The tensorial representation of the distributional stress–energy quadrupole and its dynamics

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Abstract

We investigate stress-energy tensors constructed from the covariant derivatives of delta functions on a worldline. Since covariant derivatives are used all the components transform as tensors. We derive the dynamical equations for the components, up to quadrupole order. The components do, however, depend in a non-tensorial way, on a choice of a vector along the worldline. We also derive a number of important results about general multipoles, including that their components are unique, and all multipoles can be written using covariant derivatives. We show how the components of a multipole are related to standard moments of a tensor field, by parallelly transporting that tensor field.

1 Introduction

Finite size objects can be approximated using moments. This makes sense when the object is viewed from a distance. The usual definition of moments involves integrating over a spatial hypersurface of an integrand which involves coordinates. For example one may define the quadrupole of a rank (2,0) tensor $S^{\mu\nu}$ as $\int_{\text{space}} z^a z^b S^{\mu\nu} d^3z$, where $a, b = 1, 2, 3$ and $\mu, \nu = 0, \ldots, 3$. Such an object will not, in general, be tensorial. Instead it will be highly dependent on the choice of coordinates, and there will be no simple way of transforming the expression from one coordinate system to another. In addition it is necessary to choose a foliation of spacelike hypersurfaces over which one can perform the integration. By contrast one can represent an extended object by a distribution (in the Schwartz sense) over a worldline, which may represent the "centre" of the object. This distribution is tensorial in that it acts on appropriate test tensors to give a number. Using this action one can find the transformation rules for the moments. In this article such distributions will be called **multipoles**. The components of the multipoles are intimately related to the standard moments, as we show below.

There are many extended objects one may wish to approximate. For example a scalar field concentrated at one point in space can be represented by a scalar multipole, while the current of an extended charge may be represented by a vector valued multipole. In this article we are primary interested in modelling, as a multipole, the stress-energy tensor of an extended massive object. Unlike the current which can be used directly as a source for electromagnetism, the distributional stress-energy tensor cannot be used directly in Einstein's equations. This is because, unlike Maxwell's equations, Einstein's equations are not linear. Instead one can use them as a source for linearised gravity.

Even if one is not concerned with the effect on gravity, considering the stress-energy multipole is useful. This is because the constraint that it must be divergenceless tells us information about the dynamics of the moments. For the monopole, there is just a single component, which is constant and corresponds to the total mass. For the dipole there are 10 components whose dynamics are completely determined by the Mathisson-Papapetrou-Tulczyjew-Dixon ODEs [1,2]. By contrast for the quadrupole, as well as the 40 components which have ODEs, there are 20 free components [3]. One may consider the dynamics of these free components to be constitutive relations. They could either be posited as part of the model or derived by considering the underlying matter which makes up the extended body. For example, we would expect different results for dust than for a neutron star.

Multipole distributions can be represented in a number of ways. In [3] we discuss the advantages and disadvantages of using two different expressions to represent a multipole over a worldline. These are labelled the **Ellis** [4] and **Dixon** [5–9] representations. The Ellis representation uses partial derivatives and can be applied to multipoles over any line on any manifold. However, it requires complicated coordinate transformation rules for the components, involving derivatives of the Jacobi matrix and integration over the world line [3,10]. The Dixon representation uses covariant derivatives and has the significant advantage that the components are tensors. Another advantage is that the multipole naturally splits into different orders of "poles". That is, it is a sum of a monopole, a dipole, a quadrupole and so on. We refer to this in this as the **Dixon split**.

The price one pays for this is that the manifold needs to have a connection, and one must choose a vector along the worldline, which we call the **Dixon vector**. In general changing the Dixon vector will result in a complicated transformation, which not only mixes orders but involves higher derivatives of the components.

With regards to multipoles over timelike worldlines in general relativity, these constraints are not a problem. Spacetime is endowed with a connection, usually the Levi-Civita connection, and there is a preferred vector over the worldline given by it's tangent. Other possible choices of the Dixon vector are discussed in the conclusion.

The key result of this article, given in section 4, is the derivation of the dynamics of the components of the Dixon stress-energy quadrupole. This is the generalisation of the Mathisson-Papapetrou-Tulczyjew-Dixon equations. These are derived from the divergenceless of the stress-energy tensors and shows how the quadrupole couples to the curvature and derivatives of the curvature. Similar equations have been derived by Steinhoff and Puetzfeld [11]. However their method leads to an implicit equation for the dynamics. By contrast our equations are clearer with the time derivatives of the relevant components given explicitly. The method, which involves commuting covariant derivatives can be extended to arbitrary order multipole. However, as noted in [3], the equations do not completely determine the dynamics of the quadrupole and must be augmented with 20 constitutive relations.

In his work, Dixon makes two conjectures for the dynamics of the components of a quadrupole, which we summarise in section 5. We can compare these to our dynamical equations and see that neither of them couple to the curvature. Thus they do not correspond to the divergenceless condition and are not the generalisation of the Mathisson-Papapetrou–Tulczyjew–Dixon equations for the quadrupole.

In addition to deriving the dynamical equations for stress-energy quadrupole, in this article we also establish important results about the Dixon representation of multipoles in general. In particular we show that all multipoles can be represented as Dixon multipoles and that both the Dixon split and the components are unique. We show how the components can be extracted from a multipole by letting it act on appropriate test tensors. All of these results are valid for an arbitrary tensor multipole of arbitrary order. Thus as well as the stress-energy multipole, they can also be applied to electromagnetic current multipoles. The uniqueness of the splitting and the components is an essential step in the derivation of the dynamical equations, as it is applied to the divergence of the stress-energy tensor.

This article is arranged as follows. In section 2 we give a summary of the stress-energy tensor. In section 3 we give the properties of arbitrary multipoles as described above, namely: the Dixon split

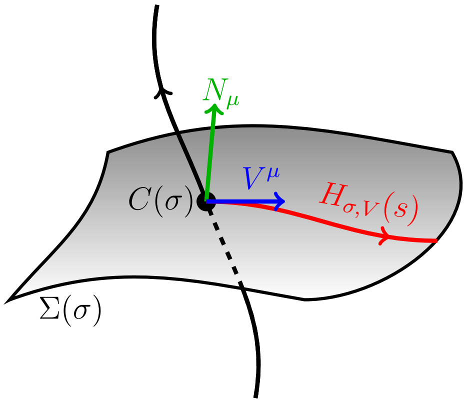


Figure 1: The worldline C (black), with the Dixon geodesic hypersurface $\Sigma(\sigma)$ (grey) which intersects the worldline at $C(\sigma)$. Emanating from $C(\sigma)$ is the geodesic $H_{\sigma,V}(s)$ (red) which lies inside $\Sigma(\sigma)$. The tangent to $H_{\sigma,V}$ at $C(\sigma)$ is the vector V^μ (blue), which is orthogonal to the Dixon vector N_μ (green). All points in the Dixon geodesic hypersurface $\Sigma(\sigma)$ can be reached by a geodesic like $H_{\sigma,V}(s)$.

(section 3.4), the formula for the components (section 3.5) and the demonstration that all multipoles can be written as Dixon multipoles (section 3.7). For this we define a natural coordinate system, which is adapted to the worldline and the Dixon vector (section 3.2). We also define the Dixon geodesic hypersurface which is the hypersurface generated from all geodesics which emanate from the worldline in a direction orthogonal to the Dixon vector (figure 1). We make explicit the two point tensor associated with parallel transport (section 3.3) and give the Taylor expansion of a test tensor which respects parallel transport and the Dixon vector (section 3.6).

As stated the definition of the spatial moments of a extended object is in terms of the integral over space with respect to some coordinate system. Fortunately, once we have chosen a Dixon vector, one can use Dixon geodesic hypersurfaces, and the adapted coordinate system. Alternatively one can parallel transport the tensor to the worldline. Thus the moments defined this way are tensorial objects with tensorial transformation properties. Although they are still non-tensorially dependent on the choice of the Dixon vector. There is a natural way of linking these moments with the components of the multipole distribution. This is done by "squeezing" a regular tensor. This is the process of reducing a tensor's spatial extent while keeping the quantity of matter constant. This is demonstrated in section 3.8.

In section 4 we derive the dynamical equations for the components of the stress-energy quadrupole. In section 5 we compare these the equations proposed by Dixon. Finally in section 6 we conclude.

2 The stress-energy distribution

We shall use the notation as defined in [3]. Let $\mathcal{M}$ be a spacetime with metric $g_{\mu\nu}$, signature $(-,+,+,+)$, and the Levi-Civita connection ∇_μ with Christoffel symbol $\Gamma^\mu_{\nu\rho}$. Here Greek indices $\mu,\nu = 0,1,2,3$ and Latin indices $a,b = 1,2,3$. Let $C : \mathcal{I} \to \mathcal{M}$ where $\mathcal{I} \subset \mathbb{R}$ is the worldline of the source[1] with components $C^\mu(\sigma)$. At this point we do not assume that σ is proper time. Here we consider stress-energy tensors $T^{\mu\nu}$ which are non-zero only on the worldline $C^\mu(\sigma)$, where it has Dirac–δ like properties. Such stress-energy tensors are called **distributional**.

Since we are dealing with distributions it is most convenient to consider $T^{\mu\nu}$ as a tensor density[2] of weight 1. Thus $\omega^{-1}T^{\mu\nu}$ is a tensor, where

$$\omega = \sqrt{-\det(g_{\mu\nu})}\,. \tag{1}$$

[1]Even using proper time in Minkowski space, one cannot assume that $\mathcal{I} = \mathbb{R}$ since it is possible to accelerate to lightlike infinity in finite proper time.

[2]An integral over $\mathcal{M}$ must contain the measure ω. There is therefore the following choice: one can choose $T^{\mu\nu}$ or $\phi_{\mu\nu}$ to be a density of weight 1, or put ω explicitly in the integrand. Here we have chosen to make $T^{\mu\nu}$ a density.

The definition of the covariant derivative of a tensor $Y^{\mu\nu\cdots}$ density of weight 1 is given by

$$\nabla_\mu Y^{\nu\rho\cdots} = \omega \nabla_\mu(\omega^{-1} Y^{\nu\rho\cdots}) = -\Gamma^\kappa_{\mu\kappa} Y^{\nu\rho\cdots} + \partial_\mu Y^{\nu\rho\cdots} + \Gamma^\nu_{\mu\kappa} Y^{\kappa\rho\cdots} + \Gamma^\rho_{\mu\kappa} Y^{\nu\kappa\cdots} + \cdots\,, \tag{2}$$

where $\Gamma^\nu_{\mu\rho}$ are the Christoffel symbols. In this article the term stress-energy tensor always refers to a stress-energy tensor density of weight 1, even if not explicitly stated. In addition the symbol $T^{\mu\nu}$ always refers to a distributional stress-energy tensor density of weight 1 over the worldline C.

As already stated the Dixon representation depends crucially on a choice of a vector field $N_\mu(\sigma)$ along the worldline C, called the **Dixon vector**. The only constraint on the choice of $N_\mu(\sigma)$ is that it is not orthogonal to the worldline C, $N_\mu\,\dot C^\mu \neq 0$. In section 4, we need to project out the spatial components. Thus we scale N_μ so that

$$N_\mu\,\dot C^\mu = 1\,. \tag{3}$$

As long as the worldline C is timelike, a natural choice of the Dixon vector is $\dot C^\mu$, i.e. $N_\mu = -g_{\mu\nu}\,\dot C^\nu$ but this is not the only choice. Having chosen N_μ, the **Dixon** representation of a multipole is given [6, Equation (1.9)] [8, Equation (4.18), (7.4), (7.5)] by

$$T^{\mu\nu} = \sum_{r=0}^{k} \frac{1}{r!} \nabla_{\rho_1} \cdots \nabla_{\rho_r} \int_{\mathcal{I}} \xi^{\mu\nu\rho_1\ldots\rho_r}(\sigma)\,\delta^{(4)}\big(x - C(\sigma)\big)\,d\sigma\,. \tag{4}$$

Tulczyjew [2] calls this the canonical form, in the case when $N_\mu = -\dot C_\mu$.

Since $T^{\mu\nu}$ is a stress-energy tensor, we have the symmetry of the indices

$$T^{\mu\nu} = T^{\nu\mu}\,, \tag{5}$$

which leads to

$$\xi^{\mu\nu\rho_1\ldots\rho_r} = \xi^{\nu\mu\rho_1\ldots\rho_r}\,. \tag{6}$$

We demand that the components $\xi^{\mu\nu\rho_1\ldots\rho_k}$ are orthogonal to the vector N_μ,

$$N_{\rho_j}\,\xi^{\mu\nu\rho_1\ldots\rho_k} = 0 \tag{7}$$

for $j = 1, \ldots, k$. The covariant derivatives do not commute. Instead they give rise to curvature terms and lower the number of derivatives. We therefore make the minimal choice and impose $\xi^{\mu\nu\rho_1\ldots\rho_k}$ are symmetric in the relevant indices.

$$\xi^{\mu\nu\rho_1\ldots\rho_k} = \xi^{\mu\nu(\rho_1\ldots\rho_k)}\,. \tag{8}$$

Since $T^{\mu\nu}$ is a tensor density this enables us to throw the covariant derivative over onto the test tensor $\phi_{\mu\nu}$, giving

$$\int_{\mathcal{M}} T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x = \sum_{r=0}^{k} (-1)^r \frac{1}{r!} \int_{\mathcal{I}} \xi^{\mu\nu\rho_1\ldots\rho_r}(\sigma)\,\big(\nabla_{\rho_1}\cdots\nabla_{\rho_r}\phi_{\mu\nu}\big)\big|_{C(\sigma)}\,d\sigma\,. \tag{9}$$

This follow since if v^μ is a vector density of weight 1 then from (2) $\nabla_\mu\,v^\mu = \partial_\mu\,v^\mu$.

At the quadrupole level the stress-energy distribution becomes

$$T^{\mu\nu} = \int_{\mathcal{M}} \xi^{\mu\nu}\,\delta^{(4)}(z - C)\,d\sigma + \nabla_\rho \int_{\mathcal{M}} \xi^{\mu\nu\rho}\,\delta^{(4)}(z - C)\,d\sigma + \tfrac{1}{2}\nabla_\rho\nabla_\sigma \int_{\mathcal{M}} \xi^{\mu\nu\rho\sigma}\,\delta^{(4)}(z - C)\,d\sigma \tag{10}$$

where from (7)

$$N_\rho \xi^{\mu\nu\rho} = 0 \qquad \text{and} \qquad N_\rho \xi^{\mu\nu\rho\sigma} = 0\,, \tag{11}$$

and from (6) and (8)

$$\xi^{\mu\nu} = \xi^{\nu\mu}, \quad \xi^{\mu\nu\rho} = \xi^{\nu\mu\rho}, \quad \xi^{\mu\nu\rho\sigma} = \xi^{\nu\mu\rho\sigma} \quad \text{and} \quad \xi^{\mu\nu\rho\sigma} = \xi^{\mu\nu\sigma\rho}\,. \tag{12}$$

In this article we assume that $T^{\mu\nu}$ is divergenceless, i.e.

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{13}$$

This gives rise to dynamical equations for the components $\xi^{\nu\mu}$, $\xi^{\nu\mu\rho}$ and $\xi^{\mu\nu\sigma\rho}$, which we give below in theorem 21.

3 Properties of the Dixon representation of arbitrary distributions over worldlines

In this section, general details of multipoles are presented, which are needed to analyse the stress-energy distribution. The most important result we will use, is that the Dixon components are unique (section 3.5). This is needed so that when we take the divergence of the stress-energy tensor and write that as a Dixon distribution, we know that all the terms must vanish.

