M)$-linear maps acting on the tensor fields. Now let us point out that, even if the expression is not $C^\infty(M)$-linear in $(e_{\lambda_{\bar{j}}})$ or u, it is $C^\infty(M)$-linear in the $\nabla^l_{()}(\phi)$ arguments. Therefore we can re-sum order by order in l defining a new set of local maps $E_{j,l} : \times^{j+1}(\Gamma_U TM)\times T^p_{(l)+q}M \to T^p_q M$

$$\sum_{l=0}^{j+1} E_{(j,l)}(e_{\lambda_{\bar{j}}},u,\nabla^l_{()}\phi) := \sum_{k=1}^{j+1} i^k([Z_{(j,k)}(e_{\lambda_{\bar{j}}},u)]\otimes\sum_{l=0}^{k} D_{(l,k)}[\nabla^l_{()}(\phi))] = \nabla^j_{\lambda_{\bar{j}}}(\nabla_u(\phi)) \tag{3.3.53}$$

and they are $C^\infty(M)$-linear. In general the explicit expression of $E_{(j,l)}$ can be very complicated but for the purpose of our theorem we just need to know that they there exist. Now we can use the statement above to manipulate the original expression obtaining:

$$[\phi, div(u\cdot\mathcal{T})] = -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\nabla^j_{\lambda_{\bar{j}}}(\nabla_u\phi)^{\mu_{\bar{p}}}_{(i)\nu_{\bar{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\bar{j}}\,\nu_{\bar{q}}}{}_{\mu_{\bar{p}}}ds = \tag{3.3.54}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\Big\{\sum_{l=0}^{j+1}E_{(j,l)}(e_{\lambda_{\bar{j}}},u,\nabla^l_{()}\phi)\Big\}^{\mu_{\bar{p}}}_{(i)\nu_{\bar{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\bar{j}}\,\nu_{\bar{q}}}{}_{\mu_{\bar{p}}}ds = \tag{3.3.55}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\Big\{\sum_{l=0}^{j+1}E_{(j,l)}(e_{\lambda_{\bar{j}}},u,\nabla^l_{(\sigma_{\bar{l}})}(\phi)^{\rho_{\bar{p}}}_{(i)\eta_{\bar{q}}}e^{\sigma_{\bar{l}}}_{(i)}\otimes e^{\eta_{\bar{q}}}_{(i)}\otimes e_{(i)\rho_{\bar{p}}}\Big\}^{\mu_{\bar{p}}}_{\nu_{\bar{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\bar{j}}\,\nu_{\bar{q}}}{}_{\mu_{\bar{p}}}ds = \tag{3.3.56}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\Big\{\sum_{l=0}^{j+1}E_{(j,l)}(e_{\lambda_{\bar{j}}},u,e^{\sigma_{\bar{l}}}_{(i)}\otimes e^{\eta_{\bar{q}}}_{(i)}\otimes e_{(i)\rho_{\bar{p}}}\Big\}^{\mu_{\bar{p}}}_{\nu_{\bar{q}}}\nabla^l_{(\sigma_{\bar{l}})}(\phi)^{\rho_{\bar{p}}}_{(i)\eta_{\bar{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\bar{j}}\,\nu_{\bar{q}}}{}_{\mu_{\bar{p}}}ds = \tag{3.3.57}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\Big\{\sum_{l=0}^{j+1}E_{(j,l)}(e_{\lambda_{\bar{j}}},u)\Big\}^{\sigma_{\bar{l}}\eta_{\bar{q}}\mu_{\bar{p}}}_{\rho_{\bar{p}}\nu_{\bar{q}}}\nabla^l_{(\sigma_{\bar{l}})}(\phi)^{\rho_{\bar{p}}}_{(i)\eta_{\bar{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\bar{j}}\,\nu_{\bar{q}}}{}_{\mu_{\bar{p}}}ds = \tag{3.3.58}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\Big\{\sum_{l=0}^{j+1}E_{(j,l)}(e_{\lambda_{\bar{j}}},u)\Big\}^{(\sigma_{\bar{l}})\eta_{\bar{q}}\mu_{\bar{p}}}_{\rho_{\bar{p}}\nu_{\bar{q}}}\nabla^l_{\sigma_{\bar{l}}}(\phi)^{\rho_{\bar{p}}}_{(i)\eta_{\bar{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\bar{j}}\,\nu_{\bar{q}}}{}_{\mu_{\bar{p}}}ds \tag{3.3.59}$$

Now it is enough to re-sum order by order in ∇^l defining a new bunch of scalar fields $\gamma_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ as appropriate linear combination of $c^\star(E_{(j,l)}(e_{\lambda_{\overline{j}}},u)^{\sigma_{\overline{l}}\eta_{\overline{q}}\mu_{\overline{p}}}_{\rho_{\overline{p}}\nu_{\overline{q}}})$ and $\alpha_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ completely symmetrised with respect the indices $\sigma_{\overline{l}}$, to conclude that:

$$[\phi, div(u\cdot\mathcal{T})] = \tag{3.3.60}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\Big\{\sum_{l=0}^{j+1} E_{(j,l)}(e_{\lambda_{\overline{j}}},u)\Big\}^{(\sigma_{\overline{l}})\eta_{\overline{q}}\mu_{\overline{p}}}_{\rho_{\overline{p}}\nu_{\overline{q}}} \nabla^l_{\sigma_{\overline{l}}}(\phi)^{\rho_{\overline{p}}}_{(i)\eta_{\overline{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds = \tag{3.3.61}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{l=0}^{N+1}\nabla^l_{\sigma_{\overline{l}}}(\phi)^{\rho_{\overline{p}}}_{(i)\eta_{\overline{q}}}\Big)c^\star(\psi_i)\gamma_{(i)}^{\sigma_{\overline{l}}\,\eta_{\overline{q}}}{}_{\rho_{\overline{p}}}ds = \tag{3.3.62}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{l=0}^{N+1}\nabla^l_{\sigma_{\overline{l}}}(\phi)^{\rho_{\overline{p}}}_{(i)\eta_{\overline{q}}}\Big)c^\star(\psi_i)\gamma_{(i)}^{\sigma_{\overline{l}}\,\eta_{\overline{q}}}{}_{\rho_{\overline{p}}}ds \tag{3.3.63}$$

By construction, since the complete symmetry is satisfied on the $\sigma_{\overline{l}}$ indices:

$$\sum_{k=0}^{N} c^\star\Big(\nabla^k_{\lambda_{\overline{k}}}(\nabla_u\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds = \sum_{k=0}^{N} c^\star\Big(\nabla^k_{(\lambda_{\overline{k}})}(\nabla_u\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds = \tag{3.3.64}$$

$$= c^\star\Big(\sum_{j=0}^{N+1}\nabla^j_{\lambda_{\overline{j}}}(\phi)\Big)c^\star(\psi_i)\gamma_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds = c^\star\Big(\sum_{j=0}^{N+1}\nabla^j_{(\lambda_{\overline{j}})}(\phi)\Big)c^\star(\psi_i)\gamma_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{3.3.65}$$

then it is enough to use the previous lemma to state that $\gamma_{(i)}^{\lambda_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ must be the local expression of a bunch of global sections $\gamma^{(j)}\in\Gamma T^{(j)+q}_{p\ \ c(\mathbb{R})}M$. □

Theorem 5: Let $c:\mathbb{R}\hookrightarrow M$. For each Dixon multipole $\mathcal{T}\in\Delta^q_p(c)$, $\forall u\in\Gamma TM$ then $L_u(\mathcal{T})\in\Delta^q_p(c)$. In other words the set $\Delta^q_p(c)$ is closed with respect the Lie derivative.

Proof. Given $\mathcal{T}\in\Delta^q_p(c)$ we have that:

$$[\phi, L_u\mathcal{T}] = -[L_u(\phi),\mathcal{T}] = -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\nabla^k_{\lambda_{\overline{k}}}(L_u\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{3.3.66}$$

Using the result of a previous lemma we can write the local expression of the Lie derivative in terms of $C^\infty(M)$-linear combinations of covariant derivatives:

$$[\nabla^k_{\lambda_{\overline{k}}}(L_v(\phi))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)} \nabla^j_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{3.3.67}$$

and then we can use the lemma about the symmetric decomposition of the higher order derivatives we can write:

$$[\nabla^k_{\lambda_{\overline{k}}}(L_v(\phi))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)} \sum_{l=0}^{j} D_{(l,j)}[\nabla^l_{(\alpha_{\overline{j}})}(\phi)]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} = \tag{3.3.68}$$

where $D_{(l,k)}$ are smooth $C^\infty(M)$-linear maps acting on the tensor fields. Now let us point out that the expression is completely $C^\infty(M)$-linear in the $\nabla^l_{()}(\phi)$ arguments. So:

$$[\nabla^k_{\lambda_{\overline{k}}}(L_v(\phi))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)} \sum_{l=0}^{j} D_{(l,j)}\left[\nabla^l_{(\delta_{\overline{l}})}(\phi))^{\eta_{\overline{p}}}_{\gamma_{\overline{q}}}(e^{\delta_{\overline{l}}} \otimes e^{\gamma_{\overline{q}}} \otimes e_{\eta_{\overline{p}}})\right]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} = \tag{3.3.69}$$

$$= \sum_{j=0}^{(k+1)} \sum_{l=0}^{j} \nabla^l_{(\delta_{\overline{l}})}(\phi))^{\eta_{\overline{p}}}_{\gamma_{\overline{q}}} D_{(l,j)}\left[(e^{\delta_{\overline{l}}} \otimes e^{\gamma_{\overline{q}}} \otimes e_{\eta_{\overline{p}}})\right]^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} = \tag{3.3.70}$$

$$= \sum_{j=0}^{(k+1)} \sum_{l=0}^{j} \nabla^l_{(\delta_{\overline{l}})}(\phi))^{\eta_{\overline{p}}}_{\gamma_{\overline{q}}} \left[D_{(l,j)}\right]^{\delta_{\overline{l}}\gamma_{\overline{q}}\sigma_{\overline{p}}}_{\eta_{\overline{p}}\rho_{\overline{q}}} h^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{3.3.71}$$

and re-sum order by order in $\nabla^l_{()}(\phi)$ we can define a new bunch of smooth local scalar fields $f^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}$ such that:

$$[\nabla^k_{\lambda_{\overline{k}}}(L_v(\phi))]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{(k+1)} \nabla^j_{(\alpha_{\overline{j}})}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} \tag{3.3.72}$$

In general the explicit expression of $g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}$ depends on v in a very complicated non linear way, but for the purpose of our theorem we just need to know that they there always exist.

Now we can use the statement above to manipulate the original expression obtaining:

$$[\phi, L_u\mathcal{T}] = -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(L_u\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{3.3.73}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{(k+1)} \nabla^j_{(\alpha_{\overline{j}})}(\phi)^{\ \sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{3.3.74}$$

$$= -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{(k+1)} \nabla^j_{(\alpha_{\overline{j}})}(\phi)^{\ \sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}\Big) c^\star\Big(g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.75}$$

Once again it is enough to re-sum order by order in ∇^j defining a new bunch of scalar fields $\gamma_{(i)}^{\gamma_{\overline{k}}\,\rho_{\overline{q}}}{}_{\sigma_{\overline{p}}}$ as an appropriate linear combination of $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ and $c^\star\Big(g^{\alpha_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}}\Big)$ completely symmetrized with respect to the indices $\alpha_{\overline{j}}$ to conclude that:

$$[\phi, L_u\mathcal{T}] = -\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N}\sum_{j=0}^{(k+1)} \nabla^j_{(\alpha_{\overline{j}})}(\phi)^{\ \sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}\Big) c^\star(\psi_i)\gamma_{(i)}^{\alpha_{\overline{j}}\,\rho_{\overline{q}}}{}_{\sigma_{\overline{p}}} ds \tag{3.3.76}$$

Since by hypothesis, $\forall\phi\in\Gamma_0 T^p_q M$ the local scalar fields $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ completely symmetric in $\lambda_{\overline{k}}$ defines the a good global smooth form on $\mathbb{R}$ with

$$c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \psi_i \sum_{k=0}^{N} L_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.77}$$

and since $\forall\phi\in\Gamma_0 T^p_q M, \forall u\in\Gamma TM \Rightarrow L_u\phi\in\Gamma_0 T^p_q M$, we have that by construction:

$$c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \psi_i \sum_{k=0}^{N} \nabla_{\lambda_{\overline{k}}}(L_u\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \psi_i \sum_{j=0}^{N+1} \nabla_{\alpha_{\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\gamma_{(i)}^{\alpha_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{3.3.78}$$

is a global smooth 1- form over $\mathbb{R}$ and:

$$c^{\star}\Big(\sum_{j=0}^{N+1}\nabla_{\alpha_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\gamma_{(i)\ \ \ \ \mu_{\overline{p}}}^{\alpha_{\overline{j}}\ \nu_{\overline{q}}}ds \tag{3.3.79}$$

can be interpreted as its local expressions. $\square$

Corollary 9: Since $\forall\mathcal{T}\in\Delta_p^q(c),\forall u\in\Gamma TM$ we have that $L_u(\mathcal{T})\in\Delta_p^q(c)$, the set $\Delta_p^q(c)$ is closed with respect the "Lie Derivative". Therefore by definition $\forall\mathcal{T}\in\Delta_q^p(c)\Rightarrow\mathcal{T}\in\Upsilon_p^q(c)$. This means that $\Delta_q^q(c)\subseteq\Upsilon_p^q(c)$ in other words the Dixon multipoles are just a subset of the Ellis multipoles.

Corollary 10: Since we proved that $\Delta_q^q(c)\subseteq\Upsilon_p^q(c)$ and $\Delta_q^q(c)\supseteq\Upsilon_p^q(c)$ at the same time, we have to conclude that $\Delta_q^q(c)=\Upsilon_p^q(c)$. Therefore the Ellis and Dixon definitions of multipoles single out the same set.

This is a fundamental result. Despite we use two very different definitions at the end there is no difference between the Ellis and the Dixon multipoles. However the two different definitions lead us to two different local expressions of the multipoles.

Definition 59: Let $c:\mathbb{R}\hookrightarrow M$ be a worldline and $\mathcal{A}=(U_i,\varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) , let ∇ be a connection on M and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given a multipole $\mathcal{T}\in\Upsilon_p^q(c)$, we define the ***local Dixon representation of the action of*** $\mathcal{T}$ the integral:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}}c^{\star}\Big(\sum_{k=0}^{N}\nabla_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^{\star}(\psi_i)\alpha_{(i)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}ds\quad,\quad\forall\phi\in\Gamma_0T_q^pM \tag{3.3.80}$$

satisfying the following condition:

1. there must exists a bunch of global sections $\alpha^{(k)}\in\Gamma T_{p\ \ c(\mathbb{R})}^{k+q}M$ completely anti-symmetric in the first k indices such that their local expression with respect to the induced trivialisation of $T_{p\ \ c(\mathbb{R})}^{k+q}M$ is given by $\alpha_{(i)\ \ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}$

2. $\forall\phi \in \Gamma_0 T^p_q M$, the action of $\mathcal{T}$ can be expressed with that integral:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} \nabla_{\lambda_{\overline{k}}}(\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds \tag{3.3.81}$$

Property 35: The existence of the Dixon representation of the action for each multipole $\mathcal{T} \in \Upsilon^q_p(c)$ is guaranteed by the previous theorem.

3.3.2 Dixon local expression of the multipoles

Since the $\mathbb{R}$-linear functionals are entirely defined by their action on the test tensor fields we can use the Dixon local expression of the action of the multipoles, to define a local expression of the multipole themselves. Furthermore since we proved that no difference occurs between the Dixon and the Ellis multipoles we denote with $\Upsilon^q_p(c)$ the set of all the multipoles induced by the embedding $c : \mathbb{R} \hookrightarrow M$ and acting on $\Gamma_0 T^p_q(M)$.

Corollary 11: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialization of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i), let ∇ be a connection on M and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. For each Dixon multipole $\mathcal{T} \in \Upsilon^q_p(c)$, there always exists at least an $N \in \mathbb{N} \mid N \geq ord(\mathcal{T})$ and bunch of smooth global sections of $\Gamma T^{(k)+q}_{p\ \ \ c(\mathbb{R})} M$ completely symmetric in the first k indices and locally expressed by $\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} \in \Gamma_{c \cap U_i}\Lambda^0\mathbb{R}$ such that, $\mathcal{T}$ can be written as:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k div^k \Big\{ \psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta \Big(\alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds \Big) \Big\} \tag{3.3.82}$$

Proof. It follows directly from the previous theorem and from the definition of action of a $\mathbb{R}$-linear functionals on the test tensor fields. $\forall\phi \in \Gamma_0 T^p_q M$, $\mathcal{T}$ acts on the local expression of ϕ as follow:

$$[\phi, \mathcal{T}] = \tag{3.3.83}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)\ \ \ \ \mu_{\overline{p}}} ds = \tag{3.3.84}$$

$$=[\phi, \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1^k) \nabla^k_{\lambda_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta \Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \Big) \Big\}] \tag{3.3.85}$$

□

Property 36: From the standard bundle theory we can state that if

$$c^\star \Big(\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \psi_i \sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}} (\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \in \Gamma \Lambda^1 \mathbb{R} \tag{3.3.86}$$

is good global smooth top form on $\mathbb{R}$, therefore it cannot depend on the choices of the smooth partition of the unity (ψ_i), therefore the multipole $\mathcal{T}$ must not depend on the chosen smooth partition of the unity.

Definition 60: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i), let ∇ a connection on M and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given a multipole $\mathcal{T} \in \Upsilon^q_p(c)$, let

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}} (\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \quad , \quad \forall \phi \in \Gamma_0 T^p_q M \tag{3.3.87}$$

be the local Dixon representation of the action of $\mathcal{T}$. Since

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k div^k \Big\{ \psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta \Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \Big) \Big\} \tag{3.3.88}$$

we define the right hand term ***the Dixon local representation of the multipole*** $\mathcal{T}$.

Property 37: The existence of the Dixon representation for each multipole $\mathcal{T} \in \Upsilon^q_p(c)$ is guaranteed by the previous corollary.

Example: It is very easy to see how this expression represent a generalisation of the

Dixon multipoles given in [1][2][3][4][7], and widely used to express the Pole-Dipole approximation of an extended body in General Relativity . Let us fix the manifold M endowed with a Lorentzian metric g and a local chart $(U_i, \varphi_{(i)})$. Let $c : \mathbb{R} \to \mathbb{R}^4$ be a worldline parametrized such that $g(\dot{c}(\tau), \dot{c}(\tau)) = -1$. Let us fix a local coordinate system $\overline{x} = (x^0, x^1, x^2, x^3)$.The action of an arbitrary multipole with support in U_i on $\phi \in \Gamma_0 T^p_q M$ test tensor field can be then expressed as:

$$\int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}} (\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} \Big) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau = \tag{3.3.89}$$

$$= \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \nabla^k_{(\lambda_1 \cdots \lambda_k)} (\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} \Big) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau = \tag{3.3.90}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} \Big[\nabla^k_{(\lambda_1 \cdots \lambda_k)} (\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} \Big]_{|c(\tau)} \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau = \tag{3.3.91}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}^4} \sqrt{g} d^4x \delta^4(\overline{x} - \overline{c}(\tau)) \Big[\nabla^k_{(\lambda_1 \cdots \lambda_k)} (\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} (\overline{x}) \Big] \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau \tag{3.3.92}$$

However, despite what is usually written, we cannot just discharge the covariant derivatives on the delta Dirac without a well posed definition of the meaning of the object "$\nabla_\lambda \delta(\overline{x})$" as a distribution. But this task is not trivial and straightforward as it can apparently appear, since given a test function ϕ on $\mathbb{R}^4$ the standard distributional definition of the derivation would be $[\phi(\overline{x}), \nabla_\lambda \delta^4(\overline{x})] = -[\nabla_\lambda \phi(\overline{x}), \delta^4(\overline{x})] = -[\partial_\lambda \phi(\overline{x}), \delta^4(\overline{x})] = [\phi(\overline{x}), \partial_\lambda \delta^4(\overline{x})]$ concluding that the connection does not contribute. But expliciting the form of the covariant derivative in the integral there is no chance that the terms related to the connections just cancel out, so discharging naively the covariant derivations on the Dirac delta inside the integral, we would have a contradiction. Because of this, in [2] (page 538) the expression is recast in a very obscure way as follow:

$$\sum_{k=0}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}^4} \sqrt{g} d^4x \delta^4(\overline{x} - \overline{c}(\tau)) \Big[\nabla^k_{(\lambda_1 \cdots \lambda_k)} (\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} (\overline{x}) \Big] \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau = \tag{3.3.93}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}^4} \Big[\int_{\mathbb{R}} \delta^4(\overline{x} - \overline{c}(\tau)) \nabla^k_{(\lambda_1 \cdots \lambda_k)} (\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} (\overline{x}) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau \Big] \sqrt{g} d^4x = \tag{3.3.94}$$

$$= \int_{\mathbb{R}^4} \phi_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} (\overline{x}) \Big[\sum_{k=0}^{N} (-1)^k \nabla^k_{(\lambda_1 \cdots \lambda_k)} \int_{\mathbb{R}} \delta^4(\overline{x} - \overline{c}(\tau)) \alpha^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{\ \ \ \mu_{\overline{p}}} d\tau \Big] \sqrt{g} d^4x = \tag{3.3.95}$$

$$= \int_{\mathbb{R}^4} \phi_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} (\overline{x}) \overset{(N)}{\mathcal{T}}{}^{\nu_{\overline{q}}}_{\mu_{\overline{p}}} \sqrt{g} d^4x \tag{3.3.96}$$

This cannot be considered more than a formal writing, because with this approach, no clear formal definition of the expression is given. However, despite no adequate investigation of the geometrical meaning of this object is usually given, the expression is commonly considered reasonable to express some linear functionals in the Pole-Dipole approximation. For example

$$\overset{(N)}{\mathcal{T}}{}^{\nu_{\overline{q}}}_{\mu_{\overline{p}}} = \sum_{k=0}^{N}(-1)^k \nabla^k_{(\lambda_1 \cdots \lambda_k)} \int_{\mathbb{R}} \delta^4(\overline{x} - \overline{c}(\tau)) \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} d\tau \tag{3.3.97}$$

If $q = 2$ and $p = 0$, and $N = 1$ and imposing that $\overset{(1)}{\mathcal{T}}{}^{\mu\nu}$ must be the "skeleton of an energy momentum tensor field" [2][3][4][7] [20] then

$$\overset{(1)}{\mathcal{T}}{}^{\mu\nu} = \nabla_\lambda \int_{\mathbb{R}} S^{\lambda(\mu} \dot{c}^{\nu)} \delta^4(\overline{x} - \overline{c}(\tau)) d\tau + \int_{\mathbb{R}} P^{(\nu} \dot{c}^{\mu)} \delta^4(\overline{x} - \overline{c}(\tau)) d\tau \tag{3.3.98}$$

As we can see, loosely speaking, the Dixon representation of a multipole consists in writing a multipole $\mathcal{T} \in \Upsilon^q_p(c)$ as a linear combination of bunch of specific multipoles with support on the patches U_i of the atlas $\mathcal{A}$ and glued together such that, for each test tensor field, they can induce a a global smooth 1-form on $\mathbb{R}$ (depending to the test tensor field and its Lie derivatives) which, if integrated, gives exactly the action of the multipole upon the test tensor field. The gluing condition are very easy and it is possible to associate to each multipole a set of global smooth test tensor field restricted to the sub-manifold $\mathbb{R}$.

Corollary 12: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i), let ∇ be a connection on M and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. A bunch of smooth global sections $\alpha^{(k)} \in T_p^{(k)+q}{}_{c(\mathbb{R})}M$ completely symmetric in the first k indices always identify also a multipole due to the Dixon local representation:

$$\mathcal{T} := \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} (-1)^k \nabla^k_{\lambda_{\overline{k}}} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta \Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \Big) \Big\} \tag{3.3.99}$$

Definition 61: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M inducing a local trivialisation of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i), let ∇ be a connection on M and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. A bunch of smooth global sections $\alpha^{(k)} \in T_p^{(k)+q}{}_{c(\mathbb{R})}M$ completely symmetric in the first k indices, locally expressed by $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ and defining a global top form on $\mathbb{R}$ by:

$$c^{\star}\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\psi_i\sum_{k=0}^{N}\nabla^{k}_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds\in\Gamma\Lambda^{1}\mathbb{R} \tag{3.3.100}$$

are called ***Dixon parameters of the multipole*** $\mathcal{T}$:

$$\mathcal{T}:=\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\sum_{k=0}^{N}(-1)^{k}\nabla^{k}_{\lambda_{\overline{k}}}\Big\{\psi_i\Big[e_{(i)}^{\mu_{\overline{p}}}\otimes e_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}\Big(\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds\Big)\Big\} \tag{3.3.101}$$

The local expression $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ related to the Dixon parameters $\alpha^{(k)}$ are called ***Dixon local parameters of the multipole*** $\mathcal{T}$

Theorem 6: Let $c:\mathbb{R}\hookrightarrow M$ be a worldline and $\mathcal{A}=(U_i,\varphi_{(i)})$ an atlas of M inducing a local trivialization of TM due to the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i), let ∇ a connection on M and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given a multipole $\mathcal{T}\in\Upsilon^{q}_{p}(c)$ the following hold:

1. Given the braiding map σ^{I}_{J} the functional $\sigma^{I}_{J}(\mathcal{T})\in\Upsilon^{q}_{p}(c)$
2. Given a global smooth vector field $v\in\Gamma TM$ then $v\cdot\mathcal{T}\in\Upsilon^{q+1}_{p}(c)$
3. Given a global smooth covector field $\alpha\in\Gamma T^{\star}M$ then $\alpha\cdot\mathcal{T}\in\Upsilon^{q}_{p+1}(c)$
4. Given a global smooth vector field $v\in\Gamma TM$ then $v\lrcorner\mathcal{T}\in\Upsilon^{q}_{p-1}(c)$
5. Given a global smooth covector field $\alpha\in\Gamma T^{\star}M$ then $\alpha\urcorner\mathcal{T}\in\Upsilon^{q-1}_{p}(c)$

Proof. We provide here a sketch of proof because it follows the same procedure adopted to prove the closure with respect to the Lie derivative. The first is trivial and it follow from the lemma about the behaviour of the braiding map with respect to the k-th covariant differential.

$$[\phi,\sigma^{I}_{J}\mathcal{T}]=[\sigma^{J}_{I}\phi,\mathcal{T}]=\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}}c^{\star}\Big(\sum_{k=0}^{N}\nabla^{k}_{\lambda_{\overline{k}}}(\sigma^{J}_{I}\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)c^{\star}(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds= \tag{3.3.102}$$

$$=\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}}c^{\star}\Big(\sum_{k=0}^{N}\nabla^{k}_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{I(\overline{q})}}^{\mu_{J(\overline{p})}}\Big)c^{\star}(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds= \tag{3.3.103}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(\phi)^{\;\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{J(\overline{q})}}_{(i)\;\;\;\;\mu_{I(\overline{p})}} ds \tag{3.3.104}$$

Defining locally a bunch of sections of $\Gamma T^{(k)+q}_{p\;\;\;\;(\mathbb{R})} M$ with the relation $\beta^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}}} = \alpha^{\lambda_{\overline{k}}\;\nu_{J(\overline{q})}}_{(i)\;\;\;\;\mu_{I(\overline{p})}}$ we have the thesis. The second can be proved easily as follow

$$[\phi, v \cdot \mathcal{T}] = [v \lrcorner \phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(v \lrcorner \phi)^{\;\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}}} ds = \tag{3.3.105}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k} A_{(k,j)}(\nabla^j(\phi)) \Big)^{\;\;\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}}} ds = \tag{3.3.106}$$

Using the generalised Leibniz rule as proved in the previous lemma. Considering that each $A_{(k,j)}$ is $C^\infty(M)$-linear in each term $\nabla^j(T)$ we can write:

$$[\phi, v \cdot \mathcal{T}] = [v \lrcorner \phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \nabla^k_{\lambda_{\overline{k}}}(v \lrcorner \phi)^{\;\;\mu_{\overline{p}\setminus\overline{1}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}\setminus\overline{1}}} ds = \tag{3.3.107}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k} \nabla^j_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} A_{(k,j)}(e^{\alpha_{\overline{j}}}_{(i)} \otimes e^{\rho_{\overline{q}}}_{(i)} \otimes e_{(i)\sigma_{\overline{p}}}) \Big)^{\;\;\mu_{\overline{p}\setminus\overline{1}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}\setminus\overline{1}}} ds = \tag{3.3.108}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k} \nabla^j_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} A^{\alpha_{\overline{j}}\rho_q\mu_{\overline{p}\setminus\overline{1}}}_{(i)\lambda_{\overline{k}}\sigma_p\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}\setminus\overline{1}}} ds = \tag{3.3.109}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{N} \sum_{j=0}^{k} \nabla^j_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} c^\star \Big(A^{\alpha_{\overline{j}}\rho_q\mu_{\overline{p}\setminus\overline{1}}}_{(i)\lambda_{\overline{k}}\sigma_p\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\;\;\;\;\mu_{\overline{p}\setminus\overline{1}}} ds \tag{3.3.110}$$

Now it is enough re-sum as usual order by order in $\nabla^j(\phi)$ to define a new bunch of local smooth scalar fields $\beta^{\alpha_{\overline{j}}\;\rho_{\overline{q}}}_{(i)\;\;\;\;\sigma_{\overline{p}}}$ as an appropriate linear combination of $A^{\alpha_{\overline{j}}\rho_q\mu_{\overline{p}\setminus\overline{1}}}_{(i)\lambda_{\overline{k}}\sigma_p\nu_{\overline{q}}}$ and

$\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}\setminus\overline{1}}}$ to end up with

$$[\phi, v\cdot\mathcal{T}] = \tag{3.3.111}$$
$$= \sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{N}\nabla^j_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big)c^\star(\psi_i)\beta_{(i)}^{\lambda_{\overline{k}}\,\rho_{\overline{q}}}{}_{\sigma_{\overline{p}}}ds \tag{3.3.112}$$

Using the previous lemma, since by construction:

$$\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\sum_{k=0}^{N}\nabla^k_{\lambda_{\overline{k}}}(v\lrcorner\phi)^{\ \mu_{\overline{p}\setminus\overline{1}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}\setminus\overline{1}}}ds = \tag{3.3.113}$$
$$= \sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\sum_{j=0}^{N}\nabla^j_{\alpha_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big)c^\star(\psi_i)\beta_{(i)}^{\lambda_{\overline{k}}\,\rho_{\overline{q}}}{}_{\sigma_{\overline{p}}}ds \tag{3.3.114}$$

is a global smooth top form on $\mathbb{R}$ we have that $\beta_{(i)}^{\alpha_{\overline{j}}\,\rho_{\overline{q}}}{}_{\sigma_{\overline{p}}}$ can be interpreted as the local expression of bunch of global smooth sections of $\beta^{(j)} \in \Gamma T_{p\ \ c(\mathbb{R})}^{(j)+q}M$. So the thesis. The third follow exactly in the same way. The fourth and the fifth follow in the same way using the definitions of contraction with a tensor and the lemma about the generalised Leibnitz rule of the higher order differential with respect to the tensor product. □

3.4 $C^\infty(\mathbb{R})$ module structure of the multipoles up to order k

It has been proved above that no difference about the Dixon and Ellis multipole definitions occurs and $\Upsilon^q_p(c) = \Delta^q_p(c)$. The only difference between the two definitions resides in the two different local expressions they lead to. Therefore it makes more sense just to talk about multipoles, eventually specifying the preferred representation when needed rather than continuing to distinguish two identical objects. Given a closed embedding $c : \mathbb{R} \hookrightarrow M$ the set of all the multipoles defined from c acting on $\Gamma_0 T^p_q M$ will be always denoted just by $\Upsilon^q_p(c)$. The symbol $\Delta^q_p(c)$ is no more needed so it is deprecated.

3.4.1 The multipoles set as a $C^\infty(\mathbb{R})$-module.

As it has been already stated, fixed an arbitrary $k \in \mathbb{N}$, the set $\overset{(k)}{\Upsilon}{}^q_p(c)$ is a module over the $C^\infty(M)$ ring. This is not the only interesting linear algebraic structure we can define upon the $\overset{(k)}{\Upsilon}{}^q_p(c)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a global smooth scalar field. We can easily define an useful operation that takes a real scalar field on $\mathbb{R}$ and a multipole and it gives us a multipole. The definition is given in a constructive way (in the very same fashion of the Lie Derivative case) fixing the explicit action on the subset $Cor(c)$, then extending them to whole set of multipoles using the other properties given in the definition.

Definition 62: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\overset{(k)}{\Upsilon}{}^q_p(c)$ the set of all the multipoles up to the k-th order. Let $C^\infty(\mathbb{R})$ be the set of all smooth scalar field defined on the domain of the embedding. We define the ***multiplication with a scalar field on*** $\mathbb{R}$ the map

$$\triangleright : C^\infty(\mathbb{R}) \times \overset{(k)}{\Upsilon}{}^q_p(c) \to \Upsilon^q_p(c) \tag{3.4.1}$$

such that:

1. the scalar product acting on $Cor(c)$ is defined as:

$$f \triangleright c_\zeta(\alpha) = c_\zeta(f \cdot \alpha) \quad , \quad \forall f \in C^\infty(\mathbb{R}), \forall \alpha \in \Gamma\Lambda^1\mathbb{R} \tag{3.4.2}$$

2. It is distributive with respect to the sum of multipoles:

$$f \triangleright (\mathcal{T} + \mathcal{S}) = f \triangleright \mathcal{T} + f \triangleright \mathcal{S} \quad , \quad \forall f \in C^\infty(\mathbb{R}), \forall \mathcal{T}, \mathcal{S} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.3}$$

3. it is distributive with respect to the sum of scalar fields:

$$(f + g) \triangleright (\mathcal{T}) = f \triangleright \mathcal{T} + g \triangleright \mathcal{T} \quad , \quad \forall f, g \in C^\infty(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.4}$$

4. it is associative with respect to the product of scalar fields:

$$(f \cdot g) \triangleright (\mathcal{T} + \mathcal{S}) = f \triangleright (g \triangleright \mathcal{T}) \quad , \quad \forall f, g \in C^\infty(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.5}$$

5. it commutes with all the other operations defined on the multipoles:

$$f \triangleright (S \cdot \mathcal{T}) = S \cdot (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c), \forall S \in \Gamma T^{p'}_{q'} M \tag{3.4.6}$$

$$f \triangleright (\sigma^I_J \cdot \mathcal{T}) = \sigma^I_J \cdot (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.7}$$

$$f \triangleright (v \lrcorner \mathcal{T}) = v \lrcorner (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c), \forall u \in \Gamma TM \tag{3.4.8}$$

$$f \triangleright (\alpha \urcorner \mathcal{T}) = \alpha \urcorner (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c), \forall \alpha \in \Gamma T^{\star} M \tag{3.4.9}$$

$$f \triangleright (i \mathcal{T}) = i (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.10}$$

$$f \triangleright (\nabla \mathcal{T}) = \nabla (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.11}$$

$$f \triangleright div(\mathcal{T}) = div(f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c) \tag{3.4.12}$$

$$f \triangleright (L_v \mathcal{T}) = L_v (f \triangleright \mathcal{T}) \quad , \quad \forall f \in C^{\infty}(\mathbb{R}), \forall \mathcal{T} \in \overset{(k)}{\Upsilon}{}^q_p(c), \forall v \in \Gamma TM \tag{3.4.13}$$

$$\tag{3.4.14}$$

Property 38: The first property defines explicitly the action of this operation upon the small subset of multipoles $Cor(c)$ and then using the other properties it is possible to extend the operation to the whole set of multipoles. This multiplication always exists and it is unique since $f \cdot \alpha \in \Gamma \Lambda^1 \mathbb{R}$, since the map c_ζ is injective and since the other operations commutes with it. This operation automatically satisfies by definition all the property needed to be interpreted as a scalar multiplication with respect to the abelian group $(\overset{(k)}{\Upsilon}{}^q_p(c), +)$. Therefore $(\overset{(k)}{\Upsilon}{}^q_p(c), +, \triangleright)$ is a $C^{\infty}(\mathbb{R})$-module. With a bit of effort, we will able to prove later that a finite linearly independent set of elements of $\overset{(k)}{\Upsilon}{}^q_p(c)$ there always exists and it is able to generate the whole space, therefore $(\overset{(k)}{\Upsilon}{}^q_p(c), +, \triangleright)$ is also a free module.

Let us stress that we do not expect the Leibniz rule from the derivatives because the derivations are operators acting on $C^{\infty}(M)$ but their action is not defined on the field with support on the domain of the map $c : \mathbb{R} \hookrightarrow M$. Although this apparently could seem quite a sterile artefact, it is actually a very interesting structure because it allows us to define upon the multipoles some concepts like basis, generators and components, coordinates and eventually some more advanced concept like a metric upon that space and eventually different topologies (with respect to the usual weak topology) induced by the existence of a local coordinate system. However for more immediate purposes, this structure is definitely needed if we want to attempt to express the intrinsic operations upon the multipoles just like simpler operations on the coefficients identifying the multipole as a linear combination of a smaller finite set of elementary generators.

Chapter 4

Concerning the Ellis Local Representation

In this chapter we are going to analyze in detail the characteristic of the Ellis local representation of the multipoles. Let us stress once again that, at this stage, the elements of $\overset{(k)}{\Upsilon}{}^q_p(c)$ are intrinsic geometrical objects and they depend just on the existence of the set $\Gamma_0 T^p_q M$ (always guaranteed on each differential manifold by the existence of the smooth cutoff or bump functions) and the closed embedding $c : \mathbb{R} \hookrightarrow M$. No other structures (like coordinates, metric, killing vectors, Connection, Adm fibration just to quote some) are needed to define and guarantee the existence of the multipoles. At this point not much can be done to investigate further, from an intrinsic point of view the properties of the multipoles. Luckily we proved that a differential manifold and a closed embedding c, have enough structure to guarantee the existence of a local expression for the action of the multipoles in terms of integration of an appropriate global form on $\mathbb{R}$ built by directly glueing together the pullback of several Lie derivatives taken with respect to the local frame induced by the trivialization of TM of the test tensor fields. Although this very primitive local expression is very useful, we will see how in general, this is not enough to fix a correspondence between the operation upon the multipoles and the operation upon the local Ellis parameters in a satisfactory way (e.g. preserving at the same time the covariance and the uniqueness of the parameters, fixing uniquely the representation of the null multipole or having a linear transformation rule of the parameters when the trivialization of TM is changed) without adding extra geometrical information. Hence, although the local Ellis representation is probably the one requiring the minimal structure, in most scenarios it does not satisfy all we need from a local expression of a geometrical object. In the very first beginning of this chapter we are going to analyse an example enlightening the problems affecting the Ellis representation, then in the later parts we are going to fix some constraints on the acceptable representations defining some specific ways to express the multipoles in a more convenient way accordingly to our needs. An intepretation is given for each different specific Ellis representation singled out as well as the analysis of the pros and cons. However some issues still remain not solved and some aspects are actual matter of research and investigation. However, Although all the issues, thanking the Ellis representation, we will be able to show that the multipoles can be interpreted as the coefficients of the asymptotic expansion related to specific one parameter families of compact support tensor fields called "squeezed tensors". Furthermore, defining a self similar squeezing of a compact support tensor field, we will see how the adapted Ellis moments related to its expansion, coincide exactly with the well known standard definition of moments for the local expression of the given tensor field.

4.1 Problems arising from the general Ellis representation

If after the big effort put in showing the existence of a local Ellis representation, we have been convinced that we are ready to use the multipoles to model problems and possibly solve them, we are making a big mistake. Although the Ellis representation is a very good starting point trying to express the multipoles using their explicit action on the local expression of the tensor fields, this approach is not satisfactory because it is affected by several non negligible issues that can be immediately expressed by some very basic examples.

4.1.1 A specific trivial example

Let us consider $\mathbb{R}^2$ as a differential manifold on itself. $\mathbb{R}^2$ always admits a global atlas where the points of $\mathbb{R}^2$ are mapped into itself due to the identity functions. Let us denote by (x^0, x^1) the coordinate expression of an arbitrary point $x \in \mathbb{R}^2$. Now let us consider a closed embedding $c : \mathbb{R} \to \mathbb{R}^2$ defined by $c(t) = (t, 0)$, $\forall t \in \mathbb{R}$. Since $\mathbb{R}^2$ is a manifold, we can build the tangent bundle $T\mathbb{R}^2$ the cotangent bundle $T^\star\mathbb{R}^2$ as well as the tangent tensor bundle $T^p_q\mathbb{R}^2$. A global natural trivialisation of $T\mathbb{R}^2$ can be fixed by $(e_0 = \frac{\partial}{\partial x^0}, e_1 = \frac{\partial}{\partial x^1})$ and this induces a global trivialisation of $T^\star\mathbb{R}^2$ and T^p_qM. Let us consider, for instance, the multipole $\mathcal{T} \in \overset{(0)}{\Upsilon}{}^1_1(c)$ defined by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{4.1.1}$$

where $\alpha^\nu_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0, 1]$ are smooth scalar fields. Therefore:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = \int_{\mathbb{R}} \{c^\star(\phi^0_0)\alpha^0_0 + c^\star(\phi^1_0)\alpha^0_1 + c^\star(\phi^0_1)\alpha^1_0 + c^\star(\phi^1_1)\alpha^1_1\} dt \tag{4.1.2}$$

But this is not the only way to express the same distribution with the Ellis representation. Let us consider:

$$[\phi, \mathcal{S}] = \int_{\mathbb{R}} \Big\{c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu\Big\} dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{4.1.3}$$

where $\beta^{\nu}_{\mu} : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ and $\beta^{\lambda\nu}_{\mu} : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ are smooth scalar fields satisfying:

$$\begin{cases} \beta^{1\nu}_{\mu} = 0 \\ -\frac{d}{dt}[\beta^{0\nu}_{\mu}] + \beta^{\nu}_{\mu} - \alpha^{\mu}_{\nu} = 0 \end{cases} \tag{4.1.4}$$

Substituting it in the integral we obtain:

$$[\phi, \mathcal{S}] = \int_{\mathbb{R}} \Big\{ c^{\star}(L_{\lambda}(\phi)^{\mu}_{\nu})\beta^{\lambda\nu}_{\mu} + c^{\star}(\phi^{\mu}_{\nu})\beta^{\nu}_{\mu} \Big\} dt = \tag{4.1.5}$$

$$= \int_{\mathbb{R}} \Big\{ c^{\star}(L_0(\phi)^{\mu}_{\nu})\beta^{0\nu}_{\mu} + c^{\star}(L_1(\phi)^{\mu}_{\nu})\beta^{1\nu}_{\mu} + c^{\star}(\phi^{\mu}_{\nu})\beta^{\nu}_{\mu} \Big\} dt = \tag{4.1.6}$$

$$= \int_{\mathbb{R}} \Big\{ \frac{d}{dt}[c^{\star}(\phi^{\mu}_{\nu})]\beta^{0\nu}_{\mu} + c^{\star}(L_1(\phi)^{\mu}_{\nu})\beta^{1\nu}_{\mu} + c^{\star}(\phi^{\mu}_{\nu})\beta^{\nu}_{\mu} \Big\} dt = \tag{4.1.7}$$

$$= \int_{\mathbb{R}} \frac{d}{dt}[c^{\star}(\phi^{\mu}_{\nu})]\beta^{0\nu}_{\mu} dt + \int_{\mathbb{R}} 0 dt + \int_{\mathbb{R}} c^{\star}(\phi^{\mu}_{\nu})\beta^{\nu}_{\mu} dt = \tag{4.1.8}$$

$$= - \int_{\mathbb{R}} c^{\star}(\phi^{\mu}_{\nu}) \frac{d}{dt}[\beta^{0\nu}_{\mu}] dt + \int_{\mathbb{R}} c^{\star}(\phi^{\mu}_{\nu})\beta^{\nu}_{\mu} dt = \tag{4.1.9}$$

$$= \int_{\mathbb{R}} c^{\star}(\phi^{\mu}_{\nu}) \{ -\frac{d}{dt}[\beta^{0\nu}_{\mu}] dt + \beta^{\nu}_{\mu} \} dt = \tag{4.1.10}$$

$$= \int_{\mathbb{R}} c^{\star}(\phi^{\mu}_{\nu})\alpha^{\mu}_{\nu} dt = [\phi, \mathcal{T}] \tag{4.1.11}$$

Another instance can be given by considering:

$$[\phi, \mathcal{S}'] = \int_{\mathbb{R}} c^{\star}(L_{\lambda_1\lambda_2}(\phi)^{\mu}_{\nu})\gamma^{\lambda_1\lambda_2\nu}_{\mu} dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1 \mathbb{R}^2 \tag{4.1.12}$$

where $\gamma^{\lambda_1\lambda_2\nu}_{\mu} : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ are smooth scalar fields satisfying:

$$\begin{cases} \gamma^{11\nu}_{\mu} = 0 \\ \gamma^{10\nu}_{\mu} + \gamma^{10\nu}_{\mu} = 0 \\ \frac{d^2}{dt^2}[\gamma^{00\nu}_{\mu}] - \alpha^{\mu}_{\nu} = 0 \end{cases} \tag{4.1.13}$$

In this case we have that:

$$[\phi, \mathcal{S}] = \int_{\mathbb{R}} \Big\{ c^\star(L_{\lambda_1\lambda_2}(\phi)^\mu_\nu)\gamma^{\lambda_1\lambda_2\nu}_\mu dt = \tag{4.1.14}$$

$$= \int_{\mathbb{R}} \Big\{ c^\star(L_{01}(\phi)^\mu_\nu)\gamma^{01\nu}_\mu + c^\star(L_{10}(\phi)^\mu_\nu)\gamma^{10\nu}_\mu + c^\star(L_{00}(\phi)^\mu_\nu)\gamma^{00\nu}_\mu + c^\star(L_{11}(\phi)^\mu_\nu)\gamma^{11\nu}_\mu \Big\} dt = \tag{4.1.15}$$

$$= \int_{\mathbb{R}} c^\star(L_{00}(\phi)^\mu_\nu)\gamma^{00\mu}_\nu dt = \int_{\mathbb{R}} c^\star(L_0 L_0(\phi)^\mu_\nu)\gamma^{00\mu}_\nu dt = \int_{\mathbb{R}} \frac{d^2}{dt^2}[c^\star((\phi)^\mu_\nu)]\gamma^{00\mu}_\nu dt = \tag{4.1.16}$$

$$= (-1)^2 \int_{\mathbb{R}} c^\star((\phi)^\mu_\nu)\frac{d^2}{dt^2}[\gamma^{00\mu}_\nu] dt = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\mu_\nu dt = [\phi, \mathcal{T}] \tag{4.1.17}$$

since $L_{01} = L_{10} - L_{[e_1,e_0]} = L_{10} - 0 = L_{10}$. Hence it is clear that even in this very trivial example the Ellis representation of a multipole is not unique and several different sets of parameters can define the same distribution. This shows explicitly why we prefer to describe α^μ_ν , $(\beta^{\lambda\nu}_\mu, \beta^\mu_\nu)$ or $\gamma^{\lambda_1\lambda_2\nu}_\mu$ just as "parameters" rather than "components". They completely define the multipoles but not in a unique $C^\infty(\mathbb{R})$ linearly independent way. Things turn even much worse if we try to change the atlas, so the charts. Let us suppose to have another global atlas defining a new coordinate system (x'^0, x'^1) linked to the old ones with:

$$\begin{cases} x'^0 = x'^0(x^0, x^1) \\ x'^1 = x'^1(x^0, y^1) \end{cases} \tag{4.1.18}$$

This automatically induces a new trivialisation of $T\mathbb{R}^2$, $T^\star\mathbb{R}^2$ and $T^p_q\mathbb{R}^2$ therefore a different set of Ellis parameters. It is extremely annoying to realise how the different choices of the Ellis parametrization defining the same multipole $\mathcal{T}$ behave in a very different way satisfying very weird transformation rules, involving integrals of the Jacobian matrix. To make matter even more problematic, if a change of local trivialisation is performed, for instance due to the change of coordinate system, we are no more able to understand if two Ellis local representation are related to the same distribution or if the Ellis parameters define the multipole in a $C^\infty(\mathbb{R})$ linearly independent way, because we are not able to integrate explicitly the components of the Lie Derivatives. Let us show it. Changing the coordinates on $\mathbb{R}^2$, we induce another global natural trivialisation of $T\mathbb{R}^2$ fixed by (e'_0, e'_1) satisfying

$$\begin{cases} e'_0 = \frac{\partial}{\partial x'^0} = \overline{J}^\mu_0 e_\mu \\ e'_1 = \frac{\partial}{\partial x'^1} = \overline{J}^\mu_1 e_\mu \end{cases} \tag{4.1.19}$$

and this induces a new global trivialisation of $T^\star\mathbb{R}^2$ and $T^p_q\mathbb{R}^2$. If we consider the first case, the multipole $\mathcal{T}$ is expressed by:

$$[\phi,\mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt \tag{4.1.20}$$

using the old trivialization and by:

$$[\phi,\mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi'^\mu_\nu)\alpha'^\nu_\mu dt \tag{4.1.21}$$

using the new one. By definition of the Ellis representation $c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt$ must be a global smooth top form over $\mathbb{R}$ independently from the chosen trivialisation, hence, $c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = c^\star(\phi'^\mu_\nu)\alpha'^\nu_\mu dt$. This fixes a constraint on the transformation rules for the local Ellis parameters:

$$c^\star(\phi'^\alpha_\beta)\alpha'^\beta_\alpha dt = c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = c^\star(\phi(e^\mu,e_\nu))\alpha^\nu_\mu dt = c^\star(\phi(J^\mu_\alpha e'^\alpha,\overline{J}^\beta_\nu e'_\beta)\alpha^\nu_\mu dt = \tag{4.1.22}$$
$$= c^\star(\phi(e'^\alpha,e'_\beta))c^\star(J^\mu_\alpha)c^\star(\overline{J}^\beta_\nu\alpha^\nu_\mu)dt \tag{4.1.23}$$

concluding that:

$$\alpha'^\beta_\alpha = c^\star(\overline{J}^\mu_\alpha)c^\star(J^\beta_\nu)\alpha^\nu_\mu \tag{4.1.24}$$

and:

$$[\phi,\mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = \int_{\mathbb{R}} c^\star(\phi'^\mu_\nu)c^\star(\overline{J}^\mu_\alpha)c^\star(J^\beta_\nu)\alpha^\nu_\mu dt = \int_{\mathbb{R}} c^\star(\phi'^\mu_\nu)\alpha'^\nu_\mu dt \tag{4.1.25}$$

If we consider the second case, the same multipole $\mathcal{T}$ can be expressed also by another Ellis representation:

$$[\phi,\mathcal{T}] = \int_{\mathbb{R}} \Big\{c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu\Big\}dt \quad , \quad \forall\phi\in\Gamma_0 T^1_1\mathbb{R}^2 \tag{4.1.26}$$

where $\beta_\mu^\nu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ and $\beta_\mu^{\lambda\nu} : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ are smooth scalar fields satisfying:

$$\begin{cases} \beta_\mu^{1\nu} = 0 \\ -\frac{d}{dt}[\beta_\mu^{0\nu}] + \beta_\mu^\nu - \alpha_\nu^\mu = 0 \end{cases} \tag{4.1.27}$$

using the old trivialisation and by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} \Big\{ c^\star(L_\lambda(\phi)'^\mu_\nu){\beta'}^{\lambda\nu}_\mu + c^\star({\phi'}^\mu_\nu){\beta'}^\nu_\mu \Big\} dt \quad , \quad \forall \phi \in \Gamma_0 T_1^1 \mathbb{R}^2 \tag{4.1.28}$$

where ${\beta'}_\mu^\nu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ and ${\beta'}_\mu^{\lambda\nu} : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ are smooth scalar fields satisfying:

$$\begin{cases} \beta_\mu^{1\nu}({\beta'}_\mu^{\lambda\nu}, {\beta'}_\nu^\mu) = 0 \\ -\frac{d}{dt}[\beta_\mu^{0\nu}({\beta'}_\mu^{\lambda\nu}, {\beta'}_\nu^\mu)] + \beta_\mu^\nu(\beta_\mu^{\lambda\nu}, \beta_\nu^\mu) = \alpha_\nu^\mu({\alpha'}_\nu^\mu) \end{cases} \tag{4.1.29}$$

using the new one. The reader can immediately notice how, although the constraint seems pretty simple when expressed on the Ellis representation induced by the first trivialisation, in general this cannot be stated when using a new Ellis representation induced by a new trivialisation of the tangent bundle. We will explicate the constraints after we explicate the transformation rules for this second representation. By definition of the Ellis representation $\Big\{ c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star({\phi'}^\mu_\nu){\beta'}^\nu_\mu \Big\} dt$ must be a global smooth top form over $\mathbb{R}$ independently from the chosen trivialisation, hence:

$$\Big\{ c^\star(L'_\lambda(\phi)'^\mu_\nu){\beta'}^{\lambda\nu}_\mu + c^\star({\phi'}^\mu_\nu){\beta'}^\nu_\mu \Big\} dt = \Big\{ c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt \tag{4.1.30}$$

must be satisfied. This automatically fixes the form for the transition functions of the local Ellis parameters:

$$\Big\{ c^\star(L'_\gamma(\phi)'^\alpha_\beta){\beta'}^{\gamma\beta}_\alpha + c^\star({\phi'}^\alpha_\beta){\beta'}^\beta_\alpha \Big\} dt = \Big\{ c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{4.1.31}$$

$$= \Big\{ c^\star(L_{J^\gamma_\lambda e'_\gamma}(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{4.1.32}$$

$$= \Big\{ c^\star(J^\gamma_\lambda L_{e'_\gamma}(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\partial_\sigma J^\gamma_\lambda \overline{J}^\mu_\gamma \phi^\sigma_\nu)\beta^{\lambda\nu}_\mu + c^\star(\partial_\nu J^\gamma_\lambda \overline{J}^\rho_\gamma \phi^\mu_\rho)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{4.1.33}$$

$$=\Big\{c^\star(J^\gamma_\lambda L'_\gamma\phi'^\alpha_\beta\overline{J}^\mu_\alpha J^\beta_\nu)\beta^{\lambda\nu}_\mu - c^\star(\partial_\sigma J^\gamma_\lambda\overline{J}^\mu_\gamma\phi'^\alpha_\beta\overline{J}^\sigma_\alpha J^\beta_\nu)\beta^{\lambda\nu}_\mu + \tag{4.1.34}$$

$$+c^\star(\partial_\nu J^\gamma_\lambda\overline{J}^\rho_\gamma\phi'^\alpha_\beta\overline{J}^\mu_\alpha J^\beta_\rho)\beta^{\lambda\nu}_\mu + c^\star(\phi'^\alpha_\beta\overline{J}^\mu_\alpha J^\beta_\nu)\beta^\nu_\mu\Big\}dt \tag{4.1.35}$$

concluding that:

$$\begin{cases}\beta'^\beta_\alpha = c^\star(\overline{J}^\mu_\alpha J^\beta_\nu)\beta^\nu_\mu - c^\star(\partial_\sigma J^\gamma_\lambda\overline{J}^\mu_\gamma\overline{J}^\sigma_\alpha J^\beta_\nu)\beta^{\lambda\nu}_\mu + c^\star(\partial_\nu J^\gamma_\lambda\overline{J}^\rho_\gamma\overline{J}^\mu_\alpha J^\beta_\rho)\beta^{\lambda\nu}_\mu\\ \beta'^{\gamma\beta}_\alpha = c^\star(J^\gamma_\lambda\overline{J}^\mu_\alpha J^\beta_\nu)\beta^{\lambda\nu}_\mu\end{cases} \tag{4.1.36}$$

To explicate the equation for the constraints for this new parameters we need to invert the transformation rules

$$\begin{cases}\beta'^\beta_\alpha = c^\star(\overline{J}^\mu_\alpha J^\beta_\nu)\beta^\nu_\mu - c^\star(\partial_\sigma J^\gamma_\lambda\overline{J}^\mu_\gamma\overline{J}^\sigma_\alpha J^\beta_\nu)\beta^{\lambda\nu}_\mu + c^\star(\partial_\nu J^\gamma_\lambda\overline{J}^\rho_\gamma\overline{J}^\mu_\alpha J^\beta_\rho)\beta^{\lambda\nu}_\mu\\ c^\star(\overline{J}^\lambda_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha = \beta^{\lambda\nu}_\mu\end{cases} \tag{4.1.37}$$

$$\begin{cases}\beta^\nu_\mu = c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^\beta_\alpha + c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)c^\star(\partial_\sigma J^\gamma_\varepsilon\overline{J}^\delta_\gamma\overline{J}^\sigma_\alpha J^\beta_\eta)\beta^{\varepsilon\eta}_\delta - c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)c^\star(\partial_\eta J^\gamma_\varepsilon\overline{J}^\rho_\gamma\overline{J}^\delta_\alpha J^\beta_\rho)\beta^{\varepsilon\eta}_\delta\\ c^\star(\overline{J}^\lambda_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha = \beta^{\lambda\nu}_\mu\end{cases} \tag{4.1.38}$$

$$\begin{cases}\beta^\nu_\mu = c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^\beta_\alpha + c^\star(\partial_\mu J^\gamma_\varepsilon\overline{J}^\delta_\gamma)\beta^{\varepsilon\nu}_\delta - c^\star(\partial_\eta J^\gamma_\varepsilon\overline{J}^\nu_\gamma)\beta^{\varepsilon\eta}_\mu\\ c^\star(\overline{J}^\lambda_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha = \beta^{\lambda\nu}_\mu\end{cases} \tag{4.1.39}$$

$$\begin{cases}\beta^\nu_\mu = c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^\beta_\alpha + c^\star(\partial_\mu J^\gamma_\varepsilon\overline{J}^\delta_\gamma)c^\star(\overline{J}^\varepsilon_\rho J^\alpha_\delta\overline{J}^\nu_\beta)\beta'^{\rho\beta}_\alpha - c^\star(\partial_\eta J^\gamma_\varepsilon\overline{J}^\nu_\gamma)c^\star(\overline{J}^\varepsilon_\rho J^\alpha_\mu\overline{J}^\eta_\beta)\beta'^{\rho\beta}_\alpha\\ c^\star(\overline{J}^\lambda_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha = \beta^{\lambda\nu}_\mu\end{cases} \tag{4.1.40}$$

$$\begin{cases}\beta^\nu_\mu = c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^\beta_\alpha + c^\star(\partial_\mu J^\alpha_\varepsilon\overline{J}^\varepsilon_\rho\overline{J}^\nu_\beta)\beta'^{\rho\beta}_\alpha - c^\star(\partial_\eta J^\gamma_\varepsilon\overline{J}^\nu_\gamma\overline{J}^\varepsilon_\rho J^\alpha_\mu\overline{J}^\eta_\beta)\beta'^{\rho\beta}_\alpha\\ c^\star(\overline{J}^\lambda_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha = \beta^{\lambda\nu}_\mu\end{cases} \tag{4.1.41}$$

and plug it in the constraint expression:

$$\begin{cases}\beta^{1\nu}_\mu(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu) = 0\\ -\frac{d}{dt}[\beta^{0\nu}_\mu(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu)] + \beta^\nu_\mu(\beta^{\lambda\nu}_\mu, \beta^\mu_\nu) - \alpha^\mu_\nu(\alpha'^\mu_\nu) = 0\end{cases} \tag{4.1.42}$$

$$\begin{cases}c^\star(\overline{J}^1_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha = 0\\ -\frac{d}{dt}[c^\star(\overline{J}^0_\gamma J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^{\gamma\beta}_\alpha] + c^\star(J^\alpha_\mu\overline{J}^\nu_\beta)\beta'^\beta_\alpha + c^\star(\partial_\mu J^\alpha_\varepsilon\overline{J}^\varepsilon_\rho\overline{J}^\nu_\beta)\beta'^{\rho\beta}_\alpha - c^\star(\partial_\eta J^\gamma_\varepsilon\overline{J}^\nu_\gamma\overline{J}^\varepsilon_\rho J^\alpha_\mu\overline{J}^\eta_\beta)\beta'^{\rho\beta}_\alpha - \alpha^\mu_\nu(\alpha'^\mu_\nu) = 0\end{cases} \tag{4.1.43}$$

We do not even try to explicate the coordinate change rules and the constraints in a new coordinate system for the third Ellis expression of $\mathcal{T}$ because involving the second order Lie derivatives would be terribly painful. However we reached our purpose showing that each multipole admits several (in general infinite) different Ellis representations and the parameters related to them do not share the same transformation rules which in general can be very complicated even in the simpler cases. Things get definitely and dramatically worse in case we are trying to consider higher order multipoles, stopping us to find any general algorithm or procedure to study in a systematic general way the properties of the multipoles and to cast functional equations involving multipoles as unique covariant equations on the parameters defining the multipoles.

4.1.2 Considerations

From the analysis of the very simple example presented above, the reader can be convinced about the fact that, despite the quite straightforward intrinsic definition, the multipoles can be very treacherous geometrical objects. Let us try to extrapolate from the examples of what we think the problems affecting the Ellis representation are and their causes.

The non-uniqueness problem.

The first relevant issue we faced in the example was given by the non unique representation of the multipoles in terms of Lie derivatives. This can be extremely problematic because the structure provided for the Ellis representation (i.e a closed embedding, an atlas, a smooth partition of the unity subordinate to the atlas and smooth local frame) is not enough to fix a one to one relationship between the multipole and its local representative. This is directly caused by the non-uniqueness of the Ellis local representation of the action of the multipoles that does not allow to single out a set of $C^\infty(\mathbb{R})$-linearly independent multipoles that generate the whole module. Although this can seem a minor issue apparently, in practice this causes the failure of a unique representation of the null multipole, for instance as a set of null scalar fields. So we must admit that at this stage we are not able to fix an isomorphism between the $C^\infty(\mathbb{R})$-module of the multipoles and the module $(C^\infty(\mathbb{R}), +, \cdot)$. Because of this, at least at this stage, we avoid the term "components" when we are referring to the Ellis parameters of a multipole. The lack of any isomorphism between a multipole and its Ellis parameters causes also the failure of the attempt to define the operation on the multipoles in terms of operations upon the local representations. For instance the sum of two multipoles can produce a null distribution, that can be expressed by a set of parameters that are not equal to the sum of two starting multipoles parameters. As it has been already widely explained, considering that a clear correspondence between the operations on the multipoles and operations upon their local expressions is essential to express intrinsic functional equations, constraints and properties of the multipoles in terms of standard $C^\infty(\mathbb{R})$-functional equations eventually solvable with known techniques, the Ellis local representation without any additional structure is not enough to satisfy our requirements.

Causes of the non-uniqueness.

The non uniqueness of the Ellis representation can be directly traced back mainly to two things: the algebra of the Lie derivatives and the Stokes theorem. We know that $L_vL_u - L_uL_v = L_{[u,v]}$ therefore fixing a local frame and a local $C^\infty(M)$-linear combination of Lie derivatives:

$$\alpha^{\lambda\mu}L_{\lambda\mu}(T) = \alpha^{\lambda\mu}L_{e_\lambda}L_{e_\mu}(T) = \alpha^{\lambda\mu}L_{e_\mu}L_{e_\lambda}(T) + \alpha^{\lambda\mu}L_{[e_\lambda,e_\mu]^\nu e_\nu}(T) = \tag{4.1.44}$$
$$=\alpha^{\lambda\mu}L_{\lambda\mu}(T) + \alpha^{\lambda\mu}[e_\lambda, e_\mu]^\nu L_\nu(T) + \tag{4.1.45}$$
$$+\alpha^{\lambda\mu}\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\nu \otimes d([e_\lambda, e_\mu]^\nu)\urcorner\sigma^{\overline{s}}(T)\Big\} + \alpha^{\lambda\mu}\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda, e_\mu]^\nu)\otimes e_\nu \lrcorner\sigma_{\overline{s}}(T)\Big\} \tag{4.1.46}$$

So as one can see, the anti-symmetric part of a linear combination of two Lie derivative taken both with respect to a local frame (e_μ) are not $C^\infty(M)$-linear independent from the lower order Lie derivatives:

$$\alpha^{\lambda\mu}L_{\lambda\mu}(T) - \alpha^{\lambda\mu}L_{\lambda\mu}(T) = \tag{4.1.47}$$
$$=\alpha^{\lambda\mu}[e_\lambda, e_\mu]^\nu L_\nu(T) + \tag{4.1.48}$$
$$+\alpha^{\lambda\mu}\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\nu \otimes d([e_\lambda, e_\mu]^\nu)\urcorner\sigma^{\overline{s}}(T)\Big\} + \alpha^{\lambda\mu}\sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda, e_\mu]^\nu)\otimes e_\nu \lrcorner\sigma_{\overline{s}}(T)\Big\} \tag{4.1.49}$$

If we take a $C^\infty(\mathbb{R})$ linear combinations of pullbacks of Lie derivatives with respect to a closed embedding $c : \mathbb{R} \hookrightarrow M$ we have that:

$$[\alpha^{\lambda\mu\nu_{\overline{p}}}_{\sigma_{\overline{p}}} - \alpha^{\mu\lambda\nu_{\overline{p}}}_{\sigma_{\overline{p}}}]c^\star\Big(L_{\lambda\mu}(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big) = \tag{4.1.50}$$
$$=\alpha^{\lambda\mu\nu_{\overline{p}}}_{\sigma_{\overline{p}}}c^\star\Big([e_\lambda, e_\mu]^\nu\Big)c^\star\Big(L_\nu(T)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}\Big) + \tag{4.1.51}$$
$$+\alpha^{\lambda\mu\nu_{\overline{p}}}_{\sigma_{\overline{p}}}c^\star\Big(\sum_{s=1}^{p}\overline{\sigma^{\overline{s}}}\Big\{e_\nu \otimes d([e_\lambda, e_\mu]^\nu)\urcorner\sigma^{\overline{s}}(T)\Big\} + \sum_{r=1}^{q}\overline{\sigma_{\overline{r}}}\Big\{d([e_\lambda, e_\mu]^\nu)\otimes e_\nu \lrcorner\sigma_{\overline{s}}(T)\Big\}\Big) \tag{4.1.52}$$

hence we can conclude that the anti-symmetric part of the pullback of Lie derivative taken with respect to the local frame (e_μ) are not $C^\infty(\mathbb{R})$-linear independent from the pullbacks of the lower order Lie derivatives. Since the Ellis representation is given by a linear combination of several compositions of higher order Lie derivatives, it is clear that it cannot be unique. Another different cause of non uniqueness of the Ellis local representation is the fact that the multipoles act with an integral on $\mathbb{R}$ on the pullback of the Lie derivatives of the tensor fields. In fact, according to the previous lemma, since

c is a closed embedding, we have that:

$$\forall s \in \mathbb{R} \,|\, c(s) \cap U \neq 0 \,,\, \forall e_0 \in \Gamma_U TM \mid e_{0|_{c(s)}} = \dot{c}(s) \quad \Rightarrow \quad c^\star(L_{e_0}(T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) = \frac{d}{ds} c^\star(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) \tag{4.1.53}$$

Let us suppose to have a local trivialisation $(e_{(i)\mu})$ such that $\forall s \in \mathbb{R} \,|\, c(s) \cap U \neq 0 \,,\, \forall e_0 \in \Gamma_U TM \mid e_{(i)0|_{c(s)}} = \dot{c}(s)$ and multipole defined by its local action:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(L_{e_{(i)0}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \sum_i \int_{\mathbb{R}} \frac{d}{ds} [c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})] c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{4.1.54}$$

$$= \sum_i \int_{\mathbb{R}} \frac{d}{ds} [c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] c^\star(\psi_i) ds - \sum_i \int_{\mathbb{R}} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \frac{d}{ds} [\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] c^\star(\psi_i) ds = \tag{4.1.55}$$

$$= \int_{\mathbb{R}} \sum_i c^\star(\psi_i) \frac{d}{ds} [c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] ds - \sum_i \int_{\mathbb{R}} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \frac{d}{ds} [\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] c^\star(\psi_i) ds = \tag{4.1.56}$$

$$= \int_{\mathbb{R}} d[c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] - \sum_i \int_{\mathbb{R}} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \frac{d}{ds} [\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] c^\star(\psi_i) ds = \tag{4.1.57}$$

$$= \int_{\mathbb{R}} d[c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] - \sum_i \int_{\mathbb{R}} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \frac{d}{ds} [\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] c^\star(\psi_i) ds \tag{4.1.58}$$

then using the Stokes theorem we conclude that:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(L_{e_{(i)0}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{4.1.59}$$

$$= 0 - \sum_i \int_{\mathbb{R}} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \frac{d}{ds} [\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] c^\star(\psi_i) ds \tag{4.1.60}$$

$$\tag{4.1.61}$$

Since the Ellis representation is given by the integration of a linear combination of several compositions of higher order Lie derivatives, it is clear that it cannot be unique. Despite some people like to regard the non uniqueness of the Ellis parameters as a "degree of freedom" interpreting it as a "gauge" and following old fashion analogies with the Classical Electromagnetism, we argue that in this case this can be misleading. Given a vector bundle or a principal bundle, a gauge transformation is an automorphism of the bundle in itself preserving the linear structure or the right action of the group respectively and that is a symmetry for the Lagrangian action. Therefore before talking about "gauge transformations", one should define a vector or a principal bundle where the Ellis parameters and the associated multipoles can be interpreted respectively as sections and actions, and finally to provide an explicit form for the action of the linear group that fixes the symmetry. This in principle can be done using the formalism of the Jet Bundles and sub-manifold Jet Bundles but it should be fully formalised first, then the compatibility between the module structure of the Distribution should be carefully checked. If other bundles are built (i.e Sub-Manifold Distribution Bundle) one should prove that the degree of freedom in the choice of the Ellis parameters can be always interpreted as a switch between compatible local trivialisation, with some transformation rules satisfying the co-cycle identities. We have already seen how all the changes of Ellis parameters induced by a change of representation is not in general compatible with the action of a $C^\infty(\mathbb{R})$-linear transformation (unless to consider the Jets Submanifold Bundle) because it involves non trivial differential equations due to the existence of an integration process, therefore the most general change of parametrization could not be even expressed as the action of a group of transformation upon the bundles, therefore it could not be interpreted as a change of trivialisation. To conclude we are not excluding a-priori the possibility that the change of parametrization in the Ellis parameters can be considered as a real gauge, we are just pointing out how one should clearly show the appropriate bundle and eventual extra structure first.

The general covariance implementation problem

We will deeply discuss later about the nature and the meaning of the general covariance and why it must be considered a fundamental requirement (especially in the framework of General Relativity) satisfied by the equation encoding physical laws. We can resume here the covariance principle: the change of local observer must be a a symmetry for the dynamical field equations. In terms of bundle theory this means that the equations must be invariant under change of all possible trivialisation. At this stage let us just assume that the general covariance of the equations is an essential condition we cannot drop. Let us suppose to have the simplest multipole equations :

$$\mathcal{T} = 0 \, , \; \mathcal{T} \in \overset{(0)}{\Upsilon}{}^q_p(c) \tag{4.1.62}$$

As it is an intrinsic equation, it automatically implements the covariance principle since it does not depend on the coordinate system, but there is nothing we can do to find an explicit form for the solution at this stage. As well as it is usually done when solving tensor field dynamical equations (i.e Maxwell's Equations, Einstein equations), the only thing we can do is try to recast the intrinsic multipole equation as a functional equation on the parameters defining the multipole itself. Doing so, we can try to solve the equations for the Ellis parameters, and so we can write the functional solving the equation simply using the Ellis local representation. We know that, since the Ellis representation in not unique, the equation $\mathcal{T} = 0$ can be recast as an equation on the parameters in several different ways depending on which parametrization is chosen. Recalling the example, we choose to express the multipole equation as follow:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu ds = 0 \quad , \quad \forall\phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{4.1.63}$$

or in a different way:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} \left\{ c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \right\} ds = 0 \quad , \quad \forall\phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{4.1.64}$$

where the parameters are coupled together by the constraint:

$$\begin{cases} \beta^{1\nu}_\mu = 0 \\ -\frac{d}{dt}[\beta^{0\nu}_\mu] + \beta^\nu_\mu - \alpha^\nu_\mu = 0 \end{cases} \tag{4.1.65}$$

If we consider the first representation of the equation, we say that the only way to have a null integral $\forall\phi \in \Gamma_0 T^1_1\mathbb{R}^2$ is to admit that all the parameters are null. Therefore we have a first local Ellis representation for the given equation:

$$\alpha^\nu_\mu = 0 \tag{4.1.66}$$

If we consider the second representation of the equation, we cannot state again that the only way to have a null integral $\forall\phi \in \Gamma_0 T^1_1\mathbb{R}^2$ is to admit that all the parameters are null, because they are not linearly independent. Luckily we obtained the detail of the link between the two parametrizations, hence knowing the representation of the equation in terms of α^ν_μ we can cast easily the equation in terms of $(\beta^{\lambda\nu}_\mu, \beta^\nu_\mu)$:

$$\begin{cases} \alpha^\nu_\mu = 0 \\ \beta^{1\nu}_\mu = 0 \\ -\frac{d}{dt}[\beta^{0\nu}_\mu] + \beta^\nu_\mu - \alpha^\nu_\mu = 0 \end{cases} \Rightarrow \begin{cases} \beta^{1\nu}_\mu = 0 \\ -\frac{d}{dt}[\beta^{0\nu}_\mu] + \beta^\nu_\mu = 0 \end{cases} \tag{4.1.67}$$

Although both the Ellis representations seem to lead both to good equations, the first one is completely invariant under transformations induced by local diffeomorphisms:

$$\alpha'^{\nu}_{\mu} = \overline{J}^{\mu}_{\alpha} J^{\nu}_{\beta} \alpha^{\beta}_{\alpha} = 0 \tag{4.1.68}$$

but the second one is not:

$$\begin{cases} c^{\star}(\overline{J}^{1}_{\gamma} J^{\alpha}_{\mu} \overline{J}^{\nu}_{\beta}) \beta'^{\gamma\beta}_{\ \ \alpha} = 0 \\ -\frac{d}{dt}[c^{\star}(\overline{J}^{0}_{\gamma} J^{\alpha}_{\mu} \overline{J}^{\nu}_{\beta}) \beta'^{\gamma\beta}_{\ \ \alpha}] + c^{\star}(J^{\alpha}_{\mu} \overline{J}^{\nu}_{\beta}) \beta'^{\beta}_{\alpha} + c^{\star}(\partial_{\mu} J^{\alpha}_{\varepsilon} \overline{J}^{\varepsilon}_{\rho} \overline{J}^{\nu}_{\beta}) \beta'^{\rho\beta}_{\ \ \alpha} - c^{\star}(\partial_{\eta} J^{\gamma}_{\varepsilon} \overline{J}^{\nu}_{\gamma} \overline{J}^{\varepsilon}_{\rho} J^{\alpha}_{\mu} \overline{J}^{\eta}_{\beta}) \beta'^{\rho\beta}_{\ \ \alpha} = 0 \end{cases} \tag{4.1.69}$$

This means that the multipole equation $\mathcal{T} = 0$, $\mathcal{T} \in \overset{(0)}{\Upsilon}{}^{q}_{p}(c)$ admits a generally covariant local representation in terms of algebraic constraints $\alpha^{\nu}_{\mu} = 0$ but equivalently admits also a non generally covariant (the equation is not invariant under transformation induced by local diffeomorphism) representation in terms of coupled algebraic and differential equations on the parameters $(\beta^{\lambda\nu}_{\mu}, \beta^{\nu}_{\mu})$. Therefore, we must conclude that not all the Ellis representations of the multipoles lead to covariant equations for the local parameters, even in the most trivial cases. One could state that this can be assumed to be a good criterion to choose acceptable Ellis parametrizations but is not trivial to prove that such a representation always exists. Although one can argue that this is just a minor problem, as we will discuss later this does not allow us to extrapolate any physical information directly from the Ellis parameters solving the equations (unless very specific and trivial cases) because the Ellis parameters defining a solution to a given multipole equation must solve a different equation in each distinct coordinate system. So concluding the analysis we can state that, in general, given a multipole equation, no direct general way to write it uniquely in terms of equations upon the Ellis local parameters exists, unless considering very specific representation. Some of them can be easily defined using a particular trivialisation of TM at the price that the local representation of the multipole equations are not invariant under diffeomorphism, thence they are not generally covariant. Taking apart very specific trivial cases, the Ellis parameters solving the local representations of non covariant multipole equations cannot be associated directly to meaningful geometrical properties (or physical observables) independently from the conventions set by the local coordinate system. An interesting geometrical interpretation of what happens can be done in terms of bundles. In fact the general covariance problem arises directly when we examine the compatibility between two structures, the action of the group of the diffeomorphism on the parameters induced by the Ellis representation and the isomorphisms (as $C^{\infty}(\mathbb{R})$-modules) of $\overset{(k)}{\Upsilon}{}^{p}_{q}(c)$ with at least one Ellis representation. Fixing a specific isomorphism could be incompatible with respect to the action of *Diff* (M) on the parameters, therefore *Diff* (M) cannot be a symmetry for the local representation of the multipole equation. In other words the request of a particular isomorphism between the multipoles and their Ellis parameters could fix too strong constraints upon the compatible trivialisation of TM defining the Ellis representation, such that the covariance is broken. On the other hand if one wants to preserve the general covariance (i.e. inducing the Ellis representation without directly linking them to a particular coordinate system), several possible isomorphisms must be excluded.

The local Ellis parameters gluing problem

In the example above we considered a very trivial case in which the manifold admits an atlas formed by one global chart. If we do not restrict ourselves to these specific cases we have to consider that the local Ellis parameters must be "glued" in an appropriate way in order to create a global smooth top form over $\mathbb{R}$ which must be integrated. This requirement follows directly from the definition of the Ellis representation of the multipoles and we proved that it is always possible to build at least one non null smooth global form defining a non null multipole via the Ellis representation. However the request that $\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ must define a smooth global 1-form on $\mathbb{R}$ via:

$$c^{\star}\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\psi_i\sum_{k=0}^{N}L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds\in\Gamma\Lambda^1\mathbb{R} \tag{4.1.70}$$

is in general a very strong constraint. In fact, in order to be a good global smooth section of $\Lambda^1\mathbb{R}$, from the bundle theory, we know that given two overlapping charts $(U_i,\varphi_{(i)})$ and $(U_j,\varphi_{(j)})$ such that $U_i\cap U_j\cap c(\mathbb{R})\neq\varnothing$ we need to satisfy the compatibility condition

$$\sum_{k=0}^{N}c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds=\sum_{k=0}^{N}c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds\quad,\quad\forall s\in c^{-1}[U_i\cap U_j\cap c(\mathbb{R})] \tag{4.1.71}$$

hence

$$\Big[\sum_{k=0}^{N}c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}\Big]_{|s}=\Big[\sum_{k=0}^{N}c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}\Big]_{|s}\quad,\quad\forall s\in c^{-1}[U_i\cap U_j\cap c(\mathbb{R})] \tag{4.1.72}$$

In general this is a non trivial condition, some obstructions could occur depending on the topology of M and the explicit form of the embedding c. Analysing the behaviour of the Lie derivatives with respect to the change of local natural frame, we concluded that the condition above fixes a sort of transition function from the local Ellis parameters in the generic form:

$$\Big[\sum_{k=0}^{N} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(j)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}\Big]_{|s}=\Big[\sum_{k=0}^{N}\sum_{s=0}^{k} c^{\star}\Big(L_{\lambda_{\overline{k}}}(\phi)_{(j)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\beta_{(ji)\gamma_{\overline{s}}\mu_{\overline{p}}\rho_{\overline{q}}}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}\sigma_{\overline{p}}}\alpha_{(i)}^{\gamma_{\overline{s}}\,\rho_{\overline{q}}}{}_{\sigma_{\overline{p}}}\Big]_{|s} \quad (4.1.73)$$

where $\beta_{(ji)\gamma_{\overline{s}}\mu_{\overline{p}}\rho_{\overline{q}}}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}\sigma_{\overline{p}}}$ is a bunch of smooth local scalar fields on $\mathbb{R}$ formed by a very tricky linear combination of Lie derivatives of the Jacobian matrices ruling the gluing of the parameters. In general there is no way to recast this constraint in a general expression due to the very awful non $C^{\infty}(M)$-linearity of the Lie Derivative. What one can do is try to explicate the condition for very specific small values of N, or alternatively check the compatibility condition of the parameters by directly substituting them inside the previous equation. It is clear that both of these approaches are not satisfactory because both of them do not offer a systematic way to cast a transformation rule for the local parameters, satisfied in order to be considered a well defined Ellis representation of a multipole. This could lead us to very serious issues when trying to define explicitly an arbitrary multipole directly through an Ellis local representation, because no explicit systematic method can be given to state how an Ellis representation is affected by the change of atlas on the manifold.Although one cannot say that it is a severe problem, this aspect is definitely problematic. In fact we would like to be very efficient when analysing the compatibility condition satisfied by an Ellis local representation, in order to check if a set of local equations fixed for the local Ellis parameters can be interpreted as the local expression of a well defined intrinsic equation of multipoles. Unfortunately a satisfactory solution to this problem is still a matter of investigation.

4.2 Isomorphism between multipoles and the Ellis representation induced by a the choice of an adapted coordinate system.

In this section we are going to show that an isomorphism between the multipoles and a very specific Ellis local representation exists. This bijection preserves the structure of $C^{\infty}\mathbb{R}$-module and it allows us to show how, for each $k \in \mathbb{N}$, the space $\overset{(k)}{\Upsilon}{}^{p}_{q}(c)$ is actually a free-module with a well defined finite dimension. We will see how this isomorphism between the multipoles and this specific Ellis representation is induced by the choice of a particular atlas $\mathcal{A}$ adapted to the embedding. The choice of an adapted atlas is needed to regularise and kill the redundancy in the Ellis representation occurring due to the integration process characterising the action of the multipoles. This approach can be very useful to single out a minimal set of $C^{\infty}(\mathbb{R})$-linearly independent generators for the multipoles up to the order k but it leads to some problems. First of all, the covariance is broken, because since the isomorphism depends on the choice of a very specific coordinate system, the action of the diffeomorphisms on the parameters (induced by the Ellis representation) is not compatible with it. We will see how this problem can be avoided using the transverse frame formalism very closely related with the "vielbein" formalism to define in a coordinate independent way the same isomorphism. The second aspect is much more subtle and concerns the choice of the the transverse frame (or equivalently the choice of an adapted coordinate system) that fix the isomorphism. In fact since there exists infinite set of transverse frames, they fix infinite different isomorphic Ellis parametrization. Hence we will see that, for the multipoles, a degree of freedom to choose appropriately different Ellis parameters induced by different transverse frames exist. The investigation of this degree of freedom is absolutely not trivial, creating a lot of problems while interpreting this representation just as particular trivialization of a specific bundle. The existence and the details of a bundle structure (possibly a vector bundle) built upon the worldine $c(\mathbb{R})$ encoding (as an actual gauge symmetry) both the invariance under the action of the local diffeomorphism of M and the different choices of transverse frames is still unknown and an actual matter of investigation.

4.2.1 Coordinate system adapted to the closed embedding and transverse space

Lemma 29: Let M be a manifold and let us suppose that $c : \mathbb{R} \hookrightarrow M$ is a closed embedding. Then at each point $t \in \mathbb{R}$ there exist at least two charts (I, ψ) and (U, φ) of $\mathbb{R}$ and M respectively, with $t \in I$ and $c(I) \subset U$, such that the local expression of $c : \mathbb{R} \hookrightarrow M$ is just the restriction on I of the standard inclusion of $i : I \subset \mathbb{R} \hookrightarrow \mathbb{R}^m$.

Proof. We provide here a sketch of proof. Let (I, ψ) and (U, ϕ) be local charts of $\mathbb{R}$ and M around t and $c(t)$ respectively such that $c(I) \subseteq U$. Let us denote by $\hat{c} : \psi(I) \to \varphi(U)$ the local expression of c defined by $\hat{c} = \varphi \circ c \circ \psi^{-1}$. The differential at $\psi(t)$ defined as $d(\hat{c})_{|_{\psi(t)}} : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^{m-1}$ always exists and is injective, therefore, up to rearranging indices, considering the canonical standard projection $\pi_1 : \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R}$ into the first factor we can define the map:

$$\pi_1 \circ d(\hat{c})_{|_{\psi(t)}} : \mathbb{R} \to \mathbb{R} \tag{4.2.1}$$

which is an isomorphism. By the inverse function theorem, by shrinking enough I, we can assume that $\pi_1 \circ \hat{c} : \psi(I) \to V_0 \subset \mathbb{R}$ is a local diffeomorphism from $\psi(I)$ to its image V_0. Let us call $(\pi_1 \circ \hat{c})^{-1} : V_0 \to \psi(I)$ the smooth inverse map. Now if we consider π_2 the canonical projection $\pi_2 : \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$ onto the second factor, then $\hat{c}(\psi(I))$ can be interpreted as the graph of:

$$\pi_2 \circ \hat{c} \circ (\pi_1 \circ \hat{c})^{-1} : V_0 \subset \mathbb{R} \to \mathbb{R}^{m-1} \tag{4.2.2}$$

It is easy to check that $\pi_1 \circ \hat{c}(\psi(I))$ is a sub-manifold of $\mathbb{R} \times \mathbb{R}^{m-1}$ and the map τ :

$V_0 \times \mathbb{R}^{m-1} \to V_0 \times \mathbb{R}^{m-1}$ such that:

$$\tau(s,y) = (s, y - \pi_2 \circ \hat{c} \circ (\pi_1 \circ \hat{c})^{-1}(s)) \tag{4.2.3}$$

is a diffeomorphism such that $\tau(\hat{c}(\psi(I)) = V_0 \times \{0\}$. It is enough to build a new set of local charts $(I, \tilde{\psi})$ and $(U, \tilde{\varphi})$ as follow:

$$\begin{cases} \tilde{\psi} = \pi_1 \circ \hat{c} \circ \psi \\ \tilde{\phi} = \tau \circ \varphi \end{cases} \tag{4.2.4}$$

and they satisfy:

$$\left[\tilde{\phi} \circ c \circ \tilde{\psi}^{-1}\right]_{|_{\tilde{\psi}(t)=s}} = \left\{\tau \circ \phi \circ c \circ \psi^{-1} \circ (\pi_1 \circ \hat{c})^{-1}\right\}(s) = (s, 0) \quad , \quad \forall t \in I \tag{4.2.5}$$

□

Definition 63: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$, a local chart (U, φ) on a manifold M is called a ***adapted chart*** if satisfies:

1. U is diffeomorphic to $I \times V \subseteq \mathbb{R} \times \mathbb{R}^{m-1}$, where I and V are open subsets of $\mathbb{R}$ and $\mathbb{R}^{m-1}$ respectively
2. there always exists two maps $\psi : U \to I \subset \mathbb{R}$ and $\phi : U \to V \subset \mathbb{R}^{m-1}$ such that $\varphi(x) = (\psi(x), \phi(x)), \forall x \in U$ is just the Cartesian product of the images of the point through $\psi(x)$ and $\phi(x)$.
3. the embedding $c : \mathbb{R} \to M$ is locally expressed by $c(s) = (s, 0^i)$ with $i \in [1, m-1] \subset \mathbb{N}$.

Definition 64: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$, an atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ of M is called ***atlas adapted to*** c if there exists a covering of the sub-manifold $c(\mathbb{R})$ defined by the embedding made by adapted local chart.

Definition 65: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$ and an adapted atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ of the manifold M, the natural local frame $(\partial_{(i)\mu})$ on the open $U_i \subseteq M$ is called an ***adapted local frame***.

Definition 66: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$, an adapted atlas $\mathcal{A}$ of M induces a trivialisation of $TM, T^\star M$ and $T^p_q M$ called ***trivialisation adapted to*** c

Property 39: The existence of an adapted local chart as well as the existence of an adapted local Atlas, an adapted local frame and an adapted trivialisation are guaranteed at each point on the image of $c(\mathbb{R})$ by the previous lemma. Furthermore it is obvious how the natural projection $\pi_1 : U \to I$ makes the open U a fibered manifold on I.

