Degenerate regimes for random growth models in the complex plane

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Abstract

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Diffusion-limited aggregation (DLA) is among the most studied models in mathematical physics, and obtaining results on the limiting geometry has proved one of the more difficult problems of the past few decades.

At the same time, techniques from complex analysis and conformal mapping theory have become popular following the definition of the Schramm-Loewner evolution by Schramm in 2000 and its subsequent use to solve scaling limit and other problems for planar models in statistical physics.

This thesis analyses the *aggregate Loewner evolution* (ALE) model, introduced in 2018 [33] to generalise versions of DLA in the complex plane. The ALE is a model of growth where a particle is added at a location on the existing cluster at a point chosen by a regularised version of harmonic measure, transformed by a parameter η. The three main chapters of this thesis examine ALE for extreme values of η, where the behaviour becomes degenerate in some sense.

In Chapter 2, we demonstrate that for large negative values, $\eta < -2$, which correspond to attachment in areas of low harmonic measure, each particle is attached near the base of the previous particle. A consequence of this is the convergence of the ALE cluster to a Schramm-Loewner evolution curve. This contributes one of the first scaling limit results with a non-deterministic limit for an aggregation model in the complex plane.

In Chapter 3, we extend the results of [33] for $\eta > 1$, demonstrating that when started from a non-trivial initial configuration, the scaling limit is the geodesic Laplacian path model [4], a model of needle growth generalising several physical models.

In Chapter 4 we examine stability of the ALE for $\eta > 1$. We find a phase transition, with increasing stability such that an additional small perturbation survives if and only if $1 < \eta < 2$.

Contents

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Declaration

This thesis is my own work, and has not been submitted for the award of a higher degree elsewhere.

The introduction and Chapter 2 are mostly drawn from a paper, *SLE scaling limits for a Laplacian random growth model* on which I am the sole author, which has been accepted for publication in the *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*.

Table of notation

We list here the notation used throughout the thesis.

Subsets of the complex plane

$\mathbb{C}_\infty$	The Riemann sphere, $\mathbb{C} \cup \{\infty\}$
$\mathbb{D}$	The open unit disc $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$.
$\overline{\mathbb{D}}$	The closed unit disc $\{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$.
Δ	The exterior disc $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$.
$\mathbb{T}$	The unit circle $\partial\Delta = \{z \in \mathbb{C} : \lvert z\rvert = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$. We will often abuse notation and identify $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ where it is not ambiguous (although we will avoid writing $1 = 0$).
∂U	The boundary of a set $U \subseteq \mathbb{C}_\infty$, defined as $\partial U = \overline{U} \setminus U^\circ$.

Conformal maps

$\mathbf{c}$	A *capacity parameter* $\mathbf{c} > 0$, which will measure the size of our basic particles, and on which all the below maps and quantities depend.
f	The conformal map $f_{\mathbf{c}} \colon \Delta \to \Delta \setminus (1, 1 + d(\mathbf{c})]$ which we say attaches a particle to the unit circle at the point 1.
f_j	Given a sequence of angles $(\theta_j)_{j\geq 1}$, f_j attaches a particle to the unit circle at the point $e^{i\theta_j}$, so $f_j(z) := e^{i\theta_j} f(e^{-i\theta_j} z)$.
β	The distance from 1 of the points which are sent to the base of the particle by f. Defined uniquely as the $\beta = \beta(\mathbf{c}) \in (0, \pi)$ such that $f_{\mathbf{c}}(e^{\pm i\beta}) = 1$, and obeys $\beta \sim 2\mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$.
d	The length of the particle attached by f, defined by $f_{\mathbf{c}}(1) = 1 + d(\mathbf{c})$. Obeys $d \sim \beta \sim 2\mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$.
Φ_n	The conformal map which attaches the entire cluster of n particles to the unit circle at the point 1. Constructed as $f_1 \circ f_2 \circ \cdots \circ f_n$.
$\Phi_{j,n}$	The conformal map which attaches only the most recent $n - j$ particles to the unit circle. Given by $\Phi_{j,n} = \Phi_j^{-1} \circ \Phi_n$.

Model parameters

η The parameter controlling the relationship between our attachment distributions and the harmonic measure on the boundary of the cluster.

T The total capacity of our cluster, fixed throughout.

$\mathbf{c}$ The capacity of each individual particle attached to the cluster. In this thesis we consider the limit $\mathbf{c} \to 0$, and the following parameters are all functions of $\mathbf{c}$.

σ A regularisation parameter, used so that we evaluate our conformal maps Φ'_n on $e^{\sigma}\mathbb{T}$ instead of on $\mathbb{T}$ where they have poles. Throughout this thesis $\sigma \to 0$ as $\mathbf{c} \to 0$.

Probabilistic objects

h_{n+1} The density of the distribution on $\mathbb{T}$ of θ_{n+1}, conditional on $\theta_1, \dots, \theta_n$. Given by $h_{n+1}(\theta) \propto |\Phi'_n(e^{\sigma+i\theta})|^{-\eta}$.

Z_n The normalising factor for h_{n+1}. Given by $Z_n := \int_{\mathbb{T}} |\Phi'_n(e^{\sigma+i\theta})|^{-\eta}\, \mathrm{d}\theta$.

$\mathbb{P}$ The law of $(\theta_n)_{n\in\mathbb{N}}$. Implicitly depends on $\mathbf{c}$ and σ.

Approximations and bounds

We will use the following notation when we have two functions depending on a parameter x which is converging to some $x_0 \in \mathbb{R} \cup \{\pm\infty\}$, and we want to say the two functions are similar in some way, or that one bounds the other.

$f(x) \sim g(x)$ The ratio $\frac{f(x)}{g(x)} \to 1$ as $x \to x_0$.

$f(x) = O(g(x))$ The ratio $\left|\frac{f(x)}{g(x)}\right|$ is bounded as $x \to x_0$, so there exists a constant $C > 0$ such that $|f(x)| \le C|g(x)|$ in a neighbourhood of x_0. The constant C (and the neighbourhood on which the bound holds) should not depend on any other parameter or variable.

$f(x) = O_\rho(g(x))$ The ratio $\left|\frac{f(x)}{g(x)}\right|$ is bounded as $x \to x_0$, but the bound depends on a given parameter ρ. Throughout each section we hold T and η fixed, so we may occasionally omit these as subscripts when the constant depends on them.

$f(x) = O_T(g(x))$ See above, for the total capacity parameter T. This is the most common form in which we will use O_ρ defined above.

$f(x) = o(g(x))$ The ratio $\left|\frac{f(x)}{g(x)}\right| \to 0$ as $x \to x_0$.

When f and g are non-negative (particularly when they are probabilities or densities), we may use the following alternative notations.

$f(x) \lesssim g(x)$ The same as $f(x) = O(g(x))$, i.e. there exists a constant $C > 0$ such that $f(x) \le Cg(x)$ in a neighbourhood of x_0.

$f(x) \ll g(x)$ The same as $f(x) = o(g(x))$, i.e. $f(x)/g(x) \to 0$ as $x \to x_0$.

$f(x) \asymp g(x)$	Both $f(x) = O(g(x))$ and $g(x) = O(f(x))$, i.e. there exists constants $C_1, C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ in a neighbourhood of x_0.

Finally, we may write $f(x) \approx g(x)$, but this will only be used informally to mean that f and g behave similarly in some sense.

Notation specific to Chapter 2

D	A bound on $\min_{\pm} \lvert\theta_{n+1} - (\theta_n \pm \beta)\rvert$ which holds with high probability. If this distance exceeds D, we stop the process. We can take $D = \mathbf{c}^{9/2}\sigma^{1/2}$.
L	The maximum distance of z from $e^{i(\theta_n \pm \beta)}$ at which we rely on the estimates for $\lvert\Phi_{j,n}(z) - e^{i\theta_{j+1}}\rvert$ we obtain in the proof of Theorem 2.9. We take L to be a function of $\mathbf{c}$ which does not decay as rapidly as σ: $L = \mathbf{c}^{2^{N+1}}$.
$\theta_j^{\top}$	The point in $\mathbb{T}$ which θ_j was "supposed to" attach nearby to, i.e. the unique choice of $\theta_{j-1} \pm \beta$ which is within D of θ_j (if θ_j is not within D of either, we will have stopped the process at time $\tau_D \leq j$).
$\theta_j^{\perp}$	The choice of $\theta_{j-1} \pm \beta$ which isn't $\theta_j^{\top}$.
$\widehat{z}_j^n$	The point on $\mathbb{T}$ corresponding to the base of the jth particle in the cluster K_n, for $1 \leq j \leq n-1$. Given by $\widehat{z}_j^n := \Phi_{j,n}^{-1}(e^{i\theta_{j+1}^{\perp}})$. See Figure 2.5 for an illustration. We refer to the points on $\mathbb{T}$ close to $\widehat{z}_j^n$ for some j as *singular points* for h_{n+1}, and points away from all $\widehat{z}_j^n$ as *regular points*.
τ_D	The first time at which some θ_{n+1} is further than D from both of $\theta_n \pm \beta$. We stop the process when this happens, but show in Section 2.3 and Section 2.4 that with high probability $\tau_D > N := \lfloor T/\mathbf{c} \rfloor$.

The notation specific to Chapter 3 and Chapter 4 will be defined in those chapters.

Chapter 1

Introduction

1.1 Conformal aggregation

Models of random aggregation, where particles are added at each time step to the existing cluster at random location, are used to model many real-world growth processes. These models are perhaps most easily defined on the lattice $\mathbb{Z}^d$, where each particle is one vertex and growth occurs in discrete time, for example diffusion-limited aggregation (DLA) [34] or the Eden model [10].

In both DLA and the Eden model, the cluster begins with a single particle at $\{0\}$. In DLA, a particle is released from a long distance, performs a random walk on the lattice, and when it first reaches a site adjacent to the existing cluster, a new particle is added to the cluster in this location, the walk ends and a new one is begun from far away again.

In the Eden model, at each step a site is chosen uniformly from those adjacent to the existing cluster, and a new particle is added there.

In each of these models the underlying anisotropy of $\mathbb{Z}^d$ may be retained by the cluster on large scales under some conditions, an obvious problem if we want physically realistic behaviour. For example, simulations of DLA [11] have suggested that it grows along the principal axes of the lattice faster than anywhere else, so its law is not invariant under rotation. Similarly, in sufficiently high dimensions it is known that the limiting shape of the Eden model is a convex compact shape, but not a Euclidean ball [5].

In two dimensions we may change to a setting without this problem; models of *conformal growth* existing in the complex plane $\mathbb{C}$ rather than $\mathbb{Z}^2$. In this thesis we will study aspects of the aggregate Loewner evolution (ALE(α, η)) model introduced in [33], which is a generalisation of the Hastings–Levitov model (HL(α)) [14].

In a conformal aggregation model, we add particles to our cluster by composing conformal maps from a fixed reference domain to smaller domains. Our initial cluster will be the closed unit disc $K_0 = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. We attach a particle to the boundary of K_0, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, by applying a map from the complement of K_0 in the Riemann sphere $\mathbb{C}_\infty$, $\Delta := \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$, to a smaller domain, and then the new cluster will be the complement of the image of Δ. We will use particles of the form $(1, 1+d]$ for $d > 0$.

Definition. For any $d > 0$, by the Riemann mapping theorem there exists a unique bijective conformal map

$$f^d \colon \Delta \to \Delta \setminus (1, 1+d]$$

such that $f^d(z) = e^{\mathbf{c}} z + O(1)$ near ∞, for some $\mathbf{c} = \mathbf{c}(d) \in \mathbb{R}$.

One advantage of slit particles over more general particle shapes is that we have an explicit expression for $f^d(z)$ [21]. We call $\mathbf{c} > 0$ the *(logarithmic) capacity* of the particle. As the name suggests, we can view $\mathbf{c}$ as measuring the "size" of a set in a certain sense. As we consider the "small-particle limit" we will parameterise the model by the particle capacity $\mathbf{c}$, which determines d.

Definition. The preimage of the particle $(1, 1+d]$ under f is $\{e^{i\theta} : -\beta \leq \theta \leq \beta\}$, where $0 < \beta(\mathbf{c}) < \pi$ is uniquely determined by $f(e^{i\beta}) = 1$.

We can explicitly relate the quantities $\mathbf{c}$, β and d using two equations found in [21] and [33]: $4e^{\mathbf{c}} = (d+2)^2/(d+1)$ and $e^{i\beta} = 2e^{-\mathbf{c}} - 1 + 2ie^{-\mathbf{c}}\sqrt{e^{\mathbf{c}} - 1}$. Asymptotically, as $\mathbf{c} \to 0$, these give us $\beta(\mathbf{c}) \sim d(\mathbf{c}) \sim 2\mathbf{c}^{1/2}$.

We have maps which can attach one particle, so now we want to be able to build a cluster with multiple particles by composing maps which attach particles in different positions. For $\theta \in \mathbb{R}$ and $\mathbf{c} > 0$, define the rotated map

$$\begin{aligned} &f^{\theta,\mathbf{c}} \colon \Delta \to \Delta \setminus e^{i\theta}(1, 1+d(\mathbf{c})], \\ &f^{\theta,\mathbf{c}}(z) = e^{i\theta} f^{d(\mathbf{c})}(e^{-i\theta} z), \end{aligned}$$

and note that it has the same behaviour $f^{\theta,\mathbf{c}}(z) = e^{\mathbf{c}} z + O(1)$ near ∞ as does $f^{d(\mathbf{c})}$. We say that $f^{\theta,\mathbf{c}}$ attaches a particle of capacity $\mathbf{c}$ at $e^{i\theta}$.

Note that we will occasionally identify $\mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}$, and so for $z \in \mathbb{T}$ we may write $f^{z,\mathbf{c}}$ to mean $f^{\arg z,\mathbf{c}}$, and may speak about "attaching a particle at θ."

Now we want to attach multiple particles.

Definition. Given a sequence of angles $(\theta_n)_{n\in\mathbb{N}}$ and of capacities $(c_n)_{n\in\mathbb{N}}$, if we write $f_j = f^{\theta_j, c_j}$ then we can define

$$\Phi_n = f_1 \circ f_2 \circ \cdots \circ f_n, \tag{1.1}$$

and define the nth cluster K_n as the complement of $\Phi_n(\Delta)$, so

$$\Phi_n \colon \Delta \to \mathbb{C}_\infty \setminus K_n.$$

Note that the total capacity is $\mathbf{c}(K_n) = \sum_{k=1}^n c_k$, i.e. $\Phi_n(z) = e^{\sum_{k=1}^n c_k} z + O(1)$ near ∞.

Remark. At first it may seem surprising that the $(n+1)$th map Φ_{n+1} is obtained from Φ_n by *pre-composing* and setting $\Phi_{n+1} = \Phi_n \circ f_{n+1}$ rather than applying the new map to set $\Phi_{n+1} = f_{n+1} \circ \Phi_n$. However, in the latter case, the image $f_{n+1}(\Phi_n(\Delta))$ is in general not a subset of $\Phi_n(\Delta)$, and so the clusters $(K_n)_{n\geq 0}$ do not form a monotone increasing sequence of sets. By pre-composing, we ensure that $K_{n+1} \supset K_n$, and so we can think of the $(n+1)$th step as simply "attaching a particle" to the nth cluster. This order of composition is the one induced by Loewner's equation, as explained in Section 1.3.

We can now use this setup to construct various models of *random* growth, by choosing the angles $(\theta_n)_{n\in\mathbb{N}}$ and capacities $(c_n)_{n\in\mathbb{N}}$ according to a stochastic process.

1.2 Aggregate Loewner evolution

The *aggregate Loewner evolution* model introduced in [33] is a conformal aggregation model as in Section 1.1, where for the $(n+1)$th particle the distribution of its attachment angle θ_{n+1} and its capacity $c_{n+1} = \mathbf{c}(P_{n+1})$ are functions of the density of *harmonic measure* on the boundary of K_n. The conditional distribution of θ_{n+1} and the way we obtain c_{n+1} are respectively controlled by the two parameters η and α.

Definition. Inductively, we choose θ_{n+1} for $n \geq 0$ conditionally on $\theta_1, \dots, \theta_n$ according to the probability density function

$$h_{n+1}(\theta) = \frac{1}{Z_n} \left| \Phi_n' \left(e^{\sigma + i\theta} \right) \right|^{-\eta}, \ \theta \in (-\pi, \pi], \tag{1.2}$$

where $Z_n = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{-\eta} \, \mathrm{d}\theta$ is a normalising factor. We have introduced a "regularisation parameter" $\sigma = \sigma(\mathbf{c}) > 0$ because the poles and zeroes of Φ_n' on the boundary mean the measure h_{n+1} is not necessarily well-defined if $\sigma = 0$, but we take $\sigma \to 0$ as $\mathbf{c} \to 0$.

Remark. Some heuristic calculations show the relationship between (1.2) and harmonic measure. Suppose we have a map $\Phi : \Delta \to \mathbb{C}_\infty \setminus K$ where the boundary is smooth enough to ensure that Φ' is defined on $\partial\Delta$ and $(\Phi^{-1})'$ defined on ∂K. For a $z \in \partial K$, let $A \subseteq \partial K$ be a connected path of arc length δz containing z. The harmonic measure of A is the probability that a Brownian motion released from infinity first hits ∂K in A. As Brownian motion is invariant in distribution under conformal maps, and harmonic measure on $\partial\Delta$ is simply normalised Lebesgue measure, the harmonic measure of A is proportional to the length of $\Phi^{-1}(A)$. For small δz, this length is approximately $|(\Phi^{-1})'(z)| \, \delta z$.

Therefore we can view $|(\Phi^{-1})'(z)|$ as the density of harmonic measure with respect to Lebesgue measure $\mathrm{d}z$ on ∂K. Then (1.2) corresponds to choosing the attachment point on ∂K_n according to a distribution proportional to $|(\Phi_n^{-1})'(z)|^{\eta+1} \, \mathrm{d}z$. Positive values of η then correspond to an attachment distribution more concentrated in areas of high harmonic measure, and negative values of η correspond to a preference for areas of low harmonic measure.

Remark. This link between harmonic measure and the parameter η allows us to define a "continuum version" of discrete models such as the Eden model or DLA. The attachment point for the $(n+1)$th particle in DLA is chosen according to harmonic measure on the boundary of the nth cluster, so we could think of ALE with $\eta = 0$ as corresponding to DLA.

In the Eden model, the $(n+1)$th particle is added at a lattice site is chosen uniformly from all those unoccupied sites adjacent to the cluster at time n. This corresponds, in the continuum, to choosing an attachment position according to (normalised) Lebesgue measure on the boundary of the nth cluster K_n. In the remark above we calculated that the attachment position on ∂K_n in the ALE model is chosen according to a measure on ∂K_n approximately proportional to the product of $|\Phi_n'|^{\eta+1}$ with Lebesgue measure, and so ALE with $\eta = -1$ can be thought of as a continuum version of the Eden model.

Remark. The rate at which $\sigma \to 0$ as $\mathbf{c} \to 0$ can affect the behaviour of the ALE model. All of the results in this thesis depend on the small-scale structure of the clusters, so a small σ is needed for (1.2) to detect these details.

A natural question is whether any sense can be made of the model if $\sigma = 0$. If $|\eta|$ is small, the poles of $|\Phi_n'(e^{i\theta})|^{-\eta}$ may still be integrable, and so we can use (1.2) to define a measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, just as the $\eta = 0$ case does not need regularisation.

For a given cluster K_n, even if $|\eta|$ is large, the measure defined by (1.2) may converge to a (not necessarily absolutely continuous) limiting measure as $\sigma \to 0$. We use this as a heuristic in Section 2.1.1, and it is a natural way to think of the regimes of the ALE process where the attachment distributions are close to atomic. However, for regimes where we expect a universal limit, such as the many regimes in which an ALE cluster converges to a Euclidean disc, $\sigma > 0$ is natural, as the overall shape of the cluster is more important than its fine structure.

By our definition the first attachment point θ_1 is chosen uniformly on $\mathbb{T}$. For convenience we work with $\theta_1 = 0$, and the random case can be recovered by applying a random rotation to the final cluster.

After choosing θ_{n+1}, we choose the capacity of the $(n+1)$th particle to be

$$c_{n+1} = \mathbf{c}|\Phi_n'(e^{\sigma + i\theta_{n+1}})|^{-\alpha} \tag{1.3}$$

where $\mathbf{c}$ is a capacity parameter and $c_1 = \mathbf{c}$, and we will later consider the limit shape of the cluster as $\mathbf{c} \to 0$.

Remark. We discussed above that different values of η make the ALE a "continuum version" of well-known lattice models because the attachment distributions for $\eta = 0$ and $\eta = -1$ correspond to those of DLA and the Eden model, respectively. In each of these lattice models, the particles are all of exactly the same size: one lattice site. To replicate this property in the ALE, only $\alpha = 2$ gives particles of approximately the same length.