Since the results are true for all tensor distributions, not simply the stress-energy tensor, we have chosen to derive the results for an arbitrary tensor of rank $(m, 0)$ and order n.

There are a number of concepts we need to define in order to show the uniqueness of the components. First we need to establish the Dixon geodesic hypersurfaces, the adapted coordinate system and the radial vector (section 3.2), the notation for parallel transport (section 3.3) and the Dixon split (section 3.4).

The next step is to show that all multipoles are Dixon multipoles (section 3.7). For this we need to be able to take a Taylor expansion of the test tensor (section 3.6). Since we are dealing with tensors it is necessary to transport the tensors around. There is no unique way of transporting tensors and different choices will lead to different Taylor expansions. The natural choice in this case is to use parallel transport along the geodesics emanating from the worldline.

The final subsection of this section relates the moments of a regular distribution with the components of a multipole. This is achieved by squeezing the distribution.

An arbitrary tensor density distribution of rank $(m, 0)$, weight 1, and order n with support on C is given by

$$J^{\mu_1\cdots\mu_m} = \sum_{k=0}^{N} \frac{1}{k!}\, \nabla_{\rho_1} \dots \nabla_{\rho_k} \int_{\mathcal{I}} \zeta^{\mu_1\cdots\mu_m\rho_1\cdots\rho_k} \delta^{(4)}\big(x - C(\sigma)\big)\, d\sigma\,, \tag{14}$$

where

$$N_{\rho_j} \zeta^{\mu_1\cdots\mu_m\rho_1\cdots\rho_k} = 0 \tag{15}$$

for $j = 1, \dots, k$ and

$$\zeta^{\mu_1\cdots\mu_m\rho_1\cdots\rho_k} = \zeta^{\mu_1\cdots\mu_m(\rho_1\cdots\rho_k)}\,. \tag{16}$$

Unlike $\xi^{\mu\nu\rho_1\cdots\rho_k}$ we do not assume any symmetry on the indices $\mu_1, \dots, \mu_m$. For convenience when dealing with arbitrary tensors we replace the indices $\mu_1 \cdots \mu_m$ with the symbol $\underline{\mu}$ so that $J^{\mu_1\cdots\mu_m} = J^{\underline{\mu}}$ and $\zeta^{\mu_1\cdots\mu_m\rho_1\cdots\rho_k} = \zeta^{\underline{\mu}\rho_1\cdots\rho_k}$.

Let us introduce the notation for the symmetric sum of multiple covariant derivatives

$$\nabla^k_{\rho_1\cdots\rho_k} = \nabla_{(\rho_1} \dots \nabla_{\rho_k)}\,. \tag{17}$$

The result of applying a test tensor $\phi_{\underline{\mu}}$ is given by

$$\int_{\mathcal{M}} J^{\underline{\mu}} \phi_{\underline{\mu}}\, d^4x = \sum_{k=0}^{n} \frac{1}{k!}\, (-1)^k \int_{\mathcal{I}} \zeta^{\underline{\mu}\rho_1\cdots\rho_k}\, (\nabla^k_{\rho_1\cdots\rho_k} \phi_{\underline{\mu}})|_{C(\sigma)}\, d\sigma\,. \tag{18}$$

Although the commutator of two covariant derivatives gives rise to curvature terms, these are of a lower order so we can always write the distribution using the symmetric sum of indices.

The key advantage of imposing the constraints (15), (16) is that they give rise to unique components $\zeta^{\underline{\mu}\rho_1\cdots\rho_k}$. We will see this below is section 3.5.

3.1 Notation and results for covariant derivatives

Given a vector V and a tensor $S^{\nu}_{\underline{\mu}}$ introduce the notation[3]

$$\nabla_V S^{\nu}_{\underline{\mu}} = V^\rho \nabla_\rho S^{\nu}_{\underline{\mu}} \tag{19}$$

and the notation

$$\nabla^r_V S^{\nu}_{\underline{\mu}} = V^{\rho_1} \cdots V^{\rho_k} \nabla^r_{\rho_1 \cdots \rho_k} S^{\nu}_{\underline{\mu}} . \tag{20}$$

Observe that in general $\nabla^r_V S^{\nu}_{\underline{\mu}} \neq (\nabla_V)^r S^{\nu}_{\underline{\mu}}$. However they do coincide when we have a geodesic.

Lemma 1. *If $H(s)$ is a geodesic and $\phi_{\underline{\mu}}$ is a tensor then*

$$\nabla^r_{\dot{H}} \phi_{\underline{\mu}} = (\nabla_{\dot{H}})^r \phi_{\underline{\mu}} . \tag{21}$$

Proof.

$$\begin{aligned}
\nabla^r_{\dot{H}} \phi_{\underline{\mu}} &= \dot{H}^{\rho_1} \cdots \dot{H}^{\rho_r} \nabla^r_{\rho_1 \cdots \rho_r} \phi_{\underline{\mu}} = \dot{H}^{\rho_1} \cdots \dot{H}^{\rho_r} \nabla_{\rho_r} \cdots \nabla_{\rho_1} \phi_{\underline{\mu}} = \dot{H}^{\rho_1} \cdots \dot{H}^{\rho_{r-1}} \nabla_{\dot{H}} \big(\nabla_{\rho_{r-1}} \cdots \nabla_{\rho_1} \phi_{\underline{\mu}} \big) \\
&= \nabla_{\dot{H}} \big(\dot{H}^{\rho_1} \cdots \dot{H}^{\rho_{r-1}} \nabla_{\rho_{r-1}} \cdots \nabla_{\rho_1} \phi_{\underline{\mu}} \big) \\
&\qquad - \Big((\nabla_{\dot{H}} \dot{H}^{\rho_1}) \dot{H}^{\rho_2} \cdots \dot{H}^{\rho_{r-1}} + \cdots + \dot{H}^{\rho_1} \cdots \dot{H}^{\rho_{r-2}} (\nabla_{\dot{H}} \dot{H}^{\rho_{r-1}}) \Big) \nabla_{\rho_{r-1}} \cdots \nabla_{\rho_1} \phi_{\underline{\mu}} \\
&= \nabla_{\dot{H}} \big(\dot{H}^{\rho_1} \cdots \dot{H}^{\rho_{r-1}} \nabla_{\rho_{r-1}} \cdots \nabla_{\rho_1} \phi_{\underline{\mu}} \big) \\
&= \cdots = (\nabla_{\dot{H}})^r \phi_{\underline{\mu}} .
\end{aligned}$$

□

3.2 The Dixon geodesic hypersurface, Dixon adapted coordinate system and the radial vector

Given $\sigma \in \mathcal{I}$, the set of vectors which are perpendicular to N_μ are denoted

$$N^\perp(\sigma) = \big\{ \text{vectors } V^\mu \text{ at the point } C(\sigma) \,\big|\, N_\mu V^\mu = 0 \big\} . \tag{22}$$

Given $V^\mu \in N^\perp(\sigma)$ let $H_{\sigma,V}(s)$ be the geodesic satisfying

$$H_{\sigma,V}(0) = C(\sigma) \qquad \text{and} \qquad \dot{H}^\mu_{\sigma,V}(0) = V^\mu , \tag{23}$$

see figure 1. Note that the parameter s is not normalised, so there is no constraint on the value of $g_{\mu\nu} \dot{H}^\mu_{\sigma,V}(s) \dot{H}^\nu_{\sigma,V}(s) = g_{\mu\nu} V^\mu V^\nu$. The domain of $H_{\sigma,V}(s)$ is distinct from the domain $\mathcal{I}$ of $C(\sigma)$. It will always contain the initial value 0. Although it may not go all the way to $\pm\infty$ it will go to the edge of the Dixon tube which is defined below.

It is useful to label the point $P(\sigma, V) \in \mathcal{M}$ reached from $C(\sigma)$ travelling along the geodesic $H_{\sigma,V}(s)$ a parameter distance 1, that is

$$P(\sigma, V) = H_{\sigma,V}(1) . \tag{24}$$

This gives the points for the geodesic $H_{\sigma,V}$ as

$$H_{\sigma,V}(s) = H_{\sigma,sV}(1) = P(\sigma, s\,V) . \tag{25}$$

All the points $P(\sigma, V)$, which are uniquely defined by (σ, V) form a neighbourhood of C. We call this the **Dixon tube**, $\mathcal{U}_{\mathrm{DT}} \subset \mathcal{M}$. Clearly $C \in \mathcal{U}_{\mathrm{DT}}$. There may be points where two different

[3] We write vectors in the usual index notation as V^μ. However when a vector is an argument to a function or is a subscript we will drop the index and just write V.

geodesics $H_{\sigma,V}$ and $H_{\sigma',V'}$, with $\sigma \neq \sigma'$ intersect. However these will be at some distance from C and outside of $\mathcal{U}_{\rm DT}$. In addition there may be points which are unreachable from C. The set $\big\{(\sigma, V^\mu)\,\big|\,\sigma\in\mathcal{I},\ V^\mu\in N^\perp(\sigma),\ P(\sigma,V)\in\mathcal{U}_{\rm DT}\big\}$, is diffeomorphic to $\mathcal{U}_{\rm DT}$.

Since we are dealing with Schwartz distributions we demand that all test forms have compact support which lie in $\mathcal{U}_{\rm DT}$. This is not a significant restriction as any other test function can be written, using partitions of unity, as the sum of two test tensors, one with support inside $\mathcal{U}_{\rm DT}$ and another test tensor with support away from C. This second test tensor, when acted upon by distributions on C will always give zero.

To define the Dixon adapted coordinate system we require a frame along C, $\{e_0, e_1, e_2, e_3\}$ where $e^\mu(\sigma)$ is a vector at the point $C(\sigma)$. We set $(e_0)^\mu = \dot{C}^\mu$ and require $(e_1)^\mu, (e_2)^\mu, (e_1)^\mu \in N^\perp(\sigma)$. Thus for any $\hat{U}^\mu \in N^\perp(\sigma)$ we can decompose it in terms of this basis, giving $\hat{U}^\mu = U^1(e_1)^\mu + U^2(e_2)^\mu + U^3(e_3)^\mu$.

The **Dixon adapted coordinate system** (σ, z^1, z^2, z^3), is given on the Dixon tube such that

$$\sigma|_{P(\sigma',V)} = P^0(\sigma',V) = \sigma' \qquad \text{and} \qquad z^a|_{P(\sigma',V)} = P^a(\sigma',V) = V^a\,. \tag{26}$$

We set $z^0 = \sigma$ so that we can label the coordinates of a point p by (p^0, p^1, p^2, p^3). We use Latin indices $a, b, \ldots = 1, 2, 3$ and use the summation convention over Latin indices to sum from 1 to 3. Any reference in the article to adapted coordinates, or whenever Latin indices are used we always mean the Dixon adapted coordinate system.

Lemma 2. *In the Dixon adapted coordinate system*

$$H^0_{\sigma,V}(s) = \sigma, \quad H^a_{\sigma,V}(s) = s\,V^a, \quad \dot{H}^0_{\sigma,V} = 0 \quad \text{and} \quad \dot{H}^a_{\sigma,V}(s) = V^a\,. \tag{27}$$

Proof. Equation (27.1) follows[4] from $H^0_{\sigma',V}(s) = H^0_{\sigma',sV}(1) = P(\sigma', sV)^0 = \sigma'$. (27.2) follows from $H^a_{\sigma,V}(s) = H^a_{\sigma,sV}(1) = P(\sigma, sV)^a = sV^a$. For (27.3) and (27.4)

$$\dot{H}^0_{\sigma,V} = \frac{d}{ds}(\sigma) = 0 \qquad \text{and} \qquad \dot{H}^a_{\sigma,V} = \frac{d}{ds}(H^a_{\sigma,sV}(s)) = \frac{d}{ds}(s\,V^a) = V^a\,.$$

□

The **radial vector** field, R^μ, is a vector field on $\mathcal{M}$ given by

$$R^\mu|_{P(\sigma,V)} = \dot{H}^\mu_{\sigma,V}(1)\,. \tag{28}$$

It is a key ingredient for the Dixon split. In [3], the radial vector was not defined completely. Instead, only some of the necessary properties of the radial vector field were given, in order to give the Dixon split up to quadrupole order. Here the specific radial vector field is defined in order to give the Dixon split to arbitrary order.

Lemma 3. *The radial vector has the properties that*

$$R^\mu|_C = 0, \qquad \nabla_{U_1} R^\mu = U_1, \qquad U_1^{\rho_1} U_2^{\rho_2} \cdots U_r^{\rho_r} \nabla^r_{\rho_1\cdots\rho_r} R^\mu = 0\,, \tag{29}$$

for $r \geq 2$, $U_i^{\rho_i} \in N^\perp(\sigma)$.

In the Dixon adapted coordinate system

$$R^0 = 0 \qquad \text{and} \qquad R^a = z^a\,. \tag{30}$$

[4](27.1) refers to the first equation in (27).

Proof. Clearly $R^\mu|_C = 0$.

Fix σ and V and let $H = H_{\sigma,V}$ so that $H(s) = P(\sigma, sV)$ and $s\dot{H}^\mu(s) = R^\mu|_{H(s)}$. Then

$$\nabla_{\dot{H}(s)} R^\mu = \nabla_{\dot{H}(s)}\big(s\dot{H}^\mu(s)\big) = \dot{H}^\mu(s)\,.$$

Hence setting $V = U_1$ and $s = 0$ we get (29.2).

For $r \geq 2$ we have from (21)

$$\nabla^r_{\dot{H}} R^\mu = (\nabla_{\dot{H}})^r R^\mu = (\nabla_{\dot{H}})^{r-1}(\nabla_{\dot{H}} R^\mu) = (\nabla_{\dot{H}})^{r-1}\dot{H}^\mu = 0\,.$$

Hence setting $s = 0$ we have $\nabla^r_V R^\mu = 0$. Now setting V equal to combinations of sums $V = U_1 \pm U_2 \pm \cdots \pm U_r$, and then taking the sums, we are left with (29.3).

From (27.3) we have (30.1). From (27.4) and (26.2) $R^a|_{P(\sigma,V)} = \dot{H}^a_{\sigma,V} = V^a = z^a|_{P(\sigma,V)}$ giving (30.2) □

We use the Dixon vector to define the **Dixon geodesic hypersurface** $\Sigma(\sigma) \subset \mathcal{U}_{\text{DT}}$ as the set

$$\Sigma(\sigma) = \big\{P(\sigma, V) \in \mathcal{U}_{\text{DT}} \,\big|\, V^\mu \in N^\perp(\sigma)\big\}\,. \tag{31}$$

This is given in figure 1.

3.3 Parallel transport

The Dixon geodesic hypersurfaces (31) are constructed from geodesics emanating from the worldline, with initial tangent orthogonal to the Dixon vector. It is necessary to parallel transport tensors along these geodesics. We therefore define a two point tensor $\Pi^\mu_\nu|^p_q$ where p and q lie along the same geodesic. This tensor can be used to parallel transport tensors along the geodesics and hence defined on the Dixon geodesics hypersurfaces. This is seen in lemmas 5, 6 and 7 below.