Property 40: It is trivial to notice that given an atlas $\mathcal{A}$ of M and a closed embedding $c : \mathbb{R} \hookrightarrow M$, it is always possible to build a non-minimal adapted atlas just adding to $\mathcal{A}$ enough adapted local charts in order to cover the sub-manifold $c(\mathbb{R})$ or alternatively just the single adapted local chart covering the whole $c(\mathbb{R})$

4.2.2 Ellis representation fixed by an adapted atlas and adapted Ellis moments of a multipole

We are going to see a possible way to fix the isomorphism between the multipoles and one Ellis representation. By choosing a specific set of local adapted coordinates covering all the manifold, we are able to kill the degree of freedom in the choice of the Ellis parameters. As we are going to see, although this approach is very straightforward and immediate, we must pay the price of introducing a new geometrical structure hidden inside the choice of adapted coordinates. Furthermore since this isomorphism between multipoles and Ellis parameters is strongly dependent just on the specific natural local trivialisation induced by a specific set of coordinates, the structure is not invariant under diffeomorphisms, so it is not covariant. From here we are going to use the split Einstein condensed convention upon the indices. The greek letters are related to indices running from 0 to $m-1$, the latin letters instead are related to indices running from 1 to $m-1$

Theorem 7: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. For each Ellis multipole $\mathcal{T} \in \Upsilon^q_p(c)$, there always exists a unique bunch of local smooth scalar fields $\alpha_{(i)}{}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in $m_{\overline{k}}$ and defining a global smooth top form:

$$c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \psi_i \sum_{k=0}^{N} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)\alpha_{(i)}{}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{4.2.6}$$

such that, $\forall\phi \in \Gamma_0 T^p_q M$, $\mathcal{T}$ acts on the local expression of ϕ as follow:

$$[\phi,\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)}{}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{4.2.7}$$

Proof. To prove the statement it is enough to show that given an adapted atlas, each multipole must be written uniquely in that way. First of all let us consider that since $e_{(i)\mu} = \partial_{(i)\mu}$ therefore the commutator is always trivially $[\partial_{(i)\mu}, \partial_{(i)\nu}] = 0$ due to the Schwartz theorem concerning the commutation of partial derivatives. Hence we can state that:

$$L_{\alpha\beta}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_\alpha L_\beta(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_\beta L_\alpha(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} - L_{[\partial_{(i)\beta}, \partial_{(i)\alpha}]}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_\beta L_\alpha(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{\beta\alpha}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \tag{4.2.8}$$

Let us suppose now that for a generic $k \in \mathbb{N}$, the composition of k Lie derivatives with respect to a natural frame $(\partial_{(i)\mu})$ is completely symmetric:

$$L_{\beta_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{\beta_{J(\overline{k})}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \quad , \quad \forall J \in \prod(k) \tag{4.2.9}$$

then we have that:

$$L_{\alpha\beta_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_\alpha L_{(\beta_1} L_{\beta_{\overline{k}\setminus\overline{1})}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{(\beta_1} L_{\underline{\alpha}} L_{\beta_{\overline{k}\setminus\overline{1})}}(\phi)^{\beta_{\overline{p}}}_{(i)\nu_{\overline{q}}} - L_{[\partial_{(i)(\beta_1}, \partial_{(i)\underline{\alpha}]}} L_{\beta_{\overline{k}\setminus\overline{1})}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.10}$$

$$= L_{(\beta_1} L_{\underline{\alpha}} L_{\mu_{\overline{k}\setminus\overline{1})}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{\beta_1\alpha\,\beta_{\overline{k}\setminus\overline{1}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \tag{4.2.11}$$

□

Iterating the same process for each index it is possible to show that α commutes with each β Therefore we must to conclude that the composition of an arbitrary number of Lie Derivatives $L_{\beta_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{e_{\beta_1}}(...(L_{e_{\beta_k}}(\phi)))^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}$ of a tensor field must be completely symmetric in the indices related to the derivations. At this point, to prove the statement it's enough to show that all $Cor^q_p(c)$ can be written in that way and then prove that the given expression is closed under the operations we used to define $\Upsilon^q_p(c)$. Let us consider a functional $\mathcal{S} \in Cor^q_p(c)$. We know by definition that there exists at least a $T \in T^q_p$ and a $\alpha \in \Gamma\Lambda^1\mathbb{R}$ such that $\mathcal{S} = T \cdot c_\zeta(\alpha)$. So we can write the action of this functional on an arbitrary test tensor field ϕ as an action on its local coordinate expression:

$$[\phi, \mathcal{S}] = [\phi, T \cdot c_\zeta(\alpha)] = \int_{\mathbb{R}} c^\star(T(\phi)) \wedge \alpha = \tag{4.2.12}$$

$$= \int_{\mathbb{R}} c^\star\Big(T\Big[\sum_{U_i \in \mathcal{A}} \psi_i \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} e^{\nu_{\overline{q}}}_{(i)} \otimes e_{(i)\mu_{\overline{p}}}\Big]\Big) \wedge \alpha = \tag{4.2.13}$$

$$= \int_{\mathbb{R}} c^{\star}\Big(\sum_{U_i \in \mathcal{A}} \psi_i \phi_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} T\Big[e_{(i)}^{\nu_{\overline{q}}} \otimes e_{(i)\mu_{\overline{p}}}\Big]\Big) \wedge \alpha = \tag{4.2.14}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\psi_i \phi_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} T_{(i)\mu_{\overline{p}}}^{\nu_{\overline{q}}}\Big) \wedge \alpha = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\phi_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) c^{\star}(\psi_i) c^{\star}(T_{(i)\mu_{\overline{p}}}^{\nu_{\overline{q}}}) \cdot \tilde{\alpha} ds \tag{4.2.15}$$

We know that $ord(\mathcal{S}) = 0$,therefore fixed arbitrarily an $N \in \mathbb{N}$ we can define a bunch of local scalar fields:

$$\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} = \begin{cases} \tilde{\alpha}_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} \quad , \quad \forall k \leq ord(\mathcal{S}) \\ 0 \quad , \quad ord(\mathcal{S}) < k \end{cases} \tag{4.2.16}$$

and rewrite the expression as follows:

$$[\phi, \mathcal{S}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) c^{\star}(\psi_i) \alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.17}$$

Since by construction, $\forall \phi \in \Gamma_0 T_q^p M$ the local scalar fields $\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}$ satisfies :

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) \alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds = c^{\star}(T(\phi)) \wedge \alpha \tag{4.2.18}$$

we have immediately that:

$$\sum_{k=0}^{ord(\mathcal{S})} c^{\star}\Big(L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) \alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.19}$$

can be interpreted as a local expression of the global smooth form $c^{\star}(T(\phi)) \wedge \alpha \in \Gamma\Lambda^1\mathbb{R}$ induced by the adapted atlas $\mathcal{A}$. Now let us assume that a generic $\mathcal{T}, \mathcal{S} \in \overset{(k)}{\Upsilon}{}_p^q(c)$ satisfies the thesis. Then we would like to prove that $\forall \phi \in \Gamma_0 T_q^p M$:

1. there always exists a bunch of local smooth scalar field $\beta_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in $m_{\overline{k}}$ defining a global smooth top form:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\beta_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \in \Gamma\Lambda^0\mathbb{R} \tag{4.2.20}$$

such that, $\mathcal{T}+\mathcal{S}$ acts on the local expressions of ϕ as follow:

$$[\phi, \mathcal{S}+\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{4.2.21}$$

2. there always exists a bunch of local smooth scalar field $\beta_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in $m_{\overline{k}}$ defining a global smooth top form:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{ord(f\cdot\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\beta_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \in \Gamma\Lambda^0\mathbb{R} \tag{4.2.22}$$

such that, $f\cdot\mathcal{T}$ acts on the local expressions of ϕ as follow:

$$[\phi, f\cdot\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(f\cdot\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{4.2.23}$$

3. For each smooth global vector field $v \in \Gamma TM$ there always exists a bunch of local smooth scalar field $\gamma_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in m_k defining a

global smooth top form:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i \sum_{k=0}^{ord(L_v\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\gamma_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \in \Gamma\Lambda^0\mathbb{R} \tag{4.2.24}$$

such that, $L_v\mathcal{T}$ acts on the local expressions of ϕ as follow:

$$[\phi, L_v\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(L_v\mathcal{T})} L_{\lambda_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\gamma_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.25}$$

Let us start with the first:

$$[\phi, \mathcal{S}+\mathcal{T}] = [\phi,\mathcal{S}] + [\phi,\mathcal{T}] = \tag{4.2.26}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds + \tag{4.2.27}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{ord(\mathcal{T})} L_{m_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\hat{\alpha}_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{j}}\ \nu_{\overline{q}}} ds = \tag{4.2.28}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} \Big[c^\star\Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} + c^\star\Big(\sum_{j=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\hat{\alpha}_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}\Big] ds \tag{4.2.29}$$

Let us suppose to have $ord(\mathcal{T}) \leq ord(\mathcal{S})$, the other case follow in the same manner. Since $ord(\mathcal{T}) \leq ord(\mathcal{S})$ we can always choose a new bunch of smooth local scalar fields:

$$\beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{\ \ \ \ \mu_{\overline{p}}} = \begin{cases} \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{\ \ \ \ \mu_{\overline{p}}} + \hat{\alpha}^{m_{\overline{k}}\ \nu_{\overline{q}}}_{\ \ \ \ \mu_{\overline{p}}}, \forall k \leq ord(\mathcal{T}) \\ \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{\ \ \ \ \mu_{\overline{p}}}\ ,\ ord(\mathcal{T}) < k \leq ord(\mathcal{S}) \\ 0 \quad , \quad ord(\mathcal{S}) < k \end{cases} \tag{4.2.30}$$

Therefore using it on the expression

$$[\phi, \mathcal{S}+\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{4.2.31}$$

Now let us consider the property of the multipoles $\forall \mathcal{T}, \mathcal{S} \in \Upsilon_p^q(c) \Rightarrow ord(\mathcal{S}+\mathcal{T}) \leq \max\{\mathcal{S}, \mathcal{T}\}$ therefore we can state that $ord(\mathcal{S}+\mathcal{T}) \leq ord(\mathcal{S})$. Then the expression can be recast as:

$$[\phi, \mathcal{S}+\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{4.2.32}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds + \tag{4.2.33}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=ord(\mathcal{T}+\mathcal{S})+1}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{4.2.34}$$

We can easily prove that the second integral must be null otherwise we are going to have a contradiction. In fact by the definition of order, $\forall j \in \mathbb{N}, \forall \phi \in \Gamma_0 T_q^p M, \forall \lambda \in C^\infty M \mid c^\star(\lambda) = 0$ we must have:

$$0 = [\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}\phi, \mathcal{T}+\mathcal{S}] \tag{4.2.35}$$

This leads us to:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}}(\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds + \tag{4.2.36}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=ord(\mathcal{T}+\mathcal{S})+1}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} \Big) c^\star(\psi_i)\beta_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = 0 \tag{4.2.37}$$

The first integral it is always null because there are not enough Lie derivatives to kill all the powers of $\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}$, and we can conclude that $\forall j \in \mathbb{N}, \forall \phi \in \Gamma_0 T^p_q M, \forall \lambda \in C^\infty M \mid c^\star(\lambda) = 0$ the second integral must vanish as well:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=ord(\mathcal{T}+\mathcal{S})+1}^{ord(\mathcal{S})} L_{m_{\overline{k}}} (\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j} \phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = 0 \tag{4.2.38}$$

Since $\beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds$ are symmetric and each term $c^\star \Big(L_{m_{\overline{k}}} (\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j} \phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big)$ is composed by derivations along linearly independent vectors with respect to $\dot{c}$ there is no chance to have a null result for each $\phi \in \Gamma_0 T^p_q M$ unless all the Ellis parameters are constrained by:

$$\beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} = 0 \quad , \quad \forall k > ord(\mathcal{S}+\mathcal{T}) \tag{4.2.39}$$

So finally we have to conclude that:

$$[\phi, \mathcal{S}+\mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}} (\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{4.2.40}$$

By construction we have that:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}} (\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \tag{4.2.41}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}} (\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds + c^\star \Big(\sum_{j=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}} (\phi)^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \hat{\alpha}^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{4.2.42}$$

Since we assumed both $\mathcal{S}$ and $\mathcal{T}$ satisfy the thesis, then both the second terms in the right hand side of the equation are well defined global smooth 1-forms on $\mathbb{R}$. Considering that the sum of two smooth global 1-forms is still a smooth one form we have immediately that

$$c^\star\Big(\sum_{k=0}^{ord(\mathcal{T}+\mathcal{S})} L_{m_{\overline{k}}}(\phi)^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\beta^{m_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}ds \tag{4.2.43}$$

can be interpreted as the local expression of a global smooth form defined on the whole $\mathbb{R}$. To prove the closure with respect to the product with scalars let us suppose that $\mathcal{T}$ satisfy the thesis, then we can use a special case of the previous lemma:

$$L_{m_{\overline{k}}}(f\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \sum_{j=0}^{k} L_{n_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{n_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)m_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}} \tag{4.2.44}$$

to manipulate the expression:

$$[\phi, f\cdot\mathcal{T}] = [f\phi, \mathcal{T}] = \tag{4.2.45}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(f\phi)^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{m_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}ds = \tag{4.2.46}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{j=0}^{k} L_{n_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{n_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)m_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{m_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}ds \tag{4.2.47}$$

Now it is enough to re-sum order by order all the terms defining a new bunch of scalar fields $\beta^{n_{\overline{j}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}$ as linear combinations of $c^\star(f^{n_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)m_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}})$ and $\alpha^{m_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}$ to end up with:

$$[\phi, f\cdot\mathcal{T}] = \tag{4.2.48}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{j=0}^{k} L_{n_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}} f^{n_{\overline{j}}\sigma_{\overline{q}}\mu_{\overline{p}}}_{(i)m_{\overline{k}}\rho_{\overline{p}}\nu_{\overline{q}}}\Big) c^\star(\psi_i)\alpha^{m_{\overline{k}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}ds = \tag{4.2.49}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{j=0}^{ord(f\cdot\mathcal{T})} L_{n_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}\Big) c^\star(\psi_i)\beta^{n_{\overline{j}}\;\nu_{\overline{q}}}_{(i)\quad\;\mu_{\overline{p}}}ds \tag{4.2.50}$$

where the property $ord(\mathcal{T}) = ord(f\cdot\mathcal{T})$ has been used. Since by hypothesis, $\forall\phi\in\Gamma_0 T^p_q M$

the local scalar fields $\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ defines a good global smooth form on $\mathbb{R}$ with

$$\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i\sum_{k=0}^{ord(f\cdot\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{4.2.51}$$

and since $\forall\phi\in\Gamma_0T_q^pM,\forall f\in C^\infty(M)\Rightarrow f\phi\in\Gamma_0T_q^pM$, we have that by construction:

$$\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(f\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds = \sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\psi_i\sum_{j=0}^{ord(f\cdot\mathcal{T})} L_{n_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\beta_{(i)}^{n_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{4.2.52}$$

is a global smooth 1- form over $\mathbb{R}$ and:

$$c^\star\Big(\sum_{j=0}^{ord(f\cdot\mathcal{T})} L_{n_{\overline{j}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)\beta_{(i)}^{n_{\overline{j}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{4.2.53}$$

can be interpreted as its local expression. The proof of the closure with respect to the "transverse" Lie derivatives is a bit painful and requires some lemmas stated in the previous chapters.

$$[\phi,L_v\mathcal{T}]=-[L_v\phi,\mathcal{T}]= \tag{4.2.54}$$

$$=-\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(L_v\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds= \tag{4.2.55}$$

$$=-\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(v^\lambda L_\lambda\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds+ \tag{4.2.56}$$

$$+\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{s=1}^{p} L_{m_{\overline{k}}}\Big\{i\Big[\sigma^{(1,s+1)}(d(v_{(i)}^\lambda)\otimes e_{(i)\lambda}\otimes\phi)\Big]\Big\}_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds+ \tag{4.2.57}$$

$$-\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{r=1}^{q} L_{m_{\overline{k}}}\Big\{i\Big[\sigma_{(1,r+1)}(d(v^\lambda_{(i)})\otimes e_{(i)\lambda}\otimes\phi)\Big]\Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds \tag{4.2.58}$$

To proceed in the proof it can be useful to analyse separately the first term.

1. Let us consider temporarily just the first term: It is possible to use the Leibnitz rule to write the expression as:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(v^\lambda_{(i)}L_\lambda\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds = \tag{4.2.59}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{j=0}^{k}\binom{k}{j} L_{m_{\overline{j}}}(v^\lambda_{(i)})L_{m_{\overline{k}\setminus\overline{j}}}L_\lambda(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds = \tag{4.2.60}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{j=0}^{k}\binom{k}{j} L_{m_{\overline{j}}}(v^n_{(i)})L_n L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds+ \tag{4.2.61}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{j=0}^{k}\binom{k}{j} L_{m_{\overline{j}}}(v^0_{(i)})L_0 L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds \tag{4.2.62}$$

Now integrating by parts the pullback of the Lie derivative taken along $e_0 = \partial_0$, since the boundary term vanishes due the compact support of ϕ:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(v^\lambda_{(i)}L_\lambda\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds = \tag{4.2.63}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\sum_{j=0}^{k}\binom{k}{j} L_{m_{\overline{j}}}(v^n_{(i)})L_n L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(\psi_i)\alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}ds+ \tag{4.2.64}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{T})} \sum_{j=0}^{k} \binom{k}{j} L_{m_{\overline{j}}}(v^0_{(i)}) L_0 L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^{\star}(\psi_i) \alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{4.2.65}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{T})} \sum_{j=0}^{k} \binom{k}{j} L_{m_{\overline{j}}}(v^n_{(i)}) L_n L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^{\star}(\psi_i) \alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds + \tag{4.2.66}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} \sum_{k=0}^{ord(\mathcal{T})} \sum_{j=0}^{k} \binom{k}{j} c^{\star}\Big(L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^{\star}(\psi_i) \frac{d}{ds} c^{\star}\Big[\Big(L_{m_{\overline{j}}}(v^0_{(i)}) \Big) \alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \Big] ds \tag{4.2.67}$$

2. Let us consider temporarily just the second term. It is possible to use the local frame to write explicitly:

$$L_{m_{\overline{k}}}\Big\{ i\Big[\sigma^{(1,s+1)}(d(v^{\lambda}_{(i)}) \otimes e_{(i)\lambda} \otimes \phi) \Big] \Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.68}$$

$$= L_{m_{\overline{k}}}\Big\{ i\Big[\sigma^{(1,s+1)}(\partial_{\alpha}(v^{\lambda}_{(i)}) e^{\alpha}_{(i)} \otimes e_{(i)\lambda} \otimes \phi) \Big] \Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.69}$$

$$= L_{m_{\overline{k}}}\Big\{ \partial_{\alpha}(v^{\lambda}_{(i)}) i\Big[\sigma^{(1,s+1)}(e^{\alpha}_{(i)} \otimes e_{(i)\lambda} \otimes \phi) \Big] \Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.70}$$

$$= \sum_{j=1}^{k} \binom{k}{j} L_{m_{\overline{j}}}\Big\{ \partial_{\alpha}(v^{\lambda}_{(i)}) \Big\} L_{m_{\overline{k}\setminus\overline{j}}}\Big\{ i\Big[\sigma^{(1,s+1)}(e^{\alpha}_{(i)} \otimes e_{(i)\lambda} \otimes \phi) \Big] \Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.71}$$

$$= \sum_{j=1}^{k} \binom{k}{j} L_{m_{\overline{j}}}\Big\{ \partial_{\alpha}(v^{\lambda}_{(i)}) \Big\} L_{m_{\overline{k}\setminus\overline{j}}}\Big\{ i\Big[\sigma^{(1,s+1)}(e^{\alpha}_{(i)} \otimes e_{(i)\lambda} \otimes \phi) \Big] \Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.72}$$

$$= \sum_{j=1}^{k} \binom{k}{j} L_{m_{\overline{j}}}\Big\{ \partial_{\alpha}(v^{\mu_s}_{(i)}) \Big\} L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} \tag{4.2.73}$$

The same can be done to obtain:

$$L_{m_{\overline{k}}}\Big\{ i\Big[\sigma_{(1,r+1)}(d(v^{\lambda}_{(i)}) \otimes e_{(i)\lambda} \otimes \phi) \Big] \Big\}^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.2.74}$$

$$= \sum_{j=1}^{k} \binom{k}{j} L_{m_{\overline{j}}}\Big\{ \partial_{\nu_r}(v^{\beta}_{(i)}) \Big\} L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.2.75}$$

Therefore considering what we have obtained it is enough to put together all the pieces composing the original expression:

$$[\phi, L_v\mathcal{T}] = \tag{4.2.76}$$

$$= - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \sum_{j=0}^{k} \binom{k}{j} L_{m_{\overline{j}}}(v^n_{(i)}) L_n L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds + \tag{4.2.77}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} \sum_{k=0}^{ord(\mathcal{T})} \sum_{j=0}^{k} \binom{k}{j} c^\star \Big(L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \frac{d}{ds} c^\star \Big[\Big(L_{m_{\overline{j}}}(v^0_{(i)}) \Big) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} \Big] ds + \tag{4.2.78}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \sum_{s=1}^{p} \sum_{j=1}^{k} \binom{k}{j} L_{m_{\overline{j}}} \Big\{ \partial_\alpha (v^{\mu_s}_{(i)}) \Big\} L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds + \tag{4.2.79}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \sum_{r=1}^{q} \sum_{j=1}^{k} \binom{k}{j} L_{m_{\overline{j}}} \Big\{ \partial_{\nu_r} (v^{\beta}_{(i)}) \Big\} L_{m_{\overline{k}\setminus\overline{j}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{4.2.80}$$

Now it is enough to re-sum order by order all the terms contracted with the Lie Derivatives to define a new appropriate bunch of scalar fields $\gamma^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}$ allowing us to recast the expression as follow:

$$[\phi, L_v\mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})+1} L_{m_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \gamma^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{4.2.81}$$

Now using the property $ord(L_v\mathcal{T}) \leq ord(\mathcal{T}+1)$ we can recast the expression as:

$$[\phi, \mathcal{S} + \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T}+1)} L_{m_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \gamma^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \tag{4.2.82}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(L_v\mathcal{T})} L_{m_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \beta^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds + \tag{4.2.83}$$

$$+\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=ord(L_v\mathcal{T})+1}^{ord(\mathcal{T})+1} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\gamma_{(i)\ \ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}ds \tag{4.2.84}$$

We can easily prove that the second integral must be null otherwise we are going to have a contradiction. In fact by the definition of order, $\forall j\in\mathbb{N},\forall\phi\in\Gamma_0T^p_qM,\forall\lambda\in C^\infty M\mid c^\star(\lambda)=0$ we must have:

$$0=[\lambda^{ord(L_v\mathcal{T})+1+j}\phi,L_v\mathcal{T}] \tag{4.2.85}$$

leading us to:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(L_v\mathcal{T})} L_{m_{\overline{k}}}(\lambda^{ord(L_v\mathcal{T})+1+j}\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\beta_{(i)\ \ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}ds+ \tag{4.2.86}$$

$$+\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=ord(L_v\mathcal{T})+1}^{ord(\mathcal{T})+1} L_{m_{\overline{k}}}(\lambda^{ord(L_v\mathcal{T})+1+j}\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\gamma_{(i)\ \ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}ds=0 \tag{4.2.87}$$

The first integral is always null because there are not enough Lie derivatives to kill all the powers of $\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}$, and we can conclude that $\forall j\in\mathbb{N},\forall\phi\in\Gamma_0T^p_qM,\forall\lambda\in C^\infty M\mid c^\star(\lambda)=0$ the second integral must vanish as well:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star\Big(\sum_{k=ord(L_v\mathcal{T})+1}^{ord(\mathcal{T})+1} L_{m_{\overline{k}}}(\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\gamma_{(i)\ \ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}ds=0 \tag{4.2.88}$$

Since $\gamma_{(i)\ \ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}ds$ are symmetric and each term $c^\star\Big(L_{m_{\overline{k}}}(\lambda^{ord(\mathcal{T}+\mathcal{S})+1+j}\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)$ is composed by derivations along linearly independent vectors with respect to $\dot{c}$, there is no chance to have a null result for each $\phi\in\Gamma_0T^p_qM$ unless all the Ellis parameters are constrained by:

$$\gamma_{(i)\ \ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}=0\quad,\quad\forall k>ord(L_v\mathcal{T}) \tag{4.2.89}$$

So finally we have to conclude that:

$$[\phi, L_v\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(L_v\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) c^{\star}(\psi_i)\gamma_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.90}$$

Since by hypothesis, $\forall\phi \in \Gamma_0 T^p_q M$ the local scalar fields $\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}$ define a good global smooth form on $\mathbb{R}$ with

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{N} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.91}$$

and since $\forall\phi \in \Gamma_0 T^p_q M, \forall v \in \Gamma TM \Rightarrow L_v\phi \in \Gamma_0 T^p_q M$, we have that by construction:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(L_v\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^{\star}\Big(\psi_i \sum_{k=0}^{ord(L_v\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\gamma_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.92}$$

is a global smooth 1- form over $\mathbb{R}$ and:

$$c^{\star}\Big(\sum_{k=0}^{ord(L_v\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\gamma_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds \tag{4.2.93}$$

can be interpreted as its local expression. To complete the proof we need finally to check the uniqueness of this Ellis representation. Let us suppose by contradiction that given an arbitrary multipole $\mathcal{S}$ there exists two distinct bunches of these specific Ellis parameters, namely $\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}$ and $\beta_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}}$ defining the same multipole. This means that $\forall\phi \in \Gamma_0 T^p_q M$ we must have that:

$$0 = [\phi, \mathcal{S}] - [\phi, \mathcal{S}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big) c^{\star}(\psi_i)\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\ \nu_{\overline{q}}} ds + \tag{4.2.94}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) c^\star(\psi_i) \beta_{(i)}^{m_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \tag{4.2.95}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{S})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) c^\star(\psi_i) \Big[\alpha_{(i)}^{m_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} - \beta_{(i)}^{m_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \Big] ds \tag{4.2.96}$$

Since the parameters are completely symmetric in the indices $m_{\overline{k}}$ and since each term $c^\star\Big(L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}}\Big)$ is composed by derivations along linearly independent vectors with respect to $\dot{c}$, there is no chance to have a null result for each $\phi \in \Gamma_0 T^p_q M$ unless all the Ellis parameters are constrained by:

$$\Big[\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - \beta^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \Big] = 0 \quad \Rightarrow \quad \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} = \beta^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \tag{4.2.97}$$

that is a contradiction since we assumed that the two sets of Ellis parameters are not equal. So we have the thesis.

Let us remark that the uniqueness strongly depends on the choices of the adapted atlas.

Definition 67: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The set of Ellis parameters $\alpha_{(i)}^{m_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in $m_{\overline{k}}$ and defining a global smooth top form:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} c^\star \Big(\psi_i L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) \alpha_{(i)}^{m_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{4.2.98}$$

such that, $\forall \phi \in \Gamma_0 T^p_q M$, $\mathcal{T} \in \Upsilon^q_p(c)$ acts on the local expression of ϕ as follow:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\;\;\mu_{\overline{p}}} \Big) c^\star(\psi_i) \alpha_{(i)}^{m_{\overline{k}}\;\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{4.2.99}$$

are called ***adapted Ellis moments*** of the multipole $\mathcal{T}$ with respect to the adapted atlas $(U_i, \varphi_{(i)})$. The local Ellis representation induced by the adapted Ellis moments

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta \Big(\alpha_{(i)}^{m_{\overline{k}}\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \Big) \Big\} \tag{4.2.100}$$

is called ***adapted Ellis local representation*** with respect to the adapted atlas $(U_i, \varphi_{(i)})$.

Corollary 13: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The adapted Ellis local representation with respect to the given adapted atlas is an isomorphism (of modules) between the $\Upsilon_p^q(c)$ and the set of the Ellis parameters.

Proof. It is quite trivial. We already proved in the previous theorem that the adapted Ellis local representation is able to express all the elements in $\Upsilon_p^q(c)$ in an unique way. It is very easy to check that the sum of distribution is mapped into the sum of parameters as well as the scalar multiplication, therefore it is a bijection preserving the linear structure, hence an isomorphism. $\square$

Let us stress once again that this particular isomorphism is strongly dependent on the choices of the adapted atlas. If another atlas is chosen then this isomorphism does not occur anymore. If the new atlas is still adapted, then a new isomorphism can be built with the same approach, however the link between two different adapted Ellis representation can be very tricky. In case the new atlas is no more adapted, then one cannot define this kind of isomorphism. This is very problematic because the way we decided to link the adapted Ellis parameter to the multipoles is not compatible with the invariance under local diffeomorphisms or equivalently for a general local coordinate transformation, therefore the covariance principle is broken.

4.2.3 The $C^\infty(\mathbb{R})$ free module structure of $\Upsilon_p^q(c)$

Since fixing an adapted atlas and using the induced Ellis local representation maps isomorphically $\Upsilon_q^p(c)$ as a $C^\infty(\mathbb{R})$ module into the set of the adapted Ellis moments, it is enough to analyse them to extrapolate some information about the algebraic structure of $\Upsilon_q^p(c)$ and its subsets $\Upsilon_q^p(c)$. Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given $\mathcal{T} \in \Upsilon_q^p(c)$ and the set of its adapted Ellis moments

$\alpha_{(i)\ \ \ \mu_{\overline{p}}}^{m_{\overline{k}}\,\nu_{\overline{q}}}$ we have that:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta \Big(\alpha_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} ds \Big) \Big\} \tag{4.2.101}$$

Using the $C^\infty(\mathbb{R})$ module scalar multiplication we can recast the expression as follow:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta (ds) \Big\} \Big) \tag{4.2.102}$$

Fixing the order k and fixing the lists of indices $m_{\overline{k}}, \mu_{\overline{p}}$ and $\nu_{\overline{q}}$, each term:

$$(-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta (ds) \Big\} \tag{4.2.103}$$

can be interpreted as a multipole belonging to $\overset{(k)}{\Upsilon}{}_p^q(c)$, defined by its action on $\Gamma_0 T_q^p M$ by:

$$\Big[\phi, (-1)^k L_{m_{\overline{k}}} \{ \psi_i [e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}}] c_\zeta (ds) \} \Big] = \int_{\mathbb{R}} c^\star \Big(L_{m_{\overline{k}}} (\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} \Big) ds = \tag{4.2.104}$$

$$= \int_{\mathbb{R}} \sum_{j=0}^{k} c^\star \Big(L_{n_{\overline{j}}} (\phi)_{(i)\rho_{\overline{q}}}^{\sigma_{\overline{p}}} \Big) c^\star (\psi_i) \delta(j,k) \delta_{m_{\overline{k}}}^{n_{\overline{j}}} \delta_{\nu_{\overline{q}}\sigma_{\overline{p}}}^{\mu_{\overline{p}}\rho_{\overline{q}}} ds \tag{4.2.105}$$

Let us then denote these multipoles with :

$$\Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} = (-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta (ds) \Big\} \tag{4.2.106}$$

then we have that an arbitrary multipole $\mathcal{T}$ can be written simply as a $C^\infty(\mathbb{R})$ combination:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}} \nu_{\overline{q}}} \triangleright \Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.2.107}$$

So we have to conclude that the set $(\Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} \,|\, k \in \mathbb{N})$ is a set of generators for the module $\Upsilon_p^q(c)$.

Lemma 30: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The multi-indexed list $(\Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} \mid k \in \mathbb{N})$ of multipoles defined as:

$$\Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} = (-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \tag{4.2.108}$$

is a basis of $\Upsilon_p^q(c)$. In the same way the sublist $(\Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} \mid k \in [0, N] \subset \mathbb{N})$ is a basis for the submodule $\overset{(N)}{\Upsilon}{}_p^q(c) \subset \Upsilon_p^q(c)$ of the multipoles up to the order N.

Proof. We already have seen how an arbitrary distribution can be written as a $C^\infty(\mathbb{R})$-linear combination of elements of the list $(\Psi_{(i) m_{\overline{k}} \nu_{\overline{q}}}^{\mu_{\overline{p}}} \,|\, k \in \mathbb{N})$. We need to check the $C^\infty(\mathbb{R})$-linear independence. This can be done by simply checking that the null distribution can be written uniquely as a linear combination of null coefficients. The uniqueness of the adapted Ellis moments with respect to an adapted atlas is guaranteed by the previous theorem, therefore it is enough to check just that the null multipole can be written via a linear combination of null coefficients. This is trivial because $\forall \phi \in \Gamma_0 T_q^p M$ we have that:

$$[\phi, \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} 0_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}} \nu_{\overline{q}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \Big)] = \tag{4.2.109}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} \Big) c^\star(\psi_i) 0_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds = \int_{\mathbb{R}} 0 \cdot ds = 0 \tag{4.2.110}$$

The same reasoning can be repeated identically for the sub-module $\overset{(N)}{\Upsilon}{}_p^q(c)$ generated by the list $(\Psi^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \mid k \in [0,N] \subset \mathbb{N})$. Thence we have the thesis. $\square$

Definition 68: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The multi-indexed list $(\Psi^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \mid k \in \mathbb{N})$ of multipoles defined as:

$$\Psi^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = (-1)^k L_{m_{\overline{k}}}\Big\{\psi_i\Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \tag{4.2.111}$$

is called ***adapted Ellis basis of the multipoles*** and the sublist $(\Psi^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \mid k \in [0,N] \subset \mathbb{N})$ is called ***adapted Ellis basis of the multipoles up to the order*** N.

Let us notice that since $\Upsilon^q_p(c)$ is a $C^\infty(\mathbb{R})$-free module and since $C^\infty(\mathbb{R})$ is not a division ring, the cardinality of the basis is no more guaranteed. At this purpose let us consider to have two adapted atlas $\mathcal{A}$ and $\mathcal{A}'$ such that they cover the sub-manifold $c(\mathbb{R})$ with a different number of local charts. Both of them induce a bijection between the multipoles and their own adapted Ellis moments and such that the linear structure is preserved. However the number of the adapted Ellis moments strongly depends on the number of charts covering $c(\mathbb{R})$, the cardinality of the adapted Ellis moments is not preserved, therefore $\Upsilon^q_p(c)$ does not own the invariant basis number property. Considering this, we must admit that $\Upsilon^q_p(c)$ has no concept of dimension. The same argument can be proposed concerning the structure of $\overset{(k)}{\Upsilon}{}^q_p(c)$ as a sub-module of $\Upsilon^q_p(c)$. Despite this behaviour could seem awful, in practice it is not a problem, and several mathematical standard objects we already defined, like the space of smooth sections of T^p_qM or $C^\infty(M)$ just to quote some, share this property.

Property 41: From the adapted local Ellis representation of the multipoles it is trivial to realise that the order of the multipole is equal to the maximum number of derivations acting on the test tensor fields before the integration process.

Considering this property, we provide the following definition.

Definition 69: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The adapted Ellis local representation with respect to the given adapted atlas is the isomorphism (of modules) between the $\Upsilon^q_p(c)$ and the set of the Ellis parameters induced by the given atlas.

4.3 Transverse basis for low order multipoles fixed by a transverse frame

We have seen how fixing an adapted atlas introduces enough structure to establish an isomorphism between the multipoles and a specific Ellis representation. Since the bijection strongly depends on the choice of a particular adapted atlas, this approach spoils the covariance. In fact the choices of this basis for $\Upsilon^q_p(c)$ is not invariant under local diffeomorphism. At this point a natural question arises: is it possible to find a isomorphism between the multipoles and a specific local Ellis representation without relying on a specific coordinate system in order to preserve the covariance? Or equivalently, is it possible to induce an Ellis basis of $\Upsilon^q_p(c)$ preserving the invariance under general changes of coordinates? For the actual state of the art, the answer is yes and no at the same time, because the question is not well posed. As far as we know, the definition of the Ellis representation does not includes enough structure allowing us to select a $C^\infty(\mathbb{R})$-linearly independent set of generators of $\Upsilon^q_p(c)$ at all. If the question is, "It is possible to establish an isomorphism between the multipoles and a particular Ellis representation of them without requiring more than the Ellis definition?" then the answer is: definitely no. At the same time we have to consider that, loosely speaking, if one introduces an adapted atlas, a unique Ellis representation not involving Lie derivatives along the direction tangent to the sub-manifold $c(\mathbb{R})$ can be singled out and the bijection between the multipoles and the adapted Ellis moments preserve the $C^\infty(M)$-linear structure. Considering this, the reader could ask: "Which geometrical additional structure is covertly introduced by choosing a particular adapted atlas? There is a chance to express this additional structure independently from the choices of a particular coordinate system?". In this section we are going to try to fix at least a starting point in the research of an exhaustive and complete mathematical formalisation of the answer. The investigation of this aspect is deeply non trivial and it is still at the early stages. A lot of work needs still to be done.

4.3.1 Transverse basis for the multipoles up to order 2

Let us start recalling the definition of adapted local charts. Given a closed embedding $c : \mathbb{R} \hookrightarrow M$, a local chart (U, φ) on a manifold M is called a ***adapted chart*** if it satisfies:

1. U is diffeomorphic to $I \times V \subseteq \mathbb{R} \times \mathbb{R}^{m-1}$, where I and V are open subsets of $\mathbb{R}$ and $\mathbb{R}^{m-1}$ respectively
2. there always exists two maps such that $\psi : U \to I \subset \mathbb{R}$ and $\phi : U \to V \subset \mathbb{R}$ such that $\varphi(x) = (\psi(x), \phi(x)), \forall x \in U$ is just the Cartesian product.
3. the embedding $c : \mathbb{R} \to M$ is locally expressed by $c(s) = (s, 0^i)$ with $i \in [1, m-1] \subset \mathbb{N}$.

A local adapted coordinate system induces naturally a local frame (∂_μ) such that:

$$(\partial_\mu)_{|c(s)} = (\frac{d}{ds}, \frac{\partial}{\partial x^i})_{|c(s)} \tag{4.3.1}$$

as well as a natural local projection $\pi_1 : U \to I = c^{-1}(U \cap c(\mathbb{R})) \subseteq \mathbb{R}$ making the open set U a fibered manifold on I. So essentially, the choice of a local adapted coordinate system introduces an additional bundle structure on each open set of the adapted atlas encoding a concept of splitting the longitudinal and transversal directions in a neighbourhood of each point of the sub-manifold $c(\mathbb{R})$. This concept seems to be very closely related to the definition of connections upon bundles but the actual possibility to encode this geometrical information as a standard vertical and horizontal splitting of the tangent space of a proper global bundle structure built upon $c(\mathbb{R})$ compatible with the invariance under diffeomorphism is still a matter of investigation. To understand why this extra structure is needed it is necessary to consider that the multipoles act on the test tensor fields with several derivations then integrated along the sub-manifold $c(\mathbb{R})$. We can realise how the contribution to the result of the integral given by higher order derivatives of the test tensor fields taken with respect to a non-longitudinal directions is identical to the one given by an appropriate $C^\infty(\mathbb{R})$-linear combination of lower order transverse derivatives due to the integration by parts. In this fashion if we are able to fix an adapted local frame, we are able to kill the ambiguity into the Ellis parameters arising by the integration process. The interesting thing is that the information we need about the transversality is completely encoded inside the adapted local frame and it plays an essential role just to define the transverse directions of the derivations. Of course this frame can be built naturally by the adapted coordinate system but since a frame is formed just by a list of $C^\infty(M)$ linearly independent smooth vector fields, that are intrinsic geometrical objects, we should not be forced to use a specific local chart to express it. So in this perspective, let us consider a multipole $\mathcal{T}$ whose action can be written uniquely with respect to an adapted local frame $(e_{(i)\mu})$ as:

$$[\phi, \mathcal{T}] = \left[\phi, \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)\mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright (-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i (e^{\mu_{\overline{p}}} \otimes e_{\nu_{\overline{q}}}) \cdot c_\zeta(ds) \Big\} \right] = \tag{4.3.2}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \int_{\mathbb{R}} c^\star \Big(L_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} \Big) c^\star(\psi_i) \alpha_{(i)\mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \tag{4.3.3}$$

Focusing on the action of $\mathcal{T}$, we can see how the adapted frame $(e_{(i)\mu})$ plays two roles here. First of all it induces the tensor frame $(e^{\mu_{\overline{p}}} \otimes e_{\nu_{\overline{q}}})$ that is responsible to extract all the components of the tensor fields $L_{m_{\overline{k}}}(\phi) \in \Gamma_0 T^p_q M$, $\forall k \leq ord(\mathcal{T})$ inside the integral. But the frame also fixes the transverse direction with respect to which the Lie derivatives are taken. As we already stated, all the problems concerning the non uniqueness of the Ellis representation are caused by the chance to integrate by parts the Lie derivative taken in non transverse directions, but no problems arises at all if the Lie derivatives are taken with respect to the same direction. So a natural way to work around the problem could simply consist to avoid any change in the directions with respect tothe derivations are taken, when a change of natural frame is induced by a change of local charts. Very easily we can interpret temporarily $L_{m_{\overline{k}}}(\phi)$ exactly as it is, a test tensor field obtained from ϕ acting with k Lie derivatives taken with respect to several directions expressed by the multi-indexed lists of local vector fields $(e_{m_1}, ..., e_{m_k})$ as follow:

$$L_{m_{\overline{k}}}(\phi) = L_{e_{m_1}}...L_{e_{m_k}}(\phi) = L_{e_{m_{\overline{k}}}}(\phi) \tag{4.3.4}$$

Let us consider arbitrary local chart $(U_i, \varphi_{(i)})$ on the manifold M inducing a natural local frame $(\partial_{(i)\mu})$ and let us consider to have an adapted local frame $(e_{(i)\mu})$ defined on U from an adapted coordinate system. The adapted local frame can be expressed as a linear combination:

$$e_{(i)\mu} = \Lambda^{\nu}_{(i)\mu}\partial_{(i)\nu} \tag{4.3.5}$$

where $\Lambda^{\nu}_{(i)\mu} : M \to GL(\mathbb{R}^m)$ are the matrices of coefficients that express the adapted frame as a linear combination of the new one. Since both of the frames are natural, it follows $\Lambda^{\nu}_{(i)\mu}$ is equal to the Jacobian of the coordinate transformation between adapted and general local charts. Without showing all the details we can simply say that $\Lambda^{\nu}_{(i)\mu}$ can be also interpreted themselves as the local section of a principal fiber bundle called the "tangent frame bundle" denoted by $L(M)$. The local smooth sections of this bundle are all the possible smooth local frames one can define on TM. This is usually the starting point to define the vielbein formalism. Starting from the expression above we can split the adapted frame into a first vector field, longitudinal with respect to $c(\mathbb{R})$, from the other $m - 1$ vector fields related to the transverse directions:

$$\begin{cases} e_{(i)0} = \Lambda^{\mu}_{(i)0}\partial_{(i)\mu} = K^{\mu}_{(i)}\partial_{(i)\mu} \\ e_{(i)m} = \Lambda^{\mu}_{(i)m}\partial_{(i)\mu} \end{cases} \tag{4.3.6}$$

Since (e_{μ}) are the members of natural frame of an adapted coordinate system they must satisfy the commutation rules:

$$\begin{cases} [e_{(i)0}, e_{(i)m}] = 0 \\ [e_{(i)m}, e_{(i)n}] = 0 \end{cases} \tag{4.3.7}$$

This fixes a very strong constraint upon the matrices $\Lambda^{\nu}_{(i)m}$ and $K^{\nu}_{(i)}$:

$$\begin{cases} [K^{\mu}_{(i)}\partial_{(i)\mu}, \Lambda^{\nu}_{(i)m}\partial_{(i)\nu}] = 0 \\ [\Lambda^{\mu}_{(i)m}\partial_{(i)\mu}, \Lambda^{\nu}_{(i)n}\partial_{(i)\nu}] = 0 \end{cases} \tag{4.3.8}$$

which tell us in an algebraic fashion that $K^{\mu}_{(i)}$ and $\Lambda^{\mu}_{(i)m}$ must be equal to the components of the Jacobian of the coordinate transformation between an adapted and a general local chart. Fixing a specific adapted frame we can just decide to perform another general change of local charts, then the transformation rules follows immediately:

$$\begin{cases} e_{(i)0} = K'^{\mu}_{(i)}\partial'_{(i)\mu} = K^{\mu}_{(i)}J^{\nu}_{\mu}\partial'_{(i)\nu} \\ e_{(i)m} = \Lambda'^{\mu}_{(i)m}\partial'_{(i)\mu} = \Lambda^{\mu}_{(i)m}J^{\nu}_{\mu}\partial'_{(i)\nu} \end{cases} \tag{4.3.9}$$

$$\begin{cases} K'^{\mu}_{(i)} = K^{\nu}_{(i)}J^{\nu}_{\mu} \\ \Lambda'^{\nu}_{(i)m} = \Lambda^{\mu}_{(i)m}J^{\nu}_{\mu} \end{cases} \tag{4.3.10}$$

In principle one could use this relations to express $L_{e_{m_{\overline{k}}}}(\phi)$ as a $C^{\infty}(M)$-linear combination of derivations with respect to the natural frame induced by an arbitrary local chart. Although this could seem trivial in appearance, in practice this process is extremely tricky and complicated due to the non $C^{\infty}(M)$-linearity of the Lie derivatives. This prevent us to find some regularities in the formulas needed defining the basis for each arbitrary local chart. At the end of the day this is a practical approach just for low order multipoles when $k = 0, 1, 2$, then it can be used explicitely to express covariantly the basis of $\overset{(k)}{\Upsilon}{}^{q}_{p}(c)$. In principle nothing forbids us to use this approach also to construct a basis for the higher order multipoles sub-sets, but the complexity of the expressions and the number of different terms involved make the life very problematic. At the actual state of the art, as far as we know, there is no way to fix a simple general covariant expression for an Ellis local representation that preserve the module structure of $\Upsilon^{q}_{p}(c)$. We are going to explicitly show now the first three simpler cases.

Transverse basis of monopoles module $\overset{(0)}{\Upsilon}{}^{q}_{p}(c)$

Let us suppose to have a generic multipole $\mathcal{T} \in \overset{(0)}{\Upsilon}{}^{p}_{q}(c)$. This is usually called a monopole. Accordingly to the adapted Ellis representation:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)}{}^{m_{\overline{k}}\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}}\Big\{\psi_i\Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}\Big) = \tag{4.3.11}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \left\{ \psi_i \left[e_{(i)}^{\mu_{\overline{p}}} \otimes e_{(i)\nu_{\overline{q}}} \right] c_\zeta(ds) \right\} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \Psi_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.12}$$

Let us try to express the same expression for a natural local frame induced by a generic coordinate system:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \left\{ \psi_i \left[\overline{\Lambda}_{(i)_{\alpha_{\overline{p}}}}^{\mu_{\overline{p}}} \Lambda_{(i)_{\nu_{\overline{q}}}}^{\beta_{\overline{q}}} dx_{(i)}^{\alpha_{\overline{p}}} \otimes \partial_{(i)\beta_{\overline{q}}} \right] c_\zeta(ds) \right\} = \tag{4.3.13}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star(\overline{\Lambda}_{(i)_{\alpha_{\overline{p}}}}^{\mu_{\overline{p}}}) c^\star(\Lambda_{(i)_{\nu_{\overline{q}}}}^{\beta_{\overline{q}}}) \alpha_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \left\{ \psi_i \left[dx_{(i)}^{\alpha_{\overline{p}}} \otimes \partial_{(i)\beta_{\overline{q}}} \right] c_\zeta(ds) \right\} = \tag{4.3.14}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \Theta_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.15}$$

where by definition we have that:

$$\Theta_{(i)\beta_{\overline{q}}}^{\alpha_{\overline{p}}} = \psi_i \left[dx_{(i)}^{\alpha_{\overline{p}}} \otimes \partial_{(i)\beta_{\overline{q}}} \right] c_\zeta(ds) \tag{4.3.16}$$

are a new set of generators and

$$\beta_{(i)\ \ \alpha_{\overline{p}}}^{\beta_{\overline{q}}} = c^\star(\overline{\Lambda}_{(i)_{\alpha_{\overline{p}}}}^{\mu_{\overline{p}}}) c^\star(\Lambda_{(i)_{\nu_{\overline{q}}}}^{\beta_{\overline{q}}}) \alpha_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \tag{4.3.17}$$

are a new set of Ellis parameters. Therefore $(\Theta_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}})$ is a set of generators for $\overset{(0)}{\Upsilon}{}_p^q(c)$. It is trivial to show $(\Theta_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}})$ is also a basis because the null multipole can be written uniquely as a linear combination of null coefficients. In fact:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \Theta_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} = 0 \Leftrightarrow [\phi, \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta_{(i)\ \ \mu_{\overline{p}}}^{\nu_{\overline{q}}} \triangleright \Theta_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}] \quad , \quad \forall \phi \in \Gamma_0 T_q^p M \tag{4.3.18}$$

This means that $\forall\phi \in \Gamma_0 T^p_q M$

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})c^\star(\psi_i)\beta^{\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} ds = 0 \tag{4.3.19}$$

that we know can be satisfied if and only if all the $\beta^{\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}}$ are null.

As one can see the monopole case is extremely trivial, since the 0-th order does not allow the existence of any derivation along a transverse directions, no adapted coordinates or adapted frames are really required to define a "transverse" direction, then the basis $(\Theta^{\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}}) = \psi_i\Big[dx^{\alpha_{\overline{p}}}_{(i)}\otimes\partial_{(i)\beta_{\overline{q}}}\Big]c_\zeta(ds)$ can be singled out without effort for each local coordinate system.

Transverse basis of dipole module $\overset{(1)}{\Upsilon}{}^q_p(c)$

The dipole case is trickier than the monopole one. First of all let us stress once again that the module of the monopoles $\overset{(0)}{\Upsilon}{}^q_p(c) \subset \overset{(1)}{\Upsilon}{}^q_p(c)$ is a submodule. Let us suppose to have a generic multipole $\mathcal{T} \in \overset{(1)}{\Upsilon}{}^p_q(c)$. This is usually called a dipole. Accordingly to the adapted Ellis representation:

$$\mathcal{T} = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}}\Big\{\psi_i\Big[e^{\mu_{\overline{p}}}_{(i)}\otimes e_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\}\Big) = \tag{4.3.20}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \alpha^{\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright \Big\{\psi_i\Big[e^{\mu_{\overline{p}}}_{(i)}\otimes e_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} - \alpha^{m\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright L_{e_m}\Big\{\psi_i\Big[e^{\mu_{\overline{p}}}_{(i)}\otimes e_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} = \tag{4.3.21}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \alpha^{\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright \Psi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} - \alpha^{m\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright \Psi^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} \tag{4.3.22}$$

Let us try to express the same expression for a natural local frame induced by a generic coordinate system but keeping attention to do not change the direction with respect to which the Lie derivatives are taken:

$$\mathcal{T} = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \alpha^{\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright \Big\{\psi_i\Big[\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}}\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}} dx^{\alpha_{\overline{p}}}_{(i)}\otimes\partial_{(i)\beta_{\overline{q}}}\Big]c_\zeta(ds)\Big\}+ \tag{4.3.23}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_m} \Big\{ \psi_i \Big[\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)_{\alpha_{\overline{p}}}} \Lambda^{\beta_{\overline{q}}}_{(i)_{\nu_{\overline{q}}}} dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.24}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)_{\alpha_{\overline{p}}}}) c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)_{\nu_{\overline{q}}}}) \alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big\{ \psi_i \Big[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.25}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)_{\alpha_{\overline{p}}}}) c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)_{\nu_{\overline{q}}}}) \alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_m} \Big\{ \psi_i \Big[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.26}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} \tag{4.3.27}$$

where by definition we have that:

$$\begin{cases} \Theta^{\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}} = \psi_i \Big[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \\ \Theta^{\alpha_{\overline{p}}}_{(i)m\beta_{\overline{q}}} = (-1) L_{e_m} \Big\{ \psi_i \Big[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \end{cases} \tag{4.3.28}$$

are a new set of generators and

$$\begin{cases} \beta^{\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}} = c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)_{\alpha_{\overline{p}}}}) c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)_{\nu_{\overline{q}}}}) \alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \\ \beta^{m\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}} = c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)_{\alpha_{\overline{p}}}}) c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)_{\nu_{\overline{q}}}}) \beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \end{cases} \tag{4.3.29}$$

are a new set of Ellis parameters. Therefore $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}})$ is a set of generators for $\overset{(0)}{\Upsilon}{}^q_p(c)$. It is trivial to show $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}})$ is also a basis because the null multipole can be written uniquely as a linear combination of null coefficients. In fact:

$$\sum_{k=0}^{1} \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = 0 \tag{4.3.30}$$

$$\Leftrightarrow [\phi, \sum_{k=0}^{1} \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}}] = 0 \quad , \quad \forall \phi \in \Gamma_0 T^p_q M \tag{4.3.31}$$

This means that $\forall \phi \in \Gamma_0 T^p_q M$

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})c^\star(\psi_i)\beta^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds + \int_{c(\mathbb{R})} c^\star(L_{e_m}\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})c^\star(\psi_i)\beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds = 0 \qquad (4.3.32)$$

which we know can be satisfied if and only if all the $\beta^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}$ are null, because the e_m are linearly independent with respect to $\dot{c} = \frac{d}{dt}c(t)$ when restricted on $c(\mathbb{R})$, so no integration by part can be performed to manipulate these derivations. Let us notice that the $\mathcal{T}$ can be expressed as a linear combination of the monopole basis and some other first order distributions. By convention the set of all $C^\infty(\mathbb{R})$ linear combination defined as:

$$\left\{\beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} \quad , \quad \forall \beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \in C^\infty(I_i) \mid I_i = c^{-1}(U_i \cap c(\mathbb{R}))\right\} \qquad (4.3.33)$$

is also a $C^\infty(\mathbb{R})$-sub-module of $\overset{(1)}{\Upsilon}{}^q_p(c)$. We call it ***the Ellis local pure dipole module with respect to the given adapted frame*** and the related adapted Ellis moments are called ***the local adapted pure dipole moments with respect to the given adapted frame***. Let us stress that this split strongly depends on the choices of the adapted local frame (e_μ). We will see later how this split is not preserved when a change in the adapted local frame is performed. Therefore a dipole being "pure" with respect to one adapted frame in general is not "pure" with respect to another adapted frame. The generators $(\Theta_{(i)\nu_{\overline{q}}}{}^{\mu_{\overline{p}}})$ related to $\overset{(0)}{\Upsilon}{}^q_p(c)$ have been already made explicit in the monopole case. The other generators $\Theta^{\alpha_{\overline{p}}}_{(i)m\beta_{\overline{q}}} = (-1)L_{e_m}\left\{\psi_i\left[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}}\right]c_\zeta(ds)\right\}$ have not been made explicit. In fact the members e_m of the new frame, should be written as a linear combination of the memebers of the natural local frame $\partial_{(i)\beta_{\overline{q}}}$. By definition we have that, $\forall \phi \in \Gamma_0 T^p_q M$ the action of $\Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}$ is:

$$[\phi, \Theta_{(i)m\nu_{\overline{q}}}{}^{\mu_{\overline{p}}}] = \int_{\mathbb{R}} c^\star\left(L_{e_m}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\right)c^\star(\psi_i)ds \qquad (4.3.34)$$

To express the derivatives along the transverse directions (e_m) as a linear combination of Lie derivatives with respect to the natural frame induced by an arbitrary local coordinate system:

$$[\phi,\Theta_{(i)m\nu_{\overline{q}}}{}^{\mu_{\overline{p}}}]=\int_{\mathbb{R}}c^{\star}\Big(L_{e_m}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^{\star}(\psi_i)ds=\int_{\mathbb{R}}c^{\star}\Big(L_{\Lambda^{\lambda}_{(i)m}\partial_{(i)\lambda}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^{\star}(\psi_i)ds= \tag{4.3.35}$$

$$=\int_{\mathbb{R}}c^{\star}\Big(\Lambda^{\lambda}_{(i)m}L_{\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}-\sum_{s=1}^{p}\partial_{\alpha}(\Lambda^{\mu_s}_{(i)m})\phi^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}}+\sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\phi^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}\Big)c^{\star}(\psi_i)ds \tag{4.3.36}$$

But we can recognise that this is a possible Ellis representation of the action of the generators $\Theta_{(i)m\nu_{\overline{q}}}{}^{\mu_{\overline{p}}}$. Therefore we can write:

$$\Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}=(-1)^k\Lambda^{\lambda}_{m}L_{\lambda}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+ \tag{4.3.37}$$

$$-\sum_{s=1}^{p}c^{\star}(\partial_{\alpha}\Lambda^{\mu_s}_{(i)m})\Big\{\psi_i\Big[dx^{\mu_{\overline{s-1}}}_{(i)}\otimes dx^{\mu_{\alpha}}_{(i)}\otimes dx^{\mu_{\overline{p}}\setminus\overline{s}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+ \tag{4.3.38}$$

$$+\sum_{r=1}^{q}c^{\star}(\partial_{\nu_r}\Lambda^{\beta}_{(i)m})\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{r}}}\otimes\partial_{(i)\beta}\otimes\partial_{(i)\nu_{\overline{q}\setminus\overline{s}}}\Big]c_{\zeta}(ds)\Big\} \tag{4.3.39}$$

Since we have explicitly chosen a specific adapted coordinate frame $(e_{(i)\mu})$ with which the Lie derivatives have been defined, the rectangular matrices $\Lambda^{\mu}_{(i)m}$ are known, as well as their derivatives $\partial_{\alpha}\Lambda^{\mu}_{(i)m}$. Therefore we can conclude that by choosing a local adapted coordinate system (or equivalently a local adapted frame) we can always induce a basis$(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}},\Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}})$ of $\overset{(1)}{\Upsilon}{}^{q}_{p}(c)$ that can be expressed as where by definition we have that:

$$\begin{cases}\Theta^{\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}}= & \psi_i\Big[dx^{\alpha_{\overline{p}}}_{(i)}\otimes\partial_{(i)\beta_{\overline{q}}}\Big]c_{\zeta}(ds)\\[2ex] \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}= & (-1)^k\Lambda^{\lambda}_{m}L_{\lambda}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+\\ & -\sum_{s=1}^{p}c^{\star}(\partial_{\alpha}\Lambda^{\mu_s}_{(i)m})\Big\{\psi_i\Big[dx^{\mu_{\overline{s-1}}}_{(i)}\otimes dx^{\mu_{\alpha}}_{(i)}\otimes dx^{\mu_{\overline{p}}\setminus\overline{s}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+\\ & +\sum_{r=1}^{q}c^{\star}(\partial_{\nu_r}\Lambda^{\beta}_{(i)m})\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{r}}}\otimes\partial_{(i)\beta}\otimes\partial_{(i)\nu_{\overline{q}\setminus\overline{s}}}\Big]c_{\zeta}(ds)\Big\}\end{cases} \tag{4.3.40}$$

Transverse basis of quadrupole module $\overset{(2)}{\Upsilon}{}^{q}_{p}(c)$

The quadrupole case is trickier than the dipole one and much more complicated than the monopole. First of all let us stress once again that the dipole module $\overset{(1)}{\Upsilon}{}^{q}_{p}(c) \subset \overset{(2)}{\Upsilon}{}^{q}_{p}(c)$ is a submodule. Let us suppose to have a generic multipole $\mathcal{T} \in \overset{(1)}{\Upsilon}{}^{p}_{q}(c)$. This is usually called a quadrupole. Accordingly to the adapted Ellis representation:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \Big) = \tag{4.3.41}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} - \alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_m} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.42}$$

$$+ \alpha^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_n} L_{e_m} \Big\{ \psi_i \Big[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.43}$$

$$\tag{4.3.44}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Psi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} - \alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Psi^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} + \alpha^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Psi^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}} \tag{4.3.45}$$

Let us try to express the same expression for a natural local frame induced by a generic coordinate system but without make any change in the directions with respect to the Lie derivatives are taken:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big\{ \psi_i \Big[\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}} \Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}} dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.46}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_m} \Big\{ \psi_i \Big[\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}} \Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}} dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.47}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \alpha^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_n} L_{e_m} \Big\{ \psi_i \Big[\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}} \Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}} dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.48}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}}) c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}}) \alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big\{ \psi_i \Big[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.49}$$

$$- \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}}) c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}}) \alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright L_{e_m} \Big\{ \psi_i \Big[dx^{\alpha_{\overline{p}}}_{(i)} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.50}$$

$$+\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}})c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}})\alpha^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\triangleright L_{e_n}L_{e_m}\Big\{\psi_i\Big[dx^{\alpha_{\overline{p}}}_{(i)}\otimes\partial_{(i)\beta_{\overline{q}}}\Big]c_\zeta(ds)\Big\}= \tag{4.3.51}$$

$$=\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\beta^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}+\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\triangleright\Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}+\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\beta^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\triangleright\Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}} \tag{4.3.52}$$

where by definition we have that:

$$\begin{cases}\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}=\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\\ \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}=(-1)L_{e_m}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\}\\ \Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}}=L_{e_n}L_{e_m}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\}\end{cases} \tag{4.3.53}$$

are a new set of generators and

$$\begin{cases}\beta^{\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}}=c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}})c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}})\alpha^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\\ \beta^{m\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}}=c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}})c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}})\alpha^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\\ \beta^{nm\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}}=c^\star(\overline{\Lambda}^{\mu_{\overline{p}}}_{(i)\alpha_{\overline{p}}})c^\star(\Lambda^{\beta_{\overline{q}}}_{(i)\nu_{\overline{q}}})\alpha^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\end{cases} \tag{4.3.54}$$

are a new set of Ellis parameters. Therefore $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}},\Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}},\Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}})$ is a set of generators for $\overset{(0)}{\Upsilon}{}^q_p(c)$. It is trivial to show $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}},\Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}},\Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}})$ is also a basis because the null multipole can be written uniquely as a linear combination of null coefficients. In fact:

$$\sum_{k=0}^{2}\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\beta^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\triangleright\Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}}=0 \tag{4.3.55}$$

$$\Leftrightarrow[\phi,\sum_{k=0}^{2}\sum_{\substack{U_i\in\mathcal{A}\\U_i\cap c(\mathbb{R})\neq\varnothing}}\beta^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}\triangleright\Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}}]=0\quad,\quad\forall\phi\in\Gamma_0T^p_qM \tag{4.3.56}$$

This means that $\forall \phi \in \Gamma_0 T^p_q M$

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i)\beta^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds + \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(L_{e_m}\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i)\beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds + \tag{4.3.57}$$

$$+ \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(L_{e_n}L_{e_m}\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i)\beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds = 0 \tag{4.3.58}$$

that we know can be satisfied if and only if all the $\beta^{\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}$, $\beta^{m\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}$, and $\beta^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}$ are null, because the $e_{(i)m}$ are linearly independents with respect to $\dot{c} = \frac{d}{dt}c(t)$ when restricted on $c(\mathbb{R})$, so no integration by part can be performed to manipulate these derivations. Furthermore since $(e_{(i)\mu})$ is an adapted frame then $[e_m, e_n] = 0$. Let us notice that the $\mathcal{T}$ can be expressed as a linear combination of the dipole basis and some other second order distributions. By convention the set of all $C^\infty(\mathbb{R})$ linear combination defined as:

$$\left\{\beta^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}} \quad , \quad \forall \beta^{nm\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \in C^\infty(I_i) \mid I_i = c^{-1}(U_i \cap c(\mathbb{R}))\right\} \tag{4.3.59}$$

is also a $C^\infty(\mathbb{R})$-submodule of $\overset{(2)}{\Upsilon}{}^q_p(c)$. We call it ***the Ellis local pure quadrupole module with respect to the given adapted frame*** and the related adapted Ellis moments are called ***the local adapted pure quadrupole moments with respect to the given adapted frame***. Let us stress that this split strongly depends on the choices of the adapted local frame (e_μ). We will see later how this split is not preserved when a change in the adapted local frame is performed, therefore a quadrupole being "pure" with respect to an adapted frame in general is no more "pure" with respect to another adapted frame. The generators $(\Theta_{(i)\nu_{\overline{q}}}{}^{\mu_{\overline{p}}}, \Theta_{(i)m\nu_{\overline{q}}}{}^{\mu_{\overline{p}}})$ related to $\overset{(1)}{\Upsilon}{}^q_p(c)$ have been already made explicit in the dipole case. The other generators $\Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}} = L_{e_n}L_{e_m}\left\{\psi_i\left[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\right]c_\zeta(ds)\right\}$ are not yet fully explicited. In fact when the directions e_m with respect to the Lie derivatives are taken, they should be written as a linear combination of the generic natural local frame as well. This time this is much more longer and difficult with respect to the dipole case. By definition we have that, $\forall \phi \in \Gamma_0 T^p_q M$ the action of $\Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}}$ is:

$$[\phi, \Theta_{(i)nm\nu_{\overline{q}}}{}^{\mu_{\overline{p}}}] = \int_{\mathbb{R}} c^\star\left(L_{e_n}L_{e_m}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\right)c^\star(\psi_i)ds \tag{4.3.60}$$

To express the derivatives along the transverse directions (e_m) as a linear combination of Lie derivatives with respect to the natural frame induced by an arbitrary local coordinate:

$$L_{e_n}L_{e_m}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_{\Lambda^\lambda_n\partial_\lambda}(L_{e_m}\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.61}$$

$$=\Lambda^\lambda_n L_\lambda(L_{e_m}\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} - \sum_{s=1}^{p}\partial_\alpha(\Lambda^{\mu_s}_n)L_{e_m}(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{\nu_{\overline{q}}} + \sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^\beta_n)L_{e_m}(\phi)^{\mu_{\overline{s}}}_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} = \tag{4.3.62}$$

$$=\Lambda^\rho_n\partial_\rho(L_{e_m}\phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) - \sum_{s=1}^{p}\partial_\alpha(\Lambda^{\mu_s}_n)L_{e_m}(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{\nu_{\overline{q}}} + \sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^\beta_n)L_{e_m}(\phi)^{\mu_{\overline{s}}}_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.63}$$

where we used that for each tensor field, locally we have $L_\lambda(T)^{\mu_{\overline{p}}}_{\nu\overline{q}} = L_{\delta^\rho_\lambda\partial_\rho}(T^{\mu_{\overline{p}}}_{\nu\overline{q}}) = \delta^\rho_\lambda\partial_\rho(T^{\mu_{\overline{p}}}_{\nu\overline{q}}) - 0 + 0 = \partial_\lambda(T^{\mu_{\overline{p}}}_{\nu\overline{q}})$ since we are considering the natural local frame. Hence:

$$L_{e_n}L_{e_m}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.64}$$

$$=\Lambda^\rho_n\partial_\rho\Big(\Lambda^\lambda_m\partial_\lambda(\phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) - \sum_{s=1}^{p}\partial_\alpha(\Lambda^{\mu_s}_m)(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{\nu_{\overline{q}}} + \sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^\beta_m)(\phi)^{\mu_{\overline{s}}}_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}\Big)+ \tag{4.3.65}$$

$$-\sum_{t=1}^{p}\partial_\gamma(\Lambda^{\mu_t}_n)\Big\{\Lambda^\lambda_m\partial_\lambda(\phi^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}_{\nu_{\overline{q}}}) - \sum_{s=1}^{t-1}\partial_\alpha(\Lambda^{\mu_s}_m)\phi^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\setminus\overline{s}}\gamma\mu_{\overline{p}\setminus\overline{t}}}_{\nu_{\overline{q}}} - \partial_\alpha(\Lambda^\gamma_m)\phi^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\setminus\overline{t}}}_{\nu_{\overline{q}}}+ \tag{4.3.66}$$

$$-\sum_{s=t+1}^{p}\partial_\alpha(\Lambda^{\mu_s}_m)\phi^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\setminus\overline{t}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{\nu_{\overline{q}}} + \sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^\beta_m)(\phi)^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}\Big\}+ \tag{4.3.67}$$

$$+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda^\delta_n)\Big\{\Lambda^\lambda_m\partial_\lambda(\phi^{\mu_{\overline{p}}}_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\setminus\overline{u}}}) - \sum_{s=1}^{p}\partial_\alpha(\Lambda^{\mu_s}_m)(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{\nu_{\overline{u-1}}\delta\nu_{\overline{q}\setminus\overline{u}}} + \sum_{r=1}^{u-1}\partial_{\nu_r}(\Lambda^\beta_m)\phi^{\mu_{\overline{p}}}_{\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\setminus r}\delta\nu_{\overline{q}\setminus\overline{u}}}+ \tag{4.3.68}$$

$$+\partial_\delta(\Lambda^\beta_m)\phi^{\mu_{\overline{p}}}_{\nu_{\overline{u}-1}\beta\nu_{\overline{q}\setminus\overline{u}}} + \sum_{r=1}^{u-1}\partial_{\nu_r}(\Lambda^\beta_m)\phi^{\mu_{\overline{p}}}_{\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\setminus u}\beta\nu_{\overline{q}\setminus\overline{r}}}\Big\} \tag{4.3.69}$$

Now distributing the sums and the partial derivatives:

$$L_{e_n}L_{e_m}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.70}$$

$$=\Lambda^\rho_n\partial_\rho\Big(\Lambda^\lambda_m\partial_\lambda(\phi^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})\Big) - \Lambda^\rho\partial_\rho\Big(\sum_{s=1}^{p}\partial_\alpha(\Lambda^{\mu_s}_m)(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{\nu_{\overline{q}}}\Big)+ \tag{4.3.71}$$

$$+\Lambda^\rho_n\partial_\rho\Big(\sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^\beta_m)(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})\Big)+ \tag{4.3.72}$$

$$-\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\Lambda_{m}^{\lambda}\partial_{\lambda}(\phi_{\nu_{\overline{q}}}^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}})+\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\sum_{s=1}^{t-1}\partial_{\alpha}(\Lambda_{m}^{\mu_s})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\backslash\overline{s}}\gamma\mu_{\overline{p}\backslash\overline{t}}}+ \tag{4.3.73}$$

$$+\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\partial_{\alpha}(\Lambda_{m}^{\gamma})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\backslash\overline{t}}}+\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\sum_{s=t+1}^{p}\partial_{\alpha}(\Lambda_{m}^{\mu_s})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\backslash\overline{t}}\alpha\mu_{\overline{p}\backslash\overline{s}}}+ \tag{4.3.74}$$

$$-\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\sum_{r=1}^{q}\partial_{\nu_r}(\Lambda_{m}^{\beta})(\phi)_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}}+ \tag{4.3.75}$$

$$+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\Lambda_{m}^{\lambda}\partial_{\lambda}(\phi_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{p}}})-\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\sum_{s=1}^{p}\partial_{\alpha}(\Lambda_{m}^{\mu_s})(\phi)_{\nu_{\overline{u-1}}\delta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}+ \tag{4.3.76}$$

$$+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\sum_{r=1}^{u-1}\partial_{\nu_r}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\backslash r}\delta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{p}}}+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\partial_{\delta}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{u}-1}\beta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{p}}}+ \tag{4.3.77}$$

$$+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\sum_{r=1}^{u-1}\partial_{\nu_r}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\backslash u}\beta\nu_{\overline{q}\backslash\overline{r}}}^{\mu_{\overline{p}}} \tag{4.3.78}$$

to end up with:

$$L_{e_n}L_{e_m}(\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}}= \tag{4.3.79}$$

$$=\Lambda_{n}^{\rho}\Lambda_{m}^{\lambda}L_{\rho}L_{\lambda}(\phi_{\nu_{\overline{q}}}^{\mu_{\overline{p}}})+\Lambda_{n}^{\rho}\partial_{\rho}(\Lambda_{m}^{\lambda})L_{\lambda}(\phi_{\nu_{\overline{q}}}^{\mu_{\overline{p}}})+ \tag{4.3.80}$$

$$-\sum_{s=1}^{p}\Lambda_{n}^{\rho}\partial_{\rho}\partial_{\alpha}(\Lambda_{m}^{\mu_s})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}-\sum_{s=1}^{p}\partial_{\alpha}(\Lambda_{m}^{\mu_s})\Lambda_{n}^{\rho}L_{\rho}(\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}+ \tag{4.3.81}$$

$$+\sum_{r=1}^{q}\Lambda_{n}^{\rho}\partial_{\rho}\partial_{\nu_r}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}^{\mu_{\overline{p}}})+\sum_{r=1}^{q}\partial_{\nu_r}(\Lambda_{m}^{\beta})\Lambda_{n}^{\rho}L_{\rho}(\phi)_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}^{\mu_{\overline{p}}}) \tag{4.3.82}$$

$$-\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\Lambda_{m}^{\lambda}L_{\lambda}(\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}}+\sum_{t=1}^{p}\sum_{s=1}^{t-1}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\partial_{\alpha}(\Lambda_{m}^{\mu_s})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\backslash\overline{s}}\gamma\mu_{\overline{p}\backslash\overline{t}}}+ \tag{4.3.83}$$

$$+\sum_{t=1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\partial_{\alpha}(\Lambda_{m}^{\gamma})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\backslash\overline{t}}}+\sum_{t=1}^{p}\sum_{s=t+1}^{p}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\partial_{\alpha}(\Lambda_{m}^{\mu_s})\phi_{\nu_{\overline{q}}}^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\backslash\overline{t}}\alpha\mu_{\overline{p}\backslash\overline{s}}}+ \tag{4.3.84}$$

$$-\sum_{t=1}^{p}\sum_{r=1}^{q}\partial_{\gamma}(\Lambda_{n}^{\mu_t})\partial_{\nu_r}(\Lambda_{m}^{\beta})(\phi)_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}}+ \tag{4.3.85}$$

$$+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\Lambda_{m}^{\lambda}L_{\lambda}(\phi)_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{p}}}-\sum_{u=1}^{q}\sum_{s=1}^{p}\partial_{\nu_u}(\Lambda_{n}^{\delta})\partial_{\alpha}(\Lambda_{m}^{\mu_s})(\phi)_{\nu_{\overline{u-1}}\delta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}+ \tag{4.3.86}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1}\partial_{\nu_u}(\Lambda_{n}^{\delta})\partial_{\nu_r}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\backslash r}\delta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{p}}}+\sum_{u=1}^{q}\partial_{\nu_u}(\Lambda_{n}^{\delta})\partial_{\delta}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{u}-1}\beta\nu_{\overline{q}\backslash\overline{u}}}^{\mu_{\overline{p}}}+ \tag{4.3.87}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1}\partial_{\nu_u}(\Lambda_{n}^{\delta})\partial_{\nu_r}(\Lambda_{m}^{\beta})\phi_{\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\backslash u}\beta\nu_{\overline{q}\backslash\overline{r}}}^{\mu_{\overline{p}}} \tag{4.3.88}$$

Substituting this into the definition:

$$[\phi, \Theta_{(i)nm\nu_{\overline{q}}}^{\ \ \ \mu_{\overline{p}}}] = \int_{\mathbb{R}} c^{\star}\Big(L_{e_n} L_{e_m} (\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i) ds \tag{4.3.89}$$

we obtain:

$$[\phi, \Theta_{(i)nm\nu_{\overline{q}}}^{\ \ \ \mu_{\overline{p}}}] = \int_{\mathbb{R}} c^{\star}\Big(\Lambda^{\rho}_{(i)n}\Lambda^{\lambda}_{(i)m} L_{\rho} L_{\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \Lambda^{\rho}_{(i)n}\partial_{\rho}(\Lambda^{\lambda}_{(i)m}) L_{\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.90}$$

$$-\sum_{s=1}^{p} \Lambda^{\rho}_{(i)n}\partial_{\rho}\partial_{\alpha}(\Lambda^{\mu_s}_{(i)m})\phi^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{q}}} - \sum_{s=1}^{p} \partial_{\alpha}(\Lambda^{\mu_s}_{(i)m})\Lambda^{\rho}_{(i)n} L_{\rho}(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.91}$$

$$+\sum_{r=1}^{q} \Lambda^{\rho}_{(i)n}\partial_{\rho}\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}} + \sum_{r=1}^{q} \partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\Lambda^{\rho}_{(i)n} L_{\rho}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}) \tag{4.3.92}$$

$$-\sum_{t=1}^{p} \partial_{\gamma}(\Lambda^{\mu_t}_{(i)n})\Lambda^{\lambda}_{(i)m} L_{\lambda}(\phi)^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{q}}} + \sum_{t=1}^{p}\sum_{s=1}^{t-1} \partial_{\gamma}(\Lambda^{\mu_t}_{(i)n})\partial_{\alpha}(\Lambda^{\mu_s}_{(i)m})\phi^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\backslash\overline{s}}\gamma\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.93}$$

$$+\sum_{t=1}^{p} \partial_{\gamma}(\Lambda^{\mu_t}_{(i)n})\partial_{\alpha}(\Lambda^{\gamma}_{(i)m})\phi^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{q}}} + \sum_{t=1}^{p}\sum_{s=t+1}^{p} \partial_{\gamma}(\Lambda^{\mu_t}_{(i)n})\partial_{\alpha}(\Lambda^{\mu_s}_{(i)m})\phi^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\backslash\overline{t}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.94}$$

$$-\sum_{t=1}^{p}\sum_{r=1}^{q} \partial_{\gamma}(\Lambda^{\mu_t}_{(i)n})\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})(\phi)^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}} + \tag{4.3.95}$$

$$+\sum_{u=1}^{q} \partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\Lambda^{\lambda}_{(i)m} L_{\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\delta\nu_{\overline{q}\backslash\overline{u}}} - \sum_{u=1}^{q}\sum_{s=1}^{p} \partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\partial_{\alpha}(\Lambda^{\mu_s}_{(i)m})(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{u-1}}\delta\nu_{\overline{q}\backslash\overline{u}}} + \tag{4.3.96}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} \partial_{\nu_u}(\Lambda^{\delta}_{n})\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\backslash r}\delta\nu_{\overline{q}\backslash\overline{u}}} + \sum_{u=1}^{q} \partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\partial_{\delta}(\Lambda^{\beta}_{(i)m})\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\beta\nu_{\overline{q}\backslash\overline{u}}} + \tag{4.3.97}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} \partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\backslash u}\beta\nu_{\overline{q}\backslash\overline{r}}}\Big) c^{\star}(\psi_i) ds \tag{4.3.98}$$

But we can recognise that this is a possible Ellis representation of the action of the generators $\Theta_{(i)nm\nu_{\overline{q}}}^{\ \ \ \mu_{\overline{p}}}$. Therefore we can write:

$$\Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}} = c^{\star}\Big(\Lambda^{\rho}_{(i)n}\Lambda^{\lambda}_{(i)m}\Big) \triangleright L_{\rho} L_{\lambda}\Big\{\psi_i(dx^{\mu_{\overline{p}}} \otimes \partial_{\nu_{\overline{q}}})c_{\zeta}(ds)\Big\} + \tag{4.3.99}$$

$$+c^{\star}\Big(\Lambda^{\rho}_{(i)n}\partial_{\rho}(\Lambda^{\lambda}_{(i)m})\Big) \triangleright L_{\lambda}\Big\{\psi_i(dx^{\mu_{\overline{p}}} \otimes \partial_{\nu_{\overline{q}}})c_{\zeta}(ds)\Big\} + \tag{4.3.100}$$

$$-\sum_{s=1}^{p} c^\star\Big(\partial_\alpha(\Lambda^{\mu_s}_{(i)m})\Lambda^\lambda_{(i)n}\Big) \triangleright L_\lambda\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.101}$$

$$+\sum_{r=1}^{q} c^\star\Big(\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\Lambda^\lambda_{(i)n}\Big) \triangleright L_\lambda\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\}+ \tag{4.3.102}$$

$$-\sum_{t=1}^{p} c^\star\Big(\partial_\gamma(\Lambda^{\mu_t}_{(i)n})\Lambda^\lambda_{(i)m}\Big) \triangleright L_\lambda\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.103}$$

$$+\sum_{u=1}^{q} c^\star\Big(\partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\Lambda^\lambda_{(i)m}\Big) \triangleright L_\lambda\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.104}$$

$$-\sum_{s=1}^{p} c^\star\Big(\Lambda^{\rho}_{(i)n}\partial_\rho\partial_\alpha(\Lambda^{\mu_s}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.105}$$

$$-\sum_{r=1}^{q} c^\star\Big(\Lambda^{\rho}_{(i)n}\partial_\rho\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\}+ \tag{4.3.106}$$

$$+\sum_{t=1}^{p}\sum_{s=1}^{t-1} c^\star\Big(\partial_\gamma(\Lambda^{\mu_t}_{(i)n})\partial_\alpha(\Lambda^{\mu_s}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\setminus\overline{s}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.107}$$

$$+\sum_{t=1}^{p} c^\star\Big(\partial_\gamma(\Lambda^{\mu_t}_{(i)n})\partial_\alpha(\Lambda^{\gamma}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.108}$$

$$+\sum_{t=1}^{p}\sum_{s=t+1}^{p} c^\star\Big(\partial_\gamma(\Lambda^{\mu_t}_{(i)n})\partial_\alpha(\Lambda^{\mu_s}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\setminus\overline{t}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.109}$$

$$-\sum_{t=1}^{p}\sum_{r=1}^{q} c^\star\Big(\partial_\gamma(\Lambda^{\mu_t}_{(i)n})\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}c_\zeta(ds)\Big\}+ \tag{4.3.110}$$

$$+\sum_{u=1}^{q}\sum_{s=1}^{p} c^\star\Big(\partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\partial_\alpha(\Lambda^{\mu_s}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{u-1}}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.111}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} c^\star\Big(\partial_{\nu_u}(\Lambda^{\delta}_{n})\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\setminus r}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.112}$$

$$+\sum_{u=1}^{q} c^\star\Big(\partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\partial_\delta(\Lambda^{\beta}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\beta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.113}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} c^\star\Big(\partial_{\nu_u}(\Lambda^{\delta}_{(i)n})\partial_{\nu_r}(\Lambda^{\beta}_{(i)m})\Big) \triangleright \Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{(i)\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\setminus u}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\} \tag{4.3.114}$$

Since we have explicitly chosen a specific adapted coordinate frame $(e_{(i)\mu})$ with which the Lie derivatives have been defined, the rectangular matrices $\Lambda^\mu_{(i)m}$ are known, as well as their derivatives $\partial_\alpha\Lambda^\mu_{(i)m}$ and $\partial_\rho\partial_\alpha\Lambda^\mu_{(i)m}$. Therefore we can conclude that by choosing a local adapted coordinate system (or equivalently a local adapted frame) we can always induce a basis $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}})$ of $\overset{(2)}{\Upsilon}{}^q_p(c)$ that can be expressed as where by definition we have that:

$$\begin{cases} \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = & \psi_i\left[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\right] c_\zeta(ds) \\ \\ \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = & (-1)^k c^\star(\Lambda^\lambda_m) \triangleright L_\lambda \Big\{\psi_i\left[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\right] c_\zeta(ds)\Big\} + \\ & - \sum_{s=1}^p c^\star(\partial_\alpha \Lambda^{\mu_s}_{(i)m}) \triangleright \Big\{\psi_i\left[dx^{\mu_{\overline{s-1}}}_{(i)} \otimes dx^{\mu_\alpha}_{(i)} \otimes dx^{\mu_{\overline{p}}\setminus\overline{s}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\right] c_\zeta(ds)\Big\} + \\ & + \sum_{r=1}^q c^\star(\partial_{\nu_r} \Lambda^\beta_{(i)m}) \triangleright \Big\{\psi_i\left[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{r}}} \otimes \partial_{(i)\beta} \otimes \partial_{(i)\nu_{\overline{q}\setminus\overline{s}}}\right] c_\zeta(ds)\Big\} \\ \\ \Theta^{\mu_{\overline{p}}}_{(i)nm\nu_{\overline{q}}} = & c^\star\Big(\Lambda^\rho_{(i)n} \Lambda^\lambda_{(i)n}\Big) \triangleright L_\rho L_\lambda \Big\{\psi_i(dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}) c_\zeta(ds)\Big\} + \cdots \end{cases} \tag{4.3.115}$$

As one can see, the explicit expression for a transverse basis of $\overset{(2)}{\Upsilon}{}^q_p(c)$ is very complicated if compared with a transverse basis of $\overset{(1)}{\Upsilon}{}^q_p(c)$.

Brief mention of higher order

As it has been already stated we do not have a general method to make the generator explicit. This is a direct consequence of the non $C^\infty(M)$-linearity of the Lie derivatives. However let us just give some hints concerning how this approach can be theoretically applied to find a basis of $\overset{(N)}{\Upsilon}{}^q_p(c)$ for each $N \in \mathbb{N}$. Let us suppose to have a generic multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^p_q(c)$. Accordingly to the adapted Ellis representation:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{\psi_i\left[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}}\right] c_\zeta(ds)\Big\}\Big) \tag{4.3.116}$$

Let us try to express the same expression for a natural local frame induced by a generic coordinate system but keeping attention to not changing the direction in which the Lie derivatives are taken:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{\psi_i\left[e^{\mu_{\overline{p}}}_{(i)} \otimes e_{(i)\nu_{\overline{q}}}\right] c_\zeta(ds)\Big\}\Big) = \tag{4.3.117}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[\overline{\Lambda}_{(i)\nu_{\overline{q}}}^{\rho_{\overline{p}}} \Lambda_{(i)\sigma_{\overline{q}}}^{\mu_{\overline{p}}} dx_{(i)}^{\sigma_{\overline{p}}} \otimes \partial_{(i)\rho_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \Big) = \quad (4.3.118)$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} c^\star(\overline{\Lambda}_{(i)\nu_{\overline{q}}}^{\rho_{\overline{p}}}) c^\star(\Lambda_{(i)\sigma_{\overline{q}}}^{\mu_{\overline{p}}}) \triangleright \Big((-1)^k L_{m_{\overline{k}}} \Big\{ \psi_i \Big[dx_{(i)}^{\sigma_{\overline{p}}} \otimes \partial_{(i)\rho_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \Big) = \quad (4.3.119)$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \beta_{(i)\ \sigma_{\overline{p}}}^{m_{\overline{k}}\rho_{\overline{q}}} \triangleright \Theta_{(i)m_{\overline{k}}\rho_{\overline{q}}}^{\sigma_{\overline{p}}} \quad (4.3.120)$$

where by definition we have that:

$$\Theta_{(i)m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} = (-1)^k L_{e_{m_1}} \ldots L_{e_{m_k}} \Big\{ \psi_i \Big[dx_{(i)}^{\mu_{\overline{p}}} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \quad (4.3.121)$$

are a new set of generators and

$$\beta_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} = c^\star(\overline{\Lambda}_{(i)_{\mu_{\overline{p}}}}^{\alpha_{\overline{p}}}) c^\star(\Lambda_{(i)_{\beta_{\overline{q}}}}^{\nu_{\overline{q}}}) \alpha_{(i)\ \alpha_{\overline{p}}}^{m_{\overline{k}}\beta_{\overline{q}}} \quad (4.3.122)$$

are a new set of Ellis parameters. Therefore $(\Theta_{(i)m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} \mid k \in [0, N])$ is a set of generators for $\overset{(N)}{\Upsilon}{}_p^q(c)$. It is trivial to show that this is also a basis because the null multipole can be written uniquely as a linear combination of null coefficients. In fact:

$$\sum_{k=0}^{ord(\mathcal{T})} \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Theta_{(i)m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} = 0 \quad (4.3.123)$$

$$\Leftrightarrow [\phi, \sum_{k=0}^{ord(\mathcal{T})} \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \beta_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Theta_{(i)m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}}] \quad , \quad \forall \phi \in \Gamma_0 T_q^p M \quad (4.3.124)$$

This means that $\forall \phi \in \Gamma_0 T^p_q M$

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} \sum_{k=0}^{ord(\mathcal{T})} c^\star (L_{m_{\overline{k}}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i) \beta^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds = 0 \tag{4.3.125}$$

which we know can be satisfied if and only if all the coefficients are null, because the e_m are linearly independent with respect to $\dot{c} = \frac{d}{dt}c(t)$ when restricted on $c(\mathbb{R})$, so no integration by part can be performed to manipulate these derivations. Let us notice that the $\mathcal{T}$ can be expressed as a linear combination of the $\overset{(N-1)}{\Upsilon^q_p}(c)$ basis and some other k order multipoles. By convention the set of all $C^\infty(\mathbb{R})$ linear combination defined as:

$$\left\{ \beta^{m_{\overline{N}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{N}}\nu_{\overline{q}}} \quad , \quad \forall \beta^{m_{\overline{N}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \in C^\infty(I_i) \mid I_i = c^{-1}(U_i \cap c(\mathbb{R})) \right\} \tag{4.3.126}$$

is also a $C^\infty(\mathbb{R})$-sub-module of $\overset{(k)}{\Upsilon^q_p}(c)$. We call it ***the Ellis local pure N-pole module with respect to the given adapted frame*** and the related adapted Ellis moments are called ***the local adapted pure N-pole moments with respect to the given adapted frame***. Let us stress that this split strongly depends on the choices of the adapted local frame $(e_(i)\mu)$. We will see later how this split is not preserved when a changing in the adapted local frame is performed. Therefore a k-pole being "pure" with respect to an adapted frame in general is no more "pure" with respect to another adapted frame. The generators $\Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = (-1)^k L_{e_m} \left\{ \psi_i \left[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \right] c_\zeta(ds) \right\}$ always exists but they have not been made fully explicit. In fact the directions $e_{(i)m}$ with respect to the Lie derivatives are taken, should be written as a linear combination of the generic natural local frame as well. This is very complicated for higher order multipoles and a general method to predict the explicit form of their Ellis local expression does not exist. The only possibility is to calculate explicitly the enormous amount of terms, expressing the transverse adapted frame with $e_{(i)m} = \Lambda^\nu_{(i)m}\partial_{(i)\nu}$, calculating how each Lie derivative taken with respect to the "transverse directions" can be written as a linear combination of lower order Lie derivatives taken with respect to the generic natural frame then finally iterating the Leibniz rule many times.

Definition 70: Let $\mathcal{T} \in \overset{(N)}{\Upsilon^q_p}(c)$ be a multipole up to order N. Given a transverse basis $\left(\Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = (-1)^k L_{e_{m_1}} ... L_{e_{m_k}} \left\{ \psi_i \left[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \right] c_\zeta(ds) \right\} \mid k \in [0, N] \right)$ for the module $\overset{(k)}{\Upsilon^q_p}(c)$

the unique set of local smooth scalar fields satisfying:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \beta_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Theta_{(i)m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.127}$$

are called ***transverse Ellis moments*** with respect to the transverse local basis $(\Theta_{(i)m_{\overline{k}}\rho_{\overline{q}}}^{\sigma_{\overline{p}}} | k \in [0, N])$.

However this problem is essentially a matter of computational power and theoretically it can be solved using enough powerful computing machines executing the algorithm needed to calculate explicitly the transverse local Ellis representation of the generator of $\overset{(k)}{\Upsilon}{}_q^p(c) \mid k \in \mathbb{N}$. Considering this we would like to explicate the algorithm that eventually can be used by a machine to perform the calculation at each order. The algorithm is inductively recursive since the basis of $\overset{(k-1)}{\Upsilon}{}_p^q(c)$ is a sub-basis for $\overset{(k)}{\Upsilon}{}_p^q(c)$:

1. ***Step 0:*** Let us choose an arbitrary atlas $\mathcal{A}$ of M. Let us fix the set of local adapted frames $e_{(i)\mu} = \Lambda_{(i)\mu}^{\nu} \partial_\nu$ defined on U_i such that $\{U_i\}$ covers the whole worldline $c(\mathbb{R})$ and let us choose ψ_i a smooth partition of the unity subordinate to the covering $\{U_i\}$.

2. ***Step 1:*** Let us calculate the generic expression of the following first order Lie derivative defined on each open set U_i :

$$L_{e_{(i)n}}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} = L_{\Lambda_{(i)n}^{\lambda}\partial_\lambda}(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} = \tag{4.3.128}$$

$$= \Lambda_{(i)n}^{\lambda} L_\lambda(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}} - \sum_{s=1}^{p} \partial_\alpha(\Lambda_{(i)n}^{\mu_s})(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}} + \sum_{r=1}^{q} \partial_{\nu_r}(\Lambda_{(i)n}^{\beta})(\phi)_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}^{\mu_{\overline{s}}} = \tag{4.3.129}$$

$$= \Lambda_{(i)n}^{\rho} \partial_\rho(\phi_{(i)\nu_{\overline{q}}}^{\mu_{\overline{p}}}) - \sum_{s=1}^{p} \partial_\alpha(\Lambda_{(i)n}^{\mu_s})(\phi)_{(i)\nu_{\overline{q}}}^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}} + \sum_{r=1}^{q} \partial_{\nu_r}(\Lambda_{(i)n}^{\beta})(\phi)_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}^{\mu_{\overline{s}}} \tag{4.3.130}$$

   Set then k=1.

3. ***Step 2:*** Let us suppose to know how to express explicitly:

$$L_{e_{(i)n_{\overline{k}}}}(\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} = L_{e_{(i)n_1}} ... L_{e_{(i)n_k}}(\phi)_{\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.131}$$

in terms of $C^\infty(M)$-linear combinations of Lie derivatives taken with respect to the local natural frame ∂_μ for the fixed value of k:

$$L_{e_{(i)n_{\overline{k}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k} \beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} L_{\lambda_1}...L_{\lambda_j}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \tag{4.3.132}$$

where $\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\sigma_{\overline{p}}}_{(i)n_{\overline{k}}\sigma_{\overline{p}}\rho_{\overline{q}}}$ are appropriate scalar fields formed by a linear combination of partial derivatives of Λ^λ_m up to the order k

4. ***Step 3***: Let let us calculate inductively:

$$L_{e_{(i)n_{\overline{k+1}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_{e_{(i)n_1}}(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.133}$$

$$=\Lambda^\lambda_{(i)n} L_\lambda(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.134}$$

$$-\sum_{s=1}^{p} \partial_\alpha(\Lambda^{\mu_s}_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.135}$$

$$+\sum_{r=1}^{q} \partial_{\nu_r}(\Lambda^{\beta}_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} = \tag{4.3.136}$$

$$=\Lambda^\lambda_{(i)n} \partial_\lambda(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})+ \tag{4.3.137}$$

$$-\sum_{s=1}^{p} \partial_\alpha(\Lambda^{\mu_s}_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.138}$$

$$+\sum_{r=1}^{q} \partial_{\nu_r}(\Lambda^{\beta}_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.139}$$

Now it is enough to substitute the explicit expression taken from the step k inside the expression above and using the Leibnitz rule:

$$L_{e_{(i)n_{\overline{k+1}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.140}$$

$$=\Lambda^\lambda_{(i)n_1} \sum_{j=0}^{k} \partial_\lambda(\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}}) L_{\lambda_1}...L_{\lambda_j}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}+ \tag{4.3.141}$$

$$+\Lambda^\lambda_{(i)n_1} \sum_{j=0}^{k} \beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}} L_\lambda(L_{\lambda_1}...L_{\lambda_j}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}})+ \tag{4.3.142}$$

$$-\sum_{s=1}^{p} \partial_\alpha(\Lambda^{\mu_s}_{(i)n_1})(\sum_{j=0}^{k} \beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}} L_{\lambda_1}...L_{\lambda_j}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}})+ \tag{4.3.143}$$

$$+\sum_{r=1}^{q}\partial_{\nu_r}(\Lambda^{\beta}_{(i)n_1})(\sum_{j=0}^{k}\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}L_{\lambda_1}...L_{\lambda_j}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}})= \tag{4.3.144}$$

Considering this, the generators are:

$$\Theta^{\mu_{\overline{p}}}_{(i)n_{\overline{k+1}}\nu_{\overline{q}}}=(-1)^k\Big\{\sum_{j=0}^{k}c^\star\Big(\Lambda^{\lambda_1}_{(i)n_1}\beta^{\lambda_{\overline{j+1}\backslash\overline{1}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}}\Big)\triangleright L_{\lambda_1}...L_{\lambda_{j+1}}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\rho_{\overline{q}}}\Big]c_\zeta(ds)\Big\}+ \tag{4.3.145}$$

$$+\sum_{j=0}^{k}c^\star\Big(\Lambda^{\lambda}_{(i)n_1}\partial_\lambda(\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}})\Big)\triangleright L_{\lambda_1}...L_{\lambda_j}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\rho_{\overline{q}}}\Big]c_\zeta(ds)\Big\}+ \tag{4.3.146}$$

$$-\sum_{j=0}^{k}\sum_{s=1}^{p}c^\star\Big(\partial_\alpha(\Lambda^{\mu_s}_{(i)n_1})\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)n_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\rho_{\overline{q}}}\Big)\triangleright L_{\lambda_1}...L_{\lambda_j}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\}+ \tag{4.3.147}$$

$$+\sum_{j=0}^{k}\sum_{r=1}^{q}c^\star\Big(\partial_{\nu_r}(\Lambda^{\beta}_{(i)n_1})\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\backslash\overline{1}}\sigma_{\overline{p}}\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}\Big)\triangleright L_{\lambda_1}...L_{\lambda_j}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\rho_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \tag{4.3.148}$$

Now increase the counter k by 1 and go to step 2.