So ALE$(\alpha = 2, \eta = 0)$ is a continuum version of DLA, and ALE$(\alpha = 2, \eta = -1)$ is a continuum version of the Eden model. Like DLA, ALE$(2, 0)$ has proved very difficult to study, with most known results covering other ranges of (α, η).

Changing the parameters α, η and σ can give a wide variety of behaviours.

If $\eta = 0$, ALE$(\alpha, 0)$ coincides with the HL(α) model introduced by Hastings and Levitov [14]. Norris and Turner [26] showed that the HL(0) model, which is completely Markovian, converges to a Euclidean disc. For this same model, Silvestri [30] showed that the fluctuations around the limiting disc are given by a Gaussian process, and described the evolution and limiting behaviour of these fluctuations.

Sola, Turner and Viklund [33] showed for $\eta > 1$ and $\sigma \leq \mathbf{c}^\gamma$ for a large positive γ that the ALE(α, η) cluster converges to a single straight line. They showed that at each step, the $(n+1)$th particle attaches near the tip of the nth particle, which is the point of highest harmonic measure.

For slowly-decaying σ, many results of convergence to a disc are known. In particular, Norris, Silvestri and Turner [25] showed that if $\alpha + \eta \leq 1$ and $\sigma \gg \mathbf{c}^{1/2}$, then the resulting ALE clusters converge to a Euclidean disc in the small-particle limit. This covers the "continuum Eden model" ALE$(2, -1)$ we discussed earlier, although with a very strong regularisation.

A recent result worth mentioning, albeit slightly outside the ALE framework we have discussed above, is the proof by Liddle and Turner [19] that under "capacity rescaling", if $0 < \alpha < 2$ then the HL(α) cluster converges to a disc. Capacity rescaling means that instead of considering the limit $\mathbf{c} \to 0$, the initial particle size is constant, but the entire cluster is rescaled after attaching each particle in order to fix the total capacity. The authors further showed that if $\alpha = 0$, then for *any* choice of attachment locations, a growth model under capacity rescaling does not converge to a disc.

Many other variations of the Hastings–Levitov model have been studied recently, including modifications of the attachment rule [1] and definition in the half-plane [2].

1.3 Loewner's equation and the Schramm–Loewner evolution

We will give a brief overview here of the Schramm–Loewner evolution, and a few useful facts from Loewner theory which we use throughout the following chapters. For a more detailed treatment, see [3], [17] and [9].

Firstly, we look at *Loewner's equation*, which encodes a growing cluster by a "driving function" taking values on the circle.

Definition. Let $\xi : [0, T] \to \mathbb{R}$ be a càdlàg function. Then there is a unique solution to *Loewner's equation*

$$\varphi_0(z) = z, \quad \frac{\partial}{\partial t}\varphi_t(z) = \varphi_t'(z) z \frac{z + e^{i\xi_t}}{z - e^{i\xi_t}}, \quad z \in \Delta, \tag{1.4}$$

corresponding to a growing cluster parameterised by capacity via $\varphi_t(\Delta) = \mathbb{C}_\infty \setminus K_t$. The *driving function* ξ_t encodes which location on the cluster boundary is growing at time t.

Remark. Loewner's equation is a partial differential equation, parameterised by both time $t \in [0, \infty)$ and space $z \in \Delta$. We will generally denote derivatives with respect to time by $\frac{\partial}{\partial t}\varphi_t(z)$, and derivatives with respect to space by $\varphi_t'(z) = \frac{\partial}{\partial z}\varphi_t(z)$. Higher and mixed derivatives will be denoted in a similar way, e.g. $\frac{\partial}{\partial t}\varphi''(z) = \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\varphi_t(z)$.

For clusters growing at more than one point, or with growth supported on arbitrary subsets of the boundary, we can consider Loewner's equation with a driving *measure*.

Definition. Given a family of finite measures $(\mu_t)_{t\geq 0}$, subject to measurability conditions on $t \mapsto \mu_t$ there is a unique solution to Loewner's equation

$$\varphi_0(z) = z, \quad \frac{\partial}{\partial t}\varphi_t(z) = \varphi_t'(z)\int_{\mathbb{T}} z\frac{z+e^{i\theta}}{z-e^{i\theta}}\,\mathrm{d}\mu_t(\theta), \quad z \in \Delta. \tag{1.5}$$

For a sequence of angles $(\theta_n)_{n\geq 1}$ and a constant capacity $\mathbf{c}$, a growth model constructed as in Section 1.1 corresponds to the cluster obtained by solving Loewner's equation with the driving function

$$\xi_t = \theta_{\lfloor t/\mathbf{c}\rfloor + 1}.$$

Then the solution of Loewner's equation $(\varphi_t)_{t\geq 0}$ satisifies

$$\varphi_{n\mathbf{c}} = \Phi_n = f_1 \circ \cdots \circ f_n$$

for each $n \geq 0$, for Φ_n as defined in (1.1).

Definition. If $(B_t)_{t\in[0,T]}$ is a standard Brownian motion, then the *Schramm–Loewner evolution* with parameter $\kappa > 0$ (SLE_κ) is the random cluster obtained by solving Loewner's equation with the driving function given by $\xi_t = \sqrt{\kappa}B_t$.

Remark. One very useful property of Loewner's equation is that the map $D[0,T] \to \mathcal{K}$ given by $\xi \mapsto K_T$ is continuous [15], where $D[0,T]$ is the usual Skorokhod space and $\mathcal{K}$ is the set of compact subsets of $\mathbb{C}$ containing 0, equipped with the Carathéodory topology described in [9].

This property of Loewner's equation means we can deduce convergence of a cluster growing by adding particles of size $\mathbf{c}$ (so with a driving function with jumps spaced $\mathbf{c}$ apart) to another cluster by showing convergence of the driving functions (or measures). For example, in Chapter 2 we show that a cluster converges to an SLE_4 by showing that its driving function is close to a simple symmetric random walk which converges as $\mathbf{c} \to 0$ to $2B$ for a standard Brownian motion B.

The Carathéodory topology is strictly weaker than the Hausdorff topology, the usual way of metrising the set of compact subsets of $\mathbb{C}$. Some results on convergence of clusters are given with respect to the Hausdorff topology, such as [15] and [33]. In each of these results the limiting shape is a deterministic straight line, and so convergence is shown by proving the cluster stays within a narrow cone with high probability.

Schramm–Loewner evolutions describe the scaling limits of many discrete models, such as the loop-erased random walk, which converges to an SLE_2 curve [18], or critical percolation, the boundaries of which has been related to SLE_6 [31]. SLEs have also been used to construct the *quantum Loewner evolution* (QLE) [23] family of clusters, which have been proposed as the scaling limits of the dielectric breakdown model on a number of random surfaces.

Loewner theory contains many alterate versions of Loewner's equation, from changes of initial domains (such as $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ in place of Δ) to a version which describes the evolution of φ_t^{-1}. We refer the reader again to [17] for more details, and here we will introduce one particularly helpful variant: the *backwards equation.* The backwards equation was used for many of the estimates in [33], the results of which are generalised in our Chapter 3 and Chapter 4.

Definition. Fix $T > 0$. Let $\xi : [0, T] \to \mathbb{R}$ be a càdlàg function, and $(\varphi_t)_{t \in [0,T]}$ the solution to Loewner's equation with driving function ξ. Then the *backwards equation* is the family of differential equations

$$u_0(z) = z, \quad \frac{\partial}{\partial t} u_t(z) = u_t(z) \frac{u_t(z) + e^{i\xi_{T-t}}}{u_t(z) - e^{i\xi_{T-t}}}, \quad z \in \Delta, \tag{1.6}$$

for $t \in [0, T]$. Then $u_T = \varphi_T$, but it is not usually true that that $u_t = \varphi_t$ for $t < T$.

Remark. One very important fact about u is that it is governed by an ordinary differential equation, which makes it easy to obtain useful estimates.

Remark. If the driving function ξ corresponds to an aggregation process, so $\xi_t = \theta_{\lfloor t/\mathbf{c} \rfloor + 1}$, then solving Loewner's (forward) equation gives $\varphi_{n\mathbf{c}} = f_1 \circ \cdots \circ f_n$. If $T = N\mathbf{c}$, certain facts about $\varphi_T(z)$, $\varphi'_T(z)$, *etc.* are easier to establish if we understand $f_N(z)$, $f_{N-1}(f_N(z))$, *etc.*. For example, $\varphi'_T(z) = \prod_{n=1}^N f'_n((f_{n+1} \circ \cdots \circ f_N)(z))$, and the argument of f'_n is difficult to understand using the forward equation. But if we solve the backward equation, we get $u_{n\mathbf{c}} = f_{N-n+1} \circ \cdots \circ f_N$, so we could write $\varphi'_T(z) = \prod_{n=1}^N f'_n(u_{(N-n)\mathbf{c}}(z))$.

Chapter 2

ALE with small η converges to SLE_4

2.1 Main results

In this chapter, we will study the ALE model defined in Section 1.2 with $\alpha = 0$ and large negative values of the parameter η, which controls the influence of harmonic measure on our attachment locations.

The case $\alpha = 2$ often gives a model in which each particle is approximately the same size. Throughout the thesis we take $\alpha = 0$, where the model can be easier to analyse as the capacities are deterministic. In this case, the distortion of particles can lead to physically unrealistic outcomes, as in [19] where the distorted size of the final particle in the cluster does not disappear in the limit. For the model we are considering here, Figure 2.1 shows that the distortion affects the shape as well as the size of the particles.

For $\eta > 0$ the density h_{n+1} in (1.2) is an exaggeration of harmonic measure, and in [33] the authors find that for $\eta > 1$ the attachment distribution is concentrated around the point of highest harmonic measure, converging to a single atom as $\mathbf{c} \to 0$. For a slit particle the point of highest harmonic measure is the tip (see Figure 2.3), so this corresponds to the growth of a straight line. In Chapter 3 we extend this to configurations made up of several slits and growing along geodesics.

In this chapter, we find the equivalent phase transition in negative η: for $\eta < -2$ the attachment distribution is concentrated around the points of lowest harmonic measure. For a slit particle the two points of lowest harmonic measure are either side of the base (see Figure 2.3 again), and so $\theta_2 \approx \theta_1 \pm \beta$ with the probability of each tending to $1/2$ as $\mathbf{c} \to 0$. We go on to find that for all n the distribution of θ_{n+1} is concentrated around $\theta_n \pm \beta$, and so the angle sequence approximates a random walk of step length $\beta \sim 2\mathbf{c}^{1/2}$.

This gives us the following statement about the *driving function* generating the cluster (see Section 1.3):

Proposition 2.1. *Fix some $T > 0$. For $\eta < -2$ and if $\sigma(\mathbf{c}) \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$ for all $\mathbf{c} < 1$ let $(\theta_n^{\mathbf{c}})_{n \geq 1}$ be the sequence of angles we obtain from the* $\text{ALE}(0, \eta)$ *process with capacity*

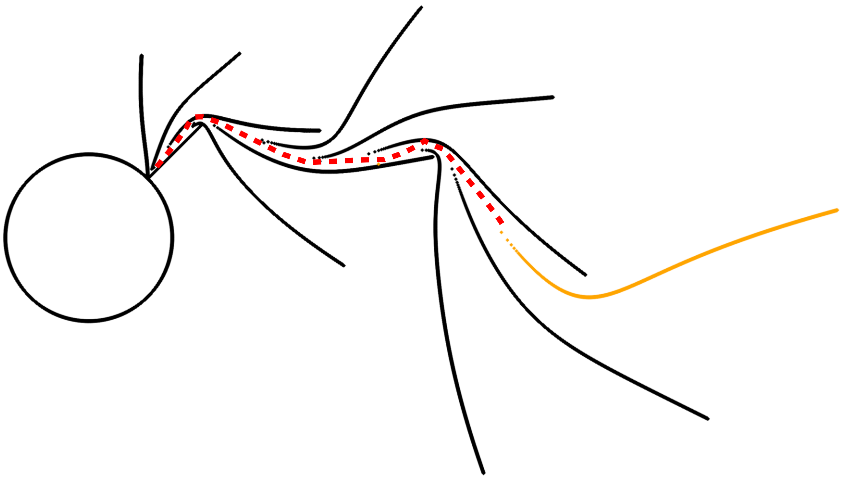

Figure 2.1: The final particle (the rightmost, in orange) of the cluster K_n is highly distorted by the application of the first $n-1$ maps $f_{n-1}, f_{n-2}, \ldots, f_1$. The distortion is much greater near the base of the particle: we have had to fill in a guess (the red dashed line) for the behaviour of the particle deep into the cluster, as the distortion is so large there that we are unable to find the exact location of enough points to draw a sensible diagram. In fact, the red dashed section corresponds to only $1/500\,000$th the length of the original, undistorted slit.

parameter $\mathbf{c}$*. Let* $D = \mathbf{c}^{9/2}\sigma^{1/2}$*. As* $\mathbf{c} \to 0$*,*

$$\mathbb{P}\left[\max_{n \leq \lfloor T/\mathbf{c} \rfloor} \min_{\pm} |\theta_n - (\theta_{n-1} \pm \beta_{\mathbf{c}})| > D\right] = O(\mathbf{c}^3).$$

Let $\xi_t^{\mathbf{c}} = \theta_{\lfloor t/\mathbf{c} \rfloor + 1}^{\mathbf{c}}$ *for all* $0 \leq t \leq T$*. Then*

$$(\xi_t^{\mathbf{c}})_{t \in [0,T]} \to (2B_t)_{t \in [0,T]} \text{ in distribution as } \mathbf{c} \to 0,$$

as a random function in the Skorokhod space $D[0,T]$*, where* B *is a standard Brownian motion.*

We explain in Section 1.3 that by using Loewner's equation we can immediately turn a result about convergence of such a driving function into a result about convergence of clusters in an appropriate space $\mathcal{K}$. The main theorem of this chapter therefore follows immediately from the proposition:

Theorem 2.2. *For* η, σ *as in Proposition 2.1, let the corresponding* ALE$(0, \eta)$ *cluster with* $N = \lfloor T/\mathbf{c} \rfloor$ *particles each of capacity* $\mathbf{c}$ *be* $K_N^{\mathbf{c}}$*. Then as* $\mathbf{c} \to 0$*,* $K_N^{\mathbf{c}}$ *converges in distribution as a random element of* $\mathcal{K}$ *to a radial* SLE$_4$ *curve of capacity* T*.*

We can see in Figure 2.2 a cluster corresponding to a random walk, which despite being visibly composed of slits resembles an SLE$_4$ curve.

Remark. We can give η a physical interpretation if we think of growth in which access to environmental resources (proportional to harmonic measure) affects the growth rate in a *non-linear* manner. For negative η we could also interpret ALE(α, η) as modelling a

Figure 2.2: One cluster of the ALE$(0, -\infty)$ process with 3,000 particles, each of capacity $\mathbf{c} = 0.0001$.

cluster in an environment which inhibits growth, so growth is concentrated in areas with the least exposure to the environment.

The most physically-relevant models are those with $\alpha = 2$, where each particle in the cluster has approximately the same size. The case we consider, $\alpha = 0$, is somewhat unphysical as the later particles have a macroscopic size (in our case the final particle has a shape approximating the whole path of the SLE_4). In this chapter the "visible" portion of each particle which is not hidden between other particles is microscopic, although the "visible" part of the later particles is still significantly longer than the first particles.

In any case, the remarkable thing about the $\eta < -2$, $\alpha = 0$ case is that it is drawn from a family of models which naturally extend DLA-type growth, and we obtain an SLE_4 scaling-limit for a whole range of parameters. To this author's knowledge no other conformal growth model in the plane has been rigorously proved to converge to a random limit such as the SLE.

Remark. The convergence of attachment distributions to atomic measures for $\eta < -2$ complements the phase transition result of [33] in which it is shown that the limiting attachment measures are atomic for $\eta > 1$. For $-2 < \eta < 1$ the distribution h_2 of the second particle is supported on all of $\mathbb{T}$ even in the limit $\mathbf{c} \to 0$, showing that we do indeed have three qualitative phases: for extreme values of η the attachment measures are degenerate, but this is not the case for $-2 < \eta < 1$.

Remark. There are several ways Theorem 2.2 could be extended. Firstly, it would not be difficult to show that $(K^{\mathbf{c}}_{\lfloor t/\mathbf{c} \rfloor})_{t \in [0,T]}$ converges as a time-dependent process to the SLE_4 process on $[0, T]$.

It is also reasonable to expect that the trace of the Nth particle itself will converge as $\mathbf{c} \to 0$ to an SLE_4 curve, as it "follows" the paths of the other particles, as can be seen in Figure 2.1. However, we have not managed to prove this.

The combination of these two possible extensions might suggest that if we grow the cluster $K^{\mathbf{c}}_N$ using Loewner's equation, then the growth of the final slit over the time interval

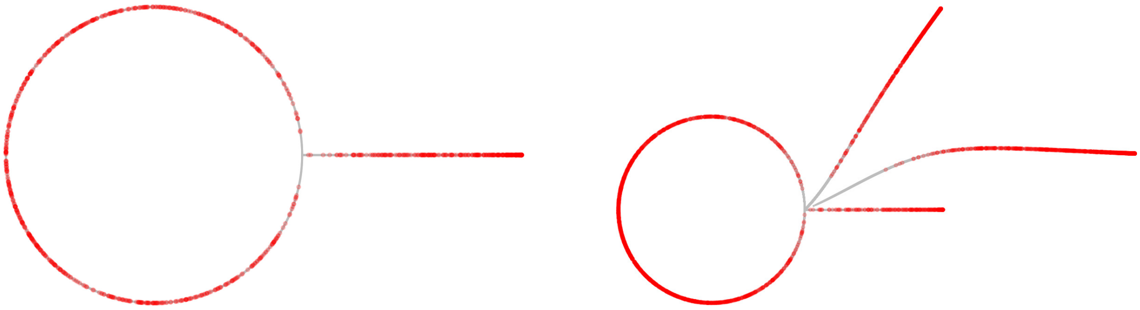

Figure 2.3: Left: the one-slit cluster of our process with 1,000 points in red sampled according to harmonic measure on the boundary. Right: the three-slit cluster of the process with 3,000 points sampled according to harmonic measure. Note in the second image that there are almost no points landing near the base of the most recent (longest) particle.

$t \in [(N-1)\mathbf{c}, N\mathbf{c})$ resembles the growth of an SLE_4 curve. However, the time change is extremely singular. The majority of the length of the Nth particle has almost no capacity, a fact which can also be seen in Figure 2.1, and so in Loewner's equation most of the Nth particle is grown in a very short time, and most of the interval $[(N-1)\mathbf{c}, N\mathbf{c})$ is spent growing the final small portion of the Nth particle.

Remark. In Section 2.6 we conjecture that similar scaling results to Proposition 2.1 and Theorem 2.2 can be obtained with particles other than the slit. Using suitable particles which have a single point of contact with the circle, we believe that the limiting cluster is always an SLE_κ for some $\kappa \in [4, \infty]$ (where by SLE_∞ we mean a uniformly growing disc).

2.1.1 Structure of proof

Our proof of Proposition 2.1 will involve showing that the distribution of θ_{n+1} conditional on the previous angles $(\theta_1, \dots, \theta_n)$ converges to $\frac{1}{2}(\delta_{\theta_n+\beta} + \delta_{\theta_n-\beta})$, and so the whole path $\xi^{\mathbf{c}}$ converges to the same limit as a simple random walk with step length $\beta \sim 2\mathbf{c}^{1/2}$.

We can use a heuristic approach to see why we might expect this to be the case. If we formally take $\eta = -\infty$ and $\sigma = 0$, so the nth attachment point θ_{n+1} is chosen uniformly from the finite set $\{\theta : \liminf_{\sigma\to 0} \inf_{w\in\mathbb{T}} |\Phi_n'(e^\sigma e^{i\theta})|/|\Phi_n'(e^\sigma w)| > 0\}$ (i.e. among the "strongest poles" of Φ_n'), and let $\tau = \inf\{n : |\theta_n - \theta_{n-1}| \neq \beta\}$, then we can calculate that for $N = \lfloor T/\mathbf{c} \rfloor$ in the limit $\mathbf{c} \to 0$ we have $\mathbb{P}[\tau \leq N] \to 0$ as $\mathbf{c} \to 0$. In other words, at each step the distribution h_{n+1} is equal to $\frac{1}{2}(\delta_{\theta_n-\beta} + \delta_{\theta_n+\beta})$.

Our approach for finite $\eta < -2$ will therefore be to find a small upper bound on $h_{n+1}(\theta)$ for θ away from the poles of Φ_n' to deduce that h_{n+1} is an approximation to a sum of atoms at the poles. Then we show separately that the contribution to $Z_n = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{-\eta}\, d\theta$ from poles other than $e^{i(\theta_n \pm \beta)}$ is small.