Given $\sigma \in \mathcal{I}$ and $V^\mu \in N^\perp(\sigma)$ and given $s_0, s_1 \in \mathbb{R}$ such that $P(\sigma, s_0V) \in \mathcal{U}_{\text{DT}}$ and $P(\sigma, s_1V) \in \mathcal{U}_{\text{DT}}$, let

$$\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,s_1V)} : \{\text{vectors at } P(\sigma, s_0V)\} \longrightarrow \{\text{vectors at } P(\sigma, s_1V)\} \tag{32}$$

be the two point tensor satisfying the differential equation

$$\frac{d}{ds}\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)} = -\Gamma^\mu_{\lambda\rho}|_{P(\sigma,sV)}\,\dot{H}^\rho_{\sigma,V}(s)\;\Pi^\lambda_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)} \qquad \text{and} \qquad \Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,s_0V)} = \delta^\mu_\nu\,. \tag{33}$$

Observe that

$$\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,s_1V)}\,\Pi^\nu_\rho\big|^{P(\sigma,s_1V)}_{P(\sigma,s_2V)} = \Pi^\mu_\rho\big|^{P(\sigma,s_0V)}_{P(\sigma,s_2V)} \qquad \text{and} \qquad \Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,s_1V)}\,\Pi^\nu_\rho\big|^{P(\sigma,s_1V)}_{P(\sigma,s_0V)} = \delta^\mu_\rho \tag{34}$$

so that from (34.2)

$$\Pi^\mu_\nu\big|^{P(\sigma,s_1V)}_{P(\sigma,s_1V)} = \delta^\mu_\nu\,. \tag{35}$$

From the definition (33) it may look like the parallel transport is dependent on the choice of parameterisation of the geodesic $H_{\sigma,V}$. We see here that this is not the case.

Lemma 4. *If $s_0V^\mu = \hat{s}_0\hat{V}^\mu$ and $s_1V^\mu = \hat{s}_1\hat{V}^\mu$ then*

$$\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,s_1V)} = \Pi^\mu_\nu\big|^{P(\sigma,\hat{s}_0\hat{V})}_{P(\sigma,\hat{s}_1\hat{V})}\,. \tag{36}$$

I.e. given $p, q \in \mathcal{U}_{\text{DT}}$ along the same $H_{\sigma,V}$, then $\Pi^\mu_\nu|^p_q$ is independent of the choice of parameterisation of $H_{\sigma,V}$.

Proof. Let $V^\mu = \kappa\hat{V}^\mu$ so that $s_0 = \hat{s}_0/\kappa$ and $s_1 = \hat{s}_1/\kappa$. Setting $\frac{d}{ds} = \kappa\frac{d}{d\hat{s}}$ then (33.1) becomes

$$\kappa\frac{d}{d\hat{s}}\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,\hat{s}V)} = -\Gamma^\mu_{\lambda\rho}\big|_{P(\sigma,\hat{s}V)}\,\kappa\frac{d}{d\hat{s}}H^\rho_{\sigma,V}(\hat{s})\;\Pi^\lambda_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,\hat{s}V)}\,,$$

and the κ's cancel. Setting $V^\mu \to \kappa\hat{V}^\mu$ and likewise for s_0 and s_1 we see the ODE (33.1) is independent of scaling of s, V, s_0, s_1. The initial condition follows from (35). □

Lemma 5. *Given (σ, V), a vector $\hat{U}^\mu$ the point $P(\sigma, s_0V)$, and vector field $U^\mu(s)$ along $H_{\sigma,V}$ then*

$$\nabla_{\dot{H}_{\sigma,V}}U^\mu = 0 \qquad \text{and} \qquad U^\mu|_{P(\sigma,s_0V)} = \hat{U}^\mu\,, \tag{37}$$

if and only if

$$U^\mu|_{P(\sigma,sV)} = \Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\;\hat{U}^\nu\,. \tag{38}$$

Proof. Let $H(s) = H_{\sigma,V}(s)$ then

$$\begin{aligned}
\nabla_{\dot{H}(s_1)}\big(\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\;\hat{U}^\nu\big) &= \frac{d}{ds}\big(\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\;\hat{U}^\nu\big)\Big|_{s=s_1} + \Gamma^\mu_{\nu\rho}|_{H(s_1)}\;\dot{H}^\rho(s_1)\;U^\nu(s_1)\\
&= \frac{d}{ds}\big(\Pi^\mu_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\big)\Big|_{s=s_1}\;\hat{U}^\nu + \Gamma^\mu_{\nu\rho}|_{H(s_1)}\;\dot{H}^\rho(s_1)\;U^\nu(s_1)\\
&= -\Gamma^\mu_{\sigma\rho}|_{H(s)}\;\dot{H}^\rho(s)\;\Pi^\sigma_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\;\Big|_{s=s_1}\;\hat{U}^\nu + \Gamma^\mu_{\nu\rho}|_{H(s_1)}\;\dot{H}^\rho(s_1)\;U^\nu(s_1)\\
&= -\Gamma^\mu_{\sigma\rho}|_{H(s_1)}\;\dot{H}^\rho(s_1)\;\Pi^\sigma_\nu\big|^{P(\sigma,s_0V)}_{P(\sigma,s_1V)}\;\hat{U}^\nu + \Gamma^\mu_{\nu\rho}|_{H(s_1)}\;\dot{H}^\rho(s_1)\;U^\nu(s_1)\\
&= 0\,.
\end{aligned}$$

Hence result. □

Lemma 6. *Given a 1–form $\hat{\phi}_\mu$ at the point $P(\sigma, s_0V)$, then*

$$\nabla_{\dot{H}_{\sigma,V}}\phi_\mu = 0 \quad \text{and} \quad \phi_\mu|_{P(\sigma,s_0V)} = \hat{\phi}_\mu\,, \tag{39}$$

if and only if

$$\phi_\mu|_{P(\sigma,sV)} = \Pi^\nu_\mu\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\;\hat{\phi}_\nu\,. \tag{40}$$

Proof. Given a vector $\hat{U} \in T_{H(s_0)}\mathcal{M}$ and a vector field $U(s)$ along H then such that (38) then

$$\begin{aligned}
\nabla_{\dot{H}(s_1)}\big(\Pi^\nu_\mu\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\;\hat{\phi}_\nu\big)\big|_{s_1}U^\mu(s_1) &= \nabla_{\dot{H}(s_1)}\big(\Pi^\nu_\mu\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\;\hat{\phi}_\nu\big)\big|_{s_1}U^\mu(s_1) + \Pi^\nu_\mu(s_1,s_0)\;\hat{\phi}_\nu\nabla_{\dot{H}(s_1)}U^\mu\\
&= \nabla_{\dot{H}(s_1)}\big(\Pi^\nu_\mu\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\;\hat{\phi}_\nu U^\mu(s)\big)\big|_{s_1}\\
&= \frac{d}{ds}\big(\Pi^\nu_\mu\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\;\hat{\phi}_\nu\;\Pi^\mu_\rho\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\;\hat{U}^\rho\big)\big|_{s_1} = \frac{d}{ds}\big(\hat{\phi}_\nu\hat{U}^\nu\big)\big|_{s_1} = 0\,.
\end{aligned}$$

Hence (40) if and only if (39). □

We need to extend the results of lemma 6 for tensors with arbitrary number of indices, i.e. $\phi_{\underline{\mu}}$. Let

$$\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)} = \Pi^{\mu_1}_{\nu_1}\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\cdots\Pi^{\mu_m}_{\nu_m}\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\,. \tag{41}$$

Lemma 7. *Given a tensor $\hat{\phi}_{\underline{\mu}}$ at the point at the point $P(\sigma, s_0V)$, then*

$$\nabla_{\dot{H}_{\sigma,V}}\phi_{\underline{\mu}} = 0 \quad \text{and} \quad \phi_{\underline{\mu}}|_{P(\sigma,s_0V)} = \hat{\phi}_{\underline{\mu}}\,, \tag{42}$$

if and only if

$$\phi_{\underline{\mu}}|_{P(\sigma,sV)} = \Pi^{\underline{\nu}}_{\underline{\mu}}\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\;\hat{\phi}_{\underline{\nu}}\,. \tag{43}$$

Proof. Set $\phi_{\underline{\mu}}$ to be the outer product of 1–forms $\phi_{\underline{\mu}} = \phi^1_{\mu_1} \cdots \phi^m_{\mu_m}$. Then apply lemma 6. $\square$

Lemma 8. *Let $\phi_{\underline{\mu}}$ be a 1–form field over $H_{\sigma,V}$. Then*

$$\frac{d^k}{ds^k}\Big(\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,sV)}_{P(\sigma,s_0V)}\phi_{\underline{\mu}}|_{P(\sigma,sV)}\Big) = \nabla^k_{\dot{H}_{\sigma,V}}\phi_{\nu}|_{P(\sigma,s_0V)}\,. \tag{44}$$

Proof. Proof by induction on r. Clearly true when $r=0$ and $r=1$, Let $H(s)=H_{\sigma,V}(s)$. Assume true for r,

$$\begin{aligned}
\frac{d^{r+1}}{ds^{r+1}}\Big(\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,s_0V)}_{P(\sigma,sV)}\phi_{\underline{\mu}}|_{P(\sigma,sV)}\Big) &= \frac{d}{ds_2}\bigg(\frac{d^r}{ds_1^r}\Big(\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,s_0V)}_{P(\sigma,s_1V)}\phi_{\underline{\mu}}|_{P(\sigma,s_1V)}\Big)\Big|_{s_1=s_2}\bigg)\bigg|_{s_2=s_0}\\
&= \frac{d}{ds_2}\bigg(\frac{d^r}{ds_1^r}\Big(\Pi^{\underline{\mu}}_{\underline{\rho}}\big|^{P(\sigma,s_0V)}_{P(\sigma,s_2V)}\;\Pi^{\underline{\rho}}_{\underline{\nu}}\big|^{P(\sigma,s_2V)}_{P(\sigma,s_1V)}\;\phi_{\underline{\mu}}|_{P(\sigma,s_1V)}\Big)\Big|_{s_1=s_2}\bigg)\bigg|_{s_2=s_0}\\
&= \frac{d}{ds_2}\bigg(\Pi^{\underline{\mu}}_{\underline{\rho}}\big|^{P(\sigma,s_0V)}_{P(\sigma,s_2V)}\;\frac{d^r}{ds_1^r}\Big(\Pi^{\underline{\rho}}_{\underline{\nu}}\big|^{P(\sigma,s_2V)}_{P(\sigma,s_1V)}\;\phi_{\underline{\mu}}|_{P(\sigma,s_1V)}\Big)\Big|_{s_1=s_2}\bigg)\bigg|_{s_2=s_0}\\
&= \frac{d}{ds_2}\bigg(\Pi^{\underline{\mu}}_{\underline{\rho}}\big|^{P(\sigma,s_0V)}_{P(\sigma,s_2V)}\;\nabla^r_{\dot{H}}\phi_{\underline{\mu}}|_{P(\sigma,s_2V)}\bigg)\bigg|_{s_2=s_0}\\
&= \nabla_{\dot{H}}\nabla^r_{\dot{H}}\phi_{\underline{\nu}}|_{P(\sigma,s_0V)} = \nabla_{\dot{H}}(\nabla_{\dot{H}})^r\phi_{\underline{\nu}}|_{P(\sigma,s_0V)} = \nabla^{r+1}_{\dot{H}}\phi_{\underline{\nu}}|_{P(\sigma,s_0V)}\,.
\end{aligned}$$

$\square$

It is necessary to define a collection of tensor fields $\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}$, which are used to define the test tensors needed to extract the components $\zeta^{\underline{\mu}\rho_1\cdots\rho_k}$ in section 3.5. These tensors are parallel and form a basis for tensors. These will be necessary to define a Taylor expansion of the test tensors and for extracting the components of a multipole.

$$\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}|_{P(\sigma,V)} = \Pi^{\underline{\nu}}_{\underline{\mu}}\big|^{P(\sigma,V)}_{C(\sigma)}\,. \tag{45}$$

The reason for placing the $\underline{\nu}$ in curly brackets is because these are tensor indices referring to the point $C(\sigma)$. Therefore the covariant derivative ∇_ρ does not produce the Christoffel symbols $\Gamma^{\nu_i}_{\rho\sigma}$. I.e.

$$\nabla_\rho\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} = \partial_\rho\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} - \Gamma^{\sigma}_{\rho\mu_1}\,\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\sigma\mu_2\cdots\mu_m} - \cdots - \Gamma^{\sigma}_{\rho\mu_m}\,\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\mu_1\cdots\mu_{m-1}\sigma}\,. \tag{46}$$

Hence from lemma 7,

$$\nabla_{\dot{H}_{\sigma,V}}\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} = 0\,. \tag{47}$$

Lemma 9. *In adapted coordinates*

$$\nabla^k_{a_1\cdots a_k}\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} = 0\,. \tag{48}$$

Proof. Fix σ and $V^\mu \in N^\perp$ and let $H = H_{\sigma,V}$ so that $H(s) = P(\sigma,sV)$. Since H is a geodesic

$$V^{a_1}\cdots V^{a_k}\nabla^k_{a_1\cdots a_k}\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}\big|_{C(\sigma)} = \nabla^k_V\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}\big|_{C(\sigma)} = \nabla^k_{\dot{H}}\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}\big|_{C(\sigma)} = 0\,.$$

Now setting V^μ equal to combinations of sums $V = U^\mu_1 \pm U^\mu_2 \pm \cdots \pm U^\mu_r$, and then taking the sums, we are left with

$$U^{a_1}_1\cdots U^{a_k}_k\nabla^k_{a_1\cdots a_k}\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}\big|_{C(\sigma)} = 0\,,$$

since $\nabla^k_{a_1\cdots a_k}$ is symmetric in $a_1,\ldots,a_k$. Since this is true for all vectors $V^\mu = U^\mu_1,\ldots,U^\mu_r$ at $C(\sigma)$ we have (48)

$\square$

3.4 The Dixon split

Given an arbitrary Dixon multipole $J^{\underline{\mu}}$, or order n, let

$$J^{\underline{\mu}} = \sum_{k=0}^{n} J^{\underline{\mu}}_{(k)}\,, \tag{49}$$

where

$$J^{\underline{\mu}}_{(k)} = \frac{1}{k!}\nabla_{\rho_1\ldots\rho_k}\int \zeta^{\underline{\mu},\rho_1\cdots\rho_k}\delta^{(4)}(z-C)\,d\sigma\,. \tag{50}$$

I.e.

$$J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] = (-1)^k\frac{1}{k!}\int \zeta^{\underline{\mu},\rho_1\cdots\rho_k}\nabla^k_{\rho_1\ldots\rho_k}\phi_{\underline{\mu}}\,d\sigma\,. \tag{51}$$

We call the set $\left\{J^{\underline{\mu}}_{(k)}\right\}$ the **Dixon split** of $J^{\underline{\mu}}$. It is the first step towards the extraction of the components.