4.3.2 Degree of freedom in the choice of the general local charts

We have proved how, given an adapted local frame $e_{(i)\mu}=\Lambda^{\nu}_{(i)\mu}\partial_{(i)\nu}$ on each open $U_i\subseteq M$ covering $c(\mathbb{R})$, it is possible to induce a transverse basis of $\overset{(0)}{\Upsilon}{}^q_p(c)$, $\overset{(1)}{\Upsilon}{}^q_p(c)$ and $\overset{(2)}{\Upsilon}{}^q_p(c)$ from an arbitrary general local chart (U_i,φ). In the same way, a basis could be found for every $\overset{(k)}{\Upsilon}{}^q_p(c)$ with $k\in\mathbb{N}$ but expliciting the generators is extremely hard and painful. It is appropriate at this point to ask ourselves what is the relationship linking different basis induced by natural frames associated to different local charts $(U_i,\varphi_{(i)})$ and how this change affects the transverse Ellis moments. Let $(U_i,\varphi_{(i)})$, $(U_i,\hat{\varphi}_{(i)})$ two local charts defining two natural local frames ∂_μ and $\hat{\partial}_\mu$. With respect to the first chart we can induce a transverse local basis

$$\Theta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}}=(-1)^kL_{e_{m_{\overline{k}}}}\Big\{\psi_i\Big[dx^{\alpha_{\overline{p}}}_{(i)}\otimes\partial_{(i)\beta_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \tag{4.3.149}$$

such that:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \beta_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Theta_{(i) m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.150}$$

while with respect to the first chart we can induce another transverse local basis:

$$\hat{\Theta}_{(i) m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} = (-1)^k L_{e_{m_{\overline{k}}}} \Big\{ \psi_i \Big[d\hat{x}_{(i)}^{\mu_{\overline{p}}} \otimes \hat{\partial}_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \tag{4.3.151}$$

such that:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \hat{\beta}_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \hat{\Theta}_{(i) m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.152}$$

We must have then:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \beta_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \Theta_{(i) m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} = \mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \hat{\beta}_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright \hat{\Theta}_{(i) m_{\overline{k}}\nu_{\overline{q}}}^{\mu_{\overline{p}}} \tag{4.3.153}$$

Hence we can conclude that

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \hat{\beta}_{(i)\ \mu_{\overline{p}}}^{m_{\overline{k}}\nu_{\overline{q}}} \triangleright (-1)^k L_{e_{m_{\overline{k}}}} \Big\{ \psi_i \Big[d\hat{x}_{(i)}^{\mu_{\overline{p}}} \otimes \hat{\partial}_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.154}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \beta_{(i)\ \alpha_{\overline{p}}}^{m_{\overline{k}}\beta_{\overline{q}}} \triangleright (-1)^k L_{e_{m_{\overline{k}}}} \Big\{ \psi_i \Big[dx_{(i)}^{\alpha_{\overline{p}}} \otimes \partial_{(i)\beta_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.155}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \beta_{(i)\ \alpha_{\overline{p}}}^{m_{\overline{k}}\beta_{\overline{q}}} \triangleright (-1)^k L_{e_{m_{\overline{k}}}} \Big\{ \psi_i \Big[\overline{J}_{\mu_{\overline{p}}}^{\alpha_{\overline{p}}} J_{\beta_{\overline{q}}}^{\nu_{\overline{q}}} dx'^{\mu_{\overline{p}}}_{(i)} \otimes \partial'_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.156}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} c^\star(\overline{J}^{\alpha_{\overline{p}}}_{\mu_{\overline{p}}}) c^\star(J^{\nu_{\overline{q}}}_{\beta_{\overline{q}}}) \beta^{m_{\overline{k}}\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}} \triangleright (-1)^k L_{e_{m_{\overline{k}}}} \Big\{ \psi_i \Big[dx'^{\mu_{\overline{p}}}_{(i)} \otimes \partial'_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \quad (4.3.157)$$

Therefore this leads us to the following transformation rules when a local change in the coordinate functions is performed:

$$\hat{\beta}^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} = c^\star(\overline{J}^{\alpha_{\overline{p}}}_{\mu_{\overline{p}}}) c^\star(J^{\nu_{\overline{q}}}_{\beta_{\overline{q}}}) \beta^{m_{\overline{k}}\beta_{\overline{q}}}_{(i)\ \alpha_{\overline{p}}} \quad (4.3.158)$$

where $J^{\alpha_{\overline{p}}}_{\mu_{\overline{p}}}$ is the Jacobian matrix of the coordinate transformation. This tells us the link occurring between the two different sets of transverse Ellis moments of a multipole when a changing of basis induced by a local coordinate transformation is performed. In the same way we can conclude that the relation between the two bases must be:

$$\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = c^\star(J^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}}) c^\star(\overline{J}^{\beta_{\overline{q}}}_{\nu_{\overline{q}}}) \Theta^{\alpha_{\overline{p}}}_{(i)m_{\overline{k}}\beta_{\overline{q}}} \quad (4.3.159)$$

for each $k \in [0, N]$. Considering this, we can conclude that all the transverse basis built with respect to a fixed adapted local frame transform very nicely with a $C^\infty(\mathbb{R})$-linear application. Since this transformation does not mix up the generators of the each "pure" multipoles modules, then the splitting of a multipole into a $C^\infty(\mathbb{R})$ linear combination of "pure" multipole terms is preserved under changing of transverse basis induced by a changing of local charts. Since the adapted local frame defining the direction of the derivations does not change, then a change in the local charts induces a $C^\infty(\mathbb{R})$-linear transformation on the transverse moments.

4.3.3 Degree of freedom in the choice of the adapted local frame defining the transverse directions

We have proved how, fixing and adapted local frame, for each general local chart it is possible to explicitely induce a basis of $\overset{(0)}{\Upsilon}{}^q_p(c)$, $\overset{(1)}{\Upsilon}{}^q_p(c)$ and $\overset{(2)}{\Upsilon}{}^q_p(c)$ that can be expressed in terms of a specific Ellis representation. In principle, in the same way, a basis could be found for every $\overset{(k)}{\Upsilon}{}^q_p(c)$ with $k \in \mathbb{N}$ but expressing the generators is extremely hard. It is natural at this point to investigate the relationship linking different bases induced by different choices in the adapted frames, for at least the simpler cases. Let us start by considering an arbitrary local chart $(U_i, \varphi'_{(i)})$ and two adapted local frames $e_{(i)\mu} = \Lambda^\nu_{(i)\mu}\partial_{(i)\nu}$ and $\hat{e}_{(i)\mu} = \hat{\Lambda}^\nu_{(i)\mu}\partial_{(i)\nu}$ defined on the same open set $U_i \subseteq M$. By definition of adapted local frame, there must exist two adapted local charts $(U_i, \varphi'_{(i)})$ and $(U_i, \hat{\varphi}'_{(i)})$ defined on the same open set such that $e_{(i)\mu} = \partial'_{(i)\mu} = \Lambda^\nu_{(i)\mu}\partial_{(i)\nu}$ and $\hat{e}_{(i)\mu} = \hat{\partial}'_\mu = \hat{\Lambda}^\nu_{(i)\mu}\partial_{(i)\nu}$.

Using this we can state:

$$\partial'_{(i)\mu} = K^\nu_\mu \hat{\partial}'_{(i)\nu} \tag{4.3.160}$$

where K^ν_μ is the Jacobian $\frac{\partial}{x^\mu}(\hat{\varphi}^\nu \circ \varphi^{-1})$ related to the change of adapted coordinates. We decided to use K^ν_μ to avoid any confusion with the role played by the Jacobian J^ν_μ related to a general change of coordinates. The K^ν_μ must satisfy very strong constraints coming from the definition of adapted coordinates. The first one is given by

$$\begin{cases} \phi'^0(c(s)) = c^\star(\hat{\phi}'^0) = s \\ \phi'^m(c(s)) = 0 \end{cases} \quad \begin{cases} \hat{\phi}'^0(c(s)) = c^\star(\hat{\phi}'^0) = s \\ \hat{\phi}'^m(c(s)) = 0 \end{cases} \quad \Rightarrow \partial'_{(i)0} = \frac{d}{ds} = \hat{\partial}'_{(i)0} \tag{4.3.161}$$

telling us that $c^\star(K^0_0) = 1$ and $c^\star(K^0_m) = 0$. The other one comes from the fact that since $(\hat{\partial}'_\mu)$ is a natural frame as well as (∂'_μ) we must have that the commutator of vector fields:

$$[K^\nu_\mu \partial'_\nu, K^\beta_\alpha \partial'_\beta] = K^\nu_\mu \partial'_\nu (K^\beta_\alpha) \partial'_\beta - K^\beta_\alpha \partial'_\beta (K^\nu_\mu) \partial'_\nu = K^\nu_\mu \partial'_\nu (K^\beta_\alpha) \partial'_\beta - K^\nu_\alpha \partial'_\nu (K^\beta_\mu) \partial'_\beta = 0 \tag{4.3.162}$$

telling us that K^ν_μ is just the Jacobian related to the transformation between two local coordinates. Considering this, theoretically we can write with a little effort the transverse generators induced by the new adapted frame $\hat{e}_\mu$ as a linear combination of the generators induced by the old adapted frame. In practice this is possible just up to the quadrupole order.

Monopole

The monopole case is trivial, no derivations are involved to define $\overset{(0)}{\Upsilon}{}^p_q(c)$ therefore a change of adapted frame does not affect the Ellis transverse representation. This is very interesting to notice, the monopole module does not need any extra information to be represented in an unique way by the transverse Ellis representation.

Dipole

The dipole case is more complicated, since there are some Lie derivatives taken along the transverse directions fixed by an adapted frame. Given two adapted local frames$(\hat{e}_{(i)\mu})$

and $(e_{(i)\mu})$, the dipoles generators can be expressed as:

$$\begin{cases} \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = & \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \\ \\ \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = & -L_{e_{(i)m}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \end{cases} \tag{4.3.163}$$

$$\begin{cases} \hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \\ \\ \hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = -L_{\hat{e}_{(i)m}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \end{cases} \tag{4.3.164}$$

respectively. Let us notice that trivially $\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}$ as we expected since they are also the generators for the monopoles. Concerning $\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}$ we can express them as a linear combination of the older generators $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}})$:

$$\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = -L_{\hat{e}_{(i)m}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = -L_{K^\lambda_m e_{(i)\lambda}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} = \tag{4.3.165}$$

$$= - c^\star(K^\lambda_m) \triangleright L_{e_{(i)\lambda}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.166}$$

$$- \sum_{s=1}^{p} c^\star(\partial_\alpha(K^{\mu_{\overline{s}}}_m)) \triangleright \Big\{ \psi_i \Big[dx^{\mu_{\overline{s-1}}}_{(i)} \otimes dx^{\mu_\alpha}_{(i)} \otimes dx^{\mu_{\overline{p}} \backslash \overline{s}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} + \tag{4.3.167}$$

$$+ \sum_{r=1}^{q} c^\star(\partial_{\nu_r}(K^\beta_m)) \triangleright \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{r}}} \otimes \partial_{(i)\beta} \otimes \partial_{(i)\nu_{\overline{q} \backslash \overline{s}}} \Big] c_\zeta(ds) \Big\} \tag{4.3.168}$$

Let us now consider just the first term $-c^\star(K^\lambda_m) \triangleright L_{e_{(i)\lambda}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\}$. Its action on the test tensor fields $\phi \in \Gamma_0 T^p_q M$ is given by:

$$\Big[\phi, -c^\star(K^\lambda_m) \triangleright L_{e_{(i)\lambda}} \Big\{ \psi_i \Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}} \Big] c_\zeta(ds) \Big\} \Big] = \tag{4.3.169}$$

$$= \int_{\mathbb{R}} c^\star \Big(L_{e_{(i)\lambda}} (\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(K^\lambda_m) c^\star(\psi_i) ds = \tag{4.3.170}$$

$$= \int_{\mathbb{R}} c^\star \Big(L_{e_{(i)n}} (\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(K^n_m) c^\star(\psi_i) ds + \int_{\mathbb{R}} c^\star \Big(L_{e_{(i)0}} (\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(K^0_m) c^\star(\psi_i) ds = \tag{4.3.171}$$

$$= \int_{\mathbb{R}} c^\star\Big(L_{e_{(i)n}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(K^n_m)c^\star(\psi_i)ds + \int_{\mathbb{R}} \frac{d}{ds}c^\star\Big((\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(K^0_m)c^\star(\psi_i)ds = \tag{4.3.172}$$

$$= \int_{\mathbb{R}} c^\star\Big(L_{e_{(i)n}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^\star(K^n_m)c^\star(\psi_i)ds - \int_{\mathbb{R}} c^\star\Big((\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\frac{d}{ds}c^\star(K^0_m)c^\star(\psi_i)ds = \tag{4.3.173}$$

$$= \Big[\phi, -c^\star(K^n_m) \triangleright L_{e_{(i)n}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} - \frac{d}{ds}c^\star(K^0_m) \triangleright \Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\}\Big] = \tag{4.3.174}$$

$$= \Big[\phi, c^\star(K^n_m) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}} - \frac{d}{ds}c^\star(K^0_m) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big] \tag{4.3.175}$$

Therefore substituting it in the previous expression we have that:

$$\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = c^\star(K^n_m) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}} - \frac{d}{ds}c^\star(K^0_m) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.176}$$

$$- \sum_{s=1}^{p} c^\star(\partial_\alpha(K^{\mu_{\overline{s}}}_m)) \triangleright \Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} + \sum_{r=1}^{q} c^\star(\partial_{\nu_r}(K^\beta_m)) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.177}$$

Therefore the expression linking the two basis of $\overset{(1)}{\Upsilon}{}^{q}_{p}$ can be given by:

$$\begin{cases} \hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = & \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \\ \\ \hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = & -c^\star(K^n_m) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}} - \frac{d}{ds}c^\star(K^0_m) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \\ & - \sum_{s=1}^{p} c^\star(\partial_\alpha(K^{\mu_{\overline{s}}}_m)) \triangleright \Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} + \sum_{r=1}^{q} c^\star(\partial_{\nu_r}(K^\beta_m)) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \end{cases} \tag{4.3.178}$$

It is very interesting to notice that the linear combination expressing the new basis mixes up the monopoles and pure dipole generators induced by the old adapted frame in a non trivial way. So the basis of the pure dipoles with respect to the adapted frame $\hat{e}_{(i)_\mu}$ is not a basis for the pure dipoles with respect to the old adapted frame $e_{(i)_\mu}$. Let us stress also that the linear combination involves the partial derivatives of K^μ_ν which is the Hessian matrix of the adapted coordinate transformation.

Quadrupole

The quadrupole case can be analysed with the same approach of the dipole case. The approach requires first of all to explicate the change of the basis of the quadrupole induced by a different choice of the local adapted frame. Given $(\hat{e}_{(i)\mu})$ and $(e_{(i)\mu})$ The dipoles generators can be expressed as:

$$\begin{cases} \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = & \psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds) \\ \Theta^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = & -L_{e_{(i)m}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \\ \Theta^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}} = & L_{e_{(i)m_1}}L_{e_{(i)m_2}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \end{cases} \tag{4.3.179}$$

$$\begin{cases} \hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = & \psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds) \\ \hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}} = & -L_{\hat{e}_{(i)m}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \\ \hat{\Theta}^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}} = & L_{\hat{e}_{(i)m_1}}L_{\hat{e}_{(i)m_2}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} \end{cases} \tag{4.3.180}$$

The generators $\hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}$ and $\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}$ have been already presented as linear combination of the new generators $(\hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}})$ in the previous analysis. We need just to expand the remaining generators $\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}}$ with respect to the new basis $(\hat{\Theta}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \hat{\Theta}^{\mu_{\overline{p}}}_{(i)m\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}})$:

$$\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}} = L_{\hat{e}_{(i)m_1}}L_{\hat{e}_{(i)m_2}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} = \tag{4.3.181}$$

$$= L_{\hat{K}^{\lambda_1}_{m_1}e_{(i)\lambda_1}}L_{\hat{K}^{\lambda_2}_{m_2}e_{(i)\lambda_2}}\Big\{\psi_i\Big[dx^{\mu_{\overline{p}}}_{(i)} \otimes \partial_{(i)\nu_{\overline{q}}}\Big]c_\zeta(ds)\Big\} = \tag{4.3.182}$$

$$= c^\star\Big(K^\rho_{m_1}K^\lambda_{m_2}\Big) \triangleright L_{e_{(i)\rho}}L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}} \otimes \partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\} + \tag{4.3.183}$$

$$+ c^\star\Big(K^\rho_{m_1}\partial_\rho(K^\lambda_{m_2})\Big) \triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}} \otimes \partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\} + \tag{4.3.184}$$

$$- \sum_{s=1}^{p} c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^\lambda_{m_1}\Big) \triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}} \otimes \partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\} + \tag{4.3.185}$$

$$+ \sum_{r=1}^{q} c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^\lambda_{m_1}\Big) \triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}} \otimes \partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\} + \tag{4.3.186}$$

$$- \sum_{t=1}^{p} c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})K^\lambda_{m_2}\Big) \triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}} \otimes \partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\} + \tag{4.3.187}$$

$$+\sum_{u=1}^{q} c^\star\Big(\partial_{\nu_u}(K^{\delta}_{(i)m_1})K^{\lambda}_{m_2}\Big)\triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.188}$$

$$-\sum_{s=1}^{p} c^\star\Big(K^{\rho}_{m_1}\partial_\rho\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.189}$$

$$-\sum_{r=1}^{q} c^\star\Big(K^{\rho}_{m_1}\partial_\rho\partial_{\nu_r}(K^{\beta}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\}+ \tag{4.3.190}$$

$$+\sum_{t=1}^{p}\sum_{s=1}^{t-1} c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\setminus\overline{s}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.191}$$

$$+\sum_{t=1}^{p} c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\gamma}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.192}$$

$$+\sum_{t=1}^{p}\sum_{s=t+1}^{p} c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\setminus\overline{t}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.193}$$

$$-\sum_{t=1}^{p}\sum_{r=1}^{q} c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_{\nu_r}(K^{\beta}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}c_\zeta(ds)\Big\}+ \tag{4.3.194}$$

$$+\sum_{u=1}^{q}\sum_{s=1}^{p} c^\star\Big(\partial_{\nu_u}(K^{\delta}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{u-1}}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.195}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} c^\star\Big(\partial_{\nu_u}(K^{\delta}_{m_1})\partial_{\nu_r}(K^{\beta}_{(m_2)})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\setminus r}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.196}$$

$$+\sum_{u=1}^{q} c^\star\Big(\partial_{\nu_u}(K^{\delta}_{m_1})\partial_\delta(K^{\beta}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\beta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.197}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} c^\star\Big(\partial_{\nu_u}(K^{\delta}_{m_1})\partial_{\nu_r}(K^{\beta}_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{(i)\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\setminus u}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\} \tag{4.3.198}$$

Now as well as it has been done for the dipole case, we can separate the Lie Derivatives taken with respect to the transverse directions with respect to the longitudinal one. The Lie derivatives taken with respect to e_0 can integrated by parts by expanding the action of the basis of the multipoles on an arbitrary test tensor field $\phi\in\Gamma_0 T^p_q M$. Let us do it term by term separately and then let us put them together later.

$$c^\star\Big(K^{\rho}_{m_1}K^{\lambda}_{m_2}\Big)\triangleright L_{e_{(i)\rho}}L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}= \tag{4.3.199}$$

$$=c^\star\Big(K^{n_1}_{m_1}K^{n_2}_{m_2}\Big)\triangleright L_{e_{(i)n_1}}L_{e_{(i)n_2}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.200}$$

$$-\frac{d}{ds}c^\star\Big(K^{0}_{m_1}K^{n}_{m_2}+K^{0}_{m_2}K^{n}_{m_1}\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.201}$$

$$+\frac{d^2}{ds^2}c^\star\Big(K^{0}_{m_1}K^{0}_{m_2}\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}= \tag{4.3.202}$$

$$=c^\star\Big(K^{n_1}_{m_1}K^{n_2}_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}}-\frac{d}{ds}c^\star\Big(K^0_{m_1}K^n_{m_2}+K^0_{m_2}K^n_{m_1}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}}+ \tag{4.3.203}$$

$$+\frac{d^2}{ds^2}c^\star\Big(K^0_{m_1}K^0_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \tag{4.3.204}$$

The terms related to the Lie derivatives are:

$$c^\star\Big(K^\rho_{m_1}\partial_\rho(K^\lambda_{m_2})\Big)\triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.205}$$

$$-\sum_{s=1}^{p}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^\lambda_{m_1}\Big)\triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.206}$$

$$+\sum_{r=1}^{q}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^\lambda_{m_1}\Big)\triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\}+ \tag{4.3.207}$$

$$-\sum_{t=1}^{p}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})K^\lambda_{m_2}\Big)\triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.208}$$

$$+\sum_{u=1}^{q}c^\star\Big(\partial_{\nu_u}(K^\delta_{(i)m_1})K^\lambda_{m_2}\Big)\triangleright L_{e_{(i)\lambda}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}= \tag{4.3.209}$$

$$=c^\star\Big(K^\rho_{m_1}\partial_\rho(K^n_{m_2})\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.210}$$

$$+\frac{d}{ds}c^\star\Big(K^\rho_{m_1}\partial_\rho(K^0_{m_2})\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.211}$$

$$-\sum_{s=1}^{p}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^n_{m_1}\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.212}$$

$$-\sum_{s=1}^{p}\frac{d}{ds}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^0_{m_1}\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.213}$$

$$+\sum_{r=1}^{q}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^n_{m_1}\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\}+ \tag{4.3.214}$$

$$+\sum_{r=1}^{q}\frac{d}{ds}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^0_{m_1}\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}})c_\zeta(ds)\Big\}+ \tag{4.3.215}$$

$$-\sum_{t=1}^{p}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})K^n_{m_2}\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.216}$$

$$-\sum_{t=1}^{p}\frac{d}{ds}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})K^0_{m_2}\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}\otimes\partial_{\nu_{\overline{q}}})c_\zeta(ds)\Big\}+ \tag{4.3.217}$$

$$+\sum_{u=1}^{q}c^\star\Big(\partial_{\nu_u}(K^\delta_{(i)m_1})K^n_{m_2}\Big)\triangleright L_{e_{(i)n}}\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}+ \tag{4.3.218}$$

$$+\sum_{u=1}^{q}\frac{d}{ds}c^\star\Big(\partial_{\nu_u}(K^\delta_{(i)m_1})K^0_{m_2}\Big)\triangleright\Big\{\psi_i(dx^{\mu_{\overline{p}}}\otimes\partial_{\nu_{\overline{u}-1}\delta\nu_{\overline{q}\setminus\overline{u}}})c_\zeta(ds)\Big\}= \tag{4.3.219}$$

$$=c^\star\Big(K^\rho_{m_1}\partial_\rho(K^n_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}}+\frac{d}{ds}c^\star\Big(K^\rho_{m_1}\partial_\rho(K^0_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.220}$$

$$-\sum_{s=1}^{p}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^n_{m_1}+\partial_\alpha(K^{\mu_s}_{m_1})K^n_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)n\nu_{\overline{q}}}+ \tag{4.3.221}$$

$$-\sum_{s=1}^{p}\frac{d}{ds}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^0_{m_1}+\partial_\alpha(K^{\mu_s}_{m_1})K^0_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.222}$$

$$+\sum_{r=1}^{q}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^n_{m_1}+\partial_{\nu_r}(K^\beta_{m_1})K^n_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus r}}+ \tag{4.3.223}$$

$$+\sum_{r=1}^{q}\frac{d}{ds}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^0_{m_1}+\partial_{\nu_r}(K^\beta_{m_1})K^0_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus r}} \tag{4.3.224}$$

The terms without any Lie derivatives can be easily recast as follow:

$$\sum_{s=1}^{p}c^\star\Big(K^\rho_{m_1}\partial_\rho\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.225}$$

$$-\sum_{r=1}^{q}c^\star\Big(K^\rho_{m_1}\partial_\rho\partial_{\nu_r}(K^\beta_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}+ \tag{4.3.226}$$

$$+\sum_{t=1}^{p}\sum_{s=1}^{t-1}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\setminus\overline{s}}\gamma\mu_{\overline{p}\setminus\overline{t}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.227}$$

$$+\sum_{t=1}^{p}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\gamma}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\setminus\overline{t}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.228}$$

$$+\sum_{t=1}^{p}\sum_{s=t+1}^{p}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\setminus\overline{t}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.229}$$

$$-\sum_{t=1}^{p}\sum_{r=1}^{q}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_{\nu_r}(K^\beta_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\setminus\overline{t}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}+ \tag{4.3.230}$$

$$+\sum_{u=1}^{q}\sum_{s=1}^{p}c^\star\Big(\partial_{\nu_u}(K^\delta_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{u-1}}\delta\nu_{\overline{q}\setminus\overline{u}}})+ \tag{4.3.231}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1}c^\star\Big(\partial_{\nu_u}(K^\delta_{m_1})\partial_{\nu_r}(K^\beta_{(m_2)})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\setminus^r}\delta\nu_{\overline{q}\setminus\overline{u}}}+ \tag{4.3.232}$$

$$+\sum_{u=1}^{q}c^\star\Big(\partial_{\nu_u}(K^\delta_{m_1})\partial_\delta(K^\beta_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\beta\nu_{\overline{q}\setminus\overline{u}}}+ \tag{4.3.233}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1}c^\star\Big(\partial_{\nu_u}(K^\delta_{m_1}\partial_{\nu_r}(K^\beta_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\setminus^u}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.234}$$

So we can put together all the terms to find the expression of $\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}}$ as a linear combination of the old basis $(\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}}, \Theta^{\mu_{\overline{p}}}_{(i)n_1n_2\nu_{\overline{q}}})$:

$$\hat{\Theta}^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}} = \tag{4.3.235}$$

$$=c^\star\Big(K^{n_1}_{m_1}K^{n_2}_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)m_1m_2\nu_{\overline{q}}} - \frac{d}{ds}c^\star\Big(K^0_{m_1}K^n_{m_2}+K^0_{m_2}K^n_{m_1}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}}+ \tag{4.3.236}$$

$$+\frac{d^2}{ds^2}c^\star\Big(K^0_{m_1}K^0_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.237}$$

$$+c^\star\Big(K^\rho_{m_1}\partial_\rho(K^n_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{q}}}+\frac{d}{ds}c^\star\Big(K^\rho_{m_1}\partial_\rho(K^0_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.238}$$

$$-\sum_{s=1}^{p}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^n_{m_1}+\partial_\alpha(K^{\mu_s}_{m_1})K^n_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)n\nu_{\overline{q}}}+ \tag{4.3.239}$$

$$-\sum_{s=1}^{p}\frac{d}{ds}c^\star\Big(\partial_\alpha(K^{\mu_s}_{m_2})K^0_{m_1}+\partial_\alpha(K^{\mu_s}_{m_1})K^0_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.240}$$

$$+\sum_{r=1}^{q}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^n_{m_1}+\partial_{\nu_r}(K^\beta_{m_1})K^n_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)n\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash r}}+ \tag{4.3.241}$$

$$+\sum_{r=1}^{q}\frac{d}{ds}c^\star\Big(\partial_{\nu_r}(K^\beta_{m_2})K^0_{m_1}+\partial_{\nu_r}(K^\beta_{m_1})K^0_{m_2}\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash r}}+ \tag{4.3.242}$$

$$+\sum_{s=1}^{p}c^\star\Big(K^\rho_{m_1}\partial_\rho\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.243}$$

$$-\sum_{r=1}^{q}c^\star\Big(K^\rho_{m_1}\partial_\rho\partial_{\nu_r}(K^\beta_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}+ \tag{4.3.244}$$

$$+\sum_{t=1}^{p}\sum_{s=1}^{t-1}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{t-1}\backslash\overline{s}}\gamma\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.245}$$

$$+\sum_{t=1}^{p}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\gamma}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{t-1}}\alpha\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.246}$$

$$+\sum_{t=1}^{p}\sum_{s=t+1}^{p}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{t-1}}\gamma\mu_{\overline{s-1}\backslash\overline{t}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{q}}}+ \tag{4.3.247}$$

$$-\sum_{t=1}^{p}\sum_{r=1}^{q}c^\star\Big(\partial_\gamma(K^{\mu_t}_{m_1})\partial_{\nu_r}(K^\beta_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{t-1}}\gamma\mu_{\overline{p}\backslash\overline{t}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\backslash\overline{r}}}+ \tag{4.3.248}$$

$$+\sum_{u=1}^{q}\sum_{s=1}^{p}c^\star\Big(\partial_{\nu_u}(K^\delta_{m_1})\partial_\alpha(K^{\mu_s}_{m_2})\Big)\triangleright\Theta^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\backslash\overline{s}}}_{(i)\nu_{\overline{u-1}}\delta\nu_{\overline{q}\backslash\overline{u}}})+ \tag{4.3.249}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1}c^\star\Big(\partial_{\nu_u}(K^\delta_{m_1})\partial_{\nu_r}(K^\beta_{(m_2)})\Big)\triangleright\Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{r}-1}\beta\nu_{\overline{u}-1\backslash r}\delta\nu_{\overline{q}\backslash\overline{u}}}+ \tag{4.3.250}$$

$$+\sum_{u=1}^{q} c^{\star}\Big(\partial_{\nu_u}(K^{\delta}_{m_1})\partial_{\delta}(K^{\beta}_{m_2})\Big) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\beta\nu_{\overline{q}\setminus\overline{u}}}+ \tag{4.3.251}$$

$$+\sum_{u=1}^{q}\sum_{r=1}^{u-1} c^{\star}\Big(\partial_{\nu_u}(K^{\delta}_{m_1})\partial_{\nu_r}(K^{\beta}_{m_2})\Big) \triangleright \Theta^{\mu_{\overline{p}}}_{(i)\nu_{\overline{u}-1}\delta\nu_{\overline{r}-1\setminus u}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.252}$$

As one can see, the expression is very long and complicated. Qualitatively, the complexity of the linear combinations rises almost factorially with respect to the order of the multipoles. It is very interesting to notice that the linear combination expressing the new basis mixes up the monopoles and pure dipoles and the pure quadrupoles induced by the old adapted frame in a non trivial way. So the basis of the pure quadrupoles with respect to the adapted frame $\hat{e}_{(i)\mu}$ is not a basis for the pure quadrupoles with respect to the old adapted frame $e_{(i)\mu}$. Let us stress also that the linear combination involves the second order partial derivatives of K^{μ}_{ν} i.e. the third order derivatives of the adapted coordinate transformation.

Brief mention to higher order

We have seen how even for the module formed by the multipoles up to the second order the expression linking different transverse Ellis basis induced by different choices of local transverse frames is very complicated. In principle the same approach could be applied for each $\overset{(k)}{\Upsilon}{}^{q}_{p}(c)\,|\,k \in \mathbb{N}$, but in practice this is extremely hard because of the combinatorics and the almost factorial growth of the number of terms involved in the linear combination. However this a matter of computational power and theoretically it can be solved using enough powerful computing machines executing the algorithm needed to calculate explicitly the linear combination. Considering this we would like to present the algorithm that eventually can be used by a machine to perform the calculation at each order:

1. ***Step 0:*** Let us choose an arbitrary atlas $\mathcal{A}$ of M. Let us fix two set of local adapted frames $\hat{e}_{(i)\mu} = K^{\nu}_{(i)\mu}e_{\nu} = K^{\nu}_{(i)\mu}\partial'_{\nu}$ defined on U_i such that $\{U_i\}$ covers the whole worldline $c(\mathbb{R})$ and let us choose ψ_i a smooth partition of the unity subordinate to the covering $\{U_i\}$.

2. ***Step 1:*** Let us calculate the generic expression of the following first order Lie derivative defined on each open set U_i :

$$L_{\hat{e}_{(i)n}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = L_{K^{\lambda}_{(i)n}\partial'_{\lambda}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \tag{4.3.253}$$

$$=K^{\lambda}_{(i)n}L_{\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} - \sum_{s=1}^{p}\partial'_{\alpha}(K^{\mu_s}_{(i)n})(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} + \sum_{r=1}^{q}\partial'_{\nu_r}(K^{\beta}_{(i)n})(\phi)^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} = \tag{4.3.254}$$

$$=K^{\rho}_{(i)n}\partial'_{\rho}(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) - \sum_{s=1}^{p}\partial'_{\alpha}(K^{\mu_s}_{(i)n})(\phi)^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} + \sum_{r=1}^{q}\partial'_{\nu_r}(K^{\beta}_{(i)n})(\phi)^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.255}$$

Then set k=1.

3. ***Step 2:*** Let us suppose to know how to express explicitly:

$$L_{e_{(i)n_{\overline{k}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_{e_{(i)n_1}}...L_{e_{(i)n_k}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{4.3.256}$$

in terms of $C^\infty(M)$-linear combinations of Lie derivatives taken with respect to the adapted local frame $e_\mu = \partial'_\mu$ for the fixed value of k:

$$L_{e_{(i)n_{\overline{k}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \sum_{j=0}^{k} \beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}} L_{e_{\lambda_1}}...L_{e_{\lambda_j}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}} \tag{4.3.257}$$

where $\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\sigma_{\overline{p}}}_{(i)n_{\overline{k}}\sigma_{\overline{p}}\rho_{\overline{q}}}$ are appropriate scalar fields formed by a linear combination of several partial derivatives $e_\nu = \partial'_\nu$ acting on K^λ_m up to the order k.

4. ***Step 3***: Let let us calculate inductively:

$$L_{e_{(i)n_{\overline{k+1}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = L_{e_{(i)n_1}}(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.258}$$

$$= K^\lambda_{(i)n} L_\lambda (L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.259}$$

$$- \sum_{s=1}^{p} \partial'_\alpha (K^{\mu_s}_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.260}$$

$$+ \sum_{r=1}^{q} \partial'_{\nu_r}(K^\beta_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} = \tag{4.3.261}$$

$$= K^\lambda_{(i)n} \partial'_\lambda (L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) + \tag{4.3.262}$$

$$- \sum_{s=1}^{p} \partial'_\alpha (K^{\mu_s}_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)\nu_{\overline{q}}} + \tag{4.3.263}$$

$$+ \sum_{r=1}^{q} \partial'_{\nu_r}(K^\beta_{(i)n_1})(L_{e_{(i)n_2}}...L_{e_{(i)n_{k+1}}}(\phi))^{\mu_{\overline{s}}}_{(i)\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}} \tag{4.3.264}$$

Now it is enough to substitute the explicit expression taken from the step k inside the expression above obtaining and using the Leibnitz rule:

$$L_{e_{(i)n_{\overline{k+1}}}}(\phi)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \tag{4.3.265}$$

$$=K^{\lambda}_{(i)n_1}\sum_{j=0}^{k}\partial'_{\lambda}(\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}})L_{e_{\lambda_1}}...L_{e_{\lambda_j}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}}+ \tag{4.3.266}$$

$$+K^{\lambda}_{(i)n_1}\sum_{j=0}^{k}\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}}L_{e_{\lambda}}(L_{e_{\lambda_1}}...L_{e_{\lambda_j}}(\phi)^{\sigma_{\overline{p}}}_{\rho_{\overline{q}}})+ \tag{4.3.267}$$

$$-\sum_{s=1}^{p}\partial'_{\alpha}(\Lambda^{\mu_s}_{(i)n_1})(\sum_{j=0}^{k}\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}}L_{e_{\lambda_1}}...L_{e_{\lambda_j}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}})+ \tag{4.3.268}$$

$$+\sum_{r=1}^{q}\partial'_{\nu_r}(K^{\beta}_{(i)n_1})(\sum_{j=0}^{k}\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}L_{e_{\lambda_1}}...L_{e_{\lambda_j}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}) = \tag{4.3.269}$$

Considering this, the generators are:

$$\hat{\Theta}^{\mu_{\overline{p}}}_{(i)n_{\overline{k+1}}\nu_{\overline{q}}}=(-1)^k\Big\{\sum_{j=0}^{k}c^{\star}\Big(K^{\lambda_1}_{(i)n_1}\beta^{\lambda_{\overline{j+1}\setminus\overline{1}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{q}}}\Big)\triangleright L_{e_{\lambda_1}}...L_{e_{\lambda_{j+1}}}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\rho_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+ \tag{4.3.270}$$

$$+\sum_{j=0}^{k}c^{\star}\Big(K^{\lambda}_{(i)n_1}\partial'_{\lambda}(\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k}}\sigma_{\overline{p}}\nu_{\overline{q}}})\Big)\triangleright L_{e_{\lambda_1}}...L_{e_{\lambda_j}}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\rho_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+ \tag{4.3.271}$$

$$-\sum_{j=0}^{k}\sum_{s=1}^{p}c^{\star}\Big(\partial'_{\alpha}(K^{\mu_s}_{(i)n_1})\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{s-1}}\alpha\mu_{\overline{p}\setminus\overline{s}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\rho_{\overline{q}}}\Big)\triangleright L_{e_{\lambda_1}}...L_{e_{\lambda_j}}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\nu_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\}+ \tag{4.3.272}$$

$$+\sum_{j=0}^{k}\sum_{r=1}^{q}c^{\star}\Big(\partial'_{\nu_r}(K^{\beta}_{(i)n_1})\beta^{\lambda_{\overline{j}}\rho_{\overline{q}}\mu_{\overline{p}}}_{(i)n_{\overline{k+1}\setminus\overline{1}}\sigma_{\overline{p}}\nu_{\overline{r-1}}\beta\nu_{\overline{q}\setminus\overline{r}}}\Big)\triangleright L_{e_{\lambda_1}}...L_{e_{\lambda_j}}\Big\{\psi_i\Big[dx^{\sigma_{\overline{p}}}_{(i)}\otimes\partial_{(i)\rho_{\overline{q}}}\Big]c_{\zeta}(ds)\Big\} \tag{4.3.273}$$

So it is enough to split each Lie derivative $(L_{e_{(i)_\lambda}}) = (L_{e_{(i)0}}, L_{e_{(i)n}})$ and integrating by partial the terms contracted with each $L_{e_{(i)0}} = L_{\partial'_{(i)0}}$ to obtain the expression of $\hat{\Theta}^{\mu_{\overline{p}}}_{(i)n_{\overline{k+1}}\nu_{\overline{q}}}$ as a $C^{\infty}\mathbb{R}$-linear combination of the old basis $(\Theta^{\mu_{\overline{p}}}_{(i)n_{\overline{k+1}}\nu_{\overline{q}}})$. Now increase the counter k by 1 and go to step 2.

4.4 Brief discussion on the Ellis top order local representation

In the previous sections we have seen how the geometrical information carried by an adapted coordinate system (or equivalently by an adapted frame) can be used to establish an isomorphism of modules between the multipoles and specific Ellis representations. It has been shown that keeping constant the "transverse direction" of the derivation, each general local chart induces a basis for the multipoles due to the natural local frame and the transformation rules are $C^\infty(\mathbb{R})$-linear. Of course, since there are an infinite number of different Ellis representations for the multipoles, the transverse and the adapted Ellis local representations are not the only possibilities that we have to express the multipoles. We have seen how the transverse Ellis representation is very convenient if we want to preserve the free module structure of $\Upsilon^q_p(c)$ independently from the local charts chosen. Unfortunately this is not the most common representation that historically has been given to the multipoles [17][19]. For the sake of completeness we discuss very briefly the usual representation given for the multipoles called here ***the Ellis top order local representation***. Very soon we are going to realise that this representation is affected by all the problems we discussed previously when the trivial examples were analysed, furthermore the very weird transformation rules induced by a local coordinate transformation can cause a lot of troubles. For these reason this representation is not able to match the requirements needed to satisfy our purposes listed above.

4.4.1 The Ellis top order local representation

It is possible to prove that given an atlas A on the Manifold M inducing a natural local trivialisation $(\partial_{(i)\lambda})$ of TM an arbitrary multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$ it can be always expressed with a specific Ellis representation:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} c^\star(L_{\lambda_{\overline{N}}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i) \beta_{(i)\ \mu_{\overline{p}}}^{\lambda_{\overline{N}}\nu_{\overline{q}}} ds = \tag{4.4.1}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} c^\star(\partial_{\lambda_1} ... \partial_{\lambda_N} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i) \beta_{(i)\ \mu_{\overline{p}}}^{\lambda_{\overline{N}}\nu_{\overline{q}}} ds \tag{4.4.2}$$

This can be easily proved considering the following lemma:

Lemma 31: Let $\alpha^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ be a bunch of local smooth scalar fields on $I_i \subseteq \mathbb{R}$ defining a global smooth form over $\mathbb{R}$. There always exists a second bunch of local smooth scalar fields $\gamma^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ defined on $I_i \subseteq \mathbb{R}$ such that:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} \sum_{k=0}^{N} c^\star(L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) c^\star(\psi_i) \alpha^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{4.4.3}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(L_{\lambda_{\overline{N}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})c^\star(\psi_i)\gamma^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}ds \tag{4.4.4}$$

Proof. Let us prove it via induction. For the step $N = 1$ we have

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} \Big\{c^\star(L_\lambda(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} + c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\Big\}c^\star(\psi_i)ds = \tag{4.4.5}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} \Big\{c^\star(L_\lambda(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} + c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\frac{d}{ds}\Big[\int_a^s \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}dt\Big]\Big\}c^\star(\psi_i)ds = \tag{4.4.6}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} \Big\{c^\star(\partial_{(i)\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - \frac{d}{ds}[c^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})]\Big[\int_a^s \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}dt\Big]\Big\}c^\star(\psi_i)ds = \tag{4.4.7}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} \Big\{c^\star(\partial_{(i)\lambda}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - [\frac{d}{ds}c^\lambda_{(i)}]c^\star(\partial_{(i)\lambda}\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\Big[\int_a^s \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}dt\Big]\Big\}c^\star(\psi_i)ds = \tag{4.4.8}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} c^\star(\partial_\lambda\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})c^\star(\psi_i)\gamma^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}ds \tag{4.4.9}$$

where $a \in I_i \subseteq \mathbb{R}$ and the fields $\gamma^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ are defined by

$$\gamma^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} = \alpha^{\lambda\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - [\frac{d}{ds}c^\lambda_{(i)}]\int_a^s \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}dt \tag{4.4.10}$$

Let us suppose the thesis holds for the case N and let us prove for the case $N+1$

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} \sum_{k=0}^{N+1} c^\star(L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}c^\star(\psi_i)ds = \tag{4.4.11}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{c(\mathbb{R})} \Big\{c^\star(L_{\lambda_{\overline{N+1}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda_{\overline{N+1}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} + \sum_{k=0}^{N} c^\star(L_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\alpha^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\Big\}c^\star(\psi_i)ds = \tag{4.4.12}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} \Big\{ c^\star (L_{\lambda_{\overline{N+1}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\lambda_{\overline{N+1}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} + c^\star (L_{\lambda_{\overline{N}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \beta^{\lambda_{\overline{N}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \Big\} c^\star(\psi_i) ds = \tag{4.4.13}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} \Big\{ c^\star (\partial_{\lambda_1} ... \partial_{\lambda_{N+1}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\lambda_{\overline{N+1}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} + c^\star (\partial_{\lambda_1} ... \partial_{\lambda_N} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \frac{d}{ds} \Big[\int_a^s \beta^{\lambda_{\overline{N}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} dt \Big] \Big\} c^\star(\psi_i) ds = \tag{4.4.14}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} \Big\{ c^\star (\partial_{\lambda_1} ... \partial_{\lambda_{N+1}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\lambda_{\overline{N+1}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - \frac{d}{ds} c^\star (\partial_{\lambda_1} ... \partial_{\lambda_N} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \Big[\int_a^s \beta^{\lambda_{\overline{N}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} dt \Big] \Big\} c^\star(\psi_i) ds = \tag{4.4.15}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} \Big\{ c^\star (\partial_{\lambda_1} ... \partial_{\lambda_{N+1}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \alpha^{\lambda_{\overline{N+1}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} + \tag{4.4.16}$$

$$- \frac{d}{ds} [c^{\lambda_{N+1}}_{(i)}] c^\star (\partial_{\lambda_{N+1}} \partial_{\lambda_1} ... \partial_{\lambda_N} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \Big[\int_a^s \beta^{\lambda_{\overline{N}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} dt \Big] \Big\} c^\star(\psi_i) ds = \tag{4.4.17}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{c(\mathbb{R})} c^\star (\partial_{\lambda_1} ... \partial_{\lambda_{N+1}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) \gamma^{\lambda_{\overline{N+1}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} c^\star(\psi_i) ds \tag{4.4.18}$$

where $a \in I_i \subseteq \mathbb{R}$ and the fields $\gamma^{\lambda \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ are defined by

$$\gamma^{\lambda_{\overline{N}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} = \alpha^{\lambda_{\overline{N+1}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - [\frac{d}{ds} c^{\lambda}_{(i)}] \int_a^s \beta^{\lambda_{\overline{N}} \nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} dt \tag{4.4.19}$$

□

By convention we are going to use the term ***Ellis top order local representation*** referring to this specific local Ellis representation. To find more details about this specific representation one can see [19].

4.4.2 Issues concerning the Ellis top order local representation

Although this is the most common representation usually given for the multipoles, we are going to see how from our point of view it is not the most convenient. We are going to show it by considering a very trivial example. Let us consider $\mathbb{R}^2$ as a differential manifold on itself. $\mathbb{R}^2$ always admits a global atlas where the points of $\mathbb{R}^2$ are mapped into itself due to the identity functions. Let us denote by (x^0, x^1) the coordinate expression of an arbitrary point $x \in \mathbb{R}^2$. Now let us consider a closed embedding $c : \mathbb{R} \to \mathbb{R}^2$ defined by $c(t) = (t, 0)$, $\forall t \in \mathbb{R}$. Since $\mathbb{R}^2$ is a manifold, we can build the tangent bundle $T\mathbb{R}^2$ the cotangent bundle $T^\star\mathbb{R}^2$ as well as the tangent tensor bundle $T^p_q\mathbb{R}^2$. A global natural trivialisation of TM can be fixed by $(e_0 = \frac{\partial}{\partial x^0}, e_1 = \frac{\partial}{\partial x^1})$ and this induces a global trivialisation of $T^\star M$ and $T^p_q M$. Let us consider for instance the multipole $\mathcal{T} \in \overset{(1)}{\Upsilon}{}^1_1(c)$ defined by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} \Big[c^\star(L_\lambda \phi^\mu_\nu) \alpha^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu) \alpha^\nu_\mu \Big] dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1 \mathbb{R}^2 \tag{4.4.20}$$

where $\alpha^\nu_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0, 1]$ are smooth scalar fields. We know that each multipole can be written also using the local top order Ellis representation :

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\partial_\lambda \phi^\mu_\nu) \gamma^{\lambda\nu}_\mu dt = \int_{\mathbb{R}} \Big[c^\star(L_\lambda \phi^\mu_\nu) \alpha^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu) \alpha^\nu_\mu \Big] dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1 \mathbb{R}^2 \tag{4.4.21}$$

where the previous lemma fixed already the relationship between the couple $(\alpha^{\lambda\nu}_\mu, \alpha^\nu_\mu)$ and $\gamma^{\lambda\nu}_\mu$ to be:

$$\gamma^{\lambda\nu}_\mu = \alpha^{\lambda\nu}_\mu - [\frac{d}{ds} c^\lambda_{(i)}] \int_a^s \alpha^\nu_\mu dt \quad , \quad a \in \mathbb{R} \tag{4.4.22}$$

Since no restriction are given on the lower bound $a \in \mathbb{R}$ of the integral defining the coefficients of this representation, one is free to choose any value without really affecting the multipole $\mathcal{T}$. Hence supposing that $a \leq b \in \mathbb{R}$ we have can define a new set of coefficients:

$$\hat{\gamma}^{\lambda\nu}_\mu = \alpha^{\lambda\nu}_\mu - [\frac{d}{ds} c^\lambda_{(i)}] \int_b^s \alpha^\nu_\mu dt = \alpha^{\lambda\nu}_\mu - [\frac{d}{ds} c^\lambda_{(i)}] \int_a^s \alpha^\nu_\mu dt + [\frac{d}{ds} c^\lambda_{(i)}] \int_a^b \alpha^\nu_\mu dt = \tag{4.4.23}$$

$$= \gamma^{\lambda\nu}_\mu + [\frac{d}{ds} c^\lambda_{(i)}] \int_a^b \alpha^\nu_\mu dt \tag{4.4.24}$$

and they still define the same multipole $\mathcal{T}$ through the local top order Ellis representation.

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\partial_\lambda \phi^\mu_\nu) \hat{\gamma}^{\lambda\nu}_\mu dt \tag{4.4.25}$$

Therefore even when a local coordinate system is fixed, this particular representation is not unique. This is quite inconvenient, in fact the 0 dipole can be expressed in an infinite number of different representations with non null coefficients, for instance:

$$\gamma_\mu^{\lambda\nu} = \frac{d}{ds}[c^\lambda]\delta_\nu^\mu \tag{4.4.26}$$

where δ_ν^μ is just the Kronecker delta. In fact:

$$[\phi, \mathcal{T}] = \int_\mathbb{R} c^\star(\partial_\lambda \phi_\nu^\mu)\gamma_\mu^{\lambda\nu} ds = \tag{4.4.27}$$

$$= \int_\mathbb{R} c^\star(\partial_\lambda \phi_\nu^\mu)\frac{d}{ds}[c^\lambda]\delta_\nu^\mu ds = \int_\mathbb{R} \frac{d}{ds} c^\star(\phi_\nu^\mu)\delta_\nu^\mu ds = -\int_\mathbb{R} c^\star(\phi_\nu^\mu)\frac{d}{ds}[\delta_\nu^\mu] ds = \tag{4.4.28}$$

$$= \int_\mathbb{R} c^\star(\phi_\nu^\mu) 0 ds = 0 \quad , \quad \forall \phi \in \Gamma_0 T_1^1 \mathbb{R}^2 \tag{4.4.29}$$

To conclude the example let us show what happens to the parameters of the top order local Ellis representation when a change of local chart is performed. Changing the coordinates on M, we induce another global natural trivialisation of TM fixed by $(\partial_0', \partial_1')$ satisfying

$$\left\{ \partial_\nu' = \frac{\partial}{\partial x'^\nu} = \overline{J}_\nu^\mu \partial_\mu \right. \tag{4.4.30}$$

and this induces a new global trivialisation of $T^\star M$ and $T_q^p M$. Then multipole $\mathcal{T}$ can be expressed by:

$$[\phi, \mathcal{T}] = \int_\mathbb{R} c^\star(\partial_\lambda \phi_\nu^\mu)\gamma_\mu^{\lambda\nu} ds \tag{4.4.31}$$

using the top order Ellis representation induced by the old trivialisation and by:

$$[\phi, \mathcal{T}] = \int_\mathbb{R} c^\star(\partial_\lambda' \phi'^\mu_\nu)\gamma'^{\lambda\nu}_\mu ds \tag{4.4.32}$$

using the new one. This fixes a constraint on the transformation rules for the parameters related to the top order local Ellis representation:

$$\int_{\mathbb{R}} c^\star(\partial'_\lambda \phi'^{\mu}_{\nu})\gamma'^{\lambda\nu}_{\mu} ds = \int_{\mathbb{R}} c^\star(\partial_\lambda \phi^{\mu}_{\nu})\gamma^{\lambda\nu}_{\mu} ds = \int_{\mathbb{R}} c^\star(\partial_\lambda(\overline{J}^{\mu}_{\alpha} J^{\beta}_{\nu} \phi'^{\alpha}_{\beta}))\gamma^{\lambda\nu}_{\mu} ds = \tag{4.4.33}$$

$$= \int_{\mathbb{R}} c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\phi'^{\alpha}_{\beta} + \overline{J}^{\mu}_{\alpha}\partial_\lambda(J^{\beta}_{\nu})\phi'^{\alpha}_{\beta} + \overline{J}^{\mu}_{\alpha}J^{\beta}_{\nu}\partial_\lambda(\phi'^{\alpha}_{\beta})\Big)\gamma^{\lambda\nu}_{\mu} ds = \tag{4.4.34}$$

$$= \int_{\mathbb{R}} c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\phi'^{\alpha}_{\beta}\Big)\gamma^{\lambda\nu}_{\mu} ds + \int_{\mathbb{R}} c^\star\Big(\overline{J}^{\mu}_{\alpha}\partial_\lambda(J^{\beta}_{\nu})\phi'^{\alpha}_{\beta}\Big)\gamma^{\lambda\nu}_{\mu} ds + \int_{\mathbb{R}} c^\star\Big(\overline{J}^{\mu}_{\alpha}J^{\beta}_{\nu}J^{\rho}_{\lambda}\partial'_\rho(\phi'^{\alpha}_{\beta})\Big)\gamma^{\lambda\nu}_{\mu} ds \tag{4.4.35}$$

Let us just consider the first integral:

$$\int_{\mathbb{R}} c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\phi'^{\alpha}_{\beta}\Big)\gamma^{\lambda\nu}_{\mu} ds = \int_{\mathbb{R}} c^\star(\phi'^{\alpha}_{\beta}) c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\Big)\gamma^{\lambda\nu}_{\mu} ds = \tag{4.4.36}$$

$$= \int_{\mathbb{R}} c^\star(\phi'^{\alpha}_{\beta})\frac{d}{ds}\Big[\int_a^s c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\Big)\gamma^{\lambda\nu}_{\mu} dt\Big] ds = \tag{4.4.37}$$

$$= -\int_{\mathbb{R}} \frac{d}{ds} c^\star(\phi'^{\alpha}_{\beta})\Big[\int_a^s c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\Big)\gamma^{\lambda\nu}_{\mu} dt\Big] ds = \tag{4.4.38}$$

$$= -\int_{\mathbb{R}} \frac{d}{ds}[c'^{\rho}] c^\star\Big(\partial'_\rho(\phi'^{\alpha}_{\beta})\Big)\Big[\int_a^s c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\Big)\gamma^{\lambda\nu}_{\mu} dt\Big] ds = \tag{4.4.39}$$

$$= -\int_{\mathbb{R}} c^\star\Big(\partial'_\rho(\phi'^{\alpha}_{\beta})\Big) c^\star(J^{\rho}_{\sigma})\frac{d}{ds}[c'^{\rho}]\Big[\int_a^s c^\star\Big(\partial_\lambda(\overline{J}^{\mu}_{\alpha}) J^{\beta}_{\nu}\Big)\gamma^{\lambda\nu}_{\mu} dt\Big] ds \tag{4.4.40}$$

The second integral follows in the same way:

$$\int_{\mathbb{R}} c^\star\Big(\overline{J}^{\mu}_{\alpha}\partial_\lambda(J^{\beta}_{\nu})\phi'^{\alpha}_{\beta}\Big)\gamma^{\lambda\nu}_{\mu} ds = \tag{4.4.41}$$

$$= -\int_{\mathbb{R}} c^\star\Big(\partial'_\rho(\phi'^{\alpha}_{\beta})\Big) c^\star(J^{\rho}_{\sigma})\frac{d}{ds}[c'^{\rho}]\Big[\int_a^s c^\star\Big(\overline{J}^{\mu}_{\alpha}\partial_\lambda(J^{\beta}_{\nu})\Big)\gamma^{\lambda\nu}_{\mu} dt\Big] ds \tag{4.4.42}$$

Thence, putting together all the terms we obtain:

$$\int_{\mathbb{R}} c^\star(\partial'_\lambda \phi'^{\mu}_{\nu})\gamma'^{\lambda\nu}_{\mu} ds = \tag{4.4.43}$$

$$= \int_{\mathbb{R}} c^\star(\partial_\lambda \phi^{\mu}_{\nu})\Big\{c^\star\Big(\overline{J}^{\mu}_{\alpha}J^{\beta}_{\nu}J^{\lambda}_{\rho}\Big)\gamma^{\rho\nu}_{\mu} - c^\star(J^{\rho}_{\sigma})\frac{d}{ds}[c'^{\lambda}]\Big[\int_a^s c^\star\Big(\partial_\rho(\overline{J}^{\mu}_{\alpha})J^{\beta}_{\nu}\Big)\gamma^{\rho\nu}_{\mu} dt\Big] + \tag{4.4.44}$$

$$-c^\star(J^\rho_\sigma)\frac{d}{ds}[c'^\lambda]\Big[\int_a^s c^\star\Big(\overline{J}^\mu_\alpha\partial_\rho(J^\beta_\nu)\Big)\gamma^{\rho\nu}_\mu dt\Big]\Big\}ds \tag{4.4.45}$$

and we can conclude that the transformation rules for the parameters are given by:

$$\gamma'^{\lambda\nu}_\mu = c^\star\Big(\overline{J}^\mu_\alpha J^\beta_\nu J^\lambda_\rho\Big)\gamma^{\rho\nu}_\mu - c^\star(J^\rho_\sigma)\frac{d}{ds}[c'^\lambda]\Big[\int_a^s c^\star\Big(\partial_\rho(\overline{J}^\mu_\alpha)J^\beta_\nu\Big)\gamma^{\rho\nu}_\mu dt\Big]+ \tag{4.4.46}$$

$$-c^\star(J^\rho_\sigma)\frac{d}{ds}[c'^\lambda]\Big[\int_a^s c^\star\Big(\overline{J}^\mu_\alpha\partial_\rho(J^\beta_\nu)\Big)\gamma^{\rho\nu}_\mu dt\Big] \tag{4.4.47}$$

The reader can immediately notice how some problems occurs:

1. The parameters are not unique then in general and the module of the multipoles is not isomorphic to the set of the top order local Ellis representation.
2. The transformation rules are not uniquely determined since $a \in \mathbb{R}$ is not fixed
3. Even if an isomorphism can be established between the multipoles and the parameters, for instance fixing the lower bound of the integral, the transformation rules are not $C^\infty(\mathbb{R})$-linear so the $C^\infty(\mathbb{R})$ module structure is not preserved by a change of local chart.
4. Although in the example above this aspect is not crucial (just a global coordinate system is considered on $\mathbb{R}^2$), in general for atlas formed by more than one local chart, the gluing condition satisfied by the local parameters $\gamma^{\lambda_{\overline{k}}\nu\overline{q}}_{(i)\mu_{\overline{p}}}$ in order to define a global smooth 1-form over $\mathbb{R}$ are extremely complicated involving higher order derivatives of the Jacobians.

Quadrupoles examples and representations can be found in [19] as well as some aspect concerning the transformation rules of this representation for a electromagnetic 3-current quadrupole.

As one can see, although this representation is the most immediate and often used in physics (expecially in covariant electromagnetism) [17][19], actually this is not a good choice from our perspective because it is affected by all the problems already singled out for the general Ellis representation. Because at this stage the relationship between the given definition of multipoles and the usual classical notion of multipoles and moments is still not clear, we are going to investigate it in the following section.

4.5 Squeezed Tensor fields, Weak Asymptotic Expansions and Adapted Ellis Moments

In this section we are going to propose the concept of "approximation" of specific one parameter families of smooth compact support tensor fields. From this we will be able to cast a specific coordinate-free definition of asymptotic expansion of smooth compact support tensor fields and we will see how the coefficients of such expansion are the $\mathbb{R}$-linear functionals defined in the third chapter generalising the concept of De Rham push-forward. We will see how, in this case, the adapted Ellis moments of the functionals characterising the expansions, coincides exactly with the common definition of "multipole moments" widely used in Mathematics, Physics, Engineering and Statistics.

4.5.1 A specific realisation

Let us consider for simplicity a compactly supported real function $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, as instance a probability distribution changing with time or a mass distribution non constant in time (a massive blob or a fluid). One could argue that this kind of functions are not able to represent anything physical, but this is a philosophical matter, since in practice we do not have any evidence that physical objects can exists for an infinite amount of time. On the other hand the support of a compactly supported function can be quite wide in time, even more then the actual age of the universe, so the support of compactly supported functions can cover any interval of time related to measurable physical phenomena happening in the universe.

The first two standard spatial raw moments about the point $(t, 0, 0, 0)$ are defined as:

$$\text{The normalization factor or total mass:} \qquad \nu(t) = \int_{-\infty}^{\infty} f(t, \overline{x}) d^3x \tag{4.5.1}$$

$$\text{The mean:} \qquad \mu^a(t) = \int_{-\infty}^{\infty} x^a f(t, \overline{x}) d^3x \tag{4.5.2}$$

$$\vdots \tag{4.5.3}$$

$$\text{The n-th raw moment:} \qquad \mu^{a_1 \dots a_n}(t) = \int_{-\infty}^{\infty} x^{a_1} \cdots x^{a_n} f(t, \overline{x}) d^3x \tag{4.5.4}$$

$$\tag{4.5.5}$$

Normalising the mean with respect to the normalization factor we can define the the baricenter:

$$\text{The baricentre:} \qquad \beta^a(t) = \frac{1}{\nu(t)} \int_{-\infty}^{\infty} x^a f(t, \overline{x}) d^3x \tag{4.5.6}$$

$$\tag{4.5.7}$$

If a translation is performed as follow:

$$\begin{cases} t = t \\ y^1 = x^1 - \mu^1(t) \\ y^2 = x^2 - \mu^2(t) \\ y^3 = x^3 - \mu^3(t) \end{cases} \tag{4.5.8}$$

the standard central spatial moments with respect to the "adapted coordinate system" of the centre of mass are the well known:

$$\text{The normalization factor or total mass:} \qquad \nu(t) = \int_{-\infty}^{\infty} f(t, \overline{y}) d^3y \tag{4.5.9}$$

$$\text{The mean:} \qquad \mu^a(t) = \int_{-\infty}^{\infty} y^a f(t, \overline{y}) d^3y = 0 \tag{4.5.10}$$

$$\text{The variance:} \qquad \mu^{ab}(t) = \int_{-\infty}^{\infty} y^a y^b f(t, \overline{y}) d^3y \tag{4.5.11}$$

$$\text{The skewness:} \qquad \mu^{abc}(t) = \int_{-\infty}^{\infty} y^a y^b y^c f(t, \overline{y}) d^3 \tag{4.5.12}$$

$$\text{The kurtosis:} \qquad \mu^{abcd}(t) = \int_{-\infty}^{\infty} y^a y^b y^c y^d f(t, \overline{y}) d^3y \tag{4.5.13}$$

$$\vdots \tag{4.5.14}$$

$$\text{The n-th central moment:} \qquad \mu^{a_1 \dots a_n}(t) = \int_{-\infty}^{\infty} y^{a_1} \cdots y^{a_n} f(t, \overline{y}) d^3y \tag{4.5.15}$$

$$\tag{4.5.16}$$

In this case the list of coefficients defining the baricenter $(\overline{\beta}(t))$ is null. Let us notice that $(t, \overline{\beta}(t))$ represent at each time a point on $\mathbb{R}^3$ so it is a curve and since f has a compact support $(t, \overline{\mu}(t))$ is a closed embedding $\mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}^3$ and the coordinates $(t, \overline{y})$ are just an adapted coordinate system for that particular embedding. In the very same fashion, let $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ a compactly supported function expressing at each time the density of a fluid flowing in a pipe, and let us consider a compactly supported vector field $V : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, expressing at each time its velocity. The current density can be defined as the compactly supported vector field $\overline{J} = \overline{v} \cdot \rho$. The first two standard spatial raw moments about the point $(t, 0, 0, 0)$ are defined as:

$$\text{The total current} \qquad I^a(t) = \int_{-\infty}^{\infty} J^a(t, \overline{x}) d^3x \tag{4.5.17}$$

$$\text{The first raw moment:} \qquad I^{ab}(t) = \int_{-\infty}^{\infty} x^a J^b(t, \overline{x}) d^3x \tag{4.5.18}$$

$$\vdots \tag{4.5.19}$$

$$\text{The n-th raw moment:} \qquad \mu^{a_1 \ldots a_n b}(t) = \int_{-\infty}^{\infty} x^{a_1} \cdots x^{a_n} J^b(t, \overline{x}) d^3x \tag{4.5.20}$$

$$\tag{4.5.21}$$

Let us notice that now there is no preferred choice to fix a specific closed embedding unless considering the "mean of the density ρ" (something quite hard if we know just the expression of $\overline{J}$) or adding an extra structure. These rules fix the standard definition of moments for scalar and vector fields, and can be easily generalised for the tensor fields depending on time $T : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3^{p+q}}$, in an obvious way. We will see how it is possible to interpret these "moments" just as a specific case of adapted "Ellis moments" of the linear functionals we defined in the third chapter.

4.5.2 Weak asymptotic expansions of "squeezed tensor fields"

Definition 71: Let M be a manifold with an atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ inducing a local trivialisation of TM with the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i). Given a smooth global top form $\omega \in \Gamma\Lambda^m M$ we define the map $\langle \quad \rangle_\omega : \Gamma T^q_p M \to \mathcal{J}^q_p(M)$ such that

$$[\phi, \langle T \rangle_\omega] = \sum_{U_i \in \mathcal{A}} \int_M \left(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \psi_i \right) \omega \tag{4.5.22}$$

Property 42: Although the definition is given in terms of fixed local frame it is very easy to show that it still holds for each different choice of atlases and local frames.

Lemma 32: Let M be a manifold with an atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ inducing a local trivialisation of TM with the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i). Given a smooth global top form $\omega \in \Gamma\Lambda^m M$ let be $\Gamma_{0\, supp\, \omega} T^q_p M \in \Gamma_0 T^p_q M$ the compact support tensor fields such that $supp(T) \subseteq supp(\omega)$. Let be

$$\langle \Gamma_{0\, supp\, \omega} T^q_p M \rangle_\omega = \{ \langle T \rangle_\omega \mid T \in \Gamma_{0\, supp\, \omega} T^q_p M \} \tag{4.5.23}$$

The restriction of the map $\langle \rangle_\omega$ upon the set $\Gamma_{0\, supp\, \omega} T^q_p M$ is an isomorphism of $C^\infty(M)$-modules between $\Gamma_{0\, supp\, \omega} T^q_p M$ and $\langle \Gamma_{0\, supp\, \omega} T^q_p M \rangle_\omega$

Proof. Let us start noting the map is an homomorphism because it preserves the C^∞-

linearity

$$[\phi, f\cdot\langle T\rangle_\omega + g\cdot\langle S\rangle_\omega] = \tag{4.5.24}$$

$$= \sum_{U_i\in\mathcal{A}}\int_M \left(f\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\psi_i\right)\omega + \sum_{U_i\in\mathcal{A}}\int_M \left(g\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} S^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\psi_i\right)\omega = \tag{4.5.25}$$

$$= \sum_{U_i\in\mathcal{A}}\int_M \left(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}[fT+gS]^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\psi_i\right)\omega = [\phi, \langle fT+gS\rangle_\omega] \tag{4.5.26}$$

By definition it is a surjection (each element of $\langle\Gamma_{0\,supp\,\omega}T^q_pM\rangle_\omega$ is defined from an element of $\Gamma_{0\,supp\,\omega}T^q_pM$). It is also injective because considering two element $\langle T\rangle_\omega$ and $\langle S\rangle_\omega$ we have:

$$\langle T\rangle_\omega = \langle S\rangle_\omega \Leftrightarrow \forall\,\phi\in\Gamma_0T^p_qM\ ,\quad [\phi,\langle T\rangle_\omega] = [\phi,\langle S\rangle_\omega] \tag{4.5.27}$$

$$\Leftrightarrow \forall\,\phi\in\Gamma_0T^p_qM\ ,\quad \sum_{U_i\in\mathcal{A}}\int_M \left(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\psi_i\right)\omega = \sum_{U_i\in\mathcal{A}}\int_M \left(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} S^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\psi_i\right)\omega \tag{4.5.28}$$

$$\Leftrightarrow \forall\,\phi\in\Gamma_0T^p_qM\ ,\quad \sum_{U_i\in\mathcal{A}}\int_{supp(T)\cup supp(S)} \left(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}[T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - S^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}]\psi_i\right)\omega = 0$$

Since ω is always non vanishing the only way to have always a null integral is to have $[T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} - S^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] = 0$ and this is true just if $T = S$ hence it is also an injection. So we have the thesis. □

Corollary 14: If ω is a global volume form upon a differential manifold then $\langle\rangle_\omega$ is an isomorphism of $C^\infty(M)$-modules between ΓT^q_pM and $\langle\Gamma T^q_pM\rangle_\omega$

Due to the previous lemma we can then state that $\Gamma_{0\,supp\,\omega}T^q_pM \cong \langle\Gamma_{0\,supp\,\omega}T^q_pM\rangle_\omega$

Definition 72: Let be M a differential manifold. Let us consider a subset $V\subset M$ and a one parameter family of closed subsets $U_\varepsilon\subset M$. We say that $\underset{\varepsilon\to 0}{slim} U_\varepsilon = V$ if and only if:

$$\exists\varepsilon_0>0 : V\subset U_\varepsilon \quad , \quad \forall\varepsilon\in(0,\varepsilon_0) \tag{4.5.29}$$

$$\forall\lambda\in\Gamma_0\Lambda^0M \mid \lambda_{|V} = 0 \Rightarrow \sup_{x\in U_\varepsilon}\{|\lambda(x)|\} = O(\varepsilon) \tag{4.5.30}$$

Definition 73: Let be $c : \mathbb{R} \hookrightarrow M$ a closed embedding and Σ a subset of M with $dim(\Sigma) = dim(M) - dim(\mathbb{R}) = m - 1$. We define a ***smooth one parameter family of embedding transverse to*** c a set of closed embedding $\hat{\Sigma} : \mathbb{R} \times \Sigma \hookrightarrow M$ such that $\hat{\Sigma}$ is smoothly dependent on the first term and $\forall s \in \mathbb{R} \Rightarrow \hat{\Sigma}_s(\Sigma) \cap c(\mathbb{R}) = c(s)$

Property 43: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$ let $(U_{(i)}, \varphi_{(i)})$ a local chart adapted to c such that $\varphi_{(i)}(x) = (s, y^m_{(i)})$ gives rise to a smooth one parameter family of transverse embeddings $\hat{\Sigma} : \mathbb{R} \times \Sigma \hookrightarrow M$ as follow:

$$\Pi_{\mathbb{R}^{m-1}}[\varphi_{(i)}(U_i)] = \Sigma_{(i)} \qquad \hat{\Sigma}_{(i)s}(y^m) = \varphi^{-1}_{(i)}(s, y^m) \forall (s, y^m) \in \varphi_{(i)}(U_{(i)}) \tag{4.5.31}$$

where $\Pi_{\mathbb{R}^{m-1}}$ is just the natural projection of $\mathbb{R}^m$ on $\mathbb{R}^{m-1}$.

Lemma 33: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$ then a smooth family of closed embedding $\hat{\Sigma} : \mathbb{R} \times \Sigma \hookrightarrow M$ gives rise to a local chart (U, ϕ) adapted to c:

Proof. We provide here a sketch of proof. Let be $\alpha : \mathbb{R} \to \Sigma$ the curve satisfying $\hat{\Sigma}_s(\alpha(s)) = c(s)$. Let (V, η) a local chart on Σ such that $\exists \varepsilon > 0 \to \alpha(s) \in V \forall s \in [s_0 - \varepsilon, s_0 + \varepsilon]$ and $\eta(\dot{\alpha}(s)) = 0 \forall s \in [s_0 - \varepsilon, s_0 + \varepsilon]$. Since $\hat{\Sigma}$ is a smooth family of transverse embeddings such a local chart must exist. If we define $U \subset M$ such that $U = \Sigma_s(V)$ and $\varphi^{-1} = \Sigma_s(\eta^{-1})$ then it is easy to check the (U, φ) are well defined local charts adapted to c. □

Definition 74: Given a closed embedding $c : \mathbb{R} \hookrightarrow M$ let us consider a one parameter family of transverse embeddings $\hat{\Sigma} : \mathbb{R} \times \Sigma \hookrightarrow M$. Denoting by (U, φ) an adapted local chart with $\varphi(x) = (\varphi^0(x), \varphi^m(x))$ we call it ***a local chart adapted to*** c ***and*** $\hat{\Sigma}$ if and only if $\varphi^0(\hat{\Sigma}_s(\Sigma)) = s$

Definition 75: Let M be a manifold with an atlas $\mathcal{A} = (U_i, \varphi_{(i)})$ inducing a local trivialisation of TM with the local frame $(e_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i). Let us consider the open interval $(0, \varepsilon_0) \subset \mathbb{R}$ and a closed embedding $c : \mathbb{R} \hookrightarrow M$. Let us consider a smooth one parameter family of compactly supported tensor fields $T : (0, \varepsilon_0) \to \Gamma_0 T^q_p M$ such that

1. $\underset{\varepsilon \to 0}{slim}\ supp(T_\varepsilon) \subseteq c(\mathbb{R})$
2. $\forall x \in M$, $\forall v_1 ... v_p \in \Gamma T^p_q M$, $\forall \alpha_1 ... \alpha_q \in \Gamma T^p_q M \Rightarrow T_\varepsilon(\alpha^{\overline{q}}, v_{\overline{p}})_{|x} \in O(\varepsilon^{-dim(M)+1})$ for $\varepsilon \to 0$
3. $\forall \phi \in \Gamma_0 T^p_q M, \forall \hat{\Sigma}$ family of transverse embedding respect to $c, \forall \alpha \in \Gamma \Lambda^{dim(M)-1} M \Rightarrow$

$\exists \{\zeta_k \in C^\infty(\mathbb{R})\ ,\ k = 0, ..., k\}$ such that:

$$\sum_{U_i \in \mathcal{A}} \int_\Sigma \hat{\Sigma}_s^\star(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\ \psi_i\ \alpha) = \sum_{k=0}^N \zeta_k(s)\varepsilon^k + O(\varepsilon^{N+1}) \tag{4.5.32}$$

We call this family a ***squeezed tensor field over*** c ***with rank q,p***. We denote with $S_p^q(c)$ the set of all the squeezed tensors fields over c with rank q,p.

Once again one could argue that this kind of compact support tensor field is not able to represent anything physical, but as stated previously, we have not any evidence that the physical objects can exists for an infinite amount of time. On the other hand the support can cover any interval of time related to each real (and measurable) physical phenomenon.

Lemma 34: The first two intrinsic requirements above are satisfied if and only if:

1. Given any arbitrary adapted local coordinate system (U, φ) such that $U \cap c \neq \varnothing$ we have:

$$\varphi^a_{|_{supp(T_\varepsilon)}} = O(\varepsilon) \tag{4.5.33}$$

2. Given any arbitrary adapted local coordinate system (U, φ) defining a natural frame (∂_μ) and coframe dx^μ we have that the components of the squeezed tensors must satisfy:

$$[T_\varepsilon]^{\nu_{\overline{q}}}_{\mu\overline{p}} = O(\varepsilon^{-dim(M)+1}) \tag{4.5.34}$$

Proof. Clearly $\underset{\varepsilon \to 0}{slim}\ supp(T_\varepsilon) \subseteq c(\mathbb{R})$ implies $\varphi^a_{|_{supp(T_\varepsilon)}} = O(\varepsilon)$ because of the definition of adapted coordinates. In the other hand let us consider $\lambda \in \Gamma\Lambda^0 M \mid c^\star(\lambda) = 0$. Then for $x \in U$ since λ is smooth, we may write

$$\lambda(\varphi^{-1}(s, y^a)) = \lambda(\varphi^{-1}(s, 0)) + y^a[\partial_a \lambda(\varphi^{-1}(s, 0)) + \xi^a(s, y^a)] \tag{4.5.35}$$

$$\lambda(\varphi^{-1}(s, y^a)) = 0 + y^a[\partial_a \lambda(\varphi^{-1}(s, 0)) + \xi^a(s, y^a)] \tag{4.5.36}$$

where ξ^a is a list of $m-1$ smooth real functions defined by Taylor's theorem and satisfying

$$\lim_{y^b \to 0} (\xi^a) = 0 \tag{4.5.37}$$

Then since $\varphi(x) = (s(x), y^a(x))$, $y^a_{|_{supp(T_\varepsilon)}} = \varphi^a_{|_{supp(T_\varepsilon)}} = O(\varepsilon)$ by hypothesis and since $\partial_a \lambda(\varphi^{-1}(s,0)) + \xi^a(s, y^a) = O(1)$ we have:

$$\lambda_{|_{supp(T_\varepsilon)}} = O(\varepsilon) \Rightarrow \underset{\varepsilon \to 0}{slim}\ supp(T_\varepsilon) \subseteq c(\mathbb{R}). \tag{4.5.38}$$

For the second statement again when $\varepsilon \to 0$ the condition $\forall x \in M, \forall v_1 ... v_p \in \Gamma T^p_q M, \forall \alpha_1 ... \alpha_q \in \Gamma T^p_q M \Rightarrow T_\varepsilon(\alpha^{\overline{q}}, v_{\overline{p}})_{|x} \in O(\varepsilon^{-dim(M)+1})$ implies automatically $[T_\varepsilon]^{\nu_{\overline{q}}}_{\mu_{\overline{p}}} = O(\varepsilon^{-dim(M)+1})$. In the other hand if $[T_\varepsilon]^{\nu_{\overline{q}}}_{\mu_{\overline{p}}} = O(\varepsilon^{-dim(M)+1})$ holds then $\forall v_1 ... v_p \in \Gamma T^p_q M$, $\forall \alpha_1 ... \alpha_q \in \Gamma T^p_q M$ then

$$T_\varepsilon(\alpha^{\overline{q}}, v_{\overline{p}})_{|x} = [T_\varepsilon]^{\nu_{\overline{q}}}_{\mu_{\overline{p}}} v^{\mu_1} ... v^{\mu_p} \alpha^{\nu_1} ... v^{\nu_q} = O(\varepsilon^{-dim(M)+1}) O(1) = O(\varepsilon^{-dim(M)+1}) \tag{4.5.39}$$

□

Lemma 35: Let M be a manifold, $c : \mathbb{R} \hookrightarrow M$ a closed embedding. let us consider an open set U such that $U \cap c(\mathbb{R}) = \varnothing$ and a compact set $K \subset U$. Given a squeezed tensor field $T_\varepsilon \in S^q_p(c)$ there always exists a small enough $\varepsilon_0 > 0$ such that,

$$supp(T_\varepsilon) \cap K = \varnothing \qquad , \qquad \forall \varepsilon \in (0, \varepsilon_0) \tag{4.5.40}$$

Proof. Since the compact K is a subset of U it is always possible to build a smooth cutoff function $\xi \in \Gamma_0 \Lambda^0 M$ such that

$$\xi = \begin{cases} \xi(x) = 1 & , \quad x \in K \\ \xi(x) = 0 & , \quad x \notin U \end{cases} \tag{4.5.41}$$

Therefore $\xi_{|_{c(s)}} = 0$ and we can use the definition of squeezed tensor to say that:

$$\sup_{x\in U_\varepsilon}\{|\xi(x)|\} = O(\varepsilon) \quad \Rightarrow \quad \lim_{\varepsilon\to 0}\sup_{x\in U_\varepsilon}\{|\xi(x)|\} = 0 \tag{4.5.42}$$

This means that:

$$\forall t>0\ ,\ \exists\delta_t \quad | \quad \forall\varepsilon\ ,\ 0<\varepsilon<\delta_t \ \Rightarrow\ \sup_{x\in U_\varepsilon}\{|\xi(x)|\} < t \tag{4.5.43}$$

Let us choose $t = 1$ and let us define $\varepsilon_0 = \delta_1$, then $\forall\varepsilon \in (0, \varepsilon_0)$ we have by definition of limit that:

$$\sup_{x\in U_\varepsilon}\{|\xi(x)|\} < 1 \tag{4.5.44}$$

So this condition can be satified $\forall\varepsilon \in (0, \varepsilon_0)$ just if $supp(T_\varepsilon) \cap K = \varnothing$ otherwise we have a contradiction. In fact, let us suppose that there exists at least a value of $\overline{\varepsilon} \in (0, \varepsilon_0)$ such that $supp(T_{\overline{\varepsilon}}) \cap K \neq \varnothing$ then by definition, $\xi(x) = 1 \quad , \quad \forall x \in supp(T_{\overline{\varepsilon}}) \cap K$, but again by definition of cutoff function the maximum value of ξ is 1 hence

$$\sup_{x\in U_{\overline{\varepsilon}}}\{|\xi(x)|\} = 1 \tag{4.5.45}$$

so we should conclude

$$\sup_{x\in U_{\overline{\varepsilon}}}\{|\xi(x)|\} = 1 < 1 \tag{4.5.46}$$

which is a contradiction. Hence we have the thesis. $\square$

Lemma 36: Let M be a manifold, $c : \mathbb{R} \hookrightarrow M$ a closed embedding and $\mathcal{A} = (U_i, \phi_{(i)})$ an atlas adapted to it. Let $\hat{\Sigma}_{(i)} : \mathbb{R} \times \Sigma_i \hookrightarrow M$ a smooth family of transverse embedding induced by an the adapted local chart satisfying $U_i \cap c(s) \neq \varnothing$. Given a squeezed tensor field $T_\varepsilon \in S^q_p(c)$ and an arbitrary global form $\alpha \in \Gamma\Lambda^{dimM-1}M$ there always exists a set of smooth scalar fields $\{\zeta^{a_{\overline{k}}\nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}} \in C^{\infty}(\mathbb{R})\ ,\ l \in [k,N]\ ,\ k \in [0,N]\}$ such that:

$$\int_{\Sigma_i} y^{a_1}_{(i)} ... y^{a_k}_{(i)} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\omega}_{(i)} dy^{\overline{m-1}}_{(i)} = \sum_{l=k}^{N} \zeta^{a_{\overline{k}}\nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}} \varepsilon^l + O(\varepsilon^{N+1}) \tag{4.5.47}$$

Proof. Since $(U_i, \varphi_{(i)})$ is an adapted chart $U_i \cap c(s) \neq \varnothing$ inducing the embedding $\hat{\Sigma}_{(i)}$ we have:

$$\varphi(U_i) = \Omega_i = I_i \times \Sigma_i \tag{4.5.48}$$

and given a $x \in U_i$ then the coordinate expression is $\varphi(x) = (s(x), y^{\overline{m-1}}(x))$ Considering that $U_i \cap c(s) \neq \varnothing$ by definition of adapted coordinates we have $\varphi^{c(s)} = (s, 0^{\overline{m-1}})$ so $0^{\overline{m-1}} \in \Sigma_i$. From the property of the standard topology defined on $\mathbb{R}^{m-1}$ we know there must exists a real number $r > 0$ such that the closed ball

$$B_{r0(i)} = \{y^{\overline{m-1}}_{(i)} \in \mathbb{R}^{m-1} \mid \sqrt{y^m_{(i)} y_{(i)m}} \leq r\} \tag{4.5.49}$$

for which the statement $0^{\overline{m-1}} \in B_{r0(i)} \subset \Sigma_i$ is true. Let us fix $s \in I$ and the closed subset $[a_i, b_i] \subset I_i \subset \mathbb{R}$ such that $s \in [a_i, b_i]$, which is a compact with respect to its standard topology, so it is $B_{r0(i)}$ with respect to the standard topology of $\mathbb{R}^{m-1}$. Let us define $W_{i\,s} = \varphi^{-1}([a_i, b_i], B_{r0(i)})$, it must be a compact set because it is the image of a compact set through the continuous map φ^{-1} and $W_{i\,s} \subset U_i$. Therefore, for each $s \in I$ it is always possible to build a smooth cutoff function $\xi_s \in \Gamma_0 \Lambda^0 M$ such that

$$\begin{cases} \xi_s(x) = 1 \quad , \quad x \in W_i \subset U_i \\ \xi_s(x) = 0 \qquad , \quad x \notin U_i \end{cases} \tag{4.5.50}$$

Let us define a set of auxiliary compact support tensor fields $\phi^{m_{\overline{k}}\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \in \Gamma_0 T^p_q M$ at fixed adapted frame:

$$\phi^{m_{\overline{k}}\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} = \varphi^{m_1}_{(i)} ... \varphi^{m_k}_{(i)} \xi_s \delta^{\nu_{\overline{q}}\alpha_{\overline{p}}}_{\mu_{\overline{p}}\beta_{\overline{q}}}\ e_{\alpha_{\overline{p}}} \otimes e^{\beta_{\overline{q}}} \tag{4.5.51}$$

$$\sum_{U_i\in\mathcal{A}}\int_{\Sigma_i}\hat{\Sigma}^{\star}_{(i)s}([\phi^{m_{\overline{k}}\mu_{\overline{p}}}_{\nu_{\overline{q}}}]^{\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}}T^{\beta_{\overline{q}}}_{\varepsilon(i)\omega_{\overline{p}}}\psi_i\alpha) = \tag{4.5.52}$$

$$= \int_{\Sigma_i}\hat{\Sigma}^{\star}_{(i)s}(\varphi^{m_1}_{(i)}...\varphi^{m_k}_{(i)}\xi_s\delta^{\nu_{\overline{q}}\alpha_{\overline{p}}}_{\mu_{\overline{p}}\beta_{\overline{q}}}T^{\beta_{\overline{q}}}_{\varepsilon(i)\alpha_{\overline{p}}}\omega) = \tag{4.5.53}$$

$$= \int_{\Sigma_i}\hat{\Sigma}^{\star}_{(i)s}(\varphi^{m_1}_{(i)}...\varphi^{m_k}_{(i)}\xi_s T^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\omega) = \tag{4.5.54}$$

$$= \int_{\Sigma_i} y^{m_1}_{(i)}...y^{m_k}_{(i)}\hat{\xi}_s\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\hat{\omega}dy^{\overline{m-1}}_{(i)} \tag{4.5.55}$$

But due to the definition of squeezed tensor field we know that there always exists a set of smooth scalar fields $\{\zeta^{a_{\overline{k}}\nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}} \in C^{\infty}(\mathbb{R})\ ,\ l\in[k,N]\ ,\ k\in[0,N]\}$ such that:

$$\sum_{U_i\in\mathcal{A}}\int_{\Sigma_i}\hat{\Sigma}^{\star}_{(i)s}([\phi^{m_{\overline{k}}\mu_{\overline{p}}}_{\nu_{\overline{q}}}]^{\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}}T^{\beta_{\overline{q}}}_{\varepsilon(i)\omega_{\overline{p}}}\psi_i\alpha) = \sum_{l=0}^{N}\zeta^{a_{\overline{k}}\nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}}\varepsilon^l + O(\varepsilon^{N+1}) \tag{4.5.56}$$

hence we can say that:

$$\int_{\Sigma_i} y^{m_1}_{(i)}...y^{m_k}_{(i)}\hat{\xi}_s\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\hat{\omega}dy^{\overline{m-1}}_{(i)} = \sum_{l=0}^{N}\zeta^{a_{\overline{k}}\nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}}\varepsilon^l + O(\varepsilon^{N+1}) \tag{4.5.57}$$

Let us suppose to be able to prove the existence of an $\varepsilon_0 > 0$ such that $\forall\varepsilon\in(0,\varepsilon_0)\Rightarrow \Sigma_s^{-1}(\xi_s T_\varepsilon)\subset B_{0r(i)}$ therefore $\forall\varepsilon\in(0,\varepsilon_0)\Rightarrow \hat{\xi}_s\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} = 1\cdot\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} = \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}$. Substituting it in the equation above we would have immediately

$$\int_{\Sigma_i} y^{m_1}_{(i)}...y^{m_k}_{(i)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\hat{\omega}dy^{\overline{m-1}}_{(i)} = \sum_{l=0}^{N}\zeta^{a_{\overline{k}}\nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}}\varepsilon^l + O(\varepsilon^{N+1}) \tag{4.5.58}$$

But such an ε_0 always exists. In fact since T_ε is a squeezed tensor field, it follows that $\xi_s T_\varepsilon$ is also a squeezed tensor field and furthermore since the smooth cutoff function ξ_s has a compact support, then by definition $supp(\xi_s T_\varepsilon)\subset U_i$ is a compact set. Considering this we can define another smooth cutoff function $\eta_s \in \Gamma_0\Lambda^0 M$ such that

$$\begin{cases} \xi_s(x) = 1 \quad , \quad x \in supp(\xi_s T_\varepsilon) \\ \xi_s(x) = 0 \qquad , \quad x \notin U_i \end{cases} \tag{4.5.59}$$

and we can use it to construct a set of compactly supported scalar field $\eta\varphi^m \in \Gamma_0\Lambda^0 M$.

By construction we have that

$$\eta\varphi^m_{(i)|_{supp(\xi_s T_\varepsilon)}} = \varphi^m_{(i)|_{supp(\xi_s T_\varepsilon)}} \tag{4.5.60}$$

both their limits must coincide:

$$\lim_{\varepsilon\to 0}\Big(\sup_{x\in supp(\xi_s T_\varepsilon)} |\varphi^m_{(i)}(x)|\Big) = \lim_{\varepsilon\to 0}\Big(\sup_{x\in supp(\xi_s T_\varepsilon)} |\eta\varphi^m_{(i)}(x)|\Big) \tag{4.5.61}$$

and both of them must be null on $U_i \cap c(\mathbb{R})$. Hence we can state from the definition of squeezed tensor field that:

$$\lim_{\varepsilon\to 0}\Big(\sup_{x\in supp(\xi_s T_\varepsilon)} |\varphi^m_{(i)}(x)|\Big) = \lim_{\varepsilon\to 0}\Big(\sup_{x\in supp(\xi_s T_\varepsilon)} |\eta\varphi^m_{(i)}(x)|\Big) = 0 \tag{4.5.62}$$

This means that $\forall t > 0 \ , \exists\delta_t > 0 \,|\, \forall\varepsilon, 0 < \varepsilon < \delta_t \Rightarrow \sup_{x\in supp(\xi_s T_\varepsilon)} |\varphi^m_{(i)}(x)| < t$

So defining $\varepsilon_0 = \delta_t$, we can state that $\forall t > 0$, since the square root is monotonic, there must exist $\varepsilon_0 > 0$ such that $\forall\varepsilon \in (0, \varepsilon_0)$ the following is satisfied:

$$\sup_{x\in supp(\xi_s T_\varepsilon)} \sqrt{\varphi^m_{(i)}(x)\varphi_{(i)m}(x)} < \sqrt{m-1}\cdot t \tag{4.5.63}$$

If we choose $t = \frac{r}{\sqrt{m-1}}$ we have that the following must be satisfied:

$$\sup_{x\in supp(\xi_s T_\varepsilon)} \sqrt{\varphi^m_{(i)}(x)\varphi_{(i)m}(x)} < r \tag{4.5.64}$$

meaning that

$$\forall x \in supp(\xi_s T_\varepsilon) \Rightarrow \Sigma_s^{-1}(x) \in B_{r,0(i)} \tag{4.5.65}$$

therefore by definition there exists a small enough $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ we have:

$$supp(T_\varepsilon) = supp(\xi_s T_\varepsilon) \tag{4.5.66}$$

So we must conclude there exists a small enough $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ we have:

$$\int_{\Sigma_i} y_{(i)}^{m_1} ... y_{(i)}^{m_k} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\omega} d y_{(i)}^{\overline{m-1}} = \sum_{l=0}^{N} \zeta^{a_{\overline{k}} \nu_{\overline{q}}}_{l(i)\mu_{\overline{p}}} \varepsilon^l + O(\varepsilon^{N+1}) \tag{4.5.67}$$

To prove the first k terms of the right hand side sum must be null, we can consider that:

$$|\int_{\Sigma_i} y_{(i)}^{m_1} ... y_{(i)}^{m_k} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\omega} d y_{(i)}^{\overline{m-1}}| \leq \int_{\Sigma_i \cap \Sigma_s^{-1}(supp(T_\varepsilon))} |y_{(i)}^{m_1} ... y_{(i)}^{m_k} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\omega} d y_{(i)}^{\overline{m-1}}| \leq \tag{4.5.68}$$

$$\leq O(\varepsilon^k) \int_{\Sigma_i} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\omega} d y_{(i)}^{\overline{m-1}} \leq O(\varepsilon^k) O(\varepsilon^0) O(\varepsilon^{-m+1}) \int_{\Sigma_i} d y_{(i)}^{m-1} \leq O(\varepsilon^k) \tag{4.5.69}$$

therefore the sum must start from $l = k$ and we have the thesis. □

Theorem 8: Let M be a manifold and $c : \mathbb{R} \hookrightarrow c$ a closed embedding. Let us denote by $\mathcal{A} = (U_i, \varphi_{(i)})$ an adapted atlas inducing an adapted local natural trivialisation of TM with the local frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i). Given a smooth global top form $\omega \in \Gamma\Lambda^m M$ let be $\Gamma_{0\,supp\,\omega} T^q_p M \subseteq \Gamma_0 T^p_q M$ the compact support tensor fields such that $supp(T) \subseteq supp(\omega)$ and let $\langle \Gamma_{0\,supp\,\omega} T^q_p M \rangle_\omega \subset \mathcal{J}^q_p(M)$ be the set of linear functionals induced by $\langle\ \rangle_\omega$. For each tensor field $T \in \Gamma_{0\,supp\,\omega} T^q_p M$ there exists a unique set of multipoles $\Psi_k \in \overset{(k)}{\Upsilon}{}^q_p(c)$ such that $\forall \phi \in \Gamma_0 T^p_q M$:

$$[\phi, \langle T_\epsilon \rangle_\omega] = \sum_{k=0}^{N} [\phi, \Psi_k] \varepsilon^k + O(\varepsilon^{N+1}) \tag{4.5.70}$$

Proof. Let us start by considering the adapted coordinate system $(U_i, \varphi^{\mu}_{(i)})$ and $\Omega_i = \varphi^{\mu}_{(i)}(U_{(i)})$. The local coordinate expression of ϕ on U_i

$$\phi = \phi_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\partial_{(i)\mu_{\overline{p}}} \otimes dx^{\nu_{\overline{q}}}_{(i)} = [\phi_{(i)} \circ \varphi^{-1}_{(i)} \circ \circ\varphi_{(i)}]^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}\partial_{(i)\mu_{\overline{p}}} \otimes dx^{\nu_{\overline{q}}}_{(i)} = \tag{4.5.71}$$

$$=\phi_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}(\varphi_{(i)})\partial_{(i)\mu_{\overline{p}}} \otimes dx^{\nu_{\overline{q}}}_{(i)} \tag{4.5.72}$$

The bunch of scalar functions $\hat{\phi}_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}} : \Omega_i \subset \mathbb{R}^{dim(M)} \to \mathbb{R}$ are the local coordinate expressions of the test tensor ϕ, so they are $C^{\infty}\mathbb{R}^m$ and thence they can be eventually expanded using the Taylor theorem. Let us consider an adapted chart in the adapted atlas $\mathcal{A}$ such that $U_i \cap c(\mathbb{R}) \neq 0$, thence from the definition of adapted chart we have that $\phi(\underline{U \cap c(\mathbb{R})}) = (s, 0^{\overline{m-1}})$ and each point $x \in U_i$ can be represented by $\phi_{(i)}(x) = (s(x), y^{\overline{m-1}}(x))$. We can then expand the local coordinate expression of ϕ using Taylor's theorem as:

$$\hat{\phi}_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}(s, y^{\overline{m}}) = \sum_{k=0}^{N} \frac{1}{k!} L_{a_{\overline{k}}}\hat{\phi}_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}(s, 0^{\overline{m-1}})y^{a_1}_{(i)}...y^{a_k}_{(i)} + \hat{A}^{\mu_{\overline{p}}}_{(i)a_{\overline{N+1}}\nu_{\overline{q}}}(s, y^{\overline{m}})y^{a_1}_{(i)}...y^{a_{N+1}}_{(i)} \tag{4.5.73}$$

Hence we can state:

$$[\phi, \langle T_{\epsilon}\rangle_{\omega}] = \sum_{U_i \in \mathcal{A}} \int_M \left(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} T^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\psi_i\right)\omega = \sum_{U_i \in \mathcal{A}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\psi}_i\hat{\omega}_{(i)} ds dy^{\overline{m-1}}_{(i)} \tag{4.5.74}$$

where $\hat{\omega}_{(i)}$, $\hat{\psi}_i$, $\hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}$ and $\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}$ are coordinate expressions of ω , ψ_i, $\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}$ and $T^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}$ respectively in the adapted coordinates $(U_i, \varphi_{(i)})$. So let us split the integral in two parts:

$$[\phi, \langle T_{\epsilon}\rangle_{\omega}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) = \varnothing}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\psi}_i\hat{\omega} ds dy^{\overline{m-1}}_{(i)} + \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\psi}_i\hat{\omega}_{(i)} ds dy^{\overline{m-1}}_{(i)} \tag{4.5.75}$$

Concerning the first term we have:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) = \varnothing}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\psi}_i\hat{\omega} ds dy^{\overline{m-1}}_{(i)} = \tag{4.5.76}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) = \varnothing}} \int_{\varphi_{(i)}(U_i \cap supp(T_\varepsilon))} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega} ds dy^{\overline{m-1}}_{(i)} = \tag{4.5.77}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) = \varnothing}} \int_{\varphi_{(i)}(supp(\psi_i) \cap supp(T_\varepsilon))} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega} ds dy^{\overline{m-1}}_{(i)} \tag{4.5.78}$$

As we stated in the previous lemma, if U_i is an open set such that $U_i \cap c(\mathbb{R}) = \varnothing$ and $supp(\psi_i) \subset U_i$ is the compact support of ψ_i, then for a small enough ε_0

$$\forall \varepsilon \in (0, \varepsilon_0) \Rightarrow supp(T_\varepsilon) \cap supp(\psi_i) = \varnothing \tag{4.5.79}$$

so the contribution of the first term must be identically null when $0 < \varepsilon < \varepsilon_0$.