In the actual ALE model with $-\infty < \eta < -2$, we can only show that h_{n+1} *approximates* $\frac{1}{2}(\delta_{\theta_n-\beta} + \delta_{\theta_n+\beta})$ as $\mathbf{c} \to 0$. However, weak convergence of these measures is not enough to prove Proposition 2.1, so we will need to introduce some extra notation to describe the possible behaviour of the process $(\theta_n)_{n\geq 1}$, and make precise the way in which its steps converge to the SSRW steps as above.

Definition. For $D = D(\mathbf{c}) = \mathbf{c}^{9/2}\sigma^{1/2}$, define the stopping time

$$\tau_D := \inf\{n \geq 2 : \min(|\theta_n - (\theta_{n-1} + \beta)|, |\theta_n - (\theta_{n-1} - \beta)|) > D\}.$$

Remark. On the event $\{n < \tau_D\}$ we have a lot of information about the angle sequence $(\theta_1, \dots, \theta_n)$, and so can derive many properties of the conditional distribution of θ_{n+1}. In particular, we can say that the distribution of $\theta_{n+1} - \theta_n$ is (approximately) symmetric and that there is a very low probability that $n + 1 = \tau_D$. The results of the following sections will be used to establish these two facts.

Theorem 2.3. *Suppose that $\eta < -2$. There exists a constant $A > 0$ depending only on η and T such that when $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, then for $D = \mathbf{c}^{9/2}\sigma^{1/2}$, on the event $\{n < N \wedge \tau_D\}$ we have*

$$\int_{F_n} h_{n+1}(\theta)\,\mathrm{d}\theta \leq A\mathbf{c}^4 \tag{2.1}$$

almost surely, where $F_n = \{\theta \in \mathbb{T} : |\theta - (\theta_n + \beta)| \geq D$ and $|\theta - (\theta_n - \beta)| \geq D\}$, and almost surely

$$\left| \int_{\theta_n+\beta-D}^{\theta_n+\beta+D} h_{n+1}(\theta)\,\mathrm{d}\theta - \int_{\theta_n-\beta-D}^{\theta_n-\beta+D} h_{n+1}(\theta)\,\mathrm{d}\theta \right| \leq A\mathbf{c}^{11/4}. \tag{2.2}$$

In Section 2.2 we prove a number of technical results about the positions of the images and preimages of points $w \in \Delta$ under the maps f_j, $\Phi_n = f_1 \circ \cdots \circ f_n$, and $\Phi_{j,n} = \Phi_j^{-1} \circ \Phi_n$ when w is close to the poles of Φ_n'. When dealing with points away from these poles, we make extensive use of results from [33]. Our estimates for the positions of these images will be useful when we find upper bounds on the derivative $|\Phi_n'(w)| = |f_n'(w)| \times |f_{n-1}'(\Phi_{n-1,n}(w))| \times \cdots \times |f_1'(\Phi_{1,n}(w))|$, using lower bounds on the distance between $\Phi_{j,n}(w)$ and the poles of f_j'.

In Section 2.3.1 we integrate the pre-normalised density $|\Phi_n'(e^{\sigma+i\theta})|^{-\eta}$ over the regions around $\theta_n \pm \beta$, and so obtain an almost-sure lower bound on

$$Z_n = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{-\eta}\,\mathrm{d}\theta$$

on the event $\{n < N \wedge \tau_D\}$.

In Section 2.3.2 and Section 2.4 we find upper bounds on $|\Phi_n'(e^{\sigma+i\theta})|$ for $\theta \in F_n$, and so using the lower bound on Z_n we can establish the bound (2.1).

In Section 2.3.3 we establish the technical results needed to prove (2.2).

Remark. In our proof of Theorem 2.3, the convergence of h_{n+1} to $\frac{1}{2}(\delta_{\theta_n+\beta} + \delta_{\theta_n-\beta})$ does not rely on the convergence of $h_1, \dots, h_n$ to these *symmetric* discrete measures, only that the event $\{n < \tau_D\}$ occurs with high probability. If we were to use the fact that the angle sequence up until time n is very close to a simple symmetric random walk, then some properties (such as the fact that the longest interval on which a SSRW is monotone has length of order $O(\log n)$) would allow us to optimise our choice of σ further than we have. However, for the convergence of our cluster to an SLE_4 curve, we do require a σ which decays at least as quickly as $\mathbf{c}^{1/\mathbf{c}}$, which is already much faster than the fixed power of $\mathbf{c}$ used in [33] and elsewhere, so we have not attempted to optimise our choice of $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$.

Remark. The results of this chapter hold when σ decays very quickly. If σ decays very slowly, $\sigma \gg \mathbf{c}^{1/2}$, then a result of Norris, Silvestri and Turner [25] shows the scaling limit of ALE with $\alpha = 0$ and $\eta < -2$ is a disc, not an SLE_4. For the intermediate regime, $\mathbf{c}^{1/\mathbf{c}} \ll \sigma \ll \mathbf{c}^{1/2}$, heuristic arguments suggest that there is a period in which the driving function is a random walk, and then a period where the growth is measurable with respect to the random walk (i.e. a period of random growth and then a period of deterministic growth).

The first period ends when a particle is attached at the base of the second-most-recent particle rather than most recent. This occurs because although the pole of $\Phi_n'(e^{i\theta})$ at the most recent basepoint is of a higher order than the poles at older basepoints, a smaller value of σ is required in order for $\Phi_n'(e^{\sigma+i\theta})$ to "see" the higher-order poles.

For example, in Figure 2.4, the red points indicate the image of the sections of $e^{\sigma}\mathbb{T}$ near the strongest poles of Φ', and the yellow points are the image of sections near weaker poles. Although the pole is stronger near the red points, the image of the red points is further from the base, indicating that $\Phi'(e^{\sigma+i\theta})$ is a poor approximation to $\Phi'(e^{i\theta})$ for these points. We do not believe the resulting cluster converges to any known object as $\mathbf{c} \to 0$.

In Section 2.6 we define a family of particles for which we believe analogous versions of our main scaling result Theorem 2.2 holds. We conjecture that suitably constructed $\mathrm{ALE}(0, \eta)$ models with $\eta < -2$ will converge to either an SLE_κ with $\kappa \geq 4$, or to a uniformly growing disc. We also believe that every $\kappa \geq 4$ is attained by this family.

2.2 Spatial distortion of points

There are several steps we need to establish our upper bound on $\int h_{n+1}(\theta)\,\mathrm{d}\theta$ in (2.1), including precise estimates for $|\Phi_n'|$ near its poles. We can decompose the derivative

$$\Phi_n'(w) = \prod_{j=0}^{n-1} f_{n-j}'(\Phi_{n-j,n}(w)) \tag{2.3}$$

where

$$\Phi_{k,n} := \Phi_k^{-1} \circ \Phi_n = f_{k+1} \circ f_{k+2} \circ \ldots \circ f_n. \tag{2.4}$$

Then we have precise estimates on $|f'|$ near to its poles $e^{\pm i\beta}$, and upper bounds away from these poles, and so we write

$$|\Phi_n'(w)| = \prod_{j=0}^{n-1} \left| f'\left(e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)\right)\right|. \tag{2.5}$$

We will show that if w is close to one of $e^{i(\theta_n \pm \beta)}$, then for each j, the point $e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)$ is close to a pole of $|f'|$, and we will derive specific estimates on the distance in terms of the distance $|w - e^{i(\theta_n \pm \beta)}|$. Conversely, we will show that the only way for *every* image

$e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)$ to be close to a pole is for w to be close to $e^{i(\theta_n \pm \beta)}$, and so the measure $\mathrm{d}h_{n+1}$ is concentrated around $\theta_n + \beta$ and $\theta_n - \beta$.

Firstly, we will establish an estimate for $|f'|$ close to its poles $e^{\pm i\beta}$, and a universal upper bound away from these two points.

Lemma 2.4. *There are universal constants $A_1, A_2 > 0$ such that for all $\mathbf{c} < 1$, for $w \in \Delta$, if $|w - e^{i\beta}| \leq \frac{3}{4}\beta$, then*

$$A_1 \frac{\beta^{1/2}}{|w - e^{i\beta}|^{1/2}} \leq |f'_{\mathbf{c}}(w)| \leq A_2 \frac{\beta^{1/2}}{|w - e^{i\beta}|^{1/2}}, \tag{2.6}$$

and similarly if $|w - e^{-i\beta}| \leq \frac{3}{4}\beta$.

Moreover, there is a third constant A_3 such that if $\min\{|w - e^{i\beta}|, |w - e^{-i\beta}|\} > \frac{3}{4}\beta$, *then*

$$|f'_{\mathbf{c}}(w)| \leq A_3.$$

Proof. Using the expression

$$f'_{\mathbf{c}}(w) = \frac{f_{\mathbf{c}}(w)}{w} \frac{w-1}{(w - e^{i\beta})^{1/2}(w - e^{-i\beta})^{1/2}} \tag{2.7}$$

from Lemma 4 of [33], the result follows from simple calculations which we omit. ☐

This lemma tells us that the derivative $|\Phi'_n(w)|$ will be large only when many of the points $e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)$ in (2.5) are close to one of the poles $e^{\pm i\beta}$. We will next introduce some technical estimates which will allow us to determine for which points w this is true.

Remark. Imagine an idealised path in which $|\theta_{i+1} - \theta_i| = \beta$ for all i, then $f_n(e^{i(\theta_n \pm \beta)}) = e^{i\theta_{n-1}}$, $f_{n-1}(e^{i\theta_{n-1}}) = e^{i\theta_{n-2}}$, and so on. Then $\Phi_{n-j,n}(e^{i(\theta_n \pm \beta)}) = e^{i\theta_{n-j+1}} = e^{i(\theta_{n-j} + s_{n-j}\beta)}$, where $s_{n-j} \in \{\pm 1\}$. So in the ALE, if a point w is close to one of $e^{i(\theta_n \pm \beta)}$ then, as f is continuous when extended to $\overline{\Delta}$, each of the points in (2.5) is close to $e^{is_{n-j}\beta}$, but continuity alone does not allow us to make precise what we mean by "w is close to $e^{i(\theta_n \pm \beta)}$", so to estimate the size of $|\Phi'_n(w)|$, we need a precise estimate for $|f(w) - f(e^{i\beta})|$ in terms of $|w - e^{i\beta}|$.

Lemma 2.5. *For $w \in \Delta$, for all $\mathbf{c} < 1$, if $|w - e^{i\beta}| \leq \beta/2$, then*

$$\begin{aligned} |f_{\mathbf{c}}(w) - 1| &= 2(e^{\mathbf{c}} - 1)^{1/4}|w - e^{i\beta}|^{1/2} \\ &\quad \times \left(1 + O\left[\frac{|w - e^{i\beta}|}{\mathbf{c}^{1/2}} \vee \mathbf{c}^{1/4}|w - e^{i\beta}|^{1/2}\right]\right). \end{aligned} \tag{2.8}$$

Proof. We will work with the half-plane slit map $\widetilde{f}_{\mathbf{c}} \colon \mathbb{H} \to \mathbb{H} \backslash (0, i\sqrt{1 - e^{-\mathbf{c}}}\,]$ by conjugating f with the Möbius map $m_{\mathbb{H}} \colon \Delta \to \mathbb{H}$ given by

$$m_{\mathbb{H}}(w) = i\frac{w-1}{w+1},$$

and its inverse

$$m_\Delta(z) := m_{\mathbb{H}}^{-1}(z) = \frac{1-iz}{1+iz}.$$

The benefit of this is that $\widetilde{f}_{\mathbf{c}}$ has a simple explicit form

$$\widetilde{f}_{\mathbf{c}}(\zeta) = e^{-\mathbf{c}/2}\sqrt{\zeta^2 - (e^{\mathbf{c}}-1)}$$

where the branch of the square root is given by $\arg\colon \mathbb{C}\setminus[0,\infty) \to (0,2\pi)$, so we write

$$f_{\mathbf{c}} = m_\Delta \circ \widetilde{f}_{\mathbf{c}} \circ m_{\mathbb{H}}$$

and will derive a separate estimate for each of the three maps.

As w is close to $e^{i\beta} = 2e^{-\mathbf{c}} - 1 + 2ie^{-\mathbf{c}}\sqrt{e^{\mathbf{c}}-1}$, we will expand each map about the images (given by a simple calculation) $m_{\mathbb{H}}(e^{i\beta}) = -\sqrt{e^{\mathbf{c}}-1}$, $\widetilde{f}_{\mathbf{c}}(-\sqrt{e^{\mathbf{c}}-1}) = 0$, and $m_\Delta(0) = 1$. Our calculations will show that m_Δ and $m_{\mathbb{H}}$ behave like scaling by a constant close to the relevant points, and that the behaviour of $f_{\mathbf{c}}$ seen in (2.8) is due to the behaviour of $\widetilde{f}_{\mathbf{c}}$ close to $\pm\sqrt{e^{\mathbf{c}}-1}$.

First, when $w = e^{i\beta} + \delta$,

$$\begin{aligned}|m_{\mathbb{H}}(w) - m_{\mathbb{H}}(e^{i\beta})| &= \left|\frac{2\delta}{(e^{i\beta}+1+\delta)(e^{i\beta}+1)}\right| \\ &= \frac{1}{2}e^{\mathbf{c}}|\delta|(1+O(|\delta|))\end{aligned} \tag{2.9}$$

since a simple calculation shows that $|e^{i\beta}+1|^2 = 4e^{-\mathbf{c}}$.

Next, we will evaluate $\widetilde{f}_{\mathbf{c}}$ at a point close to one of the two preimages of 0, $\pm\sqrt{e^{\mathbf{c}}-1}$:

$$\begin{aligned}\left|\widetilde{f}_{\mathbf{c}}(\pm\sqrt{e^{\mathbf{c}}-1}+\lambda)\right| &= e^{-\mathbf{c}/2}\left|\sqrt{\pm 2\sqrt{e^{\mathbf{c}}-1}\lambda + \lambda^2}\right| \\ &= \sqrt{2}e^{-\mathbf{c}/2}(e^{\mathbf{c}}-1)^{1/4}|\lambda|^{1/2}\left(1+O\left(\frac{|\lambda|}{\mathbf{c}^{1/2}}\right)\right).\end{aligned} \tag{2.10}$$

Finally, for a small $z \in \mathbb{H}$,

$$|m_\Delta(z) - 1| = \left|\frac{1-iz}{1+iz} - 1\right| = \left|\frac{-2iz}{1+iz}\right| = 2|z|(1+O(|z|)). \tag{2.11}$$

Then for w close to $e^{i\beta}$, applying (2.9), (2.10) and (2.11) in turn, we obtain

$$\begin{aligned}|f(w)-1| = {} & 2(e^{\mathbf{c}}-1)^{1/4}|w-e^{i\beta}|^{1/2} \\ & \times \left(1+O\left(\frac{|w-e^{i\beta}|}{\mathbf{c}^{1/2}}\right)\right)\left(1+O\left(\mathbf{c}^{1/4}|w-e^{i\beta}|^{1/2}\right)\right).\end{aligned}$$

Then for $\mathbf{c}^{3/2} \le |w - e^{i\beta}| \le \beta/2$, we have the estimate (2.8) with error term of order $\mathbf{c}^{-1/2}|w-e^{i\beta}|$, and for $|w-e^{i\beta}| \le \mathbf{c}^{3/2}$ the error term has order $\mathbf{c}^{1/4}|w-e^{i\beta}|^{1/2}$. □

Remark. Unlike most results in this chapter, we will not use the following lemma in Section

2.3, but it will be very useful in Section 2.4.2. We include it here and omit the proof as it is very similar to Lemma 2.5.

Lemma 2.6. *For all* $\mathbf{c} < 1$*, if* $z \in \Delta \setminus (1, 1 + d(\mathbf{c})]$ *has* $|z - 1| \leq \mathbf{c}$*, then*

$$\min_{\pm} |f^{-1}(z) - e^{\pm i\beta}| = \frac{|z-1|^2}{4(e^{\mathbf{c}} - 1)^{1/2}} \left(1 + O\left(|z - 1|\right)\right).$$

Now we have all the technical results we need in order to prove our lower bound on $|\Phi_n'(w)|$ when w is close to one of the two "most recent basepoints" $e^{i(\theta_n \pm \beta)}$. We will derive the bound itself in Section 2.3.1, and here we will show that each of the points $\Phi_{n-j,n}(w)$ in (2.5) is close to $e^{i\theta_{n-j+1}}$.

Proposition 2.7. *Let* $L = L(\mathbf{c}, N) = \mathbf{c}^{2^{N+1}}$*. If we condition on the event* $\{n < N \wedge \tau_D\}$*, then the following is almost surely true: If* $\delta := \min |w - e^{i(\theta_n \pm \beta)}| \leq 2L$*, and* $|w| \geq e^{\sigma}$*, then for all* $1 \leq j \leq n$*,*

$$\left|\Phi_{n-j,n}(w) - e^{i\theta^{\top}_{n-j+1}}\right| = \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-j})} \delta^{2^{-j}} (1 + O(\mathbf{c}^4)). \tag{2.12}$$

Before we begin the proof we will introduce some notation in order to make the argument easier to follow.

Definition. By definition of τ_D, on the event $\{n < \tau_D\}$, exactly one of the two angles $\theta_{n-1} \pm \beta$ is within distance D of θ_n. We will call the closer of the two angles $\theta_n^{\top}$, and the other angle $\theta_n^{\perp}$.

Proof of Proposition 2.7. We will proceed by induction on j. For $j = 1$, the estimate (2.12) follows directly from Lemma 2.5. For a given $1 \leq j \leq n - 1$, assume that

$$|\Phi_{n-j,n}(w) - e^{i\theta^{\top}_{n-j+1}}| = \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-j})} \delta^{2^{-j}} (1 + O(\mathbf{c}^4)),$$

(as $|\theta_n - \theta_n^{\top}| < D \ll \mathbf{c}^4$, this certainly holds for $j = 1$) and then by the triangle inequality, since $|e^{i\theta_{n-j}} - e^{i\theta^{\top}_{n-j}}| \leq |\theta_{n-j} - \theta^{\top}_{n-j}| < D$, we have

$$|\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| - D \leq |\Phi_{n-j-1,n}(w) - e^{i\theta^{\top}_{n-j}}| \leq |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| + D.$$

Now by Lemma 2.5,

$$\begin{aligned}
&|\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| = |f_{n-j}(\Phi_{n-j,n}(w)) - f_{n-j}(e^{i\theta^{\top}_{n-j+1}})| \\
&= |f(e^{-i\theta^{\top}_{n-j+1}}\Phi_{n-j,n}(w)) - 1| \\
&= 2(e^{\mathbf{c}} - 1)^{\frac{1}{4}} |e^{-i\theta^{\top}_{n-j+1}}\Phi_{n-j,n}(w) - 1|^{1/2} (1 + O(\mathbf{c}^{1/4}|e^{-i\theta^{\top}_{n-j+1}}\Phi_{n-j,n}(w) - 1|^{1/2})) \\
&= \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{1+(1-2^{-j})} \delta^{2^{-(j+1)}} (1 + O(\mathbf{c}^4))(1 + O(\mathbf{c}^{3/8}\delta^{2^{-(j+1)}})) \\
&= \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-(j+1)})} \delta^{2^{-(j+1)}} (1 + O(\mathbf{c}^4))
\end{aligned}$$

and the second error term is absorbed since $\delta^{2^{-(j+1)}} \leq (2L)^{2^{-(j+1)}} \leq \mathbf{c}^4$.

Now as $\delta = |w - e^{i(\theta_n \pm \beta)}| \geq |w| - 1 \geq \sigma$, and $D = \mathbf{c}^{9/2}\sigma^{1/2}$ (see the table of notation), we have

$$\begin{aligned}
|\Phi_{n-j-1,n}(w) - e^{i\theta^{\top}_{n-j}}| &= |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| \left(1 + O\left(\frac{D}{\mathbf{c}^{\frac{1}{2}(1-2^{-(j+1)})}\delta^{2^{-(j+1)}}}\right)\right) \\
&= |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| \left(1 + O\left(\mathbf{c}^4\sigma^{1/4}\right)\right),
\end{aligned}$$

and hence our result almost surely holds for all $1 \leq j \leq n$ by induction. $\square$

2.3 The newest basepoints

In the following sections we will work with the densities $h_{n+1}(\theta) := \frac{|\Phi_n'(e^{\sigma+i\theta})|^{-\eta}}{Z_n}$. Since η is negative we will simplify the notation by introducing $\nu = -\eta > 0$. It will therefore be more intuitive for the reader to see that we deduce lower bounds on $|\Phi_n'|$ in order to find lower bounds on $h_{n+1} \propto |\Phi_n'|^{\nu}$.

2.3.1 A lower bound on the normalising factor

We defined in (1.2) the density function $h_{n+1}(\theta)$ and the nth normalising factor

$$Z_n := \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{-\eta} \, \mathrm{d}\theta = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{\nu} \, \mathrm{d}\theta. \tag{2.13}$$

If we are going to find upper bounds on h_{n+1} by bounding $|\Phi_n'|$, then we will need to have some lower bound on the normalising factor Z_n. In this section, we will obtain a lower bound on Z_n, and it will give us our upper bound on h_{n+1} in Section 2.4.2. First, we will need a good estimate for $|\Phi_n'|$ around the main poles $e^{i(\theta_n \pm \beta)}$.