Lemma 10.

$$J^{\underline{\mu}}_{(k)}[\nabla^r_R\,\phi_{\underline{\mu}}] = \begin{cases} \dfrac{k!}{(k-r)!}\;J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] & \text{for}\quad r\le k\\ 0 & \text{for}\quad r>k\,. \end{cases} \tag{52}$$

Proof. This follows from two observations

$$\nabla^{r+1}_R\phi_{\underline{\mu}} = -r\nabla^r_R\phi_{\underline{\mu}} + \nabla^r_R(\nabla_R\phi_{\underline{\mu}})\,, \tag{53}$$

and

$$J^{\underline{\mu}}_{(k)}[\nabla_R\phi] = k\,J^{\underline{\mu}}_{(k)}[\phi]\,. \tag{54}$$

Equation (53) follows since in the Dixon adapted coordinates.

$$\begin{aligned}
\nabla^r_R(\nabla_R\phi_{\underline{\mu}}) &= \nabla^r_R(z^b\nabla_b\phi_{\underline{\mu}})\\
&= z^{a_1}\ldots z^{a_r}\nabla_{a_1}\ldots\nabla_{a_r}(z^b\nabla_b\phi_{\underline{\mu}})\\
&= z^{a_1}\ldots z^{a_r}\nabla_{a_1}\ldots\nabla_{a_{r-1}}(\delta^b_{a_r}\nabla_b\phi_{\underline{\mu}} + z^b\nabla_{a_r}\nabla_b\phi_{\underline{\mu}})\\
&= z^{a_1}\ldots z^{a_r}\nabla_{a_1}\ldots\nabla_{a_{r-1}}(\nabla_{a_r}\phi_{\underline{\mu}} + z^b\nabla_{a_r}\nabla_b\phi_{\underline{\mu}})\\
&= z^{a_1}\ldots z^{a_r}\nabla_{a_1}\ldots\nabla_{a_r}\phi_{\underline{\mu}} + z^{a_1}\ldots z^{a_r}\nabla_{a_1}\ldots\nabla_{a_{r-1}}(z^b\nabla_{a_r}\nabla_b\phi_{\underline{\mu}})\\
&= \nabla^r_R\phi_{\underline{\mu}} + z^{a_1}\ldots z^{a_r}\nabla_{a_1}\ldots\nabla_{a_{r-1}}(z^b\nabla_{a_r}\nabla_b\phi_{\underline{\mu}})\\
&= r\nabla^r_R\phi_{\underline{\mu}} + z^{a_1}\ldots z^{a_r}z^b\nabla_{a_1}\ldots\nabla_{a_r}\nabla_b\phi_{\underline{\mu}}\\
&= r\nabla^r_R\phi_{\underline{\mu}} + \nabla^{r+1}_R\phi_{\underline{\mu}}\,.
\end{aligned}$$

Equation (54) follows from

$$
\begin{aligned}
J^{\underline{\mu}}_{(k)}[\nabla_R\,\phi_{\underline{\mu}}] &= J^{\underline{\mu}}_{(k)}[z^b\nabla_b\phi_{\underline{\mu}}] = (-1)^k\frac{1}{k!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\nabla^k_{a_1\cdots a_k}\Big(z^b\nabla_b\phi_{\underline{\mu}}\Big)\,d\sigma \\
&= (-1)^k\frac{1}{k!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\nabla_{a_1}\ldots\nabla_{a_k}\Big(z^b\nabla_b\phi_{\underline{\mu}}\Big)\,d\sigma \\
&= (-1)^k\frac{1}{k!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\nabla_{a_1}\ldots\nabla_{a_{k-1}}\Big(\delta^b_{a_k}\nabla_b\phi_{\underline{\mu}} + z^b\nabla_{a_k}\nabla_b\phi_{\underline{\mu}}\Big)\,d\sigma \\
&= (-1)^k\frac{1}{k!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\nabla_{a_1}\ldots\nabla_{a_{k-1}}\Big(\nabla_{a_k}\phi_{\underline{\mu}} + z^b\nabla_{a_k}\nabla_b\phi_{\underline{\mu}}\Big) \\
&= (-1)^k\frac{1}{k!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\Big(\nabla_{a_1}\ldots\nabla_{a_k}\phi_{\underline{\mu}} + \nabla_{a_1}\ldots\nabla_{a_{k-1}}(z^b\nabla_{a_k}\nabla_b\phi_{\underline{\mu}})\Big)\,d\sigma \\
&= (-1)^k\frac{1}{k!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\Big(k\nabla_{a_1}\ldots\nabla_{a_k}\phi_{\underline{\mu}} + z^b\nabla_{a_1}\ldots\nabla_{a_k}\nabla_b\phi_{\underline{\mu}}\Big)\,d\sigma \\
&= (-1)^k\frac{1}{(k-1)!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\nabla_{a_1}\ldots\nabla_{a_k}\phi_{\underline{\mu}}\,d\sigma \\
&= (-1)^k\frac{1}{(k-1)!}\int \varsigma^{\underline{\mu}a_1\ldots a_k}\nabla_{a_1\cdots a_k}\phi_{\underline{\mu}}\,d\sigma \\
&= kJ^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}]\,.
\end{aligned}
$$

We now show (52) by induction. Trivial for $r=0$. Assume true for r, from (53) and (54)

$$
\begin{aligned}
J^{\underline{\mu}}_{(k)}[\nabla^{r+1}_R\,\phi_{\underline{\mu}}] &= -rJ^{\underline{\mu}}_{(k)}[\nabla^r_R\phi_{\underline{\mu}}] + J^{\underline{\mu}}_{(k)}[\nabla^r_R(\nabla_R\phi_{\underline{\mu}})] = \frac{k!}{(k-r)!}\big(-rJ^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] + J^{\underline{\mu}}_{(k)}[\nabla_R\phi_{\underline{\mu}}]\big) \\
&= \frac{k!}{(k-r)!}\big(-rJ^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] + kJ^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}]\big) = \frac{k!(k-r)}{(k-r)!}J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] = \frac{k!}{(k-(r+1))!}J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}]\,.
\end{aligned}
$$

In the case when $r=k+1$ then

$$
J^{\underline{\mu}}_{(k)}[\nabla^{k+1}_R\,\phi_{\underline{\mu}}] = \frac{k!(k-k)}{(k-r)!}J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] = 0\,.
$$

Hence (52) holds for all r. □

Theorem 11.

$$
J^{\underline{\mu}}_{(r)}[\phi_{\underline{\mu}}] = \sum_{k=r}^{n}\frac{(-1)^{k-r}}{(k-r)!\;r!}J^{\underline{\mu}}[\nabla^k_R\phi_{\underline{\mu}}]\,. \tag{55}
$$

Proof.

$$
\begin{aligned}
\sum_{k=r}^{n}\frac{(-1)^{k-r}}{(k-r)!\;r!}J^{\underline{\mu}}[\nabla^k_R\phi_{\underline{\mu}}] &= \sum_{k=r}^{n}\frac{(-1)^{k-r}}{(k-r)!\;r!}\sum_{\ell=0}^{n}J^{\underline{\mu}}_{(\ell)}[\nabla^k_R\phi_{\underline{\mu}}] = \sum_{k=r}^{n}\sum_{\ell=0}^{n}\frac{(-1)^{k-r}}{(k-r)!\;r!}\frac{\ell!}{(\ell-k)!}J^{\underline{\mu}}_{(\ell)}[\phi_{\underline{\mu}}] \\
&= \sum_{\ell=0}^{n}\sum_{k=r}^{n}(-1)^{k-r}\binom{k}{r}\binom{\ell}{k}J^{\underline{\mu}}_{(\ell)}[\phi_{\underline{\mu}}] = \sum_{\ell=0}^{n}\left(\sum_{k=0}^{n}(-1)^{k-r}\binom{k}{r}\binom{\ell}{k}\right)J^{\underline{\mu}}_{(\ell)}[\phi_{\underline{\mu}}] \\
&= \sum_{\ell=0}^{n}\delta_{r\ell}J^{\underline{\mu}}_{(\ell)}[\phi_{\underline{\mu}}] = J^{\underline{\mu}}_{(r)}[\phi_{\underline{\mu}}]\,,
\end{aligned}
$$

where we have used the standard result for the sums of products of binomial coefficients. □

Although the Dixon split, given by (49), is defined via the coordinate system, theorem 11 shows that the Dixon split is actually independent of the coordinate system, once the Dixon vector is chosen.

Corollary 12. *If $J^{\underline{\mu}}=0$ then $J^{\underline{\mu}}[\nabla^r_R\phi_{\underline{\mu}}]=0$ for all r.*

Proof. Follows trivially since if $J^{\underline{\mu}}=0$, then $J^{\underline{\mu}}_{(k)}=0$ for all $0\le k\le n$. □

3.5 Extraction of the Dixon components

In this section we show how we can extract the Dixon components $\zeta^{\underline{\mu}a_1\cdots a_k}$, in an adapted coordinate system, by applying the distribution to particular test tensors.

Let $\psi_0 : \mathbb{R} \to \mathbb{R}$ be any test function such that

$$\int_{\mathbb{R}} \psi_0(z)\, dz = 1, \qquad \psi_0(0) = 1\,,$$

and $\psi_0(z)$ is flat in an interval about 0.

In Dixon adapted coordinates, given $\sigma_0 \in \mathcal{I}$ and $\epsilon > 0$, choose a set of indices $\underline{\nu}$. Let the tensor $(\phi_{\sigma_0,\epsilon}^{\underline{\nu},a_1\cdots a_r})_{\underline{\mu}}$ be given by

$$(\phi_{\sigma_0,\epsilon}^{\underline{\nu},a_1\cdots a_r})_{\underline{\mu}}|_{(\sigma,z^1,\ldots,z^3)} = \frac{(-1)^k}{\epsilon} z^{a_1}\cdots z^{a_r}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(z^1)\psi_0(z^2)\psi_0(z^3)\overline{\mathbf{\Pi}}_{\underline{\mu}}^{\{\underline{\nu}\}}\,. \tag{56}$$

Theorem 13.

$$\zeta^{\underline{\nu}a_1\cdots a_k}(\sigma_0) = \lim_{\epsilon\to 0} J^{\underline{\mu}}_{(k)}[(\phi_{\sigma_0,\epsilon}^{\underline{\nu},a_1\cdots a_r})_{\underline{\mu}}]\,. \tag{57}$$

Proof. Since $\psi_0(z^a)$ is flat about $z^a = 0$, then $\nabla_b\psi_0(z^a) = 0$. Since $\zeta^{\underline{\nu}a_1\cdots a_k}$ is totally symmetric in $a_1 \ldots a_k$ we have from (48) $\nabla^k_{a_1\cdots a_k}\overline{\mathbf{\Pi}}_{\underline{\mu}}^{\{\underline{\nu}\}} = 0$. Also $\nabla_b z^a = \partial_b z^a = \delta^a_b$.

$$\begin{aligned}
\lim_{\epsilon\to 0} J^{\underline{\mu}}_{(k)}[(\phi_{\sigma_0,\epsilon}^{\underline{\nu},a_1\cdots a_r})_{\underline{\mu}}] &= \lim_{\epsilon\to 0}\frac{1}{k!\,\epsilon}\int_{\mathcal{I}}\zeta^{\underline{\mu}b_1\cdots b_k}\nabla^k_{b_1\cdots b_k}\Big(z^{a_1}\cdots z^{a_k}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(z^1)\psi_0(z^2)\psi_0(z^3)\overline{\mathbf{\Pi}}_{\underline{\mu}}^{\{\underline{\nu}\}}\,d\sigma\Big)\\
&= \lim_{\epsilon\to 0}\frac{1}{k!\,\epsilon}\int_{\mathcal{I}}\zeta^{\underline{\mu}b_1\cdots b_k}\nabla^k_{b_1\cdots b_k}\big(z^{a_1}\cdots z^{a_k}\big)\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(z^1)\psi_0(z^2)\psi_0(z^3)\overline{\mathbf{\Pi}}_{\underline{\mu}}^{\{\underline{\nu}\}}\big|_{z=0}\,d\sigma\\
&= \lim_{\epsilon\to 0}\frac{1}{k!\,\epsilon}\int_{\mathcal{I}}\zeta^{\underline{\mu}b_1\cdots b_k}\partial_{b_1}\ldots\partial_{b_k}\big(z^{a_1}\cdots z^{a_k}\big)\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(0)^3\delta^{\underline{\nu}}_{\underline{\mu}}\big|_{z=0}\,d\sigma\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathcal{I}}\zeta^{\underline{\nu}a_1\cdots a_k}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\,d\sigma = \zeta^{\underline{\nu}a_1\cdots a_k}(\sigma_0)\,,
\end{aligned}$$

as $\epsilon^{-1}\psi_0(\epsilon^{-1}(\sigma-\sigma_0)) \to \delta(\sigma-\sigma_0)$ as $\epsilon \to 0$. ◻

Corollary 14.

$$J^{\underline{\mu}} = 0 \qquad \text{if and only if} \qquad \zeta^{\underline{\mu}a_1\cdots a_k} = 0\,, \tag{58}$$

and hence the Dixon components are unique.

Proof. Assuming $J^{\underline{\mu}} = 0$ then from corollary 12 we have $J^{\underline{\mu}}_{(k)} = 0$, hence from (57) all the components are zero. ◻

3.6 The Taylor expansion with respect to Dixon geodesic hypersurfaces

Given a test tensor $\phi_{\underline{\mu}}$, we need to take its Taylor expansion, in a way that respects the Dixon geodesic hypersurfaces. We define the tensor field

$$\phi^{(k)}_{\underline{\mu}}|_{P(\sigma,V)} = \Pi^{\underline{\nu}}_{\underline{\mu}}\Big|^{P(\sigma,V)}_{C(\sigma)}(\nabla^k_V\phi_{\underline{\nu}}|_{C(\sigma)})\,, \tag{59}$$

by replacing V with sV this becomes

$$\phi^{(k)}_{\underline{\mu}}|_{P(\sigma,sV)} = \Pi^{\underline{\nu}}_{\underline{\mu}}\Big|^{P(\sigma,sV)}_{C(\sigma)}(\nabla^k_{sV}\phi_{\underline{\nu}}|_{C(\sigma)}) = s^k\,\Pi^{\underline{\nu}}_{\underline{\mu}}\Big|^{P(\sigma,sV)}_{C(\sigma)}(\nabla^k_V\phi_{\underline{\nu}}|_{C(\sigma)})\,. \tag{60}$$

In adapted coordinates $H_{\sigma,V}(1) = P(\sigma, V) = (\sigma, z^1, \dots, z^3)$ we have

$$\phi^{(k)}_{\underline{\mu}}|_{(\sigma,z^1,\dots,z^3)} = z^{a_1} \dots z^{a_k} \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_{a_1\cdots a_k} \phi_{\underline{\nu}}|_{C(\sigma)}\big) , \tag{61}$$

since

$$\begin{aligned}\phi^{(k)}_{\underline{\mu}}|_{(\sigma,z^1,\dots,z^3)} &= \phi^{(k)}_{\underline{\mu}}|_{P(\sigma,V)} = \Pi^{\underline{\nu}}_{\underline{\mu}}\big|^{P(\sigma,V)}_{C(\sigma)} \nabla^k_V \phi_{\underline{\nu}}|_{C(\sigma)} = \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_V \phi_{\underline{\nu}}|_{C(\sigma)}\big) = V^{a_1} \dots V^{a_k} \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_{a_1\cdots a_k} \phi_{\underline{\nu}}|_{C(\sigma)}\big) \\ &= z^{a_1} \dots z^{a_k} \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_{a_1\cdots a_k} \phi_{\underline{\nu}}|_{C(\sigma)}\big) .\end{aligned}$$

The Taylor expansion, with respect to the Dixon geodesic hypersurfaces, of a test tensor $\phi_{\underline{\mu}}$ is given by

$$\phi_{\underline{\mu}}|_{P(\sigma,sV)} = \sum_{k=0}^{N} \frac{1}{k!} \phi^{(k)}_{\underline{\mu}}|_{P(\sigma,sV)} + O(s^{N+1}) . \tag{62}$$