Let us consider now just the second term:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega}_{(j)} ds dy^{\overline{m-1}}_{(i)} = \tag{4.5.80}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_i} \sum_{U_j \in \mathcal{A}} \hat{\psi}_j(s, 0^{\overline{m}}) \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega} ds dy^{\overline{m-1}}_{(j)} = \tag{4.5.81}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_i} \sum_{U_j \in \mathcal{A}} \hat{\psi}_j(s, 0^{\overline{m}}) \Big(\sum_{k=0}^{N} \frac{1}{k!} L_{a_{\overline{k}}} \hat{\phi}^{\;\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} + \tag{4.5.82}$$

$$+ \hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s, y^{\overline{m}}_{(j)}) y^{a_1}_{(j)} ... y^{a_{N+1}}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} = \tag{4.5.83}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{U_j \in \mathcal{A}} \int_{\Omega_i} \hat{\psi}_j(s, 0^{\overline{m}}) \Big(\sum_{k=0}^{N} \frac{1}{k!} L_{a_{\overline{k}}} \hat{\phi}^{\;\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} + \tag{4.5.84}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{U_j \in \mathcal{A}} \int_{\Omega_i} \hat{\psi}_j(s, 0^{\overline{m}}) \Big(\hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s, y^{\overline{m}}) y^{a_1}_{(j)} ... y^{a_{N+1}}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} = \tag{4.5.85}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{U_j \in \mathcal{A}} \int_{\Omega_i \cap \Omega_j} \hat{\psi}_j(s, 0^{\overline{m}}) \Big(\sum_{k=0}^{N} \frac{1}{k!} L_{a_{\overline{k}}} \hat{\phi}^{\;\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \, \hat{\psi}_i \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} + \tag{4.5.86}$$

$$+\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\sum_{U_j\in\mathcal{A}}\int_{\Omega_i\cap\Omega_j}\hat{\psi}_j(s,0^{\overline{m}})\Big(\hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s,y^{\overline{m}})y^{a_1}_{(j)}...y^{a_{N+1}}_{(j)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\psi}_i\hat{\omega}_{(j)}dsdy^{\overline{m}}_{(j)}= \tag{4.5.87}$$

$$=\sum_{U_i\in\mathcal{A}}\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{\Omega_j}\hat{\psi}_j(s,0^{\overline{m}})\Big(\sum_{k=0}^{N}\frac{1}{k!}L_{a_{\overline{k}}}\hat{\phi}_{(j)\nu_{\overline{q}}}^{\ \mu_{\overline{p}}}(s,0^{\overline{m-1}})y^{a_1}_{(j)}...y^{a_k}_{(j)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\psi}_i\hat{\omega}_{(j)}dsdy^{\overline{m}}_{(j)}+ \tag{4.5.88}$$

$$+\sum_{U_i\in\mathcal{A}}\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{\Omega_j}\hat{\psi}_j(s,0^{\overline{m}})\Big(\hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s,y^{\overline{m}})y^{a_1}_{(j)}...y^{a_{N+1}}_{(j)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\psi}_i\hat{\omega}_{(j)}dsdy^{\overline{m}}_{(j)} \tag{4.5.89}$$

When $\varepsilon\to 0$ since T_ε is a squeezed tensor we can say that the integrals

$$\sum_{U_i\in\mathcal{A}}\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{\Omega_j}\hat{\psi}_j(s,0^{\overline{m}})\Big(\hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s,y^{\overline{m}}_{(j)})y^{a_1}_{(j)}...y^{a_{N+1}}_{(j)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\psi}_i\hat{\omega}_{(j)}dsdy^{\overline{m}}_{(j)}=O(\varepsilon^{N+1}) \tag{4.5.90}$$

in fact when $\varepsilon\to 0$ we have:

$$\Big|\sum_{U_i\in\mathcal{A}}\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{\Omega_j}\hat{\psi}_j(s,0^{\overline{m}})\Big(\hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s,y^{\overline{m}}_{(j)})y^{a_1}_{(j)}...y^{a_{N+1}}_{(j)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\psi}_i\hat{\omega}_{(j)}dsdy^{\overline{m}}_{(j)}\Big|\leq \tag{4.5.91}$$

$$\leq\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{\Omega_i}\hat{\psi}_j(s,0^{\overline{m}})\Big|\Big(\hat{A}^{\mu_{\overline{p}}}_{(j)a_{\overline{N+1}}\nu_{\overline{q}}}(s,y^{\overline{m}}_{(j)})y^{a_1}_{(j)}...y^{a_{N+1}}_{(j)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\psi}_i\hat{\omega}_{(j)}dsdy^{\overline{m}}_{(j)}\Big|\leq \tag{4.5.92}$$

$$\leq O(\varepsilon^{N+1})O(1)O(\varepsilon^{-dim(M)+1})\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{\varphi_{(i)}(supp(T_\varepsilon)\cap U_j)}dsdy^{\overline{m-1}}_{(j)}\leq \tag{4.5.93}$$

$$\leq O(\varepsilon^{N+1})O(1)O(\varepsilon^{-dim(M)+1})O(\varepsilon^{dim(M)-1})\leq O(\varepsilon^{N+1})=O(\varepsilon^{N+1}) \tag{4.5.94}$$

Thence for $\varepsilon \to 0$ the integral behaves like:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\psi}_i \hat{\omega}_{(i)} ds dy^{\overline{m-1}}_{(i)} = \tag{4.5.95}$$

$$= \sum_{U_i \in \mathcal{A}} \sum_{\substack{U_j \in \mathcal{A} \\ U_j \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_j} \Big(\sum_{k=0}^{N} \frac{1}{k!} \hat{\psi}_j(s, 0^{\overline{m}}) L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\psi}_i \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} + O(\varepsilon^N + 1) = \tag{4.5.96}$$

$$= \sum_{\substack{U_j \in \mathcal{A} \\ U_j \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_j} \Big(\sum_{k=0}^{N} \frac{1}{k!} \hat{\psi}_j(s, 0^{\overline{m}}) L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} + O(\varepsilon^N + 1) \tag{4.5.97}$$

Since $(U_i \phi_{(i)})$ are charts adapted to the embedding c then they define a one parameter families of smooth transverse embeddings via

$$\Pi_{\mathbb{R}^{m-1}}[\varphi_{(i)}(U_i)] = \Sigma_{(i)} \qquad \hat{\Sigma}_{(i)s}(y^m) = \varphi^{-1}_{(i)}(s, y^m) \forall (s, y^m) \in \varphi_{(i)}(U_{(i)}) \tag{4.5.98}$$

where $\Pi_{\mathbb{R}^{m-1}}$ is just the natural projection of $\mathbb{R}^m$ on $\mathbb{R}^{m-1}$. So Ω_i can be written just as $I_i \times \Sigma_i$, then the integral can be separated in the following way:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_i} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}} \hat{\psi}_i \hat{\omega} ds dy^{\overline{m-1}}_{(i)} = \tag{4.5.99}$$

$$= \sum_{k=0}^{N} \sum_{\substack{U_j \in \mathcal{A} \\ U_j \cap c(\mathbb{R}) \neq \varnothing}} \int_{\Omega_j} \Big(\frac{1}{k!} \hat{\psi}_j(s, 0^{\overline{m}}) L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} + O(\varepsilon^{N+1}) = \tag{4.5.100}$$

$$= \sum_{k=0}^{N} \sum_{\substack{U_j \in \mathcal{A} \\ U_j \cap c(\mathbb{R}) \neq \varnothing}} \int_{I_j \times \Sigma_j} \Big(\frac{1}{k!} \hat{\psi}_j(s, 0^{\overline{m}}) L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) y^{a_1}_{(j)} ... y^{a_k}_{(j)} \Big) \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\omega}_{(j)} ds dy^{\overline{m}}_{(j)} + O(\varepsilon^{N+1}) = \tag{4.5.101}$$

$$= \sum_{k=0}^{N} \sum_{\substack{U_j \in \mathcal{A} \\ U_j \cap c(\mathbb{R}) \neq \varnothing}} \int_{I_j} \hat{\psi}_j(s, 0^{\overline{m}}) L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) \Big(\frac{1}{k!} \int_{\Sigma_j} y^{a_1}_{(j)} ... y^{a_k}_{(j)} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\omega}_{(j)} dy^{\overline{m}}_{(j)} \Big) ds + O(\varepsilon^{N+1}) \tag{4.5.102}$$

Using the previous lemma we can say that

$$\int_{\Sigma_j} y^{a_1}_{(j)}...y^{a_k}_{(j)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\omega}_{(j)}dy^{\overline{m}}_{(j)} = \sum_{l=k}^{N}\varepsilon^l\alpha^{a_{\overline{k}}\nu q}_{l(j)\mu_{\overline{p}}}(s) + O(\varepsilon^{N+1}) \tag{4.5.103}$$

then using the definition of squeezed tensor we are allowed to expand the inner integral as follow:

$$\sum_{k=0}^{N}\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\int_{I_j}\hat{\psi}_j(s,0^{\overline{m}})L_{a_{\overline{k}}}\hat{\phi}^{\ \ \mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s,0^{\overline{m-1}})\sum_{l=k}^{N}\varepsilon^l\alpha^{a_{\overline{k}}\nu q}_{lj\mu_{\overline{p}}}(s)ds + O(\varepsilon^{N+1}) = \tag{4.5.104}$$

Now using the trick $\sum_{k=1}^{N}\sum_{l=k}^{N}a_{k,l} = \sum_{l=1}^{N}\sum_{k=0}^{l}a_{k,l}$ we can rewrite the integral as:

$$\sum_{l=0}^{N}\varepsilon^l\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\sum_{k=0}^{l}\int_{\mathbb{R}}c^\star\Big(L_{a_{\overline{k}}}\phi^{\ \ \mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}\Big)c^\star(\psi_j)\alpha^{a_{\overline{k}}\nu q}_{l(j)\mu_{\overline{p}}}(s)ds + O(\varepsilon^{N+1}) = \tag{4.5.105}$$

This is very close to what we want to achieve but we should still prove that these coefficients are actually able to define a multipole because nothing guarantees that they define a smooth global top form over $\mathbb{R}$. However this is not a problem because we are going to see that this expression is equal to a sum of multipoles with support on each U_j so it must be a multipole as well. Consider the expression:

$$\int_{\Omega_i}\hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\ \hat{\psi}_i\hat{\omega}_{(i)}dsdy^{\overline{m-1}}_{(i)} = \tag{4.5.106}$$

$$= \int_{\Omega_i}\Big(\sum_{k=0}^{N}\frac{1}{k!}L_{a_{\overline{k}}}\hat{\phi}^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}(s,0^{\overline{m-1}})y^{a_1}_{(i)}...y^{a_k}_{(i)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\ \hat{\psi}_i\hat{\omega}_{(i)}dsdy^{\overline{m}}_{(i)}+ \tag{4.5.107}$$

$$+\int_{\Omega_i}\Big(\hat{A}^{\mu_{\overline{p}}}_{(i)a_{\overline{N+1}}\nu_{\overline{q}}}(s,y^{\overline{m}})y^{a_1}_{(i)}...y^{a_{N+1}}_{(i)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\ \hat{\psi}_i\hat{\omega}_{(i)}dsdy^{\overline{m}}_{(i)} = \tag{4.5.108}$$

$$= \int_{\Omega_i}\Big(\sum_{k=0}^{N}\frac{1}{k!}L_{a_{\overline{k}}}\hat{\phi}^{\ \ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}(s,0^{\overline{m-1}})y^{a_1}_{(i)}...y^{a_k}_{(i)}\Big)\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(i)\mu_{\overline{p}}}\ \hat{\psi}_i\hat{\omega}_{(i)}dsdy^{\overline{m}}_{(i)} + O(\varepsilon^{N+1}) \tag{4.5.109}$$

So with the very same procedure used before we obtain:

$$\int_{\Omega_j} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\psi}_j \hat{\omega}_{(j)} ds dy^{\overline{m-1}}_{(j)} = \tag{4.5.110}$$

$$= \sum_{k=0}^{N} \int_{I_j} L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) \Big(\frac{1}{k!} \int_{\Sigma_j} y^{a_1}_{(j)} ... y^{a_k}_{(j)} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\psi}_j \hat{\omega}_{(j)} dy^{\overline{m}}_{(j)} \Big) ds + O(\varepsilon^{N+1}) \tag{4.5.111}$$

Now considering that ψ_j is a smooth compact support scalar field, we are allowed to expand the inner integral as done before obtaining:

$$\int_{\Omega_j} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\psi}_j \hat{\omega}_{(j)} ds dy^{\overline{m-1}}_{(j)} = \tag{4.5.112}$$

$$= \sum_{k=0}^{N} \int_{I_j} L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) \sum_{l=0}^{N} \varepsilon^l \beta^{a_{\overline{k}}\nu q}_{lj\mu_{\overline{p}}}(s) ds + O(\varepsilon^N + 1) = \tag{4.5.113}$$

$$= \sum_{l=0}^{N} \varepsilon^l \sum_{k=0}^{N} \int_{I_j} L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) \beta^{a_{\overline{k}}\nu q}_{lj\mu_{\overline{p}}}(s) ds + O(\varepsilon^N + 1) \tag{4.5.114}$$

But from the definition, $supp(\beta^{a_{\overline{k}}\nu q}_{lj\mu_{\overline{p}}}(s)) \subseteq I_j$ therefore defining:

$$\gamma^{a_{\overline{k}}\nu q}_{lij\mu_{\overline{p}}}(s) = \begin{cases} \beta^{a_{\overline{k}}\nu q}_{lj\mu_{\overline{p}}}(s) & , \quad i = j \\ 0 & , \quad i \neq j \end{cases} \tag{4.5.115}$$

we have that

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \gamma^{a_{\overline{k}}\nu q}_{lij\mu_{\overline{p}}}(s) L_{a_{\overline{k}}} \hat{\phi}^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}(s, 0^{\overline{m-1}}) = c^{\star} \Big(\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \psi_i \sum_{k=0}^{N} \gamma^{a_{\overline{k}}\nu q}_{lij\mu_{\overline{p}}}(s) L_{a_{\overline{k}}} \phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) \tag{4.5.116}$$

define a global smooth top form over $\mathbb{R}$, and the integral can be written as:

$$\int_{\Omega_j} \hat{\phi}^{\mu_{\overline{p}}}_{(j)\nu_{\overline{q}}} \hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}} \hat{\psi}_j \hat{\omega}_{(j)} ds dy^{\overline{m-1}}_{(j)} = \tag{4.5.117}$$

$$=\sum_{l=0}^{N}\varepsilon^l\sum_{k=0}^{N}\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}}c^\star\big(L_{a_{\overline{k}}}\phi_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\big)\gamma_{lij\mu_{\overline{p}}}^{a_{\overline{k}}\nu q}(s)ds+O(\varepsilon^N+1)=\sum_{l=0}^{N}\varepsilon^l[\phi,\Psi_{jl}]+O(\varepsilon^N+1) \tag{4.5.118}$$

So at the end for $\varepsilon\to 0$ we have:

$$\sum_{l=0}^{N}\varepsilon^l\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}\sum_{k=0}^{l}\int_{\mathbb{R}}c^\star\Big(L_{a_{\overline{k}}}\phi_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_j)\alpha_{l(j)\mu_{\overline{p}}}^{a_{\overline{k}}\nu q}ds+O(\varepsilon^{N+1})= \tag{4.5.119}$$

$$=\sum_{l=0}^{N}\varepsilon^l\sum_{\substack{U_j\in\mathcal{A}\\ U_j\cap c(\mathbb{R})\neq\varnothing}}[\phi,\Psi_{jl}]+O(\varepsilon^{N+1})=\sum_{l=0}^{N}\varepsilon^l[\phi,\Psi_l]+O(\varepsilon^{N+1}) \tag{4.5.120}$$

□

Corollary 15: The adapted Ellis moments are defined inductively by :

$$\alpha_{n(j)\mu_{\overline{p}}}^{a_{\overline{k}}\nu_{\overline{q}}}=\lim_{\varepsilon\to 0}\left(\varepsilon^{-n}\int_{\Sigma_j}y_{(j)}^{a_1}...y_{(j)}^{a_k}\hat{T}_{\varepsilon(j)\mu_{\overline{p}}}^{\nu_{\overline{q}}}\hat{\omega}_{(j)}dy_{(j)}^{\overline{m}}\quad-\sum_{l=0}^{n-1}\varepsilon^{l-n}\alpha_{l(j)\mu_{\overline{p}}}^{a_{\overline{k}}\nu_{\overline{q}}}\right)\quad,\quad n\geq k \tag{4.5.121}$$

$$\alpha_{n(j)\mu_{\overline{p}}}^{a_{\overline{k}}\nu_{\overline{q}}}=0\quad,\quad n<k \tag{4.5.122}$$

Proof. It follows directly from the previous theorem when we evaluated

$$\int_{\Sigma_j}y_{(j)}^{a_1}...y_{(j)}^{a_k}\hat{T}_{\varepsilon(j)\mu_{\overline{p}}}^{\nu_{\overline{q}}}\hat{\omega}_{(j)}dy_{(j)}^{\overline{m}}=\sum_{l=k}^{N}\varepsilon^l\alpha_{l(j)\mu_{\overline{p}}}^{a_{\overline{k}}\nu q}(s)ds+O(\varepsilon^{N+1}) \tag{4.5.123}$$

Considering that

$$\int_{\Sigma_j}y_{(j)}^{a_1}...y_{(j)}^{a_k}\hat{T}_{\varepsilon(j)\mu_{\overline{p}}}^{\nu_{\overline{q}}}\hat{\omega}_{(j)}dy_{(j)}^{\overline{m}}=O(\varepsilon^k) \tag{4.5.124}$$

splitting the sum, dividing by ε^n and taking the limit, we have the thesis. □

Corollary 16: The sum in the previous expression can be reduced to:

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n(j)\mu_{\overline{p}}} = \lim_{\varepsilon\to 0}\left(\varepsilon^{-n}\int_{\Sigma_j} y^{a_1}_{(j)}...y^{a_k}_{(j)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\omega}_{(j)}dy^{\overline{m}}_{(j)} \quad -\sum_{l=k}^{n-1}\varepsilon^{l-n}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{l(j)\mu_{\overline{p}}}\right) \quad , \quad n\geq k \tag{4.5.125}$$

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n(j)\mu_{\overline{p}}} = 0 \quad , \quad n<k \tag{4.5.126}$$

Proof. It follows trivially from the previous statement neglecting the null terms □

Property 44: Considering subjective preferences it is possible to re-scale the previous equation with a $k!$ combinatory factor defining the same moments re-scaled by a combinatory factor due to the symmetry of the higher order Lie derivations. This is just a matter of preference in the definitions and nothing affects the theory of the Ellis moments.

Property 45: Let us stress that the list of multipoles Ψ_l strongly depends on the choice of the top form ω on M

4.5.3 Transverse self-similar squeezing of a compact support tensor field

We are going to introduce now a very specific case for which one can prove how the adapted Ellis moments of a particular class of squeezed tensor fields coincide exactly with the standard definition of moments given in statistics and in physics (moments of continuous mass or charge densities). This very specific case allow us to interpret the linear functionals defined in this work as a generalisation to the manifold of the cartesian multipole moments of densities defined on $\mathbb{R}^n$

Definition 76: Let M be a manifold, $c \hookrightarrow M$ a closed embedding and $\mathcal{A} = (U_i, \varphi_{(i)})$ an adapted atlas. Let ω be a global top form. Let be $T \in \Gamma_0 T^q_p M$ a compact support tensor field on M such that $supp(T) \subset \bigcup\{U_i \in \mathcal{A} \mid U_i \cap c(\mathbb{R}) \neq \varnothing\}$. Let $V \in M$ be an open set satisfying $supp(T) \subset V$ and $\Phi : (0,1)\times V \to M$ a one parameter family of local diffeomorphism such that its adapted coordinate expression is given by:

$$\Phi_\varepsilon = \begin{cases} \tilde{s} = s \\ \tilde{y}^a_{(i)} = \frac{1}{\varepsilon} y^a_{(i)} \end{cases} \tag{4.5.127}$$

We define ***a transverse self similar squeezing of*** T ***upon*** c ***with respect to*** Φ the squeezed tensor fields $T_\varepsilon \in S^q_p(c)$ satisfying:

$$T_\varepsilon = \varepsilon^{-m}\Phi^\star_\varepsilon(T) \tag{4.5.128}$$

Property 46: It is easy to check using the adapted atlas that this one family of smooth tensor fields satisfies all the requirements to be a squeezed tensor field.

We are going to show how at least some compact support tensors fields admit a well defined self similar squeezing, so at least a case of self similar squeezing exists. However the study of the necessary condition that must be satified by a compact support tensor field to admit a well defined self similar squeezing is still a matter of investigation.

Lemma 37: Let M be a manifold, $c \hookrightarrow M$ a closed embedding and $\mathcal{A} = (U_i, \varphi_{(i)})$ an adapted atlas. Let ω be a global top form. Let be $T \in \Gamma_0 T^q_p M$ a compact support tensor field on M such that $supp(T) \subset \bigcup\{U_i \in \mathcal{A} \mid U_i \cap c(\mathbb{R}) \neq \varnothing\}$. Let $V \in M$ be an open set satisfying $supp(T) \subset V$ and $\Phi : (0,1) \times V \to M$ a one parameter family of local diffeomorphism such that its adapted coordinate expression is given by:

$$\Phi_\varepsilon = \begin{cases} \tilde{s} = s \\ \tilde{y}^a_{(i)} = \frac{1}{\varepsilon} y^a_{(i)} \end{cases} \tag{4.5.129}$$

Let be T a compact support tensor field and T_ε its self similar squeezing build up on Φ_ε, then the adapted Ellis moments related to the multipoles expansion:

$$[\phi, \langle T_\varepsilon \rangle_\omega] = \sum_{l=0}^{N} \varepsilon^l [\phi, \Psi_l] + O(\varepsilon^{N+1}) \tag{4.5.130}$$

can be easily computed as follow:

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n(j)\mu_{\overline{p}}} = \int_{\Sigma_j} \tilde{y}^{a_1}_{(j)} ... \tilde{y}^{a_k}_{(j)} \tilde{T}^{\nu_{\overline{q}}}_{(j)\mu_{\overline{p}}} \tilde{\omega}_{(j)} d\tilde{y}^{\overline{m}}_{(j)} \quad , \quad n = k \tag{4.5.131}$$

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n(j)\mu_{\overline{p}}} = 0 \quad , \quad n \neq k \tag{4.5.132}$$

where $\tilde{\omega}_{(j)}$ and $\tilde{T}^{\nu_{\overline{q}}}_{(j)\mu_{\overline{p}}}$ are the coordinate expressions in the new chart $(s, \tilde{y}^{\overline{m}})$ of T and ω respectively.

Proof. Let us consider the previous corollary fixing $n < k$ then we have that the moments must be null. In the other hand, setting $n = k$ we have:

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k(j)\mu_{\overline{p}}} = \lim_{\varepsilon\to 0}\left(\varepsilon^{-k}\int_{\Sigma_j} y^{a_1}_{(j)}...y^{a_k}_{(j)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\omega}_{(j)}dy^{\overline{m}}_{(j)} \quad - \sum_{l=k}^{k-1}\varepsilon^{l-k}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{l(j)\mu_{\overline{p}}}\right) \tag{4.5.133}$$

So we can state:

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k(j)\mu_{\overline{p}}} = \lim_{\varepsilon\to 0}\int_{\Sigma_j}\varepsilon^{-k}y^{a_1}_{(j)}...y^{a_k}_{(j)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\omega}_{(j)}dy^{\overline{m}}_{(j)} \tag{4.5.134}$$

If we consider the given diffeomorphism defining the squeezed tensor field, can be interpreted in a passive way simply as a change of local chart, so considering the transformation rules, we can easily reduce the calculation in adapted coordinates of the previous integral simply to:

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k(j)\mu_{\overline{p}}} = \lim_{\varepsilon\to 0}\int_{\Sigma_j}\varepsilon^{-k}y^{a_1}_{(j)}...y^{a_k}_{(j)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\omega}_{(j)}dy^{\overline{m}}_{(j)} = \int_{\Sigma_j}\tilde{y}^{a_1}_{(j)}...\tilde{y}^{a_k}_{(j)}\tilde{T}^{\nu_{\overline{q}}}_{(j)\mu_{\overline{p}}}\tilde{\omega}_{(j)}d\tilde{y}^{\overline{m}}_{(j)} \tag{4.5.135}$$

where $\tilde{\omega}_{(j)}$ and $\tilde{T}^{\nu_{\overline{q}}}_{(j)\mu_{\overline{p}}}$ are the coordinate expressions in the new chart $(s, \tilde{y}^{\overline{m}})$ of T and ω respectively. Now considering the case such that $n \geq k$ we have

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n(j)\mu_{\overline{p}}} = \lim_{\varepsilon\to 0}\left(\varepsilon^{-n+k-k}\int_{\Sigma_j} y^{a_1}_{(j)}...y^{a_k}_{(j)}\hat{T}^{\nu_{\overline{q}}}_{\varepsilon(j)\mu_{\overline{p}}}\hat{\omega}_{(j)}dy^{\overline{m}}_{(j)} \quad - \sum_{l=k}^{n-1}\varepsilon^{l-k}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{l(j)\mu_{\overline{p}}}\right) = \tag{4.5.136}$$

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{-n+k}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k(j)\mu_{\overline{p}}} - \sum_{l=k}^{n-1}\varepsilon^{l-k}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{l(j)\mu_{\overline{p}}}\right) \tag{4.5.137}$$

$$\tag{4.5.138}$$

Now using induction on n starting from $n = k+1$ we have :

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k+1(j)\mu_{\overline{p}}} = \lim_{\varepsilon\to 0}\left(\varepsilon^{-1}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k(j)\mu_{\overline{p}}} - \sum_{l=k}^{k}\varepsilon^{k-k-1}\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{l(j)\mu_{\overline{p}}}\right) = 0 \tag{4.5.139}$$

and supposing true the strong inductive step $\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n(j)\mu_{\overline{p}}} = 0$ for each $n > k$ we can easily prove:

$$\alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{n+1(j)\mu_{\overline{p}}} = \lim_{\varepsilon \to 0} \left(\varepsilon^{-n-1+k} \alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{k(j)\mu_{\overline{p}}} - \sum_{l=k}^{n} \varepsilon^{l-n-1} \alpha^{a_{\overline{k}}\nu_{\overline{q}}}_{l(j)\mu_{\overline{p}}} \right) = 0 \qquad (4.5.140)$$

□

Considering this result we can see how, given a compact support tensor field T, the standard calculation of the usual moments very well known in several branches of mathematics, physics and statistics, is just a very specific case of calculation of the Ellis moments of a multipole, in adapted chart, approximating a transverse self similar squeezing of the tensor field T upon a "worldline-like" curve c. These considerations include naturally the well known particular cases in which standard moments of functions $f : \mathbb{R}^n \to \mathbb{R}$ are considered as instance when the moments of a regular probability distributions are computed in Statistics or the inertia moments are calculated for a rigid body in special relativity or standard classical mechanics.

4.6 Final considerations on the Ellis local representations

We have deeply analysed how the Ellis definition of the multipoles induces naturally an explicit expression for the multipoles in terms of $C^\infty(\mathbb{R})$-linear combinations (in terms of restrictions on $c(\mathbb{R})$ of local expressions) related to the higher order Lie derivatives of the test tensor fields, then appropriately integrated on the whole worldline. We have seen how the most general Ellis representation does not own enough structure to guarantee the uniqueness of the representation or translating the module structure of the multipoles on the local parametrization. It is possible to fix some constraints on the admissible Ellis representations killing the redundancy of the parametrization but more structures (i.e. adapted local frames or charts) must be introduced. One should be very careful to fix the constraints, because a lot of them are not compatible with the covariance. Although this could seem a minor problem, in practice this would prevent us from mapping the intrinsic multipole equations into unique equations for the parameters, valid for each local charts chosen on the manifold. If we are working within a fully relativistic framework, from a physical point of view this is not acceptable, in fact the physical laws must be expressed independently from the observers, therefore the physical equations exhibit a symmetry under local charts transformations (local diffeomorphisms). The most usual way of expressing the multipole is (called here the top order local Ellis representation) seems to be not so good because it is not isomorphic to the multipole module $\overset{(k)}{\Upsilon}{}^q_p(c) | k \in \mathbb{N}$ nor does it map the multipole equation into a covariant set of equations for the parameters. An alternative representation called ***transverse local Ellis representation*** has been proposed. This specific Ellis representation derived directly from the adapted local Ellis representation is an attempt to work around the problem of fixing an unique representation preserving at the same time the module structure and the covariance of the equations like $\mathcal{T} = 0$ for each $\mathcal{T} \in \Upsilon^q_p(c)$. This is very useful because given an arbitrary map $Eq : \Upsilon^q_p(c) \to \Upsilon^{q'}_{p'}(c)$ the equation in the form $Eq(\mathcal{T}) = 0$ can be mapped into an equation on the transverse local Ellis parameters called ***transverse local Ellis moments*** such that all the local coordinates transformations are symmetries for the equation. The process needed to express explicitly a multipole using the transverse Ellis local representation is in general very complicated and can be practically performed just for very low orders, however nothing forbids the using of the computer to make the calculations for the higher order multipoles. Since the transverse Ellis local representation is fixed by choosing arbitrarily an extra structure (i.e. the adapted local frames) we investigated how this representation is affected if another structure is chosen (i.e a different adapted local frame) for the lower order cases, but in principle with enough computational power it can be done for each order $k \in \mathbb{N}$. Other alternative approaches are definitely possible. For instance, following the variational Lagrangian formalism on the fiber bundles, one could interpret the multipoles as a set of different actions taking sections of appropriate jet bundles built upon $c(\mathbb{R})$ [43]. Doing so it would be possible to interpret the degree of freedom in the choice of the adapted frame and the choices of local coordinate system as an actual gauge. In this perspective the theory of multipoles could be simply recasted in a Gauge-Natural theory built upon an appropriate bundle, where the naturality is expressed by the invariance of the multipoles (interpreted as a set of actions) under local diffeomorphism or equivalently under local chart transformation and the gauge- symmetry is given by the degree of freedom in the choice of the "transverse directions" with respect to the sub-manifold $c(\mathbb{R})$ embedded into the manifold M. If the multipole can be interpreted as an action then the 1-form inside the integral can be interpreted as a Lagrangian by following the same analogy. The way in which the local expression of the Lagrangian can be glued together defining a global Lagrangian should be able to fix the explicit form of the structure group and the compatible trivialisation of this particular bundle. This approach still needs to be formalised and it is an actual matter of investigation. We are looking forward to express the multipole theory with a single appropriate bundle structure on $c(\mathbb{R})$ encoding naturally the symmetries in the multipoles Ellis representation allowing us to interpret the non uniqueness of it as a natural gauge theory as well as other physical theories. Although the Ellis representation is quite tricky and treacherous it is very useful to formalise the concept of asymptotic expansions for specific one parameter families of smooth compactly supported tensor fields, called in order "squeezed tensor fields". In this case, the $\mathbb{R}$-linear functionals defined with the generalisation of the De Rham push-forward plays a fundamental role inside the coordinate-free weak asymptotic expansion of the families about a neighbourhood of the null value of the parameter. Considering this, it is clear now how the usual moments of the local coordinate expressions of the compactly supported tensor fields (or scalar fields equivalently) admits a clear geometrical interpretation, in fact they are just the adapted Ellis moments of the linear functionals related to the weak asymptotic expansion of the self similar squeezing of the given tensor field (or scalar fields equivalently). This closes the circle then, motivating the choice of the name "multipoles" for such a family of $\mathbb{R}$-linear functionals.

Chapter 5

Concerning the Dixon Local Representation

In this chapter we are going to analyze in detail the characteristics of the Dixon local representation of the multipoles. Despite the existence of several very detailed studies [1][2][3][4] [7] and physical interpretations [5][6][8][9][10][11][12] [15][18][23] [24][25][26] [27][28][29] [30] of this representation, we are going here to propose another different way to interpret the Dixon representation of the multipoles. This interpretation is purely geometrical and does not involve any physical consideration. First of all we are well aware that the multipoles are well defined primitive intrinsic geometrical objects, completely independent from structures like coordinates, metric, Killing vectors, connection, ADM fibration and so on. Considering this, we know that we are going to investigate just a set of possible representations for the multipoles induced by a connection on the manifold. We are going to see how the Dixon representation of the multipoles does not require necessarily all the structures and constraints used by Dixon in his work [1][2][3] to be defined (as already partially noted in [7]), but some constraints are required to fix a one to one relationship between the local representations. Other more restrictive constraints imposed by Dixon, are related to particular physical considerations holding just within General Relativity, and we will see that these constraints do not affect neither the mathematical definition of the multipoles, nor eventually the uniqueness fixed by coordinate free, model independent constraints. Clearly, separating the model-dependent "constraints" with respect the "unique-representation" constraints, we are able to represent the multipoles in a nice useful way without being forced to assume any model constraining the geometry or the existence of particular symmetries upon the considered manifold. In the very first beginning of this chapter we are going to analyse an example enlightening the behaviour of the Dixon representation, then in the later parts we are going to fix some restrictions to express the multipoles in a more convenient way accordingly to the usual Dixon one. However some issues still remains not solved and some aspect are actual matter of research and investigation.

5.1 Problems arising from the Dixon representation

As well as the Ellis representation, the "Dixon" one is affected by some issues that can became problematic when trying to express uniquely the multipoles. However, in contrast with the Ellis representation, we will see how the parameters naturally admits a covariant interpretation thanks to the affine structure fixed by the connection.

5.1.1 A specific trivial example

Let us consider $\mathbb{R}^2$ as a differential manifold on itself. $\mathbb{R}^2$ always admits a global atlas where the points of $\mathbb{R}^2$ are mapped into itself due to the identity functions. Let us denote by (x^0, x^1) the coordinate expression of an arbitrary point $x \in \mathbb{R}^2$. Now let us consider a closed embedding $c : \mathbb{R} \to \mathbb{R}^2$ defined by $c(t) = (t, 0)$, $\forall t \in \mathbb{R}$. Since $\mathbb{R}^2$ is a manifold, we can build the tangent bundle $T\mathbb{R}^2$ the cotangent bundle $T^\star\mathbb{R}^2$ as well as the tangent tensor bundle $T^p_q\mathbb{R}^2$. A global natural trivialisation of TM can be fixed by $(e_0 = \frac{\partial}{\partial x^0}, e_1 = \frac{\partial}{\partial x^1})$ and this induces a global trivialisation of $T^\star M$ and $T^p_q M$. Let us consider as instance the multipole $\mathcal{T} \in \overset{(0)}{\Upsilon}{}^1_1(c)$ defined by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{5.1.1}$$

where $\alpha^\nu_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0, 1]$ are smooth scalar fields. Therefore:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = \int_{\mathbb{R}} \{c^\star(\phi^0_0)\alpha^0_0 + c^\star(\phi^1_0)\alpha^0_1 + c^\star(\phi^0_1)\alpha^1_0 + c^\star(\phi^1_1)\alpha^1_1\}dt \tag{5.1.2}$$

But this is not the only way to express the same distribution with the Dixon representation. Let us consider:

$$[\phi, \mathcal{S}] = \int_{\mathbb{R}} \Big\{c^\star(\nabla_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu\Big\}dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{5.1.3}$$

where $\beta^\nu_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0, 1]$ and $\beta^{\lambda\nu}_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0, 1]$ are smooth scalar fields satisfying:

$$\begin{cases} \beta^{1\nu}_\mu = 0 \\ -\frac{d}{dt}[\beta^{0\nu}_\mu] + c^\star(\Gamma^\sigma_{0\mu})\beta^{0\nu}_\sigma - c^\star(\Gamma^\nu_{0\sigma})\beta^{0\sigma}_\mu + \beta^\nu_\mu - \alpha^\mu_\nu = 0 \end{cases} \tag{5.1.4}$$

Substituting it in the integral we obtain:

$$[\phi, \mathcal{S}] = \int_{\mathbb{R}} \Big\{ c^\star(\nabla_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{5.1.5}$$

$$= \int_{\mathbb{R}} \Big\{ c^\star(\nabla_0(\phi)^\mu_\nu)\beta^{0\nu}_\mu + c^\star(\nabla_1(\phi)^\mu_\nu)\beta^{1\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{5.1.6}$$

$$= \int_{\mathbb{R}} \Big\{ \frac{d}{dt}[c^\star(\phi^\mu_\nu)]\beta^{0\nu}_\mu + c^\star(\Gamma^\sigma_{0\mu}\phi^\mu_\nu)\beta^{0\nu}_\sigma - c^\star(\Gamma^\nu_{0\sigma}\phi^\mu_\nu)\beta^{0\sigma}_\mu + c^\star(L_1(\phi)^\mu_\nu)\beta^{1\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{5.1.7}$$

$$= \int_{\mathbb{R}} \Big\{ \frac{d}{dt}[c^\star(\phi^\mu_\nu)]\beta^{0\nu}_\mu + c^\star(\Gamma^\sigma_{0\mu}\phi^\mu_\nu)\beta^{0\nu}_\sigma - c^\star(\Gamma^\nu_{0\sigma}\phi^\mu_\nu)\beta^{0\sigma}_\mu \Big\} dt + \int_{\mathbb{R}} 0 dt + \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\beta^\nu_\mu dt = \tag{5.1.8}$$

$$= \int_{\mathbb{R}} \Big\{ - \frac{d}{dt}[\beta^{0\nu}_\mu] + c^\star(\Gamma^\sigma_{0\mu})\beta^{0\nu}_\sigma - c^\star(\Gamma^\nu_{0\sigma})\beta^{0\sigma}_\mu + \beta^\nu_\mu c^\star \Big\} (\phi^\mu_\nu) dt = \tag{5.1.9}$$

$$= \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\mu_\nu dt = [\phi, \mathcal{T}] \tag{5.1.10}$$

Hence it is clear that even in this very trivial example the Dixon representation of a multipole is not unique and several different sets of parameters can define the same distribution. This show explicitly why we prefer to denote α^μ_ν or $(\beta^{\lambda\nu}_\mu, \beta^\mu_\nu)$ just as "parameters" rather than "components". They completely define the multipoles but not in a unique $C^\infty(\mathbb{R})$ linearly independent way. The Dixon representation does not behave as badly as the Ellis one, when a change of coordinates is performed. Let us suppose to have another global atlas defining a new coordinate system (x'^0, x'^1) linked to the old ones with:

$$\begin{cases} x'^0 = x'^0(x^0, x^1) \\ x'^1 = x'^1(x^0, x^1) \end{cases} \tag{5.1.11}$$

This automatically induces a new trivialisation of $T\mathbb{R}^2$, $T^\star\mathbb{R}^2$ and $T^p_q\mathbb{R}^2$ therefore a different set of Dixon parameters. Let us show it. Changing the coordinates on M, we induce another global natural trivialisation of TM fixed by (e'_0, e'_1) satisfying

$$\begin{cases} e'_0 = \frac{\partial}{\partial x'^0} = \overline{J}^\mu_0 e_\mu \\ e'_1 = \frac{\partial}{\partial x'^1} = \overline{J}^\mu_1 e_\mu \end{cases} \tag{5.1.12}$$

and this induces a new global trivialisation of $T^\star M$ and $T^p_q M$. If we consider the first case, the multipole $\mathcal{T}$ is expressed by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt \tag{5.1.13}$$

using the old trivialisation and by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi'^\mu_\nu)\alpha'^\nu_\mu dt \tag{5.1.14}$$

using the new one. By definition of the Dixon representation $c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt$ must be a global smooth top form over $\mathbb{R}$ independently from the chosen trivialisation, hence, $c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = c^\star(\phi'^\mu_\nu)\alpha'^\nu_\mu dt$. This fixes a constraint on the transformation rules for the local Dixon parameters:

$$c^\star(\phi'^\alpha_\beta)\alpha'^\beta_\alpha dt = c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = c^\star(\phi(e^\mu, e_\nu))\alpha^\nu_\mu dt = c^\star(\phi(J^\mu_\alpha e'^\alpha, \overline{J}^\beta_\nu e'_\beta)\alpha^\nu_\mu dt = \tag{5.1.15}$$

$$= c^\star(\phi(e'^\alpha, e'_\beta))c^\star(J^\mu_\alpha)c^\star(\overline{J}^\beta_\nu \alpha^\nu_\mu)dt \tag{5.1.16}$$

concluding that:

$$\alpha'^\beta_\alpha = c^\star(\overline{J}^\mu_\alpha)c^\star(J^\beta_\nu)\alpha^\nu_\mu \tag{5.1.17}$$

and:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi^\mu_\nu)\alpha^\nu_\mu dt = \int_{\mathbb{R}} c^\star(\phi'^\mu_\nu)c^\star(\overline{J}^\mu_\alpha)c^\star(J^\beta_\nu)\alpha^\nu_\mu dt = \int_{\mathbb{R}} c^\star(\phi'^\mu_\nu)\alpha'^\nu_\mu dt \tag{5.1.18}$$

If we consider the second case, the same multipole $\mathcal{T}$ can be expressed also by another Dixon representation:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} \Big\{ c^\star(L_\lambda(\phi)^\mu_\nu)\beta^{\lambda\nu}_\mu + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1 \mathbb{R}^2 \tag{5.1.19}$$

where $\beta_\mu^\nu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ and $\beta_\mu^{\lambda\nu} : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ are smooth scalar fields satisfying:

$$\begin{cases} \beta_\mu^{1\nu} = 0 \\ -\frac{d}{dt}[\beta_\mu^{0\nu}] + \beta_\mu^\nu - \alpha_\nu^\mu = 0 \end{cases} \tag{5.1.20}$$

using the old trivialisation and by:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} \Big\{ c^\star(\nabla'_\lambda(\phi)'^\mu_\nu) \beta'^{\lambda\nu}_{\ \mu} + c^\star(\phi'^\mu_\nu)\beta'^{\nu}_{\ \mu} \Big\} dt \quad , \quad \forall \phi \in \Gamma_0 T^1_1 \mathbb{R}^2 \tag{5.1.21}$$

where $\beta'^\nu_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ and $\beta'^{\lambda\nu}_\mu : \mathbb{R}^2 \to \mathbb{R}, \forall \mu, \nu \in [0,1]$ are smooth scalar fields satisfying:

$$\begin{cases} \beta_\mu^{1\nu}(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu) = 0 \\ -\frac{d}{dt}[\beta_\mu^{0\nu}(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu)] + c^\star(\Gamma^\sigma_{0\mu})\beta_\sigma^{0\nu}(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu) - c^\star(\Gamma^\nu_{0\sigma})\beta_\mu^{0\sigma}(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu) + \beta_\mu^\nu(\beta'^{\lambda\nu}_\mu, \beta'^\mu_\nu) - \alpha_\nu^\mu(\alpha_\nu^\mu) = 0 \end{cases} \tag{5.1.22}$$

using the new one. We will explore the constraints after have determined the transformation rules for this second representation. By definition of the Dixon representation $\Big\{ c^\star(\nabla_\lambda(\phi)^\mu_\nu)\beta_\mu^{\lambda\nu} + c^\star(\phi'^\mu_\nu)\beta'^{\nu}_{\ \mu} \Big\} dt$ must be a global smooth top form over $\mathbb{R}$ independently from the chosen trivialisation, hence:

$$\Big\{ c^\star(\nabla'_\lambda(\phi)'^\mu_\nu) \beta'^{\lambda\nu}_{\ \mu} + c^\star(\phi'^\mu_\nu)\beta'^{\nu}_{\ \mu} \Big\} dt = \Big\{ c^\star(\nabla_\lambda(\phi)^\mu_\nu)\beta_\mu^{\lambda\nu} + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt \tag{5.1.23}$$

must be satisfied. This automatically fixes the form for the transition functions of the local Dixon parameters:

$$\Big\{ c^\star(\nabla'_\gamma(\phi)'^\alpha_\beta) \beta'^{\gamma\beta}_{\ \alpha} + c^\star(\phi'^\alpha_\beta)\beta'^{\beta}_{\ \alpha} \Big\} dt = \Big\{ c^\star(\nabla_\lambda(\phi)^\mu_\nu)\beta_\mu^{\lambda\nu} + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{5.1.24}$$

$$= \Big\{ c^\star(\nabla_{J^\gamma_\lambda e'_\gamma}(\phi)^\mu_\nu)\beta_\mu^{\lambda\nu} + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{5.1.25}$$

$$= \Big\{ c^\star(J^\gamma_\lambda \nabla_{e'_\gamma}(\phi)^\mu_\nu)\beta_\mu^{\lambda\nu} + c^\star(\phi^\mu_\nu)\beta^\nu_\mu \Big\} dt = \tag{5.1.26}$$

$$= \Big\{ c^\star(J^\gamma_\lambda \nabla'_\gamma(\phi)'^\alpha_\beta \overline{J}^\mu_\alpha J^\beta_\nu)\beta_\mu^{\lambda\nu} + c^\star(\phi'^\alpha_\beta \overline{J}^\mu_\alpha J^\beta_\nu)\beta^\nu_\mu \Big\} dt \tag{5.1.27}$$

concluding that:

$$\begin{cases} \beta'^{\beta}_{\alpha} = c^\star(\overline{J}^{\mu}_{\alpha} J^{\beta}_{\nu})\beta^{\nu}_{\mu} \\ \beta'^{\gamma\beta}_{\alpha} = c^\star(J^{\gamma}_{\lambda} \overline{J}^{\mu}_{\alpha} J^{\beta}_{\nu})\beta^{\lambda\nu}_{\mu} \end{cases} \tag{5.1.28}$$

To explicitly express the equation for the constraints for these new parameters we need to invert the transformation rules

$$\begin{cases} \beta^{\beta}_{\alpha} = c^\star(J^{\mu}_{\alpha} \overline{J}^{\beta}_{\nu})\beta'^{\nu}_{\mu} \\ \beta^{\gamma\beta}_{\alpha} = c^\star(\overline{J}^{\gamma}_{\lambda} J^{\mu}_{\alpha} \overline{J}^{\beta}_{\nu})\beta'^{\lambda\nu}_{\mu} \end{cases} \tag{5.1.29}$$

As we can see, they transform linearly just as tensors, therefore they can be interpreted as the local components of some local tensor fields with support on the worldline c. We can plug it in the constraint expression to obtain:

$$\begin{cases} \beta^{1\nu}_{\mu}(\beta'^{\lambda\nu}_{\mu}, \beta'^{\mu}_{\nu}) = 0 \\ \\ -\frac{d}{dt}[\beta^{0\nu}_{\mu}(\beta'^{\lambda\nu}_{\mu}, \beta'^{\mu}_{\nu})] + c^\star(\Gamma^{\sigma}_{0\mu})\beta^{0\nu}_{\sigma}(\beta'^{\lambda\nu}_{\mu}, \beta'^{\mu}_{\nu}) - c^\star(\Gamma^{\nu}_{0\sigma})\beta^{0\sigma}_{\mu}(\beta'^{\lambda\nu}_{\mu}, \beta'^{\mu}_{\nu}) + \\ +\beta^{\nu}_{\mu}(\beta'^{\lambda\nu}_{\mu}, \beta'^{\mu}_{\nu}) - \alpha^{\mu}_{\nu}(\alpha^{\mu}_{\nu}) = 0 \end{cases} \tag{5.1.30}$$

$$\tag{5.1.31}$$

$$\begin{cases} c^\star(\overline{J}^{1}_{\gamma} J^{\alpha}_{\mu} \overline{J}^{\nu}_{\beta})\beta'^{\gamma\beta}_{\alpha} = 0 \\ \\ -\frac{d}{dt}[c^\star(\overline{J}^{0}_{\gamma} J^{\alpha}_{\mu} \overline{J}^{\nu}_{\beta})\beta'^{\gamma\beta}_{\alpha}] + c^\star(\Gamma^{\sigma}_{0\mu} \overline{J}^{0}_{\gamma} J^{\alpha}_{\sigma} \overline{J}^{\nu}_{\beta})\beta'^{\gamma\beta}_{\alpha} - c^\star(\Gamma^{\nu}_{0\sigma} \overline{J}^{0}_{\gamma} J^{\alpha}_{\mu} \overline{J}^{\sigma}_{\beta})\beta'^{\gamma\beta}_{\alpha} + \\ +c^\star(J^{\alpha}_{\mu} \overline{J}^{\nu}_{\beta})\beta'^{\beta}_{\alpha} - \alpha^{\mu}_{\nu}(\alpha'^{\mu}_{\nu}) = 0 \end{cases} \tag{5.1.32}$$

Unlike the Ellis case, we will be able to prove that the equations expressing the constraints above, can be interpreted as the local expression of an intrinsic tensorial equation, therefore the constraints expressed by the equations above do not depend on the choices of the coordinate system. However we have showed that, even if the Dixon parameters transform just as tensor fields, the uniqueness problem still remains and we are not able to set a one to one relationship between the multipoles and the Dixon parameters. To show another issue affecting the Dixon parametrization, let us consider the following multipole:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\nabla^2_{\lambda_1\lambda_2}(\phi)^{\mu}_{\nu})\gamma^{\lambda_1\lambda_2\nu}_{\mu} dt \quad , \quad \forall\phi \in \Gamma_0 T^1_1\mathbb{R}^2 \tag{5.1.33}$$

with $\gamma_\mu^{\lambda_1\lambda_2\nu}$ a bunch of scalar fields on the worldline completely symmetric in λ_1 and λ_2. Let us suppose to be interested to express it in an equivalent way, integrating by parts all the covariant derivatives taken with respect to the direction along the worldline $e_0 = \dot{c}$ as well as it has been performed in the example above. Therefore considering our adapted coordinate system, we have that:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\nabla^2_{\lambda_1\lambda_2}(\phi)^\mu_\nu)\gamma_\mu^{\lambda_1\lambda_2\nu} dt = \tag{5.1.34}$$

$$= \int_{\mathbb{R}} \Big\{ c^\star(\nabla^2_{11}(\phi)^\mu_\nu)\gamma_\mu^{11\nu} + c^\star(\nabla^2_{01}(\phi)^\mu_\nu)\gamma_\mu^{01\nu} + c^\star(\nabla^2_{10}(\phi)^\mu_\nu)\gamma_\mu^{10\nu} + c^\star(\nabla^2_{00}(\phi)^\mu_\nu)\gamma_\mu^{00} \Big\} dt = \tag{5.1.35}$$

$$= \int_{\mathbb{R}} \Big\{ c^\star(\nabla^2_{11}(\phi)^\mu_\nu)\gamma_\mu^{11\nu} + 2c^\star(\nabla^2_{01}(\phi)^\mu_\nu)\gamma_\mu^{01\nu} - 2c^\star(\nabla^2_{[01]}(\phi)^\mu_\nu)\gamma_\mu^{01\nu} + c^\star(\nabla^2_{00}(\phi)^\mu_\nu)\gamma_\mu^{00} \Big\} dt = \tag{5.1.36}$$

$$= \int_{\mathbb{R}} \Big\{ c^\star(\nabla^2_{11}(\phi)^\mu_\nu)\gamma_\mu^{11\nu} + 2c^\star(\nabla^2_{01}(\phi)^\mu_\nu)\gamma_\mu^{01\nu} - c^\star(P_{01}(\phi)^\mu_\nu - Q_{01}(\nabla(\phi))^\mu_\nu)\gamma_\mu^{01\nu} + c^\star(\nabla^2_{00}(\phi)^\mu_\nu)\gamma_\mu^{00} \Big\} dt \tag{5.1.37}$$

with P and Q the linear maps defined in the first section satisfying:

$$\nabla^2_{[]}(T) = \frac{1}{2}P(T) - \frac{1}{2}Q(\nabla(T)) \quad , \quad \forall T \in \Gamma T^p_q M \tag{5.1.38}$$

Hence we have:

$$[\phi, \mathcal{T}] = \tag{5.1.39}$$

$$= \int_{\mathbb{R}} \Big\{ c^\star(\nabla^2_{11}(\phi)^\mu_\nu)\gamma_\mu^{11\nu} + 2c^\star(\nabla_0(\nabla_1(\phi))^\mu_\nu - \nabla_{\nabla_0(e_1)})\gamma_\mu^{01\nu} + \tag{5.1.40}$$

$$- c^\star(P_{01}(\phi)^\mu_\nu - Q_{01}(\nabla(\phi))^\mu_\nu)\gamma_\mu^{01\nu} + c^\star(\nabla_0(\nabla_0(\phi))^\mu_\nu - \nabla_{\nabla_0(e_0)})\gamma_\mu^{00\nu} \Big\} dt \tag{5.1.41}$$

As one can notice, even at the second order, the non commutativity of the covariant derivatives related to the the affine structure hidden in the higher order covariant differentials make the things much more complicated with respect the Ellis representation case. We will show that all the terms can be interpreted as the the local expression of some intrinsic tensor fields, however, as the reader can realize, the scenario is definitely tricky due to the presence of the Riemman tensor and the Torsion inside the maps P and Q. The situation get even much worse if we consider multipoles beyond the third order. In that case a lot of covariant derivatives of the Torsion and the Riemman tensor rise naturally from the manipulation of the higher order covariant derivatives making any attempt of explicit calculation very complicated.

5.1.2 Considerations

From the analysis of the very simple example presented above, the reader can be convinced about the fact that, despite the quite straightforward intrinsic definition, the multipoles can be very treacherous geometrical objects. Let us try to extract from the examples what we think the problems affecting the Dixon representation are and which are their causes.

The non-uniqueness problem.

As well as in the Ellis case, the first relevant issue was given by the non unique representation of the multipoles in terms of higher order covariant differentials. This can be extremely problematic because the structure provided for the Dixon representation (i.e a closed embedding, an atlas, a smooth partition of the unity subordinate to the atlas and smooth local frame and a global connection) is not enough to fix a one to one relationship between the multipole and its local representative. This is directly caused by the non uniqueness of the Dixon local representation of the action of the multipoles, which does not allow to single out a set of $C^\infty(\mathbb{R})$-linearly independent multipoles that are able to generate the whole module. Although this can seem a minor issue apparently, in practice this causes the failure of a unique representation of the null multipole, as instance as a set of null scalar fields. So we must admit that at this stage we are not able to fix an isomorphism between the $C^\infty(\mathbb{R})$-module of the multipoles and the module $(C^\infty(\mathbb{R}), +, \cdot)$. Because of this, at least at this stage, we avoid the term "components" when we are referring to the Dixon parameters of a multipole. The lack of any isomorphism between a multipole and its Dixon parameters causes also the failure of attempting to define the operation on the multipoles in terms of operations upon the local representations. As instance the sum of two multipoles can produce a null distribution,that can be expressed by a set of parameters that are not equals to the sum of two starting multipoles parameters. As it has been already widely explained, considering that a clear correspondence between the operations on the multipoles and operations upon their local expressions is essential to express intrinsic functional equations, constraints and properties of the multipoles in terms of standard $C^\infty(\mathbb{R})$-functional equations eventually solvable with known techniques, the Ellis local representation without any additional structure is not enough to satisfy our requirements.

Causes of the non uniqueness

The non uniqueness of the Dixon representation can be directly traced back mainly to two things: the algebra of the higher order covariant derivatives and the Stokes theorem. We know by the lemmas showed in the first chapter that each antisymmetrisation of at least two indices of some higher order covariant derivatives can be written as a linear combination of totally symmetric lower order covariant derivatives. This is not a real problem as long as we defined the Dixon representation solely in terms of Dixon parameters completely symmetric in the indices contracted with the higher order covariant derivatives. In this way any antisymmetric part of the higher order covariant derivatives does not play any role giving a null contribution to the multipoles. So the only real cause of non uniqueness affecting the Dixon representation is given by the possibility to integrate by parts some terms due to the Stokes theorem. In fact, accordingly to the previous lemma, since c is a closed embedding, we have that:

$$\forall s \in \mathbb{R} \mid c(s) \cap U \neq \varnothing, \forall e_0 \in \Gamma_U TM \mid e_{0_{|_{c(s)}}} = \dot{c}(s) \quad \Rightarrow \quad c^\star(e_0(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})) = \frac{d}{ds} c^\star(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) \tag{5.1.42}$$

Let us suppose to have a local trivialisation $(e_{(i)\mu})$ such that $\forall s \in \mathbb{R} \mid c(s) \cap U \neq \varnothing, \forall e_0 \in \Gamma_U TM \mid e_{0_{|_{c(s)}}} = \dot{c}(s)$ and a multipole defined by its local action:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\nabla_{e_{(i)0}} (\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{5.1.43}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(e_{(i)0} (\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}) + \sum_{i=1}^{p} \phi_{\nu_{\overline{q}}}^{\mu_{\overline{i-1}} \alpha \mu_{\overline{p}} \setminus \overline{i}} \Gamma^{\mu_i}_{0\alpha} - \sum_{j=1}^{q} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}} \beta \nu_{\overline{q}} \setminus \overline{j}} \Gamma^{\beta}_{0\nu_j} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{5.1.44}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} \Big\{ \frac{d}{ds} c^\star \Big(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) + c^\star \Big(\sum_{i=1}^{p} \phi_{\nu_{\overline{q}}}^{\mu_{\overline{i-1}} \alpha \mu_{\overline{p}} \setminus \overline{i}} \Gamma^{\mu_i}_{0\alpha} - \sum_{j=1}^{q} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}} \beta \nu_{\overline{q}} \setminus \overline{j}} \Gamma^{\beta}_{0\nu_j} \Big) \Big\} c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{5.1.45}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} \frac{d}{ds} c^\star \Big(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds + \tag{5.1.46}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{i=1}^{p} \phi_{\nu_{\overline{q}}}^{\mu_{\overline{i-1}} \alpha \mu_{\overline{p}} \setminus \overline{i}} \Gamma^{\mu_i}_{0\alpha} - \sum_{j=1}^{q} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}} \beta \nu_{\overline{q}} \setminus \overline{j}} \Gamma^{\beta}_{0\nu_j} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{5.1.47}$$

$$= - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \frac{d}{ds} [\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}] ds + \tag{5.1.48}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{i=1}^{p} \phi_{\nu_{\overline{q}}}^{\mu_{\overline{i-1}} \alpha \mu_{\overline{p}} \setminus \overline{i}} \Gamma^{\mu_i}_{0\alpha} - \sum_{j=1}^{q} \phi^{\mu_{\overline{p}}}_{\nu_{\overline{j-1}} \beta \nu_{\overline{q}} \setminus \overline{j}} \Gamma^{\beta}_{0\nu_j} \Big) c^\star(\psi_i) \alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds = \tag{5.1.49}$$

$$= - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(s) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \frac{D}{ds} [\alpha]^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} ds \tag{5.1.50}$$

where we have interpreted $\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ as the local components of a the tensor field α defined on the worldline since they change tensorially when a coordinate transformation is performed. Then using the Stokes theorem we conclude that:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(s)\neq\varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\nabla_{e_{(i)0}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)c^{\star}(\psi_i)\alpha^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}ds = \tag{5.1.51}$$

$$=0-\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(s)\neq\varnothing}} \int_{\mathbb{R}} c^{\star}(\phi^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}})\frac{D}{ds}[\alpha]^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}c^{\star}(\psi_i)ds \tag{5.1.52}$$

Since the Dixon representation is given by the integration of a linear combination of several of higher order covariant differentials, it is clear that it cannot be unique.

The general covariance implementation and the gluing problems

In contrast with the Ellis representation, as we have already proved, all the local Dixon parameters can be interpreted as the local expression of an appropriate bunch of tensor fields defined on the worldline c. Considering this, we will be able to show that all the constraints and equations fixed on the multipoles can be recast as intrinsic constraints and intrinsic tensorial equations. This automatically recovers the general covariance principle allowing us to write the dynamical equation on the Dixon parameters in a diffeomorphically invariant fashion. Furthermore we have a clear intrinsic coordinate free geometrical interpretation of the Dixon parameters related to a given multipole.

5.2 Isomorphism between multipoles and the Dixon representation induced by a choice of an adapted coordinate system

In this section we are going to show that an isomorphism between the multipoles and a very specific Dixon local representation occurs, translating the structure of $C^{\infty}(\mathbb{R})$-module on the Dixon moments. In this section the concept of adapted atlas is considered. It has been already defined previously at the beginning of the section concerning the adapted Ellis representation, so eventually the reader is suggested to have a look to it in case it is needed.

5.2.1 Dixon representation fixed by an adapted atlas and adapted Dixon moments of a multipole

We are going to see a possible way to fix the isomorphism between the multipoles and one Dixon representation. By choosing a specific set of local adapted coordinates covering all the manifold, we are able to kill the degree of freedom in the choice of the Ellis parameters.

As we are going to see, despite this approach is very straightforward and immediate, we must pay the price of introducing a new geometrical structure hidden inside the choice of adapted coordinates. Furthermore since this isomorphism between multipoles and Dixon parameters is strongly dependent just on the specific natural local trivialisation induced by a specific set of coordinates, the structure is not invariant under diffeomorphisms, so it is not covariant. From here we are going to use the split Einstein condensed convention upon the indices. The greek letters are related to indices running from 0 to $m-1$, the latin letters instead are related to indices running from 1 to $m-1$

Lemma 38: Let $c:\mathbb{R}\hookrightarrow M$ be a closed embedding, U an open set and $\gamma\in\Gamma T^{(k+1)+p}_{q\ c}M|supp(\gamma)\subset U$. Let us consider $\{e_\alpha\}$ an arbitrary local frame defined on U and adapted to the embedding c, so $\{e_\alpha\}_{|c}=\{\dot{c},v_m\}$ and let $\{e^\alpha\}$ the algebraic dual such that $e^\alpha(e_\beta)=\delta^\alpha_\beta$ and $\{e^\alpha\}_{|c}=\{\tilde{\dot{c}},e^m\}$. The following holds:

$$\int_c c^\star(\nabla^{k+1}_{0\,\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\gamma^{0\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds=-\int_c c^\star(\nabla^{k}_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds \tag{5.2.1}$$

Proof.

$$\int_c c^\star(\nabla^{k+1}_{0\,\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\gamma^{0\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds=\int_c c^\star(\nabla^{k+1}_{0\,\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds= \tag{5.2.2}$$

$$=\int_c c^\star(\nabla_{e_0}[\nabla^{k}_{\gamma_{\overline{k}}}(\phi)]^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds-\sum_{i=1}^{k}\int_c c^\star([\nabla^{k}_{\gamma_{\overline{i-1}}\,\nabla_0(e_{\gamma_i})\gamma_{\overline{k}\setminus\overline{i}}}(\phi)]^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds= \tag{5.2.3}$$

$$=\int_c c^\star(\nabla_{e_0}[\nabla^{k}_{\gamma_{\overline{k}}}(\phi)]^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds-\sum_{i=1}^{k}\int_c c^\star(\Gamma^{\delta}_{0\gamma_i}\nabla^{k}_{\gamma_{\overline{i-1}}\,\delta\,\gamma_{\overline{k}\setminus\overline{i}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds= \tag{5.2.4}$$

$$=\int_c\Big\{\frac{d}{ds}[c^\star(\nabla^{k}_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})]+\sum_{i=1}^{p}c^\star(\Gamma^{\delta}_{0\alpha_i}\nabla^{k}_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{i-1}}\,\delta\,\alpha_{\overline{p}\setminus\overline{i}}})+ \tag{5.2.5}$$

$$-\sum_{i=1}^{q}c^\star(\Gamma^{\beta_i}_{0\delta}\nabla^{k}_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{i-1}}\,\delta\,\beta_{\overline{q}\setminus\overline{i}}}_{\alpha_{\overline{p}}})\Big\}(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+ \tag{5.2.6}$$

$$-\sum_{i=1}^{k}\int_c c^\star(\Gamma^{\delta}_{0\gamma_i}\nabla^{k}_{\gamma_{\overline{i-1}}\,\delta\,\gamma_{\overline{k}\setminus\overline{i}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds= \tag{5.2.7}$$

$$= \int_c \frac{d}{ds}[c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})](\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds + \int_c \Big\{ \sum_{i=1}^{p} \Gamma^{\delta}_{0\alpha_i}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{i-1}}\,\delta\,\alpha_{\overline{p}\backslash\overline{i}}}) + \tag{5.2.8}$$

$$- \sum_{i=1}^{q} \Gamma^{\beta_i}_{0\delta}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{i-1}}\,\delta\,\beta_{\overline{q}\backslash\overline{i}}}_{\alpha_{\overline{p}}}) \Big\} (\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds + \tag{5.2.9}$$

$$- \sum_{i=1}^{k} \int_c \Gamma^{\delta}_{0\gamma_i}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{i-1}}\,\delta\,\gamma_{\overline{k}\backslash\overline{i}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds = \tag{5.2.10}$$

$$= -\int_c c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \frac{d}{ds}(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds + \int_c \{ \sum_{i=1}^{p} \Gamma^{\delta}_{0\alpha_i}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{i-1}}\,\delta\,\alpha_{\overline{p}\backslash\overline{i}}}) + \tag{5.2.11}$$

$$- \sum_{i=1}^{q} \Gamma^{\beta_i}_{0\delta}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{b_{\overline{i-1}}\,d\,\beta_{\overline{q}\backslash\overline{i}}}_{\alpha_{\overline{p}}}) \} (\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds + \tag{5.2.12}$$

$$- \sum_{i=1}^{k} \int_c \Gamma^{\delta}_{0\gamma_i}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{i-1}}\,d\,\gamma_{\overline{k}\backslash\overline{i}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds = \tag{5.2.13}$$

$$= -\int_c c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \frac{d}{ds}(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds + \{ \sum_{i=1}^{p} \Gamma^{\delta}_{0\alpha_i}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{i-1}}\,\delta\,\alpha_{\overline{p}\backslash\overline{i}}}) + \tag{5.2.14}$$

$$- \sum_{i=1}^{q} \Gamma^{\beta_i}_{0\delta}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{i-1}}\,\delta\,\beta_{\overline{q}\backslash\overline{i}}}_{\alpha_{\overline{p}}}) + \tag{5.2.15}$$

$$- \sum_{i=1}^{k} \Gamma^{\delta}_{0\gamma_i}{}_{|c}\, c^\star(\nabla^k_{\gamma_{\overline{i-1}}\,\delta\,\gamma_{\overline{k}\backslash\overline{i}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \} (\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds = \tag{5.2.16}$$

$$= -\int_c c^\star(\nabla^k_{\gamma_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \frac{D}{ds}(\tilde{\dot{c}}\neg\,\gamma)^{\gamma_{\overline{k}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds \tag{5.2.17}$$

□

Lemma 39: Let $c : \mathbb{R} \hookrightarrow M$ be a closed embedding, U an open set and $\gamma \in \Gamma T_{q\ c}^{(k+1)+p} M | supp(\gamma) \subset U$. Let us consider $\{e_\alpha\}$ an arbitary local frame defined on U and adapted to the embedding c, so $\{e_\alpha\}_{|c} = \{\dot{c}, v_m\}$ and let $\{e^\alpha\}$ the dual such that $e^\alpha(e_\beta) = \delta^\alpha_\beta$ and $\{e^\alpha\}_{|c} = \{\tilde{\dot{c}}, e^m\}$. There always exists a bunch of C^∞-linear map $F : \gamma \in \Gamma T_{q\ c}^{(k+1)+p} M \to$

$\Gamma T_{q\ c}^{(k-i)+p}M$ such that:

$$\int_c c^\star(\nabla^{k+1}_{m_{\overline{j}}\,0\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds =$$

$$-\int_c c^\star(\nabla^k_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds - \sum_{i=1}^{j+1}\int_c c^\star\big(\nabla^{k-i}_{\delta_{\overline{k-i}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)F^{\delta_{\overline{k-i}}\rho_{\overline{p}}}{}_{\sigma_{\overline{q}}}(\gamma^{(k+1)})ds \quad (5.2.18)$$

Proof.

$$\int_c c^\star(\nabla^{k+1}_{m_{\overline{j}}\,0\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \quad (5.2.19)$$

$$= \int_c c^\star\big(\nabla^{k+1}_{0m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}} - 2\sum_{i=1}^{j}\nabla^{k+1}_{m_{\overline{i-1}}\,[0\,m_i]\,m_{\overline{j}\setminus\overline{i+1}}\ \gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}\big)\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \quad (5.2.20)$$

$$= -\int_c c^\star(\nabla^k_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds - \int_c c^\star\big(2\sum_{i=1}^{j}\nabla^{k+1}_{m_{\overline{i-1}}\,[0\,m_i]\,m_{\overline{j}\setminus\overline{i+1}}\ \gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \quad (5.2.21)$$

$$= -\int_c c^\star(\nabla^k_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds +$$
$$-\sum_{i=1}^{j}\int_c c^\star\big(2\big[\nabla^{i-1}\nabla^2_{[\,]}\nabla^{k-i-1}\big]_{m_{\overline{i-1}}\,0\,m_i\,m_{\overline{j}\setminus\overline{i+1}}\ \gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}\big)\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds \quad (5.2.22)$$

where several lemmas in section 1 have been used. Let us analyze just the term inside the second integral:

$$\int_c c^\star\big(2\big[\nabla^{i-1}\nabla^2_{[\,]}\nabla^{k-i-1}\big]_{m_{\overline{i-1}}\,0\,m_i\,m_{\overline{j}\setminus\overline{i+1}}\ \gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}\big)\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \quad (5.2.23)$$

$$= \int_c c^\star\big(\big[\nabla^{i-1}(P(\nabla^{k-i-1}(\phi)))\big]_{m_{\overline{i-1}}\,0\,m_i\,m_{\overline{j}\setminus\overline{i+1}}\ \gamma_{\overline{k}\setminus\overline{j}}}{}^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}} - \big[\nabla^{i-1}(Q(\nabla\nabla^{k-i-1}(\phi)))\big]_{m_{\overline{i-1}}\,0\,m_i\,m_{\overline{j}\setminus\overline{i+1}}\ \gamma_{\overline{k}\setminus\overline{j}}}{}^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}\big)$$
$$\times\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \quad (5.2.24)$$

$$= \int_c c^\star\big(\big[\nabla^{i-1}(P(\nabla^{k-i-1}(\phi))) - \nabla^{i-1}(Q(\nabla\nabla^{k-i-1}(\phi)))\big]_{m_{\overline{i-1}}\,0\,m_i\,m_{\overline{j}\setminus\overline{i+1}}\;\gamma_{\overline{k}\setminus\overline{j}}\;\alpha_{\overline{p}}}^{\beta_{\overline{q}}}\big)$$

$$\times\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \tag{5.2.25}$$

$$= \int_c c^\star\Big(\sum_{l=0}^{i-1}\Big\{\binom{i-1}{l}\big[[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1-l}(P)(\nabla^{k-i-1+l}(\phi)))-[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1-l}(Q)(\nabla^{k-i+l}(\phi)))\big]\Big\}_{m_{\overline{i-1}}\,0\,m}$$

$$\times\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \tag{5.2.26}$$

$$= \int_c c^\star\Big(\sum_{l=0}^{i-1}\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1-l}(P)(\nabla^{k-i-1+l}(\phi)))\Big\}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\;\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}^{\beta_{\overline{q}}}\Big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+$$

$$-\int_c c^\star\Big(\sum_{l=0}^{j}\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\;\mathbb{I}\otimes]^l\nabla^{i-1-l}(Q)(\nabla^{k-i+l}(\phi)))\Big\}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\;\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}^{\beta_{\overline{q}}}\Big)\;\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \tag{5.2.27}$$

$$= \int_c c^\star\Big(\sum_{l=0}^{i-1}\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1-l}(P)(\nabla^{k-i-1+l}(\phi)))\Big\}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\;\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}^{\beta_{\overline{q}}}\Big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+$$

$$-\int_c c^\star\Big(\sum_{l=0}^{i-1}\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\;\mathbb{I}\otimes]^l\nabla^{i-1-l}(Q)(\nabla^{k-i+l}(\phi)))\Big\}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\;\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}^{\beta_{\overline{q}}}\Big)\;\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \tag{5.2.28}$$

Since $\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\;\mathbb{I}\otimes]^l\nabla^{i-1-l}(P)\Big\}$ and $\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\;\mathbb{I}\otimes]^l\nabla^{i-1-l}(Q)\Big\}$ are both multilinear applications we can say:

$$\Big[\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(P)\Big\}(T)\Big]^{\alpha_{\overline{p}}}_{\gamma\,\delta\,\beta_{\overline{q}}} = \Big[\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(P)\Big\}\Big]^{\alpha_{\overline{p}}\sigma_{\overline{q}}}_{\gamma\,\delta\,\beta_{\overline{q}}\rho_{\overline{p}}} T^{\rho_{\overline{p}}}_{\sigma_{\overline{q}}} \tag{5.2.29}$$

and

$$\Big[\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(Q)\Big\}(T)\Big]^{\alpha_{\overline{p}}}_{\gamma\,\delta\,\beta_{\overline{q}}} = \Big[\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(Q)\Big\}\Big]^{\alpha_{\overline{p}}\sigma_{\overline{q}}}_{\gamma\,\delta\,\beta_{\overline{q}}\rho_{\overline{p}}} T^{\rho_{\overline{p}}}_{\sigma_{\overline{q}}} \tag{5.2.30}$$

So we can use it in the previous expression to say that:

$$\int_c c^\star\big(2\big[\nabla^{i-1}\nabla^2_{[\,]}\nabla^{k-i-1}\big]_{m_{\overline{i-1}}\,0 m_{\overline{j}\setminus\overline{i}}\ \gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}\big)\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds = \tag{5.2.31}$$

$$= \sum_{l=0}^{i-1}\int_c c^\star\big(\nabla^{k-i-1+l}_{\delta_{\overline{k-i-1+l}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(P)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i-1+l}}\quad\rho_{\overline{p}}}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds +$$

$$-\sum_{l=0}^{i-1}\int_c c^\star\big(\nabla^{k-i+l}_{\delta_{\overline{k-i+l}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(Q)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i+l}}\quad\rho_{\overline{p}}}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds = \tag{5.2.32}$$

So we can substitute this expression into the main one to obtain:

$$\int_c c^\star(\nabla^{k+1}_{m_{\overline{j}}\,0\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds = \tag{5.2.33}$$

$$= -\int_c c^\star(\nabla^{k}_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds +$$

$$-\sum_{i=1}^{j}\sum_{l=0}^{i-1}\int_c c^\star\big(\nabla^{k-i-1+l}_{\delta_{\overline{k-i-1+l}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(P)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i-1+l}}\quad\rho_{\overline{p}}}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds +$$

$$+\sum_{i=1}^{j}\sum_{l=0}^{i-1}\int_c c^\star\big(\nabla^{k-i+l}_{\delta_{\overline{k-i+l}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\Big\{\binom{i-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^{l}\nabla^{i-1-l}(Q)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i+l}}\quad\rho_{\overline{p}}}_{m_{\overline{i-1}}\,0\,m_{\overline{j}\setminus\overline{i}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}} ds \tag{5.2.34}$$

Now it is possible to resum order by order the terms, first of all we consider:

$$\sum_{i=1}^{j}\sum_{l=0}^{i-1} A_{i-l}B_{i,l} = \sum_{i=1}^{j}(A_i\sum_{l=0}^{j-i} B_{i+l,l}) \tag{5.2.35}$$

that can be easily proved by induction on j, and we can use this result to re-sum each term order by order:

$$\int_c c^\star(\nabla^{k+1}_{m_{\overline{j}}\,0\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds = \tag{5.2.36}$$

$$= -\int_c c^\star(\nabla^{k}_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+$$
$$-\sum_{i=1}^{j}\int_c c^\star\big(\nabla^{k-i-1}_{\delta_{\overline{k-i-1}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\sum_{l=0}^{j-i}\Big\{\binom{i+l-1}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1}(P)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i-1}}\quad\rho_{\overline{p}}}_{m_{\overline{i+l-1}}\,0\,m_{\overline{j}\setminus\overline{i+l}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+$$
$$+\sum_{i=1}^{j}\int_c c^\star\big(\nabla^{k-i}_{\delta_{\overline{k-i}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\sum_{l=0}^{j-i}\Big\{\binom{i-1+l}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1}(Q)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i}}\quad\rho_{\overline{p}}}_{m_{\overline{i+l-1}}\,0\,m_{\overline{j}\setminus\overline{i+l}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds \tag{5.2.37}$$

$$= -\int_c c^\star(\nabla^{k}_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+$$
$$-\sum_{i=2}^{j+1}\int_c c^\star\big(\nabla^{k-i}_{\delta_{\overline{k-i}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\sum_{l=0}^{j-i+1}\Big\{\binom{i+l-2}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-2}(P)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i}}\quad\rho_{\overline{p}}}_{m_{\overline{i+l-2}}\,0\,m_{\overline{j}\setminus\overline{i-1+l}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds+$$
$$+\sum_{i=1}^{j}\int_c c^\star\big(\nabla^{k-i}_{\delta_{\overline{k-i}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)\sum_{l=0}^{j-i}\Big\{\binom{i-1+l}{l}[\sigma^{(\overline{1+p})}\mathbb{I}\otimes]^l\nabla^{i-1}(Q)\Big\}^{\beta_{\overline{q}}\quad\delta_{\overline{k-i}}\quad\rho_{\overline{p}}}_{m_{\overline{i+l-1}}\,0\,m_{\overline{j}\setminus\overline{i+l}}\ \gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}\,\sigma_{\overline{q}}}\big)\gamma^{0\,m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds \tag{5.2.38}$$

$$= -\int_c c^\star(\nabla^{k}_{m_{\overline{j}}\,\gamma_{\overline{k}\setminus\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\frac{D}{ds}[\tilde{\dot{c}}\neg\gamma]^{m_{\overline{j}}\gamma_{\overline{k}\setminus\overline{j}}\alpha_{\overline{p}}}{}_{\beta_{\overline{q}}}ds-\sum_{i=1}^{j+1}\int_c c^\star\big(\nabla^{k-i}_{\delta_{\overline{k-i}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big)F^{\delta_{\overline{k-i}}\rho_{\overline{p}}}{}_{\sigma_{\overline{q}}}(\gamma^{(k+1)})ds \tag{5.2.39}$$

□

Lemma 40: Let $c:\mathbb{R}\hookrightarrow M$ be a closed embedding, U an open set and $\gamma\in\Gamma T^{(k+1)+p}_{q\ c(s)}M|supp(\gamma)\subset U$. Let us consider $\{e_\alpha\}$ an arbitary local frame defined on U and adapted to the embedding c, so $\{e_\alpha\}_{|c}=\{\dot{c},v_m\}$ and let $\{e^\alpha\}$ the dual such that $e^\alpha(e_\beta)=\delta^\alpha_\beta$ and $\{e^\alpha\}_{|c}=\{\tilde{\dot{c}},e^m\}$. There always exists a bunch of smooth local scalar fields $\beta^{m_{\overline{j}}\mu_{\overline{p}}}_{\nu_{\overline{q}}}\in C^\infty U,\forall j\in[0,k-1]\subset\mathbb{N}$ such that:

$$\int_c c^\star(\nabla^{k}_{\lambda_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{\lambda_{\overline{k}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}ds= \tag{5.2.40}$$

$$=\int_{\mathbb{R}} c^\star(\nabla^{k}_{m_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{m_{\overline{k}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}ds+\sum_{j=0}^{k-1}\int_{\mathbb{R}} c^\star(\nabla^{j}_{m_{\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \beta^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}ds= \tag{5.2.41}$$

Proof. We provide here an hint of proof by noticing that the following split is always possible:

$$\int_c c^\star(\nabla^k_{\lambda_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{\lambda_{\overline{k}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds = \tag{5.2.42}$$

$$= \int_c c^\star(\nabla^k_{m\lambda_{\overline{k-1}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{m\lambda_{\overline{k-1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds + \int_c c^\star(\nabla^k_{0\lambda_{\overline{k-1}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{0\lambda_{\overline{k-1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds = \tag{5.2.43}$$

$$= \int_c c^\star(\nabla^k_{m\lambda_{\overline{k-1}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{m\lambda_{\overline{k-1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds - \int_c c^\star(\nabla^{k-1}_{\lambda_{\overline{k-1}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \frac{D}{ds}(\tilde{\dot{c}}\neg\gamma)^{\lambda_{\overline{k-1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds = \tag{5.2.44}$$

$$= \int_c c^\star(\nabla^k_{mn\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{mn\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds + \int_c c^\star(\nabla^k_{m0\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{m0\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds + \tag{5.2.45}$$

$$- \int_c c^\star(\nabla^{k-1}_{m\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \frac{D}{ds}(\tilde{\dot{c}}\neg\gamma)^{m\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds - \int_c c^\star(\nabla^{k-1}_{0\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \frac{D}{ds}(\tilde{\dot{c}}\neg\gamma)^{0\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds = \tag{5.2.46}$$

$$= \int_c c^\star(\nabla^k_{mn\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \gamma^{mn\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds - 2\int_c c^\star(\nabla^{k-1}_{m\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \frac{D}{ds}(\tilde{\dot{c}}\neg\gamma)^{m\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds + \tag{5.2.47}$$

$$+ \int_c c^\star(\nabla^{k-1}_{\lambda_{\overline{k-2}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}})\ \frac{D}{ds}[\tilde{\dot{c}}\neg\frac{D}{ds}(\tilde{\dot{c}}\neg\gamma)]^{\lambda_{\overline{k-2}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds - \int_c c^\star\big(\nabla^{k-1}_{\delta_{\overline{k-1}}}(\phi)^{\sigma_{\overline{q}}}_{\rho_{\overline{p}}}\big) F^{\delta_{\overline{k-1}}\rho_{\overline{p}}}{}_{\sigma_{\overline{q}}}(\gamma^{(k+1)}) ds \tag{5.2.48}$$

It is trivial to realise that it is always possible to apply the same procedure iteratively such that all the greek indices related to the higher order covariant derivatives can be split into the 0-th component and the remaining others. In general this expression can be extremely complicate but since all the 0-th components can be integrated by part using the previous lemma, it is possible to end up with an expression involving just higher order covariant derivatives taken with respect the basis vectors $(e_1, ..., e_m)$. Re-summing order by order all the surviving terms it is possible to obtain the thesis.

□

Theorem 9: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. For each multipole $\mathcal{T} \in \Upsilon^q_p(c)$, there always exists a unique a bunch of local smooth tensor field $\alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmeteric in $m_{\overline{k}}$ and defining a global smooth top form:

$$c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \psi_i \sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{m_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)\alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{5.2.49}$$

such that, $\forall \phi \in \Gamma_0 T^p_q M$, $\mathcal{T}$ acts on the local expression of ϕ as follow:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{m_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{5.2.50}$$

Proof. Let us starting to prove the existence of the expression, later we will prove the uniqueness. The first part of the thesis can be easily proven by using the previous lemma.

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{M} \nabla^k_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{5.2.51}$$

with $M \in \mathbb{N} \,|\, M \geq ord(\mathcal{T})$. So by using the lemma we can state:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{M} \int_{\mathbb{R}} c^\star \Big(\nabla^k_{\lambda_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds = \tag{5.2.52}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{M} \sum_{j=0}^{k} \int_{\mathbb{R}} c^\star \Big(\nabla^j_{m_{\overline{j}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \beta^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i,k)\quad \mu_{\overline{p}}} ds \tag{5.2.53}$$

Now it is possible to resum order by order in j defining a new set of local scalar fields $\alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}}$ as an appropriate linear combination of $\beta^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i,k)\quad \mu_{\overline{p}}}$ symmetric in $m_{\overline{j}}$. Hence we have:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{j=0}^{M} \int_{\mathbb{R}} c^\star \Big(\nabla^j_{m_{\overline{j}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{5.2.54}$$

It is possible to show how for $j > ord(\mathcal{T})$ all the terms give no contribution to the integral otherwise we lead to a contradiction with the definition of order.

Infact by definition, $\forall s \in \mathbb{N}, \forall \phi \in \Gamma_0 T^p_q M, \forall \lambda \in C^\infty M \mid c^\star(\lambda) = 0$ we must have:

$$0 = [\lambda^{ord(\mathcal{T})+1+s}, \mathcal{T}] \tag{5.2.55}$$

therefore:

$$0 = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{j=0}^{ord(\mathcal{T})} \int_{\mathbb{R}} c^\star \Big(\nabla^j_{m_{\overline{j}}} (\lambda^{ord(\mathcal{T})+1+s} \phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds + \tag{5.2.56}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{j=ord(\mathcal{T})+1}^{M} \int_{\mathbb{R}} c^\star \Big(\nabla^j_{m_{\overline{j}}} (\lambda^{ord(\mathcal{T})+1+s} \phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{5.2.57}$$

The first integral is always null because there are not enough derivation to kill all the powers of $\lambda^{ord(\mathcal{T})+1+s}$, hence we can say that $\forall s \in \mathbb{N}, \forall \phi \in \Gamma_0 T^p_q M, \forall \lambda \in C^\infty M \,|\, c^\star(\lambda) = 0$ the second integral must vanish as well:

$$\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{j=ord(\mathcal{T})+1}^{M} \int_{\mathbb{R}} c^\star \Big(\nabla^j_{m_{\overline{j}}} (\lambda^{ord(\mathcal{T})+1+s} \phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = 0 \tag{5.2.58}$$

Since $\alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds$ are completely symmetric in $m_{\overline{j}}$and each term $c^\star \Big(\nabla^j_{m_{\overline{j}}} (\lambda^{ord(\mathcal{T})+1+s} \phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big)$ is composed by derivations along linearly independent vectors with respect to $\dot{c}$ there is no chance to have a null result for each $\phi \in \Gamma_0 T^p_q M$ unless all the Ellis parameters are constrained by:

$$\alpha^{m_{\overline{j}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} = 0 \quad , \quad \forall j \geq ord(\mathcal{T}) \tag{5.2.59}$$

Therefore the action of each multipole can be written as:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{m_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \tag{5.2.60}$$

To show that the differential form

$$c^\star\Big(\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\psi_i\sum_{k=0}^{ord(\mathcal{T})}\nabla^k_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds\in\Gamma\Lambda^1\mathbb{R} \tag{5.2.61}$$

is a global smooth 1-form one can repeat exactly the same reasoning explicited for the Ellis case. The uniqueness of this representation follows exactly from the same prove given for the adapted Ellis representation.

□

Let us remark that the uniqueness strongly depends on the choices of the adapted atlas.

Definition 77: Let $c:\mathbb{R}\hookrightarrow M$ be a worldline and $\mathcal{A}=(U_i,\varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The set of Dixon parameters $\alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}}\in\Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in $m_{\overline{k}}$ and defining a global smooth top form:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\sum_{k=0}^{ord(\mathcal{T})}c^\star\Big(\psi_i\nabla^k_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)\alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds\in\Gamma\Lambda^1\mathbb{R} \tag{5.2.62}$$

such that, $\forall\phi\in\Gamma_0T^p_qM$, $\mathcal{T}\in\Upsilon^q_p(c)$ acts on the local expression of ϕ as follow:

$$[\phi,\mathcal{T}]=\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}}c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})}\nabla^k_{m_{\overline{k}}}(\phi)_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}\Big)c^\star(\psi_i)\alpha_{(i)}^{m_{\overline{k}}\ \nu_{\overline{q}}}{}_{\mu_{\overline{p}}}ds \tag{5.2.63}$$

are called ***the adapted Dixon parameters*** of the multipole $\mathcal{T}$ with respect to the adapted atlas $(U_i,\varphi_{(i)})$. The local Dixon representation induced by the adapted Dixon parameters.

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k div^k \Big\{ \psi_i \Big[e_{m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta \Big(\alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds \Big) \Big\} \tag{5.2.64}$$

is called ***adapted Dixon local representation*** with respect to the adapted atlas $(U_i, \varphi_{(i)})$.

Corollary 17: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The adapted Dixon parameters $\alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}}$ associated to the adapted Dixon local representation

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k div^k \Big\{ \psi_i \Big[e_{m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta \Big(\alpha^{m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} ds \Big) \Big\} \tag{5.2.65}$$

can be interpreted as the components with respect to the given adapted frame of some local sections $\overset{(k)}{\alpha}_{(i)} \in T_p^{(k)+q}{}_{c(s)}M$ defined on $U_i \cap c$ and satisfying $\tilde{\dot{c}}_{(i)} \neg \overset{(k)}{\alpha}_{(i)} = 0$.

Proof. It is trivial to check that the components with respect to the given adapted frame of this kind of local sections are exactly in the form of the Dixon parameters. Furthermore they both change with the same transformation rules when a change of local frame is performed, then we have the thesis. □

Definition 78: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The set of local sections $\overset{(k)}{\alpha}_{(i)} \in T_p^{(k)+q}{}_{c(s)}M$ defined on $U_i \cap c$, satisfying $\tilde{\dot{c}}_{(i)} \neg \overset{(k)}{\alpha}_{(i)}$ which components define the adapted Dixon representation via:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{m_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) \alpha^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds \tag{5.2.66}$$

are called ***the adapted local Dixon moments***.

Corollary 18: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The adapted Dixon local representation with respect to the given adapted atlas is an isomorphism (of modules) between the $\Upsilon^q_p(c)$ and the set of the Dixon parameters.

Proof. It is quite trivial. We already proved in the previous theorem that the adapted Ellis local representation is able to express all the elements in $\Upsilon^q_p(c)$ in an unique way. It is very easy to check that the sum of distribution is mapped into the sum of parameters as well as the scalar multiplication, therefore it is a bijection preserving the linear structure, hence an isomorphism. $\square$

Let us stress once again that this particular isomorphism is strongly dependent on the choices of the adapted atlas. If another atlas is chosen then this isomorphism does not occur anymore. If the new atlas is still adapted, then a new isomorphism can be built with the same approach, however the link between two different adapted Dixon representations can be very tricky. In case the new atlas is no more adapted, then one cannot define this kind of isomorphism. This is very problematic because the way we decided to link the adapted Dixon parameter to the multipoles is not compatible with the invariance under local diffeomorphisms or equivalently for a general local coordinate transformation, therefore the covariance principle seems to be broken. Luckily we will see how there is a coordinate free way to fix the very same isomorphism so at the end of the day the covariance principle is safe.

5.3 Intrinsic interpretation of the Dixon parameters

We are now going to show how the Dixon representation admits a purely coordinate-free interpretation of the Dixon parameters. This is very important considering that, for physical applications, we would like the information encoded in the multipoles not to depend on a particular coordinate system. So from this perspective, we will see how the Dixon parameters can be associated canonically to some tensor field defined on the sub-manifold $c(\mathbb{R})$ therefore eventually, some $C^\infty(M)$-linear equations on them can be immediately interpreted as purely covariant constraints, independently from the coordinate system.