Lemma 2.8. *For a given n, if we condition on the event $\{n < N \wedge \tau_D\}$ then the following is almost surely true: There are constants $A_1, A_2 > 0$ such that for any $\mathbf{c} < 1$, whenever $|\varphi| < L$,*

$$A_1^n \frac{\mathbf{c}^{\frac{1}{2}(1-2^{-n})}}{(\sigma^2 + \varphi^2)^{\frac{1}{2}(1-2^{-n})}} \leq \left|\Phi_n'\left(e^{\sigma + i(\theta_n \pm \beta + \varphi)}\right)\right| \leq A_2^n \frac{\mathbf{c}^{\frac{1}{2}(1-2^{-n})}}{(\sigma^2 + \varphi^2)^{\frac{1}{2}(1-2^{-n})}}$$

provided that $\sigma = \sigma(\mathbf{c}) \leq L$.

Proof. For $|\varphi| < L$, without loss of generality take $\theta = \theta_n + \beta + \varphi$. Since $\Phi_n = f_1 \circ \cdots \circ f_n$, by the chain rule

$$|\Phi_n'(e^{\sigma+i\theta})| = \prod_{j=0}^{n-1} \left| f'\left(e^{-i\theta_{n-j}} \Phi_{n-j,n}(e^{\sigma+i\theta})\right)\right|, \tag{2.14}$$

where $\Phi_{k,n} = \Phi_k^{-1} \circ \Phi_n = f_{k+1} \circ f_{k+2} \circ \cdots \circ f_n$.

By Proposition 2.7, if $\delta := |e^{\sigma+i\theta} - e^{i(\theta_n+\beta)}| < 2L$, then for all $1 \leq j \leq n-1$, $|\Phi_{n-j,n}(e^{\sigma+i\theta}) - e^{i\theta^{\top}_{n-j+1}}| = [2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}]^{2(1-2^{-j})}\delta^{2^{-j}}(1 + O(\mathbf{c}^4))$, and so by Lemma 2.4 (the above estimate shows that $e^{-i\theta_{n-j}}\Phi_{n-j,n}(e^{\sigma+i\theta})$ is close enough to one of $e^{\pm i\beta}$ to apply this lemma),

$$\begin{aligned}\left|f'\left(e^{-\theta_{n-j}}\Phi_{n-j,n}(e^{\sigma+i\theta})\right)\right| &\asymp \beta^{1/2}|\Phi_{n-j,n}(e^{\sigma+i\theta}) - e^{i\theta^{\top}_{n-j+1}}|^{-1/2} \\ &= \beta^{1/2}[2(e^{\mathbf{c}}-1)^{\frac{1}{4}}]^{-(1-2^{-j})}\delta^{-2^{-j-1}}(1+O(\mathbf{c}^4)) \\ &\asymp \mathbf{c}^{2^{-(j+2)}}\delta^{-2^{-(j+1)}}.\end{aligned}$$

For $j=0$, as $\Phi_{n,n}$ is the identity map,

$$\begin{aligned}|f'(e^{-i\theta_n}\Phi_{n,n}(e^{\sigma+i\theta}))| = |f'(e^{\sigma+i(\theta-\theta_n)})| &\asymp A_1\beta^{1/2}\delta^{-1/2} \\ &\asymp A\mathbf{c}^{1/4}\delta^{-1/2}.\end{aligned}$$

Now if we combine the bounds for each term in (2.14), we have

$$\begin{aligned}|\Phi_n'(e^{\sigma+i\theta})| &\geq \prod_{j=0}^{n-1}\left(A_1\mathbf{c}^{2^{-(j+2)}}\delta^{-2^{-(j+1)}}\right) \\ &= A_1^n\mathbf{c}^{\frac{1}{2}(1-2^{-n})}\delta^{-(1-2^{-n})}.\end{aligned}$$

and a similar upper bound. Finally, δ is given by

$$\begin{aligned}\delta &= |e^{\sigma+i\theta} - e^{i(\theta_n+\beta)}| \\ &= |e^{\sigma+i\varphi}-1| \\ &\asymp (\sigma^2+\varphi^2)^{1/2},\end{aligned}$$

and so, modifying the constants as necessary, we have our result. □

We can now obtain our lower bound on the normalising factor.

Proposition 2.9. *If $\eta < -2$ (so $\nu := -\eta > 2$), then there exists a constant A depending only on η such that for any fixed $T > 0$, if $N = T/\mathbf{c}$ then for sufficiently small $\mathbf{c}$, after conditioning on $\{n < N \wedge \tau_D\}$ we have the almost-sure lower bound*

$$Z_n \geq A^n\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}\sigma^{-[\nu(1-2^{-n})-1]} \tag{2.15}$$

provided that $\sigma = \sigma(\mathbf{c}) \leq L$.

Proof. The normalising factor Z_n is given by the integral $\int_{\mathbb{T}}|\Phi_n'(e^{\sigma+i\theta})|^\nu\,\mathrm{d}\theta$, and Lemma 2.8 gives us a lower bound on the integrand for θ close to $\theta_n+\beta$:

$$|\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu \geq A^n\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}(\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})}$$

when $|\varphi| < L$.

We will now integrate our lower bound over the interval $(\theta_n+\beta-L, \theta_n+\beta+L)$. First, note that

$$\begin{aligned}\int_{-L}^{L}(\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\mathrm{d}\varphi &= \int_{-L/\sigma}^{L/\sigma}(\sigma^2+\sigma^2x^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\sigma\mathrm{d}x \\ &= \sigma^{1-\nu(1-2^{-n})}\int_{-L/\sigma}^{L/\sigma}\frac{\mathrm{d}x}{(1+x^2)^{\frac{\nu}{2}(1-2^{-n})}} \\ &\geq A'\sigma^{1-\nu(1-2^{-n})}\end{aligned}$$

for a constant A', since the integral term on the right hand side increases as $\mathbf{c}\downarrow 0$ because $\sigma \ll L$. Note that this all remains true for any $\eta < 0$, and the fact that $\eta < -2$ will only be necessary in Section 2.3.2.

Finally, we can put together our bounds (and modify our constant A) to get

$$\begin{aligned}\int_{\theta_n+\beta-L}^{\theta_n+\beta+L}|\Phi_n'(e^{\sigma+i\theta})|^{\nu}\,\mathrm{d}\theta &\geq A^n\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}\int_{-L}^{L}(\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\mathrm{d}\varphi \\ &\geq A^n\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}\sigma^{1-\nu(1-2^{-n})}\end{aligned}$$

as required. □

2.3.2 Concentration about each basepoint

Most of our upper bounds on $|\Phi_n'|$ will be established in Section 2.4, but we will find one here as it uses the estimates from the previous section. Using the terminology we introduce in Section 2.4 and illustrate in Figure 2.4, in this section we look at *singular points* which are within L of one of the "main" poles $e^{i(\theta_n\pm\beta)}$ so the estimate of Lemma 2.8 is valid, but are not within D of these poles.

Proposition 2.10. *For a given n, if we condition on the event $\{n < N\wedge\tau_D\}$ then the following is almost surely true: For $\sigma(\mathbf{c}) \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, with $L = \mathbf{c}^{2^{N+1}}$ and $D = \mathbf{c}^{9/2}\sigma^{1/2} \ll L$,*

$$\frac{1}{Z_n}\int_{[-L,L]\setminus[-D,D]}|\Phi_n'(e^{\sigma+i(\theta_n\pm\beta+\varphi)})|^{\nu}\,\mathrm{d}\varphi = o(\mathbf{c}^{\gamma})$$

as $\mathbf{c}\to 0$, for any constant $\gamma > 0$, where $\nu := -\eta > 2$.

Proof. Using the symmetry of our upper bound in Lemma 2.8, it will be enough to find the upper bound $\int_D^L|\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^{\nu}\,\mathrm{d}\varphi \ll \mathbf{c}^{\gamma}Z_n$. We have, modifying the constant A_2 where necessary,

$$\begin{aligned}\int_{D}^{L}|\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^{\nu}\,\mathrm{d}\varphi &\leq A_2^n\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}\int_{D}^{L}(\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\mathrm{d}\varphi \\ &= A_2^n\frac{\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}}{\sigma^{\nu(1-2^{-n})-1}}\int_{D/\sigma}^{L/\sigma}(1+x^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\mathrm{d}x \\ &\leq A_2^n\frac{\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}}{\sigma^{\nu(1-2^{-n})-1}}\int_{D/\sigma}^{L/\sigma}x^{-\nu(1-2^{-n})}\,\mathrm{d}x \\ &\leq A_2^n\frac{\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}}{D^{\nu(1-2^{-n})-1}},\end{aligned}$$

and so, using our lower bound on Z_n,

$$\begin{aligned}\frac{\int_D^L |\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu \,\mathrm{d}\varphi}{Z_n} &\leq (A_2/A)^n \left(\frac{\sigma}{D}\right)^{\nu(1-2^{-n})-1} \\ &= (A_2/A)^n \left(\mathbf{c}^{-9/2}\sigma^{1/2}\right)^{\nu(1-2^{-n})-1}\end{aligned}$$

which, since $\nu(1-2^{-n})-1 \geq \frac{1}{2}\nu - 1 > 0$, decays faster than any power of $\mathbf{c}$ as $\mathbf{c} \to 0$. $\square$

Remark. The last line of the above proof is the only place in this chapter where we use that $\eta < -2$. If $-2 \leq \eta < 0$, then h_2 achieves its maximum around the two bases of the first particle, but does not have strong concentration around these points. For $-2 < \eta < 0$, h_2 is still supported on all of $\mathbb{T}$ as $\mathbf{c} \to 0$, so there is no concentration. If $\eta = -2$, the support of h_2 is concentrated around $\theta_1 \pm \beta$, but the event $D < |\theta_2 - (\theta_1 \pm \beta)| \ll \beta$ retains a high probability as $\mathbf{c} \to 0$. On this event, θ_2 is not close enough to $\theta_1 \pm \beta$ for our inductive arguments in Proposition 2.7 and Lemma 2.8 to apply. We can no longer guarantee that the poles of the second particle are stronger than the older pole at the base of the first particle, and so lose the SSRW-like behaviour of $(\theta_n)_{n\geq 1}$. It then becomes extremely difficult to say how the process behaves, but the scaling limit as $\mathbf{c} \to 0$ is unlikely to be described by the Schramm–Loewner evolution. Heuristic reasoning and some simulation results suggest that the behaviour may be similar to the "constrainted Hastings–Levitov" of [1], in which the cluster is not a single curve but something growing in $2\pi - o(1)$ directions for small $\mathbf{c}$.

2.3.3 Symmetry of the two most recent basepoints

There are two parts to the statement in Theorem 2.3 about convergence of h_{n+1} to the discrete measure $\frac{1}{2}(\delta_{\theta_n-\beta}+\delta_{\theta_n+\beta})$: the previous two sections and Section 2.4 establish that h_{n+1} is concentrated very tightly around $\theta_n \pm \beta$, and we will show here that the weight given to each of these two points is approximately equal.

Remark. Unlike the results from the previous two sections, the following proposition is not inductive, i.e. as long as $n < \lfloor T/\mathbf{c} \rfloor \wedge \tau_D$, the density h_{n+1} is approximately symmetric, even if the choices of the previous angles were not made symmetrically. Even in the extreme case where $(\theta_n)_{n\in\mathbb{N}}$ is close to an arithmetic progression: $\theta_2 \approx \theta_1 + \beta$, $\theta_3 \approx \theta_2 + \beta, \dots, \theta_n \approx \theta_{n-1} + \beta$, we still have an almost symmetric h_{n+1}.

Proposition 2.11. *For a given n, if we condition on the event $\{n < N \wedge \tau_D\}$ then the following is almost surely true:*

$$\sup_{|\varphi|<D} \left| \log \left(\frac{\left|\Phi_n'\left(e^{\sigma+i(\theta_n+\beta+\varphi)}\right)\right|}{\left|\Phi_n'\left(e^{\sigma+i(\theta_n-\beta-\varphi)}\right)\right|} \right) \right| \leq A\mathbf{c}^{11/4}$$

for some deterministic constant A depending only on T.

Proof. Let $z_\pm = \exp\left(\sigma + i\left[\theta_n \pm (\beta + \varphi)\right]\right)$ for $|\varphi| < D$, and write $\lambda_\pm = z_\pm - e^{i(\theta_n \pm \beta)}$. We can then write

$$\log\left(\frac{|\Phi_n'(z_+)|}{|\Phi_n'(z_-)|}\right) = \sum_{j=0}^{n-1} \log\left(\frac{|f_{n-j}'(\Phi_{n-j,n}(z_+))|}{|f_{n-j}'(\Phi_{n-j,n}(z_-))|}\right) \tag{2.16}$$

and so we can estimate each term in (2.16) separately.

The $j = 0$ term is exactly 0, by the symmetry of $|f_n'|$ about θ_n.

For $1 \leq j \leq n-1$, we will use Lemma 4 of [33], which states that

$$f'(z) = \frac{f(z)}{z} \frac{z-1}{(z-e^{i\beta})^{1/2}(z-e^{-i\beta})^{1/2}},$$

to compare the two derivatives in the jth term of (2.16). Write $z_\pm^j = \Phi_{n-j,n}(z_\pm)$, then the jth term in (2.16) is

$$|f_{n-j}'(z_\pm^j)| = \frac{|z_\pm^{j+1}|}{|z_\pm^j|} \frac{|z_\pm^j - e^{i\theta_{n-j}}|}{|z_\pm^j - e^{i\theta_{n-j+1}^\perp}|^{1/2}|z_\pm^j - e^{i\theta_{n-j+1}^\top}|^{1/2}} \tag{2.17}$$

There is some telescoping in the product which allows us to find

$$\prod_{j=1}^{n-1} \frac{|z_\pm^{j+1}|}{|z_\pm^j|} = \frac{|z_\pm^n|}{|z_\pm^1|}.$$

Then recall that in Section 2.2 we derived estimates for the distance of $z_\pm^n$ from $e^{i\theta_{n-j+1}^\top}$ in terms of $|\lambda_\pm|$. So by Proposition 2.7, as $e^{i\theta_1^\top} = 1$,

$$|z_\pm^n - 1| = \left[2(e^{\mathbf{c}}-1)^{\frac{1}{4}}\right]^{2(1-2^{-n})} |\lambda_\pm|^{2^{-n}}(1+O(\mathbf{c}^4)) = O(\mathbf{c}^{17/4})$$

since $|\lambda_\pm|^{2^{-n}} \lesssim D^{2^{-n}} \ll L^{2^{-n}} \leq \mathbf{c}^4$. Therefore $|z_\pm^n| = 1 + O(\mathbf{c}^{17/4})$, and similarly $|z_\pm^1| = 1 + O(\mathbf{c}^{17/4})$.

Having dealt with the first fraction in all derivatives (2.17) at once, we will tackle the remaining terms individually for each $1 \leq j \leq n-1$.

First note that by definition of $\theta_{n-j+1}^\top$, $|e^{i\theta_{n-j+1}^\top} - e^{i\theta_{n-j}}| = |e^{i\beta} - 1|$. Hence, using Proposition 2.7 again,

$$\begin{aligned} |z_\pm^j - e^{i\theta_{n-j}}| &= |e^{i\theta_{n-j+1}^\top} - e^{i\theta_{n-j}}| \left[1 + O\left(\frac{|z_\pm^j - e^{i\theta_{n-j+1}^\top}|}{|e^{i\theta_{n-j+1}^\top} - e^{i\theta_{n-j}}|}\right)\right] \\ &= |e^{i\beta} - 1| \left[1 + O\left(\mathbf{c}^{-2^{-(j+1)}}|\lambda_\pm|^{2^{-j}}\right)\right] \\ &= |e^{i\beta} - 1| \left[1 + O\left(\mathbf{c}^{15/4}\right)\right] \end{aligned}$$

since $|\lambda_\pm|^{2^{-j}} \ll L^{2^{-(n-1)}} \leq \mathbf{c}^4$.

Similarly,

$$|z_\pm^j - e^{i\theta_{n-j+1}^\perp}| = |e^{2i\beta} - 1|(1 + O(\mathbf{c}^{15/4})),$$

and finally, directly from Proposition 2.7,

$$|z_{\pm}^j - e^{i\theta_{n-j+1}^{\top}}| = \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-j})} |\lambda_{\pm}|^{2^{-j}} (1 + O(\mathbf{c}^4)).$$

Note that for the three estimates we just found, the only part which depends on the choice of $\pm$ is the error term (as $|\lambda_+| = |\lambda_-|$). Hence the part of the ratio of $|f_{n-j}'(z_+^j)|/|f_{n-j}'(z_-^j)|$ which comes from the second fraction in (2.17) is just $1 + O(\mathbf{c}^{15/4})$.

We can therefore find a constant A (which does not depend on n or φ) such that for each $1 \leq j \leq n-1$, $\left|\log\left(\frac{|f_{n-j}'(z_+^j)|}{|f_{n-j}'(z_-^j)|}\right)\right| \leq A\mathbf{c}^{15/4}$. As there are $O_T(\mathbf{c}^{-1})$ such terms in the product (2.16) (i.e. the number of such terms is bounded by $C\mathbf{c}^{-1}$ where the constant C may depend on T, as explained in the table of notation), we have

$$\left|\log\left(\frac{|\Phi_n'(z_+)|}{|\Phi_n'(z_-)|}\right)\right| = O_T(\mathbf{c}^{11/4})$$

as claimed. $\square$

Now we can deduce that h_{n+1} gives (asymptotically) the same measure to the sets $(\theta_n + \beta - D, \theta_n + \beta + D)$ and $(\theta_n - \beta - D, \theta_n - \beta + D)$.

Remark. Recall that earlier we used the heuristic argument that if $\eta = -\infty$ (so we choose from points with the highest-order pole), then we attach the $(n+1)$th particle to one of $\theta_n \pm \beta$, with equal probability. With finite $\eta < -2$, the derivative $|\Phi_n'|$ in fact differs slightly at each of $e^{\sigma + i(\theta_n + \beta)}$ and $e^{\sigma + i(\theta_n - \beta)}$, and so choosing to attach a particle at $e^{i\theta}$ for θ maximising $|\Phi_n'(e^{\sigma + i\theta})|$ leads to a deterministic process after the second step rather than our SLE_4 limit.

However, when we have a finite $\eta < -2$, integrating over the range $(-D, D)$ around each $\theta_n \pm \beta$ means that only the asymptotic behaviour of $|\Phi_n'|$ needs to be the same to guarantee symmetry between the two points $\theta_n \pm \beta$.

Corollary 2.12. *For a given n, if we condition on the event $\{n < N \wedge \tau_D\}$ then the following is almost surely true:*

$$\left|\int_{-D}^{D} h_{n+1}(\theta_n + \beta + \varphi)\,\mathrm{d}\varphi - \int_{-D}^{D} h_{n+1}(\theta_n - \beta - \varphi)\,\mathrm{d}\varphi\right| = O_T(\mathbf{c}^{11/4}). \tag{2.18}$$

Proof. From Proposition 2.11, we have

$$\begin{aligned}
&\int_{-D}^{D} h_{n+1}(\theta_n + \beta + \varphi)\,\mathrm{d}\varphi - \int_{-D}^{D} h_{n+1}(\theta_n - \beta - \varphi)\,\mathrm{d}\varphi \\
&\quad = \frac{1}{Z_n}\int_{-D}^{D} \left(|\Phi_n'(e^{\sigma + i(\theta_n + \beta + \varphi)})|^{\nu} - |\Phi_n'(e^{\sigma + i(\theta_n - \beta - \varphi)})|^{\nu}\right)\mathrm{d}\varphi \\
&\quad = \frac{1}{Z_n}\int_{-D}^{D} \left(|\Phi_n'(e^{\sigma + i(\theta_n + \beta + \varphi)})|^{\nu} - e^{O_T(\mathbf{c}^{11/4})}|\Phi_n'(e^{\sigma + i(\theta_n + \beta + \varphi)})|^{\nu}\right)\mathrm{d}\varphi \\
&\quad = O_T\left(\mathbf{c}^{11/4}\frac{\int_{-D}^{D}|\Phi_n'(e^{\sigma + i(\theta_n + \beta + \varphi)})|^{\nu}\,\mathrm{d}\varphi}{Z_n}\right)
\end{aligned}$$

almost surely, which is $O_T(\mathbf{c}^{11/4})$ by definition of Z_n. $\square$

2.4 Analysis of the density away from the main basepoints

In this section, we will classify the points $\theta \in \mathbb{T}$ with $|\theta - (\theta_n \pm \beta)| \geq D$ (i.e. the set F_n from Theorem 2.3) into *regular points* R_n where $h_{n+1}(\theta) \ll 1$, and *singular points* S_n where $h_{n+1}(\theta) \gtrsim 1$. We make this classification based on how close the image $\Phi_n(e^{\sigma+i\theta})$ is to the common basepoint of the cluster, which is the image of all the poles of Φ_n', as we can see in Figure 2.4.