Lemma 15. *The order $O(s^{N+1})$, as $s \to 0$, in (62) is correct.*

Proof. Taking the Taylor expansion about $C(\sigma)$ along each geodesic $H_{\sigma,V}(s)$ and using (44) we have

$$\begin{aligned}\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{C(\sigma)}_{P(\sigma,sV)} \phi_{\underline{\mu}}|_{P(\sigma,sV)} &= \sum_{k=0}^{N} \frac{s^k}{k!} \frac{d^k}{ds^k}\Big|_{s=0} \Big(\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{C(\sigma)}_{P(\sigma,sV)} \phi_{\underline{\mu}}|_{P(\sigma,sV)}\Big) + O(s^{N+1}) \\ &= \sum_{k=0}^{N} \frac{s^k}{k!} \nabla^k_V \phi_{\underline{\nu}}|_{C(\sigma)} + O(s^{N+1}) .\end{aligned}$$

Hence

$$\phi_{\underline{\mu}}|_{P(\sigma,sV)} = \sum_{k=0}^{N} \frac{s^k}{k!} \Pi^{\underline{\nu}}_{\underline{\mu}}\big|^{P(\sigma,sV)}_{C(\sigma)} \nabla^k_V \phi_{\underline{\nu}}|_{C(\sigma)} + O(s^{N+1}) = \sum_{k=0}^{N} \frac{1}{k!} \phi^{(k)}_{\underline{\mu}}|_{P(\sigma,sV)} + O(s^{N+1}) .$$

□

Lemma 16.

$$\nabla^r_R \phi^{(k)}_{\underline{\mu}} = \frac{k!}{(k-r)!} \phi^{(k)}_{\underline{\mu}} . \tag{63}$$

Proof. From (48) and (61)

$$\begin{aligned}\nabla^r_R \phi^{(k)}_{\underline{\mu}} &= z^{b_1} \dots z^{b_r} \nabla^r_{b_1\cdots b_r} \phi^{(k)}_{\underline{\mu}} \\ &= z^{b_1} \dots z^{b_r} \nabla^r_{b_1\cdots b_r} \Big(z^{a_1} \dots z^{a_k} \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_{a_1\cdots a_k} \phi_{\underline{\nu}}|_{C(\sigma)}\big)\Big) \\ &= z^{b_1} \cdots z^{b_r} \partial_{b_1} \dots \partial_{b_r} \big(z^{a_1} \dots z^{a_k}\big) \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_{a_1\cdots a_k} \phi_{\underline{\nu}}|_{C(\sigma)}\big) \\ &= \frac{k!}{(k-r)!} \big(z^{a_1} \dots z^{a_k}\big) \overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}} \big(\nabla^k_{a_1\cdots a_k} \phi_{\underline{\nu}}|_{C(\sigma)}\big) = \frac{k!}{(k-r)!} \phi^{(k)}_{\underline{\mu}} .\end{aligned}$$

□

3.7 All multipoles are Dixon multipoles

In the previous subsection we showed that the components of a Dixon multipole are unique and can be extracted using particular text functions given by (55). In this section we show that the all multipoles can be written as a Dixon multipole. Thus we see that the Dixon multipole is merely a representation of a multipole. Thus if a multipole is defined via (14) with respect to one Dixon vector, N_μ, then it is guaranteed that it can be written with respect to another Dixon vector $\hat{N}_\mu$. A similar result is available in [12] but here an explicit formula is given, which in addition respects the constraint (3).

We assume we are given an arbitrary multipole $Y^{\underline{\mu}}$, i.e. an operator which takes test tensors $\phi_{\underline{\mu}}$ to give a number $Y^{\underline{\mu}}[\phi_{\underline{\mu}}]$. In [3] we show how to construct an arbitrary distribution. This is by taking a collection of monopoles, and allowing finite sums, derivatives, contractions and products with scalar fields. In [3] we showed that all multipoles could be expressed as Ellis multipoles. Thus another interpretation is that $Y^{\underline{\mu}}$ is given in the Ellis representation. An alternative is that $Y^{\underline{\mu}}$ is given to us in the Dixon representation, but with a different Dixon vector N_μ. Even in this case it is not obvious that the multipole can be represented as a Dixon multipole with the new Dixon vector.

Our goal is to establish $Y^{\underline{\mu}}$ can be written as a Dixon multipole, but the only information we can use are the values $Y^{\underline{\mu}}[\phi_{\underline{\mu}}]$ for particular $\phi_{\underline{\mu}}$'s.

The result that all multipoles can be written as Dixon multipoles, should not be surprising as if one were to count the number of components in the Ellis representation, it is the same as the Dixon representation. However one should be careful, as the components in the Ellis representation, also have the same number of components, but they are not unique. So simply counting number of components is not sufficient.

We can deduce the order of $Y^{\underline{\mu}}$ by the statement [3, eqn. (115)]: The order of $Y^{\underline{\mu}}$ is the smallest n such that

$$Y^{\underline{\mu}}[\lambda^{n+1}\phi_{\underline{\mu}}] = 0 \quad \text{for all tensors } \phi_{\underline{\mu}} \text{ and all scalar fields } \lambda \text{ such that } \lambda|_{C(\sigma)} = 0. \tag{64}$$

Having established the order of $Y^{\underline{\mu}}$ we can now perform the Dixon split. I.e. using (55) we can create the multipoles $Y^{\underline{\mu}}_{(k)}$

$$Y^{\underline{\mu}}_{(r)}[\phi_{\underline{\mu}}] = \sum_{k=r}^{n} \frac{(-1)^{k-r}}{(k-r)!\ r!} Y^{\underline{\mu}}[\nabla_R^k \phi_{\underline{\mu}}] \,. \tag{65}$$

We can now use (57) to calculate the components of $Y^{\underline{\mu}}$,

$$\theta^{\underline{\mu} a_1\cdots a_k}(\sigma_0) = \lim_{\epsilon\to 0} Y^{\underline{\mu}}_{(k)}[(\phi^{\underline{\nu},a_1\cdots a_r}_{\sigma_0,\epsilon})_{\underline{\mu}}] \,. \tag{66}$$

We emphasise here that $Y^{\underline{\mu}}$ was not defined using $\theta^{\underline{\mu} a_1\cdots a_k}$, via (14), instead $Y^{\underline{\mu}}$ was given to us and we have used various test functions to calculate the components.

It is now necessary to establish that (14) gives us the correct distribution. To do this we define a new distribution as

$$J^{\underline{\mu}} = \sum_{k=0}^{N} \frac{1}{k!} \nabla^k_{\rho_1\ldots\rho_k} \int_{\mathcal{I}} \theta^{\underline{\mu}\rho_1\cdots\rho_k} \delta^{(4)}\big(x - C(\sigma)\big)\, d\sigma \,. \tag{67}$$

It is then necessary to show $J^{\underline{\mu}} = Y^{\underline{\mu}}$.

Theorem 17.

$$J^{\underline{\mu}} = Y^{\underline{\mu}} \,. \tag{68}$$

Hence all all multipoles over C are Dixon multipoles over C.

Proof.

$$\begin{aligned}
J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] &= \frac{(-1)^k}{k!}\int_{\mathcal{I}} \theta^{\underline{\mu}a_1\cdots a_k}(\sigma_0)(\nabla^k_{a_1\cdots a_k}\phi_{\underline{\nu}})|_{C(\sigma_0)}\ d\sigma_0\\
&= \frac{(-1)^k}{k!}\int_{\mathcal{I}} \lim_{\epsilon\to 0} Y^{\underline{\mu}}_{(k)}[(\phi^{\underline{\nu},a_1\cdots a_r}_{\sigma_0,\epsilon})_{\underline{\mu}}](\nabla^k_{a_1\cdots a_k}\phi_{\underline{\nu}})|_{C(\sigma_0)}\ d\sigma_0\\
&= \frac{(-1)^k}{k!}\lim_{\epsilon\to 0} Y^{\underline{\mu}}_{(k)}\Big[\int_{\mathcal{I}}(\phi^{\underline{\nu},a_1\cdots a_r}_{\sigma_0,\epsilon})_{\underline{\mu}}(\nabla^k_{a_1\cdots a_k}\phi_{\underline{\nu}})|_{C(\sigma_0)}\ d\sigma_0\Big]\\
&= \frac{1}{k!}\lim_{\epsilon\to 0} Y^{\underline{\mu}}_{(k)}\Big[\int_{\mathcal{I}}\frac{1}{\epsilon}z^{a_1}\cdots z^{a_r}\Big(\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(z^1)\psi_0(z^2)\psi_0(z^3)\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}\Big)(\nabla^k_{a_1\cdots a_k}\phi_{\underline{\nu}})|_{C(\sigma_0)}\ d\sigma_0\Big]\\
&= \lim_{\epsilon\to 0}\frac{1}{k!\,\epsilon} Y^{\underline{\mu}}_{(k)}\Big[\int_{\mathcal{I}}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(z^1)\psi_0(z^2)\psi_0(z^3)z^{a_1}\cdots z^{a_r}\overline{\mathbf{\Pi}}^{\{\underline{\nu}\}}_{\underline{\mu}}(\nabla^k_{a_1\cdots a_k}\phi_{\underline{\nu}})|_{C(\sigma_0)}\ d\sigma_0\Big]\\
&= \lim_{\epsilon\to 0}\frac{1}{k!\,\epsilon} Y^{\underline{\mu}}_{(k)}\Big[\int_{\mathcal{I}}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\psi_0(z^1)\psi_0(z^2)\psi_0(z^3)\phi^{(k)}_{\underline{\mu}}|_{C(\sigma_0)}d\sigma_0\Big]\\
&= \lim_{\epsilon\to 0}\frac{1}{k!\,\epsilon} Y^{\underline{\mu}}_{(k)}\Big[\int_{\mathcal{I}}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\phi^{(k)}_{\underline{\mu}}|_{C(\sigma_0)}d\sigma_0\Big]\\
&= \frac{1}{k!}Y^{\underline{\mu}}_{(k)}\Big[\lim_{\epsilon\to 0}\epsilon^{-1}\int_{\mathcal{I}}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\phi^{(k)}_{\underline{\mu}}|_{C(\sigma_0)}d\sigma_0\Big]\\
&= \frac{1}{k!}Y^{\underline{\mu}}_{(k)}[\phi^{(k)}_{\underline{\mu}}] = \frac{1}{k!}\sum_{\ell=k}^{n}\frac{(-1)^{\ell-k}}{(\ell-k)!\,k!}Y^{\underline{\mu}}[\nabla^\ell_R\phi^{(k)}_{\underline{\mu}}]\\
&= \frac{1}{k!}\sum_{\ell=k}^{k}\frac{(-1)^{\ell-k}}{(\ell-k)!\,k!}\,\frac{k!}{(k-\ell)!}\,Y^{\underline{\mu}}[\phi^{(k)}_{\underline{\mu}}] = \frac{1}{k!}Y^{\underline{\mu}}[\phi^{(k)}_{\underline{\mu}}]\,.
\end{aligned}$$

Hence

$$J^{\underline{\mu}}[\phi_{\underline{\mu}}] = \sum_{k=0}^{N} J^{\underline{\mu}}_{(k)}[\phi_{\underline{\mu}}] = \sum_{k=0}^{N}\frac{1}{k!}Y^{\underline{\mu}}\Big[\phi^{(k)}_{\underline{\mu}}\Big] = Y^{\underline{\mu}}\Big[\sum_{k=0}^{N}\frac{1}{k!}\phi^{(k)}_{\underline{\mu}}\Big] = Y^{\underline{\mu}}[\phi_{\underline{\mu}}]\,.$$

Since for any arbitrary distribution $Y^{\underline{\mu}}$ we can construct using (65), (66) and (67) $J^{\underline{\mu}}$, then (68) implies all multipoles over C are Dixon multipoles over C.

Observe that between lines 7 and 8 of this derivation we swapped $\lim_{\epsilon\to 0}$ and $Y^{\underline{\mu}}_{(k)}$. This is permitted since $Y^{\underline{\mu}}_{(k)}$ consists of only a finite number of derivatives and is hence continuous and

$$\lim_{\epsilon\to 0}\epsilon^{-1}\int_{\mathcal{I}}\psi_0\Big(\frac{\sigma-\sigma_0}{\epsilon}\Big)\phi^{(k)}_{\underline{\mu}}|_{C(\sigma_0)}d\sigma_0 \to \phi^{(k)}_{\underline{\mu}}|_{C(\sigma)}\,,$$

in the LF-topology. □

Corollary 18. *Given a multipole $J^{\underline{\mu}}$ defined by (14) with respect to one Dixon vector N_μ, and given a different Dixon vector $\hat{N}_\mu$, then there exist unique components $\hat{\zeta}^{\underline{\mu}a_1\cdots a_k}$ so that we can write $J^{\underline{\mu}}$ with respect to $\hat{N}_\mu$.*

Proof. Using theorem 17 and corollary 14. □

3.8 Parallel squeezed tensors

So far, moments have been discussed as components of the distribution. There is also the intuitive notion of moments in terms of integrals over space, multiplied by one or more coordinates.

In general relativity such objects are not covariant as they depend on both the choice of the spatial hypersurface and the choice of the coordinates on that hypersurface. The other issue is that the tensor has to be integrated, and this requires transporting the tensors to the same point.

One choice, given in [3] is to use an adapted coordinate system. This is coordinate dependent. However by squeezing the tensor field, one can produce a well defined distribution. This corresponds to the Ellis representation of the distribution.

Dixon [8] gives a natural choice of moments. This uses the Dixon geodesic hypersurfaces, the geodesic coordinates and parallel transport. Thus all the necessary structure is given to us uniquely, once we have chosen the Dixon vector.

In this section we give a method of squeezing a tensor field, such that the coefficients of the expansion are the Dixon components of the distribution.