5.3.1 The Dixon Generators

Let us start first pointing out a specific family of dipoles able to generate the whole module $\Upsilon^q_p(c)$

Lemma 41: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ atlas of M with a local natural frame $(\partial_{(i)\mu})$ inducing a trivialisation of TM and a coframe $(dx^\mu_{(i)})$ inducing a trivialization of $T^\star M$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. For each multipole $\mathcal{T} \in \Upsilon^q_p(c)$, there always exists at least a set of smooth scalar field $\gamma_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \in \Gamma_{c\cap U_i}\Lambda^0\mathbb{R}$ completely symmetric in $\lambda_{\overline{k}}$ and defining a global smooth top form:

$$c^{\star}\Big(\sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \psi_i \sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\lambda_{\overline{k}}}(\phi)^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) \gamma_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \in \Gamma\Lambda^1\mathbb{R} \tag{5.3.1}$$

such that, $\forall \phi \in \Gamma_0 T^p_q M$, $\mathcal{T}$ acts on the local expression of ϕ as follow:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\lambda_{\overline{k}}}(\phi)^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi_i) \gamma_{(i)}^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \tag{5.3.2}$$

Proof. Let us choose a local adapted atlas $\mathcal{A} = (U_i, \varphi'_{(i)})$ and a natural trivialization of TM by the local frame $\partial'_{(i)\mu}$. By the previous theorem we know that the action of each multipole can be written as:

$$[\phi, \mathcal{T}] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{m_{\overline{k}}}(\phi)'^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi'_i) \alpha'^{m_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}} ds = \tag{5.3.3}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\lambda_{\overline{k}}}(\phi)'^{\;\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^{\star}(\psi'_i) \beta'^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}} ds \tag{5.3.4}$$

where $\beta'^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}}$ are defined in the following way:

$$\beta'^{\lambda_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}} = \begin{cases} \beta'^{m_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}} = \alpha^{m_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}} \\ 0, \text{otherwise} \end{cases} \tag{5.3.5}$$

Let us stress that by construction since $\alpha^{m_{\overline{k}}\,\nu_{\overline{q}}}_{(i)}{}_{\mu_{\overline{p}}}$ is symmetric in $m_{\overline{k}}$ Now let us consider the change of local trivialisation of TM due to the general change of local charts, then $\partial_{(i)\mu} = J^{\nu}_{\mu}\partial'_{(i)\nu}$ and we have:

$$[\phi,\mathcal{T}] = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\lambda_{\overline{k}}}(\phi)'^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i')\beta'^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds = \tag{5.3.6}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\overline{J}^{\alpha_1}_{\lambda_1}e_{\alpha_1}\dots\overline{J}^{\alpha_k}_{\lambda_k}e_{\alpha_k}}(\phi)^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}} J^{\mu_1}_{\beta_1}\dots J^{\mu_k}_{\beta_k}\overline{J}^{\gamma_1}_{\nu_1}\dots\overline{J}^{\gamma_k}_{\nu_k}\Big) c^\star(\psi_i)\beta^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds = \tag{5.3.7}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\alpha_{\overline{k}}}(\phi)^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}\Big) c^\star\Big(J^{\mu_1}_{\beta_1}\dots J^{\mu_k}_{\beta_k}\overline{J}^{\alpha_1}_{\lambda_1}\dots\overline{J}^{\alpha_k}_{\lambda_k}\overline{J}^{\gamma_1}_{\nu_1}\dots\overline{J}^{\gamma_k}_{\nu_k}\Big) c^\star(\psi_i)\beta^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds = \tag{5.3.8}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \int_{\mathbb{R}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\alpha_{\overline{k}}}(\phi)^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}\Big) c^\star(\psi_i)\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{5.3.9}$$

where $\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}}$ is a set of scalar fields defined as

$$\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} = c^\star\Big(J^{\mu_1}_{\beta_1}\dots J^{\mu_k}_{\beta_k}\overline{J}^{\alpha_1}_{\lambda_1}\dots\overline{J}^{\alpha_k}_{\lambda_k}\overline{J}^{\gamma_1}_{\nu_1}\dots\overline{J}^{\gamma_k}_{\nu_k}\Big)\beta^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} \tag{5.3.10}$$

By construction we have that:

$$\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{m_{\overline{k}}}(\phi)'^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i')\alpha'^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds = \tag{5.3.11}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} c^\star\Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\lambda_{\overline{k}}}(\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big) c^\star(\psi_i)\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}} ds \tag{5.3.12}$$

therefore, since the first term must be a global smooth 1-form over $\mathbb{R}$, the local scalar fields $\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\quad \mu_{\overline{p}}}$ define a global smooth 1-form over $\mathbb{R}$. □

Corollary 19: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M and let $(e_{(i)\mu})$ and $(e^\mu_{(i)})$ be a local frame and the dual coframe trivialising respectively TM and

$T^{\star}M$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given $\mathcal{T} \in \Upsilon^p_q(c)$ and the set of its adapted Dixon moments we have that:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k div^k \Big\{ \psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_{\zeta} \Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds \Big) \Big\} \tag{5.3.13}$$

where $\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}}$ can be always interpreted as the local expressions of a set of tensor fields (completely symmetric in the first k upper indices) $\alpha^{(k)} \in \Gamma T^{(k)+q}_{p\ \ c(\mathbb{R})} M$ with respect the given frame.

Proof. The proof is trivial and follows directly form the $C^{\infty}(M)$-linearity of the higher order covariant derivatives. Let be $(e_{(i)\mu} = \Lambda^{\nu}_{(i)\mu} \partial_{(i)\nu})$ and $(e^{\nu}_{(i)} = \overline{\Lambda}^{\nu}_{(i)\mu} dx^{\nu}_{(i)})$ be local frame and the dual local coframe.

$$[\phi, \mathcal{T}] = \tag{5.3.14}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star} \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\Lambda^{\alpha_1}_{(i)\lambda_1} e_{\alpha_1} \dots \Lambda^{\alpha_k}_{(i)\lambda_k} e_{\alpha_k}} (\phi)'^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}} \overline{\Lambda}^{\mu_1}_{(i)\beta_1} \dots \overline{\Lambda}^{\mu_p}_{(i)\beta_p} \Lambda^{\gamma_1}_{(i)\nu_1} \dots \Lambda^{\gamma_q}_{(i)\nu_q} \Big) c^{\star}(\psi_i) \gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \tag{5.3.15}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star} \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\alpha_{\overline{k}}} (\phi)'^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}} \Big) \tag{5.3.16}$$

$$c^{\star}(\psi_i) c^{\star}(\Lambda^{\alpha_1}_{(i)\lambda_1}) \dots c^{\star}(\Lambda^{\alpha_k}_{(i)\lambda_k}) c^{\star}(\overline{\Lambda}^{\mu_1}_{(i)\beta_1}) \dots c^{\star}(\overline{\Lambda}^{\mu_p}_{(i)\beta_p}) c^{\star}(\Lambda^{\gamma_1}_{(i)\nu_1}) \dots c^{\star}(\Lambda^{\gamma_q}_{(i)\nu_q}) \gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \tag{5.3.17}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star} \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\alpha_{\overline{k}}} (\phi)'^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}} \Big) \tag{5.3.18}$$

$$c^{\star}(\psi'_i) c^{\star}(\Lambda^{\alpha_1}_{(i)\lambda_1}) \dots c^{\star}(\Lambda^{\alpha_k}_{(i)\lambda_k}) c^{\star}(\overline{\Lambda}^{\mu_1}_{(i)\beta_1}) \dots c^{\star}(\overline{\Lambda}^{\mu_p}_{(i)\beta_p}) c^{\star}(\Lambda^{\gamma_1}_{(i)\nu_1}) \dots c^{\star}(\Lambda^{\gamma_q}_{(i)\nu_q}) \gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \tag{5.3.19}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star} \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla^k_{\alpha_{\overline{k}}} (\phi)'^{\ \beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}} \Big) c^{\star}(\psi'_i) \gamma'^{\alpha_{\overline{k}}\ \gamma_{\overline{q}}}_{(i)\ \ \ \beta_{\overline{p}}} ds \tag{5.3.20}$$

where we defined:

$$\gamma'^{\alpha_{\overline{k}}\ \gamma_{\overline{q}}}_{(i)\qquad \beta_{\overline{p}}} = c^\star(\Lambda^{\alpha_1}_{(i)\lambda_1})...c^\star(\Lambda^{\alpha_k}_{(i)\lambda_k})c^\star(\overline{\Lambda}^{\mu_1}_{(i)\beta_1})...c^\star(\overline{\Lambda}^{\mu_p}_{(i)\beta_p})c^\star(\Lambda^{\gamma_1}_{(i)\nu_1})...c^\star(\Lambda^{\gamma_q}_{(i)\nu_q})\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} \tag{5.3.21}$$

Therefore one can see how, under the change of local frame, the set of scalar fields $\gamma^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}}$ change exactly as the components of a local section of $T^{(k)+q}_{p\quad c(s)}M$ (the $c(\mathbb{R})$-constrained $rank(p,q)$ tangent tensor bundle on M), so they are the local expression of some set of tensor fields $\alpha^{(k)} \in \Gamma T^{(k)+q}_{p\quad c(s)}M$ symmetric in the first k upper indices and constrained on $c(\mathbb{R})$ □

Definition 79: Let $c : \mathbb{R} \to M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M. Let us suppose to trivialise TM using the local frame $(e_{(i)\mu})$ and $T^\star M$ using the local coframe $e^\mu_{(i)}$.Let ψ_i be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The representation:

$$\mathcal{T} = \sum_{\substack{U_i\in\mathcal{A} \\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k div^k \Big\{\psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_\zeta\Big(\alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} ds\Big)\Big\} \tag{5.3.22}$$

is called ***truncated Dixon representation for the multipoles***

Using the $C^\infty(\mathbb{R})$ module scalar multiplication it is possible to write the truncated Dixon representation of a multipole as follow:

$$\mathcal{T} = \sum_{\substack{U_i\in\mathcal{A} \\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha^{\lambda_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\qquad \mu_{\overline{p}}} \triangleright \Big((-1)^k div^k \Big\{\psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_\zeta(ds)\Big\}\Big) \tag{5.3.23}$$

therefore in this perspective, the bunch multipoles

$$(-1)^k div^k \Big\{\psi_i \Big[e_{(i)\lambda_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_\zeta(ds)\Big\} \tag{5.3.24}$$

defined by the action:

$$\Big[\phi,(-1)^k div^k\Big\{\psi_i\Big[e_{(i)\lambda_{\overline{k}}}\otimes e_{(i)\nu_{\overline{q}}}\otimes e^{\mu_{\overline{p}}}_{(i)}\Big]c_\zeta(ds)\Big\}\Big]=\int_{\mathbb{R}}c^\star\Big(\nabla^k_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)ds= \tag{5.3.25}$$

$$=\int_{\mathbb{R}}\sum_{j=0}^{k}c^\star\Big(\nabla^j_{\eta_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}\Big)c^\star(\psi_i)\delta(j,k)\delta^{\eta_{\overline{j}}}_{\lambda_{\overline{k}}}\delta^{\mu_{\overline{p}}\rho_{\overline{q}}}_{\nu_{\overline{q}}\sigma_{\overline{p}}}ds \tag{5.3.26}$$

are the generators of the whole modules of the multipoles.

Definition 80: Let $c:\mathbb{R}\to M$ be a worldline and $A=(U_i,\phi_{(i)})$ an atlas of M. Let us suppose to trivialise TM using the local frame $(e_{(i)\mu})$ and $T^\star M$ using the local coframe $(e^{\mu}_{(i)})$ Let ψ_i be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The multi-indexed list $\Delta^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}}\mid k\in\mathbb{N}$ of multipoles defined as:

$$\Delta^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}}=(-1)^k div^k\Big\{\psi_i\Big[e_{\lambda_{\overline{k}}}\otimes e_{(i)\nu_{\overline{q}}}\otimes e^{\mu_{\overline{p}}}_{(i)}\Big]c_\zeta(ds)\Big\} \tag{5.3.27}$$

is called ***Dixon generators of the multipoles*** and the sub-list $\Delta^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}}\mid k\in[0,N]\subset\mathbb{N}$ is called ***Dixon generators of the multipoles up to the order*** N

Property 47: If a change of local frame is performed it is trivial to check from the definition that a new set of Dixon generators is induced. The relationship between the old set of generator and the new one can be easily found:

$$\Big[\phi,\Delta'^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}}\Big]=\Big[\phi,(-1)^k div^k\Big\{\psi_i\Big[e'_{(i)\lambda_{\overline{k}}}\otimes e'_{(i)\nu_{\overline{q}}}\otimes e'^{\mu_{\overline{p}}}_{(i)}\Big]c_\zeta(ds)\Big\}\Big]= \tag{5.3.28}$$

$$=\int_{\mathbb{R}}c^\star\Big(\nabla^k_{\lambda_{\overline{k}}}(\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}}\Big)ds= \tag{5.3.29}$$

$$=\int_{\mathbb{R}}c^\star\Big(\nabla^k_{\alpha_{\overline{k}}}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}\Big)c^\star(\overline{\Lambda}^{\alpha_1}_{\lambda_1})...c^\star(\overline{\Lambda}^{\alpha_k}_{\lambda_k})c^\star(\overline{\Lambda}^{\gamma_1}_{\nu_1})...c^\star(\overline{\Lambda}^{\gamma_q}_{\nu_q})c^\star(\Lambda^{\beta_1}_{\mu_1})....c^\star(\Lambda^{\beta_p}_{\mu_p})ds= \tag{5.3.30}$$

$$=\Big[\phi,c^\star(\overline{\Lambda}^{\alpha_1}_{\lambda_1})...c^\star(\overline{\Lambda}^{\alpha_k}_{\lambda_k})c^\star(\overline{\Lambda}^{\gamma_1}_{\nu_1})...c^\star(\overline{\Lambda}^{\gamma_q}_{\nu_q})c^\star(\Lambda^{\beta_1}_{\mu_1})....c^\star(\Lambda^{\beta_p}_{\mu_p})\Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}}\Big] \tag{5.3.31}$$

therefore we have that:

$$\Delta'^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}}=c^\star(\overline{\Lambda}^{\alpha_1}_{\lambda_1})...c^\star(\overline{\Lambda}^{\alpha_k}_{\lambda_k})c^\star(\overline{\Lambda}^{\gamma_1}_{\nu_1})...c^\star(\overline{\Lambda}^{\gamma_q}_{\nu_q})c^\star(\Lambda^{\beta_1}_{\mu_1})....c^\star(\Lambda^{\beta_p}_{\mu_p})\Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} \tag{5.3.32}$$

It is very interesting to notice that, in contrast with the Ellis representation case, an arbitrary change of frame does not mix up together all the generators, therefore the structure of the linear combination expressing a multipole with a truncated Dixon representation is preserved under general coordinate transformation or under a generic change of trivialisation of the bundle TM

5.3.2 Adapted Dixon basis for the multipoles

Since fixing an adapted atlas the induced Dixon local representation maps isomorphically $\Upsilon^p_q(c)$ as a $C^\infty(\mathbb{R})$ module into the set of the adapted Ellis moments, it is enough to analyse them to extrapolate some information about the algebraic structure of $\Upsilon^p_q(c)$ and its subsets $\Upsilon^p_q(c)$. Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. Given $\mathcal{T} \in \Upsilon^p_q(c)$ and the set of its adapted Dixon moments $\alpha_{(i)}^{m_{\overline{k}}\,\nu_{\overline{q}}}{}_{\mu_{\overline{p}}}$ we have that:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} (-1)^k div^k \Big\{ \psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta \Big(\alpha_{(i)}^{m_{\overline{k}}\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} ds \Big) \Big\} \tag{5.3.33}$$

Using the $C^\infty(\mathbb{R})$ module scalar multiplication we can recast the expression as follow:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha_{(i)}^{m_{\overline{k}}\nu_{\overline{q}}}{}_{\mu_{\overline{p}}} \triangleright (-1)^k div^k \Big\{ \psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta(ds) \Big\} \tag{5.3.34}$$

Fixing the order k and fixing the lists of indices $m_{\overline{k}}, \mu_{\overline{p}}$ and $\nu_{\overline{q}}$, each term:

$$(-1)^k div^k \Big\{ \psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta(ds) \Big\} \tag{5.3.35}$$

can be interpreted as a multipole belonging to $\overset{(k)}{\Upsilon}{}^q_p(c)$, defined by its action on $\Gamma_0 T^p_q M$ by:

$$\Big[\phi, (-1)^k div^k \Big\{ \psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)} \Big] c_\zeta(ds) \Big\} \Big] = \int_{\mathbb{R}} c^\star \Big(\nabla_{m_{\overline{k}}} (\phi)^{\mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) ds = \tag{5.3.36}$$

$$= \int_{\mathbb{R}} \sum_{j=0}^{k} c^{\star}\Big(\nabla_{n_{\overline{j}}}(\phi)^{\sigma_{\overline{p}}}_{(i)\rho_{\overline{q}}}\Big) c^{\star}(\psi_i)\delta(j,k)\delta^{n_{\overline{j}}}_{m_{\overline{k}}}\delta^{\mu_{\overline{p}}\rho_{\overline{q}}}_{\nu_{\overline{q}}\sigma_{\overline{p}}} ds \tag{5.3.37}$$

Let us then denote these multipoles with :

$$\Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = (-1)^k div^k \Big\{\psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_{\zeta}(ds)\Big\} \tag{5.3.38}$$

then we have that an arbitrary multipole $\mathcal{T}$ can be written simply as a $C^{\infty}(\mathbb{R})$ combination:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} \alpha^{\ m_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \tag{5.3.39}$$

So we have to conclude that the set $(\Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \,|\, k \in \mathbb{N})$ is a set of generators for the module $\Upsilon^q_p(c)$.

Lemma 42: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma\Lambda^1\mathbb{R}$. The multi-indexed list $(\Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \mid k \in \mathbb{N})$ of multipoles defined as:

$$\Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} = (-1)^k div^k \Big\{\psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_{\zeta}(ds)\Big\} \tag{5.3.40}$$

is a basis of $\Upsilon^q_p(c)$. In the same way sublist the $(\Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \mid k \in [0,N] \subset \mathbb{N})$ is a basis for the submodule $\overset{(N)}{\Upsilon}{}^q_p(c) \subset \Upsilon^q_p(c)$ of the multipoles up to the order N.

Proof. We already have seen how the list an arbitrary distribution can be written as a $C^{\infty}(\mathbb{R})$-linear combination of elements of the list $(\Delta^{\mu_{\overline{p}}}_{(i)m_{\overline{k}}\nu_{\overline{q}}} \,|\, k \in \mathbb{N})$. We need to check the $C^{\infty}(\mathbb{R})$-linear independence. This can be done simply checking that the null distribution can be written uniquely as a linear combination of null coefficients. The uniqueness of the adapted Dixon moments with respect an adapted atlas is guaranteed by the previous theorem, therefore it is enough to check just that the null multipole can be written via a linear combination of null coefficients. This is trivial because $\forall \phi \in \Gamma_0 T^p_q M$ we have that:

$$[\phi, \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{ord(\mathcal{T})} 0^{m_{\overline{k}} \nu_{\overline{q}}}_{(i)\ \mu_{\overline{p}}} \triangleright \Big((-1)^k div^k \Big\{\psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_\zeta(ds)\Big\}\Big)] = \tag{5.3.41}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^\star \Big(\sum_{k=0}^{ord(\mathcal{T})} \nabla_{m_{\overline{k}}} (\phi)^{\ \mu_{\overline{p}}}_{(i)\nu_{\overline{q}}} \Big) c^\star(\psi_i) 0^{m_{\overline{k}}\ \nu_{\overline{q}}}_{(i)\ \ \ \mu_{\overline{p}}} ds = \int_{\mathbb{R}} 0 \cdot ds = 0 \tag{5.3.42}$$

the same reasoning can be repeated identically for the submodule $\overset{(N)}{\Upsilon}{}^q_p(c)$ generated by the list $(\Delta^{\mu_{\overline{p}}}_{(i) m_{\overline{k}} \nu_{\overline{q}}} \mid k \in [0, N] \subset \mathbb{N})$. Thence we have the thesis. $\square$

Definition 81: Let $c : \mathbb{R} \hookrightarrow M$ be a worldline and $\mathcal{A} = (U_i, \varphi_{(i)})$ an atlas of M adapted to c inducing a local adapted trivialisation of TM due to the local adapted frame $(\partial_{(i)\mu})$. Let (ψ_i) be a smooth partition of the unity subordinate to (U_i) and let ds be a global coframe of $\Gamma \Lambda^1 \mathbb{R}$. The multi-indexed list $(\delta^{\mu_{\overline{p}}}_{(i) m_{\overline{k}} \nu_{\overline{q}}} \mid k \in \mathbb{N})$ of multipoles defined as:

$$\Delta^{\mu_{\overline{p}}}_{(i) m_{\overline{k}} \nu_{\overline{q}}} = (-1)^k div^k \Big\{\psi_i \Big[e_{(i)m_{\overline{k}}} \otimes e_{(i)\nu_{\overline{q}}} \otimes e^{\mu_{\overline{p}}}_{(i)}\Big] c_\zeta(ds)\Big\} \tag{5.3.43}$$

is called ***adapted Dixon basis of the multipoles*** and the sub-list $(\Delta^{\mu_{\overline{p}}}_{(i) m_{\overline{k}} \nu_{\overline{q}}} | k \in [0, N] \subset \mathbb{N})$ is called ***adapted Dixon basis of the multipoles up to the order*** N.

Let us notice that since $\Upsilon^q_p(c)$ is a $C^\infty(\mathbb{R})$-free module and since $C^\infty(\mathbb{R})$ is not a division ring, the cardinality of the basis is no more guaranteed. At this purpose let us consider to have two adapted atlas $\mathcal{A}$ and $\mathcal{A}'$ such that they cover the sub-manifold $c(\mathbb{R})$ with a different number of local charts. Both of them induce a bijection between the multipoles and their own adapted Dixon moments and such that the linear structure is preserved, however the number of the adapted Dixon moments strongly depends on the number of charts covering $c(\mathbb{R})$, the cardinality of the adapted Dixon generators is not preserved, therefore $\Upsilon^q_p(c)$ does not own the invariant basis number property. Considering this, we must admit that $\Upsilon^q_p(c)$ has no concept of dimension. The same argument can be proposed concerning the structure of $\overset{(k)}{\Upsilon}{}^q_p(c)$ as a sub-module of $\Upsilon^q_p(c)$. Despite this behaviour could seem awful, in practice it is not a problem, and several mathematical standard objects we already defined like the space of smooth sections of $T^p_q M$ or $C^\infty(M)$ just to quote some share this property.

Property 48: From the adapted local Ellis representation of the multipoles it is trivial to realise that the order of the multipole is equal to the maximum number of derivations acting on the test tensor fields before the integration process.

5.3.3 Dixon parameters of the multipoles as tensor field restricted on a worldline

We have already proven how the set of Dixon generators of the multipoles strongly depends on the trivialisation of TM. If a change of trivialisation is performed then the new set of generators are linked with the old one with the action of the linear group:

$$\Delta'^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} = c^{\star}(\overline{\Lambda}^{\alpha_1}_{\lambda_1})...c^{\star}(\overline{\Lambda}^{\alpha_k}_{\lambda_k})c^{\star}(\overline{\Lambda}^{\gamma_1}_{\nu_1})...c^{\star}(\overline{\Lambda}^{\gamma_q}_{\nu_q})c^{\star}(\Lambda^{\beta_1}_{\mu_1})....c^{\star}(\Lambda^{\beta_p}_{\mu_p})\Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} \tag{5.3.44}$$

where Λ is the linear map defining locally the change of frames on TM. Considering this it is easy to show that the transformation rules for the local Dixon parameters coincides with the rules for the local sections of $T^{(k)+q}_{p\ \ c(s)}M$. Let us consider an arbitrary multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$ as follow:

$$\mathcal{T} = \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{N} \alpha'^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright \Delta'^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} = \tag{5.3.45}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{N} \alpha'^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} \triangleright c^{\star}(\overline{\Lambda}^{\alpha_1}_{\lambda_1})...c^{\star}(\overline{\Lambda}^{\alpha_k}_{\lambda_k})c^{\star}(\overline{\Lambda}^{\gamma_1}_{\nu_1})...c^{\star}(\overline{\Lambda}^{\gamma_q}_{\nu_q})c^{\star}(\Lambda^{\beta_1}_{\mu_1})....c^{\star}(\Lambda^{\beta_p}_{\mu_p})\Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} = \tag{5.3.46}$$

$$= \sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}} \sum_{k=0}^{N} \alpha^{\alpha_{\overline{k}}\gamma_{\overline{q}}}_{(i)\ \ \beta_{\overline{p}}} \triangleright \Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} \tag{5.3.47}$$

Having that:

$$\alpha^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}} c^{\star}(\Lambda^{\alpha_1}_{\lambda_1})...c^{\star}(\Lambda^{\alpha_k}_{\lambda_k})c^{\star}(\Lambda^{\gamma_1}_{\nu_1})...c^{\star}(\Lambda^{\gamma_q}_{\nu_q})c^{\star}(\overline{\Lambda}^{\beta_1}_{\mu_1})....c^{\star}(\overline{\Lambda}^{\beta_p}_{\mu_p}) = \alpha'^{\alpha_{\overline{k}}\gamma_{\overline{q}}}_{(i)\ \ \beta_{\overline{p}}} \tag{5.3.48}$$

Thence considering what we know about the bundle theory we have to admit that the Dixon parameters $\alpha^{\lambda_{\overline{k}}\nu_{\overline{q}}}_{(i)\ \ \mu_{\overline{p}}}$ are the local expressions of an appropriate set of global sections $\overset{(k)}{\alpha} \in \Gamma T^{(k)+q}_{p\ \ c(s)}M \,|\, k \in [0, N] \subset \mathbb{N}$. In fact since the local expression satisfies the appropriate cocycle rules we know that it is possible to glue together all the local sections into a single global section.

Definition 82: Let $c : \mathbb{R} \to M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M. Let us suppose to trivialize TM using the local frame $(e_{(i)\mu})$ and $T^\star M$ using the local coframe $(e^\mu_{(i)})$. Fixing a smooth partition of the unity ψ_i subordinate to (U_i) and a global coframe of $\Gamma\Lambda^1\mathbb{R}$ denoted by ds, let $(\Delta^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} \mid k \in [0, N] \subset \mathbb{N})$ be the set of Dixon generators for the multipoles up to order N. Given a multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \overset{(k)}{\alpha}{}^{\alpha_{\overline{k}}\gamma_{\overline{q}}}_{(i)\ \beta_{\overline{p}}} \triangleright \Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} \tag{5.3.49}$$

we define the ***Dixon tensor parameters*** related to the multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$ the n-tuple of global sections:

$$(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M \tag{5.3.50}$$

such that:

$$\overset{(k)}{\alpha}(e_{(i)\beta_{\overline{p}}}, e^{\alpha_{\overline{k}}}_{(i)}, e^{\gamma_{\overline{q}}}_{(i)}) = \alpha^{\alpha_{\overline{k}}\gamma_{\overline{q}}}_{(i)\ \beta_{\overline{p}}} \qquad \forall U_i \in \mathcal{A} \mid U_i \cap c(\mathbb{R}) \neq \varnothing \qquad \forall k \in [0, N] \subset \mathbb{N} \tag{5.3.51}$$

Property 49: Let us stress that by definition, each Dixon tensor parameter $\overset{(k)}{\alpha}$ related to a multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$ must be completely symmetric in the first k upper indices.

Property 50: Let $c : \mathbb{R} \to M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M. Let us suppose to trivialise TM using the local frame $(e_{(i)\mu})$ and $T^\star M$ using the local coframe $(e^\mu_{(i)})$. Fixing a smooth partition of the unity ψ_i subordinate to (U_i) and a global coframe of $\Gamma\Lambda^1\mathbb{R}$ denoted by ds, let $(\Delta^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} \mid k \in [0, N] \subset \mathbb{N})$ be the set of Dixon generators for the multipoles $\Upsilon^q_p(c)$. A set of Dixon tensor parameters

$$(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M \tag{5.3.52}$$

always defines canonically a multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$ via:

$$\mathcal{T} = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \overset{(k)}{\alpha}\big(e_{(i)\beta_{\overline{p}}}, e^{\alpha_{\overline{k}}}_{(i)}, e^{\gamma_{\overline{q}}}_{(i)}\big) \triangleright \Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} \tag{5.3.53}$$

Proof. The proof trivially follows from the property of the Dixon tensor parameters and their local expression with respect to the given frame. Acting with the give expression with an arbitrary test tensor field one can see how it is a multipole. To prove that each bunch of Dixon tensor parameters $\overset{(k)}{\alpha}$ defines canonically a multipole $\mathcal{T}$ it is enough to check that the local expression is preserved under change of local frames, therefore despite the given definition of $\mathcal{T}$ is invariant under the change of local frame. $\square$

Considering the previous properties one can realise that, it is always possible to associate a set of Dixon tensor parameters $\overset{(k)}{\alpha} \in \Gamma T^{(k)+q}_{p\;c(s)} M \,|\, k \in [0, N] \subset \mathbb{N}$ to a multipole $\mathcal{T} \in$ using the Dixon representation. Although in appearance this link should strongly depend on the frame used to induce a specific set of local Dixon generators, at the end of the day, it does not depend on the chosen trivialisation of TM. This result is quite important because it shows how the Dixon tensor parameters and the multipoles are linked by an intrinsic correspondence of geometrical objects rather than just correspondence of local coordinate expressions.

Definition 83: Let $c : \mathbb{R} \to M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M, and let $(\overset{0}{\alpha}, ..., \overset{N}{\alpha}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\;c(s)} M$ be a set of Dixon parameters.

Choosing arbitrarily a local frame $(e_{(i)\mu})$, the local coframe $(e^{\mu}_{(i)})$, a smooth partition of the unity ψ_i subordinate to (U_i) and a global coframe of $\Gamma\Lambda^1\mathbb{R}$ denoted by ds, let $(\Delta^{\mu_{\overline{p}}}_{(i)\lambda_{\overline{k}}\nu_{\overline{q}}} \mid k \in [0, N] \subset \mathbb{N})$ be the set of Dixon generators for the multipoles $\overset{(N)}{\Upsilon}{}^q_p(c)$. We define ***the Dixon Tensorial Parametrization*** the $C^\infty(\mathbb{R})$-linear canonical map $\overset{(N)}{\Omega} : \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\;c(s)} M \to \overset{(N)}{\Upsilon}{}^q_p(c)$ such that:

$$\overset{(N)}{\Omega}\big(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}\big) = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \overset{(k)}{\alpha}\big(e_{(i)\beta_{\overline{p}}}, e^{\alpha_{\overline{k}}}_{(i)}, e^{\gamma_{\overline{q}}}_{(i)}\big) \triangleright \Delta^{\beta_{\overline{p}}}_{(i)\alpha_{\overline{k}}\gamma_{\overline{q}}} = \mathcal{T} \tag{5.3.54}$$

Using the definition of the Dixon generator, the action of such a multipole is written as:

$$[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^{\star}\left(\nabla_{\alpha_{\overline{k}}}(\phi)^{\beta_{\overline{p}}}_{\gamma_{\overline{q}}}\right) \overset{(k)}{\alpha}(e_{(i)\beta_{\overline{p}}}, e^{\alpha_{\overline{k}}}_{(i)}, e^{\gamma_{\overline{q}}}_{(i)}) c^{\star}(\psi_i) ds \tag{5.3.55}$$

Property 51: It is trivial to notice that the map is $C^{\infty}(\mathbb{R})$-linear by definition as well as surjective, therefore $\overset{(N)}{\Omega} : \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M \to \overset{(N)}{\Upsilon}{}^{q}_{p}(c)$ is a surjective homomorphisms of modules (epimorphism) by construction. It is enough to check the example given previously at the beginning of the chapter to realise that in general the Dixon Tensorial Parametrization is not injective because several different Dixon tensor Parameters can define the same multipole, however it represents a canonical way to map canonically an appropriate set of tensors field defined upon the image of a closed embedding into the space of the multipoles preserving the linear structure.

Definition 84: Given the equality of funtionals "=", we define the equivalence relation of Dixon Tensor Parameters "∼":

$$(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \sim (\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \Leftrightarrow \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) = \overset{(N)}{\Omega}(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \tag{5.3.56}$$

with $(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M$ and $(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M$. We define then the equivalence class of Dixon tensor parameters:

$$[(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})] = \left\{(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M \mid \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) = \overset{(N)}{\Omega}(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta})\right\} \tag{5.3.57}$$

Property 52: From the property of the surjective maps it is obvious to conclude that the Dixon tensorial parametrization $\overset{(N)}{\Omega}$ restricted upon the quotien set $\bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_{\sim}$ is a bijective map.

Definition 85: The quotient set $\bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_{\sim}$ is called ***set of footprints of*** $\overset{(N)}{\Upsilon}{}^{q}_{p}(c)$ ***induced by the Dixon representation*** or simply ***set of footprints of*** $\overset{(N)}{\Upsilon}{}^{q}_{p}(c)$. The elements belonging to it, called ***footprints***, define uniquely all the multipoles in $\overset{(N)}{\Upsilon}{}^{q}_{p}(c)$ due to the morphism $\overset{(N)}{\Omega}$.

Property 53: The following property for the Dixon Tensorial Parametrization follows immediately from the definition:

$$\overset{(N+1)}{\Omega}(\overset{(0)}{\alpha},...,\overset{(N+1)}{\alpha}) = \overset{(N+1)}{\Omega}(0,...,0,\overset{(N+1)}{\alpha}) + \overset{(N)}{\Omega}(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha}) \tag{5.3.58}$$

Since $\overset{(N)}{\Upsilon}{}^q_p(c) \subset \overset{(N+1)}{\Upsilon}{}^q_p(c)$ the sum of a order N multipole with an $N+1$ order multipole is well defined.

5.3.4 Covariant choice of the Dixon moments induced by a covector field on the worldline

We have seen how the Dixon Parametrization fixes a canonical covariant bijective map between $\bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_\sim$ and $\overset{(N)}{\Upsilon}{}^q_p(c)$. It is very interesting at this stage to investigate the structure of the footprints interpreted as equivalence classes of n-tuples of tensor fields and to choose explicitly at least one representative of the footprints. We start with the lemma:

Lemma 43: Let $\overset{(N)}{\Upsilon}{}^q_p(c)$ and $\bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_\sim$ the module of the multipoles and the set of the footprints. Let us denote with $[(\overset{(0)}{0},...,\overset{(N)}{0})] \in \Gamma T^{(k)+q}_{p\ c(s)} M/_\sim$ the footprint related to the null multipole. Given an arbitrary footprint $[(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})] \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_\sim$ let be $(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})$ an arbitrary representative. We have that $(\overset{(0)}{\beta},...,\overset{(N)}{\beta}) \in [(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})]$ if and only if there exists $(\overset{(0)}{\gamma},...,\overset{(N)}{\gamma}) \in [(\overset{(0)}{0},...,\overset{(N)}{0})]$ such that:

$$(\overset{(0)}{\beta},...,\overset{(N)}{\beta}) = (\overset{(0)}{\alpha},...,\overset{(N)}{\alpha}) + (\overset{(0)}{\gamma},...,\overset{(N)}{\gamma}) \tag{5.3.59}$$

In other words each footprint can be obtained just by summing (with the standard sum) a representative (chosen arbitrarily) with all the members belonging to the footprint of the distribution. We can then write with an abuse of notation:

$$\forall [(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})] \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_\sim \Rightarrow [(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})] = \left\{(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha}) + [(\overset{(0)}{0},...,\overset{(N)}{0})]\right\} \tag{5.3.60}$$

Proof. Given an arbitrary footprint $[(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})] \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_\sim$ let $(\overset{(0)}{\alpha},...,\overset{(N)}{\alpha})$ be an arbitrary representative. For each $(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in [(\overset{(0)}{0}, ..., \overset{(N)}{0})]$ we have:

$$\left[\phi, \overset{(N)}{\Omega}[(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) + (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})]\right] = \left[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) + \overset{(N)}{\Omega}(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})\right] = \tag{5.3.61}$$

$$= \left[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})\right] + \left[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})\right] = \left[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})\right] + 0 = \tag{5.3.62}$$

$$= \left[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})\right] \tag{5.3.63}$$

therefore, from the definition, we have to conclude that

$$\forall(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in [(\overset{(0)}{0}, ..., \overset{(N)}{0})] \Rightarrow (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \sim (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) + (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \tag{5.3.64}$$

By contrast let us suppose that there exists some Dixon parameters $(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \in [(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})]$ such that

$$(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \neq (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) + (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \quad , \quad \forall(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in [(\overset{(0)}{0}, ..., \overset{(N)}{0})] \tag{5.3.65}$$

Hence $\forall(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in [(\overset{(0)}{0}, ..., \overset{(N)}{0})]$ we can state:

$$(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) - (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \neq (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \tag{5.3.66}$$

and we must conclude that:

$$(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) - (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \notin [(\overset{(0)}{0}, ..., \overset{(N)}{0})] \tag{5.3.67}$$

But since $(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \in [(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})]$ we must have that $(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) \sim (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})$ that means $\overset{(N)}{\Omega}(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) = \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})$, so we must admit that

$$0 = \left[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) - \overset{(N)}{\Omega}(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})\right] = \left[\phi, \overset{(N)}{\Omega}[(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) - (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})]\right] \tag{5.3.68}$$

leading to the contradiction:

$$(\overset{(0)}{\beta}, ..., \overset{(N)}{\beta}) - (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \in [(\overset{(0)}{0}, ..., \overset{(N)}{0})] \tag{5.3.69}$$

$\square$

Considering the lemma we can then give another equivalent definition of footprint

Definition 86: The footprint associated to the null distribution $0 \in \overset{(N)}{\Upsilon}{}^q_p(c)$ can be defined as:

$$[(\overset{(0)}{0}, ..., \overset{(N)}{0})] = Ker(\overset{(N)}{\Omega}) \tag{5.3.70}$$

The footprint represented by the Dixon moments $(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M$ is spanned by

$$[(\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha})] = \left\{ (\overset{(0)}{\alpha}, ..., \overset{(N)}{\alpha}) + (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \mid (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in Ker(\overset{(N)}{\Omega}) \right\} \tag{5.3.71}$$

This is very useful because it is enough to investigate the structure of the footprint related to the null distribution to get the information we need to define all the footprints of each multipole. Unfortunately the structure of $Ker(\overset{(N)}{\Omega})$ is very complicated. For an arbitrary order multipole is beyond our possibilities to provide an explicit form of the equivalence class, but we will see how there always exists at least a way to choose a representative for each footprint in a completely covariant fashion.

Property 54: It is easy to check that $\bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_{\sim}$ is a $C^{\infty}(\mathbb{R})$-module, in fact the equivalence relation is $C^{\infty}(\mathbb{R})$-linear, so it preserves $C^{\infty}(\mathbb{R})$-linear combinations. However, if this consideration is not enough,it is sufficient to chose the representative of each equivalence class by fixing a representative of the null distribution, and to check that the sum and multiplication by a scalar defined on the representatives satisfy the condition required by the $C^{\infty}(\mathbb{R})$-module definition.

Property 55: Since the Dixon tensorial parametrization $\overset{(N)}{\Omega}$ restricted upon the module $\bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M/_{\sim}$ is a bijective $C^{\infty}(\mathbb{R})$-linear map therefore it is an isomorphism of modules. In this perspective we can state that the set of multipoles is isomorphic to the set of its footprints.

Property 56: This is a key result, since the set of all the Dixon moments is isomorphic with respect the set of multipoles, so a Dixon moments is able to identify uniquely a multipole and the $C^\infty(\mathbb{R})$-linear operations on the multipoles can be directly cast in $C^\infty(\mathbb{R})$-linear operations upon the Dixon moments.

We are going to see how there always exists at least one coordinate free choice to fix uniquely the representative of the footprints.

Theorem 10: Let $c : \mathbb{R} \to M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M. Let us suppose to trivialize TM using an adapted local frame $(e_{(i)\mu})$ and let us denote with $(e_{(i)\mu|_{c(s)}}) = (\dot{c}, v_{(i)\overline{m}})$ its restriction on the image of the worldline. Let us define the induced adapted local coframe $(e^\mu_{(i)})$ using the relationship $e^\mu_{(i)}(e_{(i)\nu}) = \delta^\mu_\nu$ and let us denote with $(e^\mu_{(i)|_{c(s)}}) = (\tilde{\dot{c}}_{(i)}, \tilde{v}^{\overline{m}}_{(i)})$ its restriction on the image of the worldline. Let $n \in \Gamma T^\star_{c(s)} M$ be a smooth covector field defined upon the whole image of the worldline $c(s)$ satisfying $n(\dot{c})_{|_{c(s)}} \neq 0 \quad , \quad \forall s \in \mathbb{R}$. For each multipole $\mathcal{T} \in \overset{(N)}{\Upsilon}{}^q_p(c)$ there always exists a unique set of Dixon moments $(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in \bigoplus_{k=0}^N \Gamma T^{(k)+q}_{p\ c(s)} M$ satisfying

$$\overset{(N)}{\Omega}(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) = \mathcal{T} \tag{5.3.72}$$

such that

$$n \urcorner \overset{(k)}{\gamma} = 0 \quad , \quad \forall k \in [1, N] \subset \mathbb{N} \tag{5.3.73}$$

Proof. To prove the thesis we need to show that $\overset{(N)}{\Omega}$ is a bijection between the set of the Dixon moments satisfying the condition and the set of the multipoles $\overset{(N)}{\Upsilon}{}^q_p(c)$. Let us start with some preliminary considerations. Let us denote by $u^\mu_{(i)}$ the n-tuple $(\tilde{\dot{c}}_{(i)}, \tilde{v}^{\overline{a}}_{(i)})$ and let us consider the n-tuple $w^\mu_{(i)} = (n, \tilde{v}^{\overline{a}}_{(i)})$ of local covector fields defined on $c(s)$. We can then define a local linear map Λ such that $w^\mu_{(i)} = \Lambda^\mu_\nu u^\nu_{(i)}$ in the following explicit way

$$\begin{cases} n = n_0 \tilde{\dot{c}}_{(i)} + n_a \tilde{v^a}_{(i)} \\ \tilde{v^b}_{(i)} = \delta^b_a \tilde{v^a}_{(i)} \end{cases} \tag{5.3.74}$$

This defines a triangular matrix at each point of $c(s)$, linking $(n, \tilde{v}^{\overline{a}}_{(i)})$ with $(\tilde{\dot{c}}_{(i)}, \tilde{v}^{\overline{a}}_{(i)})$. Since $n(\dot{c})_{|_{c(s)}} \neq 0 \quad , \quad \forall s \in \mathbb{R}$ we have that $n_0 \neq 0$ for each value of s, it is obvious to notice that this map is a maximum rank linear map due to this non null determinant at each point belonging to the worldline $c(s)$. Therefore since $(\tilde{\hat{c}}_{(i)}, \tilde{v}^{\overline{a}}_{(i)})$ is a local coframe of $T^{\star}_{c(s)}M$, the n-tuple $(n, \tilde{v}^{\overline{a}}_{(i)})$ is still local coframe of $T^{\star}_{c(s)}M$ well defined on $U_i \cap c(s)$. Because of the duality relationship we can then define a local frame $w_{(i)\mu} = (\hat{n}_{(i)}, v_{(i)\overline{m}})$ as usual as $w^{\mu}_{(i)}(w_{(i)\nu}) = \delta^{\mu}_{\nu}$. Now let us consider some arbitrary Dixon moments $\overset{(k)}{\gamma} \in (\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})$ satisfying $n \neg \overset{(k)}{\gamma} = 0$ for each $k \in [1, N] \subset \mathbb{N}$. Considering the hypothesis, they can be expressed locally with respect $w^{\mu}_{(i)} = (n, \tilde{v}^{\overline{a}}_{(i)})$ and $w_{(i)\mu} = (\hat{n}_{(i)}, v_{(i)\overline{m}})$:

$$\overset{(k)}{\gamma} = \overset{(k)}{\gamma}{}^{l_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \quad w_{(i)l_{\overline{k}}} \otimes w_{(i)\nu_{\overline{q}}} \otimes w^{\mu_{\overline{p}}}_{(i)} \tag{5.3.75}$$

and they must be completely symmetric in the first k upper indices. They can also be expressed locally with respect $u^{\mu}_{(i)} = (n, \tilde{v}^{\overline{a}}_{(i)})$ and $u_{(i)\mu} = (\hat{n}_{(i)}, v_{(i)\overline{m}})$:

$$\overset{(k)}{\gamma} = \overset{(k)}{\gamma}{}^{l_{\overline{k}}\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \Lambda^{\alpha_1}_{l_1} ... \Lambda^{\alpha_k}_{l_k} \Lambda^{\gamma_1}_{\nu_1} ... \Lambda^{\gamma_q}_{\nu_q} \overline{\Lambda}^{\mu_1}_{\beta_1} ... \Lambda^{\mu_p}_{\beta_p} \quad u_{(i)\alpha_{\overline{k}}} \otimes u_{(i)\gamma_{\overline{q}}} \otimes u^{\beta_{\overline{p}}}_{(i)} = \tag{5.3.76}$$

$$= \overset{(k)}{\gamma}{}'^{\alpha_{\overline{k}}\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} \quad u_{(i)\alpha_{\overline{k}}} \otimes u_{(i)\gamma_{\overline{q}}} \otimes u^{\beta_{\overline{p}}}_{(i)} \tag{5.3.77}$$

and they must be still completely symmetric in the first k upper indices.

At this stage we have all we need to perform the proof via induction upon the order N. First, let us prove the thesis for the multipoles belonging in $\overset{(1)}{\Upsilon}{}^{q}_{p}(c)$. Given an arbitrary set of Dixon multipoles $(\overset{(0)}{\gamma}, \overset{(1)}{\gamma})$ such that $n \neg \overset{(1)}{\gamma} = 0$ we have $\forall \phi \in \Gamma_0 T^p_q M$:

$$[\phi, \overset{(1)}{\Omega}(\overset{(0)}{\gamma}, \overset{(1)}{\gamma})] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\nabla_{\alpha}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(1)}{\gamma}{}'^{\alpha\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds + \tag{5.3.78}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\phi^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(0)}{\gamma}{}'^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds = \tag{5.3.79}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\nabla_{a}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(1)}{\gamma}{}'^{a\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds + \tag{5.3.80}$$

$$+ \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\phi^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(0)}{\gamma}{}'^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds - \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\phi^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \frac{D}{ds} [\tilde{\hat{c}}_{(i)} \neg \overset{(1)}{\gamma}]'^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds = \tag{5.3.81}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\nabla_{a}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(1)}{\gamma}{}^{l\,\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \Lambda^{a}_{l} \Lambda^{\gamma_1}_{\nu_1} ... \Lambda^{\gamma_q}_{\nu_q} \overline{\Lambda}^{\mu_1}_{\beta_1} ... \overline{\Lambda}^{\mu_p}_{\beta_p} ds + \tag{5.3.82}$$

$$+\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star(\phi^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}})c^\star(\psi_i)\{\overset{(0)}{\gamma}-\frac{D}{ds}[\tilde{\dot{c}}_{(i)}\neg\overset{(1)}{\gamma}]\}^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\Lambda^{\gamma_1}_{\nu_1}...\Lambda^{\gamma_q}_{\nu_q}\overline{\Lambda}^{\mu_1}_{\beta_1}...\overline{\Lambda}^{\mu_p}_{\beta_p}ds= \tag{5.3.83}$$

$$=\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star(\nabla_a(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}})c^\star(\psi_i)\alpha^{a\,\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}ds+\sum_{\substack{U_i\in\mathcal{A}\\ U_i\cap c(\mathbb{R})\neq\varnothing}}\int_{\mathbb{R}} c^\star(\phi^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}})c^\star(\psi_i)\alpha^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}ds \tag{5.3.84}$$

But this is an adapted Dixon representation of a multipole induced by the adapted local frame $e_{(i)\mu}$ where the adapted Dixon moments are defined as:

$$\begin{cases}\overset{(1)}{\gamma}{}^{a\,\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\Lambda^{\gamma_1}_{\nu_1}...\Lambda^{\gamma_q}_{\nu_q}\overline{\Lambda}^{\mu_1}_{\beta_1}...\overline{\Lambda}^{\mu_p}_{\beta_p}=\alpha^{a\,\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}\\ \{\overset{(0)}{\gamma}-\frac{D}{ds}[\tilde{\dot{c}}_{(i)}\neg\overset{(1)}{\gamma}]\}^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\Lambda^{\gamma_1}_{\nu_1}...\Lambda^{\gamma_q}_{\nu_q}\overline{\Lambda}^{\mu_1}_{\beta_1}...\overline{\Lambda}^{\mu_p}_{\beta_p}=\alpha^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}\end{cases} \tag{5.3.85}$$

$$\begin{cases}\overset{(1)}{\gamma}{}^{a\,\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}=\alpha^{a\,\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}\overline{\Lambda}^{\nu_1}_{\gamma_1}...\overline{\Lambda}^{\nu_q}_{\gamma_q}\Lambda^{\beta_1}_{\mu_1}...\Lambda^{\beta_p}_{\mu_p}\\ \overset{(0)}{\gamma}{}^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}=\alpha^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}\overline{\Lambda}^{\nu_1}_{\gamma_1}...\overline{\Lambda}^{\nu_q}_{\gamma_q}\Lambda^{\beta_1}_{\mu_1}...\Lambda^{\beta_p}_{\mu_p}+\frac{D}{ds}[\tilde{\dot{c}}_{(i)}\neg\overset{(1)}{\gamma}]\}^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}\end{cases} \tag{5.3.86}$$

Since each bunch of adapted Dixon moments defines uniquely a multipole then we can conclude that the components $\overset{(1)}{\gamma}{}^{a\,\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ there always exist and must be unique for each multipole in $\overset{(1)}{\Upsilon}{}^q_p(c)$, so it is $\overset{(1)}{\gamma}$. Since $\overset{(1)}{\gamma}$ and $\alpha^{\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}}$ the components $\overset{(0)}{\gamma}{}^{\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}}$ exist and are unique. So $\overset{(0)}{\gamma}$ exists and it is unique. Hence we must admit that the Dixon moments $(\overset{(0)}{\gamma},\overset{(1)}{\gamma})$ such that $n\neg\overset{(1)}{\gamma}=0$ define uniquely all multipoles in $\overset{(1)}{\Upsilon}{}^q_p(c)$

Now let us suppose the thesis holds for the multipoles belonging to $\overset{(N)}{\Upsilon}{}^q_p(c)$ and let us prove it for the multipoles belonging to $\overset{(N+1)}{\Upsilon}{}^q_p(c)$.

$$[\phi,\overset{(N+1)}{\Omega}(\overset{(0)}{\gamma},...,\overset{(N+1)}{\gamma})]= \tag{5.3.87}$$

$$=[\phi,\overset{(N)}{\Omega}(\overset{(0)}{\gamma},...,\overset{(N)}{\gamma})+\overset{(N+1)}{\Omega}(0,...,0,\overset{(N+1)}{\gamma})]= \tag{5.3.88}$$

$$=[\phi,\overset{(N)}{\Omega}(\overset{(0)}{\gamma},...,\overset{(N)}{\gamma})]+[\phi,\overset{(N+1)}{\Omega}(0,...,0,\overset{(N+1)}{\gamma})] \tag{5.3.89}$$

Let us consider the second term in the expression:

$$[\phi, \overset{(N+1)}{\Omega}(0, ..., 0, \overset{(N+1)}{\gamma})] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\nabla_{\alpha_{\overline{N+1}}}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(N+1)}{\gamma'}{}^{\alpha_{\overline{N+1}}\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds \tag{5.3.90}$$

Using the preliminary considerations we can recast the expression as follow:

$$[\phi, \overset{(N+1)}{\Omega}(0, ..., 0, \overset{(N+1)}{\gamma})] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\nabla_{\alpha_{\overline{N+1}}}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(N+1)}{\gamma'}{}^{\alpha_{\overline{N+1}}\gamma_{\overline{q}}}_{(i)\beta_{\overline{p}}} ds = \tag{5.3.91}$$

$$= \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \int_{\mathbb{R}} c^{\star}(\nabla_{\alpha}(\phi)^{\beta_{\overline{p}}}_{(i)\gamma_{\overline{q}}}) c^{\star}(\psi_i) \overset{(N+1)}{\gamma}{}^{l_{\overline{N+1}}\,\nu_{\overline{q}}}_{(i)\mu_{\overline{p}}} \Lambda^{\alpha_1}_{l_1} ... \Lambda^{\alpha_{N+1}}_{l_{N+1}} \Lambda^{\gamma_1}_{\nu_1} ... \Lambda^{\gamma_q}_{\nu_q} \overline{\Lambda}^{\mu_1}_{\beta_1} ... \overline{\Lambda}^{\mu_p}_{\beta_p} ds \tag{5.3.92}$$

From the previous lemma we can state that there always exists a bunch of smooth local scalar fields $\beta^{m_{\overline{j}}\mu_{\overline{p}}}_{\nu_{\overline{q}}} \in C^{\infty}U, \forall j \in [0, k-1] \subset \mathbb{N}$ such that:

$$\int_{c} c^{\star}(\nabla^{N+1}_{\lambda_{\overline{N+1}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \; \overset{(N+1)}{\gamma'}{}^{\lambda_{\overline{N+1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds = \tag{5.3.93}$$

$$= \int_{\mathbb{R}} c^{\star}(\nabla^{N+1}_{m_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \; \overset{(N+1)}{\gamma'}{}^{m_{\overline{N+1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds + \sum_{j=0}^{N} \int_{\mathbb{R}} c^{\star}(\nabla^{j}_{m_{\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \; \beta'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds \tag{5.3.94}$$

The smooth local scalar fields $\beta'^{m_{\overline{j}}\mu_{\overline{p}}}_{\nu_{\overline{q}}} \in C^{\infty}U$ can be defined from the components of $\overset{(N+1)}{\gamma}$ acting with several Sums, Covariant Derivations, Contractions and $C^{\infty}\mathbb{R}$-linear maps. Despite the explicit form of each $\beta'^{m_{\overline{j}}\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ is extremely complicated, at least we can say that all of them must be functions of the components of $\overset{(N+1)}{\gamma}$. So once $\overset{(N+1)}{\gamma}$ is fixed, then all the $\beta^{m_{\overline{j}}\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ are fixed as well uniquely. Let us take now the first term: we can use the adapted Dixon representation to express it:

$$[\phi, \overset{(N)}{\Omega}(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{j=0}^{N} \int_{\mathbb{R}} c^{\star}(\nabla^{j}_{m_{\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \; \alpha'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds \tag{5.3.95}$$

Where the $\alpha'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}$ are the adapted Dixon moments. Let us stress once again the adapted Dixon moments are associated uniquely to each multipole in $\overset{(N)}{\Upsilon}{}^{q}_{p}(c)$. Furthermore taking into account the inductive hypothesis, each set of Dixon moments $(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})$ is also associated uniquely to each multipole in $\overset{(N)}{\Upsilon}{}^{q}_{p}(c)$. Therefore there must exist a bijection between $(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})$ and $(\alpha'^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}}, ..., \alpha'^{m_{\overline{N+1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}})$ and the choice of $(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})$ fixes uniquely $(\alpha'^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}}, ..., \alpha'^{m_{\overline{N+1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}})$ and vice-versa. Hence putting together the expressions we have:

$$\int_{\mathbb{R}} c^{\star}(\nabla^{N+1}_{m_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \; \overset{(N+1)}{\gamma'}{}^{m_{\overline{N}+1}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} ds + \sum_{j=0}^{N} \int_{\mathbb{R}} c^{\star}(\nabla^{j}_{m_{\overline{j}}}(\phi)^{\beta_{\overline{q}}}_{\alpha_{\overline{p}}}) \; \{\alpha'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} + \beta'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}\} ds \qquad (5.3.96)$$

where each term $\{\alpha'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} + \beta'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}\}$ is uniquely fixed by the set $\overset{(N)}{\Omega}(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}, \overset{(N+1)}{\gamma})$. But this is an adapted Dixon representation therefore for each multipole this must be unique. So the components $\overset{(N+1)}{\gamma'}{}^{m_{\overline{N}+1}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}$ there always exist and must be unique for each multipole in $\overset{(N+1)}{\Upsilon}{}^{q}_{p}(c)$ so with the same reasoning performed for the $N = 1$ case we can say that for each multipole the Dixon moment $\overset{(N+1)}{\gamma}$ such that $n\neg\overset{(N+1)}{\gamma} = 0$ always exists and it is unique. But since $\overset{(N+1)}{\gamma}$ is fixed uniquely for each multipole, then each set of scalar fields $\beta'^{m_{\overline{j}}\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}}$ are uniquely determined for each multipole. Likewise, each set of scalar fields $\alpha'^{m_{\overline{j}}\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}}$ is uniquely determined because $\{\alpha'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}} + \beta'^{m_{\overline{j}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}}\}$ is related to the adapted Dixon representation. But due the existence of the bijection, fixing all the n-tuple $(\alpha'^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}}, ..., \alpha'^{m_{\overline{N+1}}\alpha_{\overline{p}}}_{\beta_{\overline{q}}})$ we select uniquely the set of Dixon moments $(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \mid n\neg\overset{(k)}{\gamma} = 0, \forall k \in [1, N] \subset \mathbb{N}$. Therefore we can conclude that for each multipole $\mathcal{T} \in \overset{(N+1)}{\Upsilon}{}^{q}_{p}(c)$ the set $(\overset{(0)}{\gamma}, ..., \overset{(N+1)}{\gamma})$ such that $\mathcal{T} = \overset{(N)}{\Omega} + 1(\overset{(0)}{\gamma}, ..., \overset{(N+1)}{\gamma})$ satisfying $n\neg\overset{(k)}{\gamma} = 0, \forall k \in [1, N+1] \subset \mathbb{N}$ always exists and it is unique. $\square$

So finally we reached our achievement: we are now able to associate uniquely all the elements of $\overset{(N)}{\Upsilon}{}^{q}_{p}(c)$ to a subset of $\Gamma T^{(k)+q}_{p\ c(s)}M$ through the map $\overset{(N)}{\Omega}$. Let us remark that this bijection is completely covariant (is a bijection between well defined geometrical objects and does not depend on the choices of atlas on M) and completely canonical (does not depend on the local frames used to trivialise the tangent bundle TM). The bijection is however depending on the choice of the covector field n defining the polarisation of the Dixon moments. This choice is in principle completely arbitrary and it represents a degree of freedom.

Definition 87: Let $c : \mathbb{R} \to M$ be a worldline and $A = (U_i, \phi_{(i)})$ an atlas of M. Let $n \in \Gamma T^{\star}_{c(s)}M$ be a smooth covector field defined upon the whole image of the worldline

$c(s)$ satisfying $n(\dot{c})_{|_{c(s)}} \neq 0 \quad , \quad \forall s \in \mathbb{R}$. The set of Dixon moments

$$(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma}) \in \bigoplus_{k=0}^{N} \Gamma T^{(k)+q}_{p\ c(s)} M \tag{5.3.97}$$

satisfying:

$$n \lrcorner \overset{(k)}{\gamma} = 0, \forall k \in [1, N] \subset \mathbb{N} \tag{5.3.98}$$

are called ***transverse polarised Dixon moments with respect to*** n or just ***n-transverse Dixon moments***. Each Dixon moment $\overset{(k)}{\gamma}$ is then called ***Dixon k-pole moment transverse to*** n.

This very important result closes the circle, allowing us to write all the multipoles using the transverse polarised Dixon representation. Therefore, given a smooth covector field n such that $n(\dot{c})_{c(s)} \neq 0$ for each $s \in \mathbb{R}$ the generic linear functionals we called multipoles always admits a local expression of the action on the tensor fields as follow :

$$[\phi, \mathcal{T}] = [\phi, \overset{(N)}{\Omega}(\overset{(0)}{\gamma}, ..., \overset{(N)}{\gamma})] = \sum_{\substack{U_i \in \mathcal{A} \\ U_i \cap c(\mathbb{R}) \neq \varnothing}} \sum_{k=0}^{N} \int_{\mathbb{R}} c^{\star}(\nabla^k_{\mu_{\overline{k}}}(\phi)^{\beta_{\overline{q}}}_{(i)\alpha_{\overline{p}}}) \ \gamma^{\mu_{\overline{k}}\alpha_{\overline{p}}}_{(i)\beta_{\overline{q}}} ds \tag{5.3.99}$$

with

$$n \lrcorner \overset{(k)}{\gamma} = 0 \qquad , \qquad \forall k \in [1, N] \subset \mathbb{N} \tag{5.3.100}$$

that coincides exactly with the result given by Dixon in its work [1][2][3] except for some combinatoric factors overall easily absorbed in the definition of $\overset{(k)}{\gamma}$. However despite what Dixon claimed in this method, the construction of this unique set of moments does depend just on the existence of an affine connection on the manifold, the existence of a closed embedding $c : \mathbb{R} \hookrightarrow M$ and the choice of a global covector field n on $c(\mathbb{R})$ satisfying the previous constraint. In general no metric is required as already noted by [7].

5.4 Final considerations on the Dixon local representations

In this chapter we have proved how the Dixon representation of the multipoles naturally induces an explicit expression for the multipoles, as linear combination of higher order covariant differentials of the test tensor fields, integrated on a worldline. We have seen how, the most general Dixon representation, as well as the Ellis representation, is not endowed with enough structure to provide an isomorphism of modules with the set of the moments but despite the Ellis case, a clear immediate coordinate free interpretation of the Dixon parameters as tensor fields defined on the the image of the worldline can be easily achieved. If from a point of view this is very convenient, on the other hand we have that performing the calculations accounting on the Dixon representation is much more treacherous than relying on the Ellis representation, due to the non trivial commutation algebra of the higher order covariant differentials and their split in linear combination of symmetric parts. However it is still the most convenient known representation when the multipoles are used within fully relativistic frameworks as weak asymptotic expansions of regular compact support tensor fields, because it guarantees naturally a geometric coordinate free interpretation of the moments so possibly they are able to encode some physical information independently from the choices made by the observers to map the events on the space-time into $\mathbb{R}^m$ with a particular local chart. Some proposal concerning how to fix the uniqueness of the moments are provided. The Dixon representation induced by an adapted local chart can be very convenient for the calculations, but it is strongly dependent on the choice of a particular local chart so it is not compatible with the principle of general covariance. Another method independent from the choices of a particular local chart have been shown, involving the concept of transverse polarised frame with respect to a covector field defined on $c(\mathbb{R})$. This method leads us to find for the multipoles the very same local representation used by Dixon to interpret his definition of moments in terms of functionals acting on the local expression of the test tensor fields, widely used in the Pole-Dipole expansion of the energy-momentum of extended objects in General Relativity, ruled by the Mathisson-Papapetrou-Dixon equation. Although the standard method used by Dixon to achieve a decent mathematical definition of moments is strongly dependent on a particular choice of classes of space-times (the connection is the Levi Civita connection of the Lorentzian metric tensor) as well as some other physical choices (centre of mass), it has been proved here that the Dixon representation does not require any of that structures and can be still used within more general Relativistic and Geometrical Theories (e.g. f(R)-Palatini or other modified gravity theories). Furthermore this approach is not constrained to a particular model of tensor field, therefore in principle it can be applied also to define, in the same very coordinate free fashion, the multipole expansions of currents and sources related to other interaction fields play fundamental roles in the physical theories, for instance in Electromagnetism and other Gauge theories.

Chapter 6

Specific applications of the multipoles to the Relativistic Dynamics

In this chapter we are going to see how, using the self similar squeezing tensor field technology, it is possible to cast the equations for the moments of the multipoles related to the first and second order of the weak asymptotic expansion of the Energy-Momentum tensor fields of a free falling extended body both in General Relativity and when the connection is not fixed to be the Levi Civita connection of a specific metric. This approach provides also a clear coordinate-free interpretation of the geometrical meaning of Pole-Dipole approximation (and possibly beyond) for the dynamics of an extended object within a relativistic framework. In General Relativity several predictions are based on the dynamics of the "moments" in Pole-Dipole approximation [4][5][6][8][9][10][11][12][15][22] [23][24][25][26][27][28][29] [30], some of them attempting to extend the "moments" to non standard General Relativity [7][15][16]. Here the physics is not discussed since the only purpose is to show just that theoretically there is no mathematical limitation to the Dixon representation even for an arbitrary conection and without a metric. We want to stress that the physical meaning of the given generalisation is not discussed so it should not be considered as a realistic physical model. Let be M a manifold , $c : \mathbb{R} \to M$ a worldline. Let $T \in \Gamma_0 T^2 M$ be a smooth compact support tensor field associated to an extended free falling object. Let us also assume that T satisfy two strong requirements.

$$\begin{cases} \sigma^{(1,2)}T = T \\ div(T) = 0 \end{cases} \tag{6.0.1}$$

Usually, within General Relativity, the Energy-Momentum tensor field associated to an extended free falling object satisfies these two constraints, so in this perspective T can play the role of an Energy-Momentum tensor field. However at this stage, we are completely ignoring the physical nature of it. This can be an issue since in some other physical models outside of G.R. the divergence-less symmetric condition required on the Energy-Momentum tensor can be too strong (i.e. Einstein-Cartan theory). However the following derivation can be interpreted more as a methodological procedure rather than a full consistent physical model. The reader, if not completely comfortable with that, can interpret this approach just as a way to derive the dynamical equation of the multipoles related to a generic divergence-less symmetric rank two tensor field, without being forced to interpret it as the Energy-Momentum tensor. Obviously in that case the interpretation related to the free fall of spinning test particles is completely lost. However, in case the Energy-Momentum tensor cannot be symmetric, one can drop the constraint and follow the same procedure to find different equations of motion for the multipoles. Let us suppose that there exists on M a smooth top form ω such that $\nabla(\omega) = 0$ and $supp(T) \subset supp(\omega)$. Therefore we can easily prove using the definition of $\langle \quad \rangle_\omega$ that:

$$0 = [\phi, \langle div(T) \rangle_\omega] = [\nabla\phi, \langle T \rangle_\omega] \tag{6.0.2}$$

Let us suppose to build an appropriate one smooth parameter family of symmetric and divergenceless squeezed tensor fields T_ε with support on a worldline c, such that $div(T_\varepsilon) = 0 \quad , \quad \forall \varepsilon \in (0,1)$ and for $\varepsilon = \varepsilon_0 \in (0,1) \quad \Rightarrow \quad T = T_{\varepsilon_0}$. Let be M a manifold , $c : \mathbb{R} \to M$ a worldline. At this point we can perform a weak asymptotic expansion of the one parameter family in a neighbourhood of the parameter $\varepsilon = 0$ as proven in the previous chapter obtaining:

$$0 = [\phi, \langle div(T_\varepsilon) \rangle_\omega] = [\nabla\phi, \langle T_\varepsilon \rangle_\omega] = \sum_{k=0}^{N} [\nabla\phi, \mathcal{T}_k] \varepsilon^k + O(\varepsilon^{N+1}) = \sum_{k=0}^{N} [\nabla\phi, \mathcal{T}_k] \varepsilon^k + O(\varepsilon^{N+1}) \tag{6.0.3}$$

for each $\varepsilon \in (0, \varepsilon_0)$, for each $\phi \in \Gamma_0 T^p_q M$ smooth test tensor field, with $\mathcal{T}_k \in \overset{(k)}{\Upsilon}{}^2(c)$. Therefore we must conclude that for each order $k \in [0, N]$ we must satisfy:

$$[\nabla\phi, \mathcal{T}_k] = 0 \tag{6.0.4}$$

therefore:

$$[\phi, div(\mathcal{T}_k)] = 0 \tag{6.0.5}$$

Furthermore considering the symmetry of the tensor T, and the linearity of the braiding map we can state that:

$$\sum_{k=0}^{N}[\phi, \mathcal{T}_k]\varepsilon^k + O(\varepsilon^{N+1}) = [\phi, \langle T_\varepsilon\rangle_\omega] = [\phi, \langle\sigma^{(1,2)}(T_\varepsilon)\rangle_\omega] = [\sigma_{(1,2)}\phi, \langle T_\varepsilon\rangle_\omega] = \tag{6.0.6}$$

$$= \sum_{k=0}^{N}[\sigma_{(1,2)}\phi, \mathcal{T}_k]\varepsilon^k + O(\varepsilon^{N+1}) = \sum_{k=0}^{N}[\phi, \sigma^{(1,2)}\mathcal{T}_k]\varepsilon^k + O(\varepsilon^{N+1}) \tag{6.0.7}$$

for each $\varepsilon \in (0, \varepsilon_0)$, for each $\phi \in \Gamma_0 T^p_q M$ smooth test tensor field, with $\mathcal{T}_k \in \overset{(k)}{\Upsilon}{}^2(c)$. Hence this leads us to conclude that:

$$[\phi, \mathcal{T}_k] = [\phi, \sigma^{(1,2)}\mathcal{T}_k] \tag{6.0.8}$$

Considering this analogy, given $\mathcal{T} \in \overset{(k)}{\Upsilon}{}^2(c)$ such that:

$$\begin{cases} \sigma^{(1,2)}\mathcal{T} = \mathcal{T} \\ div(\mathcal{T}) = 0 \end{cases} \tag{6.0.9}$$

If T can be interpreted as an Energy-Momentum tensor field we call such multipole the ***Energy-Momentum k-pole*** because in this perspective it satisfies the constraints required to be eventually interpreted as the k-th term in the weak asymptotic expansion of a one parameter family of squeezed Energy-Momentum tensor fields.

6.1 Dynamics of a free falling test particle modelled as a Dixon monopole

Let be M a manifold , $c : \mathbb{R} \to M$ a worldline and let us suppose for simplicity that there exists an adapted $A = (U_i, \varphi_{(i)})$ such that $c \subset U_1$ and φ_1 is an adapted local chart. Let us consider $\{e_\alpha\}$ an arbitrary local frame defined on U and adapted to the embedding c (for instance ∂_α from an adapted local chart), so $\{e_\alpha\}_{|c} = \{\dot{c}, v_m\}$. Let $\{e^\alpha\}$ the algebraic dual defined by $e^\beta(e_\alpha) = \delta^\beta_\alpha$ (for instance dx^β from the local frame ∂_α) and let us denote $\{e^\alpha\}_{|c} = \{\tilde{\dot{c}}, e^m\}$ the restriction of the local coframe on the image of the embedding. Let

$\mathcal{T} \in \overset{(0)}{\Upsilon}{}^2(c)$ be a monopole satisfying two conditions:

$$\begin{cases} \sigma^{(1,2)}\mathcal{T} = \mathcal{T} \\ div(\mathcal{T}) = 0 \end{cases} \tag{6.1.1}$$

in case T is an Energy-Momentum tensor field, such a monopole is an Energy-Momentum Monopole. Being a an Energy-Momentum Monopole is a very restrictive requirement and it fixes very strong constraints not just on the transverse Dixon monopole moments but even upon the closed embedding c on which the monopole can be defined. Let us express the monopole using the adapted Dixon representation induced by the adapted local chart.The symmetry condition tells us:

$$\int_{\mathbb{R}} c^\star(\phi_{\mu\nu})\alpha^{\mu\nu} ds = [\phi, \mathcal{T}] = [\phi, \sigma^{(1,2)}\mathcal{T}] = [\sigma_{(1,2)}\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\phi_{\mu\nu})\alpha^{\nu\mu} ds \tag{6.1.2}$$

So we have to conclude that $\alpha^{\mu\nu} = \alpha^{\nu\mu}$. So the symmetry condition is recast as a symmetry of the local adapted Dixon moments. The divergenceless condition tells us:

$$0 = [\phi, div(\mathcal{T})] = -[\nabla\phi, \mathcal{T}] = -\int_{\mathbb{R}} c^\star(\nabla_\mu\phi_\nu)\alpha^{\mu\nu} ds = \tag{6.1.3}$$

$$= -\int_{\mathbb{R}} c^\star(\nabla_m\phi_\nu)\alpha^{m\nu} ds + \int_{\mathbb{R}} c^\star(\phi_\nu)\frac{D}{ds}[\tilde{\dot{c}} \neg\, \alpha]^\nu ds \tag{6.1.4}$$

The only way to have a null result for each test tensor field ϕ is to have:

$$\begin{cases} \alpha^{\mu\nu} = \alpha^{\nu\mu} \\ \alpha^{m\nu} = 0 \\ \frac{D}{ds}[\tilde{\dot{c}} \neg\, \alpha]^\nu = 0 \end{cases} \tag{6.1.5}$$

from the first two condition we have to conclude that the only non null components of adapted Dixon moments must be the local scalar field . Let us suppose that $\alpha_{00} > 0$ and let us denote it with m. Therefore the adapted Dixon monopole moment can be written as $m[\dot{c} \otimes \dot{c}]$ and the condition can be recast as:

$$\begin{cases} \alpha = m[\dot{c} \otimes \dot{c}] \\ \frac{D}{ds}[m\dot{c}] = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha = m[\dot{c} \otimes \dot{c}] \\ \frac{D}{ds}[\dot{c}] = -\frac{d}{ds}(\ln(m))\dot{c} \end{cases} \tag{6.1.6}$$

The second condition is very interesting because it is telling us the image of the embedding must be a geodesic trajectory and the parametrization must be related to the variation of the positive scalar field m along the worldline. Considering this we can say that it is always possible to choose an appropriate reparametrization of the worldline defined as:

$$t(s) = -\int \frac{1}{\ln[m(s)]} ds \tag{6.1.7}$$

and the condition became:

$$\begin{cases} \alpha = m[\dot{c} \otimes \dot{c}] \\ \frac{d}{dt}m = 0 \\ \frac{D}{dt}[\dot{c}] = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha = m[\dot{c} \otimes \dot{c}] \\ m(t) = const \\ \nabla_{\dot{c}}(\dot{c}) = 0 \end{cases} \tag{6.1.8}$$

It is very interesting to see how the condition $m(t) = const$ is strongly related to the chosen parametrization of the worldline, so is strongly related to the definition of the congruence of clocks chosen in a neighbourhood of the worldline c. Let us remark how the constraints on the monopoles are not affected by any Torsion contribution. In fact the worldline must satisfy the tensorial equation:

$$\frac{D}{ds}[\dot{c}] = -\frac{d}{ds}(\ln(m))\dot{c} \tag{6.1.9}$$

expressed in local coordinates by:

$$\dot{c}^{\nu}\partial_{\nu}\dot{c}^{\lambda} + \Gamma^{\lambda}_{\nu\mu}\dot{c}^{\nu}\dot{c}^{\mu} + \frac{d}{ds}(\ln(m))\dot{c}^{\lambda} = 0 \tag{6.1.10}$$

We can always split the Connection in symmetric and antisymmetric part obtaining:

$$\dot{c}^\nu \partial_\nu \dot{c}^\lambda + \Gamma^\lambda_{(\nu\mu)} \dot{c}^\nu \dot{c}^\mu + Tor^\lambda_{\nu\mu} \dot{c}^\nu \dot{c}^\mu + \frac{d}{ds}(\ln(m))\dot{c}^\lambda = 0 \tag{6.1.11}$$

and since the Torsion is completely antisymmetric we can end up with:

$$\dot{c}^\nu \partial_\nu \dot{c}^\lambda + \Gamma^\lambda_{(\nu\mu)} \dot{c}^\nu \dot{c}^\mu + \frac{d}{ds}(\ln(m))\dot{c}^\lambda = 0 \tag{6.1.12}$$

so the equation fixing the worldline on which a symmetric divergenceless monopole is defined does not involve the contribution of the Torsion.

In this perspective, when the symmetry and divergenceless conditions are compatible with the physical model, our proposal is to model a free falling test particle simply as a divergenceless Energy-Momentum monopole. Then the trajectory related to a free falling test particle is just the support c of the monopole so a geodesic trajectory, the vector $\dot{c}$ tangent to the trajectory is the velocity of the test particle. The trajectory can always be reparametrized appropriately such that the Energy-Momentum monopole can be generated just by a constant m over the worldline, usually interpreted as the mass of the considered test particle. In this case the trajectory of the particle is also a geodesic curve satisfying $\nabla_{\dot{c}}(\dot{c}) = 0$. It is extremely interesting to notice how different values of m (even the non constant values) define always the same geodesics trajectory, recovering automatically the "Einstein Equivalence Principle" on the universality of the free falling motion. Accepting this interpretation, we can automatically state that the dynamics of a free falling test particle should not depends on the torsion, so just observing the motion of free falling monopole test particles we should be not able to distinguish a spacetime with Torsion from a Torsion free spacetime.

6.2 Dynamics of a free falling spinning test particles modelled as a Dixon dipole

The dynamics of a spinning free falling object within the General Relativity theory, is usually believed to be well modelled by the Mathisson Papapetrou Dixon Equation. Usually the procedure involved to find this equation is very tricky and complicated. The Dixon approach is fully given within standard General Relativity model and it is not very clear which structures are required for the proofs, which are assumed because of the physical evidences and which are taken because of the standard conventions set usually in General Relativity. We are going to see how these equations are direct consequences of the symmetry and divergenceless conditions imposed on the dipoles. To find the equations in their common form, it will be also necessary to fix some simple covariant constraints on the moments. Some of them depends on the usual convention adopted in G.R. (i.e.

the fixing of a specific worldline parametrization), others are consequences of the physical interpretation of them (i.e. the momentum and the angular momentum). We are not investigating the physical meaning of them, since here we are interested just to show how to recover the standard equations from the adopted formalism. However, the relevant result is that, just asking for a torsionless affine connection, we are able to reproduce the Mathisson Papapetrou Dixon equation, even in case the connection is not Levi Civita or a Lorentzian metric does not exist. So from this perspective, one could state that the dynamics of a massive free falling particle does not depend a priori on the choice of a specific metric, as found by Ehlers, Pirani and Schild in their work. In fact just requiring with the extra axiom (motivated by the experiments on the propagation of the electromagnetic waves) that the light trajectories must be also free falling trajectories (compatibility condition) then we can partially relate the metric to the affine connection. Let be M a manifold , $c : \mathbb{R} \to M$ a worldline and let us suppose for simplicity that there exists an adapted atlas $A = (U_i, \varphi_{(i)})$ such that $c \subset U_1$ and φ_1 is an adapted local chart. Let us consider $\{e_\alpha\}$ an arbitrary local frame defined on U and adapted to the embedding c (for instance ∂_α from an adapted local chart), so $\{e_\alpha\}_{|c} = \{\dot{c}, v_m\}$. Let $\{e^\alpha\}$ be the algebraic dual defined by $e^\beta(e_\alpha) = \delta^\beta_\alpha$ (for instance dx^β from the local frame ∂_α) and let us denote $\{e^\alpha\}_{|c} = \{\tilde{\dot{c}}, e^m\}$ the restriction of the local coframe on the image of the embedding. Let $\mathcal{T} \in \overset{(1)}{\Upsilon}{}^2(c)$ be a dipole satisfying two conditions:

$$\begin{cases} \sigma^{(1,2)}\mathcal{T} = \mathcal{T} \\ div(\mathcal{T}) = 0 \end{cases} \tag{6.2.1}$$

Such a dipole is called Energy Momentum Dipole. Being an Energy-Momentum Monopole is a very restrictive requirement and it fixes very strong constraints not just on the transverse Dixon monopole moments. In contrast with the monopole case this condition are not strong enought to fix the closed embedding c on which the dipole can be defined. So we will see how this condition is not able to fix the kinematics of the trajectory of the spinning objects. Let us express the Dipole using the adapted Dixon representation induced by the adapted local chart.

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^\star(\nabla_\lambda(\phi)_{\mu\nu})\beta^{\lambda\mu\nu} ds + \int_{\mathbb{R}} c^\star(\phi_{\mu\nu})\alpha^{\mu\nu} ds \tag{6.2.2}$$

where β is the adapted Dixon dipole moment and α is the adapted Dixon monopole moment. Both together (α, β) are the representative of the footprint of the Dipole. Let us stress that by definition automatically $\tilde{\dot{c}} \neg \beta = 0$ so using the adapted frame $\beta^{0\mu\nu} = 0$

and eventually the dipole can be written as:

$$[\phi, \mathcal{T}] = \int_{\mathbb{R}} c^{\star}(\nabla_l(\phi)_{\mu\nu})\beta^{l\mu\nu} ds + \int_{\mathbb{R}} c^{\star}(\phi_{\mu\nu})\alpha^{\mu\nu} ds \tag{6.2.3}$$

The symmetry condition tells us:

$$\int_{\mathbb{R}} c^{\star}(\nabla_\lambda(\phi)_{\mu\nu})\beta^{\lambda\mu\nu} ds + \int_{\mathbb{R}} c^{\star}(\phi_{\mu\nu})\alpha^{\mu\nu} ds = [\phi, \mathcal{T}] = [\phi, \sigma^{(1,2)}\mathcal{T}] = [\sigma_{(1,2)}\phi, \mathcal{T}] = \tag{6.2.4}$$

$$= \int_{\mathbb{R}} c^{\star}(\nabla_\lambda(\phi)_{\nu\mu})\beta^{\lambda\nu\mu} ds + \int_{\mathbb{R}} c^{\star}(\phi_{\nu\mu})\alpha^{\mu\nu} ds \tag{6.2.5}$$

So we have to conclude that $\alpha^{\mu\nu} = \alpha^{\nu\mu}$ and $\beta^{\lambda\mu\nu} = \beta^{\lambda\nu\mu}$. So the symmetry condition is recast as a symmetry with respect to the last two indices of the local adapted Dixon moments. The divergenceless condition tells us:

$$0 = [\phi, div(\mathcal{T})] = -[\nabla\phi, \mathcal{T}] = -\int_{\mathbb{R}} c^{\star}(\nabla^2_{\lambda\mu}(\phi)_\nu)\beta^{\lambda\mu\nu} ds - \int_{\mathbb{R}} c^{\star}(\nabla_\mu(\phi)_\nu)\beta^{\lambda\mu\nu} ds \tag{6.2.6}$$

As one can notice, since $\mathcal{T} \in \overset{(1)}{\Upsilon^2}(c)$, then $\mathcal{T} \in \overset{(2)}{\Upsilon^2}(c)$ so it belongs to the quadrupole space. If we want to solve this multipolar equation we must recast it using the adapted Dixon representation and find the value of the adapted Dixon moments that are solutions to it.

$$0 = \int_{\mathbb{R}} c^{\star}(\nabla^2_{\lambda\mu}(\phi)_\nu)\beta^{\lambda\mu\nu} ds + \int_{\mathbb{R}} c^{\star}(\nabla_\mu(\phi)_\nu)\alpha^{\lambda\mu\nu} ds = \tag{6.2.7}$$

$$= \int_{\mathbb{R}} c^{\star}(\nabla^2_{\lambda\mu}(\phi)_\nu)\beta^{\lambda\mu\nu} ds + \int_{\mathbb{R}} c^{\star}(\nabla_\mu(\phi)_\nu)\alpha^{\mu\nu} ds \tag{6.2.8}$$

$$= \int_{\mathbb{R}} c^{\star}(\nabla^2_{(\lambda\mu)}(\phi)_\nu)\beta^{(\lambda\mu)\nu} ds + \int_{\mathbb{R}} c^{\star}(\nabla^2_{[\lambda\mu]}(\phi)_\nu)\beta^{[\lambda\mu]\nu} ds + \int_{\mathbb{R}} c^{\star}(\nabla_\mu(\phi)_\nu)\beta^{\lambda\mu\nu} ds = \tag{6.2.9}$$

$$= \int_{\mathbb{R}} c^{\star}(\nabla^2_{(\lambda\mu)}(\phi)_\nu)\beta^{(\lambda\mu)\nu} ds - \frac{1}{2}\int_{\mathbb{R}} c^{\star}(R^{\sigma}_{\lambda\mu\nu}(\phi)_\sigma)\beta^{[\lambda\mu]\nu} ds + \tag{6.2.10}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^{\star}(Tor^{\sigma}_{\lambda\mu}(\nabla_\sigma\phi)_\nu)\beta^{[\lambda\mu]\nu} ds + \int_{\mathbb{R}} c^{\star}(\nabla_\mu(\phi)_\nu)\alpha^{\mu\nu} ds \tag{6.2.11}$$

Now we must perform the $3+1$ split of the first term taking account of the condition

$\beta^{0\mu\nu} = 0$. The first term can be split as:

$$\int_{\mathbb{R}} c^\star(\nabla^2_{(\lambda\mu)}(\phi)_\nu)\beta^{(\lambda\mu)\nu} ds = \int_{\mathbb{R}} c^\star(\nabla^2_{\lambda\mu}(\phi)_\nu)\beta^{(\lambda\mu)\nu} ds = \tag{6.2.12}$$

$$= \int_{\mathbb{R}} c^\star(\nabla^2_{00}(\phi)_\nu)\beta^{(00)\nu} ds + \int_{\mathbb{R}} c^\star(\nabla^2_{0a}(\phi)_\nu)\beta^{(0a)\nu} ds + \tag{6.2.13}$$

$$+ \int_{\mathbb{R}} c^\star(\nabla^2_{a0}(\phi)_\nu)\beta^{(a0)\nu} ds + \int_{\mathbb{R}} c^\star(\nabla^2_{ba}(\phi)_\nu)\beta^{(ba)\nu} ds = \tag{6.2.14}$$

$$= 0 + \int_{\mathbb{R}} c^\star(\nabla^2_{0a}(\phi)_\nu)\frac{1}{2}\beta^{a0\nu} ds + \tag{6.2.15}$$

$$+ \int_{\mathbb{R}} c^\star(\nabla^2_{a0}(\phi)_\nu)\frac{1}{2}\beta^{a0\nu} ds + \int_{\mathbb{R}} c^\star(\nabla^2_{ba}(\phi)_\nu)\beta^{(ba)\nu} ds = \tag{6.2.16}$$

$$= \int_{\mathbb{R}} c^\star(\nabla^2_{0a}(\phi)_\nu)\beta^{a0\nu} ds + 2\int_{\mathbb{R}} c^\star(\nabla^2_{[a0]}(\phi)_\nu)\frac{1}{2}\beta^{a0\nu} ds + \int_{\mathbb{R}} c^\star(\nabla^2_{ba}(\phi)_\nu)\beta^{(ba)\nu} ds = \tag{6.2.17}$$

$$= -\int_{\mathbb{R}} c^\star(\nabla_a(\phi)_\nu)\frac{D}{ds}[\tilde{\dot{c}}\neg\sigma^{(1,2)}\beta]^{a\nu} ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{a0\nu}(\phi)_\sigma)\beta^{a0\nu} ds + \tag{6.2.18}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^\sigma_{a0}(\nabla_\sigma(\phi)_\nu)\beta^{a0\nu} ds + \int_{\mathbb{R}} c^\star(\nabla^2_{ba}(\phi)_\nu)\beta^{(ba)\nu} ds \tag{6.2.19}$$

We can notice that the term

$$\int_{\mathbb{R}} c^\star(\nabla^2_{ba}(\phi)_\nu)\beta^{(ba)\nu} ds \tag{6.2.20}$$

is the higher order term of the adapted Dixon representation of $div(\mathcal{T})$. So to have $div(\mathcal{T}) = 0$, we must conclude that $\beta^{(ba)\nu} = 0$.

So we have the constraints for the Dipole Dixon adapted moments with respect an adapted frame:

$$\begin{cases} \beta^{\lambda\mu\nu} = \beta^{\lambda\nu\mu} \\ \beta^{0\mu\nu} = 0 \\ \beta^{(lm)\nu} = 0 \end{cases} \tag{6.2.21}$$

The entire expression for a divergenceless symmetric dipole will be reduced then to:

$$0 = -\int_{\mathbb{R}} c^\star(\nabla_a(\phi)_\nu)\frac{D}{ds}[\dot{\tilde{c}}\neg\sigma^{(1,2)}\beta]^{a\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{a0\nu}(\phi)_\sigma)\beta^{a0\nu}ds + \tag{6.2.22}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^\sigma_{a0}(\nabla_\sigma(\phi)_\nu)\beta^{a0\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{\lambda\mu\nu}(\phi)_\sigma)\beta^{[\lambda\mu]\nu}ds + \tag{6.2.23}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^\sigma_{\lambda\mu}(\nabla_\sigma\phi)_\nu)\beta^{[\lambda\mu]\nu}ds + \int_{\mathbb{R}} c^\star(\nabla_\mu(\phi)_\nu)\alpha^{\mu\nu}ds \tag{6.2.24}$$

Now performing again the $3+1$ split on the remaining terms:

$$0 = -\int_{\mathbb{R}} c^\star(\nabla_a(\phi)_\nu)\frac{D}{ds}[\dot{\tilde{c}}\neg\sigma^{(1,2)}\beta]^{a\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{a0\nu}(\phi)_\sigma)\beta^{a0\nu}ds + \tag{6.2.25}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^a_{b0}(\nabla_a(\phi)_\nu)\beta^{b0\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^0_{a0}(\nabla_0(\phi)_\nu)\beta^{a0\nu}ds + \tag{6.2.26}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{\lambda\mu\nu}(\phi)_\sigma)\beta^{[\lambda\mu]\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^a_{\lambda\mu}(\nabla_a\phi)_\nu)\beta^{[\lambda\mu]\nu}ds + \tag{6.2.27}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^0_{\lambda\mu}(\nabla_0\phi)_\nu)\beta^{[\lambda\mu]\nu}ds + \int_{\mathbb{R}} c^\star(\nabla_a(\phi)_\nu)\alpha^{a\nu}ds + \tag{6.2.28}$$

$$+\int_{\mathbb{R}} c^\star(\nabla_0(\phi)_\nu)\alpha^{0\nu}ds \tag{6.2.29}$$

$$0 = -\int_{\mathbb{R}} c^\star(\nabla_a(\phi)_\nu)\frac{D}{ds}[\dot{\tilde{c}}\neg\sigma^{(1,2)}\beta]^{a\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{a0\nu}\phi_\sigma)\beta^{a0\nu}ds + \tag{6.2.30}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^a_{b0}(\nabla_a(\phi)_\nu)\beta^{b0\nu}ds + \frac{1}{2}\int_{\mathbb{R}} c^\star(\phi_\sigma)\frac{D}{ds}[\dot{\tilde{c}}\neg Tor(e_a, \dot{c})\dot{\tilde{c}}\neg e^a\neg\beta]^\sigma ds + \tag{6.2.31}$$

$$-\frac{1}{2}\int_{\mathbb{R}} c^\star(R^\sigma_{\lambda\mu\nu}\phi_\sigma)\beta^{[\lambda\mu]\nu}ds - \frac{1}{2}\int_{\mathbb{R}} c^\star(Tor^a_{\lambda\mu}(\nabla_a\phi)_\nu)\beta^{[\lambda\mu]\nu}ds + \tag{6.2.32}$$

$$+\frac{1}{2}\int_{\mathbb{R}} c^\star(\phi_\sigma)\frac{D}{ds}[\dot{\tilde{c}}\neg Tor(e_\lambda, e_\mu)e^\mu\neg e^\lambda\neg\beta]^\sigma ds + \int_{\mathbb{R}} c^\star(\nabla_a(\phi)_\nu)\alpha^{a\nu}ds + \tag{6.2.33}$$

$$-\int_{\mathbb{R}} c^\star(\phi_\sigma)\frac{D}{ds}[\dot{\tilde{c}}\neg\alpha]^\sigma ds \tag{6.2.34}$$

This leads us directly to the full set of constraints for a divergenceless symmetric Dipole at fixed adapted frame:

$$\begin{cases} \beta^{\lambda\mu\nu} = \beta^{\lambda\nu\mu} \\ \\ \beta^{0\mu\nu} = 0 \\ \\ \beta^{(lm)\nu} = 0 \\ \\ \alpha^{a\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\sigma^{(1,2)}\beta]^{a\nu} - \frac{1}{2}Tor^{a}_{b0}\beta^{b0\nu} - \frac{1}{2}Tor^{a}_{\lambda\mu}\beta^{\lambda\mu\nu} = 0 \\ \\ \frac{1}{2}\frac{D}{ds}[\tilde{\dot{c}}\neg Tor(e_a, \dot{c})\tilde{\dot{c}}\neg e^{a}\neg\beta]^{\sigma} + \frac{1}{2}\frac{D}{ds}[\tilde{\dot{c}}\neg Tor(e_\lambda, e_\mu)e^{\mu}\neg e^{\lambda}\neg\beta]^{\sigma} + \\ \qquad\qquad - \frac{1}{2}R^{\sigma}_{a0\nu}\beta^{a0\nu} - \frac{1}{2}R^{\sigma}_{\lambda\mu\nu}\beta^{\lambda\mu\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\alpha]^{\sigma} = 0 \end{cases} \tag{6.2.35}$$

As one can notice, as well as in case of the Monopoles, the equations fix a relationship between the parametrization of the worldline c and the adapted Dixon moments. However, in contrast with the monopole case, the equations do not impose any condition on the image of the embedding, so the support of a divergenceless symmetric dipole can be in principle any arbitrary worldline. The chosen parametrization of the worldline fixes the form of the constraints. Standard General Relativity is a very particular relativistic theory due to the fact that a very specific assumption on the observer's conventions are fixed. The existence of a global Lorentzian metric determining the distances on the spacetime is assumed and the connection determinig the free falling is assumed to be the Levi Civita connection of that particular metric. This implies that the worldlines solving the geodesic curve equation are parametrized by the arc length induced by the metric, so it fixes a very strong relationship between the free falling and the clocks used to defining the proper times of the physical objects. As showed in [31][32][33] this assumption can be too restrictive with respect to the physical evidence and possibly non compatible with the actual protocols used to measure the proper time with the atomic transitions (atomic clocks). The effects of the discrepancy between the proper time fixed by an arbitrary congruence of clocks (i.e atomic processes) and the "gravitational" clocks (i.e. celestial object motion) is deeply investigated in [31][32][33] and in some cases it is able to affect the observations in the same manner as a Cosmological Constant. Because of this reason we did not assume anything about the nature of the connection needed to define the Dixon Multipoles, so the given equations are as general as possible.

However it is interesting to show that, under the very specific assumptions of Standard General Relativity it is possible to show how the Dixon Moments of a divergenceless symmetric dipole must satisfy the Mathisson Papaetrou Dixon Equations.

If the following extra conditions are assumed:

1. The dipole moment $\beta^{\lambda\mu\nu} = S^{\lambda(\mu}\dot{c}^{\nu)}$
2. The tensor $S^{\lambda\mu} = -S^{\mu\lambda}$.
3. The tensor $S^{0\mu}\dot{c}^{\nu} = -S^{0\nu}\dot{c}^{\mu}$.