In Section 2.4.1 we make this classification explicit and establish a bound on h_{n+1} for the regular points. In Section 2.4.2 we analyse the singular points more carefully and establish an upper bound on $\int_{S_n} h_{n+1}(\theta)\,\mathrm{d}\theta$ using similar techniques as in Section 2.3.1.

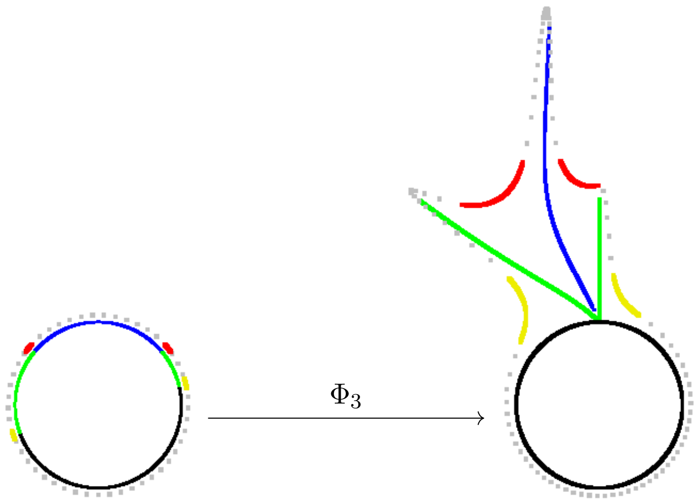

Figure 2.4: We can see on the left the three types of points in $e^{\sigma}\mathbb{T}$ for the three-slit cluster: we have the *singular points* in red and yellow and the *regular points* in grey dots. The right hand side of the diagram shows that a point on $e^{\sigma}\mathbb{T}$ is classified as regular if its image under Φ_n is far from the common basepoint (Proposition 2.13 in Section 2.4.1 shows that this implies $h_{n+1} \ll 1$), and the singular points are further classified into the two main (red) arcs containing $e^{i(\theta_n \pm \beta)}$, and the other (yellow) singular points. We have $h_{n+1} \gtrsim 1$ for all singular points, but we obtained a lower bound on the integral of $|\Phi_n'|^{-\eta}$ over the red regions in Section 2.3.1, and we will find an upper bound on the integral of this derivative over the yellow regions in Section 2.4.2. Note that the choice of σ we have used for this diagram is around $\mathbf{c}^2$ rather than the much smaller $\mathbf{c}^{2^{1/\mathbf{c}}}$, which is necessary to make the envelope $\Phi_3(e^{\sigma}\mathbb{T})$ clear, but does mean that some "regular" points are closer to the common basepoints than the red "singular" points. With a sufficiently small σ this isn't the case.

2.4.1 Regular points

In this section, we will establish a criterion for $\theta \in \mathbb{T}$ to be in our set of *regular* points for which $h_{n+1}(\theta) \ll 1$, based on the position of $\Phi_n(e^{\sigma+i\theta})$, as shown in Figure 2.4.

We will first derive an upper bound on $|\Phi_n'(w)|$ in terms of $|\Phi_n(w) - 1|$, so we can classify $w \in \Delta$ as a regular point using the distance of its image $\Phi_n(w)$ from 1.

Proposition 2.13. *Let* $N(\mathbf{c}) = \lfloor T/\mathbf{c} \rfloor$. *For* $\theta \in \mathbb{R}$, *let* $w = \exp(\sigma + i\theta)$. *On the event* $\{n < N(\mathbf{c}) \wedge \tau_D\}$, *the following is almost surely true:*

For any function $a : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $D^{2^{-N}}/\beta \le a(\mathbf{c}) \le \mathbf{c}^{3/2}$ *for all* $0 < \mathbf{c} < 1$, *if*

$$|\Phi_n(w) - 1| \ge \beta a(\mathbf{c}) \tag{2.19}$$

then, for sufficiently small $\mathbf{c}$,

$$|\Phi_n'(w)| \le A^n \beta^{n/2} \left(\frac{a(\mathbf{c})}{8}\right)^{-\frac{1}{2}(2^n-1)} \tag{2.20}$$

where A *is a universal constant independent of* a.

Proof. We will use the estimate (2.8) from Lemma 2.5. For convenience, let $z = \Phi_n(w)$, and we will estimate $|\Phi_n'(w)| = |(\Phi_n^{-1})'(z)|^{-1}$ by using (2.5) and estimating each term separately, using Lemma 2.5 to obtain estimates on $\Phi_{n-j,n}(w) = \Phi_{n-j}^{-1}(z)$ by induction on j.

First we claim that for $A(\mathbf{c}) \le \mathbf{c}^{1/2}$, and $\zeta \in \Delta \setminus (1, 1+d(\mathbf{c})]$, if we have $|\zeta - 1| \ge \beta A(\mathbf{c})$, then

$$\min_{\pm}(|f^{-1}(\zeta) - e^{\pm i\beta}|) \ge \frac{1}{4}\beta A(\mathbf{c})^2 \tag{2.21}$$

for all $\mathbf{c} < c_0$, where $c_0 > 0$ is a universal constant which doesn't depend on A.

To see this, suppose that $|f^{-1}(\zeta) - e^{i\beta}| < \frac{1}{4}\beta A(\mathbf{c})^2$. Then by Lemma 2.5, setting $\varepsilon = 2^{1/4} - 1 > 0$, for sufficiently small $\mathbf{c}$,

$$\begin{aligned}
|\zeta - 1| &= |f(f^{-1}(\zeta)) - f(e^{i\beta})| \\
&= 2(e^{\mathbf{c}} - 1)^{1/4} |f^{-1}(\zeta) - e^{i\beta}|^{1/2} \left(1 + O\left(A(\mathbf{c})^2 \vee \mathbf{c}^{1/2} A(\mathbf{c})\right)\right) \\
&< 2(\beta/2)^{1/2}(1+\varepsilon)\frac{1}{2}\beta^{1/2} A(\mathbf{c})(1+\varepsilon) \\
&= \beta A(\mathbf{c}),
\end{aligned}$$

so we have shown the contrapositive for our claim.

The derivative $|\Phi_n'(w)|$ is decomposed in (2.5) into the product of n terms of the form $\left|f'(e^{-i\theta_k}\Phi_{k,n}(w))\right|$, and so we can find an upper bound on $|\Phi_n'(w)|$ by obtaining lower bounds on each $|\Phi_{k,n}(w) - e^{i(\theta_k \pm \beta)}| = |\Phi_k^{-1}(z) - e^{i(\theta_k \pm \beta)}|$ for $0 \le k \le n-1$ and applying Lemma 2.4.

We claim that, for each $0 \le k \le n-1$,

$$|\Phi_k^{-1}(z) - e^{i\theta_{k+1}}| \ge \beta \times 8 \left(\frac{a(\mathbf{c})}{8}\right)^{2^k} \tag{2.22}$$

and we will show this using induction. For $k = 0$, (2.22) is exactly the assumption (2.19) of this proposition. For $k \ge 1$, we assume as the induction step that

$$|\Phi_{k-1}^{-1}(z) - e^{i\theta_k}| \ge \beta \times 8 \left(\frac{a(\mathbf{c})}{8}\right)^{2^{k-1}}$$

and aim to obtain (2.22) by applying (2.21).

Taking $A(\mathbf{c}) = 8\left(\frac{a(\mathbf{c})}{8}\right)^{2^{k-1}}$ in (2.21) gives us

$$|\Phi_k^{-1}(z) - e^{i\theta_{k+1}^\top}| \geq \beta \times 16\left(\frac{a(\mathbf{c})}{8}\right)^{2^k},$$

and so since $8\beta\left(\frac{a(\mathbf{c})}{8}\right)^{2^k} \geq 2D$ when $k < N \wedge \tau_D$ (for $\mathbf{c}$ sufficiently small),

$$\begin{aligned}|\Phi_k^{-1}(z) - e^{i\theta_{k+1}}| &\geq |\Phi_k^{-1}(z) - e^{i\theta_{k+1}^\top}| - |e^{i\theta_{k+1}} - e^{i\theta_{k+1}^\top}| \\ &\geq 16\beta\left(\frac{a(\mathbf{c})}{8}\right)^{2^k} - 2D \\ &\geq 8\beta\left(\frac{a(\mathbf{c})}{8}\right)^{2^k},\end{aligned}$$

verifying (2.22).

Then (2.22) tells us, using (2.21), that for each $0 \leq k \leq n-1$,

$$|\Phi_k^{-1}(z) - e^{i(\theta_k \pm \beta)}| \geq \beta \times 16\left(\frac{a(\mathbf{c})}{8}\right)^{2^k}, \tag{2.23}$$

and so, by Lemma 2.4, for $\mathbf{c}$ sufficiently small,

$$\begin{aligned}|\Phi_n'(w)| &= \prod_{k=0}^{n-1} |f_{k+1}'(\Phi_k^{-1}(z))| \\ &\leq A^n \beta^{n/2} \prod_{k=1}^{n-1}\left(\beta^{1/2}\left[\beta \times 16\left(\frac{a(\mathbf{c})}{8}\right)^{2^k}\right]^{-1/2}\right) \\ &= (A/4)^n \beta^{n/2}\left(\frac{a(\mathbf{c})}{8}\right)^{-\frac{1}{2}(2^n-1)}\end{aligned}$$

for a universal constant A. □

In the next section we will use these results with $a(\mathbf{c})$ equal to $\frac{L}{4\beta}$. We can easily check now that if we use this choice of a in Proposition 2.13 then, comparing (2.20) with (2.15), if σ decays as fast as $\mathbf{c}^{2^{2^N}}$ then $|\Phi_n'(z)|^\nu$ is far smaller than $\mathbf{c}Z_n$, for z away from the preimages of $e^{i\theta_1}$, and so if we classify our regular points as those θ for which $|\Phi_n(e^{\sigma+i\theta}) - 1| \geq L/4$ then we do have $\sup_{\theta \in R_n} h_{n+1}(\theta) \ll 1$.

2.4.2 Old singular points

In Section 2.3, we established a lower bound on the nth normalising factor Z_n. So to show that it is unlikely for the $(n+1)$th particle to be attached at a point in $E \subseteq \mathbb{T}$, we need to find an upper bound on $\int_E |\Phi_n'(e^{\sigma+i\theta})|^\nu \, \mathrm{d}\theta$, where $\nu := -\eta > 2$.

We did this over certain regions in Section 2.4.1 by finding a bound $|\Phi_n'(e^{\sigma+i\theta})|^\nu \ll \mathbf{c}Z_n$. In this section we will consider *singular* points where we can have $|\Phi_n'(e^{\sigma+i\theta})|^\nu \gg Z_n$.

However, if we look at Figure 2.4 we can see that not all singular points are close to the preimages $\theta_n \pm \beta$ of the base of the most recent particle; there are also singular points at the preimages of the base of older particle. We will therefore need to estimate the integrand $|\Phi_n'|^\nu$ more carefully, and show that when integrated over the singular points around these old bases and normalised by Z_n, the resulting probability is small.

The first thing we need to do is to describe precisely which points we are integrating over. We have previously classified our points into regular points R_n and singular points S_n by looking at the distance $|\Phi_n(w) - 1|$. Points are singular when $|\Phi_n(w) - 1| < \beta a(\mathbf{c})$ (for an $a(\mathbf{c})$ we will specify later), and we will find a way of differentiating between the "new" singular points around the preimages of the nth particle's base and the "older" singular points around the preimages of the other particles' bases. To make this clear, we will first give names to all of these preimages.

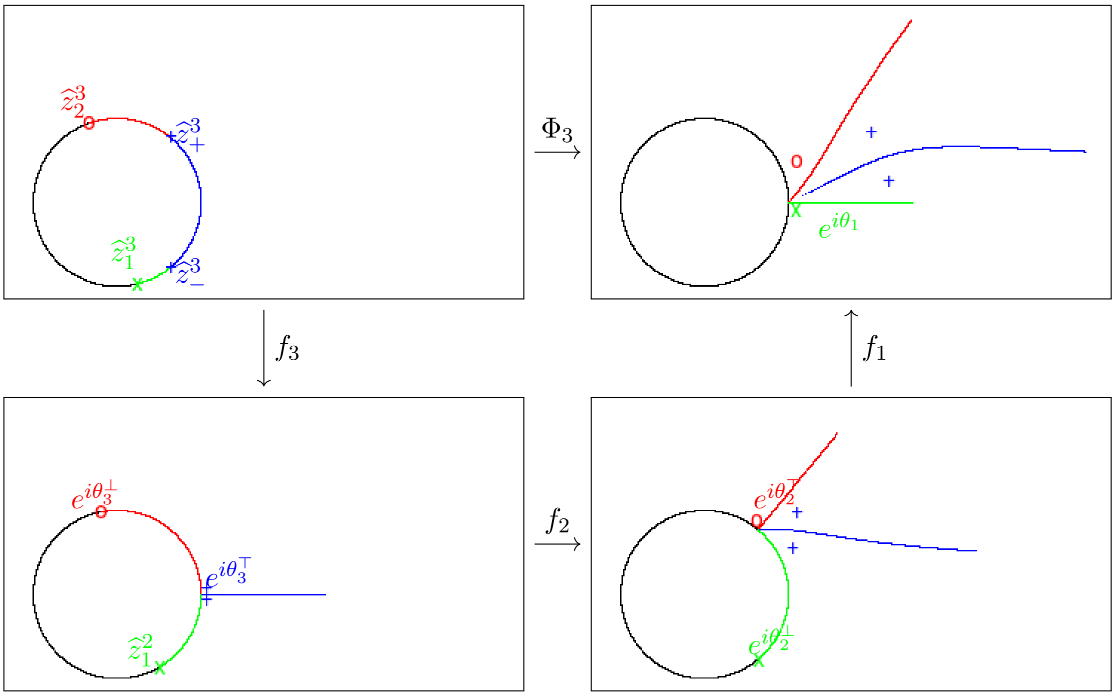

Figure 2.5: The construction of a cluster with three particles by composing the three maps f_3, f_2 and f_1. The top left diagram has labelled the four poles $\widehat{z}_\pm^3$, $\widehat{z}_2^3$ and $\widehat{z}_1^3$ of Φ_3' with text, and the markers $+$, $\times$ and $\circ$ have been used to track the images of $e^\sigma \widehat{z}$ for each pole $\widehat{z}$. By following the preimages of each point in the upper-right diagram through each map f_1, f_2 and f_3, we can see how we defined the "lesser" poles $\widehat{z}_2^3$ and $\widehat{z}_1^3$: for example, in the lower-right diagram $e^{i\theta_2^\perp}$ is a pole of f_1', its preimage under f_2 is $\widehat{z}_1^2$, and the preimage of $\widehat{z}_1^2$ under f_3 is $\widehat{z}_1^3$. Note that the three indicated intervals may overlap slightly, or have gaps between them, but these defects are too small to be seen in this diagram, and these $\widehat{z}$ points are well-defined in both the "$\eta = -\infty$" case where the intervals coincide perfectly, and the case of finite $\eta < -2$.

Firstly, we have the two "most attractive" points: the preimages of the base of the most recent (nth) slit. We will call these two points $\widehat{z}_\pm^n = e^{i(\theta_n \pm \beta)}$. Now the other points correspond to the bases of the $n-1$ other slits in the cluster, and we will denote them by $\widehat{z}_j^n$ for $1 \le j \le n-1$. The base of the first slit is the image under f_1 of the choice of $e^{i(\theta_2 \pm \beta)}$ which is *not* close to $e^{i\theta_2}$. We defined this earlier to be $e^{i\theta_2^\perp}$, and so the point sent to the base of the first slit by Φ_n is the preimage under $f_2 \circ \cdots \circ f_n = \Phi_{1,n}$ of $e^{i\theta_2^\perp}$, so set $\widehat{z}_1^n = \Phi_{1,n}^{-1}(e^{i\theta_2^\perp})$.

In general, when the jth slit is attached to the cluster by f_j, there are two points which are mapped to the base of the slit: $e^{i\theta_{j+1}^\top}$ (where the later slits are also attached), and $e^{i\theta_{j+1}^\perp}$, which has nothing else attached to it. Therefore, the point sent to the base of the jth slit by Φ_n is the preimage of $e^{i\theta_{j+1}^\perp}$ under $f_{j+1} \circ \cdots \circ f_n$. We can see this illustrated in Figure 2.5.

Definition. The base of the jth slit for $1 \le j \le n-1$ is the image of

$$\widehat{z}_j^n := \Phi_{j,n}^{-1}\left(e^{i\theta_{j+1}^\perp}\right) \tag{2.24}$$

under Φ_n.

Note that on the event $\{n < N \wedge \tau_D\}$, for all $1 \le j \le n-1$, we have

$$f_n(\widehat{z}_j^n) = \widehat{z}_j^{n-1}, \tag{2.25}$$

where we adopt the convention that $\widehat{z}_{n-1}^{n-1} = e^{i\theta_n^\perp}$.

Remark. We will bound $|\Phi_n'(w)|$ above when w is close to $\widehat{z}_j^n$, so first we will have to show that these points $\widehat{z}_j^n$ for $1 \le j \le n-1$ are not close to the points $e^{i(\theta_n \pm \beta)}$ where we have already shown $|\Phi_n'|$ is large.

Lemma 2.14. *Condition on the event $\{n < N \wedge \tau_D\}$, and let $1 \le j \le n-1$. Then, almost surely,*

$$|e^{i(\theta_n \pm \beta)} - \widehat{z}_j^n| \ge \mathbf{c}^{2^{n-j}},$$

when $\mathbf{c}$ is sufficiently small.

Proof. Assume for contradiction that $|e^{i(\theta_n+\beta)} - \widehat{z}_j^n| < \mathbf{c}^{2^{n-j}}$. By Lemma 2.5,

$$\begin{aligned}
|e^{i\theta_n} - \widehat{z}_j^{n-1}| &= |f_n(e^{i(\theta_n+\beta)}) - f_n(\widehat{z}_j^n)| \\
&= 2(e^{\mathbf{c}} - 1)^{1/4}\mathbf{c}^{2^{n-j-1}}\left(1 + O\left(\mathbf{c}^{1/4}\mathbf{c}^{2^{n-j-1}}\right)\right) \\
&< \frac{1}{2}\mathbf{c}^{2^{n-j-1}}
\end{aligned}$$

for $\mathbf{c}$ smaller than some universal c_0 (with $(c_0-1)^{1/4} < 1/4$, and small enough to make the error term irrelevant), and so

$$|e^{i\theta_n^\top} - \widehat{z}_j^{n-1}| \le |e^{i\theta_n^\top} - e^{i\theta_n}| + |e^{i\theta_n} - \widehat{z}_j^{n-1}| < \mathbf{c}^{2^{n-j-1}}, \tag{2.26}$$

since $|e^{i\theta_n^\top} - e^{i\theta_n}| \lesssim D \ll \mathbf{c}^{2^{n-j-1}}$. Then, as $\theta_n^\top = \theta_{n-1} \pm \beta$ for some choice of $\pm$, we can apply this argument repeatedly until we arrive at $|e^{i\theta_{j+1}^\top} - \widehat{z}_j^j| < \mathbf{c}^{2^{j-j}} = \mathbf{c}$. But as we noted after (2.25), $\widehat{z}_j^j = e^{i\theta_{j+1}^\perp}$, and $|e^{i\theta_{j+1}^\top} - e^{i\theta_{j+1}^\perp}| \sim 4\mathbf{c}^{1/2} \gg \mathbf{c}$, and so we have our contradiction. ∎

Remark. In fact the lower bound in Lemma 2.14 is fairly generous; it would take only a small amount of extra work in the proof above to get a tighter bound of $\mathbf{c}^{2^{n-j-1}}$, and we could improve this even further as we used the weak bound $(e^{\mathbf{c}}-1)^{1/4} < \frac{1}{4}$ in the initial calculation. However, all we need from Lemma 2.14 is a bound which decays more slowly than $L = \mathbf{c}^{2^{N+1}}$, and so we have chosen the bound which leads to the simplest possible proof.

Remark. The following corollary (which we will not prove) is not used in the proof of this chapter's main results, but does answer a question we may worry about: if we know that w is within L of some $\widehat{z}_j^n$, then is that j uniquely determined?

Corollary 2.15. *On the event $\{n < N \wedge \tau_D\}$, if $1 \le j < k \le n-1$, then almost surely*

$$|\widehat{z}_j^n - \widehat{z}_k^n| \ge \mathbf{c}^{2^{n-j}}$$

for sufficiently small $\mathbf{c}$.