Let $U^{\underline{\mu}}$ be a tensor density (of weight 1) on $\mathcal{M}$. From this we can construct the squeezed tensor density $U_\epsilon^{\underline{\mu}} \in \Gamma TM$ given by

$$U_\epsilon^{\underline{\mu}}|_{P(\sigma,V)} = \epsilon^{-3}\, \Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,\epsilon^{-1}V}_{P(\sigma,V)}\, U^{\underline{\nu}}|_{P(\sigma,\epsilon^{-1}V)}\,. \tag{69}$$

Let

$$\xi^{\underline{\mu}\rho_1\cdots\rho_k}(\sigma) = (-1)^k \int_{N^\perp(\sigma)} V^{\rho_1}\cdots V^{\rho_k}\, \Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,V)}_{C(\sigma)}\, U^{\underline{\nu}}|_{P(\sigma,V)}\, d^3V\,. \tag{70}$$

These components clearly satisfy (14) and (15). In Dixon geodesics coordinates this becomes

$$\xi^{\underline{\mu}\rho_1\cdots\rho_k}(\sigma) = (-1)^k \int_{\Sigma(\sigma)} z^{\rho_1}\cdots z^{\rho_k}\, U^{\underline{\mu}}|_{(\sigma,z^1,z^2,z^3)}\, d^3z\,. \tag{71}$$

Lemma 19.

$$\int_{\mathcal{M}} U_\epsilon^{\underline{\mu}}\, \phi_{\underline{\mu}}\, d^4x = \sum_{k=0}^{N} (-1)^k \frac{\epsilon^k}{k!} \int_{\mathcal{I}} \xi^{\underline{\mu}\rho_1\cdots\rho_k}\, \nabla^k_{\rho_1\cdots\rho_k} \phi_{\underline{\mu}}\, d\sigma + O(\epsilon^{N+1})\,. \tag{72}$$

Thus

$$U_\epsilon^{\underline{\mu}} = \sum_{k=0}^{N} \frac{\epsilon^k}{k!} \nabla^k_{\rho_1\cdots\rho_k} \int_{\mathcal{I}} d\sigma\, \xi^{\underline{\mu}\rho_1\cdots\rho_k}\, \delta^{(4)}(x-C) + O(\epsilon^{N+1})\,. \tag{73}$$

Proof. From (60) and (62) we have

$$\Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{C(\sigma)}_{P(\sigma,sV)} \phi_{\underline{\mu}}|_{P(\sigma,sV)} = \sum_{k=0}^{N} \frac{s^k}{k!}\, \nabla^k_V \phi_{\underline{\nu}}|_{C(\sigma)} + O(s^{N+1})\,.$$

In Dixon geodesic coordinates (σ, z^1, z^2, z^3). We set the vector field $V^\mu = \epsilon W^\mu$ and do a Taylor expansion in ϵ.

$$
\begin{aligned}
&\int_{\mathcal{M}} U_\epsilon^{\underline{\mu}}\, \phi_{\underline{\mu}}\, d\sigma\, d^3z \\
&\quad = \int_{\mathcal{I}} d\sigma \int_{\Sigma(\sigma)} U_\epsilon^{\underline{\mu}}\, \phi_{\underline{\mu}}\, d^3z \\
&\quad = \int_{\mathcal{I}} d\sigma \int_{N^\perp(\sigma)} U_\epsilon^{\underline{\mu}}|_{P(\sigma,V)}\, \phi_{\underline{\mu}}|_{P(\sigma,V)}\, d^3V \\
&\quad = \epsilon^{-3} \int_{\mathcal{I}} d\sigma \int_{N^\perp(\sigma)} \Pi^{\underline{\mu}}_{\underline{\nu}}\big|^{P(\sigma,\epsilon^{-1}V)}_{P(\sigma,V)}\, U^{\underline{\nu}}|_{P(\sigma,\epsilon^{-1}V)}\, \phi_{\underline{\mu}}|_{P(\sigma,V)}\, d^3V \\
&\quad = \epsilon^{-3} \int_{\mathcal{I}} d\sigma \int_{N^\perp(\sigma)} \Pi^{\underline{\mu}}_{\underline{\lambda}}\big|^{C(\sigma)}_{P(\sigma,V)}\, \Pi^{\underline{\lambda}}_{\underline{\nu}}\big|^{P(\sigma,\epsilon^{-1}V)}_{C(\sigma)}\, U^{\underline{\nu}}|_{P(\sigma,\epsilon^{-1}V)}\, \phi_{\underline{\mu}}|_{P(\sigma,V)}\, d^3V \\
&\quad = \int_{\mathcal{I}} d\sigma \int_{N^\perp(\sigma)} \Pi^{\underline{\lambda}}_{\underline{\nu}}\big|^{P(\sigma,W)}_{C(\sigma)}\, U^{\underline{\nu}}|_{P(\sigma,W)}\, \Pi^{\underline{\mu}}_{\underline{\lambda}}\big|^{C(\sigma)}_{P(\sigma,\epsilon W)}\, \phi_{\underline{\mu}}|_{P(\sigma,\epsilon W)}\, d^3W \\
&\quad = \int_{\mathcal{I}} d\sigma \int_{N^\perp(\sigma)} \Pi^{\underline{\lambda}}_{\underline{\nu}}\big|^{P(\sigma,W)}_{C(\sigma)}\, U^{\underline{\nu}}|_{P(\sigma,W)}\, d^3W \bigg(\sum_{k=0}^{N} \frac{\epsilon^k}{k!} \nabla^k_W \phi_{\underline{\lambda}}|_{C(\sigma)}\bigg) + O(\epsilon^{N+1}) \\
&\quad = \sum_{k=0}^{N} \frac{\epsilon^k}{k!} \int_{\mathcal{I}} d\sigma \int_{N^\perp(\sigma)} \Pi^{\underline{\lambda}}_{\underline{\nu}}\big|^{P(\sigma,W)}_{C(\sigma)}\, U^{\underline{\nu}}|_{P(\sigma,W)}\, W^{\rho_1}\cdots W^{\rho_k}\, d^3W \big(\nabla^k_{\rho_1\cdots\rho_k} \phi_{\underline{\lambda}}|_{C(\sigma)}\big) + O(\epsilon^{N+1}) \\
&\quad = \sum_{k=0}^{N} (-1)^k \frac{\epsilon^k}{k!} \int_{\mathcal{I}} d\sigma\, \xi^{\underline{\lambda}\rho_1\cdots\rho_k}(\sigma) \big(\nabla^k_{\rho_1\cdots\rho_k} \phi_{\underline{\lambda}}|_{C(\sigma)}\big) + O(\epsilon^{N+1})\,.
\end{aligned}
$$

□

Thus we have shown the link between the components of the multipoles and the moments of a regular distribution.

4 Dynamical equations in Dixon language

In this section we derive the key result of this article, namely the dynamical equations of the stress-energy quadrupole, in the Dixon representation. These follow from the divergenceless condition (13). As stated this has the advantage that they are tensorial. The tensorial expression is given below in theorem 21. We first derive the equation in Dixon adapted coordinates.

Theorem 20. *In Dixon adapted coordinates the divergenceless condition (13) corresponds to the following dynamical equations for the components*

$$
\begin{aligned}
\nabla_0 \xi^{\mu 0bc} &= -2\xi^{\mu(bc)}\,, && (74)\\
\nabla_0 \xi^{\mu 0a} &= -\xi^{\mu a} + \tfrac{1}{2}\xi^{\mu 0bc} R^a{}_{b0c} + \xi^{\rho 0ac} R^\mu{}_{\rho 0c} + \tfrac{1}{2}\xi^{\rho cba} R^\mu{}_{\rho cb} + \tfrac{1}{6}\xi^{\mu dbc} R^a{}_{bdc}\,, && (75)\\
\nabla_0 \xi^{\mu 0} &= \xi^{\rho 0b} R^\mu{}_{\rho 0b} + \tfrac{1}{2}\xi^{\rho ab} R^\mu{}_{\rho ab} + \tfrac{1}{2}\nabla_0\big(\xi^{\mu 0bc} R^0{}_{b0c}\big) + \tfrac{1}{2}\xi^{\rho 0bc}\left(\nabla_c R^\mu{}_{\rho 0b}\right) + \tfrac{1}{6}\nabla_0\big(\xi^{\mu abc} R^0{}_{bac}\big) \\
&\quad - \tfrac{1}{3}\xi^{\rho abc}\left(\nabla_c R^\mu{}_{\rho ab}\right)\,, && (76)
\end{aligned}
$$

together with the constraint

$$\xi^{\mu(abc)} = 0\,. \tag{77}$$

Proof. For this proof, all integrals are implicitly over $\mathcal{I}$. Let

$$F^\mu = \nabla_\sigma T^{\sigma\mu} = F^\mu_{(1)} + F^\mu_{(2)} + F^\mu_{(3)}\,,$$

where

$$F^\mu_{(1)} = \nabla_\nu \int \xi^{\mu\nu}\delta(x-C)d\sigma, \quad F^\mu_{(2)} = \nabla_\nu\nabla_\rho \int \xi^{\mu\nu\rho}\delta(x-C)d\sigma, \quad F^\mu_{(3)} = \tfrac{1}{2}\nabla_\nu\nabla^2_{\rho\sigma} \int \xi^{\mu\nu\rho\sigma}\delta(x-C)d\sigma\,. \tag{78}$$

Manipulating $F^\mu_{(1)}$, $F^\mu_{(2)}$ and $F^\mu_{(3)}$ in turn.

$$\begin{aligned} F^\mu_{(1)}[\phi_\mu] &= -\int \xi^{\mu\nu}\,\nabla_\nu\,\phi_\mu\,d\sigma = -\int \xi^{\mu a}\,\nabla_a\,\phi_\mu\,d\sigma - \int \xi^{\mu 0}\,\nabla_0\phi_\mu\,d\sigma \\ &= -\int \xi^{\mu a}\,\nabla_a\phi_\mu\,d\sigma + \int (\nabla_0\xi^{\mu 0})\,\phi_\mu\,d\sigma\,, \end{aligned}$$

which using (52) in lemma 10 gives

$$F^\mu_{(1)}[\nabla^3_R\phi_\mu] = F^\mu_{(1)}[\nabla^2_R\phi_\mu] = 0, \tag{79}$$

$$F^\mu_{(1)}[\nabla_R\phi_\mu] = -\int \xi^{\mu a}\,\nabla_a\phi_\mu\,d\sigma, \tag{80}$$

and
$$F^\mu_{(1)}[\phi_\mu] = F^\mu_{(1)}[\nabla_R\phi_\mu] + \int (\nabla_0\xi^{\mu 0})\,\phi_\mu\,d\sigma\,. \tag{81}$$

For $F^\mu_{(2)}[\phi_\mu]$ we have

$$\begin{aligned} F^\mu_{(2)}\big[\phi_\mu\big] &= \int \xi^{\mu\nu\rho}\,\nabla_\rho\nabla_\nu\,\phi_\mu\,d\sigma = \int \xi^{\mu 0b}\,\nabla_b\nabla_0\,\phi_\mu\,d\sigma + \int \xi^{\mu ab}\,\nabla_b\nabla_a\,\phi_\mu\,d\sigma \\ &= \int \xi^{\mu 0b}\,\nabla_0\nabla_b\,\phi_\mu\,d\sigma - \int \xi^{\mu 0b}\,R^\rho{}_{\mu 0b}\,\phi_\rho\,d\sigma + \int \xi^{\mu(ab)}\,\nabla_a\nabla_b\,\phi_\mu\,d\sigma - \tfrac{1}{2}\int \xi^{\mu ab}\,R^\rho{}_{\mu ab}\,\phi_\rho\,d\sigma \\ &= -\int (\nabla_0\xi^{\mu 0b})\nabla_b\,\phi_\mu\,d\sigma - \int \xi^{\mu 0b}\,R^\rho{}_{\mu 0b}\,\phi_\rho\,d\sigma + \int \xi^{\mu(ab)}\,\nabla_{ab}\phi_\mu\,d\sigma - \tfrac{1}{2}\int \xi^{\mu ab}\,R^\rho{}_{\mu ab}\,\phi_\rho\,d\sigma\,. \end{aligned}$$

Using equation (52) we can split $F^\mu_{(2)}[\phi_\mu]$ to give

$$F^\mu_{(2)}\big[\nabla^3_R\phi_\mu\big] = 0 \tag{82}$$

$$F^\mu_{(2)}\big[\nabla^2_R\phi_\mu\big] = 2\int \xi^{\mu(ab)}\,\nabla_{ab}\phi_\mu\,d\sigma \tag{83}$$

$$F^\mu_{(2)}\big[\nabla_R\phi_\mu\big] = \tfrac{1}{2}F^\mu_{(2)}\big[\nabla^2_R\phi_\mu\big] - \int (\nabla_0\xi^{\mu 0a})\nabla_a\,\phi_\mu\,d\sigma \tag{84}$$

$$F^\mu_{(2)}\big[\phi_\mu\big] = F^\mu_{(2)}\big[\nabla_R\phi_\mu\big] - \int \xi^{\rho 0b}\,R^\mu{}_{\rho 0b}\,\phi_\mu\,d\sigma - \tfrac{1}{2}\int \xi^{\rho ab}\,R^\mu{}_{\rho ab}\,\phi_\mu\,d\sigma\,. \tag{85}$$

Now consider $F^\mu_{(3)}[\phi_\mu]$.

$$F^\mu_{(3)}\big[\phi_\mu\big] = -\frac{1}{2}\int \xi^{\mu\nu\rho\sigma}\,\nabla_\sigma\nabla_\rho\nabla_\nu\,\phi_\mu\,d\sigma = -\frac{1}{2}\int \xi^{\mu abc}\,\nabla_c\nabla_b\nabla_a\,\phi_\mu\,d\sigma - \frac{1}{2}\int \xi^{\mu 0bc}\,\nabla_c\nabla_b\nabla_0\,\phi_\mu\,d\sigma\,. \tag{86}$$

The second term on the right hand side of (86) gives

$$
\begin{aligned}
&\int \xi^{\mu 0bc}\, \nabla_c \nabla_b \nabla_0\, \phi_\mu\, d\sigma \\
&= \int \xi^{\mu 0bc}\, \nabla_c \nabla_0 \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu 0bc}\, \nabla_c (R^\rho{}_{\mu 0b}\, \phi_\rho)\, d\sigma \\
&= \int \xi^{\mu 0bc}\, \nabla_0 \nabla_c \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu 0bc} \Big(R^\rho{}_{b0c} \nabla_\rho\, \phi_\mu + R^\rho{}_{\mu 0c} \nabla_b\, \phi_\rho + \nabla_c R^\rho{}_{\mu 0b}\, \phi_\rho + R^\rho{}_{\mu 0b}\, \nabla_c \phi_\rho \Big)\, d\sigma \\
&= \int \xi^{\mu 0bc}\, \nabla_0 \nabla_c \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu 0bc} \Big(R^0{}_{b0c} \nabla_0\, \phi_\mu + R^d{}_{b0c} \nabla_d\, \phi_\mu + R^\rho{}_{\mu 0c} \nabla_b\, \phi_\rho \\
&\qquad\qquad + \nabla_c R^\rho{}_{\mu 0b}\, \phi_\rho + R^\rho{}_{\mu 0b}\, \nabla_c \phi_\rho \Big)\, d\sigma \\
&= -\int \nabla_0 (\xi^{\mu 0bc})\, \nabla^2_{cb}\, \phi_\mu\, d\sigma + \int \nabla_0 \Big(\xi^{\mu 0bc}\, R^0{}_{b0c} \Big)\, \phi_\mu\, d\sigma \\
&\quad - \int \xi^{\mu 0bc} \Big(R^d{}_{b0c} \nabla_d\, \phi_\mu + R^\rho{}_{\mu 0c} \nabla_b\, \phi_\rho + \nabla_c R^\rho{}_{\mu 0b}\, \phi_\rho + R^\rho{}_{\mu 0b}\, \nabla_c \phi_\rho \Big)\, d\sigma \\
&= -\int \nabla_0 (\xi^{\mu 0bc})\, \nabla^2_{cb}\, \phi_\mu\, d\sigma + \int \Big(\nabla_0 \big(\xi^{\mu 0bc}\, R^0{}_{b0c} \big) + \xi^{\rho 0bc}\, (\nabla_c R^\mu{}_{\rho 0b}) \Big) \phi_\mu\, d\sigma \\
&\quad - \int \Big(\xi^{\mu 0bc}\, R^a{}_{b0c} + 2\xi^{\rho 0ac}\, R^\mu{}_{\rho 0c} \Big) \nabla_a \phi_\mu\, d\sigma\, .
\end{aligned}
$$

For the first term on the right hand side of (86), it is necessary to symmetrise the abc of ξ^{abc}.

$$
\begin{aligned}
\int \xi^{\mu abc}\, \nabla_c \nabla_b \nabla_a\, \phi_\mu\, d\sigma &= \int \xi^{\mu abc}\, \nabla_c \nabla_a \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu abc}\, \nabla_c (R^\rho{}_{\mu ab}\, \phi_\rho)\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_c \nabla_a \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu abc} \Big((\nabla_c R^\rho{}_{\mu ab}) \phi_\rho + R^\rho{}_{\mu ab}\, \nabla_c \phi_\rho \Big)\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_c \nabla_a \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\rho cba} \Big((\nabla_a R^\mu{}_{\rho cb}) \phi_\mu + R^\mu{}_{\rho cb}\, \nabla_a \phi_\mu \Big)\, d\sigma\, .
\end{aligned}
$$