4. The monopole moment $\alpha^{\lambda\mu} = -P^{(\lambda}\dot{c}^{\mu)}$

5. The scalar $m = \tilde{\dot{c}} \neg P$ satisfies $\frac{D}{ds}(m\dot{c}) = 0$

we are going to show that is possible to recover the usual Mathisson Papapetrou Dixon Equations. Despite these assumptions could seem quite an artifact, they are strongly motivated in several works [1][2][3] and it is widely proved they are needed to extend the standard non relativistic definition of Momentum and Spin of an extended object to the relativistic cases, being consistent with the Standard General Relativity framework. Therefore in this perspective, as shown in [3] we are allowed to interpret the following quantities as:

1. The tensor field S is the total Spin of the Relativistic Extended object

2. The vector field P is the total Momentum of the Relativistic Extended object

3. The scalar $m = \tilde{\dot{c}} \neg P$ is the non-rotating rest mass of the Relativistic Extended object

4. The vector field $\frac{D}{ds}(m\dot{c}) = 0$ is the contribution to the total Energy of the non-rotating rest mass.

Other extensions can be made changing slightly the given assumptions but without affecting the given interpretation of S and P as Spin and Momentum as shown by Dixon in his works. If these extensions of the Spin and Momentum are the only one consistent with the Standard General Relativity prescriptions, this is still a matter of investigation. For sure we can say that they are definitely non strictly necessary when no assumptions on the parametrization of the worldlines are made. In case of Extended Relativistic Theories, when no relationships between the Lorentzian metric inducing distances and the Connection determining the free falling are a priori assumed, things get even messier and the interpretation of the Dipole and Monopole moments are much more tricky. In general it seems that there exist an infinite number of different ways with which a Multipole can be associated to a regular compact support tensor field such that it contain some "information" about the structure of the regular compact support tensor field. In some cases it is even possible to show that the Multipoles can be considered weak approximations of one parameter families of regular compact support tensor fields, in the very same way it is possible to interpret some distributions on $\mathbb{R}$ as the weak approximations of the one parameter families of real functions. However, how to link a multipole to an extended regular tensor field, how many different ways there exists, which choices determine them, which links and which representation of the multipoles offer the best useful way to express physical and geometrical information we are interested in and how to interpret them consistently with the prescription fixed by the considered physical and mathematical assumptions is still a matter of investigation. We can say that in general a unique answer cannot be provided because it depends on the different frameworks needed to create the mathematical or physical model. However, despite this is fundamental crucial aspects, we are not interested here to validate or reject the motivation or the interpretation provided above. We are settling here to show just that, assuming them, the constraints on the divergence-less symmetric dipole become simply the well known Mathisson-Papapetrou-Dixon Equations. If the 6 extra conditions are assumed, at fixed adapted frame, the first 3 constraints become just identities :

$$\begin{cases} 0 = 0 \\ \\ 0 = 0 \\ \\ S^{lm}\dot{c}^{\nu} + S^{l\nu}\dot{c}^{m} + S^{ml}\dot{c}^{\nu} + S^{m\nu}\dot{c}^{l} = 0 \end{cases} \tag{6.2.36}$$

$$\begin{cases} 0 = 0 \\ \\ 0 = 0 \\ \\ 0 = 0 \end{cases} \tag{6.2.37}$$

The two remaining equations then can be reduced to the Mathisson-Papapetrou-Dixon Equations. First of all let us set to 0 the Torsion contribution, as assumed in Standard General Relativity. Therefore

$$\begin{cases} \alpha^{a\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\sigma^{(1,2)}\beta]^{a\nu} = 0 \\ \\ \frac{1}{2}R^{\sigma}_{a0\nu}\beta^{a0\nu} + \frac{1}{2}R^{\sigma}_{\lambda\mu\nu}\beta^{\lambda\mu\nu} + \frac{D}{ds}[\tilde{\dot{c}}\neg\alpha]^{\sigma} = 0 \end{cases} \tag{6.2.38}$$

$$\begin{cases} \alpha^{a\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\sigma^{(1,2)}\beta]^{a\nu} = 0 \\ \\ \frac{1}{2}R^{\sigma}_{a0\nu}\beta^{a0\nu} + \frac{1}{2}R^{\sigma}_{a\mu\nu}\beta^{a\mu\nu} + \frac{D}{ds}[\tilde{\dot{c}}\neg\alpha]^{\sigma} = 0 \end{cases} \tag{6.2.39}$$

$$\begin{cases} -P^{(a}\dot{c}^{\nu)} - \frac{D}{ds}[S^{\lambda(0}\dot{c}^{\nu)} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0}\dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.40}$$

$$\begin{cases} -P^{a}\dot{c}^{\nu} - \frac{D}{ds}[S^{\lambda 0}\dot{c}^{\nu} e_\lambda \otimes e_\nu + S^{\lambda\nu}\dot{c}^{0} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0}\dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.41}$$

$$\begin{cases} -P^{a}\dot{c}^{\nu} - \frac{D}{ds}[S^{a0}\dot{c}^{\nu} e_a \otimes e_\nu + S^{00}\dot{c}^{\nu} e_0 \otimes e_\nu + S^{\lambda\nu}\dot{c}^{0} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0}\dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.42}$$

$$\begin{cases} -P^{a}\dot{c}^{\nu} - \frac{D}{ds}[-S^{0a}\dot{c}^{\nu} e_a \otimes e_\nu + 0 + S^{\lambda\nu}\dot{c}^{0} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0}\dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.43}$$

$$\begin{cases} -P^{a}\dot{c}^{\nu} - \frac{D}{ds}[-S^{0\nu}\dot{c}^{a} e_a \otimes e_\nu + 0 + S^{\lambda\nu}\dot{c}^{0} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0}\dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.44}$$

$$\begin{cases} -P^{a}\dot{c}^{\nu} - \frac{D}{ds}[0 + 0 + S^{\lambda\nu}\dot{c}^{0} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0}\dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.45}$$

$$\begin{cases} -P^a \dot{c}^\nu - \frac{D}{ds}[S^{\lambda\nu} e_\lambda \otimes e_\nu]^{a\nu} = 0 \\ \\ \frac{1}{2} R^\sigma_{a0\mu} S^{a(0} \dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu} \dot{c}^{\nu)} - \frac{D}{ds}[P^{(0} \dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \end{cases} \tag{6.2.46}$$

Let us consider the first equation. Since $\dot{c}^a = 0$ at fixed adapted frame, we can add the term $-P^0 \dot{c}^a$ without affecting the equation,

$$P^a \dot{c}^\nu - P^\nu \dot{c}^a + \frac{D}{ds}[S]^{a\nu} = 0 \tag{6.2.47}$$

$$2P^{[a} \dot{c}^{\nu]} + \frac{D}{ds}[S]^{a\nu} = 0 \tag{6.2.48}$$

Now considering that $2P^{[0} \dot{c}^{0]} = \frac{D}{ds}[S]^{[00]} = 0$ we can state that another trivial identity must be automatically satisfied

$$2P^{[0} \dot{c}^{0]} + \frac{D}{ds}[S]^{00} = 0 \tag{6.2.49}$$

Therefore:

$$\begin{cases} 2P^{[a} \dot{c}^{\nu]} - \frac{D}{ds}[S]^{[a\nu]} = 0 \\ 2P^{[0} \dot{c}^{0]} - \frac{D}{ds}[S]^{[00]} = 0 \end{cases} \Rightarrow 2P^{[\lambda} \dot{c}^{\mu]} - \frac{D}{ds}[S]^{\lambda\mu} = 0 \tag{6.2.50}$$

The second equation is then:

$$\frac{1}{2} R^\sigma_{a0\mu} S^{a(0} \dot{c}^{\mu)} + \frac{1}{2} R^\sigma_{a\mu\nu} S^{a(\mu} \dot{c}^{\nu)} - \frac{D}{ds}[P^{(0} \dot{c}^{\alpha)} e_\alpha]^\sigma = 0 \tag{6.2.51}$$

Appliying the condition onthe Energy conservation of the un-rotating rest mass:

$$\frac{1}{2}R^{\sigma}_{a0\mu}S^{a0}\dot{c}^{\mu}+\frac{1}{2}R^{\sigma}_{a0\mu}S^{a\mu}\dot{c}^{0}+\frac{1}{2}R^{\sigma}_{a\mu\nu}S^{a\mu}\dot{c}^{\nu}+\frac{1}{2}R^{\sigma}_{a\mu\nu}S^{a\nu}\dot{c}^{\mu}-\frac{D}{ds}[P]^{\sigma}=0 \qquad (6.2.52)$$

Now using $S^{0\mu}\dot{c}^{\nu}=-S^{0\nu}\dot{c}^{\mu}$ we have

$$+\frac{1}{2}R^{\sigma}_{a0b}S^{ab}\dot{c}^{0}+\frac{1}{2}R^{\sigma}_{ab0}S^{ab}\dot{c}^{0}+\frac{1}{2}R^{\sigma}_{a0b}S^{ab}\dot{c}^{0}-\frac{D}{ds}[P]^{\sigma}=0 \qquad (6.2.53)$$

Since the assumption of Standard General Relativity is the Levi-Civita connection, the first Bianchi Identity holds, so $R^{\lambda}_{\mu\nu\sigma}=-R^{\lambda}_{\mu\sigma\nu}$ therefore:

$$-\frac{1}{2}R^{\sigma}_{ab0}S^{ab}\dot{c}^{0}+\frac{1}{2}R^{\sigma}_{ab0}S^{ab}\dot{c}^{0}-\frac{1}{2}R^{\sigma}_{ab0}S^{ab}\dot{c}^{0}-\frac{D}{ds}[P]^{\sigma}=0 \qquad (6.2.54)$$

to end up with the constraint at fixed adapted frame:

$$\frac{1}{2}R^{\sigma}_{ab0}S^{ab}\dot{c}^{0}+\frac{D}{ds}[P]^{\sigma}=0 \qquad (6.2.55)$$

Since $\dot{c}^{a}=0$ and $S^{0\mu}\dot{c}^{\nu}=-S^{0\nu}\dot{c}^{\mu}$, we can add some extra null terms into the equation to end up with

$$\frac{D}{ds}[P]^{\sigma}+\frac{1}{2}R^{\sigma}_{\lambda\mu\nu}S^{\lambda\mu}\dot{c}^{\nu}=0 \qquad (6.2.56)$$

So under the prescription of Standard General Relativity, if we assume the Dixon generalisation and interpretation of the standard concepts of momentum and spin associated to an extended relativistic object, the divergence-less symmetric multipole admits the adapted Dixon moments satisfying:

$$\begin{cases} \frac{D}{ds}[S]^{\mu\nu} - 2P^{[\mu}\dot{c}^{\nu]} = 0 \\ \\ \frac{D}{ds}[P]^{\sigma} + \frac{1}{2}R^{\sigma}_{\lambda\mu\nu}S^{\lambda\mu}\dot{c}^{\nu} = 0 \end{cases} \tag{6.2.57}$$

Although we performed the calculation using an adapted local frame, the equations are completely covariant because both of the right hand sides transform linearly when a change of frame is performed.

So from this perspective, our proposal is to model a free falling point-like spinning particle simply as divergence-less Energy Momentum dipole. Additional constraints needed to extract and interpret the physical quantities in several different relativistic frameworks beyond Standard General Relativity are still matter of investigation

6.3 Dynamics of a divergenceless symmetric Dixon dipole when Torsion occur

We have already found the constraints for a divergence-less symmetric multipoles

$$\begin{cases} \beta^{\lambda\mu\nu} = \beta^{\lambda\nu\mu} \\ \\ \beta^{0\mu\nu} = 0 \\ \\ \beta^{(lm)\nu} = 0 \\ \\ \alpha^{a\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\sigma^{(1,2)}\beta]^{a\nu} - \frac{1}{2}Tor^{a}_{b0}\beta^{b0\nu} - \frac{1}{2}Tor^{a}_{\lambda\mu}\beta^{\lambda\mu\nu} = 0 \\ \\ \frac{1}{2}\frac{D}{ds}[\tilde{\dot{c}}\neg Tor(e_a,\dot{c})\tilde{\dot{c}}\neg e^{a}\neg\beta]^{\sigma} + \frac{1}{2}\frac{D}{ds}[\tilde{\dot{c}}\neg Tor(e_\lambda,e_\mu)e^{\mu}\neg e^{\lambda}\neg\beta]^{\sigma} + \\ \qquad -\frac{1}{2}R^{\sigma}_{a0\nu}\beta^{a0\nu} - \frac{1}{2}R^{\sigma}_{\lambda\mu\nu}\beta^{\lambda\mu\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\alpha]^{\sigma} = 0 \end{cases} \tag{6.3.1}$$

If no assumptions on the relationship between connection and metric are given, then it is possible to find a generalisation of the Mathisson-Papapetrou-Dixon Equations. However, before showing the derivation of such equations a clear disclaimer is needed. First of all, at this stage, we are completely ignoring the physical interpretation of the divergenceless symmetric tensor field T from which the Dixon dipole is coming from. The interpretation of the originary T as an Energy-Momentum tensor field could be an issue since in some physical models beyond G.R. the divergence-less symmetric condition required on it can be too strong (i.e. Einstein-Cartan theory). However the following derivation can be interpreted as a methodological procedure rather than a full consistent physical model. The reader, if not completely comfortable with that, can interpret this approach just as a way to derive the dynamical equation of the multipoles related to a generic divergenceless symmetric rank two tensor field, without be forced to interpret it as the Energy-Momentum tensor. Obviously in that case the interpretation related to the free fall of spinning test particles is completely lost. However in the case when the Energy-Momentum tensor cannot be symmetric, one can drop the constraint and follow the same procedure to find different equations of motion for the multipoles. Let us suppose to be happy to have a divergenceless symmetric energy momentum tensor, a second possible issue could be the interpretation of S and P. Usually the tensors S and P are interpreted as the spin and the momentum of some extended object, because the rigorous definition of these two tensor fields coincides with the usual definition of spin and momentum given in Special Relativity so in this perspective they are considered as a well defined generalisation of the concept of spin and momentum of an extended relativistic object within the Standard General Relativity theory. As far as we know, nothing has been done yet to investigate and interpret these physical quantities in other relativistic models for instance when no relationship about metric and connection is assumed, when parametrizations other than the arc length are used to express the worldline or when a Torsion contribution occurs. Even in the simplest extensions of Standard General Relativity the interpretation of simpler dynamical quantities is usually very tricky and a very careful analysis on the foundation of even the most elementary common physical concept should be performed to avoid any unwanted inner contradictions of the theories [31] [32][33]. The validity of the statements and the properties concerning the object of relativistic theories slightly differing from Standard General Relativity (even the most natural and fundamental) should be then proved again from scratch and cannot be taken for granted[34][35][36] [37][38][39][40][41]. In this perspective, outside the strong boundaries fixed in Standard General Relativity, we have no elements to say that S and P exists and can be interpreted as the Spin and the momentum associated to an extended relativistic object. As instance the presence of a non null Torsion contribution to the connection could eventually affect the definition of S and P so they can be no more interpreted as the usual Spin and the Momentum. The possibility to provide a well defined quantities S and P interpreted as Spin and momentum in other relativistic models without leading to contradictions is very tricky and it is an actual matter of investigation.

However, let us suppose that there exists at least a relativistic model in which the connection is not fixed to be Levi Civita, the parametrization is not fixed necessarily to the arc length of the worldline but it is possible to define coherently a concept of Spin and Momentum satisfying the following condition

1. The dipole moment $\beta^{\lambda\mu\nu} = S^{\lambda(\mu}\dot{c}^{\nu)}$
2. The tensor $S^{\lambda\mu} = -S^{\mu\lambda}$.
3. The monopole moment $\alpha^{\lambda\mu} = -P^{(\lambda}\dot{c}^{\mu)}$

Let us stress once again that this is an extremely strong conjecture and one should check the consistency of these assumption within a model that could be quite different to standard General Relativity. One again we state than that rather than a concrete physical model this procedure should be interpreted more like an inspiring example on how there are no mathematical or geometrical limitation to this approach. However in this scenario it is possible to derive the generalisation of the Mathisson-Papapetrou-Dixon Equations. Let us substitute the assumptions on the equations of the dipole using the adapted local frame:

$$\begin{cases} 0 = 0 \\ \\ 0 = 0 \\ \\ S^{0\mu}\dot{c}^{\nu} = -S^{0\nu}\dot{c}^{\mu} \\ \\ \alpha^{a\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\sigma^{(1,2)}\beta]^{a\nu} - \frac{1}{2}Tor^{a}_{b0}\beta^{b0\nu} - \frac{1}{2}Tor^{a}_{\lambda\mu}\beta^{\lambda\mu\nu} = 0 \\ \\ \frac{1}{2}\frac{D}{ds}[\tilde{\dot{c}}\neg Tor(e_a, \dot{c})\tilde{\dot{c}}\neg e^{a}\neg\beta]^{\sigma} + \frac{1}{2}\frac{D}{ds}[\tilde{\dot{c}}\neg Tor(e_\lambda, e_\mu)e^{\mu}\neg e^{\lambda}\neg\beta]^{\sigma} + \\ \qquad\qquad - \frac{1}{2}R^{\sigma}_{a0\nu}\beta^{a0\nu} - \frac{1}{2}R^{\sigma}_{\lambda\mu\nu}\beta^{\lambda\mu\nu} - \frac{D}{ds}[\tilde{\dot{c}}\neg\alpha]^{\sigma} = 0 \end{cases} \tag{6.3.2}$$

where two constraints are automatically reduced to identities. We can use the first equation into the others to find:

$$\begin{cases} S^{0\mu}\dot{c}^{\nu} = -S^{0\nu}\dot{c}^{\mu} \\ \\ -P^{a}\dot{c}^{\nu} - \frac{D}{ds}[S^{\lambda\nu}e_\lambda \otimes e_\nu]^{a\nu} - Tor^{a}_{b0}S^{b(0}\dot{c}^{\nu)} - Tor^{a}_{\lambda\mu}S^{\lambda(\mu}\dot{c}^{\nu)} = 0 \\ \\ \frac{1}{2}\frac{D}{ds}[Tor^{0}_{a0}S^{a(0}\dot{c}^{\nu)}e_\nu]^{\sigma} + \frac{1}{2}\frac{D}{ds}[Tor^{0}_{\lambda\mu}S^{\lambda(\mu}\dot{c}^{\nu)}e_\nu]^{\sigma} + \\ \qquad\qquad + \frac{1}{2}R^{\sigma}_{a0\mu}S^{a(0}\dot{c}^{\mu)} + \frac{1}{2}R^{\sigma}_{a\mu\nu}S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)}e_\alpha]^{\sigma} = 0 \end{cases} \tag{6.3.3}$$

Now keeping in mind the first equation we can manipulate the second as follow:

$$-P^{a}\dot{c}^{\nu} - \frac{D}{ds}[S^{\lambda\nu}e_\lambda \otimes e_\nu]^{a\nu} - \frac{1}{2}Tor^{a}_{\lambda\mu}S^{\lambda\nu}\dot{c}^{\mu} - \frac{1}{2}Tor^{a}_{\lambda\mu}S^{\lambda\mu}\dot{c}^{\nu} - \frac{1}{2}Tor^{a}_{\lambda\mu}S^{\lambda\nu}\dot{c}^{\mu} = 0 \tag{6.3.4}$$

Considering that using an adapted frame $\dot{c}^{a} = 0$ we can say that $P^{\nu}\dot{c}^{a} = 0$ as well as $-\frac{1}{2}Tor^{\nu}_{\lambda\mu}S^{\lambda\mu}\dot{c}^{a} = 0$. So we can add it to the equation without affecting it:

$$-P^a\dot{c}^\nu + P^\nu\dot{c}^a - \frac{D}{ds}[S^{\lambda\nu}e_\lambda \otimes e_\nu]^{a\nu} - Tor^a_{\lambda\mu}S^{\lambda\nu}\dot{c}^\mu - \frac{1}{2}Tor^a_{\lambda\mu}S^{\lambda\mu}\dot{c}^\nu - \frac{1}{2}Tor^\nu_{\lambda\mu}S^{\lambda\mu}\dot{c}^a = 0 \tag{6.3.5}$$

$$-2P^{[a}\dot{c}^{\nu]} - \frac{D}{ds}[S^{\lambda\nu}e_\lambda \otimes e_\nu]^{a\nu} - Tor^a_{\lambda\mu}S^{\lambda\nu}\dot{c}^\mu - Tor^{(a}_{\lambda\mu}\dot{c}^{\nu)}S^{\lambda\mu} = 0 \tag{6.3.6}$$

and splitting the symmetric and antisymmetric parts:

$$\begin{cases} -2P^{[a}\dot{c}^{\nu]} - \frac{D}{ds}[S^{\lambda\nu}e_\lambda \otimes e_\nu]^{a\nu} - Tor^{[a}_{\lambda\mu}S^{\underline{\lambda}\nu]}\dot{c}^\mu = 0 \\ -Tor^{(a}_{\lambda\mu}S^{\underline{\lambda}\nu)}\dot{c}^\mu - Tor^{(a}_{\lambda\mu}\dot{c}^{\nu)}S^{\lambda\mu} = 0 \end{cases} \tag{6.3.7}$$

$$\begin{cases} -2P^{[\lambda}\dot{c}^{\nu]} - \frac{D}{ds}[S]^{\lambda\nu} - Tor^{[\lambda}_{\sigma\mu}S^{\underline{\sigma}\nu]}\dot{c}^\mu = 0 \\ Tor^{(a}_{\lambda\mu}S^{\underline{\lambda}\nu)}\dot{c}^\mu + Tor^{(a}_{\lambda\mu}\dot{c}^{\nu)}S^{\lambda\mu} = 0 \end{cases} \tag{6.3.8}$$

Ending up with

$$\begin{cases} -2P^{[\lambda}\dot{c}^{\nu]} - \frac{D}{ds}[S]^{\lambda\nu} - Tor^{[\lambda}_{\sigma\mu}S^{\underline{\sigma}\nu]}\dot{c}^\mu = 0 \\ [Tor - \tilde{\dot{c}}\neg Tor \otimes \dot{c}]^{(\lambda}_{\sigma\mu}S^{\underline{\sigma}\nu)}\dot{c}^\mu + [Tor - \tilde{\dot{c}}\neg Tor \otimes \dot{c}]^{(\lambda}_{\sigma\mu}S^{\underline{\sigma\mu}}\dot{c}^{\nu)} = 0 \end{cases} \tag{6.3.9}$$

it is interesting to notice how the dynamics depends just on the antisymmetric part of the equation. The symmetric part is an additional algebraic constraint on the dipole moments:

$$[Tor - \tilde{\dot{c}}\neg Tor \otimes \dot{c}]^{(\lambda}_{\sigma\mu}\beta^{\underline{\sigma\mu}\nu)} = 0 \tag{6.3.10}$$

We believe this additional non dynamical constraint can be directly satisfied by casting an appropriate definition of the Spin of an extended object in presence of Torsion. So once again we remark the need of a more fundamental approach to define and interpret the moments of a multipole as physical quantities beyond the standard General Relativity. However, let us suppose to be able to define the spin S in a way it satisfy automatically the algebraic constraint we have still another dynamical equation.

$$\frac{1}{2}\frac{D}{ds}[Tor^0_{a0}S^{a(0}\dot{c}^{\nu)}e_\nu]^\sigma + \frac{1}{2}\frac{D}{ds}[Tor^0_{\lambda\mu}S^{\lambda(\mu}\dot{c}^{\nu)}e_\nu]^\sigma + \tag{6.3.11}$$
$$+\frac{1}{2}R^\sigma_{a0\mu}S^{a(0}\dot{c}^{\mu)} + \frac{1}{2}R^\sigma_{a\mu\nu}S^{a(\mu}\dot{c}^{\nu)} - \frac{D}{ds}[P^{(0}\dot{c}^{\alpha)}e_\alpha]^\sigma = 0 \tag{6.3.12}$$

$$\frac{1}{2}\frac{D}{ds}[Tor^0_{a0}S^{a(0}\dot{c}^{\nu)}e_\nu]^\sigma + \frac{1}{2}\frac{D}{ds}[Tor^0_{\lambda\mu}S^{\lambda(\mu}\dot{c}^{\nu)}e_\nu]^\sigma + \tag{6.3.13}$$
$$+\frac{1}{2}R^\sigma_{a0\mu}S^{a(0}\dot{c}^{\mu)} + \frac{1}{2}R^\sigma_{a\mu\nu}S^{a(\mu}\dot{c}^{\nu)} - \frac{1}{2}\frac{D}{ds}[P]^\sigma - \frac{1}{2}\frac{D}{ds}[(\tilde{\dot{c}}\neg P)\dot{c}]^\sigma = 0 \tag{6.3.14}$$

Manipulating the torsion bit as done before we obtain:

$$-\frac{1}{2}\frac{D}{ds}[P]^\sigma - \frac{1}{2}\frac{D}{ds}[(\tilde{\dot{c}}\neg P)\dot{c}]^\sigma + \frac{D}{ds}[Tor^0_{\lambda\nu}S^{\lambda\mu}\dot{c}^\nu e_\nu]^\sigma + \frac{D}{ds}[(\tilde{\dot{c}}\neg Tor)_{\lambda\mu}S^{\lambda\mu}\dot{c}]^\sigma + \tag{6.3.15}$$
$$+\frac{1}{2}R^\sigma_{a0\mu}S^{a(0}\dot{c}^{\mu)} + \frac{1}{2}R^\sigma_{a\mu\nu}S^{a(\mu}\dot{c}^{\nu)} = 0 \tag{6.3.16}$$

hence

$$-\frac{1}{2}\frac{D}{ds}[P]^\sigma - \frac{1}{2}\frac{D}{ds}[\{(\tilde{\dot{c}}\neg P) - i^2[(\tilde{\dot{c}}\neg Tor)\otimes S\}\dot{c}]^\sigma + \frac{D}{ds}[Tor^0_{\lambda\nu}S^{\lambda\mu}\dot{c}^\nu e_\nu]^\sigma + \tag{6.3.17}$$
$$+\frac{1}{2}R^\sigma_{a0\mu}S^{a(0}\dot{c}^{\mu)} + \frac{1}{2}R^\sigma_{a\mu\nu}S^{a(\mu}\dot{c}^{\nu)} = 0 \tag{6.3.18}$$

that can be recast using $\dot{c}^a = 0$ as well as $S^{0\mu}\dot{c}^\nu = -S^{0\nu}\dot{c}^\mu$

$$-\frac{1}{2}\frac{D}{ds}[P]^\sigma - \frac{1}{2}\frac{D}{ds}[\{(\tilde{\dot{c}}\neg P) - i^2[(\tilde{\dot{c}}\neg Tor)\otimes S\}\dot{c}]^\sigma + \frac{D}{ds}[Tor^0_{\lambda\nu}S^{\lambda\mu}\dot{c}^\nu e_\nu]^\sigma + \tag{6.3.19}$$
$$+\frac{1}{4}R^\sigma_{\lambda\mu\nu}S^{\lambda\mu}\dot{c}^\nu + \frac{1}{2}R^\sigma_{\lambda\mu\nu}S^{\lambda\nu}\dot{c}^\mu = 0 \tag{6.3.20}$$

Finally using the antisymmetry of S and R in the first two indices we can write:

$$-\frac{1}{2}\frac{D}{ds}[P]^{\sigma}-\frac{1}{2}\frac{D}{ds}[\{(\tilde{\dot{c}}\neg P)-i^{2}[(\tilde{\dot{c}}\neg Tor)\otimes S\}\dot{c}]^{\sigma}+\frac{D}{ds}[Tor^{0}_{\lambda\nu}S^{\lambda\mu}\dot{c}^{\nu}e_{\nu}]^{\sigma}+ \tag{6.3.21}$$

$$-\frac{1}{4}R^{\sigma}_{\lambda\mu\nu}S^{\lambda\mu}\dot{c}^{\nu}+R^{\sigma}_{\lambda(\mu\nu)}S^{\lambda\mu}\dot{c}^{\nu}=0 \tag{6.3.22}$$

So we have the dynamical equations generalising the Mathisson-Papapetrou-Dixon Equations:

$$\begin{cases} -2P^{[\lambda}\dot{c}^{\nu]}-\frac{D}{ds}[S]^{\lambda\nu}-Tor^{[\lambda}_{\sigma\mu}S^{\underline{\sigma}\nu]}\dot{c}^{\mu}=0 \\ \\ -\frac{1}{2}\frac{D}{ds}[P]^{\sigma}-\frac{1}{2}\frac{D}{ds}[\{(\tilde{\dot{c}}\neg P)-i^{2}[(\tilde{\dot{c}}\neg Tor)\otimes S\}\dot{c}]^{\sigma}+\frac{D}{ds}[Tor^{0}_{\lambda\nu}S^{\lambda\mu}\dot{c}^{\nu}e_{\nu}]^{\sigma}+ \\ \qquad -\frac{1}{4}R^{\sigma}_{\lambda\mu\nu}S^{\lambda\mu}\dot{c}^{\nu}+R^{\sigma}_{\lambda(\mu\nu)}S^{\lambda\mu}\dot{c}^{\nu}=0 \end{cases} \tag{6.3.23}$$

and the additional algebraic constraint:

$$[Tor-\tilde{\dot{c}}\neg Tor\otimes\dot{c}]^{(\lambda}_{\sigma\mu}S^{\underline{\sigma}\nu)}\dot{c}^{\mu}+[Tor-\tilde{\dot{c}}\neg Tor\otimes\dot{c}]^{(\lambda}_{\sigma\mu}S^{\underline{\sigma\mu}}\dot{c}^{\nu)}=0 \tag{6.3.24}$$

Chapter 7

Conclusions

In this work we saw how the usual definition of "moments", already well known and applied in several branches of Mathematics, Physics and Statistics (just to quote some), can be interpreted as the local coordinate expression of a geometrical object defined intrinsically by a generalisation of the De Rham push-forward [13][14]taken with respect a given closed embedding $c : \mathbb{R} \hookrightarrow M$ (worldline).

Since the natural action on the class of smooth compact support tensor fields (test tensor fields) of such objects is $\mathbb{R}$-linear, they can be interpreted as $\mathbb{R}$-linear functionals.

Using specific representations of them, it has been proved how these geometrical objects play an essential role during the asymptotic expansions of specific one parameter families of smooth compact support tensor fields called squeezed tensor fields, therefore we definitely prefer to refer to them as "multipoles" rather than "tensorial currents", in order to emphasise the very strong correspondence occurring with the weak asymptotic expansions and to emphasise the analogy with the multipole expansion of the electromagnetic sources, already well known in Classical Electromagnetism.

It has been proved as well that it is possible to define the multipoles in two different coordinate free ways, and although the two definitions could seem very different, they are completely equivalent and they define the very same set of $\mathbb{R}$-linear functionals acting on the test tensor fields.

Due to the $\mathbb{R}$-linearity of the action, it is possible to extend to the set of the multipoles all the operations defined usually upon the smooth tensor fields (sum, product with scalar field, internal contraction, contractions with vector and covector fields, covariant and Lie derivative and divergence just to quote some). Other more advanced operations are specifically built on them considering they are functionals (e.g. product with tensor fields).

We have seen how the multipoles have got three main intrinsic characteristics: the rank, the support and the order. The first two can be considered just as a mere transposition in terms of functionals of the standard concepts of rank and support of tensor fields, the second one is closed related with the "number of derivations" acting on the test tensor fields through the given multipole.

Unlike the rank and the support, the order of the multipoles exhibits an odd behaviour when even the most trivial operations are performed, for instance the sum of two order two multipoles can be eventually an order one multipole. This aspect, combined with other properties owned by the multipoles, can be problematic when trying to achieve a unique local coordinate expression of these functionals.

Considering this, the inclusion chain that constrains the set of all the multipoles with order N to be included in the set of all multipoles with order $N+1$ has been studied, as well as the algebraic structure of $C^\infty(M)$-module that these sets naturally exhibit. This is very convenient, allowing us to express each multipole as a $C^\infty(M)$-linear combination.

The two equivalent definitions of multipoles are then analysed, leading us to two different classes of representations for these objects, in order the Ellis [17][18] and the Dixon [1] [2] [3] representations. The first one represents the actions of the multipoles in terms of the coordinate expression of linear combinations of several Lie derivatives of the test tensor fields, summed together and integrated on the image of the embedding $c : \mathbb{R} \hookrightarrow M$.

This representation is very minimal and general, and does not require other than the differential structure owned naturally by the considered differential manifold, so it always exists without any constraint. The second one represents the actions of the multipoles in terms of the coordinate expressions of linear combinations of several covariant derivatives of the test tensor fields, summed together and integrated on the image of the embedding. This representation is still minimal but it requires the existence of an affine structure (affine connection) defined on the manifold. In general this is not a problem, since no assumptions on the affine connection (i.e. Levi Civita) are needed.

Let us stress that the two definitions of multipoles, and so the two classes of representations, do not rely on the existence of any metric, or other extra structures on the manifold, so in principle there are no geometrical restrictions on the existence of such objects on arbitrary space-times with non trivial geometry (non torsionless spaces, bimetric spaces and Weyl spaces just to quote some). Both of these two representations have some pros and cons and, as widely discussed, the Dixon and the Ellis representations provide a natural way to express the multipoles (and their actions) in terms of operations upon the local coordinates expression of the test tensor fields, so they induce a local expression for the multipoles.

This is very important, since the manipulation and the study of the geometrical objects upon the manifolds often can be done in practice by just manipulating and studying their local coordinate expression, using standard analysis tools defined on $\mathbb{R}^m$ and relaying on the local diffeomorphism of it with subsets of the manifolds. However, as showed and widely discussed, the achievement of a satisfactory local coordinate expressions can be extremely problematic when dealing with the multipoles.

Let us summarise briefly what has been found crucial both in the Ellis and in the Dixon representations.

The first relevant issue we faced is that, even fixing the standard differential structure on the manifold (i.e a closed embedding, an atlas, a smooth partition of the unity subordinate to the atlas and smooth local frame), this is not enough to fix a unique Ellis local representation for each multipole. This is directly caused by the failure in finding a unique Ellis representation of the null multipole. As consequence, the lack of any isomorphism between a multipole and its Ellis parameters causes both the failure of the attempt to create local expressions of the operations upon the multipoles and the impossibility to bound the equality of multipoles to the equality of their local expressions. Another non-negligible problem is that, by construction, the Ellis local representation does not exhibit a linear transformation rule like vectors or tensor fields, therefore even the simpler mathematical expression equating the Ellis local components of the multipoles to zero, is not invariant under diffeomorphims so it is not compatible with the general covariance principle.

Several approaches are discussed to working around these two problems. Adding extra structures makes it possible to work around the non uniqueness problem saving the covariance or not, but in general it seems quite clear that the Ellis representation is not so useful for practical applications when the invariance under local diffeomorphisms is required (i.e. relativistic models). On the contrary, a specific Ellis representation called adapted Ellis representation is an essential tool to prove specific intrinsic mathematical properties of the multipoles, to study the algebraic structure and to link them to the weak asymptotic expansion of the compact support tensor fields.

As showed, given a one parameter family of smooth compact support tensor fields satisfying very specific conditions (in order squeezed tensor fields), its weak asymptotic expansion can be cast as a coordinate-free asymptotic expansion in a neighbourhood of the null value of the parameter, and the coefficients can be identified intrinsically just as multipoles. Furthermore, if an adapted local coordinate system is chosen, then the adapted Ellis representation of such multipoles coincides just with the usual well known moments defined commonly in Statistics, Mathematics, Classical Mechanics and Classical Electromagnetism, computed by simply integrating, in an appropriate way, the local expression of the given tensor field multiplied by powers of the coordinate functions. This provides a clear geometrical coordinate-free interpretation of what a multipole expansion of a compact support tensor field is (so the geometrical interpretation of the multipole expansion of a physical source), and how to interpret geometrically the standard moments related to it.

In the other hand, the very same linear functionals, can be represented also using the Dixon method which, from a covariant point of view, is much more preferable to deal with when trying to extract and classify physical information within a fully Relativistic Theory.

As well as the Ellis representation, the standard differential structure on the manifold is not enough to fix a unique Dixon local representation for each multipole, but despite the Ellis case, because of the affine structure brought by the affine connection on the manifold, the Dixon representation produces a local representation that changes with the very same linear rules characterising the local expressions of vector and tensor fields.

In contrast with respect to the Ellis case, it is possible to easily kill the redundancy in the local representation in a completely coordinate-free way, therefore with this representation it is possible to fix an isomorphism between the multipoles and their local expressions preserving the invariance under local diffeomeorphism.

Furthermore, despite in the Ellis case, since the transformation rules for the Dixon local representation are linear with respect to the Jacobian of the coordinate transformations, it is possible to interpret the Dixon moments in a coordinate-free manner just as the local expressions of specific smooth tensor fields restricted on the worldline. So they can be eventually interpreted like physical quantities in a fully relativistic framework, as it is already implicitly done in the standard pole dipole approximation within General Relativity[1][2][3][4][5][6][7][8][9][10][11][12].

Let us stress once again that the given approach does not require a specific metric, or a specific connection, therefore in principle it is possible to represent the multipoles in a nice useful way without being forced to assume any model subtending the physics and the geometry or the existence of particular symmetries upon the considered manifold.

Considering this, the "transverse polarised Dixon representation", can be eventually used to provide a unique local expression of the multipoles even within relativistic models beyond General Relativity.

To conclude the analysis, we showed a possible specific application of the multipoles to the relativistic models, the dynamics of the monopoles and the dipoles, respectively related to the first two terms in the asymptotic expansion of a divergence-less symmetric Energy Momentum tensor field about a worldline.

It has been shown that in this scenario, for an assumed Levi Civita connection, the monopole support must satisfy the geodesic equation and the dipole term must satisfy the Mathisson-Papapetrou-Dixon equations. For a non metric connection, the dynamical equations of the monopole is affected just by a reparametrization factor without affecting the geodesic trajectory, while the dynamics of the dipole is coupled with an additional torsion term.

As already stated in the introductory section, because of the non negligible back reaction problem, even if the generalised Mathisson-Papapetrou-Dixon equations for the Pole-Dipole approximation is achieved, one should consider this work more as a methodological introduction to the problem of approximating the dynamics of extended objects in Relativistic Models (possibly beyond General Relativity) rather than a concrete physical proposal. However with this work we showed that in principle there is a clear geometrical meaning in terms of De Rham push-forward and linear functionals upon the differential manifolds, subtending the calculation of both the standard "moments" [13][14] and the ones used in the Pole-Dipole [1][2][3][7] approximation to describe the dynamics of a relativistic free falling spinning particle.

These geometrical objects, that seem to be good candidates to model the test particles within relativistic frameworks, do not depend on any extra structure defined on the manifold except the closed embedding $c \hookrightarrow M$, but their local representations do. It is clear now that different representations exhibits different pros and cons and the choice of a particular way to express the multipoles, in general, strongly depends on the prescriptions and the axioms characterising the theories as well as the general purpose subtending the use of such geometrical objects.

In conclusion, with this work, a general framework characterising in a model-independent way the pure geometrical meaning of the multipoles upon a differential manifold has been provided, showing that a slight generalisation of the De Rham push-forward [13][14] can be linked to several well known mathematical tools, apparently unrelated until now.

The strength of the coordinate-free and model-independent approach shows how, theoretically, the moments are not just some useful quantities computed within specific models but they can be fully interpreted as the local expression of some well founded geometrical objects defined on the differential manifolds. The moments of the extended objects or sources, rising naturally in Classical Field Theory, Statistics, Classical Mechanics and in Special Relativity now can be easily interpreted just as specific local expressions of a very particular set of $\mathbb{R}$-linear functionals defined upon the manifolds by the De Rham push-forward and approximating the tensor fields at a deep coordinate-free geometrical level, allowing us to fully understand in a pure covariant way, the meaning of "multipole approximation" of compact support tensor fields within relativistic models eventually beyond General Relativity.

Since in General Relativity the dynamics of the first two multipoles of an energy momentum tensor field coincides exactly with the dynamics of a free falling point-like test particle and a free falling spinning test particle respectively, we believe that there is a nice chance that multipoles are able to represent eventually an effective pure geometrical way to model test particles and test charges. They provide an interpretation of them just as a "geometrical weak asymptotic expansion" of the extended sources of the interaction fields in the relativistic theories.

Appendix A

Conventions and Notation

In this chapter the conventions and notations used in this work are collected and shown. Some of them are pretty standard and commonly used in maths and physics. Others are created from scratch specifically to deal as straightforwardly as possible with some required calculations and analysis concerning the mathematical objects we are investigating.

A.1 Indices and Lists and Multi-indexed Lists

Considering we massively manipulate indices and lists related to the coordinate expression of tensors and multipoles, a clear multi-index notation is required. It should be simple to use, compact, quite intuitive and straightforward to be able to implement comfortably the standard operations about arbitrary rank tensor field and multipoles local coordinate expressions.

A.1.1 Introduction

Let us consider $\mathbb{N}^+$ the set of all non-negative natural numbers. Given $a, b \in \mathbb{N} \,|\, a < b$ we denote with $[a, b] = \{x \in \mathbb{N} | a \leq x \leq b\}$ a generic interval. Given a set U and $l \in \mathbb{N}^+$, a list I of elements in U with length l is a function $I : [a, a+l-1] \to U$. Hence a list is isomorphic to an indexed n-tuple of elements in U. We can use the standard round bracket notation $(u_a, ...u_{a+l-1})$ to denote the indexed n-tuples $I = \{(u(\mu), \mu) \,|\, u(\mu) \in U, \forall \mu \in [a, a+l-1]\}$. The letter μ is called index and it point uniquely to an element inside the list, therefore given a list I we can denote uniquely an element of it just specifying the name of the list and the corresponding index. Let us remark that the indices of a list with length l must cover all the values in the interval $[a, a+l-1]$, for instance the n-tuple $(t_3, t_4, t_5, t_6, t_7, t_8)$ is a good list but $(t_1, t_3, t_4, t_5, t_6, t_7)$ is not. Given a list I we can define a sub-list J a subset of I such that it is a list. For instance given the list $(t_3, t_4, t_5, t_6, t_7, t_8)$ a good sub-list is given by (t_4, t_5, t_6). A natural generalisation of a list is the multi-indexed list. Given a set U and $l \in \mathbb{N}^+$, a multi-indexed list I of elements in U is a function

$$I : [a_1, a_1 + l_1 - 1] \times [a_2, a_2 + l_2 - 1] \times ... \times [a_n, a_n + l_n - 1] \to U$$

Hence a generic multi-indexed list can be written as:

$$I = \{(u(\mu_1, \mu_2, ..., \mu_n), (\mu_1, \mu_2, ..., \mu_n)) \mid u(\mu_1, \mu_2, ..., \mu_n) \in U, \forall \mu_i \in [a_i, a_i + l_i - 1]\}$$

As one can easily realise, the multi-indexed lists can be interpreted just as normal lists for which each element can be uniquely identified by a list of indices rather than a single index. The using of a multi-indexed list can be very useful in case one need to separate a list in several sub-list and operating separately on each of them (i.e. linear algebra, tensor calculus or cycle decomposition of permutations). There are several different lists and multi-indexed list in our work, all of them are used for different purposes. Often it is mandatory to face lists of lists, lists of indices or lists with unfixed length. There is no way to single out just one specific notation for the lists adapted to all the needs in a satisfactory way. We decided to be pragmatic, prioritising the attempt to make the operations and manipulations on tensors and multipoles easier as possible, but at the same time we tried to be minimal and conservative, keeping the standard conventions whenever a new notation is not strictly necessary and providing the reader an user-friendly environment. The result is a blend of some different notations based on the role the lists play in the game.

A.1.2 Notation about the lists indexed by positive natural numbers

The choice of a good notation is very important to avoid as much as possible confusion and misunderstandings, making easy the lists easy to manipulate. As stated above we tried to be pragmatic, defining a notation adapted to our purposes. A list of natural numbers starting from 1 and ending at $a \in \mathbb{N}^+$ is denoted by $\overline{a}$. Hence accordingly to this notation $\overline{a} = (1, ..., a)$. A list of natural number starting from $a \in \mathbb{N}^+$ and ending in $b \in \mathbb{N}^+, b > a$ is denoted by $\overline{b}\backslash\overline{a}$. The length of the list is just $b-a$. By convention the empty list can be denoted both by $\overline{0}$. This is compatible with the previous notation in fact $\overline{a} \setminus \overline{a} = \overline{0} = \varnothing$. Accordingly to this,when it is convenient, a list $(u_1, ..., u_a)$ of elements in U, can be denoted just with $u_{\overline{a}}$. For list starting not from 1 for instance $(u_{a+1}, u_{a+2}, ..., u_{b-1}, u_b)$ we use the following notation $u_{\overline{b}\backslash\overline{a}}$. By convention a list as $u_{\overline{0}}$ is the empty list as well as $u_{\overline{a}\backslash\overline{a}}$. In our work we are going to use just specific multi-indexed lists. As stated before a multi-indexed list can be interpreted just as a list in which each element is pointed by a list of indices rather than just one index. Therefore a compact notation for the list of indices is required. The lists of indices by convention start from a number greater than 0 making the counting of the of indices more intuitive, hence it could be something like this $(\mu_{a+1}, \mu_{a+2} ..., \mu_{b-1}, \mu_b)$. To express it in a compact way also for a list of indices of unfixed length we decided to use this notation $(\mu_{a+1}, ..., \mu_b) = \mu_{\overline{b}\backslash\overline{a}}$ accordingly with the previous notation set up for the list of natural numbers. The most common lists of indices start from 1 and end in a, making the notation very easy: $(\mu_1, ..., \mu_a) = \mu_{\overline{a}}$. By convention a list as $\mu_{\overline{0}}$ is the empty list as well as $\mu_{\overline{a}\backslash\overline{a}}$ As we can see in the following sections, to express the geometrical objects we are interested in, all the greek indices of the multi-indexed lists must run from 0 to a fixed constant value $l \in \mathbb{N}^+$. Hence is not important to pay attention to the range of each index, since it is shared by all of them.

A.1.3 Local coordinate expressions of points on the manifold

To denote a list of object concerning the local coordinate expressions of standard geometrical objects like curves, functions, vectors, tensors and connection, we use the usual standard Einstein summation convention. Let M be the differential manifold we are working on, let $dim(M) = m$ be the dimension of it and let (U, ϕ) with $\phi : U \subset M \to \mathbb{R}^m$ a local chart. Then the image of a generic point $x \in U \subset M$ through ϕ is a list of real numbers. The convention provides a list of real number for the local expression of points on the manifold, it is indexed from 0 to $m-1$: $\phi(x) = (\phi(x)_{(0)}, ..., \phi(x)_{(m-1)})$. By convention, an element of the list related to the local expression of a point x can be singled out just naming the point and fixing the index on the top of it as follow: $x^0 = x(0)$ therefore we can cast the coordinate expression for a point as $\phi(x) = (x^0, ..., x^{m-1})$. If we switch fom the local coordinate system (U, ϕ) to a new one (U', ϕ') then by convention we have the new list: $\phi'(x) = (x'^0, ..., x'^{m-1})$. Sometimes it can be very useful to express an arbitrary element into the list, it can be done just specifying the element accordingly the previous convention, but without fixing explicitly the value of the index. Therefore, when we write x^μ with a greek letter as index, we mean an arbitrary element inside the list $(x^0, ..., x^{m-1})$. A lot of time it is very convenient to express an arbitrary element inside the the list, which is not the first element. In this case we write x^i with the latin index, and we mean an arbitrary element of the sub-list $(x^1, ..., x^{m-1}) \subset (x^0, ..., x^{m-1})$. Due to the properties of the coordinate system and of the differential manifolds, we can say that x is diffeomorphic to its coordinate expression, hence it is diffeomorphic to the list $(x^0, ..., x^{m-1})$. Sometime it is useful to identify the lists of the local expression of a point with the set of it's own elements but without specifying them explicitly. In that case accordingly to this notation we write a generic list $(x^0, ..., x^{m-1})$ as (x^μ). Then the expression (x^μ) can also be used to denote the whole list of coordinates related to the point x.

A.1.4 Local coordinate expression for tangent vectors

To perform easily the operations with vectors, and low rank tensors we use the standard Einstein notation. It was developed by Einstein to make very easy the multi-linear operations due to the action of tensors on tangent vectors. Given M a differential manifold with $dim(M) = m$ and x a point on it, let T_xM be the vector space tangent to M at the point x. We can always single out lists of length m of linearly independent vectors that span the whole space called basis. It is known by theorem that given a local coordinate system (x^μ) a particular basis called "natural basis" always exists and its coordinate expression is given by the list of partial derivatives $(\frac{\partial}{\partial x^0}, ..., \frac{\partial}{\partial x^{m-1}})$ also denoted as (∂_μ). The existence of a natural basis induces a natural set of coordinate on T_xM and a generic vector $v \in T_xM$ can be always expressed by an $\mathbb{R}$-linear combination of partial derivatives as follow: $v = v^0\partial_0 + ... + v^{m-1}\partial_{m-1} = \sum_{\mu=0}^{m-1} v^\mu \partial_\mu$ where for each value of μ, v^μ is an element of a list of real numbers (v^μ) called "components" representing the natural local coordinate expression of v. By this convention we have some advantages. First of all, the local expression of a vector can be written using exactly the same conventions on the lists used to indicate the local expression of the points on M. As second instance, a natural basis is denoted using lower indices, accordingly to the standard notation used for the partial derivatives. For this reason we fix our notation extending the convention of lower indices to each basis of T_xM to remark that each element acts as a derivation on $C^\infty M$. Hence an arbitrary basis of T_xM can be expressed by the list of linearly independent vectors (e_μ). We can see how the fundamental property that makes this notation so useful and powerful rises combining the two convention above and the local expression of a vector. In fact, fixed a local basis (e_μ) on T_xM, we can see how the valid linear combinations of vectors must be always express by a sum of m terms which are composed multiplying an element of the list of components v^μ with an element of the basis with same index e_μ. Since the index μ runs over all its possible values and the sum is commutative and associative we can decide just to omit the symbol of sum letting it be implicitly implied by the presence of the index μ in both upper and lower position inside the expression. Hence the vector $v = \sum_{\mu=0}^{m-1} v^\mu \partial_\mu$ can be expressed in a much more simple way as follows $v = v^\mu \partial_\mu$. Hence by convention when an index is repeated twice both in upper and lower position inside an expression, then a sum over all the possible values of the index is implicitly meant. In this case the index is defined contracted or "dummy index" because it cannot anymore be fixed arbitrarily to single out a specific element inside the related list, but it is forced to run all over the range of its possible values. Sometimes, at fixed frame it is useful to identify the lists of the local expression of a vector with the set of its own elements but without specifying them explicitly. In that case accordingly to this notation we write a generic list $(v^0, ..., v^{m-1})$ as (v^μ). Then the expression (v^μ) can also be used to denote the whole list of coordinate related to the vector v.

A.1.5 Local coordinate expression for covectors.

We can see how the Einstein notation for vectors fixed above is able to induce an useful convention for the $\mathbb{R}$-linear functionals acting on tangent vectors also called covectors or 1-forms. Let $\alpha \in T_x^\star M$ be a 1-form. Given (e_μ) a local basis of T_xM we can always induces the local expression of α in the following way $\alpha(\sum_{\mu=0}^{m-1} v^\mu e_\mu) = \sum_{\mu=0}^{m-1} v^\mu \alpha(e_\mu) = \sum_{\mu=0}^{m-1} v^\mu \alpha_\mu$ using the $\mathbb{R}$-linearity. The list $\{(\alpha(e_\mu), \mu) \mid e_\mu \in (e_\mu), \forall \mu \in [0, m-1]\}$ is called local expression of the 1-form and it is by definition the action α on each vector of the basis (e_μ), therefore it is a list of real numbers. A generic element in this list can be written as α_μ, with lower index. The convention of lower index concerning the list associated to the local expression of covectors is consistent with the definition of α and the convention about the basis of T_xM. Furthermore fixed (e_μ), the action of α on v is always expressed by a linear combination of elements in (a_μ) with elements in (v^μ) therefore using the convention of lower indices we gain for free the "implicit sum" and "dummy indices" rules also for the local expression of actions of 1-forms over vectors. By theorem we know that fixed a basis on T_xM we can always induce canonically a basis of $T_x^\star M$ in the following way:

$$\{(f(\mu), \mu) \mid f(\mu) \in T_x^\star M, f(\mu)(e_\nu) = \delta(\mu, \nu),\ \forall \mu, \nu \in [0, m-1]\}$$

where $\delta(\mu, \nu)$ is the standard Kronecker delta. A generic element can be denoted by f^μ. We use the upper index notation for the basis of $T^\star M$ and once again this is consistent with the previous notation. Furthermore we gain for free the Einstein convention for the covectors as a linear combination of basis covectors of $T_x^\star M$: $\alpha = \sum_{\mu=0}^{m-1} \alpha_\mu f^\mu = \alpha_\mu f^\mu$. So, the action of a one form over a vector at fixed frame, usually expressed quite pedantically by:

$$\alpha(v) = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} \alpha_\mu f^\mu (v^\nu e_\nu) = \sum_{\mu=0}^{m-1} \alpha_\mu v^\mu$$

can be recast in a much more compact way as:

$$\alpha(v) = \alpha_\mu f^\mu (v^\nu e_\nu) = \alpha_\mu v^\mu$$

Considering that fixed a basis (e_μ) there always exists just one canonically induced basis of $T_x^\star M$ which we denote with (e^μ) rather that (f^μ) and no ambiguity rises from that. Finally we can state the coordinate expression of a 1-form can be written simply as: $\alpha_\mu e^\mu$ Let us remark that the action of α on v induces a dual action of v on α due to $v(\alpha) := \alpha(v)$. From the local expressions point of view this isomorphism is even more obvious because their actions are indistinguishable. Sometimes, at fixed frame it is useful to identify the lists of the local expression of a covector with the set of its own elements but without specifying them explicitly. In that case accordingly to this notation we write a generic list $(\alpha_0, ..., \alpha_{m-1})$ as (α_μ). Then the expression (α^μ) can also be used to denote the whole list of coordinate related to the covector α.

A.1.6 Local expression for small rank tensors

Let us consider $T \in T^p_{q\,x} M$ a tensor. A tensor is a $\mathbb{R}$ multi-linear map acting on vectors and covectors. Due to canonical isomorphism we can say that vectors and covectors can be interpreted as a very special tensors too. Let us start from the simplest case: $T : T_x^\star M \times T_x M$. Let us fix a basis of TM (e_μ) and let us induce a basis of $T^\star M$ (e^μ). The action of T at fixed frame will be:

$$T(\alpha, v) = T(\sum_{\mu=0}^{m-1} \alpha_\mu e^\mu, \sum_{\nu=0}^{m-1} v^\nu e_\nu) = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} \alpha_\mu v^\nu T(e^\mu, e_\nu) = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} \alpha_\mu v^\nu T^\mu_\nu =$$

$$= \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} T^\mu_\nu \alpha_\mu v^\nu = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} v^\nu T^\mu_\nu \alpha_\mu = etc...$$

using the $\mathbb{R}$-multilinearity. The multi-indexed list:

$$\{(T(e^\mu, e_\nu), (\mu, \nu)) \mid e^\mu \in (e^\mu), e_\nu \in (e_\nu), \forall \mu, \nu \in [0, m-1]\}$$

is called local expression of T and it is given by definition, acting with T on each ordered couple (e^μ, e_ν) of elements belonging to the two bases, therefore it is a multi-indexed list of real numbers. A generic element of the multi-indexed list is denoted by T^μ_ν. Let us remark that with this notation both the indices run from 0 to $m-1$ covering all the values. One can see how, using the convention of lower and upper indices we gain for free the "implicit sum" and "dummy indices" rules also for the local expression of actions of tensors over vectors and covectors.

By the property of the tensor product and due to the isomorphism between rank one tensors, vectors and covectors, we know that fixing a basis on $T_x M$ expressed as (e_μ) we are always able to induce a basis on $T^p_{q\,x} M$ via the tensor product $\otimes$. Let us consider our case, $T \in T^1_{1\,x} M$. A basis of that space is given by a list $(E(k)) = (e_0 \otimes e^0, ..., e_0 \otimes e^{m-1}, e_1 \otimes e^0, ..., e_1 \otimes e^{m-1}, ..., e_{m-1} \otimes e^{m-1})$, the length of the list is now m^2. Considering this, it is much more convenient to use a multi-indexed list expressing the basis of $T^1_{1\,x} M$ in the following way

$$\{(E(\mu,\nu),(\mu,\nu)) \mid E(\mu,\nu) \in T^1_{1\,x} M, E(\mu,\nu)(e^\lambda, e_\rho) = \delta(\mu,\lambda) \cdot \delta(\nu,\rho)\}$$

where $\delta(\mu,\lambda)$ and $\delta(\nu,\rho)$ are the standard Kronecker deltas, rather than expressing the basis of the tensor space just a single list. A generic element of the multi-indexed list can be denoted as $E^\mu_\nu = e_\nu \otimes e^\mu$. We use the upper index notation for the basis of $T^1_1 M$ and once again this is consistent with the previous notation. Furthermore we gain for free the Einstein convention for the tensors as a $\mathbb{R}$-linear combination of the basis of $T^1_{1\,x} M$: $T = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} T^\mu_\nu \; e_\mu \otimes e^\nu = T^\mu_\nu \; e_\mu \otimes e^\nu$. So, the action of T over a vector and a covector at fixed frame, usually expressed quite pedantically by:

$$T(\alpha, v) = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} T^\mu_\nu [e_\mu \otimes e^\nu] (\sum_{\lambda=0}^{m-1} \alpha_\lambda e^\lambda, \sum_{\rho=0}^{m-1} v^\rho e_\rho) = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} \sum_{\lambda=0}^{m-1} \sum_{\rho=0}^{m-1} \alpha_\lambda v^\rho T^\mu_\nu [e_\mu \otimes e^\nu](e^\lambda, e_\rho) =$$

$$= \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} \sum_{\lambda=0}^{m-1} \sum_{\rho=0}^{m-1} \alpha_\lambda v^\rho T^\mu_\nu \delta^\lambda_\mu \delta^\nu_\rho = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{m-1} \alpha_\mu v^\nu T^\mu_\nu$$

can be recast in a much more compact way as:

$$T(\alpha, v) = T^\mu_\nu \; [e_\mu \otimes e^\nu](\alpha_\lambda e^\lambda, v^\rho e_\rho) = \alpha_\lambda v^\rho T^\mu_\nu \; [e_\mu \otimes e^\nu](e^\lambda, e_\rho) =$$

$$= \alpha_\lambda v^\rho T^\mu_\nu \delta^\lambda_\mu \delta^\nu_\rho = \alpha_\mu v^\nu T^\mu_\nu$$

Given a basis (e_μ) there always exists just one canonically induced basis of $T^p_{q\,x} M$ which we denote it with $(e_\mu \otimes e^\nu)$. Finally we can state the coordinate expression of T can be written simply as: $T^\mu_\nu \; e_\mu \otimes e^\nu$. Sometimes, at fixed frame it is useful to identify the lists of the local expression of a tensor with the set of its own elements but without specifying them explicitly. In that case accordingly to this notation we write a generic multi-indexed list $(T_0^0, ..., T_{m-1}^{m-1})$ as (T_ν^μ). Then the expression (T_ν^μ) can also be used to denote the whole multi-indexed list of coordinates related to the covector T.

A.1.7 Recalling the standard Einstein notation

Considering all we have seen in the previous section we can cast the set of rules known as Einstein convention allowing us to perform easily multi-linear operation on vectors,covectors and tensors:

1. Local coordinates for a points x on M must be expressed by a list of real numbers with upper index (x^μ). The length is always m and the index runs from 0 to $m-1$. A single element can be expressed by x^μ.

2. A basis of T_xM must be expressed by a list of vectors with lower index. The length is always m and the index runs from 0 to $m-1$.

3. Given a basis (e_μ) of T_xM the components of a vector v with respect (e_μ) should be expressed by a list of real numbers with upper index (v^μ). The length is always m and the index runs from 0 to $m-1$.

4. A basis of $T_x^\star M$ must be expressed by a list of covectors with upper index. The length is always m and the index runs from 0 to $m-1$.

5. Given the dual basis (e^μ) of T_xM the components of a covector α with respect to (e_μ) should be expressed by a list of real numbers with lower index (v^μ). The length is always m and the index runs from 0 to $m-1$.

6. Given (e^μ) and (e_ν), the components of a tensor T can be expressed by a multi-indexed list of real numbers with upper and lower indices accordingly to their action on vectors and covectors of the given basis. The number of components are $m^{rank(T)}$ and all the indices run from 0 to $m-1$.

7. In a local expression an index appearing just once, is called "free index" and it can assume an arbitrary value from 0 to $m-1$, an index appearing two times both in upper and lower position is called "dummy index". A dummy index implicitly implies a sum over all the possible values of the index it can be renamed arbitrarily without changing the meaning of the expression.

8. Valid local expressions of intrinsic geometrical objects allow just the use of free or dummy indices.

9. The notation still works even to express sum over indices of non-tensorial objects, but one has to keep in mind that for local expressions of non tensorial equations the covariant meaning could be lost.

A.1.8 Generalisation for arbitrary rank tensors and condensed Einstein notation

The Einstein convention for tensor calculus is very powerful and very reliable to both perform very complicated tensorial calculations and check the accuracy of the local coordinate expression for tensorial equations but unfortunately it is not able to deal efficiently with arbitrary rank tensors contraction and calculation. Usually for standard purposes in General Relativity and Electromagnetism this is not a real problem because the ranks of the involved tensors are quite small and always fixed. But if one want to perform some calculation or proofs about sequences and series of increasing rank tensors (i.e. asymptotic expansions of tensor fields or linear functional analysis), the standard Einstein notation becomes pedantic and inefficient very quickly. Luckily, with a very little effort, it is possible to generalise the Einstein convention adapting it to the situation in which the ranks of the tensors involved in the calculation is arbitrary and unspecified. Without this improved notation, performing the proofs about the multipoles at each order would not have been possible. The main key here is to merge the Einstein notation with the notation on the lists of indices defined in the section above, generating what we call the "condensed Einstein notation". It is very interesting to notice that the new notation inherits all the nice property of the both previous notations and no collisions of conventions rises from that.

Let us start by considering an expression with all dummy indices:

$$T = T^{\mu\nu\lambda}_{\sigma\rho} e_\mu \otimes e_\nu \otimes e_\lambda \otimes e^\sigma \otimes e^\rho \tag{A.1.1}$$

We would like to provide a notation for it whose length and complexity does not depend linearly on the rank of the considered tensor. First of all, knowing that the dummy indices can be renamed, we can decide to use the same greek letter indexed by a a positive natural number to distinguish between different indices obtaining:

$$T = T^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2} e_{\mu_1} \otimes e_{\mu_2} \otimes e_{\mu_3} \otimes e^{\nu_1} \otimes e^{\nu_2} \tag{A.1.2}$$

Now we can perform the following translation of notations:

$$\begin{cases} T^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2} = T^{\mu_{\overline{3}}}_{\nu_{\overline{2}}} \\ e_{\mu_1} \otimes e_{\mu_2} \otimes e_{\mu_3} \otimes e^{\nu_1} \otimes e^{\nu_2} = e_{\mu_{\overline{3}}} \otimes e^{\nu_{\overline{2}}} \end{cases} \tag{A.1.3}$$

inspired by the list of indices notation used previously. Hence we can write:

$$T = T^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2} e_{\mu_1} \otimes e_{\mu_2} \otimes e_{\mu_3} \otimes e^{\nu_1} \otimes e^{\nu_2} = T^{\mu_{\overline{3}}}_{\nu_{\overline{2}}} \ e_{\mu_{\overline{3}}} \otimes e^{\nu_{\overline{2}}} \tag{A.1.4}$$

This notation is very compact and the information about the index structure of T is not lost. Considering this, we can write the local expression of a generic tensor $T \in T^p_q{}_x M$ in a very compact way, without being forced to fix a priori the value of p and q:

$$T = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}} = T^{\mu_1\mu_2...\mu_{p-1}\mu_p}_{\nu_1\nu_2...\nu_{q-1}\nu_q} e_{\mu_1} \otimes e_{\mu_2} \otimes ... \otimes e_{\mu_{p-1}} \otimes e_{\mu_p} \otimes e^{\nu_1} \otimes e^{\nu_2} \otimes ... \otimes e^{\nu_{q-1}} \otimes e^{\nu_{\overline{q}}} \tag{A.1.5}$$

We define a "dummy list of indices" a list composed just by dummy indices, accordingly we define a a "free list of indices" a list composed just by dummy indices. At this point the definition of two operations on the lists of indices is required. Let us consider two lists of indices $(\mu_a ... \mu_i)$ and $(\mu_{i+1} ... \mu_b)$. We can see that if we merge the elements of the first with the element of the second we can create a new indexed n-tuple: $(\mu_a, ..., \mu_i, \mu_{i+1}, ..., \mu_b)$. It is trivial to notice that this new collection of objects satisfies all the properties required to be a good list of indices. Two lists satisfying this property are called ***adjoining lists***. Given a generic list of indices $(\mu_a, ..., \mu_b)$ and given $i \in [a, b]$ we can always extract two adjoining sub-lists of indices $(\mu_a ... \mu_i)$ and $(\mu_{i+1} ... \mu_b)$ due to the "choice axiom".

Therefore we can define the ***split*** of a list of indices the operation which extracts two adjoining sub-lists from a given list and we define the ***join*** of two adjoining list the operation which merges two adjoining list.

It is quite obvious to notice that the splitting lists of indices and joining adjoining lists of indices does not affect at all the meaning of our local coordinate expression. In fact:

$$T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = T^{\mu_1...\mu_p}_{\nu_1...\nu_q} = T^{\mu_1...\mu_i\mu_{i+1}...\mu_p}_{\nu_1...\nu_j\nu_{j+1}...\nu_q} = T^{\mu_{\overline{i}}\mu_{\overline{p}\setminus\overline{i}}}_{\nu_{\overline{j}}\nu_{\overline{q}\setminus\overline{j}}} \tag{A.1.6}$$

In the same way we have that:

$$e^{\mu_{\overline{p}}} \otimes e_{\nu_{\overline{q}}} = e^{\mu_1} \otimes ... \otimes e^{\mu_p} \otimes e_{\nu_1} \otimes ... \otimes e_{\nu_q} = \tag{A.1.7}$$

$$= e^{\mu_1} \otimes ... \otimes e^{\mu_i} \otimes e^{\mu_{i+1}} \otimes ... \otimes e^{\mu_p} \otimes e_{\nu_1} \otimes ... \otimes e_{\mu_j} \otimes e_{\mu_{j+1}} \otimes ... \otimes e_{\nu_q} = \tag{A.1.8}$$

$$= e^{\mu_{\overline{i}}} \otimes e^{\mu_{\overline{p}\setminus\overline{i}}} \otimes e_{\nu_{\overline{j}}} \otimes e_{\nu_{\overline{q}\setminus\overline{j}}} \tag{A.1.9}$$

This property leads naturally us to deal easily with more complicated expressions involving lists having both free and dummy indices.

For instance let us consider the following equation:

$$A_{\zeta}^{\alpha\delta\varepsilon} = T_{\zeta\eta\theta\iota}^{\alpha\beta\gamma\delta\epsilon} S_{\beta\gamma}^{\eta\theta} v^{\iota} \tag{A.1.10}$$

where it is clear that both dummy and free indices are involved. Now the translation into a more compact notation is not as straightforward as before. Let us analyse the expression trying to guess some general rules. Let us start from the side of the expression with the greater number of contractions, in this case the right hand side and let us start to rename all the dummy indices accordingly to the previous method. If we decide to use the same greek letter indexed by a a positive natural number to distinguish between different indices then we obtain:

$$A_{\zeta}^{\alpha\delta\varepsilon} = T_{\zeta\lambda_2\lambda_3\lambda_4}^{\alpha\mu_2\mu_3\delta\epsilon} S_{\mu_2\mu_3}^{\lambda_2\lambda_3} v^{\lambda_4} \tag{A.1.11}$$

Hence performing the following translation of notations inspired by the lists of indices:

$$\begin{cases} T_{\zeta\lambda_2\lambda_3\lambda_4}^{\alpha\mu_2\mu_3\delta\epsilon} = T_{\zeta\lambda_{\overline{4}\setminus\overline{1}}}^{\alpha\mu_{\overline{3}\setminus\overline{1}}\delta\varepsilon} \\ \\ S_{\mu_2\mu_3}^{\lambda_2\lambda_3} = S_{\mu_{\overline{3}\setminus\overline{1}}}^{\lambda_{\overline{3}\setminus\overline{1}}} \\ \\ v^{\lambda_3} = v^{\lambda_{\overline{4}\setminus\overline{3}}} \end{cases} \tag{A.1.12}$$

we can write the equation as:

$$A_{\zeta}^{\alpha\delta\varepsilon} = T_{\zeta\lambda_{\overline{4}\setminus\overline{1}}}^{\alpha\mu_{\overline{3}\setminus\overline{1}}\delta\varepsilon} S_{\mu_{\overline{3}\setminus\overline{1}}}^{\lambda_{\overline{3}\setminus\overline{1}}} v^{\lambda_{\overline{4}\setminus\overline{3}}} \tag{A.1.13}$$

This is better than the original expression but we have still to deal with the free indices. Hence we can rename the free indices in both sides of the equation obtaining:

$$A_{\lambda_1}^{\mu_1\mu_4\mu_5} = T_{\lambda_1\lambda_{\overline{4}\setminus\overline{1}}}^{\mu_1\mu_{\overline{3}\setminus\overline{1}}\mu_4\mu_5} S_{\mu_{\overline{3}\setminus\overline{1}}}^{\lambda_{\overline{3}\setminus\overline{1}}} v^{\lambda_{\overline{4}\setminus\overline{3}}} \tag{A.1.14}$$

Now it is enough to join the adjoining sublist of indices to achieve the final expression:

$$A^{\mu_1\mu_{\overline{5}\setminus\overline{3}}}_{\lambda_1} = T^{\mu_{\overline{5}}}_{\lambda_{\overline{4}}} S^{\lambda_{\overline{3}\setminus\overline{1}}}_{\mu_{\overline{3}\setminus\overline{1}}} v^{\lambda_4} \tag{A.1.15}$$

As one can easily check that no information about the free and dummy indices is lost and one can keep easily trace of the contractions. Since we gave some examples of common scenarios and we were able to deal easily with them, we are able to introduce some general rules for the condensed Einstein notation. The fundamental prescription is: *We use the convention given on lists to express in a compact way whole bunches of indices. But in each list of indices, each index must follow the usual Einstein convention. If we expand the condensed convention we must recall the standard Einstein notation.* This prescription is very strong and leads us to conclude the following rules concerning the condensed Einstein notation:

1. Lists of indices are denoted by the name of the index with an overlined subscript denoting the range of the list: $\mu_{\overline{a}} = (\mu_1, ..., \mu_a)$ and $\mu_{\overline{b}\setminus\overline{a}} = (\mu_{a+1}, ..., \mu_b)$
2. Given $T \in T^p_q{}_x M$ the local expression is given by the condensed representation $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ of $T^{\mu_1 ... \mu_p}_{\nu_1 ... \nu_q}$. A basis of $T^p_q{}_x M$ is expressed by:$e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}}$ the condensed representation of $e^{\nu_1} \otimes ... \otimes e^{\mu_q} \otimes e_{\mu_1} \otimes ... \otimes e_{\mu_p}$
3. In each expression the same list of indices can appear just once or twice. If it is present twice it must appear both as an upper and a lower list of indices.
4. A list can be split into two adjoining sublist without affect the expression.
5. Two adjoining sublists can be joined together without affect the expression.
6. A dummy sublist implies implicit summation over all the values of all their indices. They can be renamed without affecting the expression.
7. Expanding the condensed notation we must be able to obtain the standard Einstein notation.
8. The notation still works even to express sums over indices of non-tensorial objects, but one has to keep in mind that for local expressions of non tensorial equations the covariant meaning could be lost.

A.1.9 Split notation for multipoles

Analysing deeply the multipole structure one can see how it is necessary sometimes to split the list related to the local expression of the geometrical object's distinct part. For our purposes it is enough to separate the terms pointed to by the index $\mu = 0$ from the others. Hence we decided to use the standard convention about greek and latin indices to separate the list of real numbers representing the local expressions in several sub-lists.

Hence by convention if the greek indices run from 0 to $m-1$ then the latin indices run from 1 to $m-1$. The result is the following:

$$\begin{cases} (x^\mu) = (x^0, (x^i)), i \in [1, m-1] \text{ for points} \\ (v^\mu) = (v^0, (v^i)), i \in [1, m-1] \text{ for vectors} \\ (\alpha_\mu) = (\alpha_0, (\alpha^i)), i \in [1, m-1] \text{ for covectors} \\ (T^{\mu\lambda_{\overline{p}}}) = (T^{0\lambda_{\overline{p}}}, (T^{i\lambda_{\overline{p}}})) \text{ for tensors and other indexed objects} \end{cases} \tag{A.1.16}$$

This is useful if we want to single out a specific contribution of a specific component in a linear combination:

$$T^{\mu\nu}\alpha_\mu\omega_\nu = \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} T^{ij}\alpha_i\omega_j + T^{00}\alpha_0\omega_0 + \sum_{i=1}^{m-1} T^{i0}\alpha_i\omega_0 + \sum_{j=1}^{m-1} T^{0j}\alpha_0\omega_j \tag{A.1.17}$$

The prescription to perform this splitting is that the latin indices must not be contracted with the greek indices otherwise we have a mismatch of the number of terms in the the sums. Considering that the greek and latin indices never interact, it is possible to use both the Einstein and condensed Einstein convention on the latin indices, always reminding that the lists of latin indices must be separated from the list of greek indices. The previous expression can be written as follow

$$T^{\mu\nu}\alpha_\mu\omega_\nu = T^{\mu 0}\alpha_\mu\omega_0 + T^{\mu j}\alpha_\mu\omega_j = T^{ij}\alpha_i\omega_j + T^{00}\alpha_0\omega_0 + T^{i0}\alpha_i\omega_0 + T^{0j}\alpha_0\omega_j \tag{A.1.18}$$

where the standard Einstein convention is applied to the latin indices. Furthermore we have always to keep in mind that this split is made just for algebraic purposes, a linear combination of terms indexed just by a latin indices is not covariant hence it does not admit in general an intrinsic geometrical meaning. Let us remark that this convention is applied just for local expression and not for lists of intrinsic geometrical objects.

A.1.10 Recalling the "Split Condensed Einstein Convention"

Considering all we have seen in the previous section we can cast a complete set of rules known as ***split condensed Einstein convention*** allowing us to perform easily multilinear operation on vectors, covectors, tensors as well as tensor fields and $\mathbb{R}$-linear funtionals on test tensor fields:

1. Local coordinates for a points x on M must be expressed by a list of real number with upper index (x^μ). The length is always m and the index runs from 0 to $m-1$. A single element can be expressed by x^μ.

2. A basis of $T_x M$ must be expressed by a list of vectors with lower index. The length is always m and the index runs from 0 to $m-1$.

3. Given a basis (e_μ) of $T_x M$ the components of a vector v with respect (e_μ) should be expressed by a list of real number with upper index (v^μ). The length is always m and the index runs from 0 to $m-1$.

4. A basis of $T_x^\star M$ must be expressed by a list of covectors with upper index. The length is always m and the index runs from 0 to $m-1$.

5. Given a basis (e^μ) of $T_x M$ the components of a covector α with respect (e_μ) should be expressed by a list of real number with lower index (v^μ). The length is always m and the index runs from 0 to $m-1$.

6. In a local expression greek indices can assume values from 0 to $m-1$, the latin indices from 1 to $m-1$. The latin indices and the greek never interact and they cannot be packed into the same list.

7. The lists of indices are denoted by the name of the index with an bar subscript denoting the range of the list: $\mu_{\overline{a}} = (\mu_1, ..., \mu_a)$ and $\mu_{\overline{b}\setminus\overline{a}} = (\mu_{a+1}, ..., \mu_b)$

8. Given $T \in T^p_{q\,x}M$ the local expression is given by $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ the condensed representation of $T^{\mu_1...\mu_p}_{\nu_1...\nu_q}$. A basis of $T^p_{q\,x}M$ is expressed by:$e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}}$ the condensed representation of $e^{\nu_1} \otimes ... \otimes e^{\mu_q} \otimes e_{\mu_1} \otimes ... \otimes e_{\mu_p}$

9. In each expression the same list of indices can appear just once or twice. If it is present twice it must appear both as an upper and a lower list of indices. In that case it is called "dummy list".

10. A list can be split in two adjoining sub-lists without affecting the expression.

11. Two adjoining sub-lists can be joined together without affecting the expression.

12. Dummy sub-lists implies implicitly sums over all the values of all their indices. They can be renamed without affect the expression.

13. Valid local expressions of intrinsic geometrical objects allow just the use of free or dummy sub-lists of greek indices.

14. Expanding the condensed notation we must be able to obtain the standard Einstein notation.

15. The notation still works even to express sum over indices of non-tensorial objects, but one has to keep in mind that for local expressions of non tensorial equations the covariant meaning could be lost.

Appendix B

Fiber Bundles, Tangent Tensors and Tensors Fields

As it has been outlined in the introduction, one of the main purposes of this work is to provide a satisfactory geometrical intrinsic interpretation of what a multipole on a differential manifold is. The definition of tangent tensors fields and intrinsic operations acting on them is definitely needed to build step by step all the elements needed to achieve our purposes. In this chapter we assume the reader is familiar with the standard structures that can be built on a differential manifolds as curves, vectors, covectors, tangent and cotangent spaces at a point of a manifold, however a brief review of this common structures is given. In this chapter both the standard and the condensed Einstein notations are widely used, the reader can find details and explanations about this convention in the appendix.

B.1 Tangent and cotangent bundles

It is very convenient to use the Fiber Bundles technology for defining, from scratch, what a tensor field on a manifold M is and which are their standard properties. It allow us to show the geometrical structure subtending the concept of field on a manifold.

B.1.1 Elements of fiber bundles and fields

Let M be a smooth manifold. A fiber bundle over the base M is, loosely speaking, a differential manifold B admitting locally the product topology $U_j \times F$, where $(U_j) \,|\, j \in I$ is an open covering of M, F a manifold called standard fiber and I any set of indices. The local models $(U_j \times F)$, in general, can be "glued together" with some prescriptions in a very non-trivial way such that the topology of B may be different from the topology of a global Cartesian product $M \times F$. The mathematical meaning of "glueing together" consists of choosing at each point of each intersection $U_{jk} = U_j \cup U_k \neq \varnothing$ a group of transformations on the fiber $G \subset \mathit{Diff}(F)$, that establishes how to switch from a local model to another, called structure group.

Definition 88: A ***fiber bundle*** is a quadruple $\mathcal{B} = (B, M, \pi, F)$ such that:

1. B, M, F are differential manifolds called respectively ***total space***, ***base*** and ***standard fiber***
2. $\pi : B \to M$ is a maximal rank surjective map called ***projection*** and for each $x \in M$ we call $\pi^{-1}(x)$ a ***fiber*** at x
3. there must exist an open covering (U_j) of the base M such that $\forall U_i \in (U_j)$ there exists a diffeomorphism $t_i : \pi^{-1}(U_i) \to U_i \times F$. The pair $(U_i, t_{(i)})$ is called ***local trivialisation***. The set of all local trivialisation is called simply ***trivialisation*** of $\mathcal{B}$ and $(U_i \times F)$ are the ***local models***.

For simplicity, when we have to denote the bundle $\mathcal{B}$ with its total space B explicitly, we are talking about a bundle. Let us remark that a trivialisation is simply required to exist, in general each bundle admits infinite equivalent trivialisations defining the same bundle structure.

Definition 89: Let us fix the trivialization $(U_i, t_{(i)})$ on a fiber bundle (B, M, π, F) and let us denote by $U_{ij} = U_i \cap U_j$ the overlaps of the covering (U_j) on the base M. Let $t_{(i)}(x) : \pi^{-1}(x) \to x \times F \simeq F$ be the restriction of $t_{(j)}$ on the fiber $\pi^{-1}(x)$. We define the ***transition functions*** the map $\hat{g}_{(ij)} : U_{ij} \to \mathit{Diff}(F)$ such that:

$$\hat{g}_{(ij)}(x) = t_{(i)}(x) \circ t^{-1}_{(j)}(x) \tag{B.1.1}$$

where $\mathit{Diff}(F)$ denotes the group of diffeomorphisms on the fiber F.

Sometimes the trivialisation induces automatically a set of charts on the bundle B but we stress however that the trivialisation domain need not be a coordinate domain. For instance the cylinder allows a global trivialisation $S^1 \times \mathbb{R}$ even if there is no global coordinate on the circle.

Property 57: Given a bundle B, let $\{(U_i, t_{(i)}\}$ be a trivialisation and let $g_{ij} = t_{(i)}(x) \circ t^{-1}_{(j)}(x)$ be the transition functions. Then it is easy to check the following cocycle identity holds:

$$\begin{cases} \hat{g}_{(jj)}(x) = id_F \\ \hat{g}_{(ji)}(x) = [\hat{g}_{(ij)}(x)]^{-1} \\ \hat{g}_{(ij)}(x) \circ \hat{g}_{(jk)}(x) \circ \hat{g}_{(ki)}(x) = id_F \end{cases} \tag{B.1.2}$$

where id_F is the identity transformation on the fiber.

Proposition 1: Let us remark a very useful and interesting property of bundles, for which just a sketch of proof is provided. Let F and M be two manifolds, let (U_j) be a covering of M that overlaps in $U_{ij} = U_i \cap U_j$ and let $\hat{g}_{(ij)} : U_{ij} \to \mathit{Diff}(F)$. There always exists a unique bundle (B, M, π, F) (modulo isomorphisms) which admits $(U_i \times F)$ as local models and $\hat{g}_{(ij)}$ as transition functions.

Proof. The sketch of proof can be performed as follow. Let F and M be two manifolds with $x, x' \in M$ and $\phi, \phi' \in F$. Let us define the space A as the disjoint union $A = \bigsqcup_{i \in I} (U_i \times F)$ and the equivalence relation:

$$(i, x, \phi) \sim (j, x', \phi') \Leftrightarrow \begin{cases} x = x' \\ \phi = [\hat{g}_{(ij)}(x)](\phi') \end{cases} \quad . \tag{B.1.3}$$

We denote the equivalence classes by $[x, \phi]_{(i)}$. Now it is possible to define the quadruple (B, M, π, F) where:

1. $B = A_{/\sim}$
2. $\pi([x, \phi]_{(i)}) = x$
3. $t_{(i)}([x, \phi]_i) \to (x, \phi)$

One can check that this is a good fiber bundle. To show the uniqueness it is sufficient to consider a fundamental property from the fiber bundle theory: two bundles having the same base M and standard fiber F admitting the same transition functions with respect to the same open covering U_i are isomorphic. Therefore we must conclude that all the bundles defined above must be isomorphic then they define the same bundle structure. $\square$

Definition 90: Let us consider a bundle (B, M, π, F) and let U be an open set of M. A ***local section*** of the bundle is a map $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = id_U$ where $id(U)$ is the identity transformation on U. If $U = M$ then the map σ is called a ***global section***

The existence of local sections is guaranteed in any fibered bundle due to the existence of the local trivialisation, in fact each map $\alpha : U_i \to F$ induces a local section:

Definition 91: Given a local trivialisation $(U_i, t_{(i)})$ and a map $\alpha : U_i \to F$ we can define the ***induced local section*** the map:

$$\sigma_\alpha : x \to t_{(i)}^{-1}(x, \alpha(x)) = [x, \alpha(x)]_{(i)} \tag{B.1.4}$$

Given a local section $\sigma_{(i)}$, we define its ***local representative*** the map

$$t_{(i)} \circ \sigma_{(i)} : U_i \to U_i \times F \tag{B.1.5}$$

In contrast, global sections may not exists, depending on the topology of the considered bundle.

Definition 92: Given a bundle B we denote by ΓB the set of all sections.Let U be an open set of the base M, we denote by $\Gamma_U B$ all the local sections with domain on U. Accordingly to this we denote the set of all the global sections with $\Gamma_M B$.

The local sections of a bundle (B, M, π, F) form a sheaf. In fact, let us consider two open sets U_i and U_j on the base M such that $U_i \cap U_j = U_{ij} \neq \varnothing$, and let $\sigma_{(i)} : U_i \to B$ and $\sigma_{(j)} : U_j \to B$ be two local sections defined on them. If the sections satisfy the constraint

$$\sigma_{(i)|_{U_{ij}}} = \sigma_{(j)|_{U_{ij}}} \tag{B.1.6}$$

called ***compatibility condition*** then obviously there exists a unique local section $\sigma_{(ij)}$ defined on $U_i \cup U_j$ such that:

$$\begin{cases} \sigma_{(ij)|_{U_i}} = \sigma_{(i)} \\ \sigma_{(ij)|_{U_j}} = \sigma_{(j)} \end{cases} \tag{B.1.7}$$

Given a bundle B let us denote by $(U_i, t_{(i)})$ and $(U_j, t_{(j)})$ two local trivializations such that $U_{ij} = U_i \cap U_j \neq \varnothing$. Let $f : U_i \to F$ and $g : U_j \to F$ be two maps, then we can induce two local sections $\sigma_{f(i)} : U_i \to B$ and $\sigma_{g(j)} : U_j \to B$ written as:

$$\sigma_{f(i)}(x) = [x, f(x)]_{(i)} = t_{(i)}^{-1}(x, f(x)) \qquad \sigma_{g(j)}(x) = [x, g(x)]_{(j)} = t_{(j)}^{-1}(x, g(x)) \tag{B.1.8}$$

The two sections are compatible if they satisfy the compatibility condition $\sigma_{f(i)|_{U_{ij}}} = \sigma_{g(j)|_{U_{ij}}}$, therefore we can recast the compatibility condition directly upon the local representative of the two sections:

$$\sigma_{g(j)|_{U_{ij}}} = t_{(j)}^{-1}(x, g(x))_{|U_{ij}} = t_{(i)}^{-1}(x, f(x))_{|U_{ij}} = \sigma_{f(i)|_{U_{ij}}} \Leftrightarrow \tag{B.1.9}$$

$$\Leftrightarrow \quad (x, g(x))_{|U_{ij}} = t_{(j)} \circ t_{(i)}^{-1}(x, f(x))_{|U_{ij}} = \hat{g}_{ji}(x, f(x)) \tag{B.1.10}$$

Property 58: Two local sections are "glued together" (in other words are compatible) if their local representative satisfy the compatibility condition stated above:

$$(x, g(x))_{|U_{ij}} = \hat{g}_{(ji)}(x, f(x))_{|U_{ij}} \tag{B.1.11}$$

Definition 93: Let B be a bundle and $\{(U_i, t_{(i)})\}$ a trivialisation. Let us suppose that B admits a global section $\sigma : M \to B$. The restrictions of σ to each set U_i in the open cover of the base M define uniquely a family of local compatible sections, therefore a family of local representatives $\{(x, f_{\sigma(i)})\}$ (depending on the trivialisation) related to σ. The family of local representatives $\{(x, f_{\sigma(i)})\}$ is called the ***local representative of the global section*** σ

Property 59: Let B be a bundle and $\{(U_i, t_{(i)})\}$ a trivialisation. Due to the compatibility condition, each local expression of a section must satisfy :

$$(x, f_{\sigma(i)}(x))_{|U_{ij}} = \hat{g}_{(ji)}(x, f_{\sigma(j)}(x))_{|U_{ij}} \tag{B.1.12}$$

Property 60: Given a bundle B with a trivialisation $\{(U_i, t_{(i)})\}$ and a family of maps $f_{(i)}(x) : U_i \to B$ for each open U_i.The family $\{(x, f_{\sigma(i)}(x))\}$ can be interpreted as the local representative of a global section if and only if:

$$(x, f_{\sigma(i)}(x))_{|U_{ij}} = \hat{g}_{(ji)}(x, f_{\sigma(j)}(x))_{|U_{ij}} \tag{B.1.13}$$

Proof. We proved already that due the compatibility condition, given a section, each local representative must satisfy the condition. In the other hand one can ask when, given a bundle B with a trivialisation $\{(U_i, t_{(i)})\}$, a family of local representative $(x, f_{\sigma(i)}(x))$ can define a global section. It can be performed only gluing inductively on each set U_i all the local sections related to each local representative. But "gluing" sections means satisfying the compatibility condition, hence the family $\{(x, f_{\sigma(i)}(x))\}$ can be glued into a global section just if the condition is satisfied. $\square$

Depending on the particular form of the transition function, gluing together local representatives can be very difficult. Sometimes for each arbitrary $(x, f_{\sigma(i)}(x))$, satisfying it is just impossible, therefore the non trivial topological structure of B encoded by the transition functions, fixes very strong topological constraints about the properties and the existence of global sections on B. Let (B, M, π, F) be a bundle and let $b \in B$, $x \in M$, $y \in F$ be arbitrary points. Let us fix a local trivialisation $(U_i, t_{(i)})$ such that $t(b) = (x, y)$ and denote briefly $b = [x, y]_{(i)}$. We can choose the open neighbourhood U_i of $x \in M$ to be the domain of a local chart $\phi : U_i \subset M \to \mathbb{R}^{dim(M)}$ such that $x^\mu = \phi^\mu(x)$. In the same way we can choose the open neighbourhood W of $y \in F$ to be the domain of a local chart

$\phi : U_i \subset M \to \mathbb{R}^{dim(M)}$ such that $y^A = \phi^A(x)$. Hereafter, unless an explicit warning is given, greek indices will be used to label coordinates in the base manifold and capital latin indices will label coordinates in the standard fiber, both with the usual conventions on lists. In this way one can produce a local coordinate system (x^μ, y^A) on B supported in the product $U_i \times W$ due to the local diffeomorphisms between B and $M \times F$ fixed by the trivialisation. In general given a base M one can build many different type of bundles considering different standard fibers F and different trivialisations. Here we are going to introduce three very important classes of fiber bundles:

Definition 94: We define a ***vector bundle*** a bundle (B, M, π, A) where the standard fiber A is an affine space and exists at least one trivialisation $\{(U_i, t_{(i)})\}$ such that the transition functions $\hat{g}_{(ij)} = t_{(i)} \circ t^{-1}_{(j)}$ fill a subgroup of the affine transformations $GA(A)$.

Definition 95: We define a ***vector bundle*** a bundle (B, M, π, V) where the standard fiber V is a vector space and there exists at least one trivialization $\{(U_i, t_{(i)})\}$ such that the transition functions $\hat{g}_{(ij)} = t_{(i)} \circ t^{-1}_{(j)}$ fill a subgroup of $GL(V)$.

Property 61: Each vector bundle admits global sections.

Proof. Given a trivialisation $\{(U_i, t_(i))\}$ of a vector bundle one can define the null local sections as:

$$0_{(i)} = t^{-1}_{(i)}(x, 0) \tag{B.1.14}$$

The family of local null sections are compatible because the null element in V is preserved by the linear transformations and since $\hat{g}_{(ij)} : U_{i,j} \to GL(V)$ the compatibility condition:

$$(x, 0)_{|U_{ij}} = \hat{g}_{(ij)}(x, 0)_{|U_{ij}} \tag{B.1.15}$$

is always satisfied. Hence there exists at least a global null section 0. By smoothly deforming the null section on a compact support one can define many non null global sections. □

Given a trivialisation of a finite dimensional vector bundle, if we choose a basis (e_A) on the fiber V we are inducing a set of fiber coordinates on it and in that case we have the local expression of the transition functions:

$$(x, v_{(i)}(x))_{|U_{ij}} = \hat{g}_{(ji)}(x, v_{(j)}(x))_{|U_{ij}} \tag{B.1.16}$$

$$\Leftrightarrow \quad \big(x, \{v^A_{(i)} e_{(i)A}\}(x)\big)_{|U_{ij}} = \hat{g}_{(ji)}\big(x, \{v^B_{(j)} e_{(j)B}\}(x)\big)_{|U_{ij}} \tag{B.1.17}$$

$$\Leftrightarrow \quad \big(x, \{v^A_{(i)} e_{(i)A}\}(x)\big)_{|U_{ij}} = \big(x, v_{(j)}(x)^B \hat{g}_{(ji)}{}^A_B(x) e_{(i)A}(x)\big)_{|U_{ij}} \tag{B.1.18}$$

therefore the local expression of the transition functions are just standard matrices depending on the point x.

Definition 96: Let (B, M, π, F) be a fiber bundle with a local trivialization $(U_i, t_{(i)})$ and let $((\phi, \Phi), (V, W))$ a local chart on $M \times F$ with $V \subseteq U_i$. Let pr_M and pr_F the canonical projection on $M \times F$. We define a ***fiber coordinate*** a local coordinate $\big((\phi \circ pr_M \circ t_{(i)}, \Phi \circ pr_F \circ t_{(i)}), t^{-1}_{(i)}(V \times W)\big)$ on B induced by the local trivialization $(U_i, t_{(i)})$.

Definition 97: We define a ***principal bundle*** a bundle (B, M, π, G) where the standard fiber G is Lie Group and exists at least one trivialisation $\{(U_i, t_{(i)})\}$ such that the transition functions $\hat{g}_{(ij)} = t_{(i)} \circ t^{-1}_{(j)}$ act on G via the left translation $L_g : G \to G \mid L_g(h) = gh$.