Remark. The next result will be useful in telling us for which points $\theta \in \mathbb{T}$ we can bound $|\Phi_n'(e^{\sigma+i\theta})|$ above using Proposition 2.13, and will later help us locate those points for which Proposition 2.13 does not provide an upper bound.

Lemma 2.16. *Condition on the event $\{n < N \wedge \tau_D\}$. Let $w \in \Delta$. For all $\mathbf{c}$ sufficiently small, if $|\Phi_n(w) - 1| \le \frac{L}{4}$, then either $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \le L$, or there exists some $1 \le j \le n-1$ such that*

$$|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}| \le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j},$$

almost surely.

Proof. Suppose that there is no such j. We will show that $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \le L$. We now claim that $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\top}}| \le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$ for all $0 \le j \le n-1$ (where $\Phi_{0,n} = \Phi_n$ and $\theta_1^{\top} = \theta_1 = 0$). For $j = 0$ the claim is the true by assumption, and if the claim is true for $0 \le j < n-1$, then by Lemma 2.6, as $|\Phi_{j,n} - e^{i\theta_{j+1}}| \le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j} + |e^{i\theta_{j+1}^{\top}} - e^{i\theta_{j+1}}| \le \frac{\beta}{2}\left(\frac{L}{\beta}\right)^{2^j}$, for sufficiently small $\mathbf{c}$,

$$\begin{aligned}
\min(|\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\top}}|, |\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\perp}}|) &\le \frac{\frac{1}{4}\beta^2\left(\frac{L}{\beta}\right)^{2^{j+1}}}{4(e^{\mathbf{c}}-1)^{1/2}}\left(1 + \frac{1}{2}\right) \\
&= \frac{3\beta/2}{4(e^{\mathbf{c}}-1)^{1/2}} \times \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j} \\
&\le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}
\end{aligned}$$

since $\beta \sim 2(e^{\mathbf{c}}-1)^{1/2}$ for small $\mathbf{c}$. But we supposed that $|\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\perp}}| > \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^{j+1}}$, and so the above shows that $|\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\top}}| \le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^{j+1}}$ and by induction our claim holds. Finally, one more application of Lemma 2.6 after the $j = n-1$ case of our claim,

$|\Phi_{n-1,n}(w) - e^{i\theta_n}| \le \frac{\beta}{2}\left(\frac{L}{\beta}\right)^{2^{n-1}}$, tells us that $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \le \frac{3\beta/2}{16(e^{\mathbf{c}}-1)^{1/2}}\beta\left(\frac{L}{\beta}\right)^{2^n} \ll L$, as required. $\square$

Remark. We intend to use this lemma to find a precise expression for our set S_n of singular points and then we can make a precise estimate on the size of $|\Phi_n'(e^{\sigma+i\theta})|$ for $\theta \in S_n$ as we did in Lemma 2.8. For a singular point w, Lemma 2.16 tells us that for some j, $\Phi_{j,n}(w)$ is close to $e^{i\theta_{j+1}^{\perp}}$, and we now need to turn that into an estimate for the distance between w and $\Phi_{j,n}^{-1}(e^{i\theta_{j+1}^{\perp}}) = \hat{z}_j^n$.

Corollary 2.17. *On the event $\{n < N \wedge \tau_D\}$, the following is almost surely true: For all $\mathbf{c}$ sufficiently small, for any $w \in \Delta$, if $|\Phi_n(w) - 1| \le \frac{L}{4}$ then either $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \le L$ or there exists some $1 \le j \le n-1$ such that*

$$|w - \hat{z}_j^n| \le A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j},$$

where A is some deterministic universal constant.

Proof. To deduce this from Lemma 2.16, we need only show that there is some constant A such that $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}| \le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j} \implies |w - \hat{z}_j^n| \le A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$. Fix some $1 \le j \le n-1$. We will show that for $j \le k \le n-1$, $|\Phi_{k+1,n}(w) - \hat{z}_j^{k+1}| \le A|\Phi_{k,n}(w) - \hat{z}_j^k|$.

Fix a path $\gamma : (0,1] \to \Delta$ with $\lim_{\varepsilon \downarrow 0} \gamma(\varepsilon) = \hat{z}_j^k$, $\gamma(1) = \Phi_{k,n}(w)$, and $|\gamma(t) - \hat{z}_j^k| \le |\Phi_{k,n}(w) - \hat{z}_j^k|$ for all $t \in (0,1]$. We can also choose γ in such a way that it has arc length $\ell := \int_\gamma |\mathrm{d}z| \le 2|\Phi_{k,n}(w) - \hat{z}_j^k|$. By the fundamental theorem of calculus,

$$\begin{aligned}
|\Phi_{k+1,n}(w) - \hat{z}_j^{k+1}| &= |f_{k+1}^{-1}(\Phi_{k,n}(w)) - f_{k+1}^{-1}(\hat{z}_j^k)| \\
&= \left|\int_\gamma (f_{k+1}^{-1})'(\zeta)\,\mathrm{d}\zeta\right| \\
&\le \ell \times \sup_{\zeta \in \gamma(0,1]} |(f_{k+1}^{-1})'(\zeta)| \\
&= \frac{\ell}{\inf_{\omega \in f_{k+1}^{-1}(\gamma(0,1])} |f_{k+1}'(\omega)|}.
\end{aligned}$$

Now there must be some constant $M \ge 1$ such that $|\omega - e^{i\theta_{k+1}}| \ge \beta/M$ for all $\omega \in f_{k+1}^{-1}(\gamma(0,1])$. Otherwise, if $|\omega - e^{i\theta_{k+1}}| < \beta/M$, then it is easy to check using the explicit form of $f_{\mathbf{c}}$ from [21] that $|f_{k+1}(\omega) - e^{i\theta_{k+1}}(1+d)| = O(\beta/M^2)$, and so

$$|\hat{z}_j^k - e^{i\theta_{k+1}}(1+d)| \le |f_{k+1}(\omega) - e^{i\theta_{k+1}}(1+d)| + |f_{k+1}(\omega) - \hat{z}_j^k| \le \frac{1}{2}d$$

for sufficiently large M, contradicting $|\hat{z}_j^k| = 1$. Hence by Lemma 2.4, there is a constant A such that

$$\inf_{\omega \in f_{k+1}^{-1}(\gamma(0,1])} |f_{k+1}'(\omega)| \ge 2A^{-1}.$$

We therefore obtain

$$|\Phi_{k+1,n}(w) - \widehat{z}_j^{k+1}| \le A|\Phi_{k,n}(w) - \widehat{z}_j^k| \tag{2.27}$$

for all $j \le k \le n-1$, and so

$$|w - \widehat{z}_j^n| = |\Phi_{n,n}(w) - \widehat{z}_j^n| \le A^{n-j}|\Phi_{j,n}(w) - \widehat{z}_j^j| \le A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j},$$

as required. □

If we let L_j^n be the upper bound in Corollary 2.17, then we now have a necessary condition for points to be singular, based only on their location: if $e^{\sigma+i\theta} \in \Delta$ is not within L_j^n of $\widehat{z}_j^n$ for some j, then θ is regular. The set of singular points S_n is therefore contained in the union of only $n+1$ intervals centred around $e^{\theta_n \pm \beta}$ and each $\widehat{z}_j^n$.

We can now find a precise estimate for $|\Phi_n'|$ on S_n as we did in Lemma 2.8. The proof will also be similar to that of Lemma 2.8.

Lemma 2.18. *Condition on the event $\{n < N \wedge \tau_D\}$, and let $1 \le j \le n-1$. If $\mathbf{c}$ is sufficiently small, then for all $w \in \Delta$ with $|w| = e^\sigma$ and $|w - \widehat{z}_j^n| \le A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$, for A as in Corollary 2.17, we have*

$$|\Phi_n'(w)| \le B^n \mathbf{c}^{\frac{n-j}{4}+1}\mathbf{c}^{\frac{1}{2}(1-2^{-j})}\frac{1}{\mathbf{c}^{2^{n-j}}}|w - \widehat{z}_j^n|^{-(1-2^{-j})}$$

where B is a universal constant.

Proof. We will complete the proof by finding bounds on $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^\pm}|$; an upper bound to show $|\Phi_{j,n}'(w)|$ is small, and a lower bound to show $|\Phi_j'(\Phi_{j,n}(w))|$ is small. The rest of the proof will be similar to the way we deduced Lemma 2.8 from Proposition 2.7.

First, we will estimate the positions of $\Phi_{n-1,n}(w), \Phi_{n-2,n}(w), \dots, \Phi_{j,n}(w)$. As in the proof of Corollary 2.17, for $j+1 \le k \le n$,

$$\begin{aligned}|\Phi_{k-1,n}(w) - \widehat{z}_j^{k-1}| &= |f_k(\Phi_{k,n}(w)) - f_k(\widehat{z}_j^k)| \\ &\le 2|\Phi_{k,n}(w) - \widehat{z}_j^k| \times \sup_{|\zeta - \widehat{z}_j^k| \le |\Phi_{k,n}(w) - \widehat{z}_j^k|} |f_k'(\zeta)|,\end{aligned} \tag{2.28}$$

so we need only bound $|f_k'(\zeta)|$ for ζ close to $\widehat{z}_j^k$. We will also need inductively that $|\Phi_{k,n}(w) - \widehat{z}_j^k|$ is small in order to say that ζ *is* close to $\widehat{z}_j^k$.

Claim. For $j+1 \le k \le n$, $|\Phi_{k,n}(w) - \widehat{z}_j^k| \le A^{n-j}\mathbf{c}^{3\times 2^n}$ for sufficiently small $\mathbf{c}$.

The claim is true for $k = n$, as

$$|w - \widehat{z}_j^n| \le A^{n-j}\mathbf{c}^{1/2}\left(\frac{1}{2}\mathbf{c}^{2^{n+1}-1/2}\right)^{2^j} \le A^{n-j}\mathbf{c}^{2^{n+j+1}-2^{j-1}} \le A^{n-j}\mathbf{c}^{2^{n+2}-2^n}.$$

Then, if the claim holds for all $l \geq k$, we have

$$|\Phi_{l,n}(w) - \widehat{z}_j^l| \leq A^{n-j}\mathbf{c}^{3\times 2^n} \leq \frac{1}{2}\mathbf{c}^{2^{l-j}}$$

for all sufficiently small $\mathbf{c}$, and so, by Lemma 2.14 and the triangle inequality, for all ζ such that $|\zeta - \widehat{z}_j^l| \leq |\Phi_{l,n}(w) - \widehat{z}_j^l|$, we have $\min_{\pm} |\zeta - e^{i(\theta_l \pm \beta)}| \geq \frac{1}{2}\mathbf{c}^{2^{l-j}}$. Hence by Lemma 2.4,

$$|f_k'(\zeta)| \leq A_2 \frac{\mathbf{c}^{1/2}}{\mathbf{c}^{2^{l-j-1}}}$$

Therefore, by (2.28),

$$\begin{aligned}
|\Phi_{k-1,n}(w) - \widehat{z}_j^{k-1}| &\leq 2^{n-k+1}|\Phi_{n,n}(w) - \widehat{z}_j^n| \times \prod_{l=k}^{n}\left(A_2\mathbf{c}^{\frac{1}{2}-2^{l-j-1}}\right)\\
&\leq (2A_2)^{n-k+1}A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}\mathbf{c}^{\frac{n-k+1}{4}}\mathbf{c}^{-\sum_{l=k-j-1}^{n-j-1}2^l}\\
&\leq \left[(2A_2)^{n-k+1}\mathbf{c}^{\frac{n-k+1}{4}}\right]A^{n-j}\left(\mathbf{c}^{2^{n+1}-\frac{1}{2}}\right)^{2^j}\mathbf{c}^{-(2^{n-j}-2^{k-j-1})}\\
&\leq A^{n-j}\mathbf{c}^{2^{n+j+1}-2^{j-1}-2^{n-j}+2^{k-j-1}}\\
&\leq A^{n-j}\mathbf{c}^{2^{n+2}-2^{n-1}-2^{n-1}}\\
&= A^{n-j}\mathbf{c}^{3\times 2^n},
\end{aligned}$$

and so our claim holds by induction.

We can also see, from the same computation, that

$$|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}| = |\Phi_{j,n}(w) - \widehat{z}_j^j| \leq \mathbf{c}^{3\times 2^n}. \tag{2.29}$$

Then for each $j+1 \leq k \leq n$, as $\mathbf{c}^{3\times 2^n} \leq \frac{1}{2}\mathbf{c}^{2^{k-j}}$, we have by the triangle inequality and Lemma 2.14 that $|\Phi_{k,n}(w) - e^{i(\theta_k \pm \beta)}| \geq \frac{1}{2}\mathbf{c}^{2^{k-j}}$, and so by Lemma 2.4,

$$\begin{aligned}
|\Phi_{j,n}'(w)| &= \prod_{k=j+1}^{n}|f_k'(\Phi_{k,n}(w))|\\
&\leq \prod_{k=j+1}^{n}A_2\frac{\beta^{1/2}}{(\frac{1}{2}\mathbf{c}^{2^{k-j}})^{1/2}}\\
&\leq (2A_2)^{n-j}\mathbf{c}^{\frac{n-j}{4}-\sum_{k=0}^{n-j-1}2^k}\\
&= (2A_2)^{n-j}\mathbf{c}^{\frac{n-j}{4}-2^{n-j}+1}
\end{aligned} \tag{2.30}$$

for sufficiently small $\mathbf{c}$.

We will next establish an upper bound on $|\Phi_j'(\Phi_{j,n}(w))|$. By the arguments used to prove Corollary 2.17, we have a lower bound on $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}|$ as well as the upper bound we just established:

$$|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}| \geq A^{-(n-j)}|w - \widehat{z}_j^n|, \tag{2.31}$$

where A is a constant. The upper bound in (2.29) is less than $\mathbf{c}^{2^{n+1}}$, and so we can apply (the proof of) Lemma 2.8 to say

$$|\Phi_j'(\Phi_{j,n}(w))| \le \frac{(A')^j}{A^{\frac{n-j}{2}}} \frac{\mathbf{c}^{\frac{1}{2}(1-2^{-j})}}{|w-\widehat{z}_j^n|^{1-2^{-j}}}, \tag{2.32}$$

and so we can combine (2.30) and (2.32) to obtain

$$\begin{aligned}|\Phi_n'(w)| &= |\Phi_{j,n}'(w)| \times |\Phi_j'(\Phi_{j,n}(w))| \\ &\le \left(\frac{2A_2}{\sqrt{A}}\right)^{n-j} (A')^j \mathbf{c}^{\frac{n-j}{4}-2^{n-j}+1}\mathbf{c}^{\frac{1}{2}(1-2^{-j})}|w-\widehat{z}_j^n|^{-(1-2^{-j})} \\ &\le (A'')^n \mathbf{c}^{\frac{n-j}{4}+1}\mathbf{c}^{\frac{1}{2}(1-2^{-j})}\frac{1}{\mathbf{c}^{2^{n-j}}}|w-\widehat{z}_j^n|^{-(1-2^{-j})}\end{aligned}$$

where $A'' = \max(\frac{2A_2}{\sqrt{A}}, A')$ is a constant. ☐

Corollary 2.19. *On the event $\{n < N \wedge \tau_D\}$, for $1 \le j \le n-1$ the following is almost surely true: for $L_j^n := A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$, we have*

$$\int_{-L_j^n}^{L_j^n} |\Phi_n'(\widehat{z}_j^n e^{\sigma+i\varphi})|^\nu \,\mathrm{d}\varphi \le B_\nu^n \frac{\mathbf{c}^{\nu(\frac{n-j}{4}+1)}}{\mathbf{c}^{\nu 2^{n-j}}}\mathbf{c}^{\frac{\nu}{2}(1-2^{-j})}\sigma^{-\left[\nu(1-2^{-j})-1\right]} \tag{2.33}$$

where B_ν is a constant depending only on $\nu := -\eta$.

Proof. As $|\widehat{z}_j^n e^{\sigma+i\varphi} - \widehat{z}_j^n| \asymp (\sigma^2+\varphi^2)^{1/2}$, the bound follows immediately from Lemma 2.18 (in the same way as we obtained Proposition 2.9 from Lemma 2.8). ☐

2.5 Proof of chapter's main results

With the results of the previous sections, we are finally ready to prove our main scaling limit result, that the cluster $K_N^{\mathbf{c}}$ converges in distribution, as $\mathbf{c} \to 0$, to an SLE$_4$ cluster. To help picture the sets $S_{n,j}$ and R_n, it may be useful to refer to Figure 2.4.

Proof of Theorem 2.3. We want to show that $h_{n+1}(F_n) = \int_{F_n} h_{n+1}(\theta)\,\mathrm{d}\theta$ is small, and so we will decompose F_n into several sets.

Let $R_n = \{\theta \in \mathbb{T} : |\Phi_n(e^{\sigma+i\theta}) - 1| > \frac{L}{4}\}$, $S_n = F_n \setminus R_n$. We will further decompose S_n: first define

$$T_n = \{\theta \in S_n : D < \min_{\pm} |e^{\sigma+i\theta} - e^{i(\theta_n \pm \beta)}| \le L\},$$

and for $1 \le j \le n-1$ define

$$S_{n,j} = \{\theta \in S_n : |e^{\sigma+i\theta} - \widehat{z}_j^n| \le L_j^n\},$$

where L_j^n is the bound appearing in Corollary 2.17, then Corollary 2.17 tells us that

$S_n = T_n \cup \left(\bigcup_{j=1}^{n-1} S_{n,j}\right)$. We can then split the integral as

$$h_{n+1}(F_n) \leq h_{n+1}(R_n) + h_{n+1}(T_n) + \sum_{j=1}^{n-1} h_{n+1}(S_{n,j}). \tag{2.34}$$

We showed in Section 2.3.2 that $h_{n+1}(T_n) = o(\mathbf{c}^\gamma)$ for any fixed $\gamma > 0$, and so we only need to bound $h_{n+1}(R_n)$ and each $h_{n+1}(S_{n,j})$. Bounding $h_{n+1}(R_n)$ is simple using Proposition 2.13, and Proposition 2.9, as for any $\theta \in R_n$, we have

$$\frac{|\Phi_n'(e^{\sigma+i\theta})|^\nu}{Z_n} \leq \frac{A^{\nu n}\beta^{\nu n/2}(L/32\beta)^{-\frac{\nu}{2}(2^n-1)}}{A^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}\sigma^{-[\nu(1-2^{-n})-1]}}.$$

Since $L := \mathbf{c}^{2^{N+1}}$, and $32\beta \leq 1$ for sufficiently small $\mathbf{c}$, we have

$$\left(\frac{L}{32\beta}\right)^{-\frac{\nu}{2}(2^n-1)} \leq \mathbf{c}^{-\frac{\nu}{2}2^{N+1}(2^n-1)} \leq \mathbf{c}^{-\nu 2^{N+n}} \leq \mathbf{c}^{-\nu 2^{2N}}.$$

Since $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, we also have $\sigma^{\nu(1-2^{-n})-1} \leq \sigma^{\frac{\nu}{2}-1} \leq \mathbf{c}^{(\frac{\nu}{2}-1)2^{2^{1/\mathbf{c}}}}$. Hence

$$\frac{(L/32\beta)^{-\frac{\nu}{2}(2^n-1)}}{\sigma^{-[\nu(1-2^{-n})-1]}} \leq \mathbf{c}^{(\frac{\nu}{2}-1)2^{2^{1/\mathbf{c}}}-\nu 2^{2N}},$$

which decays extremely quickly compared to the growth of the exponential term $A^{(\nu-1)n}$ and the remaining powers of $\mathbf{c}$. Hence we can write

$$|\Phi_n'(e^{\sigma+i\theta})|^\nu \leq \mathbf{c}^4 Z_n$$

for sufficiently small $\mathbf{c}$, and so $h_{n+1}(R_n) = o(\mathbf{c}^4)$. Finally, we will bound $h_{n+1}(S_{n,j})$. Using the bounds from Proposition 2.9 and Corollary 2.19, we have

$$\begin{aligned}
h_{n+1}(S_{n,j}) &\asymp \frac{1}{Z_n}\int_{-L_j^n}^{L_j^n} |\Phi_n'(\hat{z}_j^n e^{\sigma+i\varphi})|^\nu \,\mathrm{d}\varphi \\
&\leq \frac{B_\nu^n \frac{\mathbf{c}^{\nu(\frac{n-j}{4}+1)}}{\mathbf{c}^{\nu 2^{n-j}}}\mathbf{c}^{\frac{\nu}{2}(1-2^{-j})}\sigma^{-\left[\nu(1-2^{-j})-1\right]}}{A^n\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}\sigma^{-[\nu(1-2^{-n})-1]}} \\
&= \left(\frac{B_\nu}{A}\right)^n \underbrace{\mathbf{c}^{\nu(\frac{n-j}{4}+1)}\mathbf{c}^{-\frac{\nu}{2}(2^{-j}-2^{-n})}}_{o(\mathbf{c}^5)}\mathbf{c}^{-\nu 2^{n-j}}\sigma^{\nu(2^{-j}-2^{-n})} \\
&\ll \mathbf{c}^5\left(\frac{B_\nu}{A}\right)^n \mathbf{c}^{-\nu 2^{n-j}}\sigma^{\nu 2^{-n}},
\end{aligned}$$

then as $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, we have $\mathbf{c}^{-\nu 2^{n-j}}\sigma^{\nu 2^{-n}} \leq \mathbf{c}^{\nu\left(2^{2^{1/\mathbf{c}}-n}-2^{n-j}\right)} \leq \mathbf{c}^{\nu\left(2^{2^{1/\mathbf{c}}-N}-2^N\right)}$ which decays faster than exponentially in N. Therefore $h_{n+1}(S_{n,j}) = o_T(\mathbf{c}^5)$, and so we have $\sum_{j=1}^{n-1} h_{n+1}(S_{n,j}) = o_T(\mathbf{c}^4)$, establishing (2.1). The second bound, (2.2), comes immediately from Corollary 2.12. $\square$

Remark. We have now seen that $(\theta_n^{\mathbf{c}})_{n\leq\lfloor T/\mathbf{c}\rfloor}$ is very close to a simple symmetric random walk with step length $\beta \sim 2\mathbf{c}^{1/2}$, and so we expect $(\xi_t^{\mathbf{c}})_{t\in[0,T]} = (\theta_{\lfloor t/\mathbf{c}\rfloor})_{t\in[0,T]}$ will converge in distribution to $(2B_t)_{t\in[0,T]}$, where B is a standard Brownian motion. We now state a result by McLeish [22] which gives conditions for near-martingales to converge to a diffusive limit.