Continuing

$$
\begin{aligned}
&\int \xi^{\mu abc}\, \nabla_c \nabla_b \nabla_a\, \phi_\mu\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_a \nabla_c \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu abc} \Big(R^\rho{}_{bac} \nabla_\rho\, \phi_\mu + R^\rho{}_{\mu ac} \nabla_b\, \phi_\rho + (\nabla_c R^\rho{}_{\mu ab}) \phi_\rho + R^\rho{}_{\mu ab}\, \nabla_c \phi_\rho \Big)\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_a \nabla_c \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu abc} \Big(R^\rho{}_{bac} \nabla_\rho\, \phi_\mu + 2R^\rho{}_{\mu ac} \nabla_b\, \phi_\rho + (\nabla_c R^\rho{}_{\mu ab}) \phi_\rho \Big)\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_a \nabla_c \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\mu abc} \Big(R^0{}_{bac} \nabla_0\, \phi_\mu + R^d{}_{bac} \nabla_d\, \phi_\mu + 2R^\rho{}_{\mu ac} \nabla_b\, \phi_\rho + (\nabla_c R^\rho{}_{\mu ab}) \phi_\rho \Big)\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_a \nabla_c \nabla_b\, \phi_\mu\, d\sigma + \int \nabla_0 \Big(\xi^{\mu abc}\, R^0{}_{bac} \Big)\, \phi_\mu\, d\sigma \\
&\quad - \int \xi^{\mu abc} \Big(R^d{}_{bac} \nabla_d\, \phi_\mu + 2R^\rho{}_{\mu ac} \nabla_b\, \phi_\rho + (\nabla_c R^\rho{}_{\mu ab}) \phi_\rho \Big)\, d\sigma \\
&= \int \xi^{\mu abc}\, \nabla_a \nabla_c \nabla_b\, \phi_\mu\, d\sigma + \int \Big(-\, \xi^{\mu dbc}\, R^a{}_{bdc} - 2\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \int \Big(\nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) - \xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma\, .
\end{aligned}
$$

Hence

$$
\begin{aligned}
3 \int \xi^{\mu abc}\, \nabla_c \nabla_b \nabla_a\, \phi_\mu\, d\sigma
&= \int \xi^{\mu abc}\, \nabla_c \nabla_b \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \int \xi^{\mu abc}\, \nabla_c \nabla_a \nabla_b\, \phi_\mu\, d\sigma - \int \xi^{\rho cba} \Big((\nabla_a R^\mu{}_{\rho cb}) \phi_\mu + R^\mu{}_{\rho cb}\, \nabla_a \phi_\mu \Big)\, d\sigma \\
&\quad + \int \xi^{\mu abc}\, \nabla_a \nabla_c \nabla_b\, \phi_\mu\, d\sigma + \int \Big(- \xi^{\mu dbc}\, R^a{}_{bdc} - 2\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \int \Big(\nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) - \xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma \\
&= 3 \int \xi^{\mu (abc)}\, \nabla_{(abc)} \phi_\mu\, d\sigma + \int \Big(- \xi^{\rho cba}\, R^\mu{}_{\rho cb} - \xi^{\mu dbc}\, R^a{}_{bdc} - 2\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \int \Big(- \xi^{\rho cba}\, (\nabla_a R^\mu{}_{\rho cb}) + \nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) - \xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma \\
&= 3 \int \xi^{\mu (abc)}\, \nabla_{(abc)} \phi_\mu\, d\sigma + \int \Big(- \xi^{\mu dbc}\, R^a{}_{bdc} - 3\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \int \Big(\nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) - 2\xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma\, .
\end{aligned}
$$

Hence (86) becomes

$$
\begin{aligned}
F^\mu_{(3)}[\phi_\mu] &= \tfrac{1}{2} \int \nabla_0 (\xi^{\mu 0bc})\, \nabla^2_{cb}\, \phi_\mu\, d\sigma - \tfrac{1}{2} \int \Big(\nabla_0 \big(\xi^{\mu 0bc}\, R^0{}_{b0c} \big) + \xi^{\rho 0bc}\, (\nabla_c R^\mu{}_{\rho 0b}) \Big) \phi_\mu\, d\sigma \\
&\quad + \tfrac{1}{2} \int \Big(\xi^{\mu 0bc}\, R^a{}_{b0c} + 2\xi^{\rho 0ac}\, R^\mu{}_{\rho 0c} \Big) \nabla_a \phi_\mu\, d\sigma \\
&\quad - \tfrac{1}{2} \int \xi^{\mu (abc)}\, \nabla^3_{abc} \phi_\mu\, d\sigma + \tfrac{1}{6} \int \Big(\xi^{\mu dbc}\, R^a{}_{bdc} + 3\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \tfrac{1}{6} \int \Big(- \nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) + 2\xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma \\
&= -\tfrac{1}{2} \int \xi^{\mu (abc)}\, \nabla^3_{abc} \phi_\mu\, d\sigma + \tfrac{1}{2} \int \nabla_0 (\xi^{\mu 0bc})\, \nabla^2_{cb}\, \phi_\mu\, d\sigma \\
&\quad + \int \Big(\tfrac{1}{2} \xi^{\mu 0bc}\, R^a{}_{b0c} + \xi^{\rho 0ac}\, R^\mu{}_{\rho 0c} R^\mu{}_{\rho cb} + \tfrac{1}{6} \xi^{\mu dbc}\, R^a{}_{bdc} + \tfrac{1}{2} \xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma \\
&\quad + \int \Big(- \tfrac{1}{2} \nabla_0 \big(\xi^{\mu 0bc}\, R^0{}_{b0c} \big) - \tfrac{1}{2} \xi^{\rho 0bc}\, (\nabla_c R^\mu{}_{\rho 0b}) - \tfrac{1}{6} \nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) + \tfrac{1}{3} \xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma\, .
\end{aligned}
$$

Splitting $F^\mu_{(3)}[\phi_\mu]$ gives

$$
F^\mu_{(3)}[\nabla^3_R \phi_\mu] = -3 \int \xi^{\mu (abc)}\, \nabla^3_{abc} \phi_\mu\, d\sigma\, , \tag{87}
$$

$$
F^\mu_{(3)}[\nabla^2_R \phi_\mu] = \tfrac{1}{3} F^\mu_{(3)}[\nabla^3_R \phi_\mu] + \int \nabla_0 (\xi^{\mu 0bc})\, \nabla^2_{cb}\, \phi_\mu\, d\sigma\, , \tag{88}
$$

$$
F^\mu_{(3)}[\nabla_R \phi_\mu] = \tfrac{1}{2} F^\mu_{(3)}[\nabla^2_R \phi_\mu] + \int \Big(\tfrac{1}{2} \xi^{\mu 0bc}\, R^a{}_{b0c} + \xi^{\rho 0ac}\, R^\mu{}_{\rho 0c} + \tfrac{1}{6} \xi^{\mu dbc}\, R^a{}_{bdc} + \tfrac{1}{2} \xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu\, d\sigma\, , \tag{89}
$$

$$
\begin{aligned}
F^\mu_{(3)}[\phi_\mu] = F^\mu_{(3)}[\nabla_R \phi_\mu] + \int \Big(&- \tfrac{1}{2} \nabla_0 \big(\xi^{\mu 0bc}\, R^0{}_{b0c} \big) - \tfrac{1}{2} \xi^{\rho 0bc}\, (\nabla_c R^\mu{}_{\rho 0b}) - \tfrac{1}{6} \nabla_0 \big(\xi^{\mu abc}\, R^0{}_{bac} \big) \\
&+ \tfrac{1}{3} \xi^{\rho abc}\, (\nabla_c R^\mu{}_{\rho ab}) \Big) \phi_\mu\, d\sigma\, .
\end{aligned} \tag{90}
$$

Since $F^\mu = 0$, then from corollary 12,

$$0 = F^\mu[\nabla_R^3 \phi_\mu] = F^\mu_{(3)}[\nabla_R^3 \phi_\mu]\,, \tag{91}$$
$$0 = F^\mu[\nabla_R^2 \phi_\mu] = F^\mu_{(3)}[\nabla_R^2 \phi_\mu] + F^\mu_{(2)}[\nabla_R^2 \phi_\mu]\,, \tag{92}$$
$$0 = F^\mu[\nabla_R \phi_\mu] = F^\mu_{(3)}[\nabla_R \phi_\mu] + F^\mu_{(2)}[\nabla_R \phi_\mu] + F^\mu_{(1)}[\nabla_R \phi_\mu]\,, \tag{93}$$
$$0 = F^\mu[\phi_\mu] = F^\mu_{(3)}[\phi_\mu] + F^\mu_{(2)}[\phi_\mu] + F^\mu_{(1)}[\phi_\mu]\,. \tag{94}$$

From (91) and (87) we get (77).

From (92), (83) and (88) we can calculate $\nabla_0 \xi^{\mu 0ab}$

$$0 = F^\mu_{(3)}[\nabla_R^2 \phi_\mu] + F^\mu_{(2)}[\nabla_R^2 \phi_\mu] = 2 \int \xi^{\mu(ab)}\, \nabla_{ab}\phi_\mu \, d\sigma + \int \nabla_0(\xi^{\mu 0ab})\, \nabla^2_{ab}\, \phi_\mu \, d\sigma\,,$$

which gives (74).

From (93), (80), (84) and (89) we can calculate $\nabla_0 \xi^{\mu 0a}$

$$\begin{aligned}
0 &= F^\mu[\nabla_R \phi_\mu] = F^\mu_{(3)}[\nabla_R \phi_\mu] + F^\mu_{(2)}[\nabla_R \phi_\mu] + F^\mu_{(1)}[\nabla_R \phi_\mu] \\
&= -\int \xi^{\mu a}\, \nabla_a \phi_\mu \, d\sigma + \tfrac{1}{2} F^\mu_{(2)}\big[\nabla_R^2 \phi_\mu\big] - \int (\nabla_0 \xi^{\mu 0a}) \nabla_a\, \phi_\mu \, d\sigma \\
&\quad + \tfrac{1}{2} F^\mu_{(3)}[\nabla_R^2 \phi_\mu] + \int \Big(\tfrac{1}{2}\xi^{\mu 0bc}\, R^a{}_{b0c} + \xi^{\rho 0ac}\, R^\mu{}_{\rho 0c} + + \tfrac{1}{6}\xi^{\mu dbc}\, R^a{}_{bdc} + \tfrac{1}{2}\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu \, d\sigma \\
&= \int \Big(-\xi^{\mu a} - (\nabla_0 \xi^{\mu 0a}) + \tfrac{1}{2}\xi^{\mu 0bc}\, R^a{}_{b0c} + \xi^{\rho 0ac}\, R^\mu{}_{\rho 0c} + \tfrac{1}{6}\xi^{\mu dbc}\, R^a{}_{bdc} + \tfrac{1}{2}\xi^{\rho bac}\, R^\mu{}_{\rho bc} \Big) \nabla_a\, \phi_\mu \, d\sigma\,,
\end{aligned}$$

which gives (75).

From (94), (81), (85) and (90) we have

$$\begin{aligned}
0 &= F^\mu[\phi_\mu] = F^\mu_{(3)}[\phi_\mu] + F^\mu_{(2)}[\phi_\mu] + F^\mu_{(1)}[\phi_\mu] \\
&= F^\mu_{(1)}[\nabla_R \phi_\mu] + \int (\nabla_0 \xi^{\mu 0})\, \phi_\mu \, d\sigma \\
&\quad + F^\mu_{(2)}\big[\nabla_R \phi_\mu\big] - \int \xi^{\rho 0b}\, R^\mu{}_{\rho 0b}\, \phi_\mu \, d\sigma - \tfrac{1}{2} \int \xi^{\rho ab}\, R^\mu{}_{\rho ab}\, \phi_\mu \, d\sigma \\
&\quad + F^\mu_{(3)}[\nabla_R \phi_\mu] + \int \Big(-\tfrac{1}{2}\nabla_0\big(\xi^{\mu 0bc}\, R^0{}_{b0c}\big) - \tfrac{1}{2}\xi^{\rho 0bc}\,(\nabla_c R^\mu{}_{\rho 0b}) - \tfrac{1}{6}\nabla_0\big(\xi^{\mu abc}\, R^0{}_{bac}\big) \\
&\qquad\qquad + \tfrac{1}{3}\xi^{\rho abc}\,(\nabla_c R^\mu{}_{\rho ab})\Big) \phi_\mu \, d\sigma \\
&= \int \Big((\nabla_0 \xi^{\mu 0}) - \xi^{\rho 0b}\, R^\mu{}_{\rho 0b} - \tfrac{1}{2}\xi^{\rho ab}\, R^\mu{}_{\rho ab} - \tfrac{1}{2}\nabla_0\big(\xi^{\mu 0bc}\, R^0{}_{b0c}\big) - \tfrac{1}{2}\xi^{\rho 0bc}\,(\nabla_c R^\mu{}_{\rho 0b}) - \tfrac{1}{6}\nabla_0\big(\xi^{\mu abc}\, R^0{}_{bac}\big) \\
&\qquad\qquad + \tfrac{1}{3}\xi^{\rho abc}\,(\nabla_c R^\mu{}_{\rho ab})\Big) \phi_\mu \, d\sigma\,,
\end{aligned}$$

which gives (76)

□

As stated the key advantage with using the Dixon representation is that the components are tensors. Therefore writing (74)-(77) using spacetime indices, results in tensor equations which are valid for all coordinate systems. We introduce the spatial projection tensor

$$\pi^\rho_\alpha = \delta^\rho_\alpha - \dot{C}^\rho\, N_\alpha\,. \tag{95}$$

Theorem 21. *The divergenceless condition (13) corresponds to the following tensor equations for the components*

$$\nabla_{\dot{C}}(N_\nu\,\xi^{\mu\nu\rho\sigma}) = -2\,\pi^\rho_\beta\,\pi^\sigma_\alpha\,\xi^{\mu(\beta\alpha)}\,, \tag{96}$$

$$\nabla_{\dot{C}}(N_\nu\,\xi^{\mu\nu\rho}) = \pi^\rho_\alpha\big(-\xi^{\mu\alpha} + \tfrac{1}{2}N_\nu\dot{C}^\beta\,\xi^{\mu\nu\lambda\sigma}R^\alpha{}_{\lambda\beta\sigma} + (N_\nu\dot{C}^\beta + \tfrac{1}{2}\pi^\beta_\nu)\,\xi^{\lambda\nu\sigma\alpha}R^\mu{}_{\lambda\beta\sigma} + \tfrac{1}{6}\pi^\beta_\nu\,\xi^{\alpha\nu\lambda\sigma}R^\mu{}_{\lambda\beta\sigma}\big)\,, \tag{97}$$

$$\begin{aligned}\nabla_{\dot{C}}(N_\nu\,\xi^{\mu\nu}) = {} & (N_\nu\dot{C}^\beta + \tfrac{1}{2}\pi^\beta_\nu)\xi^{\rho\nu\lambda}R^\mu{}_{\rho\beta\lambda} + \nabla_{\dot{C}}\big((\tfrac{1}{2}N_\nu\dot{C}^\beta + \tfrac{1}{6}\pi^\beta_\nu)N_\zeta\,\xi^{\mu\nu\lambda\sigma}\,R^\zeta{}_{\lambda\beta\sigma}\big)\\ & + (\tfrac{1}{2}N_\nu\dot{C}^\beta - \tfrac{1}{3}\pi^\beta_\nu)\,\xi^{\rho\nu\lambda\sigma}\,(\nabla_\lambda R^\mu{}_{\rho\beta\sigma})\,,\end{aligned} \tag{98}$$

together with the constraint

$$\pi^\beta_\nu\,\pi^\alpha_\rho\,\pi^\lambda_\sigma\,\xi^{\mu(\nu\rho\sigma)} = 0\,. \tag{99}$$

Proof. The equations (74)-(77) are replaced by (96)-(99) as follows. ∇_0 is replaced by $\nabla_{\dot{C}}$. Each lower index 0 it is necessary to contract with $\dot{C}^\beta$ and each upper index 0 it is necessary to contract with N_ν. Each spatial index $a, b, \ldots$ it is necessary to project out using π^β_ν. However if the spatial index is one the third or fourth index of $\xi^{\rho\nu a}$ or $\xi^{\rho\nu ab}$ the projection is not necessary.