Both the affine and principal bundles have some very interesting properties and very peculiar characteristics. A deep investigation of them is beyond our purposes. The reader can find more details in [21]. In physics, fiber bundles often come with a preferred group of transformations, usually the symmetry group of the system. This group as a fundamental structure which should be implemented from a very beginning. This endows bundles with a further structure. As any manifold structure is fully defined by a maximal atlas, geometric bundles are associated with the concept of "maximal trivialisation". However one can decide to restrict the allowed local trivialisation such that the same geometrical bundle can be trivialised using a smaller class of local representatives. From a geometrical point of view it means to impose a further structure on the bare bundle. Examples have been already introduced: vector bundles are characterised to allow a linear local trivialisation, affine bundles must admit an affine local trivialisation, principal bundles require the existence of a trivialisation with transition functions valued in the left action of the fiber group. Further examples come from physics: gauge transformations are used as transition functions from a configuration bundle of any gauge theory. It is clear that for physical applications the concept of bare bundle is not enough, we need to enrich the concept of bundle with some information about the allowed trivialisation.

Definition 98: A ***fiber bundle with a structure group G*** is a sex-tuple $(B, M, \pi, F, \lambda, G)$ such that:

1. (B, M, π, F) is a fiber bundle. The ***structure group*** is a Lie group G and $\lambda : G \to \mathit{Diff}(F)$ defines the left action of G on the standard fiber F.
2. There must exist a family of preferred trivialisation $\{(U_i, t_{(i)})\}$ of the bundle (B, M, π, F) and a family of maps $g_{(ij)} : U_{ij} \to G$ such that they satisfy the following:
   (a) defining as usual the transition functions $\hat{g}_{(ij)} : U_{ij} \to \mathit{Diff}(F) \,|\, \hat{g}_{(ij)} = t_{(i)} \circ t^{-1}_{(j)}$ on the overlaps U_{ij} we have that

$$\hat{g}_{(ij)}(x) = \lambda(g_{(ij)}(x)) \tag{B.1.19}$$

(b) defining e the neutral element of G then $\forall x \in U_{ijk} = U_i \cap U_j \cap U_k$ the cocycle identity is satisfied:

$$\begin{cases} g_{(ii)}(x) = e \\ g_{(ji)}(x) = [g_{(ij)}(x)]^{-1} \\ g_{(ij)}(x) \cdot g_{(jk)}(x) \cdot g_{(ki)}(x) = e \end{cases} \tag{B.1.20}$$

This preferred trivialisation is said to be ***compatible with the structure***.

Property 62: Given a fiber bundle with structure group $(B, M, \pi, F, \lambda, G)$, there always exists a related principal bundle (P, M, p, G).

Proof. Given a fiber bundle with structure group $(B, M, \pi, F, \lambda, G)$, let us fix a compatible trivialization $\{(U_i, t_{(i)})\}$ and let $\hat{g}_{ij}$ be the transition function valued in G. Due to the left action of the transformation group on itself we can define the maps $\hat{L}_{g_{(ij)}} : U_{ij} \to \mathit{Diff}(G)$ as:

$$\hat{L}_{g_{(ij)}}(x) = L_{g_{(ij)}(x)} \tag{B.1.21}$$

One can check easily that they satisfies the cocycle identity hence they can be interpreted as the transition functions between local representatives $\{U_i \times G\}$. We know by a property of the fiber bundles that each bundle can be uniquely defined modulo isomorphism just specifying the local representatives $\{U_i \times G\}$ and the transition functions. Let us define the space A as the disjoint union $A = \bigsqcup_{i \in I}(U_i \times G)$ the equivalence relation:

$$(i, x, h) \sim (j, x', h') \Leftrightarrow \begin{cases} x = x' \\ h = [\hat{L}_{g_{(ij)}}(x)](h') \end{cases} . \tag{B.1.22}$$

We denote the equivalence classes by $[x, \phi]_{(i)}$. Now it is possible to define the quadruple (P, M, p, G) where:

1. $P = A_{/\sim}$
2. $p([x, h]_{(i)}) = x$
3. $t_{(i)}([x, h]_i) \to (x, h)$

This is by definition for a principal bundle. $\square$

Definition 99: Given a fiber bundle with structure group $(B, M, \pi, F, \lambda, G)$ the related principal bundle (P, M, p, G) built above is called ***structure bundle***.

One can prove that each automorphism of the structure bundle induces a transformation on the related bundle with structure group. In this way $(B, M, \pi, F, \lambda, G)$ is endowed with a preferred group of transformations represented by the canonical action on it due to the group automorphisms of its structure bundle. These transformations are called ***generalised gauge transformations***. Showing the details of these features is once again beyond the purpose of this work. The reader can find them in [21].

B.1.2 Tangent bundle of a smooth manifold

Let us recall briefly how to build the tangent space of a manifold. We assume the details are already known and the reader is familiar with them. This is needed to introduce the concept of the tangent bundle of a differential manifold. Given a differential manifold M with $dim(M) = m$, let $U \subseteq M$ be an arbitrary open subset. Let $\mathcal{F}$ be the sheaf of local functions $f : U \to \mathbb{R}$ and $\mathcal{F}(U)$ be the set of global functions over U which can be identified as the global section of the trivial bundle $U \times \mathbb{R}$. For any U, $\mathcal{F}(U)$ forms an infinite dimensional real algebra with respect to the point-wise sum $[f+g](x) = f(x)+g(x)$ and the point-wise product $fg(x) = f(x)g(x)$. We can denote by $\mathcal{F}_{loc}(M)$ the class of all local function defined on an any arbitrary open subset U of M. When speaking of a local function $f \in \mathcal{F}_{loc}(M)$ we shall just write for simplicity $f : M \to M$ keeping in mind the local character of f. Let $C^{\infty}_{loc}(M) \subset \mathcal{F}_{loc}(M)$ the subset of all the smooth local functions.

Definition 100: For each point $x \in M$ we can define the ***tangent space at*** x denoted by T_xM the set of all equivalence classes of curves based on x defined by:

$$\gamma \sim \gamma' \tag{B.1.23}$$

$$\Leftrightarrow \quad [f \circ \gamma](0) = [f \circ \gamma'](0)\ , \ \frac{d}{dt}[f \circ \gamma](t)_{|t=0} = \frac{d}{dt}[f \circ \gamma'](t)_{|t=0}\ , \ \forall f \in C^{\infty}_{loc}(M) \mid x \in dom(f) \tag{B.1.24}$$

Definition 101: The tangent space of a manifold M, denoted by TM is the set:

$$TM = \bigsqcup_{x \in M} T_xM \tag{B.1.25}$$

The tangent space of a manifold can be regarded just as the collection of all the tangent spaces at each point of it, but with a very little effort we can prove that TM admits naturally a bundle structure called ***tangent bundle*** of M.

Definition 102: Due to the property of the disjoint union, given TM we can always define a canonical surjective map $\tau_M : TM \to M$ such that:

$$\tau_M(\dot{\gamma}_x) = x \tag{B.1.26}$$

Property 63: Let M be a m-dimensional manifold, let TM be the tangent space and $\tau_M : TM \to M$ be the projection defined above . The quadruple $(TM, M, \tau_M, \mathbb{R}^m)$ is a good fiber bundle with standard fiber $\mathbb{R}^m$.

Proof. Let us fix an atlas on M denoted by $\{(U_i, \varphi_{(i)})\}$. Each local chart $(U_i, \varphi_{(i)})$ induces a local coordinates expression $x^\mu_{(i)} = \varphi^\mu_{(i)}(x)$ for each point $x \in U_i \subseteq M$. Let us consider now a list of curves $\hat{c}_\mu : \mathbb{R} \to \mathbb{R}^m \mid \hat{c}_\mu = x^\mu_0 + \delta^\mu_\nu t$ (where δ^μ_ν is a Kronecker delta) based on $x_0 \in U_i \subseteq M$, due to the local charts it induces a list of local curves $c_{\mu(i)} : \mathbb{R} \to U_i \mid c(t) = \varphi^{-1}_{(i)}(\hat{c}_\mu) = \varphi^{-1}_{(i)}(x^\mu_0 + \delta^\mu_\nu t)$ called ***coordinate curves of the chart***. Let be $\partial_{\mu(i)_{|x_0}} = [c_{\mu(i)}] = \dot{c}_{\mu(i)}$ the equivalence class associated to each coordinate curve, therefore by definition $\partial_{\mu(i)_{|x_0}} \in TM$ and we have that $\tau_M(\partial_{\mu(i)_{|x_0}}) = x_0$. Each $\partial_{\mu(i)_{|x_0}}$ acts on $C^\infty_{loc}(M)$ as follows:

$$\partial_{\mu(i)_{|x_0}}(f) = \frac{d}{dt}(f \circ \varphi^{-1}_{(i)} \circ \varphi_{(i)} \circ c_{\mu(i)})_{|t=0} = \tag{B.1.27}$$

$$= \frac{\partial}{\partial x^\nu_{(i)}}(f \circ \varphi^{-1}_{(i)})_{|\varphi_{(i)}(x_0)} \frac{d}{dt}(x^\nu_0 + \delta^\nu_\mu t) = \frac{\partial}{\partial x^\mu_{(i)}}(f \circ \varphi^{-1}_{(i)})_{|\varphi_{(i)}(x_0)} \tag{B.1.28}$$

so we can say that the equivalence classes of the coordinate curves act as the partial derivatives on the coordinate expression of the functions. The preimage $\tau^{-1}(x_0)$ is just the set of all equivalence classes of curves based in x_0, therefore by definition $\tau^{-1}(x_0) = T_{x_0}M$ hence it is a real m-dimensional vector space. An arbitrary element $\dot{\gamma}_{x_0} \in T_{x_0}M$ acts on $C^\infty_{loc}(M)$ as follows:

$$v_{x_0}(f) = \dot{\gamma}_{x_0}(f) = \frac{d}{dt}(f \circ \varphi^{-1}_{(i)} \circ \varphi_{(i)} \circ \gamma_{x_0})_{|t=0} = \tag{B.1.29}$$

$$= \frac{\partial}{\partial x^\nu_{(i)}}(f \circ \varphi^{-1}_{(i)})_{|\varphi_{(i)}(x_0)} \frac{d}{dt}(\varphi_{(i)} \circ \gamma_{x_0})^\nu_{|t=0} = v^\mu \partial_{\mu(i)_{|x_0}}(f) \tag{B.1.30}$$

therefore we can say that for each point $x_0 \in U_i \subset M$ the equivalence classes of coordinate curves based on x_0 denoted by $\partial_{\mu(i)}$ is a $\mathbb{R}$-linear set of independent generators of $T_{x_0}M$ because it spans all the derivations and the 0 derivation can be written uniquely. From this we can conclude that T_xM must be a m-dimensional real vector spaces. This means that $(\partial_{\mu(i)|_{x_0}})$ is a basis for $T_{x_0}M, \forall x_0 \in U_i$ called ***natural basis***. One can easily check that, fixing a natural basis, the map $t_{(i)} : \tau_M^{-1}(U_i) \to U_i \times \mathbb{R}^m$ defined as:

$$t_{(i)}(v_{x_0}) \to (x_0, v^\mu) \tag{B.1.31}$$

is invertible and differentiable, hence it is a diffeomorphism and defines a local trivialisation (one for each open set U_i in the atlas of M). Let us remark that this trivialisation depends just on the chosen charts $(U_i, \varphi_{(i)})$, they are thence called ***natural trivialisation*** because no extra structure (i.e symmetry conditions or gauge transformation) is needed to induce this trivialisation. □

Corollary 20: As a consequence of the trivialisation process, we proved that T_xM must be a real m-dimensional vector space. Given two vectors the sum and the multiplication by a scalar can be induced directly from the sum and multiplication by a scalar defined on $\mathbb{R}^m$

Definition 103: The quadruple $(TM, M, \tau_M, \mathbb{R}^m)$ is called ***tangent bundle of*** M.

Property 64: Given a smooth manifold M, a tangent bundle is vector bundle because the natural trivialization admits transition functions in $GL(\mathbb{R}^m) \subset \mathit{Diff}(\mathbb{R}^M)$.

Proof. Let us suppose to have two local charts $(U_i, \varphi_{(i)})$ and $(U_j, \varphi_{(j)})$ overlapping in $U_{ij} = U_i \cap U_j$. As proved before the two charts induce a natural local trivialization of the tangent bundle TM denoted by $(U_i, t_{(i)})$ and $(U_j, t_{(j)})$. Let $c_{(i)\mu}$ and $c_{(j)\mu}$ be the coordinate curves induced respectively by the first and the second charts. For each point $x_0 \in U_{ij}$ we have:

$$\begin{cases} \partial_{\mu(i)|_{x_0}}(f) = \frac{\partial}{\partial x^\mu_{(i)}}(f \circ \varphi_{(i)}^{-1})_{|\varphi_{(i)}(x_0)} \\ \partial_{\mu(j)|_{x_0}}(f) = \frac{\partial}{\partial x^\mu_{(j)}}(f \circ \varphi_{(j)}^{-1})_{|\varphi_{(j)}(x_0)} \end{cases} \tag{B.1.32}$$

that leads to:

$$\partial_{\mu(j)|_{x_0}}(f) = \frac{d}{dt}(f \circ \varphi_{(j)}^{-1} \circ \varphi_{(j)} \circ c_{\mu(j)})_{|t=0} = \tag{B.1.33}$$

$$= \frac{d}{dt}(f \circ \varphi_{(j)}^{-1} \circ \varphi_{(j)} \circ \varphi_{(i)}^{-1} \circ \varphi_{(i)} \circ \varphi_{(j)}^{-1} \circ \varphi_{(j)} \circ c_{\mu(j)})_{|t=0} = \tag{B.1.34}$$

$$= \frac{\partial}{\partial x^\nu_{(i)}}(f \circ \varphi_{(j)}^{-1} \circ \varphi_{(j)} \circ \varphi_{(i)}^{-1})_{|\varphi_{(i)}(x_0)} \frac{d}{dt}[\varphi_{(i)} \circ \varphi_{(j)}^{-1}(x_0^\lambda + \delta^\lambda_\mu t)]^\nu = \tag{B.1.35}$$

$$=\frac{\partial}{\partial x^\nu_{(i)}}(f\circ\varphi^{-1}_{(i)})_{|\varphi_{(i)}(x_0)}\frac{\partial}{\partial x^\lambda_{(j)}}[\varphi_{(i)}\circ\varphi^{-1}_{(j)}]^\nu_{|\varphi_{(i)}(x_0)}\frac{d}{dt}(x^\lambda_0+\delta^\lambda_\mu t)=\partial_{\nu(i)}{}_{|x_0}\overline{J}^\nu_{\mu_{|\varphi_{(i)}(x_0)}} \tag{B.1.36}$$

where $J^\nu_{\mu_{|\varphi_{(i)}(x_0)}}$ is used to denote the Jacobian of the coordinates transformations at a point x_0. Therefore on the overlaps U_{ij} we have:

$$v_{x_0}=v^\mu_{x_0(i)}\partial_{\mu(i)}{}_{|x_0}=v^\mu_{x_0(i)}J^\nu_{\mu_{|\varphi_{(i)}(x_0)}}\overline{J}^\lambda_{\nu_{|\varphi_{(i)}(x_0)}}\partial_{\lambda(i)}{}_{|x_0}= \tag{B.1.37}$$

$$=v^\nu_{x_0(i)}J^\mu_{\nu_{|\varphi_{(i)}(x_0)}}\partial_{\mu(j)}{}_{|x_0}=v^\mu_{x_0(j)}\partial_{\mu(j)}{}_{|x_0} \tag{B.1.38}$$

and we can conclude the transition functions that glue the standard fibers $\mathbb{R}^m$ are:

$$\begin{cases} x_0=x_0\\ v^\nu_{x_0(j)}=v^\mu_{x_0(i)}J^\nu_{\mu_{|\varphi_{(i)}(x_0)}}\end{cases} \tag{B.1.39}$$

□

Let us remark that the cocycle is by definition completely determined by the atlas $\{U_i,\varphi_{(i)}\}$. In other words, the tangent bundle encodes just the information already encoded on the base manifold M. This is a key feature of the tangent bundle.

Definition 104: A bundle completely defined by the differential structure of the base M is called ***natural bundle***.

Definition 105: A local section of the tangent bundle $\sigma: U\subseteq M\to TM$ is called ***local vector field***. The set of all the section of TM defined on the open U are denoted by $\Gamma_U TM$. If $U=M$ then σ is called ***global vector field***.

Let us suppose to have a vector field $v\in\Gamma TM$, the restriction of $v_{|x}$ to each point is a tangent vector which have a natural action on $C^\infty(U)$ defined previously. Thus given a smooth function $f\in C^\infty(M)$ it is possible to define a new function $v(f)\in\mathcal{F}(U)$ such that:

$$v(f): x\to v_{|x}(f) \tag{B.1.40}$$

Therefore a vector field $v\in\Gamma_U TM$ can be interpreted as a map $v: C^\infty(U)\to\mathcal{F}(U)$.

Property 65: Given a vector field $v: C^\infty(U)\to\mathcal{F}(U)$, one can easily check the following rules hold:

1. $v(\lambda f+\mu g)=\lambda v(f)+\mu v(g)\quad,\quad\forall\mu,\lambda\in\mathbb{R}\,,\,\forall f,g\in C^\infty(U)$

2. $v(fg) = v(f)g + fv(g) \quad , \quad \forall f, g \in C^\infty(U)$

If the vector field $v \in \Gamma_U TM$ is also a smooth section then $v : C^\infty(U) \to C^\infty(U)$, then in this way the smooth vector fields can be regarded as a derivation on the infinite-dimensional real algebra $C^\infty(U)$ because they are $\mathbb{R}$-linear and satisfy the Leibniz rule. Since smooth vector fields are differential operators, it is possible to define the commutator of them.

Definition 106: Given two smooth vector fields $v, w \in \Gamma TM$ we can define a binary $\mathbb{R}$-linear operation $[\,,\,] : \Gamma TM \times \Gamma TM \to \Gamma TM$ called ***commutator*** such that:

$$[v, w](f) = v(w(f)) - w(v(f)) \quad , \quad \forall f \in C^\infty(M) \tag{B.1.41}$$

Property 66: It is easy to check just using the natural trivialisation that the commutator of two vector fields is a good vector field and satisfies the Jacobi identity, hence the set of all smooth vector fields with the commutator forms a good Lie algebra.

Definition 107: On $\Gamma_U TM$ we can define two useful operations:

1. ***Sum***: $+$: $\Gamma_U TM \times \Gamma_U TM \to \Gamma_U TM$ such that:

$$[v + w](f) = v(f) + w(f) \quad \forall f \in C^\infty(U), \forall v, w \in \Gamma_U TM \tag{B.1.42}$$

2. ***Product by a scalar field***: $\cdot : \mathcal{F}(U) \times \Gamma_U TM \to \Gamma_U TM$ such that:

$$[fv](g) = f \cdot v(g) \quad \forall g \in C^\infty(U)\ ,\ \forall v \in \Gamma_U TM, \forall f \in \mathcal{F}(U) \tag{B.1.43}$$

Property 67: One can easily check that $(\Gamma_U TM, +, \cdot)$ satisfies all the conditions to be a module on the ring of functions $(\mathcal{F}(U), +, \cdot)$. As we proved before if there exists a smooth global frame on U then $\Gamma_U TM$ can be spanned by a unique $\mathcal{F}(U)$-linear combination of sections belonging to the chosen frame, therefore it is a free module.

For our purposes in this work we are going to consider mainly smooth vector fields, therefore unless it is explicitly specified we assume from here all the considered vector fields are smooth sections of the tangent bundle.

Let us conclude our brief review on the tangent bundles analysing the case in which the tangent bundle is trivialised in a slightly different way still compatible with the vector bundle structure of TM.

Definition 108: Given U an open set of the base M we define a ***local frame*** on TM the list (e_μ) of local sections $e_\mu : U \subseteq M \to \tau_M^{-1}(U) \subseteq TM$ such that $\forall x \in U \Rightarrow (e_{\mu_{|x}})$ is a basis of T_xM. If for each μ the section e_μ is smooth then (e_μ) is called ***smooth local frame***.

The existence of at least a smooth local frame for each open set in the atlas associated to a manifold is guaranteed by the existence of a point-wise natural basis. It is very easy to check from the definition that since the charts $\{(U_i, \varphi_{(i)})\}$ are smooth, then the sections $\partial_{\mu(i)} : U_i \to \tau_M^{-1}(U_i)$ are smooth local frames. Let us suppose to have a manifold M endowed with an atlas $\{(U_i, \varphi_{(i)})\}$. Let $(e_{\mu(i)})$ be a smooth local frame (we know that at least the natural one exists), any other tangent vectors $v_x \in \tau_M^{-1}(U_i)$ may be uniquely written as $v_x = \hat{v}_x^\mu e_{\mu(i)|x}$, so a local trivialisation can be defined by:

$$\hat{t}_{(i)} : \tau_M^{-1}(U_i) \to U_i \times \mathbb{R}^m \quad | \quad \hat{t}_{(i)}(v) = (x, \hat{v}_{(i)}^\mu) \tag{B.1.44}$$

Let us suppose to have $(e_{\mu(j)})$ another smooth local frame defined on U_j inducing in the same way another local trivialisation $(U_j, \hat{t}_{(j)})$. The transition functions between the trivialisations $(U_i, \hat{t}_{(i)})$ and $(U_j, \hat{t}_{(j)})$ can be easily computed considering that on the overlaps $U_{ij} = U_i \cap U_j$ each tangent vector $e_{\mu(j)|x} \in \tau_M^{-1}(U_{ij})$ can be expressed as a point-wise $\mathbb{R}$-linear combination of the vector basis $e_{\mu(i)|x}$ such that

$$e_{\mu(i)|x} = \Lambda^\nu_{\mu(ij)_{|x}}\, e_{\nu(j)|x} \tag{B.1.45}$$

where $\Lambda^\nu_{\mu(ij)_{|x}}$ is the image of the point x through the map $\Lambda_{(ij)} : U_{ij} \to GL(\mathbb{R}^m)$. Therefore:

$$v_x = v^\mu_{(j)} e_{\mu(j)} = v^\mu_{(i)} e_{\mu(i)} = v^\mu_{(i)} \Lambda^\nu_{\mu(ij)_{|x}}\, e_{\nu(j)|x} \tag{B.1.46}$$

and we can conclude that:

$$v^\nu_{(j)} = v^\mu_{(i)} \Lambda^\nu_{\mu(ij)_{|x}} \tag{B.1.47}$$

Since one can check that $\Lambda^\nu_{\mu(ij)}$ satisfies all requirements to be a cocycle we can conclude that it can be interpreted as a good transition function valued in $GL(\mathbb{R}^m)$, hence the local trivialisations induced by the choice of some local smooth frames are still compatible with the vector bundle structure $(TM, M, \tau_M, \mathbb{R}^m, \lambda, GL(\mathbb{R}^m))$.

Definition 109: Let M be a manifold and $\phi : \mathbb{R} \to \mathit{Diff}(M)$ a smooth one family of diffeomorphisms such that:

1. $\phi_0 = id_U$
2. $\phi_t \circ \phi_s = \phi_{t+s}$

then by definition this family is an abelian subgroup of diffeomorphisms and it is called ***flow on*** M.

Definition 110: Given a flow $\phi : \mathbb{R} \to \mathit{Diff}(M)$ and a point $x \in M$ the restriction $\phi_{|x} : \mathbb{R} \to M$ is a local smooth curve called ***integral curve of*** ϕ ***based in*** x. The equivalence class of integral curves

$$\dot{\phi}_{|x}(f) = \frac{d}{dt}[f \circ \phi_{|x}]_{|t=0} \tag{B.1.48}$$

at each point is a tangent vector called ***infinitesimal generator of*** ϕ ***at*** x.

Definition 111: Given a local smooth flow $\phi : \mathbb{R} \to \mathit{Diff}(M)$ we can define a smooth local vector field denoted by $\dot{\phi} : U \to TM$ such that

$$[\dot{\phi}(x)](f) = \dot{\phi}_{|x}(f) = \frac{d}{dt}[f \circ \phi_{|x}]_{|t=0} \tag{B.1.49}$$

The vector field is called ***infinitesimal generator of*** ϕ

Definition 112: Let be M a manifold and let $v : M \to TM$ be a smooth global vector field on U. We define the ***integral curve*** the maps $\gamma_x : I_x \subset \mathbb{R} \to M$ solving the ordinary differential equation:

$$\begin{cases} \gamma_x(0) = x \quad , \quad \forall x \in M \\ \frac{d}{dt}(f \circ \gamma_x(t)) = v(f)_{|\gamma_x(t)} \quad , \quad \forall f \in C^\infty M, \forall x \in M \end{cases} \tag{B.1.50}$$

If $I_x = \mathbb{R}, \forall x \in M$ then the vector field is called ***complete***.

Property 68: The existence and the uniqueness of the integral curves is guaranteed by the Picard-Lindeloff theorem, furthermore using the theorem one can state that since the vector field is smooth its integral curves must depend smoothly on the initial condition $v_{|x}$. Therefore, as one can easily check, a complete vector field can be associated uniquely to flow on M defined as:

$$\begin{cases} \phi_0(x) = x \quad , \quad \forall x \in M \\ \frac{d}{dt}(f \circ \phi_t(x)) = v(f)_{|_{\phi_t(x)}} \quad , \quad \forall f \in C^\infty M, \forall x \in M \end{cases} \tag{B.1.51}$$

Definition 113: Given a smooth complete vector field $v \in \Gamma TM$ a flow $\phi : \mathbb{R} \to \mathit{Diff}(M)$ satisfying the equation:

$$\begin{cases} \phi_0(x) = x \quad , \quad \forall x \in M \\ \frac{d}{dt}(f \circ \phi_t(x)) = v(f)_{|_{\phi_t(x)}} \quad , \quad \forall f \in C^\infty M, \forall x \in M \end{cases} \tag{B.1.52}$$

is called ***integral flow of*** v

B.1.3 Cotangent bundle

There another important natural vector bundle that one can define on a smooth differential manifold called ***cotangent bundle***. Loosely speaking the cotangent bundle is the set of all the covectors tangent to each point on the manifold. Due to the duality relation between vectors and covectors, one can prove that the tangent bundle and the cotangent bundle are completely isomorphic. As we will prove, the choice of a compatible trivialisation on the tangent vector bundle induces immediately a trivialisation on the cotangent bundle compatible with the vector structure. Let us begin recalling the definition of cotangent space at a point:

Definition 114: Let M be a smooth m-dimensional differential manifold. Given T_xM the tangent space at x we can define the ***cotangent space*** $T_x^\star M$ as follows:

$$T_x^\star M := \{\alpha : T_xM \to \mathbb{R} | \alpha(\lambda v + \mu w) = \lambda\alpha(v) + \mu\alpha(w), \forall v, w \in T_xM, \forall \lambda, \mu \in \mathbb{R}\} \tag{B.1.53}$$

In other words $T_x^\star M$ is the set of all the $\mathbb{R}$-linear functionals acting on tangent vectors at x.

Definition 115: Given $T_x^\star M$ we can define trivially two operations due to the linear action of covectors on vectors:

1. ***Sum***: $+ : T_x^\star M \times T_x^\star M \to T_x^\star M$ such that

$$[\alpha + \beta](v) = \alpha(v) + \beta(v) \quad , \quad \forall \alpha, \beta \in T_x^\star M \, , \, \forall v \in T_x M \tag{B.1.54}$$

2. ***Multiplication by a scalar***: $\cdot : \mathbb{R} \times T_x^\star M \to T_x^\star M$ such that

$$[\lambda\alpha](v) = \lambda \cdot \alpha(v) \quad , \quad \forall \alpha \in T_x^\star M \, , \, \forall v \in T_x M \, , \, \forall \lambda \in \mathbb{R} \tag{B.1.55}$$

Property 69: It is very easy to check that $(T_x^\star M, +, \cdot)$ is a good real vector space. The 0 covector is defined as

$$0(v) = 0 \quad , \quad \forall v \in T_x M \tag{B.1.56}$$

Property 70: Let M be a smooth m-dimensional differential manifold, a choice of basis on $T_x M$ induces an isomorphisms between $T_x^\star M$ and $\mathbb{R}^m$, therefore since $T_x M$ and $T^\star M$ are both isomorphic to $\mathbb{R}^m$, they must be isomorphic.

Proof. Let us consider $\alpha \in T_x^\star M$ and let (e_μ) be a basis of $T_x M$. Due the linearity we can write:

$$\alpha(v) = \alpha(v^\mu e_\mu) = v^\mu \alpha(e_\mu) = v^\mu \alpha_\mu \quad , \quad \forall \alpha \in T_x^\star M \, , \, \forall v \in T_x M \tag{B.1.57}$$

where α_μ is a list of m real number defined as the action of α on the basis $\alpha_\mu := \alpha(e_\mu)$

Now let us define the list of covectors (e^μ) such that:

$$v = e^\mu(v) e_\mu \quad , \quad \forall v \in T_x M \tag{B.1.58}$$

This list always exist because it is formed just by the linear maps that associate to a vector its components with respect the basis (e_μ). This list must satisfy trivially the ***duality relation***:

$$e^\mu(e_\nu) = \delta^\mu_\nu \tag{B.1.59}$$

where δ^μ_ν is just the standard Kronecker delta. It is easy to show that (e^μ) is a basis of $T_x^\star M$ therefore is called ***dual basis***. First of all let us prove that (e^μ) span the cotangent space at x:

$$\alpha(v) = v^\mu \alpha_\mu = v^\mu \delta^\nu_\mu \alpha_\nu = v^\mu e^\nu(e_\mu)\alpha_\nu = \tag{B.1.60}$$
$$= e^\nu(v^\mu e_\mu)\alpha_\nu = \alpha_\nu e^\nu(v) \quad , \quad \forall \alpha \in T^\star_x M \, , \, \forall v \in T_x M \tag{B.1.61}$$

Then considering that the 0 covector can be written as a linear combination $0_\nu v^\nu$, its definition

$$0 = 0(v) = 0_\nu e^\nu(v) = 0_\nu v^\nu \quad , \quad \forall v \in T_x M \tag{B.1.62}$$

is satisfied if and only if $0_\nu = (0, .., 0)$ in other words its components must be all null. □

Definition 116: Given a basis (e_μ) on $T_x M$ we define ***the dual basis*** the basis on $T^\star_x M$ satisfing the ***duality relation***:

$$e^\mu(e_\nu) = \delta^\mu_\nu \tag{B.1.63}$$

The natural $\mathbb{R}$-linear action of covectors on vectors induces an $\mathbb{R}$-linear action of vectors on covectors, therefore the tangent vectors can be interpreted as linear functionals acting on the covectors and one can easily prove $T^{\star\star}_x M = T_x M$

Definition 117: Given $\alpha \in T^\star_x M$ and $v \in T_x M$ we define the $\mathbb{R}$-linear action of v on α as follow:

$$v(\alpha) = \alpha(v) \tag{B.1.64}$$

Definition 118: The ***cotangent space of a manifold*** M, denoted by TM is the set:

$$T^\star M = \bigsqcup_{x \in M} T^\star_x M \tag{B.1.65}$$

The cotangent space of a manifold can be regarded just as the collection of all the cotangent spaces at each point of it, but with a very little effort we can prove that $T^\star M$ admits naturally a bundle structure called ***cotangent bundle*** of M.

Definition 119: Due to the property of the disjoint union, given $T^\star M$ we can always define a canonical surjective map $\hat{\tau}_M : T^\star M \to M$ such that:

$$\hat{\tau}_M(\alpha_x) = x \tag{B.1.66}$$

Property 71: Let M be a m-dimensional manifold, let $T^\star M$ be the cotangent space and $\hat{\tau}_M : T^\star M \to M$ be the projection defined above . The quadruple $(T^\star M, M, \hat{\tau}_M, \mathbb{R}^m)$ is a good fiber bundle with standard fiber $\mathbb{R}^m$.

Proof. Let us consider a smooth manifold M with an atlas $(U_i, \varphi_{(i)})$, let $T^\star M$ be the cotangent space to M, let TM be the tangent bundle of M and let $(U_i, t_{\mu(i)})$ a local trivialisation of the tangent bundle induced by the choice of a smooth local frame $(U_i, e_{\mu(i)})$. For each U_i, for each $x \in U_i$, for each $\alpha_x \in \hat{\tau}^{-1}(U_i) \mid \hat{\tau}_M(\alpha_x) = x$ and for each $v_x \in \tau^{-1}(U_i) \mid \tau_M(v_x) = x$ as showed before we can write:

$$\alpha_x(v_x) = \alpha_x(v_x^\mu e_{\mu(i)|_x}) = v_x^\mu \alpha_\mu \tag{B.1.67}$$

Then considering that a basis on T_xM induce a basis on $T_x^\star M$ we can use the frame $e_{\mu(i)}$ to induce a basis on $T_x^\star M$ at each point of U_i via the point-wise relation:

$$e^\mu_{(i)|_x}(v_x) e_{\mu(i)|_x} = v_x \quad , \quad \forall v_x \in \tau^{-1}(U_i) \tag{B.1.68}$$

To each point, the existence of the dual basis is guaranteed as proven before hence one can write:

$$\alpha_x(v_x) = v_x^\mu \alpha_\mu = \alpha_\mu e^\mu_{(i)|_x}(v_x) \tag{B.1.69}$$

where α_μ is an n-tuple of real number. One can easily check that, fixing a smooth local frame $(e_{\mu(i)})$, the map $\hat{t}_{(i)} : \hat{\tau}^{-1}(U_i) \to U_i \times \mathbb{R}^m$ defined as:

$$\hat{t}_{(i)}(\alpha_x) \to (x, \alpha_\mu) \tag{B.1.70}$$

is invertible and differentiable, hence it is a diffeomorphisms and it defines a local trivialisation of $T^\star M$ (one for each open set U_i in the atlas of M). □

Definition 120: The quadruple $(T^\star M, M, \hat{\tau}_M, \mathbb{R}^m)$ is called ***cotangent bundle of*** M.

Property 72: Given a smooth manifold M, a cotangent bundle is a vector bundle because the given trivialization admits transition functions in $GL(\mathbb{R}^m) \subset \mathit{Diff}(\mathbb{R}^m)$

Proof. Let TM be the tangent bundle of M and let us suppose to have two local trivialisation $(U_i, t_{(i)})$ and $(U_j, t_{(j)})$ induced by two local smooth frames $(e_{\mu(i)})$ and $(e_{\mu(j)})$ such that $U_i \cap U_j = U_{ij} \neq \varnothing$. We know that these induce two local trivialisations of $T^\star M$,$(U_i, \hat{t}_{(i)})$ and $(U_j, \hat{t}_{(j)})$ respectively. As proven previously we know that the transition functions related to these trivialisation of TM are characterised by:

$$v^{\nu}_{(j)} = v^{\mu}_{(i)}\Lambda^{\nu}_{\mu(ij)|x} \tag{B.1.71}$$

therefore on the overlaps U_{ij} the following holds:

$$\alpha_x(v_x) = \alpha_{\mu(i)}e^{\mu}_{(i)|x}(v_x) = \alpha_{\mu(j)}e^{\mu}_{(j)|x}(v_x) = \alpha_{\mu(j)}v^{\mu}_{(j)} = \alpha_{\mu(j)}v^{\nu}_{(i)}\Lambda^{\mu}_{\nu(ij)|x} = \alpha_{\mu(j)}\Lambda^{\mu}_{\nu(ij)|x}e^{\nu}_{(j)|x}(v_x) \tag{B.1.72}$$

and we must conclude the transition functions are:

$$\alpha_{\mu(i)} = \alpha_{\nu(j)}\Lambda^{\mu}_{\nu(ij)|x} \quad \Leftrightarrow \quad \alpha_{\mu(j)} = \alpha_{\nu(i)}\Lambda^{\mu}_{\nu(ji)|x} = \alpha_{\mu(i)}\overline{\Lambda}^{\mu}_{\nu(ij)|x} \tag{B.1.73}$$

□

The local trivialization of the cotangent bundle induced by the tangent bundle is compatible with the vector bundle structure $(T^\star M, M, \hat{\tau}_M, \mathbb{R}^m, \lambda, GL(\mathbb{R}^m))$.

Definition 121: A local section of the cotangent bundle $\sigma : U \subset M \to TM$ is called ***local covector field*** or **1*-form***. The set of all sections defined on the open set U are denoted by $\Gamma_U T^\star M$. If $U = M$ then σ is called ***global covector field***.

Property 73: Covector fields can be interpreted as specific maps on vector fields and and vice-versa. Given a 1-form $\alpha \in \Gamma_U T^\star M$ and a vector field $v \in \Gamma_U TM$ then we can define the function $\alpha(v) : U \to \mathbb{R}$ as follows:

$$\alpha(v)_{|x} = \alpha_{|x}(v_{|x}) \quad , \quad \forall x \in U \tag{B.1.74}$$

Hence we can state:

1. $\alpha \in \Gamma_U T^\star M$ can be interpreted as a map $\alpha : \Gamma_U TM \to \mathcal{F}(U)$
2. $v \in \Gamma_U TM$ can be interpreted as a map $v : \Gamma_U T^\star M \to \mathcal{F}(U)$

These maps are both $\mathcal{F}(U)$-linear in their arguments in fact:

$$\alpha(fv + gw)(x) = \alpha_{|x}((fv)_{|x} + (gw)_{|x}) = \alpha_{|x}(f_{|x}v_{|x} + g_{|x}w_{|x}) = \tag{B.1.75}$$

$$= f_{|x}\alpha_{|x}(v_{|x}) + g_{|x}\alpha_{|x}(w_{|x}) = f(x)\alpha(v)(x) + g(x)\alpha(w)(x) \quad , \quad \forall x \in U \tag{B.1.76}$$

Definition 122: Given a local smooth frame $(e_{\mu(i)})$ on $U_i \subseteq TM$ we define ***the local dual frame*** or simply ***local coframe*** the list of local sections $e^{\mu}_{(i)} : U \to \hat{\tau}^{-1}(U_i) \subseteq T^\star_x M$ satisfing the pointwise duality relation:

$$e^{\mu}_{(i)|_x}(e_{\nu(i)|_x}) = \delta^{\mu}_{\nu} \tag{B.1.77}$$

Property 74: Since at fixed basis, the function mapping a vector in its own components are smooth then the local coframe of a smooth frame is smooth as well.

Definition 123: On $\Gamma_U TM$ we can define two useful operations:

1. ***Sum***: $+$: $\Gamma_U TM \times \Gamma_U TM \to \Gamma_U TM$ such that:

$$[v + w](f) = v(f) + w(f) \quad \forall f \in C^\infty(U), \forall v, w \in \Gamma_U TM \tag{B.1.78}$$

2. ***Product by a scalar field***: $\cdot : \mathcal{F}(U) \times \Gamma_U TM \to \Gamma_U TM$ such that:

$$[f\alpha](v) = f \cdot \alpha(v) \quad \forall \alpha \in \Gamma_U T^\star M \ , \ \forall v \in \Gamma_U TM, \forall f \in \mathcal{F}(U) \tag{B.1.79}$$

Property 75: One can easily check that $(\Gamma_U T^\star M, +, \cdot)$ satisfies all the conditions to be a module on the ring of functions $(\mathcal{F}(U), +, \cdot)$. As we proved before if there exists a smooth global frame on U then $\Gamma_U T^\star M$ can be spanned by a unique $\mathcal{F}(U)$-linear combination of sections belonging to the chosen frame, therefore it is a free module.

Given a smooth function defined on $U \in M$, it is possible to define a canonical $\mathbb{R}$-linear map that induces a local section of $T^\star M$.

Definition 124: Let M be a smooth manifold, $C^\infty(U)$ the set of the smooth functions defined on the open set $U \subseteq M$ and $T^\star M$ be the cotangent bundle. We define a ***differential*** the map $d : C^\infty(U) \to \Gamma_U T^\star M$ such that:

$$[d(f)](v) = v(f) \quad , \quad \forall v \in \Gamma_U TM \tag{B.1.80}$$

Property 76: Since the vector fields are derivations on the $C^\infty U$ algebra, it is trivial to check that d is $\mathbb{R}$-linear and it satisfies the Leibniz rule.

Property 77: Let M be a smooth manifold with an atlas $\{U_i, \varphi_i\}$. The smooth charts guarantee the existence of a local smooth coframe $(dx^\mu_{(i)})$ on $T^\star M$ such that $(dx^\mu_{(i)})$ is the local dual frame with respect $(\partial_{\mu(i)})$

Proof. As we proved before, for each U_i the smooth local charts guarantee the existence of a local smooth frame $(\partial_{\mu(i)})$ on TM. Introduce the local trivialisation $(U_i, \varphi_{(i)})$ where $\varphi^\mu_{(i)}(x) = x^\mu_{(i)}$ are the coordinate functions. Then using the definition of differential:

$$dx^\mu_{(i)|_x}(\partial_{\nu(i)|_x}) = \partial_{\nu(i)|_x}(x^\mu_{(i)|_x}) = \frac{\partial}{\partial x^\nu}_{(i)|_x}(x^\mu_{(i)|_x}) = \delta^\nu_\mu \quad , \quad \forall x \in U_i \tag{B.1.81}$$

□

Considering this proof, we must admit that $(T^M, M, \hat{\tau}_M, \mathbb{R}^m, \lambda, GL(\mathbb{R}^m))$ is a natural vector bundle over M.

For our purposes in this work we are going to consider mainly smooth covector fields, therefore unless it is explicitly specified we assume from here all the considered 1-forms are smooth sections of the cotangent bundle.

B.2 Tangent tensors at a point

B.2.1 Introduction to tangent tensors

Let us consider a differential manifold M with dimension $dim(M) = m$. Let us denote by $x \in M$ a generic point on the manifold and let $T_x M$ and $T^\star_x M$ respectively the tangent and the cotangent spaces to M at the point x.

Definition 125: A ***tangent tensor*** T at x with rank $p, q \in \mathbb{N}^+$ is a map:

$$T : (\times^p T^\star_x M) \times (\times^q T_x M) \to \mathbb{R} \tag{B.2.1}$$

such that it is multi-linear: $\forall i \in [1,p]$, $\forall j \in [1,q]$, $\forall \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$, $\forall \alpha, \beta \in T^\star_x M$, $\forall w, u \in T_x M$, $\forall \omega^{\overline{i-1}} \in \times^{i-1} T^\star M$, $\forall \omega^{\overline{p} \setminus \overline{i}} \in \times^{p-i} T^\star M$, $\forall v_{\overline{j-1}} \in \times^{j-1} TM$, $\forall v_{\overline{j} \setminus \overline{q}} \in$

$\times^j TM \quad \Rightarrow$

$$T(\omega^{\overline{i-1}}, \lambda_1\alpha + \mu_1\beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, \lambda_2 w + \mu_2 u, v_{\overline{q}\backslash\overline{j}}) = \tag{B.2.2}$$
$$=\lambda_1\lambda_2 T(\omega^{\overline{i-1}}, \alpha, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, w, v_{\overline{q}\backslash\overline{j}}) + \lambda_1\mu_2 T(\omega^{\overline{i-1}}, \alpha, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, u, v_{\overline{q}\backslash\overline{j}}) + \tag{B.2.3}$$
$$+\mu_1\lambda_2 T(\omega^{\overline{i-1}}, \beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, w, v_{\overline{q}\backslash\overline{j}}) + \mu_1\mu_2 T(\omega^{\overline{i-1}}, \beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, u, v_{\overline{q}\backslash\overline{j}}) \tag{B.2.4}$$

Definition 126: The ***set of all tangent tensors*** at x with rank p, q is denoted by ${T^p_q}_x M$.

Considering that all the vectors, covectors and tensors used in this work are tangent to M, there is no need to specify they are tangent. From here, each time we refer to vectors, covectors and tensors we mean implicitly they are tangent to M. Due to the multi-linearity ot the action of its elements upon vectors and covectors, ${T^p_q}_x M$ naturally inherits two operations:

Definition 127: Given two tensors $T, S \in {T^p_q}_x M$ we define a ***sum of tensors*** the map $+ : {T^p_q}_x M \times {T^p_q}_x M \to {T^p_q}_x M$ such that:

$$[T + S](\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha^{\overline{p}}, v_{\overline{q}}) + S(\alpha^{\overline{p}}, v_{\overline{q}}) \quad , \quad \forall \alpha^{\overline{p}} \in \times^p T^\star_x M \ , \ \forall v_{\overline{q}} \in \times^q T_x M \tag{B.2.5}$$

Definition 128: Given a tensor $T \in {T^p_q}_x M$ and a scalar $\lambda \in \mathbb{R}$ we define a ***multiplication by a scalar*** the map $\cdot : \mathbb{R} \times {T^p_q}_x M \to {T^p_q}_x M$ such that:

$$[\lambda T](\alpha^{\overline{p}}, v_{\overline{q}}) = \lambda[T(\alpha^{\overline{p}}, v_{\overline{q}})] \quad , \quad \forall \alpha^{\overline{p}} \in \times^p T^\star_x M \ , \ \forall v_{\overline{q}} \in \times^q T_x M \tag{B.2.6}$$

Property 78: It is very easy to check that the algebraic structure $({T^p_q}_x M, +, \cdot)$ satisfies all the requirements to be a good vector space on the field $\mathbb{R}$. The null vector of the tensor space is identifed with the null map $0 \in {T^p_q}_x M$ such that

$$0(\alpha^{\overline{p}}, v_{\overline{q}}) = 0 \quad , \quad \forall \alpha^{\overline{p}} \in \times^p T^\star_x M \ , \ \forall v_{\overline{q}} \in \times^q T_x M \tag{B.2.7}$$

The algebraic operations defined above are enough to endow $(T^p_{q\,x}M)$ with a linear structure that characterise it as a vector space, but these are not the only useful operation we are able to define on tensors. The standard multiplication on $\mathbb{R}$ induces another very important binary operation called tensor product. The definition of tensor product of vector spaces is very fundamental in mathematics, it can be given in a very general way and it is deeply rooted in the category theory. However it is beyond our purposes to analyse in detail how to establish general canonical universal isomorphism and correspondences between algebraic structures. We settle here to give a definition of tensor product that may be a bit simplistic but definitely very effective for achieving our purposes.

Definition 129: Given two tensors $T \in T^p_{q\,x}M$ and $S \in T^{p'}_{q'\,x}M$ we define a ***tensor product*** the map $\otimes : T^p_{q\,x}M \times T^{p'}_{q'\,x}M \to T^{p+p'}_{q+q'\,x}M$ such that:

$$[T \otimes S](\alpha^{\overline{p}}, \beta^{\overline{p}'}, v_{\overline{q}}, u_{\overline{q}'}) = T(\alpha^{\overline{p}}, v_{\overline{q}})S(\beta^{\overline{p}'}, u_{\overline{q}'}) \,, \tag{B.2.8}$$

$$\forall \alpha^{\overline{p}} \in \times^p T^\star_x M \,,\ \forall v_{\overline{q}} \in \times^q T_x M \quad , \quad \forall \beta^{\overline{p}} \in \times^p T^\star_x M \,,\ \forall u_{\overline{q}} \in \times^q T_x M \tag{B.2.9}$$

Considering the tensors are multi-linear maps, they must act on n-tuples of vectors and covectors. Therefore we have a natural action of the group of permutations on $T^p_{q\,x}M$ induced by the permutations on the arguments.

Definition 130: Let I and J be two of permutations of p and q elements respectively. Let P_I and P_J be their representation acting respectively on the n-tuples $\alpha^{\overline{p}} \in \times^p T^\star_x M$ and $v_{\overline{q}} \in \times^q T_x M$ as following:

$$I(\alpha^{\overline{p}}) = (\alpha^{P_I(\overline{p})}) \tag{B.2.10}$$

$$J(\alpha^{\overline{p}}) = (v_{P_J(\overline{q})}) \tag{B.2.11}$$

Given a tensor $T \in T^p_{q\,x}M$ we define a ***braiding map*** the map $\sigma^I_J : T^p_{q\,x}M \to T^p_{q\,x}M$ such that

$$[\sigma^I_J T](\alpha^{\overline{p}}, v_{\overline{q}}) = T(I(\alpha^{\overline{p}}), J(v_{\overline{q}})) = T(\alpha^{P_I(\overline{p})}, v_{P_J(\overline{q})}) \tag{B.2.12}$$

Of course anyone is free to choose their own notation to express the permutation I and J, however we decided to use the standard cycle decomposition because it offers a direct representation of their action on the list of indices related to the coordinate representation of the tensors.

Example: We give here some very useful particular cases of braiding maps which are widely used in standard tensor manipulation. Let us suppose to have a transposition $(i, j) \mid i, j \in \mathbb{N}^+\ ,\ i \leq j$:

$$\sigma^{(ij)}T(\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha^{\overline{i-1}}, \alpha^j, \alpha^{\overline{j}\setminus\overline{i}}, \alpha^i, \alpha^{\overline{p}\setminus\overline{j}}, v_{\overline{q}}) \tag{B.2.13}$$

$$\sigma_{(ij)}T(\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha^{\overline{p}}, v_{\overline{i-1}}, v_j, v_{\overline{j}\setminus\overline{i}}, v_i, v_{\overline{q}\setminus\overline{j}}) \tag{B.2.14}$$

Let us suppose to have a cycle of subsequent elements $(\overline{j} \setminus \overline{i}) \mid i, j \in \mathbb{N}^+\ ,\ i \leq j$:

$$\sigma^{(\overline{j}\setminus\overline{i})}T(\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha^{\overline{i}}, \alpha^j, \alpha^{\overline{j-1}\setminus\overline{i}}, \alpha^{\overline{p}\setminus\overline{j}}, v_{\overline{q}}) \tag{B.2.15}$$

$$\sigma_{(\overline{j}\setminus\overline{i})}T(\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha^{\overline{p}}, v_{\overline{i}}, v_j, v_{\overline{j-1}\setminus\overline{i}}, v_{\overline{p}\setminus\overline{j}}) \tag{B.2.16}$$

It is very interesting to notice how the action of tensors upon vectors and covectors induces canonically an action of vectors and covectors on the tensors themselves:

Definition 131: Given a tensor $T \in T^p_{q\,x}M$, with $q \geq 1$ we define a ***contraction with a vector*** the map $\lrcorner : T_xM \times T^p_{q\,x}M \to T^p_{q-1\,x}$ such that

$$u \lrcorner T(\alpha^{\overline{p}}, v_{\overline{q-1}}) = T(\alpha^{\overline{p}}, u, v_{\overline{q-1}}) \tag{B.2.17}$$

Given a tensor $T \in T^p_{q\,x}M$, with $p \geq 1$ we define a ***contraction with a covector*** the map $\urcorner : T^\star_xM \times T^p_{q\,x}M \to T^{p-1}_{q\,x}M$ such that:

$$\beta \urcorner T(\alpha^{\overline{p-1}}, v_{\overline{q-1}}) = T(\beta, \alpha^{\overline{p-1}}, u, v_{\overline{q-1}}) \tag{B.2.18}$$

Definition 132: Let (e_μ) and (e^μ) be respectively the basis of T_xM and $T^\star_xM$. Given a tensor $T \in T^p_{q\,x}M$, with $p, q \geq 1$ we define an ***internal contraction*** the map $i : T^p_{q\,x}M \to T^{p-1}_{q-1\,x}M$ such that:

$$[iT](\alpha^{\overline{p-1}}, v_{\overline{q-1}}) = T(e^\mu, \alpha^{\overline{p-1}}, e_\mu, v_{\overline{q-1}}) \tag{B.2.19}$$

where the dummy index μ implies a sum accordingly to the Einstein notation. As it is proved in the following sections, even if the definition is basis dependent, the operator i does not depend on a specific choice of it.

Since $T^p_{q\,x}M$ is a vector space, it is natural to define the $\mathbb{R}$-linear maps acting on them.

Definition 133: Given two tensor spaces $T^p_{q\,x}M$ and $T^{p'}_{q'\,x}M$ we define a $\mathbb{R}$-linear map $\mathcal{L} : T^p_{q\,x}M \to T^{p'}_{q'\,x}M$ such that:

$$\mathcal{L}(\lambda T + \mu S) = \lambda\mathcal{L}(T) + \mu\mathcal{L}(S) \quad , \quad \forall\lambda, \mu \in \mathbb{R} \quad , \quad \forall T, S \in T^p_{q\,x}M \tag{B.2.20}$$

It is very interesting and useful to notice that there is a canonical isomorphism between the linear maps on tensors and the tensors themselves. Let $T^p_{q\,x}M$ and $T^{p'}_{q'\,x}M$ be two tensor spaces, and $Lin(p, q, p', q') = \{\mathcal{L} : T^p_{q\,x}M \to T^{p'}_{q'\,x}M\}$ be the space of the linear map between them. There always exists an unique tensor $L \in T^{q+p'}_{p+q'\,x}M$ such that:

$$\mathcal{L}(T) = [i]^{p+q}[\sigma^{\overline{(p+q)}}]^p(T \otimes L) \tag{B.2.21}$$

therefore $Lin(p, q, p', q')$ is isomorphic to $T^{q+p'}_{p+q'\,x}M$ as a vector space and we can perform on them all the operations defined on tensors. The proof will be provided in the following section because extra structures are needed.

Property 79: The sum, multiplication by a scalar, tensor product, braiding maps, contractions and internal contractions are all $\mathbb{R}$-linear maps.

Proof. The sum and the multiplication by scalar are trivially $\mathbb{R}$-linear by definition. The tensor product is $\mathbb{R}$-linear due to its definition and by the distributivity of the multiplication with respect the sum. The braiding map is $\mathbb{R}$-linear because of the commutativity of the sum and the multiplication, the contractions are linear by definition as well as the internal contraction that is a sum of contractions. □

B.2.2 Coordinate expressions induced by the choice of basis

Let be M a smooth manifold with $dim(M) = m$, then $(T^p_{q\,x}M, +, \cdot)$ is a m^{p+q} dimensional vector space. Fixed a basis (e_μ) on T_xM we induce a basis $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ on $T^p_{q\,x}M$. Then we can write uniquely $T = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}$. To show it, let us recall briefly some fundamental concepts of differential geometry. We assume the reader is already familiar with the details. First of all let us remark that due to the action of a covector on vectors we are able to induce an action of a vector on the covectors, hence there is a canonical isomorphism between the vectors and the $\mathbb{R}$ linear functionals acting on the covectors. Hence given $\alpha \in T^\star_x M$ and $v \in T_x M$, the action of v on α is given by definition as $v(\alpha) := \alpha(v)$. Keeping this in mind, let us start by considering the fact that $T_x M$ and $T^\star_x M$, are both vector spaces with dimension M. We know also that the choice of a basis (e_μ) of $T_x M$ induces naturally a basis on $T^\star M$ denoted by (e^μ) due to the relation $e^\mu(e_\nu) = e_\nu(e^\mu) = \delta^\mu_\nu$ (where δ^μ_ν is just the standard Kronecker delta). Hence given a tensor $T \in T^p_{q\,x} M$ using the multi-linearity and the definition of tensor product, one can write:

$$T(\alpha^{\overline{p}}, v_{\overline{q}}) = T(\alpha_{\mu_{\overline{p}}} e^{\mu_{\overline{p}}}, v^{\nu_{\overline{q}}} e_{\nu_{\overline{q}}}) = \alpha_{\mu_{\overline{p}}} v^{\nu_{\overline{q}}} T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \alpha_{\mu_{\overline{p}}} v^{\nu_{\overline{q}}} = \tag{B.2.22}$$

$$= T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \delta^{\lambda_{\overline{p}}}_{\mu_{\overline{p}}} \delta^{\nu_{\overline{q}}}_{\rho_{\overline{q}}} \alpha_{\lambda_{\overline{p}}} v^{\rho_{\overline{q}}} = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e_{\mu_{\overline{p}}}(e^{\lambda_{\overline{p}}}) e^{\nu_{\overline{q}}}(e_{\rho_{\overline{q}}}) \alpha_{\lambda_{\overline{p}}} v^{\rho_{\overline{q}}} = \tag{B.2.23}$$

$$= T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} [e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}](e^{\lambda_{\overline{p}}}, e_{\rho_{\overline{q}}}) \alpha_{\lambda_{\overline{p}}} v^{\rho_{\overline{q}}} = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} [e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}](\alpha_{\lambda_{\overline{p}}} e^{\lambda_{\overline{p}}}, v^{\rho_{\overline{q}}} e_{\rho_{\overline{q}}}) = \tag{B.2.24}$$

$$= T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} [e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}](\alpha^{\overline{p}}, v_{\overline{q}}) \tag{B.2.25}$$

where $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} := T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}})$ is a multi-indexed list of real numbers produced by acting with T upon the list of basis vectors $(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}})$. This means each tensor of $T^p_{q\,x} M$ can be written as an $\mathbb{R}$-linear combination of tensors singled out from the list $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ by fixing the value of each index μ_i and ν_j. Hence the multi-indexed list $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ is a set of generators for $T^p_{q\,x} M$. It is easy to prove that the given set of generator is linearly independent, in fact if they are, the linear combination expressing the 0 tensor must be uniquely written as a linear combination of null coefficients $0^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}$. This requirement is satisfied, in fact:

$$\forall \alpha^{\overline{p}} \in \times^p T^\star_x M \ , \ \forall v_{\overline{q}} \in \times^q T_x M \ \Rightarrow \tag{B.2.26}$$

$$0 = 0(\alpha^{\overline{p}}, v_{\overline{q}}) = 0^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} [e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}](\alpha^{\overline{p}}, v_{\overline{q}}) = 0^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \alpha_{\mu_{\overline{p}}} v^{\nu_{\overline{q}}} \tag{B.2.27}$$

and this equation is satisfied for all the n-tuples $\alpha^{\overline{p}}$ and $v_{\overline{q}}$ if and only if:

$$0^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = (0, ..., 0) \ , \ \forall \mu_i, \nu_j \in [0, m-1], \forall i \in [1, p], \forall j \in [1, q] \tag{B.2.28}$$

Therefore we can conclude that given a basis (e_μ) on $T_x M$ we are able to induce canonically a basis $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ on $T^p_{q\,x} M$. From a geometrical perspective, the choice of a basis on $T_x M$ induces a unique way to map diffeomorphically and globally the vector space $T^p_{qx} M$ into $\mathbb{R}^{m^{p+q}}$ due to:

$$T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} := T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) \tag{B.2.29}$$

hence $T^p_{qx} M$ is a vector space but also a differential manifold globally diffeomorphic to $\mathbb{R}^{m^{p+q}}$ whose coordinates functions are exactly the vectors belonging to the basis.

Definition 134: Let $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ a basis on $T^p_{q\,x}M$. The ***coordinate expression of a tensor*** $T \in T^p_{q\,x}M$ with respect the chosen basis is given by the multi-indexed list of real scalars $(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}})$ defined by $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}})$. At fixed basis the coordinate expression is unique and it can be used to write the tensor via a linear combination $T = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}}$.

B.2.3 Pull-back and Push-forward of tensors

When one has two manifolds and a smooth map between them there is a canonical natural way to transport back and forward tangent structures between them called pull-back and push-forward. As usual there are different ways to interpret the pull-back and push-forward, some of them are very sophisticated involving functor and category theory. Once again we must be pragmatic and we settle here to providing an operative definition of pull-back and push-forward without investigating in detail all the properties in terms of maps between categories.

Definition 135: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local generic map between them. We define the ***pull-back of functions*** the map $\phi^\star : \mathcal{F}(V) \to \mathcal{F}(U)$ such that:

$$\phi^\star(f) = f \circ \phi \quad , \quad \forall f \in \mathcal{F}(V) \tag{B.2.30}$$

Let us stress that any function can be pulled back along any map. On the contrary, this is not the case for push-forward. Given a function on M there is no general way to define a function on N. For pushing forward functions, one has to either restrict functions or restrict maps.

Definition 136: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local invertible map between them. We define the ***push-forward of functions*** the map $\phi_\star : \mathcal{F}(U) \to \mathcal{F}(V)$ such that:

$$\phi_\star(f) = f \circ \phi^{-1} \quad , \quad \forall f \in \mathcal{F}(U) \tag{B.2.31}$$

It is trivial to check that for an invertible map ϕ the push-forward is just the inverse map of the pull-back, furthermore $\phi^\star = (\phi^{-1})_\star$

Definition 137: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local smooth map between them. Let $x \in U$ a point, then we define the ***push-forward of vectors at*** x the map $\phi_\star : T_xM \to T_{\phi(x)}N$ such that:

$$[\phi_\star(v)](f) = v(\phi^\star(f)) = v(f \circ \phi) \quad , \quad \forall v \in T_xM \, , \, \forall f \in \mathcal{F}(V) \tag{B.2.32}$$

Sometimes the push-forward of vectors $\phi_\star : T_xM \to T_{\phi(x)}N$ is denoted by $d\phi_{|x}$ and is called the ***differential of the map*** ϕ ***at the point*** x

Let us stress that any vector can be pushed forward along any smooth map. On the contrary, this is not the case for the pull-back.

Definition 138: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a diffeomorphism between them. Let $y \in V$ a point, then we define the ***pull-back of vectors at*** y the map $\phi_\star : T_{\phi(x)}N \to T_xM$ such that:

$$[\phi^\star(v)](f) = v(\phi_\star(f)) = v(f \circ \phi^{-1}) \quad , \quad \forall v \in T_{\phi(x)}N \, , \, \forall f \in \mathcal{F}(U) \tag{B.2.33}$$

Again for an invertible map ϕ the push-forward is just the inverse map of the pull-back, furthermore $\phi_\star = (\phi^{-1})^\star$.

Definition 139: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local smooth map between them. Let $x \in U$ be a point, we define the ***pull-back of covectors*** the map $\phi^\star : T^\star_{\phi(x)}N \to T^\star_xM$ such that:

$$[\phi^\star(\alpha)](v) = \alpha(\phi_\star(v)) \quad , \quad \forall \alpha \in T^\star_{\phi(x)}N \, , \, \forall v \in T_xM \tag{B.2.34}$$

Let us stress once again that any covector can be pulled back along any smooth map but this is not the case for the push-forward.

Definition 140: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a diffeomorphism between them. Let $x \in U$ be a point, we define the ***push-forward of***

covectors the map $\phi^\star : T^\star_x M \to T^\star_{\phi(x)} N$ such that:

$$[\phi_\star(\alpha)](v) = \alpha(\phi^\star(v)) \quad , \quad \forall \alpha \in T^\star_{\phi(x)} N \, , \, \forall v \in T_x M \tag{B.2.35}$$

Again for an invertible map ϕ the push-forward is just the inverse map of the pull-back, and the relation $\phi^\star = (\phi^{-1})_\star$ can be easily checked.

Definition 141: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local smooth map between them. Let $x \in U$ be a point, then we define the ***push-forward of covariant tensors at*** x the map $\phi_\star : T^p_x M \to T^p_{\phi(x)} N$ such that:

$$[\phi_\star(T)](\alpha^{\overline{p}}) = T([\phi^\star(\alpha)]^{\overline{p}}) \quad , \quad \forall \alpha^{\overline{p}} \in \times^p T^\star_{\phi(x)} M \, , \, \forall T \in T^p_x M \tag{B.2.36}$$

In the same way we define the ***pull-back of contravariant tensors at*** x the map $\phi^\star : T_{q\,\phi(x)} N \to T_{q\,x} M$ such that:

$$[\phi^\star(T)](v_{\overline{q}}) = T([\phi_\star(v)]_{\overline{q}}) \quad , \quad \forall v_{\overline{q}} \in \times^p T_x M \, , \, \forall T \in T_{q\,\phi(x)} M \tag{B.2.37}$$

Definition 142: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local diffeomorphism between them. Let $x \in U$ be a point, then we define the ***pull-back of covariant tensors at*** x the map $\phi_\star : T^p_{\phi(x)} N \to T^p_x M$ such that:

$$[\phi^\star(T)](\alpha^{\overline{p}}) = T([\phi_\star(\alpha)]^{\overline{p}}) = T([\phi^{-1\star}(\alpha)]^{\overline{p}}) \quad , \quad \forall \alpha^{\overline{p}} \in \times^p T^\star_x M \, , \, \forall T \in T^p_{\phi(x)} M \tag{B.2.38}$$

In the same way we define a ***push-forward of contravariant tensors at*** x the map $\phi^\star : T_{q\,\phi(x)} N \to T_{q\,x} M$ such that:

$$[\phi_\star(T)](v_{\overline{q}}) = T([\phi^\star(v)]_{\overline{q}}) = T([\phi^{-1}{}_\star(v)]^{\overline{p}}) \quad , \quad \forall v_{\overline{q}} \in \times^p T_{\phi(x)} M \, , \, \forall T \in T_{q\,x} M \tag{B.2.39}$$

Definition 143: Let be M and N two manifolds and $\phi : U \subseteq M \to V \subseteq N$ be a local diffeomorphism between them. Let $x \in U$ be a point, then we define the ***push-forward of tensors at*** x the map $\phi_\star : T^p_x M \to T^p_{\phi(x)} N$ such that $\forall \alpha^{\overline{p}} \in \times^p T^\star_{\phi(x)} M$, $\forall v^{\overline{q}} \in \times^q T_{\phi(x)} M$, $\forall T \in T^p_{q\,x} M$:

$$[\phi_\star(T)](\alpha^{\overline{p}}, v_{\overline{q}}) = T([\phi^\star(\alpha)]^{\overline{p}}, ([\phi^\star(v)]_{\overline{q}}) = T([\phi^\star(\alpha)]^{\overline{p}}, [\phi^{-1}{}_\star(v)]_{\overline{q}}) \tag{B.2.40}$$

In the same way we define a ***pull-back of tensors at*** x the map $\phi^\star : T_{q\,\phi(x)} N \to T_{q\,x} M$ such that $\forall \alpha^{\overline{p}} \in \times^p T^\star_x M$, $\forall v^{\overline{q}} \in \times^q T_x M$, $\forall T \in T^p_{q\,\phi(x)} M$:

$$[\phi^\star(T)](\alpha^{\overline{p}}, v_{\overline{q}}) = T([\phi_\star(\alpha)]^{\overline{p}}, [\phi_\star(v)]_{\overline{q}}) = T([\phi^{-1\star}(\alpha)]^{\overline{p}}, [\phi_\star(v)]_{\overline{q}}) \tag{B.2.41}$$

Let us remark that due to the isomorphisms between rank $0, 1$ tensors and covectors, the pull-back of covectors is just a particular case of push-forward of contravariant tensors. In the same way due to the isomorphisms between rank 1,0 tensors and vectors, we can state that the push-forward of vectors is just a specific case of pull-back of covariant tensors.

Property 80: If ϕ is a diffeomorphism then the pull-back and push-forward of vectors, covectors and tensor is a maximum rank map therefore they map basis into basis.

Proof. Let ϕ be a diffeomorphism and ϕ^μ the local coordinate expression. Given an arbitrary basis of $T_x M$denoted by e_μ and the natural one $\partial_{\nu|_x}$ we know that there must exists a maximum rank matrix of real numbers Λ^ν_μ such that $e_\mu = \Lambda^\nu_\mu \partial_{\nu|_x}$, therefore

$$\phi^\star e_\nu(f) = \phi^\star(\Lambda^\mu_\nu \partial_{\mu|_x})(f) = d\phi_{|_x}(\Lambda^\mu_\nu \partial_{\mu|_x})(f) = \Lambda^\mu_\nu d\phi_{|_x}(\partial_{\mu|_x})(f) = \tag{B.2.42}$$

$$= \Lambda^\mu_\nu \partial_{\mu|_x}(f \circ \phi) = \Lambda^\mu_\nu \frac{\partial \phi^\lambda}{\partial x^\mu}_{|_x} \frac{\partial}{\partial y^\lambda}_{|_{\phi(x)}}(f) = \Lambda^\mu_\nu d\phi^\lambda_{\mu|_x} \frac{\partial}{\partial y^\lambda}_{|_{\phi(x)}}(f). \tag{B.2.43}$$

To be a diffeomorphism ϕ must be invertible and the inverse must be differentiable, therefore the matrix $d\phi^\lambda_{\mu|_x} = \frac{\partial \phi^\lambda}{\partial x^\mu}_{|_x}$ must be invertible then its rank is maximum. Comparing the first and the last element we have

$$\phi^\star e_\nu = \Lambda^\mu_\nu d\phi^\lambda_{\mu|_x} \frac{\partial}{\partial y^\lambda}_{|_{\phi(x)}} \tag{B.2.44}$$

Since Λ and $d\phi$ are maximum rank linear operators and since $\frac{\partial}{\partial y^\lambda}_{|\phi(x)}$ is a basis of $T_{\phi(x)M}$ we can conclude the thesis. For covectors and tensors the proof follows in the same way. □

Corollary 21: The differential $d\phi$ of a diffeomorphism ϕ acts on the natural basis with the Jacobian matrix related to the map ϕ:

$$\phi^\star(\partial_{\mu|_x}) = d\phi^\lambda_{\mu|_x} \frac{\partial}{\partial y^\lambda}_{|\phi(x)} = \frac{\partial \phi^\lambda}{\partial x^\mu}_{|x} \frac{\partial}{\partial y^\lambda}_{|\phi(x)} \tag{B.2.45}$$

therefore $\frac{\partial \phi^\lambda}{\partial x^\mu}_{|x}$ is the coordinate expression of $d\phi$ with respect the natural basis $\partial_{\mu|_x}$ and $\partial_{\mu|_{\phi(x)}}$.

Property 81: The pull-back of contravariant tensors and the push-forward of covariant tensors satisfy the following properties:

1. they are $\mathbb{R}$-linear and commute with the multiplication by a scalar
2. they are distributive with respect to the tensor product
3. they commute with the braiding map

Proof. All of them are trivial due to the definitions:

1.
$$\phi^\star(\lambda T)(v_{\overline{q}}) = \lambda T(\phi_\star(v)_{\overline{q}}) = \lambda(T(\phi_\star(v)_{\overline{q}})) = \lambda[\phi^\star(T)](v_{\overline{q}}) \tag{B.2.46}$$
$$\phi_\star(\lambda T)(\alpha^{\overline{p}}) = \lambda T(\phi^\star(\alpha)^{\overline{p}}) = \lambda(T(\phi^\star(\alpha)^{\overline{p}})) = \lambda[\phi_\star(T)](\alpha^{\overline{p}}) \tag{B.2.47}$$

2.

$$\phi^\star(T\otimes S)(v_{\overline{q}}, u_{\overline{q}'}) = [T\otimes S](\phi_\star(v)_{\overline{q}}\,, \phi_\star(u)_{\overline{q}'}) = T(\phi_\star(v)_{\overline{q}})S(\phi_\star(u)_{\overline{q}'}) = \tag{B.2.48}$$
$$=\phi^\star(T)(v_{\overline{q}})\phi^\star(S)(u_{\overline{q}'}) = [\phi^\star(T)\otimes\phi^\star(S)](v_{\overline{q}}\,, u_{\overline{q}'}) \tag{B.2.49}$$

$$\phi_\star(T\otimes S)(\alpha^{\overline{p}}, \beta_{\overline{p}'}) = [T\otimes S](\phi^\star(\alpha)^{\overline{p}}, \phi^\star(\beta)^{\overline{p}'}) = T(\phi^\star(\alpha)^{\overline{p}})S(\phi^\star(\beta)^{\overline{p}'}) = \tag{B.2.50}$$
$$=\phi_\star(T)(\alpha^{\overline{p}})\phi_\star(S)(\beta^{\overline{q}'}) = [\phi_\star(T)\otimes\phi_\star(S)](\alpha_{\overline{p}}\,, \beta_{\overline{p}'}) \tag{B.2.51}$$

3.

$$\phi^\star(\sigma_J[T])(v_{\overline{q}}) = \sigma_J[T](\phi_\star(v)_{\overline{q}}) = \tag{B.2.52}$$
$$=[T](\phi_\star(v)_{P_J(\overline{q})}) = [\phi^\star T](v_{P_J(\overline{q})}) = \sigma_J[\phi^\star[T])](v_{\overline{q}}) \tag{B.2.53}$$

$$\phi_\star(\sigma^I[T])(\alpha^{\overline{p}}) = \sigma^J[T](\phi^\star(\alpha)^{\overline{p}}) = \tag{B.2.54}$$
$$=[T](\phi^\star(\alpha)^{P_I(\overline{p})}) = [\phi_\star T](\alpha^{P_I(\overline{p})}) = \sigma^I[\phi^\star[T])](\alpha^{\overline{p}}) \tag{B.2.55}$$

□

Property 82: Let $\phi : U \subseteq M \to V \subseteq N$ be a diffeomorphism, then the following hold:

1. $\phi^\star \circ \phi_\star = \phi_\star \circ \phi^\star = id$
2. $\phi^\star$ and $\phi_\star$ are distributive with both the contractions
3. since the map is a difffeomorphism $\phi^\star$ and $\phi_\star$ commute with the internal contraction

Proof. 1. The first is trivial starting from the definition of pull-back and push-forward of functions:

$$\phi_\star \circ \phi^\star(f) = \phi_\star(\phi^\star(f)) = \phi_\star(f \circ \phi) = \phi^{-1\star}(f \circ \phi) = f \circ \phi \circ \phi^{-1} = f \tag{B.2.56}$$

$$[\phi_\star \circ \phi^\star(v)](f) = [\phi_\star(\phi^\star(v))](f) = [(\phi^\star(v))](\phi^\star f) = v(\phi_\star(\phi^\star f)) = v(f) \tag{B.2.57}$$

$$[\phi_\star \circ \phi^\star(\alpha)](v) = [\phi_\star(\phi^\star(\alpha))](v) = [(\phi^\star(\alpha))](\phi^\star v) = \alpha(\phi_\star(\phi^\star v)) = \alpha(v) \tag{B.2.58}$$

$$[\phi_\star \circ \phi^\star(T)](\alpha^{\overline{p}}, v_{\overline{q}}) = [\phi_\star(\phi^\star(T))](\alpha^{\overline{p}}, v_{\overline{q}}) = [(\phi^\star(T))](\phi^\star(\alpha)^{\overline{q}}, \phi^\star(v)_{\overline{q}}) = \tag{B.2.59}$$
$$=T(\phi_\star(\phi^\star(\alpha))^{\overline{q}}, \phi_\star(\phi^\star(v))_{\overline{q}}) = T(\alpha^{\overline{p}}, v_{\overline{q}}) \tag{B.2.60}$$

2. It is enough to calculate explicitly:

$$[\phi_\star(u \lrcorner T)](\alpha^{\overline{p}}, v_{\overline{q}}) = u \lrcorner T(\phi^\star(\alpha)^{\overline{p}}, \phi^\star(v)_{\overline{q}}) = T(\phi^\star(\alpha)^{\overline{p}}, u, \phi^\star(v)_{\overline{q}}) = \tag{B.2.61}$$

$$=T(\phi^\star(\alpha)^{\overline{p}}, \phi^\star(\phi_\star(u)), \phi^\star(v)_{\overline{q}}) = \phi_\star[T](\alpha^{\overline{p}}, \phi_\star(u), v_{\overline{q}}) = \tag{B.2.62}$$
$$=\phi_\star(u) \lrcorner \phi_\star[T](\alpha^{\overline{p}}, v_{\overline{q}}) \tag{B.2.63}$$

$$[\phi_\star(\beta\urcorner T)](\alpha^{\overline{p}}, v_{\overline{q}}) = \beta\urcorner T(\phi^\star(\alpha)^{\overline{p}}, \phi^\star(v)_{\overline{q}}) = T(\beta, \phi^\star(\alpha)^{\overline{p}}, \phi^\star(v)_{\overline{q}}) = \tag{B.2.64}$$
$$=T(\phi^\star(\phi_\star(\beta)), \phi^\star(\alpha)^{\overline{p}}, \phi^\star(\phi_\star(u)), \phi^\star(v)_{\overline{q}}) = \phi_\star[T](\phi_\star(\beta), \alpha^{\overline{p}}, v_{\overline{q}}) = \tag{B.2.65}$$
$$=\phi_\star(\beta)\urcorner\phi_\star[T](\alpha^{\overline{p}}, v_{\overline{q}}) \tag{B.2.66}$$

The same calculation can be performed to prove this property in case of pull-backs.

3. Using the previous properties we have:

$$\phi_\star[i(T)](\alpha^{\overline{p}}, v_{\overline{q}}) = (iT)(\phi^\star(\alpha)^{\overline{p}}, \phi^\star(v)_{\overline{q}}) = T(e^\mu, \phi^\star(\alpha)^{\overline{p}}, e_\mu, \phi^\star(v)_{\overline{q}}) = \tag{B.2.67}$$
$$=T(\phi^\star(\phi_\star(e^\mu)), \phi^\star(\alpha)^{\overline{p}}, \phi^\star(\phi_\star(e_\mu)), \phi^\star(v)_{\overline{q}}) = \tag{B.2.68}$$
$$=\phi^\star(T)(\phi_\star(e^\mu), \alpha^{\overline{p}}, \phi^\star(e_\mu), v_{\overline{q}}) = \phi^\star(e_\mu)\lrcorner\phi_\star(e^\mu)\urcorner\phi^\star(T)(\alpha^{\overline{p}}, v_{\overline{q}}) = \tag{B.2.69}$$
$$=i[\phi_\star(T)](\alpha^{\overline{p}}, v_{\overline{q}}) \tag{B.2.70}$$

because it does not depend on the choice of the basis.

□

B.2.4 Coordinate expression for standard operations on tensors

Since at fixed basis there is a one to one relation between a tensor and its coordinate expression, we can ask ourselves how the operations defined in the previous section affect the coordinates of a tensors. This is very useful because it allows us to single out for each operation defined above, the rules to manipulate directly the coordinate expressions for the tensors. Finding the coordinate manipulation rules for the coordinates is very easy, it is enough to fix a basis on T_xM and apply both the definition of tensor coordinates and the definition of the operations:

1. ***Sum***:

$$(T+S)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = [T+S](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) + S(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} + S^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{B.2.71}$$

Example: $(g+h)_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$

2. ***Multiplication by a scalar***:

$$(\lambda T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = [\lambda T](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = \lambda \cdot [T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}})] = \lambda T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{B.2.72}$$

Example: $(\lambda g)_{\mu\nu} = \lambda g_{\mu\nu}$

3. ***Tensor product***:

$$(T \otimes S)^{\mu_{\overline{p}}\alpha_{\overline{p}'}}_{\nu_{\overline{q}}\beta_{\overline{q}'}} = [T \otimes S](e^{\mu_{\overline{p}}}, e^{\alpha_{\overline{p}'}}, e_{\nu_{\overline{q}}}, e_{\beta_{\overline{q}'}}) = T(e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) S(e^{\alpha_{\overline{p}'}}, e_{\beta_{\overline{q}'}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} S^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}} \tag{B.2.73}$$

Example: $(g \otimes h)_{\mu\nu\alpha\beta} = g_{\mu\nu} h_{\alpha\beta}$

4. ***Braiding***:

$$(\sigma^I_J T)^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = [\sigma^I_J T](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q}}}) = T(e^{\mu_{P_I(\overline{p})}}, e_{\nu_{P_J(\overline{q})}}) = T^{\mu_{P_I(\overline{p})}}_{\nu_{P_J(\overline{q})}} \tag{B.2.74}$$

Example: $(\sigma_{(12)} g)_{\mu\nu} = g_{\nu\mu}$

5. ***Contractions***:

$$(v \lrcorner T)^{\mu_{\overline{p}}}_{\nu_{\overline{q-1}}} = [v \lrcorner T](e^{\mu_{\overline{p}}}, e_{\nu_{\overline{q-1}}}) = T(e^{\mu_{\overline{p}}}, v, e_{\nu_{\overline{q-1}}}) = v^\alpha T(e^{\mu_{\overline{p}}}, e_\alpha, e_{\nu_{\overline{q-1}}}) = v^\alpha T^{\mu_{\overline{p}}}_{\alpha\nu_{\overline{q-1}}} \tag{B.2.75}$$

$$(\alpha \urcorner T)^{\mu_{\overline{p-1}}}_{\nu_{\overline{q}}} = [\alpha \urcorner T](e^{\mu_{\overline{p-1}}}, e_{\nu_{\overline{q}}}) = T(\alpha, e^{\mu_{\overline{p-1}}}, e_{\nu_{\overline{q}}}) = \alpha^\alpha T(e^\alpha, e^{\mu_{\overline{p-1}}}, e_\alpha, e_{\nu_{\overline{q}}}) = \alpha^\alpha T^{\alpha\mu_{\overline{p-1}}}_{\nu_{\overline{q}}} \tag{B.2.76}$$

Example: $(v \lrcorner g)_\nu = v^\mu g_{\mu\nu}$

6. ***Internal contraction***:

$$(iT)^{\mu_{\overline{p-1}}}_{\nu_{\overline{q-1}}} = iT(e^{\mu_{\overline{p-1}}}, e_{\nu_{\overline{q-1}}}) = iT(e^{\alpha}, e^{\mu_{\overline{p-1}}}, e_{\alpha}, e_{\nu_{\overline{q-1}}}) = T^{\alpha\mu_{\overline{p-1}}}_{\alpha\nu_{\overline{q-1}}} \tag{B.2.77}$$

Example: $(i\,Tor)_{\mu} = Tor^{\lambda}_{\lambda\mu}$

Lemma 44: Let $T^{p}_{q\,x}M$ and $T^{p'}_{q'\,x}M$ be two tensor spaces, and $Lin(p,q,p',q') = \{\mathcal{L} : T^{p}_{q\,x}M \to T^{p'}_{q'\,x}M\}$ be the space of the linear map between them, there always exists an unique tensor $L \in T^{q+p'}_{p+q'\,x}M$ such that:

$$\mathcal{L}(T) = [i]^{p+q}[\sigma^{\overline{(p+q)}}]^{p}(T \otimes L) \tag{B.2.78}$$

therefore $Lin(p,q,p',q')$ is isomorphic to $T^{q+p'}_{p+q'\,x}M$ as a vector space and we can perform on them all the operations defined on tensors.

Proof. Let us fix the basis on T_xM, and let us induce from it the basis on $T^{p}_{q\,x}M$ and on $T^{p'}_{q'\,x}M$. Then we can write:

$$\mathcal{L}(T) = \mathcal{L}(T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}}) \tag{B.2.79}$$

Considering the definition of $\mathcal{L}$ and since at fixed indices μ and ν we have a tensor, we can write $\mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}}) = [\mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}\ e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{q}'}}$ where $[\mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}$ is a multi-indexed list of real numbers. Hence defining $L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}} = [\mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}$ we can recast the expression as follow:

$$\mathcal{L}(T) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} [\mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}\ e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{q}'}} = \tag{B.2.80}$$

$$= T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}}\ e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{q}'}} \tag{B.2.81}$$

In the other hand we have that:

$$[i]^{p+q}[\sigma^{\overline{(p+q)}}]^{p}(T \otimes L) = \{[i]^{p+q}[\sigma^{\overline{(p+q)}}]^{p}(T \otimes L)\}^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}}\ e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{q}'}} = \tag{B.2.82}$$

$$= \{[i]^{p+q}[\sigma^{\overline{(p+q)}}]^p(T \otimes L)\}(e^{\alpha_{\overline{p}'}}, e_{\beta_{\overline{q}'}})\ e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{q}'}} = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}}\ e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{q}'}} \tag{B.2.83}$$

Since $L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}}$ is the coordinate expression of the tensor L it must be unique hence there must be a one to one relation between $\mathcal{L} : T^p_{q\,x}M \to T^{p'}_{q'\,x}M$ and the tensor $L \in T^{q+p'}_{p+q'x}M$, given explicit at fixed basis by the relation:

$$L^{\nu_{\overline{q}}\alpha_{\overline{p}'}}_{\mu_{\overline{p}}\beta_{\overline{q}'}}\ e_{\nu_{\overline{q}'}} \otimes e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{p}'}} \otimes e^{\alpha_{\overline{q}'}} = [\mathcal{L}(e^{\nu_{\overline{q}}} \otimes e_{\mu_{\overline{p}}})]^{\alpha_{\overline{p}'}}_{\beta_{\overline{q}'}} e_{\nu_{\overline{q}'}} \otimes e_{\alpha_{\overline{p}'}} \otimes e^{\beta_{\overline{p}'}} \otimes e^{\alpha_{\overline{q}'}}. \tag{B.2.84}$$

Although the proof is given at fixed basis, this hold for an arbitrary basis, hence the bijection does not depend on the basis. To prove the relation is an isomorphism of vector spaces one should prove that the sum and multiplication by scalars are preserved. This can be trivially verified from the definition of L, $\mathcal{L}$ and the bijection. □

Property 83: Since the pull-back and push-forward of tangent tensors are $\mathbb{R}$-linear maps they are tensors itself. In particular we have that $\phi_\star : T^p_{x\,q}M \to T^p_{\phi(x)\,q}N$ is isomorphic to the tensor $(\otimes^p d(\phi)_{|x}) \otimes (\otimes^q d(\phi^{-1})_{|\phi(x)})$ and $\phi^\star : T^p_{\phi(x)\,q}N \to T^p_{x\,q}M$ is isomorphic to the tensor $(\otimes^p d(\phi^{-1})_{|\phi(x)}) \otimes (\otimes^q d(\phi)_{|x})$

Proof. Let x and $y = \phi(x)$ be two point on M and N respectively and x^μ and y^μ be the coordinates of the two points. It is enough to use the definitions and to use the chain rules for the derivations:

$$[\phi_\star(v)](f) = v(\phi_\star(f)) = v^\mu e_\mu(f \circ \phi) = v^\mu d\phi^\nu_{\mu|x} e_\nu(f) \tag{B.2.85}$$

$$[\phi^\star(\alpha)](v) = \alpha(\phi_\star(v)) = \alpha(d\phi(v^\mu e_\nu)) = \alpha(v^\mu d\phi^\nu_{\mu|x} e_\nu) = \tag{B.2.86}$$

□

B.2.5 Coordinate transformations and basis changes.

Inspired by the differential manifolds theory, it is natural to ask ourselves how the coordinate expressions transform when a change of coordinate is performed, therefore how the coordinate expression of a tensor is affected by the change of basis (e_μ) on T_xM. To do it, let us recall briefly some theory about the basis of vector spaces. We know from the fundamental theorem of the linear applications, that given two arbitrary basis (e'_μ) and (e_ν) of T_xM we can express each vector of the first as a linear combination of the vectors of the second basis obtaining:

$$e'_\mu = e^\nu(e'_\mu)e_\nu = \overline{\Lambda}^\nu_\mu e_\nu \tag{B.2.87}$$

where $\overline{\Lambda}^\nu_\mu$ is a multi-indexed list of real numbers expressing the new basis as a linear combination of the old one. In this case the multi-index list is just called the ***inverse transformation matrix*** and it is defined as the action of the (e^μ) on (e'_μ). There is also another completely equivalent intrinsic way to interpret $\overline{\Lambda}^\mu_\nu$ as a tensor. Let us fix given two basis (e'_μ) and (e_ν) of T_xM with the respective basis (e'^μ) and (e^ν) on $T^\star_xM$ and let $v \in T_xM$ and $\alpha \in T^\star_xM$ be a vector and a covector respectively. Given the identity map $Id : T_xM \to T_xM \mid Id(v) = v \quad , \quad \forall v \in T_xM$ due to the canonical isomorphism, we can define uniquely an isomorphic tensor $\delta \in T^1_{1x}M$ such that

$$\delta(\alpha, v) = [Id(v)](\alpha) = v(\alpha) = \alpha(v) \tag{B.2.88}$$

Its coordinate expression can be obtained from the tensor coordinates definition:

$$\delta^\mu_\nu = \delta(e^\mu, e_\nu) = [Id(e^\mu)](e_\nu) = e^\mu(e_\nu) \tag{B.2.89}$$

hence we can conclude that the coordinate expression of the δ tensor must be just the Kronecker delta. Let us remark that this property is completely independent from the choice of basis, in fact it holds for each generic basis arbitrarily chosen. From the δ tensor we are able to characterise easily the inverse transformation matrix $\overline{\Lambda}^\mu_\nu$ in fact:

$$e'_\nu \lrcorner (e^\mu \urcorner \delta) = \delta(e^\mu, e'_\nu) = e^\mu(e'_\nu) = \overline{\Lambda}^\mu_\nu \tag{B.2.90}$$

Hence we can conclude that the transformation matrix $\overline{\Lambda}^\mu_\nu$ is just the δ tensor contracted with the old basis (e^μ) and the new basis (e'_ν). Considering this, we can rewrite the change of basis as operations with the delta tensor:

$$e'_\mu = e^\nu(e'_\mu)e_\nu = \overline{\Lambda}^\nu_\mu e_\nu = \delta(e^\mu, e'_\nu)e_\mu = e'_\nu \lrcorner (e^\mu \urcorner \delta \otimes e_\mu) \tag{B.2.91}$$

Let us remark that a change of basis on T_xM induces a change on the basis on $T^\star_xM$. If $\overline{\Lambda}^\mu_\nu$ is the inverse transformation matrix then:

$$e'_\mu = e_\nu(e'^\mu)e^\mu = e'^\mu(e_\nu)e_\mu = \Lambda^\nu_\mu e^\mu \tag{B.2.92}$$

To satisfy the duality relation for the new basis $e'^\mu(e'_\nu) = \delta^\mu_\nu$ we must have

$$\delta^\mu_\nu = e'^\mu(e'_\nu) = e'^\mu(\overline{\Lambda}^\lambda_\nu e_\lambda) = \overline{\Lambda}^\lambda_\nu e'^\mu(e_\lambda) = \overline{\Lambda}^\lambda_\nu \Lambda^\mu_\lambda \tag{B.2.93}$$

From this we can conclude that the matrix Λ^μ_λ must be the inverse of $\overline{\Lambda}^\mu_\lambda$, therefore we call it ***transformation matrix***. The transformation matrix Λ^μ_ν can be associated to the δ tensor as well as $\overline{\Lambda}^\mu_\nu$:

$$e_\nu \lrcorner (e'^\mu \neg \delta) = \delta(e'^\mu, e_\nu) = e'^\mu(e_\nu) = \Lambda^\mu_\nu \tag{B.2.94}$$

Considering this, we can rewrite the change of basis as operations with the delta tensor:

$$e'^\mu = e_\nu(e'^\mu)e^\nu = e'^\mu(e_\nu)e^\nu = \Lambda^\mu_\nu e^\nu = \delta(e'^\mu, e_\nu)e^\nu = e_\nu \lrcorner (e'^\mu \neg \delta \otimes e^\nu) \tag{B.2.95}$$

Property 84: Let (e_μ) be a basis on T_xM and let (e^μ) and $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ be respectively the induced basis on $T^\star_xM$ and $T^p_{q\,x}M$. Then a change of basis $e'_\mu = \overline{\Lambda}^\nu_\mu e_\nu$ induces a change of basis on $T^p_{q\,x}M$ as follows:

$$e'_{\mu_{\overline{p}}} \otimes e'^{\nu_{\overline{q}}} = e'_{\mu_1} \otimes ... \otimes e'_{\mu_p} \otimes e'^{\nu_1} \otimes ... \otimes e'^{\nu_q} = \tag{B.2.96}$$

$$= \overline{\Lambda}^{\lambda_1}_{\mu_1} e_{\lambda_1} \otimes ... \otimes \overline{\Lambda}^{\lambda_p}_{\mu_p} e_{\lambda_p} \otimes \Lambda^{\nu_1}_{\rho_1} e'^{\rho_1} \otimes ... \otimes \Lambda^{\nu_q}_{\rho_q} e'^{\rho_q} = \tag{B.2.97}$$

$$= \overline{\Lambda}^{\lambda_{\overline{p}}}_{\mu_{\overline{p}}} \Lambda^{\nu_{\overline{q}}}_{\rho_{\overline{q}}} e'_{\lambda_{\overline{p}}} \otimes e'^{\rho_{\overline{q}}} \tag{B.2.98}$$

where $\Lambda^{\nu_{\overline{q}}}_{\rho_{\overline{q}}}$ and $\overline{\Lambda}^{\lambda_{\overline{p}}}_{\mu_{\overline{p}}}$ are just the condensed Einstein expressions for $\Lambda^{\nu_1}_{\rho_1}\Lambda^{\nu_2}_{\rho_2}...\Lambda^{\nu_{q-1}}_{\rho_{q-1}}\Lambda^{\nu_q}_{\rho_q}$ and $\Lambda^{\lambda_1}_{\mu_1}\Lambda^{\lambda_2}_{\mu_2}...\Lambda^{\lambda_{p-1}}_{\mu_{p-1}}\Lambda^{\lambda_p}_{\mu_p}$.

Property 85: Let (e_μ) and (e'_μ) be two bases on T_xM linked by an inverse transformation matrix Λ^ν_μ, let (e^μ) and (e'^μ) be respectively the two induced basis on $T^\star_x M$ and let $(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})$ and $(e'_{\mu_{\overline{p}}} \otimes e'^{\nu_{\overline{q}}})$ be the two induced basis on $T^p_{q\,x}M$. When a change of basis is performed $e'_\mu = \overline{\Lambda}^\nu_\mu e_\nu$, the coordinate expression for vectors, covectors, and tensors change as follows:

$$\begin{cases} v'^\mu = v(e'^\mu) = e'^\mu(v) = \Lambda^\mu_\nu e^\nu(v) = \Lambda^\mu_\nu v^\nu \\ \alpha'_\mu = \alpha(e'_\mu) = \alpha(\overline{\Lambda}^\nu_\mu e_\nu) = \overline{\Lambda}^\nu_\mu \alpha(e_\nu) = \overline{\Lambda}^\nu_\mu \alpha^\nu \\ T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = T(e'^{\mu_{\overline{p}}}, e'_{\nu_{\overline{q}}}) = T(\Lambda^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} e^{\lambda_{\overline{p}}}, \overline{\Lambda}^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} e_{\rho_{\overline{q}}}) = \Lambda^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} \overline{\Lambda}^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} T(e^{\lambda_{\overline{p}}}, e_{\rho_{\overline{q}}}) = \Lambda^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} \overline{\Lambda}^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \end{cases} \tag{B.2.99}$$

Lemma 45: Let $T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}}$ be a multi-index list and (e_μ) be a basis on $T_x M$, then there must exists a tensor $T \in T^p_q M$ such that $T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} = T(e^{\lambda_{\overline{p}}}, e_{\rho_{\overline{q}}})$ if and only if for any change of basis $e'_\mu = \overline{\Lambda}^\nu_\mu e_\nu$ the multi-index list $T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}}$ changes as follow:

$$T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \Lambda^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} \overline{\Lambda}^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \tag{B.2.100}$$

Proof.