Proposition 2.20 (Corollary 3.8 of [22])**.** *Let $(X_{n,i})_{n,i\in\mathbb{N}}$ be an array of random variables, $J = [0,T]$ for $0 < T < \infty$, and $(k_n)_{n\in\mathbb{N}}$ a sequence of right-continuous functions $J \to \mathbb{N}\cup\{0\}$. Write $W_n(t) = \sum_{i=1}^{k_n(t)} X_{n,i}$ for $t \in J$, and assume the following three limits hold in probability as $n \to \infty$:*

$$\sum_{j=1}^{k_n(t)} \mathbb{E}\left[X_{n,j}^2 1[|X_{n,j}| > \varepsilon] | X_{n,1},\dots,X_{n,j-1}\right] \to 0 \text{ for all } \varepsilon > 0, \tag{2.35}$$

$$\sum_{j=1}^{k_n(t)} \mathbb{E}\left[X_{n,j}^2 | X_{n,1},\dots,X_{n,j-1}\right] \to t, \tag{2.36}$$

$$\sum_{j=1}^{k_n(t)} \left|\mathbb{E}\left[X_{n,j} | X_{n,1},\dots,X_{n,j-1}\right]\right| \to 0, \tag{2.37}$$

for all $t \in J$. Then $W_n \to B$ weakly in $D(J)$ as $n \to \infty$, where B is a standard Brownian motion.

Proof of Proposition 2.1. The bound $\mathbb{P}[\tau_D \le \lfloor T/\mathbf{c}\rfloor] = O_T(\mathbf{c}^3)$ is obtained immediately from Theorem 2.3, by observing for $1 \le j \le \lfloor T/\mathbf{c}\rfloor$ that

$$\mathbb{P}[\tau_D \le j] \le A\mathbf{c}^4 + \mathbb{P}[\tau_D \le j-1].$$

For the convergence of the driving function, we will apply Proposition 2.20, replacing $n \to \infty$ by $\mathbf{c} \to 0$ (this can be justified by showing the limit holds for any sequence of capacities $\mathbf{c}_n$ tending to zero as $n \to \infty$) and $k_n(t)$ by $\lfloor t/\mathbf{c}\rfloor$. Then $X_{\mathbf{c},j} = \theta_j - \theta_{j-1}$. Note that we will have $4t$ rather than t as the limit in (2.36), corresponding to a limit of $2B$ instead of B.

The expectation of the jth term in (2.35) is

$$\begin{aligned}\mathbb{E}\int_{-\pi}^{\pi} \varphi^2 h_j(\theta_{j-1}+\varphi) 1[|\varphi| > \varepsilon]\,\mathrm{d}\varphi &\le \pi^2 \mathbb{E}(\mathbb{P}(|\theta_j - \theta_{j-1}| > \varepsilon \,|\, \theta_1,\dots,\theta_{j-1})) \\ &\le \pi^2 \mathbb{P}(\tau_D \le j)\end{aligned}$$

when $\mathbf{c}$ is sufficiently small so $\beta + D < \varepsilon$. Using our bound on $\mathbb{P}[\tau_D \le \lfloor T/\mathbf{c}\rfloor]$, we see (2.35) tends to zero in L^1 and hence also in probability.

Next, since h_j approximates $\frac{1}{2}(\delta_{\theta_{j-1}-\beta}+\delta_{\theta_{j-1}+\beta})$, the jth term in (2.36) is

$$\begin{aligned}\int_{-\pi}^{\pi}\varphi^2 h_j(\theta_{j-1}+\varphi)\,\mathrm{d}\varphi &= \int_{\beta-D}^{\beta+D}\varphi^2 h_j(\theta_{j-1}+\varphi)\,\mathrm{d}\varphi + \int_{-\beta-D}^{-\beta+D}\varphi^2 h_j(\theta_{j-1}+\varphi)\,\mathrm{d}\varphi + E_j \\ &= (\beta+O(D))^2\int_{\mathbb{T}\setminus F_{j-1}} h_j(\theta)\,\mathrm{d}\theta + E_j \\ &= \beta^2 + O(\beta D) + E'_j\end{aligned}$$

where E'_j is the sum of two terms:

$$\int_{F_{j-1}}\theta^2 h_j(\theta)\,\mathrm{d}\theta \le \pi^2 1[\tau_D \le \lfloor t/\mathbf{c}\rfloor] + \pi^2 A\mathbf{c}^4$$

and

$$(\beta^2+O(\beta D))\int_{F_{j-1}} h_j(\theta)\,\mathrm{d}\theta \le 2\beta^2 1[\tau_D \le \lfloor t/\mathbf{c}\rfloor] + 4A\mathbf{c}^5$$

(both bounds come from Theorem 2.3). Hence (2.36) is

$$\sum_{j=1}^{\lfloor t/\mathbf{c}\rfloor}\int_{-\pi}^{\pi}\varphi^2 h_j(\theta_{j-1}+\varphi)\,\mathrm{d}\varphi = \lfloor t/\mathbf{c}\rfloor\,\beta^2 + O\left(\frac{\beta D}{\mathbf{c}}\right) + \sum_{j=1}^{\lfloor t/\mathbf{c}\rfloor} E'_j.$$

Then

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{\lfloor t/\mathbf{c}\rfloor}|E'_j|\right] &\le \left\lfloor\frac{t}{\mathbf{c}}\right\rfloor\left((\pi^2+2\beta^2)\mathbb{P}[\tau_D \le \lfloor t/\mathbf{c}\rfloor] + \pi^2 A\mathbf{c}^4 + 4A\mathbf{c}^5\right) \\ &= O_T(\mathbf{c}^2),\end{aligned}$$

so (2.36) converges in L^1 to $\lim_{\mathbf{c}\to 0}(\lfloor t/\mathbf{c}\rfloor\,\beta^2) = 4t$ for any $t \in [0,T]$ as $\mathbf{c}\to 0$.

Finally, for the symmetry condition we can combine (2.1) and (2.2) to bound the jth term in (2.37):

$$\begin{aligned}\left|\int_{-\pi}^{\pi}\varphi h_j(\theta_{j-1}+\varphi)\,\mathrm{d}\varphi\right| &\le \left|\int_{\beta-D}^{\beta+D}\varphi(h_j(\theta_{j-1}+\varphi)-h_j(\theta_{j-1}-\varphi))\,\mathrm{d}\varphi\right| + \pi h_j(F_{j-1}) \\ &\le \pi 1[\tau_D \le \lfloor t/\mathbf{c}\rfloor] + (\beta+O(D))A\mathbf{c}^{11/4} + A\mathbf{c}^4,\end{aligned}$$

so as with (2.35), taking expectations it is simple to show that (2.37) tends to zero in L^1 and hence in probability as $\mathbf{c}\to 0$. $\square$

2.6 Alternative particle shapes

We believe that the results obtained above when using particles of the form $(1, 1+d]$ can be extended to a more general family of particles. In this case, depending on the form of the particles chosen, we believe an SLE_κ cluster can be obtained as the limit of an $\mathrm{ALE}(0,\eta)$ for $\eta < -2$ for any $\kappa \in [4,\infty]$ (where SLE_∞ is the growing disc $t \mapsto e^t\overline{\mathbb{D}}$).

We present below a few definitions and statements to make this conjecture precise, and some sketch arguments to support our claims.

Definition. Let $\mathcal{P}$ be a family of subsets of Δ, with $P \in \mathcal{P}$ if and only if:

(i) $P \cup \overline{\mathbb{D}}$ is closed and bounded,

(ii) for all $z \in P$, we have $z^* \in P$,

(iii) $\overline{P} \cap \overline{\mathbb{D}} = \{1\}$, and

(iv) P is convex.

Note that for every $P \in \mathcal{P}$, there is a unique map $f^P : \Delta \to \Delta \setminus P$ of the form $f^P(z) = e^{\mathbf{c}} z + O(1)$ near ∞ for some $\mathbf{c} = \mathbf{c}(P) > 0$. As with the case $P = (1, 1+d]$ there is also a unique $0 < \beta(P) < \pi$ such that $f^P(e^{\pm i\beta(P)}) = 1$.

Condition (iii) is necessary to obtain an SLE scaling result. If the particle has a non-trivial base, then the basepoints no longer sit in increasingly deep "fjords" of low harmonic measure, so the most recent basepoints are no longer signficiantly more attractive than the older basepoints.

Condition (iv) ensures the basepoints of each particle are the areas of lowest harmonic measure. For example the particle $P_{\theta,\ell} = (1, 1+e^{i\theta}] \cup (1, 1+e^{-i\theta}]$ satisfies (i), (ii) and (iii), but $(f^P)'$ has an additional singularity at 1 as well as at $e^{\pm i\beta}$ if $0 < \theta < \pi$. For small values of θ the singularity at 1 is in fact stronger than those at $e^{\pm i\beta}$.

We conjecture that condition (iv) could be substantially weakened to the condition that the two basepoints $e^{\pm i\beta}$ maximise the *local dimension of harmonic measure* on $\partial(\Delta \setminus P)$. As defined in [20], for z on the boundary, let $B_\delta(z)$ be the set of points on the boundary at distance $\leq \delta$ from z (measured without crossing the boundary, so for example the two "sides" of a slit particle are measured separately). Then if ω is harmonic measure on the boundary, i.e. the pushforward of (normalised) Lebesgue measure by f^P, the lower local dimension of harmonic measure at z is

$$\liminf_{\delta \to 0} \frac{\log \omega B_\delta(z)}{\log \delta}.$$

The upper local dimension is defined similarly, with lim sup. Note that a higher local dimension corresponds to lower harmonic measure near a point. Then we could replace condition (iv) with the condition that the lower local dimension at $e^{\pm i\beta}$ is strictly greater than the upper local dimension at any other points. This introduces a large number of extra technicalities, including the fact that there may be any number of preimages of 1 under f^P (so we define β as the largest positive argument of a preimage of 1) and that the two additional examples of particles we have given below have infinite lower local dimension.

Aside from particles of the form $(1, 1+d]$, examples of particles in this family are discs D_r of radius $r > 0$ and centre $1 + r$, and line segments tangent to $\mathbb{T}$, of the form $T_\ell = [1 - i\ell, 1 + i\ell]$ for $\ell > 0$.

Definition. Given a family $(P_{\mathbf{c}})_{\mathbf{c}>0}$ of particles from $\mathcal{P}$, indexed by capacity so that $\mathbf{c}(P_{\mathbf{c}}) = \mathbf{c}$, we will call the family *κ-stable* for $\kappa \in [0, \infty]$ if $\beta(P_{\mathbf{c}})^2/\mathbf{c} \to \kappa$ as $\mathbf{c} \to 0$.

We can compute the maps f^{D_r} and f^{T_ℓ} by elementary methods, and so establish that both families are stable and compute their respective κs. We write both maps here so that the reader can satisfy themselves that they have the same important properties as the map $f^{(1,1+d]}$.

For $r > 0$ we have $\beta_r = \frac{\pi r}{1+r}$ and define $m_r : \Delta \to \mathbb{H}$ by

$$m_r(z) = e^{i\beta_r} \frac{z - e^{-i\beta_r}}{z - e^{i\beta_r}}, \tag{2.38}$$

and $\psi_r : \mathbb{H} \to \Delta \setminus D_r$ by

$$\psi_r(w) = \frac{\log w + i\beta}{\log w - i\beta}, \tag{2.39}$$

where the logarithm is defined by $0 < \arg w < \pi$. Then we have $f^{D_r} \colon \Delta \to \Delta \setminus D_r$ given by $f^{D_r} = \psi_r \circ m_r$. It is then relatively easy to compute that the capacity of D_r, $\mathbf{c}(D_r) \sim \frac{1}{6}\pi^2 r^2$ and so, suitably reparameterised, $(f^{D_r})_{r>0}$ is 6-stable.

The map for T_ℓ is somewhat more complicated. Following the Schwarz–Christoffel computations in [28] (adapted for a symmetric tangent), there are two important quantities as $\ell \to 0$: $e_\ell \sim \ell$ (closely related to β_{T_ℓ}) and $y_\ell = 2 - \frac{1}{6\pi} e_\ell^3 + o(e_\ell^3)$ (related to the capacity). Using these, we can define maps $m_\ell : \Delta \to \mathbb{H}$, $\psi_\ell : \mathbb{H} \to \mathbb{H} \setminus$ (two arcs), and $\varphi_\ell : \mathbb{H} \setminus$ (two arcs) $\to \Delta \setminus T_\ell$, given by

$$m_\ell(z) = i y_\ell \frac{z-1}{z+1}, \tag{2.40}$$

$$\psi_\ell(w) = \frac{1}{2\pi} \log\left(\frac{w - e_\ell}{w + e_\ell}\right) - \frac{1 - e_\ell/\pi}{w}, \tag{2.41}$$

$$\varphi_\ell(\zeta) = \frac{2\zeta + i}{2\zeta - i}. \tag{2.42}$$

Then $f^{T_\ell} = \varphi \circ \psi_\ell \circ m_\ell$. Some calculations then give $\beta_\ell^2/\mathbf{c}(T_\ell) \sim \frac{12\pi}{\ell}$ as $\ell \to 0$, and so, again reparameterised by capacity, $(T_\ell)_{\ell>0}$ is ∞-stable.

Our main conjecture is that we have a version of Proposition 2.1 for every family of κ-stable particles, and so the resulting cluster converges in distribution to an SLE_κ.

To grow most of the particles in $\mathcal{P}$ it is necessary to use Loewner's equation (1.4) with a driving *measure* on $\mathbb{T}$ rather than a driving function. We will not go into detail of this here, but refer the reader to [17] or the following chapter of this thesis. For a given particle P with capacity $\mathbf{c}$, we denote the driving probability measure (evolving in time) by $(\mu_t^P)_{0 \le t \le \mathbf{c}}$.

Conjecture. *Fix $T > 0$ and let $\eta < -2$. Suppose $(P_{\mathbf{c}})_{\mathbf{c}>0}$ is a κ-stable family of particles from $\mathcal{P}$ for $\kappa \in [4, \infty]$. Let $(\theta_n^{\mathbf{c}})_{n \ge 1}$ be the sequence of angles we obtain from the* $\mathrm{ALE}(0, \eta)$ *process using particle $P_{\mathbf{c}}$ and let $\sigma \le c_0(P_{\mathbf{c}})$, some function which decays quickly as $\mathbf{c} \to 0$.*

Let $\tau_D = \inf\{n \ge 2 : \min_\pm |\theta_n - (\theta_{n-1} \pm \beta_{\mathbf{c}})| > D\}$, where D is a suitable function of

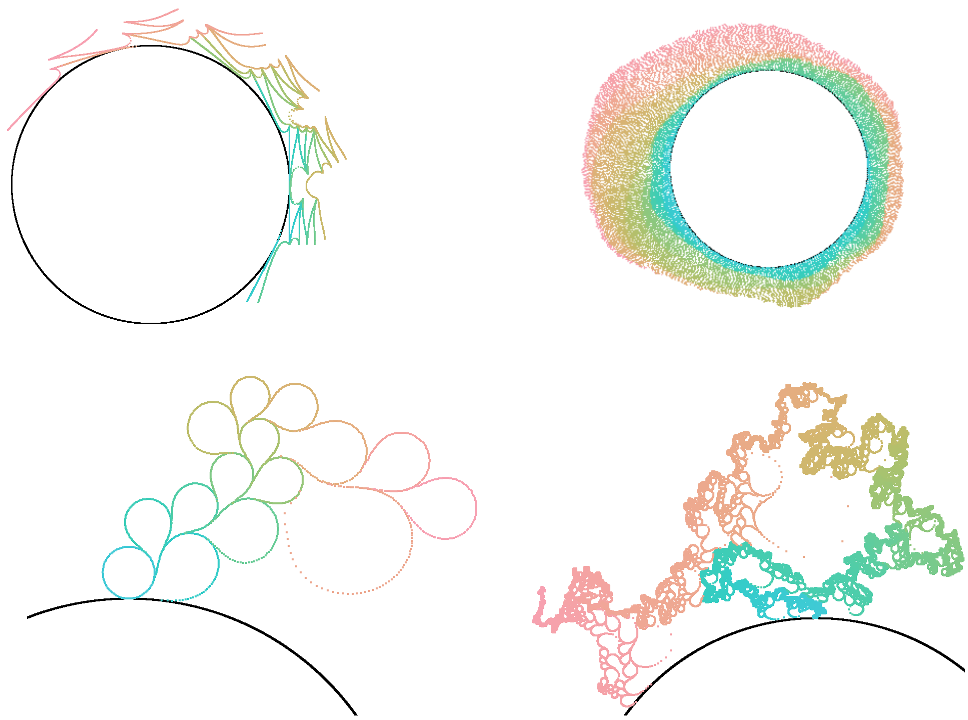

Figure 2.6: Clusters composed of tangent particles T_ℓ (top) and disc particles D_r (bottom), generated with an angle sequence $\theta_k = \beta X_k$, for a simple symmetric random walk X_k, coloured according to the order of attachment (the earliest particles in blue and the latest in red). Note that these are *not* simulations of an ALE process, but illustrations of what we conjecture their behaviour to be. For the tangent and disc particles (and even for the slit), the σ necessary for convergence to an SLE is far too small to make simulating ALE practical in the regime this paper considers. The clusters on the right have 8,000 particles each and a total capacity around 0.2. The bottom-right cluster is close to an SLE_6, and the top-right cluster approximates an SLE_κ with κ around 377.

σ and $\mathbf{c}$.

As $\mathbf{c} \to 0$,

$$\mathbb{P}[\tau_D \leq \lfloor T/\mathbf{c} \rfloor] = O(\mathbf{c}^\gamma)$$

for some $\gamma > 1$.

The driving measure for the whole cluster is $\mathrm{d}\xi_t^{\mathbf{c}}(\varphi) = \mathrm{d}\mu^{P_{\mathbf{c}}}_{t-\mathbf{c}\lfloor t/\mathbf{c}\rfloor}(\theta_{\lfloor t/\mathbf{c}\rfloor+1} + \varphi)$ for $0 \leq t \leq T$. Then if $\kappa < \infty$,

$$(\xi_t^{\mathbf{c}})_{t\in[0,T]} \to (\delta_{\sqrt{\kappa}B_t})_{t\in[0,T]} \text{ in distribution as } \mathbf{c} \to 0,$$

as a random element of the space of finite measures on $S = \mathbb{T} \times [0, T]$ (equipped with the bounded Wasserstein metric), and if $\kappa = \infty$ then $(\xi_t^{\mathbf{c}})_{t\in[0,T]}$ converges in the same sense to Lebesgue measure $\frac{1}{2\pi}\,\mathrm{d}\varphi\,\mathrm{d}t$ on S.

Conjecture (Generalisation of Theorem 2.2, simple corollary of the previous conjecture)**.** *For η, σ, κ and $(P_{\mathbf{c}})_{\mathbf{c}>0}$ as in the previous conjecture, let the $\mathrm{ALE}(0, \eta)$ cluster with $N = \lfloor T/\mathbf{c} \rfloor$ particles of capacity $\mathbf{c}$ be $K_N^{\mathbf{c}}$. As $\mathbf{c} \to 0$, if $\kappa < \infty$ then $K_N^{\mathbf{c}}$ converges in distribution as a random element of $\mathcal{K}$ to a radial SLE_κ cluster of capacity T. If $\kappa = \infty$ then $K_N^{\mathbf{c}}$*

converges in $\mathcal{K}$ to the disc $e^T\overline{\mathbb{D}}$.