As a result, in the adapted coordinate system where $\dot{C}^\beta = \delta^\beta_0$ and $N_\nu = \delta^0_\nu$ then (96)-(99) become (74)-(77). However, (96)-(99) are clearly tensorial equations and therefore true for all coordinate systems. □

As was observed in [3], the equations for the components arising from the divergenceless condition of the stress-energy tensor are insufficient to completely determine the dynamics of the components. To see this, from (12) we have (10+40+100=150) components. From (11) this reduces to (150-10-40=100) components. Using either (77) or (99) gives us 40 constraints, hence there are 60 components. Equations (74)-(76) or (96)-(98) give 40 ODEs. Hence there are 20 free components. These still have to be determined via constitutive relations from a model of the underlying material.

These are similar to the equations have been found by Steinhoff and Puetzfeld [11]. However their equations are implicit. On the right (98) we see that there is a covariant derivative of $\xi^{\rho\nu\lambda\sigma}$. However this term can be expanded out to identified $\nabla_{\dot{C}}\xi^{\rho\nu\lambda\sigma}$. One can then substitute in (96) and the appropriate constitutive relations.

Corollary 22. *As a simple consistency check we see if we reduce to a dipole. Set*

$$\xi^{\rho\nu\lambda\sigma} = 0, \quad \xi^{\rho\nu\lambda} = X^\lambda\dot{C}^\rho\dot{C}^\nu + S^{\lambda(\rho}\dot{C}^{\nu)} \quad \text{and} \quad \xi^{\rho\nu} = 2P^{(\rho}\dot{C}^{\nu)} - 2m\dot{C}^\rho\dot{C}^\nu\,, \tag{100}$$

where $N_\nu = -\dot{C}_\nu$, m is the rest mass, X^μ is the displacement vector, P^μ is the rate of change of the displacement vector and $S^{\mu\nu}$ is the spin tensor satisfy

$$X_\mu\,\dot{C}^\mu = 0, \quad P_\mu\,\dot{C}^\mu = 0, \quad \dot{C}_\mu\,S^{\mu\nu} = 0 \quad \text{and} \quad S^{\mu\nu} + S^{\nu\mu} = 0\,. \tag{101}$$

Then we get the Mathisson-Papapetrou-Tulczyjew-Dixon for a dipole along a geodesics.

$$\dot{m} = 0\,, \quad \nabla_{\dot{C}}X^\mu = -P^\mu\,, \quad \nabla_{\dot{C}}P^\mu = \tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,S^{\kappa\rho} + R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,\dot{C}^\rho\,X^\kappa \quad \text{and} \quad \nabla_{\dot{C}}S^{\mu\nu} = 0\,, \tag{102}$$

Proof. Since C is a geodesic and $N_\nu = -\dot{C}_\nu$ then $\nabla_{\dot{C}}N_\nu = 0$. From (100) we have

$$\nabla_{\dot{C}}(N_\nu\,\xi^{\mu\nu\rho}) = \nabla_{\dot{C}}\Big(N_\nu\big(X^\rho\dot{C}^\mu\dot{C}^\nu + S^{\rho(\mu}\dot{C}^{\nu)}\big)\Big) = \nabla_{\dot{C}}\big(X^\rho\dot{C}^\mu + \tfrac{1}{2}S^{\rho\mu}\big) = \dot{C}^\mu\nabla_{\dot{C}}X^\rho + \tfrac{1}{2}\nabla_{\dot{C}}S^{\rho\mu}$$

While from (97) we have

$$\nabla_{\dot{C}}(N_\nu\,\xi^{\mu\nu\rho}) = -\pi^\rho_\alpha\xi^{\mu\alpha} = 2\pi^\rho_\alpha\big(m\dot{C}^\mu\dot{C}^\alpha - P^{(\mu}\dot{C}^{\alpha)}\big) = -P^\rho\dot{C}^\mu$$

giving

$$\dot{C}^\mu\nabla_{\dot{C}}X^\rho + \tfrac{1}{2}\nabla_{\dot{C}}S^{\rho\mu} = -P^\rho\dot{C}^\mu$$

Projecting this out with respect to $\dot{C}_\mu$ and π^α_μ gives (102.2) and (102.4).

From (100.3) we have

$$\nabla_{\dot{C}}(N_\nu \xi^{\mu\nu}) = \nabla_{\dot{C}}\Big(N_\nu\big(P^{(\mu}\dot{C}^{\nu)} - 2m\dot{C}^\mu\dot{C}^\nu\big)\Big) = \nabla_{\dot{C}}P^\mu - 2\dot{C}^\mu\nabla_{\dot{C}}m = \nabla_{\dot{C}}P^\mu - 2\dot{C}^\mu\dot{m}\nabla_{\dot{C}}P^\mu$$

Substituting (100.1) into (98) gives

$$\begin{aligned}\nabla_{\dot{C}}(N_\nu\xi^{\mu\nu}) &= (N_\nu\dot{C}^\beta + \tfrac{1}{2}\pi^\beta_\nu)\xi^{\rho\nu\lambda}R^\mu{}_{\rho\beta\lambda} = (N_\nu\dot{C}^\beta + \tfrac{1}{2}\pi^\beta_\nu)\left(X^\lambda\dot{C}^\rho\dot{C}^\nu + S^{\lambda(\rho}\dot{C}^{\nu)}\right)R^\mu{}_{\rho\beta\lambda} \\ &= \dot{C}^\beta X^\lambda\dot{C}^\rho R^\mu{}_{\rho\beta\lambda} + \frac{1}{2}N_\nu\dot{C}^\beta\dot{C}^\nu S^{\lambda\rho}R^\mu{}_{\rho\beta\lambda} + \frac{1}{2}N_\nu\dot{C}^\beta\dot{C}^\rho S^{\lambda\nu}R^\mu{}_{\rho\beta\lambda} \\ &\quad + \tfrac{1}{2}\pi^\beta_\nu\left(X^\lambda\dot{C}^\rho\dot{C}^\nu + S^{\lambda(\rho}\dot{C}^{\nu)}\right)R^\mu{}_{\rho\beta\lambda} \\ &= X^\lambda\dot{C}^\beta\dot{C}^\rho R^\mu{}_{\rho\beta\lambda} + \frac{1}{2}\dot{C}^\beta S^{\lambda\rho}R^\mu{}_{\rho\beta\lambda} + \tfrac{1}{2}\pi^\beta_\nu\left(X^\lambda\dot{C}^\rho\dot{C}^\nu + S^{\lambda(\rho}\dot{C}^{\nu)}\right)R^\mu{}_{\rho\beta\lambda} \\ &= X^\lambda\dot{C}^\beta\dot{C}^\rho R^\mu{}_{\rho\beta\lambda} + \frac{1}{2}\dot{C}^\beta S^{\lambda\rho}R^\mu{}_{\rho\beta\lambda} + \frac{1}{4}\dot{C}^\rho S^{\lambda\beta}R^\mu{}_{\rho\beta\lambda} \\ &= X^\lambda\dot{C}^\beta\dot{C}^\rho R^\mu{}_{\rho\beta\lambda} + \frac{1}{4}\dot{C}^\rho S^{\lambda\beta}R^\mu{}_{\rho\beta\lambda} + \frac{1}{4}\dot{C}^\rho S^{\lambda\beta}R^\mu{}_{\rho\beta\lambda} \\ &= X^\lambda\dot{C}^\beta\dot{C}^\rho R^\mu{}_{\rho\beta\lambda} + \frac{1}{2}\dot{C}^\rho S^{\lambda\beta}R^\mu{}_{\rho\beta\lambda}\,.\end{aligned}$$

hence

$$\nabla_{\dot{C}}P^\mu - 2\dot{C}^\mu\dot{m} = X^\lambda\dot{C}^\beta\dot{C}^\rho R^\mu{}_{\rho\beta\lambda} + \frac{1}{2}\dot{C}^\rho S^{\lambda\beta}R^\mu{}_{\rho\beta\lambda}$$

Projecting this out with respect to $\dot{C}_\mu$ and π^α_μ gives (102.1) and (102.3). □

5 Comparison with Dixon's results

In his work, Dixon makes two conjectures for the dynamics of the components of a quadrupole. Neither of these equations correspond to the divergenceless condition. Thus they are not the generalisation of the Mathisson-Papapetrou–Tulczyjew–Dixon equations for the quadrupole.

Recall from [3], that we can identify $\xi^{\mu\nu} = I^{\mu\nu}$, $\xi^{\mu\nu\kappa} = -I^{\kappa\mu\nu}$ and $\xi^{\mu\nu\kappa\lambda} = I^{\kappa\lambda\mu\nu}$. Using [9, eqn. (1.37)] and [6, eqn. (2.4)] we can[5] write $J^{\kappa\lambda\mu\nu} = I^{[\kappa[\lambda\mu]\nu]} = \frac{1}{4}(I^{\kappa\lambda\mu\nu} - I^{\mu\lambda\kappa\nu} - I^{\kappa\nu\mu\lambda} + I^{\mu\nu\kappa\lambda})$.

In [7, eqns. (7.34)-(7.37)] Dixon proposes a simple rotational dynamics. Here he introduces a connection (7.18), $\overset{M}{\nabla}_{\dot{C}}$ which he writes as $\overset{(m)}{\frac{\delta}{ds}}$

$$\overset{M}{\nabla}_{\dot{C}}B^\kappa = \nabla_{\dot{C}}B^\kappa - (\dot{u}^\kappa\,u^\lambda - \dot{u}^\lambda\,u^\kappa)B^\lambda\,, \qquad (103)$$

where u^κ is described as the body's dynamical velocity. Using this connection we can define a rotation tensor $\chi\Omega_{\lambda\kappa}$ where (7.34)

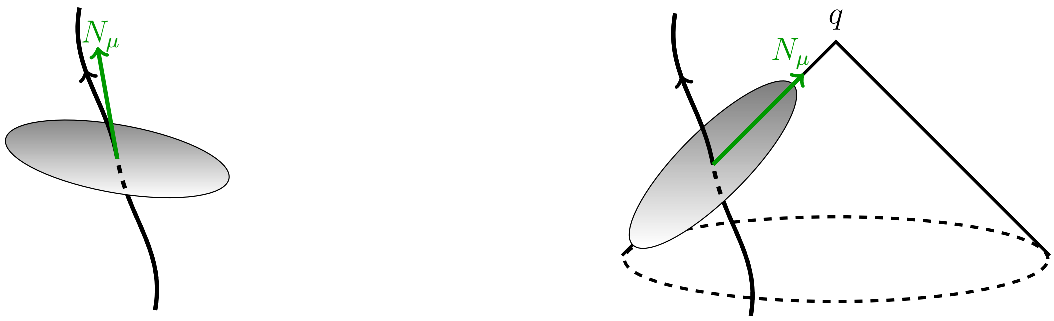
$$\overset{M}{\nabla}_{\dot{C}}B^\kappa = \chi\Omega^\kappa{}_\lambda\,B^\lambda, \quad \Omega_{(\kappa\lambda)} = 0 \quad \text{and} \quad \Omega^\kappa{}_\lambda\,u^\lambda = 0\,. \qquad (104)$$

From this the dynamical equation for a non-rotating quadrupole is given by (7.36)

$$\overset{M}{\nabla}_{\dot{C}}J^{\kappa\lambda\mu\nu} = -\chi\Omega^\kappa{}_\rho J^{\lambda\rho\mu\nu} + \chi\Omega^\lambda{}_\rho J^{\kappa\rho\mu\nu} - \chi\Omega^\nu{}_\rho J^{\kappa\lambda\rho\mu} + \chi\Omega^\mu{}_\rho J^{\kappa\lambda\rho\nu}\,. \qquad (105)$$

It is clear that such a non-rotating quadrupole would not satisfy (96)–(99) and therefore not correspond to a divergenceless stress-energy tensor.

[5]In [3], we identified $J^{\kappa\lambda\mu\nu}$ incorrectly.



(a) The Dixon vector given by the tangent to the worldline.

(b) The Dixon vector given by the backward lightcone of a distant observer event q.

Figure 2: The Dixon vector N_μ and the corresponding Dixon geodesic hypersurfaces, for two different scenarios.

By contrast in [8, eqn. (4.11)] Dixon proposes a non-dynamical equation based on symmetry. In our language this becomes

$$\xi^{\mu(\nu\rho\sigma)} = 0 \,. \tag{106}$$

This is very similar to the constraint (99) but without the projections. Interestingly such a constraint gives the same number of free components as the dynamical equations arising from the divergenceless condition. Again such conditions on $J^{\mu\nu\rho\sigma}$ do not correspond to the divergenceless condition.

6 Discussion and Conclusion

The key result of this article is derivation of the dynamics of the Dixon quadrupole, as given in section 4. It is clear looking at the method used to derive the equations, that this could be extended to arbitrary order multipoles, giving rise to higher covariant derivatives of the curvature. In section 5 we compared these equations with two which had be conjectured by Dixon. We observe that since Dixon's equations do not couple to curvature as they were not compatible with the divergenceless property of the stress-energy tensor.

In order to get to derive the dynamical equations it was necessary to establish many properties of general Dixon multipoles, presented in section 3. These include the fact that all multipoles can be represented as Dixon multipoles, that the components were unique and that they can be extracted using particular test tensors. We concluded this section showing the link between the moments of regular tensors and multipoles.

As noted throughout this article, all the results depend on the choice of the Dixon vector. There are multiple choices of such Dixon vectors, and there may not be a natural one. In figure 2 we see two such choice. In figure 2a we show the Dixon vector given by the tangent of the worldline. This is a natural choice if one is interested in the multipole dynamics as observed by the particle itself.

Alternatively, one may wish to model the dynamics of the multipole as we, as distant observers, see it. As seen in figure 2b, this involves constructing the backward lightcone from each event q in our worldline. This give rise to a lightlike Dixon vector which points from the multipole worldline, C to q. The corresponding Dixon geodesic hypersurface is then tangent to backward lightcone. It does not, however, coincide with the backward lightcone. For a very distant object the two would be very close and this may be a sufficiently good approximation. However if the object and the observer were closer this discrepancy may become important. It would then be necessary to use the Ellis representation of multipoles and a coordinate system adapted to backward lightcones. As was observed, the Dixon vector N_μ in figure 2b is lightlike. This does not affect any of the calculation as the only constraint on the Dixon vector is (3).

As noted in section 4, there are 20 free components of quadrupole. The equations for these components, also known as constitutive relations, will need to use additional information. This could be knowledge of the underlying matter which makes up the extended object, or they could arise from observation. Now that the tensorial expression of the dynamical equations for the moments are known, it will help in establishing the constitutive relations for different objects.

Once the additional equations have been chosen, one could ask how several quadrupoles gravitationally interact. As stated in [3] this interaction can only be perturbative, where the quadrupoles interact via gravitational waves. There is still the issue of gravitational radiation reaction. However, this can be avoided if we only consider the action of one quadrupoles by the fields generated by all the other quadrupoles. This is similar to the electromagnetic case considered in [13].

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