1. $\exists T \in T^p_q M \mid T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} = T(e^{\lambda_{\overline{p}}}, e_{\rho_{\overline{q}}}) \Rightarrow T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \Lambda^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} \Lambda^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}}$
   This is already showed above.
2. $T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \Lambda^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} \Lambda^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \Rightarrow \exists T \in T^p_q M \mid T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} = T(e^{\lambda_{\overline{p}}}, e_{\rho_{\overline{q}}})$
   To prove this let us consider a generic basis (e'_μ) linked with the old one by $e'_\mu = \overline{\Lambda}^\nu_\mu e_\nu$. We can always define a tensor T to be a valid linear combination of the new induced basis $(e'_{\mu_{\overline{p}}} \otimes e'^{\nu_{\overline{q}}})$ on $T^p_{qx}M$ with coefficients $T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ as follow:

$$T = T'^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}}\ e'_{\lambda_{\overline{p}}} \otimes e'^{\rho_{\overline{q}}} \tag{B.2.101}$$

   Therefore

$$T(e_{\mu_{\overline{p}}}, e^{\nu_{\overline{q}}}) = [T'^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}}\ e'_{\lambda_{\overline{p}}} \otimes e'^{\rho_{\overline{q}}}](e_{\mu_{\overline{p}}}, e^{\nu_{\overline{q}}}) = \tag{B.2.102}$$
$$= [\Lambda^{\lambda_{\overline{p}}}_{\alpha_{\overline{p}}} \overline{\Lambda}^{\beta_{\overline{q}}}_{\rho_{\overline{q}}} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}}\ e'_{\lambda_{\overline{p}}} \otimes e'^{\rho_{\overline{q}}}](e_{\mu_{\overline{p}}}, e^{\nu_{\overline{q}}}) = \Lambda^{\lambda_{\overline{p}}}_{\alpha_{\overline{p}}} \overline{\Lambda}^{\beta_{\overline{q}}}_{\rho_{\overline{q}}} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} \overline{\Lambda}^{\mu_{\overline{p}}}_{\lambda_{\overline{p}}} \Lambda^{\rho_{\overline{q}}}_{\nu_{\overline{q}}} = \tag{B.2.103}$$
$$= T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}} \delta^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}} \delta^{\beta_{\overline{q}}}_{\nu_{\overline{q}}} = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{B.2.104}$$

□

Due to this lemma we are able to state that a generic multi-index list of real numbers is a coordinate expression for a tensor if and only if it changes with a very specific rule when a change of basis is performed. This is very important because it allow us to distinguish the multi-index list related to the coordinate expression of a tensor from the others, just observing the way they transform.

Property 86: As a consequence of the lemma, a multi-indexed $T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}}$ is the coordinate expression for a tensor $T \in T^p_{q_x} M$ if and only if the coordinate transformation rule is:

$$T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \Lambda^{\mu\overline{p}}_{\lambda\overline{p}} \overline{\Lambda}^{\rho\overline{q}}_{\nu\overline{q}} T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \tag{B.2.105}$$

where $\Lambda^{\mu\overline{p}}_{\lambda\overline{p}}$ and $\overline{\Lambda}^{\rho\overline{q}}_{\nu\overline{q}}$ are the condensed Einstein expressions for $\Lambda^{\mu_1}_{\lambda_1}...\Lambda^{\mu_p}_{\lambda_p}$and $\overline{\Lambda}^{\rho_1}_{\nu_1}...\overline{\Lambda}^{\rho_p}_{\nu_p}$, with Λ^μ_ν the transformation matrix related to change of basis.

Property 87: As a consequence of the lemma, if T is an unknown geometrical object for which the coordinate expression $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ is provided, then T is a tensor if and only if the coordinate expression satisfies:

$$T'^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = \Lambda^{\mu\overline{p}}_{\lambda\overline{p}} \overline{\Lambda}^{\rho\overline{q}}_{\nu\overline{q}} T^{\lambda_{\overline{p}}}_{\rho_{\overline{q}}} \tag{B.2.106}$$

when we perform a change of basis $e_\mu = \Lambda^\nu_\mu e_\nu$ on $T_x M$.

Property 88: The internal contraction does not depend on basis, although its definition does:

Proof.

$$[iT](\alpha^{\overline{p-1}}, v_{\overline{q-1}}) = T(e'^{\mu}, \alpha^{\overline{p-1}}, e'_\mu, v_{\overline{q-1}}) = T(\Lambda^\mu_\nu e^\nu, \alpha^{\overline{p-1}}, \overline{\Lambda}^\sigma_\mu e_\sigma, v_{\overline{q-1}}) = \tag{B.2.107}$$

$$= \Lambda^\mu_\nu \overline{\Lambda}^\sigma_\mu T(e^\nu, \alpha^{\overline{p-1}}, e^\sigma, v_{\overline{q-1}}) = \delta^\sigma_\nu T(e^\nu, \alpha^{\overline{p-1}}, e_\sigma, v_{\overline{q-1}}) = T(e^\nu, \alpha^{\overline{p-1}}, e_\nu, v_{\overline{q-1}}) \tag{B.2.108}$$

□

Property 89: One can check very easily the following relations between the tensor product and the contractions:

$$i(v \otimes T) = v \lrcorner T \tag{B.2.109}$$

$$i(\alpha \otimes T) = \alpha \urcorner T \tag{B.2.110}$$

B.3 Tensor fields

B.3.1 The tangent tensor bundle

Definition 144: The ***rank*** p, q ***tangent tensor space of a manifold*** M, denoted by $T^p_q M$ is the set:

$$T^p_q M = \bigsqcup_{x \in M} T^p_{q\,x} M \tag{B.3.1}$$

Property 90: The tangent and cotangent spaces are particular cases of tangent tensor spaces. Since the definition coincide we can say that of $T^\star_x M = T_{1\,x} M$ hence $T^\star M = T_{1\,x} M$. Due the actions of vectors on covectors we have stated that $T^{\star\star}_x M = T_x M$ therefore $TM = T^1 M$.

The rank p, q tangent tensor space of a manifold can be regarded just as the collection of all the rank p, q tensor spaces tangent at each point of M, but with a very little effort we can prove that $T^p_q M$ admits naturally a bundle structure called ***rank*** p, q ***tangent tensor bundle*** of M.

Definition 145: Due to the property of the disjoint union, given $T^p_q M$ we can always define a canonical surjective map $\tilde{\tau}_M : T^p_q M \to M$ such that:

$$\tilde{\tau}_M(T_x) = x \tag{B.3.2}$$

Property 91: Let M be a m-dimensional manifold, let $T^p_q M$ be the cotangent space and $\hat{\tau}_M : T^\star M \to M$ be the projection defined above . The quadruple $(T^p_q M, M, \hat{\tau}_M, \mathbb{R}^{m^{p+q}})$ is a good fiber bundle with standard fiber $\mathbb{R}^{m^{p+q}}$.

Proof. The proof is just a generalisation of what has been done previously to build the cotangent bundle. Let us consider a smooth manifold M with an atlas $(U_i, \varphi_{(i)})$, let $T^\star M$ be the cotangent space to M, let TM and $T^\star M$ be the tangent and cotangent bundles of M respectively and let $(U_i, t_{\mu(i)})$, $(U_i, \hat{t}_{\mu(i)})$ and $(U_i, e^\mu_{(i)})$ be respectively the local trivialisation of TM and $T^\star M$, and the cotangent local frame induced by the choice of a smooth local frame $(U_i, e_{\mu(i)})$. Then can state that

1. For each U_i, for each $x \in U_i$,
2. for each $T_x \in \tilde{\tau}_M^{-1}(U_i) \mid \tilde{\tau}_M(T_x) = x$,
3. for each $\alpha_x^{\overline{p}} \in \times^p[\hat{\tau}^{-1}(U_i)] \mid \hat{\tau}_M(\alpha^s_x) = x$, $\forall s \in [1, p]$
4. and for each $v_{x\,\overline{q}} \in \times^q[\tau^{-1}(U_i) \mid \tau_M(v_{xr}) = x$, $\forall r \in [1, q]$

the following holds:

$$T_x(\alpha_x^{\overline{p}}, v_{x\overline{q}}) = \alpha_{x\,\mu_{\overline{p}}} v_x^{\nu_{\overline{q}}} T(e_{\nu_{\overline{p}}(i)|_x}, e^{\mu_{\overline{p}}}_{(i)|_x}) = \alpha_{x\,\mu_{\overline{p}}} v_x^{\nu_{\overline{q}}} T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} \tag{B.3.3}$$

Then considering that a basis on T_xM induce a basis on $T_x^\star M$ and $T^p_{qx}M$ we can use the frame $e_{\mu(i)}$ to induce a basis on $T^p_{q\,x}M$ at each point of U_i via the pointwise relation:

$$(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})_x = e_{\mu_1(i)|_x} \otimes ... \otimes e_{\mu_p(i)|_x} \otimes e^{\nu_1}_{(i)|_x} \otimes ... \otimes e^{\nu_q}_{(i)|_x} = e_{\mu_{\overline{p}}(i)|_x} \otimes e^{\nu_{\overline{q}}}_{(i)|_x} \tag{B.3.4}$$

To each point, the existence of this basis is guaranteed as proven before hence one can write:

$$T_x(\alpha_x^{\overline{p}}, v_{x\overline{q}}) = \alpha_{x\,\mu_{\overline{p}}} v_x^{\nu_{\overline{q}}} T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}} = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}\ [e_{\mu_{\overline{p}}(i)|_x} \otimes e^{\nu_{\overline{q}}}_{(i)|_x}](\alpha_x^{\overline{p}}, v_{x\overline{q}}) \tag{B.3.5}$$

where $T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}$ is multi-indexed list of real numbers. One can easily check that, fixing a smooth local frame $(e_{\mu(i)})$, the map $\tilde{t}_{(i)} : \hat{\tau}^{-1}(U_i) \to U_i \times \mathbb{R}^{m^{p+q}}$ defined as:

$$\hat{t}_{(i)}(T_x) \to (x, T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}) \tag{B.3.6}$$

is invertible and differentiable, hence it is a diffeomorphism and it defines a local trivialisation of T^p_qM (one for each open set U_i, in the atlas of M) $\square$

Definition 146: The quadruple $(T^\star M, M, \hat{\tau}_M, \mathbb{R}^{m^{p+q}})$ is called ***rank*** p, q ***tangent tensor bundle of*** M.

Property 92: Given a smooth manifold M, a cotangent bundle is a vector bundle because the given trivialisation admits transition functions in $GL(\mathbb{R}^{m^{p+q}}) \subset \mathit{Diff}(\mathbb{R}^{m^{p+q}})$

Proof. Let TM be the tangent bundle of M and let us suppose to have two local trivialisations $(U_i, t_{(i)})$ and $(U_j, t_{(j)})$ induced by two local smooth frames $(e_{\mu(i)})$ and $(e_{\mu(j)})$ such that $U_i \cap U_j = U_{ij} \neq \varnothing$. We know that these induce two local trivialisations on $T^\star M$ and T^p_qM, denoted by $(U_i, \hat{t}_{(i)})$, $(U_j, \hat{t}_{(j)})$ and $(U_i, \tilde{t}_{(i)})$, $(U_j, \tilde{t}_{(j)})$ respectively. As proven previously we know that the transition functions related to these trivialisations of TM ans $T^\star M$ are characterised by:

$$\begin{cases} v^\mu_{(j)} = v^\nu_{(i)} \Lambda^\mu_{\nu(ij)|_x} \\ \alpha_{\mu(j)} = \alpha_{\nu(i)} \overline{\Lambda}^\nu_{\mu(ij)|_x} \end{cases} \tag{B.3.7}$$

therefore on the overlaps U_{ij} the following holds:

$$T_x(\alpha_x^{\overline{p}}, v_{x\overline{q}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}\ [e_{\mu_{\overline{p}}(i)|_x} \otimes e^{\nu_{\overline{q}}}_{(i)|_x}](\alpha_x^{\overline{p}}, v_{x\overline{q}}) = T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}}\ [e_{\mu_{\overline{p}}(j)|_x} \otimes e^{\nu_{\overline{q}}}_{(j)|_x}](\alpha_x^{\overline{p}}, v_{x\overline{q}}) = \tag{B.3.8}$$

$$=\alpha_{x\ \mu_{\overline{p}}(j)} v^{\nu_{\overline{q}}}_{x(j)} T^{\mu_{\overline{p}}}_{\nu_{\overline{q}}(j)} = \alpha_{x\ \mu_{\overline{p}}(i)} \Lambda^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}(ij)|_x} v^{\nu_{\overline{q}}}_{x(j)} \Lambda^{\beta_{\overline{q}}}_{\nu_{\overline{q}}(ij)|_x} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}(j)} = \tag{B.3.9}$$

$$=\Lambda^{\beta_{\overline{q}}}_{\nu_{\overline{q}}(ij)|_x} \overline{\Lambda}^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}(ij)|_x} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}(j)} [e_{\mu_{\overline{p}}(i)|_x} \otimes e^{\nu_{\overline{q}}}_{(i)|_x}](\alpha_x^{\overline{p}}, v_{x\overline{q}}) \tag{B.3.10}$$

and we must conclude the transition functions are:

$$\overline{\Lambda}^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}(ij)|_x} \Lambda^{\beta_{\overline{q}}}_{\nu_{\overline{q}}(ij)|_x} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}(j)} = T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}(i)} \quad \Leftrightarrow \quad T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}(j)} = \Lambda^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}(ij)|_x} \overline{\Lambda}^{\beta_{\overline{q}}}_{\nu_{\overline{q}}(ij)|_x} T^{\alpha_{\overline{p}}}_{\beta_{\overline{q}}(i)} \tag{B.3.11}$$

where $\Lambda^{\mu_{\overline{p}}}_{\alpha_{\overline{p}}(ij)|_x} = \Lambda^{\mu_1}_{\alpha_1(ij)|_x} \Lambda^{\mu_2}_{\alpha_2(ij)|_x} ... \Lambda^{\mu_{p-1}}_{\alpha_{p-1}(ij)|_x} \Lambda^{\mu_p}_{\alpha_p(ij)|_x}$ according to the condensed Einstein notation. □

The given local trivialisation of $T^p_q M$ is thence compatible with the vector bundle structure $(T^p_q M, M, \tilde{\tau}_M, \mathbb{R}^{m^{p+q}}, \lambda, GL(\mathbb{R}^{m^{p+q}}))$

Definition 147: A local section of the rank p, q tangent tensor bundle $\sigma : U \subset M \to T^p_q M$ is called ***local rank*** p, q ***tensor field***. The set of all sections defined on the open set U are denoted by $\Gamma_U T^p_q M$. If $U = M$ then σ is called ***global rank*** p, q ***tensor field***.

Property 93: Tensor fields can be interpreted as specific maps on n-tuples of vector and covector fields . Given $\alpha^{\overline{p}} \in \times^p \Gamma_U T^\star M$ and $v_{\overline{q}} \in \times^q \Gamma_U TM$ then we can define the function $T(\alpha^{\overline{p}}, v_{\overline{q}}) : U \to \mathbb{R}$ as follow:

$$T(\alpha^{\overline{p}}, v_{\overline{q}})_{|_x} = T_{|_x}(\alpha^{\overline{p}}_{|_x}, v_{\overline{q}|_x}) \quad , \quad \forall x \in U \tag{B.3.12}$$

Hence we can state that a tensor field $T \in \Gamma_U T^p_q M$ can be interpreted as a map

$$T : (\times^p \Gamma_U T^\star M) \times (\times^q \Gamma_U TM) \to \mathcal{F}(U) \tag{B.3.13}$$

Tensor fields are then $\mathcal{F}(U)$-multi-linear in their arguments because:

1. $\forall i \in [1, p]$, $\forall j \in [1, q]$
2. $\forall f_1, f_2, g_1, g_2 \in C^\infty M$
3. $\forall \alpha, \beta \in \Gamma T^\star M, \forall w, u \in \Gamma TM$

$$
\begin{aligned}
&T(\omega^{\overline{i-1}}, f_1\alpha + g_1\beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, f_2 w + g_2 u, v_{\overline{q}\backslash\overline{j}})(x) = && \text{(B.3.14)}\\
=&T_{|x}(\omega^{\overline{i-1}}_{|x}, f_{1|x}\alpha_{|x} + g_{1|x}\beta_{|x}, \omega^{\overline{p}\backslash\overline{i}}_{|x}, v_{\overline{j}|x}, f_{2|x}w_{|x} + g_{2|x}u_{|x}, v_{\overline{q}\backslash\overline{j}|x}) && \text{(B.3.15)}\\
=&f_{1|x}f_{2|x}T_{|x}(\omega^{\overline{i-1}}_{|x}, \alpha_{|x}, \omega^{\overline{p}\backslash\overline{i}}_{|x}, v_{\overline{j}|x}, w_{|x}, v_{\overline{q}\backslash\overline{j}|x}) + && \text{(B.3.16)}\\
+&f_{1|x}g_{2|x}T(\omega^{\overline{i-1}}_{|x}, \alpha_{|x}, \omega^{\overline{p}\backslash\overline{i}}_{|x}, v_{\overline{j}|x}, u_{|x}, v_{\overline{q}\backslash\overline{j}|x}) + && \text{(B.3.17)}\\
+&g_{1|x}f_{2|x}T_{|x}(\omega^{\overline{i-1}}_{|x}, \beta_{|x}, \omega^{\overline{p}\backslash\overline{i}}_{|x}, v_{\overline{j}|x}, w_{|x}, v_{\overline{q}\backslash\overline{j}|x}) + && \text{(B.3.18)}\\
+&g_{1|x}g_{2|x}T_{|x}(\omega^{\overline{i-1}}_{|x}, \beta_{|x}, \omega^{\overline{p}\backslash\overline{i}}_{|x}, v_{\overline{j}|x}, u_{|x}, v_{\overline{q}\backslash\overline{j}|x}) = && \text{(B.3.19)}\\
=&f_1(x)f_2(x)T(\omega^{\overline{i-1}}, \alpha, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, w, v_{\overline{q}\backslash\overline{j}})(x) && \text{(B.3.20)}\\
+&f_1(x)g_2(x)T(\omega^{\overline{i-1}}, \alpha, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, u, v_{\overline{q}\backslash\overline{j}})(x) + && \text{(B.3.21)}\\
+&g_1(x)f_2(x)T(\omega^{\overline{i-1}}, \beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, w, v_{\overline{q}\backslash\overline{j}})(x) + && \text{(B.3.22)}\\
+&g_1(x)g_2(x)T(\omega^{\overline{i-1}}, \beta, \omega^{\overline{p}\backslash\overline{i}}, v_{\overline{j}}, u, v_{\overline{q}\backslash\overline{j}})(x) \qquad , \qquad \forall x \in U && \text{(B.3.23)}
\end{aligned}
$$

Definition 148: Given a local smooth frame $(e_{\mu(i)})$ on $U_i \subseteq TM$ we define ***the local tensor frame*** the multi-indexed list of local sections $e_{\mu_{\overline{p}}(i)} \otimes e^{\nu_{\overline{q}}}_{(i)} : U_i \to \hat{\tau}^{-1}(U_i) \subseteq T^{\star}_x M$ satisfying the point-wise duality relation:

$$
(e_{\mu_{\overline{p}}} \otimes e^{\nu_{\overline{q}}})_x = e_{\mu_1(i)|x} \otimes ... \otimes e_{\mu_p(i)|x} \otimes e^{\nu_1}_{(i)|x} \otimes ... \otimes e^{\nu_q}_{(i)|x} = e_{\mu_{\overline{p}}(i)|x} \otimes e^{\nu_{\overline{q}}}_{(i)|x} \quad , \quad \forall x \in U_i \quad \text{(B.3.24)}
$$

Property 94: Due to the definition of tensor product, it is easy to check that since $e_{\mu(i)}$ is smooth, then the previous relation is differentiable, therefore the local tensor frames induced by a smooth frame is smooth as well.

Property 95: Let M be a smooth manifold with an atlas $\{U_i, \varphi_i\}$. The smooth charts guarantee the existence of a local smooth tensor frame $(\partial_{\mu_{\overline{p}}(i)} \otimes dx^{\nu_{\overline{q}}}_{(i)})$ on $T^p_q M$. Considering this, we must admit that $(T^M, M, \hat{\tau}_M, \mathbb{R}^m, \lambda, GL(\mathbb{R}^m))$ is a natural vector bundle over M.

Definition 149: On $\Gamma_U T^p_q M$ we can introduce two useful operations which will be analyzed in details later:

1. ***Sum***: $+$: $\Gamma_U T^p_q M \times \Gamma_U T^p_q M \to \Gamma_U T^p_q M$ such that:

$$
[v + w](f) = v(f) + w(f) \quad \forall f \in \mathcal{F}(U), \forall v, w \in \Gamma_U T^p_q M \qquad \text{(B.3.25)}
$$

2. ***Product by a scalar field***: $\cdot : \mathcal{F}(U) \times \Gamma_U T^p_q M \to \Gamma_U T^p_q M$ such that:

$$[fv](g) = f \cdot v(g) \quad \forall g \in \mathcal{F}(U) \; , \; \forall v \in \Gamma_U T^p_q M, \forall f \in \mathcal{F}(U) \tag{B.3.26}$$

Property 96: One can easily check that $(\Gamma_U TM, +, \cdot)$ satisfies all the conditions to be a module on the ring of functions $(\mathcal{F}(U), +, \cdot)$. As we proved before if there exists a smooth global frame on U then $\Gamma_U TM$ can be spanned by a unique $\mathcal{F}(U)$-linear combination of sections belonging to the chosen frame, therefore it is a free module.

For our purposes in this work we are going to consider mainly smooth tensor fields, therefore unless it is explicitly specified we assume from here all the considered 1-forms are smooth sections of the cotangent bundle.

B.3.2 The $c(\mathbb{R})$-constrained tangent tensor bundle $T^p_{q\,c(\mathbb{R})}M$

We are going now to build a very specific bundle that is fundamental foor understanding the multipoles. In fact we in the main body of the thesis how this geometrical structure encodes pieces of information carried by the multipoles in a complete covariant coordinate-free way. We show there exists at least one trivialisation that is characterised by linear transformations of the fiber, therefore it is a vector bundle. This feature is very useful as we see in the main body of the thesis.

Definition 150: Let $c : \mathbb{R} \hookrightarrow M$ a closed embedding. We define the ***rank (p,q) tangent tensor space of*** M ***restricted to the sub-manifold*** $c(\mathbb{R})$ the set:

$$T^p_{q\,c(\mathbb{R})}M = \bigsqcup_{x \in c(\mathbb{R})} T^p_{q\,x}M \tag{B.3.27}$$

Property 97: Since $c : \mathbb{R} \hookrightarrow M$ is an embedding then it is an injective map, therefore distinct elements are mapped into distinct elements. Therefore c is a bijection between $\mathbb{R}$ and the image set $c(\mathbb{R})$. So we have automatically that:

$$T^p_{q\,c(\mathbb{R})}M = \bigsqcup_{x \in c(\mathbb{R})} T^p_{q\,x}M = \bigsqcup_{s \in \mathbb{R}} T^p_{q\,c(s)}M \tag{B.3.28}$$

Property 98: Trivially we have that:

$$T^p_{q\,c(\mathbb{R})}M = \bigsqcup_{x\in c(\mathbb{R})} T^p_{q\,x}M \subset \bigsqcup_{x\in M} T^p_{q\,x}M = T^p_q M \tag{B.3.29}$$

therefore the "rank (p,q) tangent tensor space of M restricted to the sub-manifold $c(\mathbb{R})$" is a subset of the "rank (p,q) tangent tensor space of M".

Definition 151: Due to the property of the disjointed union given $T^p_{q\,c(\mathbb{R})}M$ we can always define a canonical surjective map $\pi : T^p_{q\,c(\mathbb{R})}M \to \mathbb{R}$ such that

$$\pi(T_{c(s)}) = s \quad , \quad \forall T_{c(s)} \in T^p_{q\,c(s)} \subset T^p_{q \quad c(\mathbb{R})} \tag{B.3.30}$$

Let us remark that π is a good surjective map just because $c : \mathbb{R} \hookrightarrow M$ is injective.

Property 99: Let $T^p_q M$ be the tangent tensor space at M and $\hat{\tau}_M : T^p_q M \to M$ the canonical projection induced by the disjointed union. We have that from the definition:

$$\pi = c^{-1} \circ \hat{\tau}_M|_{T^p_{q\,c(\mathbb{R})}M} \tag{B.3.31}$$

Proof. Since τ_M is a good map defined globally on the whole space $T^p_q M$ then its restriction to the subset $T^p_{q\,c(\mathbb{R})}M$ is still a well defined projection. By definition of τ_M and $T^p_{q\,c(\mathbb{R})}M$ we have that $\forall T_x \in T^p_{q\,c(\mathbb{R})}M \Rightarrow \tau_M(T_x) \in c(\mathbb{R})$ therefore the map $c^{-1} \circ \hat{\tau}_M|_{T^p_{q\,c(\mathbb{R})}M}$ is well defined. One can check then that $\forall s \in \mathbb{R}$ we have:

$$c^{-1} \circ \hat{\tau}_M|_{T^p_{q\,c(\mathbb{R})}M} T_{c(s)} = c^{-1}(\hat{\tau}_M|_{T^p_{q\,c(\mathbb{R})}M} T_{c(s)}) = c^{-1}(c(s)) = s \tag{B.3.32}$$

□

Property 100: . Let M be a m-dimensional manifold, let $c \hookrightarrow M$ a closed embedding, let $T^p_{q\,c(\mathbb{R})}M$ be the tangent tensor space restricted to $c(\mathbb{R})$ and let $\pi : T^p_{q\,c(\mathbb{R})}M \to \mathbb{R}$ be the projection defined above . The quadruple $(T^p_{q\,c(\mathbb{R})}M, \mathbb{R}, \pi, \mathbb{R}^{m^{p+q}})$ is a good fiber bundle with standard fiber $\mathbb{R}^{m^{p+q}}$.

Proof. Let us suppose to have fixed an arbitrary trivialisation of $T^p_q M$ denoted by $(U_i, e_{(i)\mu_{\overline{p}}}\otimes$

$e^{\nu\bar{q}}_{(i)}$). Due to the closed embedding c we are able to build easily a trivialisation of $T^{p}_{q\,c(\mathbb{R})}M$ in the following way:

1. Since (U_i) is a covering of M and c is a closed embedding, therefore

$$(I_i) = \{c^{-1}(U_i \cap c(\mathbb{R}))\} \tag{B.3.33}$$

   is a well defined open covering of $\mathbb{R}$.

2. Since $e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)} : U_i \to \tau^{-1}(U_i) \subseteq T^p_q M$ we can define

$$(e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)}) \circ c : I_i \to \pi^{-1}(I_i) \subseteq T^{p}_{q\,c(\mathbb{R})}M \tag{B.3.34}$$

Let us prove that $\left(I_i, (e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)}) \circ c\right)$ induces a well defined trivialization. As stated already (I_i) is a covering of $\mathbb{R}$. For each I_i, for each $s \in I_i \subseteq \mathbb{R}$, we can use $(e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)}) \circ c$ to induce a basis of $T^{p}_{q\,c(s)}M$. Then for each $T_s \in \pi^{-1}(I_i) \mid \pi(T_s) = s$ we have that:

$$T_s = T^{\ \ \mu_{\bar{p}}}_{(i)\nu_{\bar{q}}} [(e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)}) \circ c]_{|_s} \tag{B.3.35}$$

where $T^{\mu_{\bar{p}}}_{(i)\nu_{\bar{q}}}$ is an n-tuple of real number. One can easily check how, fixing $(e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)}) \circ c$ induced by the smooth frame $(e_{(i)\mu_{\bar{p}}} \otimes e^{\nu\bar{q}}_{(i)})$, the map $t_{(i)} : \pi^{-1}(I_i) \to I_i \times \mathbb{R}^{m^{p+q}}$defined as:

$$t_{(i)}(T_s) \to (s, T^{\mu_{\bar{p}}}_{(i)\nu_{\bar{q}}}) \tag{B.3.36}$$

is invertible and differentiable, hence it is a diffeomorphism and it defines a local trivialisation of $T^{p}_{q\,c(s)}M$ (one for each open set U_i in the atlas of M). Therefore by definition of a fiber bundle we have the thesis. □

Definition 152: We define the $c(\mathbb{R})$***-constrained rank (p,q) tangent tensor bundle of*** M to be the quadruple $(T^{p}_{q\,c(\mathbb{R})}M, \mathbb{R}, \pi, \mathbb{R}^{m^{p+q}})$

Property 101: Let $U \subseteq M$ to be an open set such that $U \cap c(\mathbb{R}) \neq \varnothing$. For each local section $\sigma_U : M \to \hat{\tau}^{-1}_M(U) \subseteq T^p_q M$ the closed embedding c induces a local section on

$T^p_{q\,c(\mathbb{R})}M$ as follows:

$$\rho_I : I = c^{-1}(U \cap c(\mathbb{R})) \subset \mathbb{R} \longrightarrow \pi^{-1}(I) \subseteq T^p_q M \tag{B.3.37}$$

such that

$$\rho_I = \sigma_U \circ c \tag{B.3.38}$$

The smoothness of c guarantees that smooth local sections always induce smooth local sections.

Property 102: Given a smooth manifold M and a closed embedding $c : \mathbb{R} \hookrightarrow M$. The $c(\mathbb{R})$-constrained tangent tensor bundle of M admits transition functions in $GL(\mathbb{R}^{m^{p+q}}) \subset$ *Diff* $(\mathbb{R}^{m^{p+q}})$.

Proof. Let $T^p_q M$ be the tangent tensor bundle of M and let us suppose to have the trivialisation $(U_i, \hat{t}_{(i)})$ induced by the choice of some local frames $(e_{(i)\mu_{\overline{p}}} \otimes e^{\nu\overline{q}}_{(i)})$. As we have seen this induces a trivialisation $(I_i, t_{(i)})$ of the bundle $T^p_{q\,c(s)}M$ due to $(e_{(i)\mu_{\overline{p}}} \otimes e^{\nu\overline{q}}_{(i)}) \circ c$, the composition of the local frame with the embedding c. Let us consider two local trivialisations $(I_i, t_{(i)})$ and $(I_j, t_{(j)})$ such that $I_i \cap I_j \neq \varnothing$. Then $\forall s \in I_i \cap I_j$ we have by definition that:

$$T_{(j)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}[(e_{(j)\mu_{\overline{p}}} \otimes e^{\nu\overline{q}}_{(j)}) \circ c]_{|_s} = T_s = T_{(i)\nu_{\overline{q}}}^{\ \ \mu_{\overline{p}}}[(e_{(i)\mu_{\overline{p}}} \otimes e^{\nu\overline{q}}_{(i)}) \circ c]_{|_s} \tag{B.3.39}$$

We showed in the previous section that given two local smooth frames they must be linked with $(e_{(j)\mu_{\overline{p}}} \otimes e^{\nu\overline{q}}_{(j)})_{|x} = \overline{\Lambda}_{(ji)\mu_{\overline{p}}}^{\ \ \alpha_{\overline{p}}} \Lambda_{(ji)\beta_{\overline{q}}}^{\ \ \nu_{\overline{q}}} (e_{(j)\alpha_{\overline{p}}} \otimes e^{\beta\overline{q}}_{(j)})_{|x}$ therefore we have that:

$$[(e_{(j)\mu_{\overline{p}}} \otimes e^{\nu\overline{q}}_{(j)}) \circ c]_{|_s} = \{[\overline{\Lambda}_{(ji)\mu_{\overline{p}}}^{\ \ \alpha_{\overline{p}}} \Lambda_{(ji)\beta_{\overline{q}}}^{\ \ \nu_{\overline{q}}} (e_{(i)\alpha_{\overline{p}}} \otimes e^{\beta\overline{q}}_{(i)})] \circ c\}_{|_s} = \tag{B.3.40}$$

$$= c^\star(\overline{\Lambda}_{(ji)\mu_{\overline{p}}}^{\ \ \alpha_{\overline{p}}})_{|_s} c^\star(\Lambda_{(ji)\beta_{\overline{q}}}^{\ \ \nu_{\overline{q}}})_{|_s} \{[(e_{(i)\alpha_{\overline{p}}} \otimes e^{\beta\overline{q}}_{(i)})] \circ c\}_{|_s} \tag{B.3.41}$$

The given local trivialisation of $T^p_q\ {}_{c(\mathbb{R})}M$ is thence compatible with the vector bundle structure $(T^p_q\ {}_{c(\mathbb{R})}M, \mathbb{R}, \pi, \mathbb{R}^{m^{p+q}}, \lambda, GL(\mathbb{R}^{m^{p+q}}))$ □

Appendix C

K-forms, operations and integration

Forms are a very specific class of tensor field. It turns out that they play a prominent role in geometry, encoding the information of volume of a space and are related to the integral calculus on a manifold. The constraints concerning the existence and the globality of particular classes and chains of forms on a manifold provides a lot of pieces of information related to the topology of the manifold itself. We are not pretending here to provide a detailed description of all the aspects concerning the forms expecially those related to the topological properties, however they are very important both in physics and maths, and an interested reader can find some details on standard textbooks. As usual trying to be pragmatic, we settle here to recap briefly the main concept subtending the wide world of the differential forms, focusing on the element needed to define the multipoles on the differential manifold.

C.1 K-forms as antisymmetric tensor fields

C.1.1 Definitions

Definition 153: Let be M an m dimensional manifold and $U \subseteq M$ an open subset. A scalar field $f \in \mathcal{F}(U)$ is called 0***-form***, a covector field $\alpha \in \Gamma T_1 M$ is called 1***-form***.

Definition 154: Let be M an m dimensional manifold. We define a ***k-form*** (with $k > 2, k \in \mathbb{N}$) a tensor field $\omega \in \Gamma T_k M$ such that

$$\sigma_J(\omega) = (-1)^{\sharp(J)}\omega \quad , \quad \forall J \in \prod(k) \tag{C.1.1}$$

where $\sharp(J)$ is the sign of the permutation J, and $\prod(k)$ is the set of all permutations of k elements. In other words a k-forms ω is a completely antisymmetric $C^\infty(M)$-multilinear map such that $\omega : \times^k \Gamma TM \to C^\infty(M)$

A smooth k-form is usually called k differential form. The set of all the smooth k-form defined on arbitrary subsets of the manifold M is denoted by $\Omega^k(M)$.

Definition 155: Given a k-form $\omega \in \Omega^k(M)$, k is called the ***degree*** of ω. If the degree is not immediately specified ω can be just called ***differential form***. From here we are going to consider just smooth forms, therefore, if not specified, when we refer simply to a form we mean "differential form".

Property 103: Given a form $\omega \in \Omega^k(M)$, with $k > 2$ the contraction with the same vector more than once is always 0.

$$u \lrcorner u \lrcorner \omega = 0 \quad , \quad \forall u \in \Gamma TM \tag{C.1.2}$$

Proof.

$$u \lrcorner u \lrcorner \omega = u \lrcorner u \lrcorner [\sigma_{(12)}(\omega)] = (-1)^{\sharp(12)} u \lrcorner u \lrcorner \omega = -u \lrcorner u \lrcorner \omega \tag{C.1.3}$$

□

Lemma 46: A list of fixed indices $(\hat{\mu}_1, ..., \hat{\mu}_k)$ satisfying the property:

$$\hat{\mu}_i, \hat{\mu}_j \in [0, m-1] \subset \mathbb{N} \, , \; \hat{\mu}_i \neq \hat{\mu}_j \quad \forall i, j \in [1, k] \subset \mathbb{N}^+ \, , \; i \neq j \tag{C.1.4}$$

exists just for $k \leq m$

Proof. Let us suppose by contradiction that such a list can exists for $k > m$. Let us consider two generic distinct elements $\hat{\mu}_{i_1}$ and $\hat{\mu}_{i_2}$ in the list. Since they are distinct elements then $i_2 \neq i_1$ and we have that $\hat{\mu}_{i_1} \neq \hat{\mu}_{i_2}, \forall i_2 \in [1, k]$. This means that:

$$\mu_{i_2} \in \Big\{ [0, m-1] \setminus \{\mu_{i_1}\} \Big\} \tag{C.1.5}$$

If we consider a third arbitrary element μ_{i_3} distinct with respect to the first two, then $i_3 \neq i_2 \neq i_1$, and following the same reasoning we must conclude that:

$$\mu_{i_3} \in \Big\{ [0, m-1] \setminus \{\mu_{i_1}, \mu_{i_2}\} \Big\} \tag{C.1.6}$$

Since $k > m$, iterating the reasoning $m+1$ times we are able to state that for the arbitrary

$(m+1)$-th element in the list:

$$\mu_{i_{m+1}} \in \Big\{ [0, m-1] \setminus \{\mu_{i_1}, ..., \mu_{i_m}\} \Big\} \tag{C.1.7}$$

Now let us remark the set $\{\mu_{i_1}, ..., \mu_{i_m}\}$ by construction is formed by m distinct element picked from the set $[0, m-1]$ hence we have to conclude that:

$$\{\mu_{i_1}, ..., \mu_{i_m}\} \subseteq [0, m-1] \tag{C.1.8}$$

At the same time the number of elements of $\{\mu_{i_1}, ..., \mu_{i_m}\}$ and $[0, m-1]$ are both equal to m and therefore each element belonging to $[0, m-1]$ must belong to $\{\mu_{i_1}, ..., \mu_{i_m}\}$ too. This leads us to:

$$\mu_{i_{m+1}} \in \Big\{ [0, m-1] \setminus \{\mu_{i_1}, ..., \mu_{i_m}\} \Big\} = \varnothing \tag{C.1.9}$$

and we must conclude that the $m+1$-th arbitrary element of this list does not exist, so the list can just admit at most m distinct elements. □

Property 104: Let be M an m dimensional manifold. For $k > m$ we have $\Omega^k = \{0\}$

Proof. Let us suppose by contradiction that for $k > m$ there must exist at least an $\omega \in \Omega^k(M) \,|\, \omega \neq 0$ therefore this means that there must exists at least a bunch of vector fields $u_{\overline{k}} \in \times^k \Gamma TM$ such that $\omega(u_{\overline{k}})$. Hence fixing arbitrarily a local frame $(e_{\mu_{\overline{k}}})$ we can state locally:

$$0 \neq \omega(u_{\overline{k}}) = u^{\mu_{\overline{k}}} \omega(e_{\mu_{\overline{k}}}) = \sum_{\mu_1=0}^{m-1} u^{\mu_1} \cdots \sum_{\mu_k=0}^{m-1} u^{\mu_k} [\omega(e_{\mu_1}, ..., e_{\mu_k})] \tag{C.1.10}$$

where we have explicitely stated the the sum over the dummy indices imposed by the condensed Einstein notation. Considering this, we can notice that the only way to have a non null result is to admits that there must exist at least a list of fixed value indices

$$(\hat{\mu}_1, ..., \hat{\mu}_k) \,|\, \hat{\mu}_i \in [0, m-1]\,,\ \forall i \in [1, k] \tag{C.1.11}$$

such that:

$$\omega(e_{\hat{\mu}_1}, ..., e_{\hat{\mu}_k}) \neq 0 \tag{C.1.12}$$

Now considering the properties of the forms, we can state that the list of fixed indices $(\hat{\mu}_1, ..., \hat{\mu}_k)$ must satisfy the property:

$$\hat{\mu}_i \neq \hat{\mu}_j \quad \forall i, j \in [1, k]\ ,\ i \neq j \tag{C.1.13}$$

otherwise:

$$\omega(e_{\hat{\mu}_1}, ..., e_{\hat{\mu}_i}, ..., e_{\hat{\mu}_i}, ..., e_{\hat{\mu}_k}) = \tag{C.1.14}$$
$$= e_{\hat{\mu}_k} \lrcorner ... \lrcorner e_{\hat{\mu}_{j+1}} \lrcorner e_{\hat{\mu}_{j-1}} \lrcorner ... \lrcorner e_{\hat{\mu}_{i+1}} \lrcorner e_{\hat{\mu}_{i-1}} \lrcorner ... e_{\hat{\mu}_1} \lrcorner e_{\hat{\mu}_i} \lrcorner e_{\hat{\mu}_i} \lrcorner \sigma_{\bar{i}} \sigma_{\bar{j}} \omega = \tag{C.1.15}$$
$$= e_{\hat{\mu}_k} \lrcorner ... \lrcorner e_{\hat{\mu}_{j+1}} \lrcorner e_{\hat{\mu}_{j-1}} \lrcorner ... \lrcorner e_{\hat{\mu}_{i+1}} \lrcorner e_{\hat{\mu}_{i-1}} \lrcorner ... e_{\hat{\mu}_1} \lrcorner e_{\hat{\mu}_i} \lrcorner e_{\hat{\mu}_i} \lrcorner \{(-1)^{\sharp(\bar{i})\sharp(\bar{j})} \omega\} = \tag{C.1.16}$$
$$= (-1)^{\sharp(\bar{i})\sharp(\bar{j})} e_{\hat{\mu}_k} \lrcorner ... \lrcorner e_{\hat{\mu}_{j+1}} \lrcorner e_{\hat{\mu}_{j-1}} \lrcorner ... \lrcorner e_{\hat{\mu}_{i+1}} \lrcorner e_{\hat{\mu}_{i-1}} \lrcorner ... e_{\hat{\mu}_1} \lrcorner \{e_{\hat{\mu}_i} \lrcorner e_{\hat{\mu}_i} \lrcorner \omega\} = \tag{C.1.17}$$
$$= (-1)^{\sharp(\bar{i})\sharp(\bar{j})} e_{\hat{\mu}_k} \lrcorner ... \lrcorner e_{\hat{\mu}_{j+1}} \lrcorner e_{\hat{\mu}_{j-1}} \lrcorner ... \lrcorner e_{\hat{\mu}_{i+1}} \lrcorner e_{\hat{\mu}_{i-1}} \lrcorner ... e_{\hat{\mu}_1} \lrcorner \{0\} = 0 \tag{C.1.18}$$

Therefore to have a non null k-form there must exist at least a list of fixed value indices such that:

$$(\hat{\mu}_1, ..., \hat{\mu}_k) \mid \hat{\mu}_i, \hat{\mu}_j \in [0, m-1]\ ,\ \hat{\mu}_i \neq \hat{\mu}_j \quad \forall i, j \in [1, k]\ ,\ i \neq j \tag{C.1.19}$$

But this is clearly a contradiction because it has been already proven in the previous lemma that such a list cannot exists for $k > m$ □

Property 105: It is easy to check that $\Omega^k(M)$ is closed with respect all the $\mathbb{R}$-linear operations on tensor fields preserving the rank and commuting with the "lower" braiding maps. For this reason we can state that: sum, product by scalar fields, pullback and pushforward along diffeomorphisms, Lie derivative, covariant derivative and higher order covariant derivatives of forms are forms.

Proof. Let be $O : \Gamma T_k M \to \Gamma T_k M$ a generic map on the tensor fields such that it commutes with the "lower" braiding maps. Therefore given a form $\omega \in \Omega^k$ for each permutation $J \in \prod(k)$ we have:

$$\sigma_J[O(\omega)] = O(\sigma_J[\omega]) = O[(-1)^{\sharp(J)}\omega] = (-1)^{\sharp(J)}O[\omega] \tag{C.1.20}$$

and we must conclude that $O(\omega)$ is still a form. $\square$

C.1.2 Specific operations on forms

The space $\Omega^k(M)$ is not closed with respect the tensor product, the covariant differential, the higher order covariant differentials and the contraction by a vector. It is however possible to define on the forms specific operations that are very close related to those mentioned above but are able to map $\Omega^k(M)$ into $\Omega^{k'}(M)$ preserving the antisymmetric structure that characterize the forms.

Definition 156: Given $\Omega^k(M)$ (with $k > 1$) the space of k-form on the manifold M, we define ***the contraction of a form with a vector field*** the map $\lrcorner : \Gamma TM \times \Omega^k(M) \to \Omega^{k-1}(M)$ such that

$$[u \lrcorner \omega](v_{\overline{k-1}}) = \omega(u, v_{\overline{k-1}}) \quad , \quad \forall u \in \Gamma TM \ , \ \forall v_{\overline{k-1}} \in \times^k \Gamma TM \tag{C.1.21}$$

Often the contraction of a form with a vector field u is denoted by i_u

Property 106: Let us notice that the definition of contraction for the forms is a good definition and it coincides with the contraction of tensor in case of antisymmetric contravariant tensor fields. Therefore it inherits all the good properties of $\lrcorner$ i.e. $C^\infty(M)$-linearity

Proof. To prove that the contraction maps k-forms to $k-1$-forms it is enough to check the antisymmetry is preserved. This is always true since the permutations of $k-1$ elements form a subgroup of the permutations of k elements.

$$\sigma_J\{[u \lrcorner \omega]\}(v_{\overline{k-1}}) = \omega(u, v_{P_J(\overline{k-1})}) = (-1)^{\sharp(J)}\omega(u, v_{\overline{k-1}}) \tag{C.1.22}$$

Since it is just the restriction of the usual $\lrcorner$ on the class of antisymmetric tensor fields then it satisfies all the properties of the standard contraction by vector fields. $\square$

Property 107: It is trivial to check due to the antisymmetry that:

$$u \lrcorner v \lrcorner \omega = i_u(i_v \omega) = -i_v(i_u \omega) = -v \lrcorner u \lrcorner \omega \tag{C.1.23}$$

In the same way we can conclude that:

$$i_v i_v \omega = v \lrcorner v \lrcorner \omega = 0 \tag{C.1.24}$$

Inspired by the tensor product we can define a binary operation on forms acting in the same way as the standard tensor product.

Definition 157: Given $\Omega^p(M)$ and $\Omega^q(M)$ the space of k-forms on the manifold M, we define ***the wedge product*** the map $\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$ such that:

$$\alpha \wedge \beta = \frac{1}{p!\, q!} \sum_{K \in \prod(p+q)} (-1)^{\sharp(K)} \sigma_K[\alpha \otimes \beta] \tag{C.1.25}$$

Property 108: It is easy to check the wedge product of two forms is by construction a good antisymmetric covariant tensor and hence it is a form. The $\wedge$ inherits the $C^\infty(M)$ linearity in both the arguments as well as the associativity. Furthermore the following holds:

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \tag{C.1.26}$$

Proof. The first and second are trivial. Tho prove the formula $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ we can consider the explicit definition of the wedge. We have that $\forall u_{\overline{p+q}} \in \times^{p+q}\Gamma TM$:

$$\{\alpha \wedge \beta\}(u_{\overline{p+q}}) = \{\frac{1}{p!\, q!} \sum_{K \in \prod(p+q)} (-1)^{\sharp(K)} \sigma_K[\alpha \otimes \beta]\}(u_{\overline{p}}, u_{\overline{p+q} \setminus \overline{p}}) = \tag{C.1.27}$$

$$= \frac{1}{p!\, q!} \sum_{K \in \prod(p+q)} (-1)^{\sharp(K)} \alpha(u_{P_K(\overline{p})}) \beta(u_{P_K(\overline{p+q} \setminus \overline{p})}) = \tag{C.1.28}$$

$$= \frac{1}{p!\, q!} \sum_{K \in \prod(p+q)} (-1)^{\sharp(K)} \beta(u_{P_K(\overline{q+p} \setminus \overline{p})}) \alpha(u_{P_K(\overline{p})}) = \tag{C.1.29}$$

$$=\{\frac{1}{p!\,q!}\sum_{K\in\prod(p+q)}(-1)^{\sharp(K)}\sigma_K[\beta\otimes\alpha]\}(u_{\overline{q+p}\setminus\overline{p}},u_{\overline{p}})= \tag{C.1.30}$$

$$=[\sigma_{\overline{q+p}\setminus\overline{p}}]^q\Big\{\frac{1}{p!\,q!}\sum_{K\in\prod(p+q)}(-1)^{\sharp(K)}\sigma_K[\beta\otimes\alpha]\Big\}(u_{\overline{p}},u_{\overline{p+q}\setminus\overline{p}})= \tag{C.1.31}$$

$$=(-1)^{\sharp([\sigma_{\overline{p+q}\setminus\overline{p}}]^q)}\Big\{\frac{1}{p!\,q!}\sum_{K\in\prod(p+q)}(-1)^{\sharp(K)}\sigma_K[\beta\otimes\alpha]\Big\}(u_{\overline{p+q}})= \tag{C.1.32}$$

$$=(-1)^{pq}\Big\{\frac{1}{p!\,q!}\sum_{K\in\prod(p+q)}(-1)^{\sharp(K)}\sigma_K[\beta\otimes\alpha]\Big\}(u_{\overline{p+q}})=(-1)^{pq}\{\beta\wedge\alpha\}(u_{\overline{p+q}}) \tag{C.1.33}$$

□

As we stated before, it is easy to check how the forms are not closed with respect the covariant differential. A possibility is to take account just of the total antisymmetric part of the covariant differential of a form:

Definition 158: We define ***the exterior covariant differential*** the map $d^\nabla:\Omega^k\to\Omega^{k+1}$ such that:

$$d^\nabla\omega=\frac{1}{k!}\sum_{K\in\prod(k+1)}(-1)^{\sharp(K)}\sigma_K(\nabla\omega)=\sum_{i=1}^{k+1}(-1)^{i-1}\sigma_{(1i)}(\nabla\omega) \tag{C.1.34}$$

Lemma 47: Given $\omega\in\Omega^k(M)$ the exterior covariant differential $d^\nabla(\omega)$ can always be written as the sum of two $k+1$ forms: the first denoted by $d\omega$ is not dependent on the choice of ∇ while the second one is a $C^\infty(M)$-linear combination of ω depending on Tor

Proof. The proof can be performed by fixing a local frame and then glueing together the results. Let (∂_μ) be the natural local frame induced on the open set $U\subseteq M$ by a local chart.

$$d^\nabla(\omega)_{\mu\alpha_{\overline{k}}}=\frac{1}{k!}\sum_{K\in\prod(k+1)}(-1)^{\sharp(K)}\sigma_K(\nabla\omega)_{\mu\alpha_{\overline{k}}}= \tag{C.1.35}$$

$$=\frac{k+1}{k+1!}\sum_{K\in\prod(k+1)}(-1)^{\sharp(K)}\sigma_K(\nabla\omega)_{\mu\alpha_{\overline{k}}}=(k+1)\nabla\omega_{[\mu\alpha_{\overline{k}}]}= \tag{C.1.36}$$

$$=(k+1)\{\partial_{[\mu}\omega_{\alpha_{\overline{k}}]}-\sum_{i=1}^{k}\Gamma^\beta_{[\mu\alpha_i}\omega_{\alpha_{\overline{i-1}}\underline{\beta}\alpha_{\overline{k}\setminus\overline{i}}]}\}= \tag{C.1.37}$$

$$=(k+1)\{\partial_{[\mu}\omega_{\alpha_{\overline{k}}]}-\sum_{i=1}^{k}\Gamma^\beta_{\big[[\mu\alpha_i]}\omega_{\alpha_{\overline{i-1}}\underline{\beta}\alpha_{\overline{k}\setminus\overline{i}}\big]}\}= \tag{C.1.38}$$

$$=(k+1)\{\partial_{[\mu}\omega_{\alpha_{\overline{k}}]} - \sum_{i=1}^{k}\frac{1}{2}Tor^{\beta}_{[\mu\alpha_i}\omega_{\alpha_{\overline{i-1}}\underline{\beta}\alpha_{\overline{k}\setminus\overline{i}}]}\} = \tag{C.1.39}$$

$$=(k+1)\{\partial_{[\mu}\omega_{\alpha_{\overline{k}}]} - \frac{1}{2}\sum_{i=1}^{k}(-1)^{i-1}Tor^{\beta}_{[\mu\alpha_i}\omega_{\underline{\beta}\alpha_{\overline{i-1}}\alpha_{\overline{k}\setminus\overline{i}}]}\} = \tag{C.1.40}$$

$$=(k+1)\{\partial_{[\mu}\omega_{\alpha_{\overline{k}}]} - \frac{1}{2}\sum_{i=1}^{k}(-1)^{i-1+i}Tor^{\beta}_{[\mu\alpha_1}\omega_{\underline{\beta}\alpha_{\overline{i-1}}\alpha_i\alpha_{\overline{k}\setminus\overline{i}}]}\} = \tag{C.1.41}$$

$$=(k+1)\{\partial_{[\mu}\omega_{\alpha_{\overline{k}}]} + \frac{k}{2}Q(\omega)_{[\mu\alpha_{\overline{k}}]}\} \tag{C.1.42}$$

$$=d\omega_{\mu\alpha_{\overline{k}}} + \frac{k(k+1)}{2}Q(\omega)_{[\mu\alpha_{\overline{k}}]} \tag{C.1.43}$$

where we defined $d\omega_{\mu\alpha_{\overline{k}}} = (k+1)\partial_{[\mu}\omega_{\alpha_{\overline{k}}]}$. The term $\frac{k(k+1)}{2}Q(\omega)_{[\mu\alpha_{\overline{k}}]}$ is the coordinate expression of a completely antisymmetric $C^{\infty}(M)$-linear application acting on ω, therefore can be interpreted as the local expression of a $k+1$ form. The term $d\omega_{\mu\alpha_{\overline{k}}}$ can be interpreted as the local expression of a $k+1$-form as well, in fact it is completely antisymmetric and it changes with the correct transformation rules when another trivialization is chosen (because it is a subtraction of two tensorial quantities). In fact considering the previous calculation:

$$d\omega_{\mu\alpha_{\overline{k}}} = (k+1)\partial_{[\mu}\omega_{\alpha_{\overline{k}}]} = d^{\nabla}(\omega)_{\mu\alpha_{\overline{k}}} - \frac{k(k+1)}{2}Q(\omega)_{[\mu\alpha_{\overline{k}}]} \tag{C.1.44}$$

Therefore there must exist a local section $d\omega$ for each local natural trivialization of the tangent tensor bundle and they glue together with the right tensorial transition function. Hence since $\omega \in \Omega^k(M)$ we can say that $d\omega \in \Omega^{k+1}(M)$. □

Definition 159: We define the ***the differential of forms*** the map

$$d : \Omega^k \to \Omega^{k+1} \tag{C.1.45}$$

such that:

1. It is the usual differential when applied to the scalar function:

$$[d(f)](v) = v(f) \quad , \quad \forall v \in \Gamma TM \, , \forall f \in \Omega^0(M) = C^{\infty}(M) \tag{C.1.46}$$

2. It is $\mathbb{R}$-linear:

$$d(\lambda\omega + \mu\eta) = \lambda d(\omega) + \mu d(\eta) \quad , \quad \forall \mu, \lambda \in \mathbb{R} \, , \, \forall \omega, \eta \in \Omega^k(M) \, , \, \forall k \in [0, m] \subset \mathbb{N} \tag{C.1.47}$$

3. It satisfies the graded Leibniz rule with respect to the $\wedge$. We have $\forall \omega \in \Omega^p(M), \forall \eta \in \Omega^q(M)$, $\forall p, q \in [0, m] \subset \mathbb{N}$:

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta) \tag{C.1.48}$$

4. It is nilpotent:

$$d(d(\omega)) = 0 \quad , \quad \forall \omega \in \Omega^k(M) \, \forall k \in [0, m] \tag{C.1.49}$$

Property 109: Using the definition of differential is very easy to check that fixing the natural trivialization ($e_\mu = \partial_\mu$), the following holds

$$d(\omega)_{\mu_{\overline{k+1}}} = d[\omega](e_{\mu_{\overline{k+1}}}) = (k+1)\partial_{[\mu_1}\omega_{\mu_{\overline{k+1}\setminus\overline{1}}]} \tag{C.1.50}$$

Definition 160: We define ***the n-th exterior covariant differential*** the map $d^{\nabla^n} : \Omega^k \to \Omega^{k+n}$ such that:

$$d^{\nabla^n}\omega = \frac{1}{k!} \sum_{K \in \prod(k+n)} (-1)^{\sharp(K)} \sigma_K(\nabla^n \omega) \tag{C.1.51}$$

Property 110: Given the higher order differential $d^{\nabla^n} : \Omega^k \to \Omega^{k+n}$ the following holds:

$$d^{\nabla}(d^{\nabla^k}(\omega)) = d^{\nabla^{k+1}}(\omega) \tag{C.1.52}$$

Proof.

$$d^{\nabla}[d^{\nabla^n}\omega] = \frac{1}{(k+n)!}\sum_{J\in\prod(k+n+1)}(-1)^{\sharp(J)}\sigma_J\Big(\nabla\Big\{\frac{1}{k!}\sum_{K\in\prod(k+n)}(-1)^{\sharp(K)}\sigma_K(\nabla^n\omega)\Big\}\Big) \tag{C.1.53}$$

If we define a new permutation $K' \in \prod(k+n+1)$ such that:

$$\begin{cases} K'(1) = (1) \\ K'(i+1) = K(i) \quad , \quad \forall i \in [1, k+n] \end{cases} \tag{C.1.54}$$

we can write:

$$d^{\nabla}[d^{\nabla^n}\omega] = \frac{1}{(k+n)!}\sum_{J\in\prod(k+n+1)}(-1)^{\sharp(J)}\sigma_J\Big(\nabla\Big\{\frac{1}{k!}\sum_{K\in\prod(k+n)}(-1)^{\sharp(K)}\sigma_K(\nabla^n\omega)\Big\}\Big) = \tag{C.1.55}$$

$$=\frac{1}{(k+n)!}\sum_{J\in\prod(k+n+1)}(-1)^{\sharp(J)}\sigma_J\Big(\frac{1}{k!}\sum_{K'\in\prod(k+n+1)\,|\,K'(1)=1}(-1)^{\sharp(K')}\sigma'_K\nabla\Big\{(\nabla^n\omega)\Big\}\Big) = \tag{C.1.56}$$

$$=\frac{(k+n)!}{(k+n)!}\frac{1}{k!}\Big(\sum_{K'\in\prod(k+n+1)\,|\,K'(1)=1}(-1)^{\sharp(K')}\sigma'_K\Big(\nabla^{n+1}\omega\Big)\Big)+ \tag{C.1.57}$$

$$+\frac{(k+n)!}{(k+n)!}\frac{1}{k!}\sum_{J\in\prod(k+n+1)\,|\,J(1)\neq 1}(-1)^{\sharp(J)}\sigma_J\Big(\nabla^{n+1}\omega\Big) = \tag{C.1.58}$$

$$=\frac{1}{k!}\sum_{K\in\prod(k+n+1)}(-1)^{\sharp(K)}\sigma_K(\nabla^{n+1}\omega) = d^{\nabla^{k+1}}(\omega) \tag{C.1.59}$$

□

Property 111: Given a connection ∇ such that $Tor = 0$ the following holds:

$$\begin{cases} d^{\nabla^1}\omega = d\omega \\ d^{\nabla^2}\omega = d(d\omega) = 0 \\ d^{\nabla^k}\omega = d^{k-2}(d(d\omega)) = 0 \quad , \quad \forall k > 2 \end{cases} \tag{C.1.60}$$

Proof. Since $Tor = 0$, we have $Q = 0$, then from the previous definition of d^{∇} we can easily realize the thesis. The other statments follow immediately from the property of d and from the property proved above. $\square$

Property 112: Since the forms are antisymmetric tensor fields, it is possible to take the Lie derivative of them. it is possible to show the following properties:

1. the Cartan formula holds: $L_v(\alpha) = (i_v d + d i_v)(\alpha)\ ,\ \forall\alpha \in \Gamma\Lambda M$
2. it commutes with respect to the differential: $L_v(d\alpha) = dL_v(\alpha)$, $\forall\alpha \in \Gamma\Lambda M$
3. the Cartan formula holds: $L_{fv}(\alpha) = fL_v(\alpha) + df \wedge v\lrcorner\alpha$, $\forall\alpha \in \Gamma\Lambda M$, $\quad\forall f \in \Gamma\Lambda^0 M$
4. satisfies the Leibniz rule with respect to the wedge product:$L_v(\alpha\wedge\beta) = L_v(\alpha)\wedge\beta + \alpha\wedge L_v(\beta)$
5. it satisfies:$L_v L_u(\alpha) - L_u L_v(\alpha) = L_{[v,u]}(\alpha)$
6. it satisfies:$L_v(u\lrcorner(\alpha)) - u\lrcorner(L_v(\alpha)) = [v,u]\lrcorner(\alpha)$

C.1.3 Elements concerning orientation and integration of differential forms

The concept of orientability is a fundamental concept subtending the topological and geometrical properties of the differential manifolds. The existence of some structures and operation on $\mathbb{R}^n$ are strongly dependent on the fact that $\mathbb{R}^n$ is an orientable vector space.

Definition 161: Let V be an m-dimensional vector space. Given an arbitrary non null totally antisymmetric maximum rank multilinear map $\omega : \times^m V \to \mathbb{R}$ we can induce an equivalence relation of basis called ***orientation of*** ω:

$$(e_\mu) \overset{\omega}{\sim} (e'_\mu) \Leftrightarrow \frac{\omega(e_0,...,e_{m-1})}{|\omega(e_0,...,e_{m-1})|} = \frac{\omega(e'_0,...,e'_{m-1})}{|\omega(e'_0,...,e'_{m-1})|} \tag{C.1.61}$$

Property 113: Since $\frac{\omega(e_0,...,e_{m-1})}{|\omega(e_0,...,e_{m-1})|} \in \{+1,-1\}$ there are two equivalence classes of bases.

Definition 162: Let be V an m-dimensional vector space and ω a non null totally antisymmetric maximum rank multilinear map. A basis (e_μ) satisfying:

$$\frac{\omega(e_0,...,e_{m-1})}{|\omega(e_0,...,e_{m-1})|} = 1 \tag{C.1.62}$$

is defined a ***right handed basis with respect to*** ω. Its equivalence class is then denoted by $[\omega, R]$. A basis (e_μ) satisfying:

$$\frac{\omega(e_0, ..., e_{m-1})}{|\omega(e_0, ..., e_{m-1})|} = -1 \tag{C.1.63}$$

is defined a ***left handed basis with respect to*** ω. Its equivalence class is then denoted by $[\omega, L]$.

At this point one can ask themmself how many different orientations can be induced by different non null totally antisymmetric maximum rank multilinear maps

Property 114: Two non null totally antisymmetric maximum rank multilinear map define the same orientation if and only if there exists a basis (e_μ) such that:

$$\frac{\omega(e_0, ..., e_{m-1})}{|\omega(e_0, ..., e_{m-1})|} = \frac{\alpha(e_0, ..., e_{m-1})}{|\alpha(e_0, ..., e_{m-1})|} \tag{C.1.64}$$

Proof. Let be (e'_μ) and (e_μ) two arbitrary bases. We know that there must exist a maximum rank linear map $\Lambda : V \to V$ such that $e'_\mu = \Lambda^\nu_\mu e_\nu$. We already proved previously that for a maximum rank totally antisymmetric multilinear map we have $\omega(e'_0, ..., e'_{m-1})det(\Lambda) = \omega(e_0, ..., e_{m-1})$ therefore we can state:

$$\frac{\omega(e'_0, ..., e'_{m-1})}{|\omega(e'_0, ..., e'_{m-1})|} = \frac{det(\Lambda)\omega(e_0, ..., e_{m-1})}{|det(\Lambda)||\omega(e'_0, ..., e_{m-1})|} = \tag{C.1.65}$$

$$= \frac{det(\Lambda)\alpha(e_1, ..., e_m)}{|det(\Lambda)||\alpha(e_0, ..., e_{m-1})|} = \frac{\alpha(e'_0, ..., e'_{m-1})}{|\alpha(e'_0, ..., e'_{m-1})|} \tag{C.1.66}$$

Therefore we have:

$$\frac{\omega(e'_0, ..., e'_{m-1})}{|\omega(e'_0, ..., e'_{m-1})|} = \frac{\omega(e_0, ..., e_{m-1})}{|\omega(e_0, ..., e_{m-1})|} \Leftrightarrow \frac{\alpha(e_0, ..., e_{m-1})}{|\alpha(e_0, ..., e_{m-1})|} = \frac{\alpha(e'_0, ..., e'_{m-1})}{|\alpha(e'_1, ..., e'_m)|} \tag{C.1.67}$$

and

$$(e'_\mu) \overset{\omega}{\sim} (e_\mu) \Leftrightarrow (e'_\mu) \overset{\alpha}{\sim} (e_\mu) \tag{C.1.68}$$

$\square$

Definition 163: Let V be an m-dimensional vector space. Given an arbitrary basis (e_μ) we can define the equivalence relation of non null totally antisymmetric maximum rank multilinear maps inducing the same orientation:

$$\alpha \sim \omega \Leftrightarrow \exists \text{ a basis } (e_\mu) \text{ such that } \frac{\omega(e_0, ..., e_{m-1})}{|\omega(e_0, ..., e_{m-1})|} = \frac{\alpha(e_0, ..., e_{m-1})}{|\alpha(e_0, ..., e_{m-1})|} \tag{C.1.69}$$

This equivalence relation is called ***orientation of*** (e_μ)

Property 115: Fixed a basis $\frac{\omega(e_1,...,e_m)}{|\omega(e_1,...,e_m)|} \in \{+1, -1\}$ therefore there exists two equivalence classes of non null totally antisymmetric maximum rank multilinear maps inducing the same orientation.

Definition 164: Let V be an m-dimensional vector space and (e_μ) a basis. A non null totally antisymmetric maximum rank multilinear map ω satisfying:

$$\frac{\omega(e_1, ..., e_m)}{|\omega(e_1, ..., e_m)|} = 1 \tag{C.1.70}$$

is called ***positive oriented with respect the basis*** (e_μ). Its equivalence class is then denoted by $[(e_\mu), +]$. A non null totally antisymmetric maximum rank multilinear map ω satisfying:

$$\frac{\omega(e_1, ..., e_m)}{|\omega(e_1, ..., e_m)|} = -1 \tag{C.1.71}$$

is defined a ***negative oriented with respect the basis*** (e_μ). Its equivalence class is then denoted by $[(e_\mu), -]$.

Property 116: Let be V a vector space, (e_μ) a basis and (e^μ) the natural induced dual basis of $V^\star$ such that $e^\mu(e_\nu) = \delta^\mu_\nu$. The standard duality relation between vector and covectors fixes a one to one relationship between the equivalence classes $([(e_\mu), +], [(e_\mu), -])$ and $([\omega, L], [\omega, R])$, therefore they share equivalently the same geometrical information about the structures built upon V (i.e. basis and top forms).

Proof. Given a basis (e_μ) there always exist the dual basis (e^μ) and the basis of the top forms $e^0 \wedge ... \wedge e^{m-1}$. The top form $e^0 \wedge ... \wedge e^{m-1}$ induce an orientation. Considering the duality and the wedge definition, the basis (e_μ) belongs to $[e^0 \wedge ... \wedge e^{m-1}, R]$ and $e^0 \wedge ... \wedge e^{m-1}$ belongs to $[(e_\mu), +]$. Due to this, considering that each top form can be expressed in the top form basis as follow $\omega = \tilde{\omega}\, e^0 \wedge ... \wedge e^{m-1}$ we can see there is a one to one relationship (i.e. duality) between the two equivalence classes. □

Considering this there is no difference between the orientation of a top form and the orientation of a basis, therefore on a vector spaces there is no need to distinguish them and we will use just the word ***orientation*** keeping in mind that we can express it as an equivalence class of bases or as an equivalence class of top forms according to our purposes. We can also unify the notation of the equivalence classes of forms and bases to avoid redundant notation identifying the equivalence class fixed by $[e^0 \wedge ... \wedge e^{m-1}, R]$ with the equivalence class fixed by $[e_\mu, +]$. This does not mean that we are identifying the element $e^0 \wedge ... \wedge e^{m-1}$ with the element e_μ. Since there is just the same single concept of orientation both on forms and vectors, then we use $[e_\mu, +]$ to identify it.

Definition 165: Given $\mathbb{R}^m$ the existence of the standard basis (E_μ) fixes a preferred orientation $[(E_\mu), +]$ called ***standard orientation***. The ***standard representative*** for the positive oriented forms with respect to $[(E_\mu), +]$ is given by $E^0 \wedge ... \wedge E^{m-1}$.

Definition 166: Let be U and V two vector spaces and $\phi : U \to V$ a linear invertible map (i.e. an isomoprhism). Given a non null top form ω on V there is a natural way to induce an orientation on U via the pullback of ω

$$\phi^\star(\omega) \tag{C.1.72}$$

Property 117: Since the pullback along an isomorphism of a non null form is always non null and preserves the degree of the form then $\phi^\star(\omega)$ is a good non null top form on U. Therefore it induces equivalence class of bases of U and hence it induces an orientation on U.

Property 118: Le U be an arbitrary p-dimensional subspace of V and π_U each totally antisymmetric multilinear map $\omega : \times^p V \to \mathbb{R} \mid \omega_{|U} \neq 0$ induces an orientation on U.

Proof. Since U is a p-dimensional subspace of V therefore U must be also a p-dimensional vector space. The restriction $\omega_{|U}$ is by definition a non null antisymmetric maximum rank multilinear map on the vector space U therefore it induces equivalent classes of basis of U, hence it induces an orientation on U. □

Property 119: Let U and W be two vector spaces. Let us denote by $V = U \oplus W$ the direct sum vector space endowed with the standard projections $\pi_U : V \to U$ and $\pi_W : V \to W$. Given two non null top forms θ and η on U and W respectively, we can induce naturally an orientation on the direct sum $V = U \oplus W$ via the pullback and the wedge product $\omega = \pi_U^\star\theta \wedge \pi_W^\star\eta$

Proof. This follow directly from the properies of the direct sum, pullback and wedge product Let $(e_1, ..., e_{p-1})$ and $(f_0, ..., f_{q-1})$ be two arbitrary bases of U and W respectively. By definition of direct sum we can say that $(e_0, ..., e_{p-1}, f_0, ..., f_{q-1})$ must be a basis of V. Since θ and η are not non null top form the pull back along the projections $\pi_U^\star\theta$ and $\pi_W^\star\eta$ are non null forms. Then

$$\pi_U^\star\theta \wedge \pi_W^\star\eta = \tilde{\pi}_U^\star\theta\, \tilde{\pi}_W^\star\eta\, e^0 \wedge ... \wedge e^{p-1} \wedge f^0 \wedge ... \wedge f^{p-1} \tag{C.1.73}$$

is a non null top form over V. Therefore it induces equivalence class of bases and forms on V and therefore it fixes an orientation on V. □

Property 120: Let be U and W two vector spaces. Let us denote by $V = U \oplus W$ the direct sum vector space endowed with the standard projections $\pi_U : V \to U$ and $\pi_W : V \to W$ and the standard coprojection $\alpha_U : U \hookrightarrow V$ and $\alpha_W : W \hookrightarrow V$. Fixing a positive oriented basis of U denoted by $(e_0, ..., e_{p-1})$ and ω a non null top form on V, we can induce an orientation on W defining a representative as follow:

$$\alpha_w^\star[i_{\alpha_{U^\star}(e_{p-1})} ... i_{\alpha_{U^\star}(e_0)}(\omega)] \tag{C.1.74}$$

Proof. Trivially one can check that $\alpha_w^\star[i_{\alpha_{U^\star}(e_{p-1})} ... i_{\alpha_{U^\star}(e_0)}\pi_U^\star(\omega)]$ the defined form on W is a good non null top form on W and therefore it induces an orientation in W □

It is very interesting to notice what happes when a vector space U can be decomposed as the direct sum of two identical copies of the same vector space U. The orientation fixed on U induces naturally an orientation on $U \oplus U$ and an orientation on the subspaces embedded into it.

Property 121: Let be U a vector spaces. Let us denote by $V = U \oplus U$ the direct sum vector space endowed with the standard projections $\pi_1 : V \to U$ and $\pi_2 : V \to U$ and the standard coprojection $\alpha_1 : U \hookrightarrow V$ and $\alpha_2 : U \hookrightarrow V$. Fixing a positive oriented basis $(e_0, ..., e_{p-1})$ on U, we induce the ***the orientation of*** $\alpha_1(U)$ ***as a subspace of*** V via the form:

$$\pi_1^\star(e^0 \wedge ... \wedge e^{p-1}) \tag{C.1.75}$$

and we induce the ***the orientation of*** $\alpha_2(U)$ ***as a subspace of*** V

$$i_{\alpha_{U^\star}(e_{p-1})}...i_{\alpha_{U^\star}(e_0)}\Big(\pi_1^\star(e^0\wedge...\wedge e^{p-1})\wedge\pi_2^\star(e^0\wedge...\wedge e^{p-1})\Big) \tag{C.1.76}$$

Proof. Trivially one can check that all of them are good non null top forms on $\alpha_2(U)$ and $\alpha_2(U)$ □

Let us notice that this induced orientation is completely compatible with the natural inclusion and natural projections defined on $U\oplus U$. Since $\mathbb{R}^m = \overset{m}{\oplus}\mathbb{R}$, given an orientation on $\mathbb{R}$ we can induce naturally an orientation compatible with all the projections and coprojections naturally defined on the chains:

$$\mathbb{R}\hookrightarrow\mathbb{R}^2\hookrightarrow...\hookrightarrow\mathbb{R}^{m-1}\hookrightarrow\mathbb{R}^m \tag{C.1.77}$$

$$\mathbb{R}^m\to\mathbb{R}^{m-1}\to...\to\mathbb{R}^2\to\mathbb{R} \tag{C.1.78}$$

Definition 167: Given $\mathbb{R}^m$ and given an orientation $[(e_\mu),+]$ we define the ***orientation operator*** the map $O:\times^m\mathbb{R}^m\to\{+1,-1\}$ such that:

$$\begin{cases} O(x_1,...,x_m)=1 \quad , \quad (x_1,...,x_m)\in[(e_\mu),+] \\ O(x_1,...,x_m)=-1 \quad , \quad (x_1,...,x_m)\in[(e_\mu),+] \\ O(x_1,...,x_m)=0 \quad , \quad \text{otherwise (i.e they are not linearly independent)} \end{cases} \tag{C.1.79}$$

We are going now to introduce here the concept of orientable manifold. There are several equivalent definitions of it, some of them very interesting, deep rooted in the vector bundle theory. Again we are forced to be pragmatic, therefore we are going to give just the definition of orientability propaedeutic to our purposes. The motivation that leads us to define the orientability of a manifold can be quite obscure here, in fact for a full understanding of it one should prove the equivalence between this definition and the more advanced fundamental definition involving advanced branches of maths i.e. homology and fiber bundle. Intuitively one can say that a manifold is orientable if it can be mapped locally into $\mathbb{R}^m$ preserving the concept of orientation when we glue together local representations since $\mathbb{R}^m$ is a vector space. This is enough to motivate the following definition.

Definition 168: A m-dimensional manifold M is defined ***orientable*** if and only if it admits at least an atlas $(U_i, \varphi_{(i)})$ such that, $\forall i, j \mid U_i \cap U_j \neq 0$, and the determinant of the Jacobian $J_{(ij)}$ of the transition functions $\varphi_{(i)} \circ \varphi_{(j)}^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is always positive .

Property 122: An m-dimensional manifold M is defined ***orientable*** if and only if there exists a global smooth from $\omega \in \Gamma_M \Lambda^k M$ such that:

$$\omega_{|x}(v_1, ..., v_m) \neq 0 \quad \forall x \in M, \forall v_{\overline{k}} \in \times^m T_x M \tag{C.1.80}$$

In other words the above holds if and only if there exists a non vanishing smooth global top form on the whole manifold.

Proof. We provide here just a sketch of proof. Let us remark that since $\mathbb{R}$ can always be mapped globally on itself by the identity map, and the determinant Jacobian of the identity map is n so $\mathbb{R}$ is an orientable manifold. Let us suppose that such an atlas exists. Let us denote that by $(U_i, \varphi_{(i)})$. Since $\mathbb{R}^m$ is a vector space let us denote by (E_μ) an arbitary basis and (E^μ) the natural cobasis. The natural cobasis induced by the local chart is then $(dx^\mu_{(i)}) = (\phi^\star_{(i)}(E^\mu))$, therefore a non null smooth local m-form $(dx^1_{(i)} \wedge ... \wedge dx^m_{(i)})$ can be built in the same way we previously built the basis of the local m-forms on $\Lambda^m_U M$. Let us suppose to have a smooth partition of the unity (ψ_i) subordinate to the family of open sets (U_i). We can always define the following top form:

$$\omega = \sum_i \psi_i \big(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)}\big) = \sum_i \psi_i \big(\varphi^\star_{(i)}(E^0 \wedge ... \wedge E^{m-1})\big) \tag{C.1.81}$$

This form is clearly smooth because it is built using a sum of locally smooth forms, pulled back along a local diffeomorphism and modulated with a smooth partition of unity. The only thing we have to check is this form does not vanish in each point of the manifold. For each U_i composing the atlas, let us suppose to have $x \in U_i$ but $x \notin U_i \cap U_j$, $\forall j \neq i$ therefore the restriction of ω to x can be reduced to $\omega_{|x} = (dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})_{|x}$ due to the definition of the partition of the unity subordinate to (U_i). Furthermore by the definition of pullback along a local chart, $\varphi^\star_{(i)}$ is an isomorphism from $\mathbb{R}^{m^{p+q}}$ and $T^p_{qx}M$ for each $x \in U_i$, hence $(dx^1_{(i)} \wedge ... \wedge dx^n_{(i)})_{|x} = \varphi^\star_{(i)}(E^0 \wedge ... \wedge E^{m-1})_{|x}$ is null if and only if $(E^0 \wedge ... \wedge E^{m-1})_{|x^\mu}$ is null. So we have to conclude that $\omega_{|x} = (dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})_x \neq 0$ for each U_i in the atlas, for each point $x \in U_i$, $x \notin U_i \cap U_j$, $\forall j \neq i$. Let us consider now, for each U_i in the atlas, that remaining points $x \in U_i \mid \exists U_j$, $j \neq i \Rightarrow x \in U_i \cap U_j$. In this case we have that

$$\omega_{|x} = \psi_i[(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})]_{|x} + \psi_j[(dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)})]_{|x} = \tag{C.1.82}$$

$$= \psi_i[(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})]_{|x} + \psi_j[\phi^\star_{(i)}\phi_{(i)\star}(dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)})]_{|x} = \tag{C.1.83}$$

$$=\psi_i[(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})]_{|x} + \psi_j[\phi^\star_{(i)}[\phi^{-1}_{(i)}\phi_{(j)}]^\star(E^0 \wedge ... \wedge E^m - 1)]_{|x} = \tag{C.1.84}$$
$$=\psi_i[(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})]_{|x} + \psi_j[\phi^\star_{(i)}(det(J_{(ij)})\, E^0 \wedge ... \wedge E^{m-1})]_{|x} = \tag{C.1.85}$$
$$=\psi_i[(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})]_{|x} + \psi^\star_j[(det(J_{(ij)})dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)})]_{|x} = \tag{C.1.86}$$
$$=\{\psi_i + det(J_{(ij)})\psi_j\}_{|x}(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})_{|x} \tag{C.1.87}$$

By the property of the partition of the unity if $det(J_{(ij)}) > 0$ we have that $\{\psi_i + det(J_{(ij)})\psi_j\}_{|x} \neq 0 \quad , \quad \forall x \in U_i \cap U_j$ and since $[\phi^\star_{(i)}(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})]_{|x}$ is non vanishing $\forall x \in U_i \cap U_j$ we have to conclude that:

$$\omega_{|x} = \{\psi_i + det(J_{(ij)})\psi_j\}_{|x}(dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)})_{|x} \neq 0 \quad , \quad \forall x \in U_i \cap U_j \tag{C.1.88}$$

In the other way we can prove that given a global non vanishing top form on the manifold M we can always build an atlas $(U_i, \phi_{(i)})$ such that $det(J_{(ij)}) > 0$. First of all let us recall that for a scalar function $f \in C^0(\mathbb{R}^m)$ the theorem of the sign permanence implies automatically that

$$f(x) \neq 0, \forall x \in D \subseteq \mathbb{R}^m, \exists x_0 \in \ | \ f(x_0) > 0 \Rightarrow f(x) > 0\, ,\ \forall x \in D \tag{C.1.89}$$

Therefore given the Jacobian of the coordinate changes between two charts $\phi_{(i)}$ and $\phi_{(j)}$, since it is a diffeomorphism, its determinant must be non zero on the whole $U_i \cap U_j$ and we can state that $det(J)$ cannot change its sign on the whole set $U_i \cap U_j$. At this point let us suppose to have a global form ω never vanishing on M, this means that $\forall x \in U_i \cap U_j$ we must satisfy the proper transformation rules for the top forms:

$$\omega = \tilde{\omega}_{(j)}dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)} = \tilde{\omega}_{(j)}det(J_{(ij)})dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)} \quad , \quad \forall x \in U_i \cap U_j \tag{C.1.90}$$

Since $dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)}$ is a basis for the local top forms on $U_{(j)}$, we have $\omega \neq 0 \Leftrightarrow \tilde{\omega}_j \neq 0$ therefore we can conclude that:

$$dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)} = det(J_{(ij)})dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)} \quad , \quad \forall x \in U_i \cap U_j \tag{C.1.91}$$

Since $dx^0_{(j)} \wedge ... \wedge dx^{m-1}_{(j)}$ and $dx^0_{(i)} \wedge ... \wedge dx^{m-1}_{(i)}$ are both valid natural bases induced by the charts $\varphi_{(i)}$ and $\varphi_{(j)}$ on $U_i \cap U_j$ they cannot vanish $\forall x \in U_i \cap U_j$ therefore to satisfy the equation we have that $det(J_{(ij)}) \neq 0 \forall x \in U_i \cap U_j$. Let us suppose there exists at least a point $x_0 \in U_i \cap U_j$ such that $det(J_{ij})_{|x_0} > 0$ therefore due to the sign permanence we can say that $det(J_{(ij)})_{|U_i \cap U_j} > 0$. In the other hand, let us suppose that exists at least a point $x_0 \in U_i \cap U_j$ such that $det(J_{ij})_{|x_0} < 0$, for the sign permanence we can state that $det(J_{ij})_{|U_i \cap U_j} < 0$ but we can always define a new chart $(U_j, \varphi'_{(j)})$ such that:

$$\begin{cases} \varphi'^{0}_{(j)} = -\varphi^{0}_{(j)} \\ \varphi'^{k}_{(j)} = \varphi^{k}_{(j)} \quad , \forall k \in [1, m-1] \end{cases} \tag{C.1.92}$$

The Jacobian of this transformation between $(U_j, \varphi'_{(j)})$ and $(U_j, \varphi_{(j)})$ is just $J_{(j'j)} = diag(-1, 1, ..., 1)$ so $det(J_{(i'i)}) = -1$. Therefore we can define the transformation between the new chart $(U_j, \varphi'_{(j)})$ and the old one $(U_i, \varphi_{(i)})$ just composing the two transformations and the compatibility rule states:

$$dx'^{0}_{(j)} \wedge ... \wedge dx'^{m-1}_{(j)} = det(J_{(jj')})det(J_{(ij)})dx^{0}_{(i)} \wedge ... \wedge dx^{m-1}_{(i)} \quad , \quad \forall x \in U_i \cap U_j \tag{C.1.93}$$

and we can conclude that:

$$dx'^{0}_{(j)} \wedge ... \wedge dx'^{m-1}_{(j)} = -det(J_{(ij)})dx^{0}_{(i)} \wedge ... \wedge dx^{m-1}_{(i)} \quad , \quad \forall x \in U_i \cap U_j \tag{C.1.94}$$

Considering this, the transformation from $(U_j, \varphi'_{(j)})$ to $(U_i, \phi_{(i)})$ must admit a positive determinant, and we can define a new atlas substituting $(U_j, \varphi_{(j)})$ with $(U_j, \varphi'_{(j)})$.

Iterating the process over the charts in the atlas we are able to define a new atlas such that the determinant of the change of charts is always positive. □

From the property above, we can then state that the existence of a global non vanishing top form on a manifold is constrained by the "orientability" of a manifold as well as the existence of a global frame is constrained by the "parallelizability". Inspired by the standard linear algebra we can define now a concept of "volume" on T_xM.

Definition 169: Let be T_xM the tangent space at x of an m-dimensional manifold M. Given a list of vectors $v_{\overline{m}} \in \times^m T_xM$ and a non null top form $\omega \in \Lambda_x M$, we define the ***oriented volume with respect*** ω or equivalently the ***volume charge with respect*** ω bounded by the vectors as:

$$\Omega = \omega(v_1, ..., v_m) = \omega(v_{\overline{m}}) \tag{C.1.95}$$

Definition 170: Let M be an orientable manifold, a global non vanishing top form is called ***volume form***.

Definition 171: Let M be a m-dimensional manifold and $D \subset M$ a k dimensional orientable subset of M. Let $\mathcal{A} = (U_i, \varphi_{(i)})$ be an atlas such that it is a natural restriction on D and is a positive oriented atlas of D . Given $(\psi_{(i)})$ a smooth partition of the unity subordinate to the atlas, and a k-form $\omega \in \Omega^k$ we can define the operation of integration of the k- form upon D, to be:

$$\int_D \omega = \sum_{U_i \in A} \int_{\varphi(D \cap U_i)} (\varphi_{(i)}^{-1})^{\star}(\psi_{(i)}\omega) = \sum_{U_i \in A} \int_{\varphi(D \cap U_i)} \hat{\omega}_{(i)} dx_{(i)}^{\overline{k}} \tag{C.1.96}$$

where

$$\int_{\varphi(D \cap U_i)} \hat{\omega}_{(i)} dx_{(i)}^{\overline{k}} \tag{C.1.97}$$

is the standard Lebesgue integration process defined on subsets of $\mathbb{R}^k$

Property 123: It is possible to show that the given definition does not depend on a specific choice of the used partion of the unity or from the choices of local charts [44].

Property 124: The integration on manifold, share all the well known properties of the standard Lebesgue integration upon $\mathbb{R}^k$. For instance:

1. Linearity:
$$\int_D \omega + \eta = \int_D \omega + \int_D \eta$$
2. Additivity: given D_1 and D_2 such that $D_1 \cap D_2 = \varnothing$ we have:
$$\int_{D_1 \cup D_2} \omega = \int_{D_1} \omega + \int_{D_2} \omega$$
3. Stokes's theorem: given D let ∂D be the oriented boundary we have:
$$\int_D d\omega = \int_{\partial D} \omega$$

It is clear from the given definition that the integration of a k-form can be performed just on k dimensional orientable domain. Even if much more advanced definitions are casted (involving sequences of chains on the manifolds) one is forced to realise that there is no way to perform an integration of differential forms without accounting on the concept of orientability. However, it is possible to define other geometrical objects very closely-related to the differential forms that can be integrated over non-orientable domain called ***densities***. A complete analysis of the theory of the densities is beyond the goal of this work, therefore we settle here just to recall the definition and the main properties.

C.1.4 Densities

Let us remark that, loosely speaking, an orientable manifold admits a smooth definition of volume at each point, or equivalently that each fiber of the tangent bundle admits a concept of volume compatible with the trivialisation. It is very interesting to notice that this definition would be a good definition of "volume" unless the sign. In fact, while the usual standard concept of volume spanned by a set of linearly independent vectors is greater than 0 and null otherwise, the volume form is positive when acting on a righ-handed lists of linearly independent vectors but negative on the left-handed lists. This is the reason because the definition of "charged volume" has been introduced. A top form define a volume on T_xM but taking account on the orientation of the domain bounding that volume by a sign overall. However, the standard volume defined usually on an m-dimensional vector space V (or simply on $\mathbb{R}^m$) must satisfiy $Vol(v_{\overline{m}}) \geq 0 \quad , \quad \forall v_{\overline{m}} \in \times^m V$ and this property is fundamental as long as it is used to define a measure on finite dimensional topological vector spaces. It is possible then to define in a geometrical way some new objects called densities that define in terms of maps the concept of (uncharged) volume on T_xM.

Definition 172: Let us consider an m-dimensional manifold M. Given T_xM the tangent vector space. We define a ***density*** the map:

$$\tilde{\mu} : \times^m (T_xM) \to M \tag{C.1.98}$$

such that: $\forall i \in [1, m]$, $\forall \lambda \in \mathbb{R}$, $\forall v_{\overline{m}} \in \times^m T_xM \Rightarrow$

1. it scales with the absolute value of the scalar:

$$\tilde{\mu}(v_{\overline{i-1}}, \lambda v_i, v_{\overline{m}\setminus\overline{i}}) = |\lambda| \, \tilde{\mu}(v_{\overline{i-1}}, v_i, v_{\overline{m}\setminus\overline{i}}) \tag{C.1.99}$$

2. Given a linear map $\Lambda : T_xM \to T_xM$

$$\tilde{\mu}(\Lambda(v_1), ...\Lambda(v_m)) = |det(\Lambda)| \tilde{\mu}(v_1, ..., v_m) \tag{C.1.100}$$

Observe that a density is not linear in any of its arguments therefore is not a tensor. However we will see that it is possible to induce densities from top forms, in fact they are very closely related.

Property 125: Given M a m-dimensional manifold, let $\omega \in \Lambda^m_x M$ be a tangent top form at $x \in M$. The map $|\omega| : \times^m T_x M \to \mathbb{R}$ such that:

$$|\omega|(v_{\overline{m}}) = |\omega(v_{\overline{m}})| \quad , \quad \forall v_{\overline{m}} \in \times^m T_x M \tag{C.1.101}$$

is a good density tangent at x

Proof. The proof is trivial, it is enough to combine the property of the modulus with the behaviour of the top forms under a generic endomorphism Λ to show that $|\omega|$ satisfies all the requirements to be a good density. □

Definition 173: Let us consider an m-dimensional manifold M. Given $T_x M$ the tangent vector space. We denote with $\tilde{\Lambda}_x M$ the set of all densities tangent at x

Two operations can be naturally induced on densities from the standard sum and multiplication on $\mathbb{R}$.

Definition 174: We define the ***sum of densities*** the map

$$+ : \tilde{\Lambda}_x M \times \tilde{\Lambda}_x M \to \tilde{\Lambda}_x M \tag{C.1.102}$$

such that:

$$[\tilde{\alpha} + \tilde{\beta}](v_m) = \tilde{\alpha}(v_m) + \tilde{\beta}(v_m) \tag{C.1.103}$$

Definition 175: We define the ***multiplication by a scalar*** the map

$$\cdot : \mathbb{R} \times \tilde{\Lambda}_x M \to \tilde{\Lambda}_x M \tag{C.1.104}$$

such that:

$$[\lambda\tilde{\omega}](v_m) = \lambda\tilde{\omega}(v_m) \tag{C.1.105}$$

Property 126: Let us consider an m-dimensional manifold M. Given T_xM the tangent vector space the following properties for the densities holds:

1. $(\tilde{\Lambda}_x M, +, \cdot)$ is a real vector space
2. Let (e_μ) and (e^μ) be a basis on T_xM and $T_x^\star M$ respectively. A basis for $\tilde{\Lambda}_x M$ is given by $|e^1 \wedge ... \wedge e^m|$.
3. $dim(\tilde{\Lambda}_x M) = 1$

Given a k-form over a k-orientable domain it is always possible to associate to it a density at each point of the domain in a following way:

Property 127: Given D a k-orientable domain, let $\omega \in \Lambda_x^k D$ be a tangent top form at $x \in D \subset M$. The map $|\omega| : \times^m \Gamma TD \to \mathbb{R}$ such that:

$$\{|\omega|(v_{\overline{m}})\}_{|x} = |\omega(v_{\overline{m}})|_x \quad , \quad \forall x \in D, \quad \forall v_{\overline{m}} \in \times^m TD \tag{C.1.106}$$

is a good density field over D

Definition 176: Let M be a m-dimensional manifold and $D \subset M$ a k dimensional subset of M. Let $\mathcal{A} = (U_i, \varphi_{(i)})$ be an atlas. Given $(\psi_{(i)})$ a smooth partition of the unity subordinate to the atlas, and a k-form $\omega \in \Omega^k$, we can define the operation of integration of the k- density $|\omega|$ associate to ω upon D, to be:

$$\int_D |\omega| = \sum_{U_i \in A} \int_{\varphi(D \cap U_i)} |(\varphi_{(i)}^{-1})^\star (\psi_{(i)}\omega)| = \sum_{U_i \in A} \int_{\varphi(D \cap U_i)} |\hat{\omega}_{(i)} dx_{(i)}^{\overline{k}}| \tag{C.1.107}$$

where

$$\int_{\varphi(D \cap U_i)} |\hat{\omega}_{(i)}||dx_{(i)}^{\overline{k}}| \tag{C.1.108}$$

is the standard Lebesgue volume integration process defined on subsets of $\mathbb{R}^k$

Property 128: Let M be a m-dimensional manifold and $D \subset M$ a k orientable subset of M. Let D^+ and D^- be the set D endowed respectively with a positive and negative orientation with respect to the one induced by a k-form ω. The following holds:

$$\int_{D^+} \omega = \int_D |\omega| = -\int_{D^-} \omega \tag{C.1.109}$$

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