We believe the proof of the first conjecture is fairly straightforward for particles where the map $f^{P_{\mathbf{c}}}$ is known explicitly, such as T_ℓ and D_r. As the support of $\mu_t^{P_{\mathbf{c}}}$ is $o(1)$ as $\mathbf{c} \to 0$, proving convergence of the driving measure is reduced to proving the angle sequence approximates a symmetric random walk. This follows quite simply if we can prove similar bounds to those in Theorem 2.3, which we believe is simply a matter of carefully verifying the type of explicit calculations we were able to do for $f^{(1,1+d]}$.

A proof for general κ-stable families will require more generalised estimates of the maps and their derivatives for particles in the class $\mathcal{P}$, which we have not yet developed.

Remark. One question which naturally arises is the significance of the $\kappa = 4$ appearing in Theorem 2.2 for the slit particle. In fact we strongly believe that this is the minimal attainable κ for our ALE$(0, \eta < -2)$ models. Geometrically, slits $(1, 1 + d]$ are the only particles with "zero width", and $\kappa = 4$ marks a phase transition for SLE, since SLE$_4$ is a simple curve, and SLE$_\kappa$ for $\kappa > 4$ is never a simple curve.

Proposition 2.21. *For $0 \leq \kappa < 4$ there is no family of κ-stable particles in $\mathcal{P}$.*

Proof idea. First note that the family of slit particles $(Q_{\mathbf{c}})_{\mathbf{c}>0} = ((1, 1 + d(\mathbf{c})])_{\mathbf{c}>0}$ is 4-stable. For any particle $P \in \mathcal{P}$, we can express $(f^P)^{-1}$ as the solution to the reverse Loewner equation with a symmetric driving measure, and then $e^{i\beta_P} = \lim_{\varepsilon\downarrow 0}(f^P)^{-1}(e^{i\varepsilon})$. An explicit calculation shows that if P has capacity $\mathbf{c}$ then $\beta_P \geq \beta_{Q_{\mathbf{c}}}$. □

Remark. We are confident that an SLE$_\kappa$ can be realised as the limit of an ALE$(0, \eta)$ model for every $\kappa \in [4, \infty]$. For example, isoceles triangular particles joined to the circle at the apex, with vertex angle θ, can interpolate between the slit particle $(1, 1 + d]$ (the $\theta \to 0$ limit) and the tangent T_ℓ the $\theta \to \pi$ limit). We can therefore interpolate between $\kappa = 4$ and $\kappa = \infty$, realising every value in $(4, \infty)$ as θ varies in $(0, \pi)$.

Chapter 3

ALE with large η and the Laplacian path model

In this chapter we will demonstrate convergence of the ALE with $\eta > 1$ started from a non-trivial initial configuration to the (geodesic) *Laplacian path model* (LPM).

3.1 Laplacian path model

The LPM was defined in 2002 by Carleson and Makarov [4] to generalise several models of needle-like growth in mathematical physics. A number of properties of the model and related models were derived in [4] and the PhD thesis of Göran Selander [29].

Definition. Let K_0 be the union of rays of the form $e^{i\theta_j}(1, 1 + d_j]$ for $j = 1, \ldots, k$. Let $\Delta_0 = \Delta \setminus K_0$, and Φ_0^{LPM} the unique conformal map $\Phi_0^{\mathrm{LPM}} : \Delta \to \Delta_0$ satisfying $\Phi_0^{\mathrm{LPM}}(z) = e^{c_0} z + O(1)$ as $z \to \infty$ for some positive c_0.

We define the LPM cluster with parameter η, which has k growing slits whose tips at time $t > 0$ are at $a_t^j \in \Delta \setminus K_0$, via Loewner's equation. We have the driving measure

$$\mu_t^{\mathrm{LPM}} = \sum_{j=1}^{k} p_t^j \delta_{\phi_t^j},$$

where $\delta_{\phi_t^j}$ is the preimage of a_t^j under Φ_t^{LPM}, and $p_t^j = \frac{|(\Phi_t^{\mathrm{LPM}})''(e^{i\phi_t^j})|^{-\eta}}{Z_t}$, and $Z_t = \sum_{j=1}^{k} |(\Phi_t^{\mathrm{LPM}})''(e^{i\phi_t^j})|^{-\eta}$.

Remark. We can think of the growth of each slit in the geodesic LPM at time t as being in the direction of the hyperbolic geodesic from a_t^j to ∞ in $\Delta \setminus K_t$, at a speed proportional to $|(\Phi_t^{\mathrm{LPM}})''(e^{i\phi_t^j})|^{-(\eta-1)}$.

Remark. In the definition of p_t^j we use the second rather than first derivative of the cluster map Φ_t^{LPM}. In [4], Carleson and Makarov obtain this definition by considering the Laplacian field ∇G_t, where G_t is the Green's function of $\Delta \setminus K_t$, given by $G_t(z) = \log |(\Phi_t^{\mathrm{LPM}})'(z)|$. Equally, we could derive the expression for p_t^j by considering the limit of the regularised ALE density with $\eta > 1$. For $\sigma > 0$, let μ_t^σ be the ALE measure on

$\mathbb{T}$, i.e. the measure with density $h^\sigma(\theta) \propto |(\Phi_t^{\mathrm{LPM}})'(e^{\sigma+i\theta})|^{-\eta}$. If $\eta > 1$ then as $\sigma \to 0$ the density becomes concentrated around the zeroes of $(\Phi_t^{\mathrm{LPM}})'$ at ϕ_t^j for $j = 1, \ldots, k$, and $|(\Phi_t^{\mathrm{LPM}})'(e^{\sigma+i\phi_t^j})| \sim \sigma|(\Phi_t^{\mathrm{LPM}})''(e^{i\phi_t^j})|$. Hence $\mu_t^\sigma \to \mu_t^{\mathrm{LPM}}$ as $\sigma \to 0$.

Remark. The η above corresponds to $\eta - 1$ in [4]; we have shifted it to match the η in the corresponding ALE.

Almost all known results about the Laplacian path model appear in [4] and [29].

In [4], Carleson and Makarov defined both the geodesic LPM above and the *needle LPM*, in which the cluster is a collection of straight slits which grow at speed proportional to $|(\Phi_t^{\mathrm{LPM}})''(e^{i\phi_t^j})|^{-(\eta-1)}$.

Selander examined in his PhD thesis [29] a version of the geodesic LPM in which the weights p_t^j are fixed constants, and the needle LPM with growth speeds proportional to harmonic measure at the tips, which corresponds to $\eta = 3/2$. Each of these can be viewed as a simplification of a "non-branching DLA".

For the needle LPM with $\eta = 3/2$, Selander proved results about the stability of stationary solutions: slit configurations in which the ratio of lengths remains constant for all $t \geq 0$. For his modified geodesic model, he proved convergence as $t \to \infty$ to a stationary configuration determined by the weights, when started from any initial configuration.

For a simplified "chordal" geodesic LPM, in which finitely many curves grow from the tip of an infinite half-line, Carleson and Makarov proved an analogue of the result in Chapter 4. Starting the process from a two-arm configuration, they showed that if $\eta < 2$ then both arms always survive, while there are configurations in which only one arm survives if $\eta > 2$. A number of the techniques they developed for treating the chordal case are used in later parts of this thesis to analyse the ALE and non-chordal LPM.

In the non-simplified geodesic LPM, they also showed that the completely symmetric three-armed cluster was stable for $1 < \eta < \eta_c$ and unstable for $\eta > \eta_c$, for a critical value $\eta_c = \frac{18}{3+4\log 2} \approx 3.11815$.

They also proved limiting results for the simpler "needle" LPM, and made a number of conjectures about the geodesic LPM.

Although little progress has been made since 2002 on analysing the geodesic LPM, various modifications have been studied and applied. In [12], Gubiec and Szymczak apply a similar construction to model finger growth in the half-plane.

The LPM and similar models have also been used to model the development of cracks in materials and formation of systems of rivers and streams. In particular, the chordal geodesic LPM with two needles at the tip of a half-line can be used to model the bifurcation of a stream. Carleson and Makarov proved that the angle between the two resulting streams in the chordal LPM must be $2\pi/5$ (also predicted by other authors using conformal mappings [13] [7]), which agrees with the average angle of $72°$ measured in a Florida stream system [6] [35].

3.2 Main result

The main theorem of this chapter concerns convergence of the ALE to the Laplacian path model's cluster started from the same initial conditions.

Theorem 3.1. *For a fixed $T > 0$, let $(\Phi_t^{\mathrm{ALE}})_{t\in[0,T]}$ be the ALE(α, η) map started from initial cluster $K_0 \cup \overline{\mathbb{D}}$, where $K_0 = \bigcup_{j=1}^k e^{i\phi_0^j}(1, 1+d_j]$ for $d_j > 0$ and distinct $\phi_0^j \in [0, 2\pi)$. Let μ_t^{ALE} be the driving measure for Φ^{ALE}, i.e. $\mu_t^{\mathrm{ALE}} = \delta_{\theta_{\lfloor t/\mathbf{c}\rfloor+1}}$. Let the parameters be $\alpha = 0$, $\eta > 1$ and $\sigma = \mathbf{c}^\gamma$ for a $\gamma = \gamma(\eta)$ we will specify later.*

Let $(\Phi_t^{\mathrm{LPM}})_{t\in[0,T]}$ be the map for the LPM started from the same initial conditions, and let μ_t^{LPM} be the driving measure $\mu_t^{\mathrm{LPM}} = \sum_j \bar{p}_t^j \delta_{\bar{\phi}_t^j}$.

Then as $\mathbf{c} \to 0$, $\mu_t^{\mathrm{ALE}} \otimes m_{[0,T]}$ converges in distribution to $\mu_t^{\mathrm{LPM}} \otimes m_{[0,T]}$ as random elements of the space of measures $S = \mathbb{T} \times [0, T]$, where $m_{[0,T]}$ is normalised Lebesgue measure on $[0, T]$.

In particular this means if K_t^{ALE} and K_t^{LPM} are the respective clusters at time t, we have $K_T^{\mathrm{ALE}} \to K_T^{\mathrm{LPM}}$ weakly as a random element of the space of compact subsets of $\mathbb{C}$ (equipped with the Carathéodory topology).

To show the convergence of the ALE model to the LPM, we introduce two intermediate models which we will call the *auxiliary* ALE, and the multinomial model.

The auxiliary ALE will have growth exactly at the tips, essentially reducing the ALE process, which is supported on $\mathbb{T}$ at each step, to a finite dimensional process supported on k atoms like the LPM.

The multinomial model will be introduced later to replace the regularised derivative in the ALE with the second derivative *on* the boundary.

Definition. Let K_0 be of the same form as above, and $\Phi_0^* = \Phi_0^{\mathrm{LPM}}$. We will choose an attachment point θ as we do for the ALE, but before attaching a particle to that point, we rotate the entire cluster so that θ lies exactly at the tip of one of the slits. More precisely, at step n we will still have a configuration of k curves. Let the preimage of the jth tip under Φ_n^* be $\bar{\phi}_n^j$. Choose θ_{n+1}^* according to the conditional density

$$h(\theta \mid \theta_1, \ldots, \theta_n) = \frac{1}{Z_n^*} |(\Phi_t^*)'(e^{\sigma+i\theta})|^{-\eta},$$

and if $j_n = \operatorname{argmin}_j |e^{i\theta_{n+1}^*} - e^{i\bar{\phi}_n^j}|$, let $\hat{\theta}_{n+1}^* = \bar{\phi}_n^{j_n}$, and $\delta_{n+1} = \theta_{n+1}^* - \hat{\theta}_{n+1}^*$.

Then if $R_\theta(z) = e^{i\theta} z$, set

$$\Phi_{n+1}^* = R_{\delta_{n+1}} \circ \Phi_n^* \circ R_{-\delta_{n+1}} \circ f_{\theta_{n+1}^*}.$$

Since $f_\theta \circ R_{-\delta} = R_{-\delta} \circ f_{\theta+\delta}$, we can write Φ_n^* in two ways:

$$\begin{aligned}\Phi_n^* &= R_{\delta_1+\cdots+\delta_n} \circ \Phi_0 \circ R_{-\delta_1} \circ f_{\theta_1^*} \circ R_{-\delta_2} \circ \cdots \circ R_{-\delta_n} \circ f_{\theta_n^*} \\ &= R_{\delta_1+\cdots+\delta_n} \circ \Phi_0 \circ R_{-(\delta_1+\cdots+\delta_n)} \circ f_{\theta_1^*+\delta_2+\delta_3+\cdots+\delta_n} \circ f_{\theta_2^*+\delta_3+\cdots+\delta_n} \circ \cdots \circ f_{\theta_n^*}, \qquad (3.1)\end{aligned}$$

and this latter expression is also equal to

$$R_{\delta_1+\cdots+\delta_n} \circ \Phi_0 \circ R_{-(\delta_1+\cdots+\delta_n)} \circ f_{\hat{\theta}_1^*+\delta_1+\cdots+\delta_n} \circ f_{\hat{\theta}_2^*+\delta_2+\cdots+\delta_n} \circ f_{\hat{\theta}_n^*+\delta_n}.$$

3.3 Convergence of models

3.3.1 ALE to auxiliary model

We will frequently use a lemma from [33], so we state it here.

Lemma 3.2 (Lemma 11 of [33])**.** *Suppose $z_0 \in \Delta$, $T > 0$ and $\xi^0 : (0,T] \to \mathbb{R}$ are given and let*

$$\Lambda_t = \int_0^t \frac{2|u_s^0(z_0)|^2 \mathrm{d}s}{|(u_s^0)'(z_0)||u_s^0(z_0)e^{-i\xi^0_{T-s}} - 1|^2}.$$

There exists some absolute constant A such that, for all $|z| > 1$ satisfying

$$|z - z_0| \leq A^{-1} \inf_{0 \leq t \leq T} \left(\frac{|u_t^0(z_0)e^{-i\xi^0_{T-t}} - 1|}{|(u_t^0)'(z_0)|} \wedge \left(\int_0^t \frac{|(u_s^0)'(z_0)|}{|u_s^0(z_0)e^{-i\xi^0_{T-s}} - 1|^3} \mathrm{d}s \right)^{-1} \right), \tag{3.2}$$

we have, for all $0 \leq t \leq T$,

$$\left| \log \frac{u_t^0(z) - u_t^0(z_0)}{(z - z_0)(u_t^0)'(z_0)} \right| \leq A|z - z_0| \int_0^t \frac{|(u_s^0)'(z_0)|\mathrm{d}s}{|u_s^0(z_0)e^{-i\xi^0_{T-s}} - 1|^3} \tag{3.3}$$

(where we interpret the left hand side as being equal to 0 if $z = z_0$) and

$$\left| \log \frac{(u_t^0)'(z)}{(u_t^0)'(z_0)} \right| \leq A|z - z_0| \int_0^t \frac{|(u_s^0)'(z_0)|\mathrm{d}s}{|u_s^0(z_0)e^{-i\xi^0_{T-s}} - 1|^3}. \tag{3.4}$$

Furthermore, A can be chosen so that if, in addition, $\xi^1 : (0,T] \to \mathbb{R}$ satisfies

$$\|\xi^1 - \xi^0\|_T \leq A^{-1} \inf_{0 \leq t \leq T} \left(\frac{|u_t^0(z_0)e^{-i\xi^0_{T-t}} - 1|}{|(u_t^0)'(z_0)|\Lambda_t + |u_t^0(z_0)|} \wedge \left(\int_0^t \frac{\Lambda_s|(u_s^0)'(z_0)| + |u_s^0(z_0)|}{|u_s^0(z_0)e^{-i\xi^0_{T-s}} - 1|^3} \mathrm{d}s \right)^{-1} \right), \tag{3.5}$$

then, for all $0 \leq t \leq T$,

$$\left| u_t^1(z) - u_t^0(z) \right| \leq A|(u_t^0)'(z_0)|\|\xi^1 - \xi^0\|_T \Lambda_t \tag{3.6}$$

and

$$\left| \log \frac{(u_t^1)'(z)}{(u_t^0)'(z)} \right| \leq A\|\xi^1 - \xi^0\|_T \int_0^t \frac{\Lambda_s|(u_s^0)'(z_0)| + |u_s^0(z_0)|}{|u_s^0(z_0)e^{-i\xi^0_{T-s}} - 1|^3} \mathrm{d}s. \tag{3.7}$$

Definition. We denote the preimages of the jth tip by $e^{i\phi_n^j}$ in the ALE model, and by

$e^{i\bar{\phi}_n^j}$ in the auxiliary model. Define

$$j_n = \operatorname*{argmin}_j |e^{i\theta_n} - e^{i\phi_n^j}|,$$
$$j_n^* = \operatorname*{argmin}_j |e^{i\theta_n^*} - e^{i\bar{\phi}_n^j}|.$$

If we construct a version of each of the ALE and auxiliary models on a common probability space, then we can define the stopping time

$$\tau_{\text{coupling}} = \min\{n \geq 1 : j_n \neq j_n^*\}. \tag{3.8}$$

To show that the ALE and auxiliary models are close, we will show that with high probability they land near the same slit, and the ALE lands very near the tip so looks similar to the auxiliary model.

First we will examine the regions near the tips in each model, and find that the probability of landing near each tip is similar. Then we will prove in each case that the probability of not landing near any tip is $o(1)$, generalising the main result of [33], with a similar argument.

Definition. Given that we will be defining both the ALE and auxiliary model on the same probability space in order to show that a good coupling exists, we will define a stopping time to reflect that fact that the coupling will fail if *either* model attaches a particle too far from a tip. Let

$$\tau_D = \inf\{n \geq 1 : |\theta_n^* - \hat{\theta}_n^*| > D \text{ or } \min_j |\theta_n - \phi_n^j| > D\},$$

where $D = \sqrt{\sigma}$.

Remark. The ALE and auxiliary models will have a common weak limit if with high probability $\tau_D \wedge \tau_{\text{coupling}} > T/\mathbf{c}$. This event means that every particle in the ALE model has been attached within D of a "main" tip, and every particle in the auxiliary model has chosen the same slit to attach to as the ALE model.

Proposition 3.3. *There exists a constant $A = A(K_0, k, \eta, T)$ such that the ALE and auxiliary models can be constructed on a common probability space and on this space*

$$\mathbb{P}(\tau_D \wedge \tau_{\text{coupling}} \leq \lfloor T/\mathbf{c} \rfloor) \leq A\mathbf{c}$$

provided $\sigma < \mathbf{c}^\gamma$ for $\gamma = \frac{2(\eta+2)}{\eta-1} \vee \frac{5\eta+10}{2\eta} \vee 8$.

Proof. Follows immediately from Proposition 3.9. □

Remark. The three terms determining γ come from three separate requirements in the proof of this section's result: we require $\sigma < \mathbf{c}^8$ so that the derivatives of each model look similar near the "main" tips, as in the following lemma. We require that $\sigma < \mathbf{c}^{\frac{5\eta+10}{2\eta}}$ so that "old" particles do not contribute in the ALE model, and we require that $\sigma < \mathbf{c}^{\frac{2(\eta+2)}{\eta-1}}$

so that the measures are concentrated very tightly around the tip of each particle, with each particle attached within distance $\sqrt{\sigma}$ of a main tip.

Lemma 3.4. *Suppose $\sigma < \mathbf{c}^8$. Then on the event $\{n < \tau_D \wedge \tau_{\text{coupling}}\}$,*

$$\sup_{|\theta|<\mathbf{c}^2} \left| \log \frac{(\Phi_n^{\text{ALE}})'(e^{\sigma+i\theta} e^{i\phi_n^j})}{(\Phi_n^*)'(e^{\sigma+i\theta} e^{i\bar{\phi}_n^j})} \right| \le AT\mathbf{c}^{3/2}$$

almost surely.

Proof. Denote the preimages of the k tips under Φ_n^{ALE} in the ALE model by $\phi_n^1, \dots, \phi_n^k$. For each $j \in \{1, 2, \dots, k\}$ denote the subsequence of $(\theta_\ell^{\text{ALE}})_{\ell=1}^n$ consisting of the times at which a particle is attached to the jth slit by

$$\theta_{n_j(1)}^{\text{ALE}}, \theta_{n_j(2)}^{\text{ALE}}, \dots, \theta_{n_j(N_j)}^{\text{ALE}}.$$

Near ϕ_n^j we can decompose Φ_n^{ALE} as

$$\Phi_n^{\text{ALE}} = \Phi_0^{\text{ALE}} \circ \Psi_0^j \circ f_{n_j(1)}^{\text{ALE}} \circ \Psi_1^j \circ f_{n_j(2)}^{\text{ALE}} \circ \Psi_2^j \circ \cdots \circ f_{n_j(N_j)}^{\text{ALE}} \circ \Psi_{N_j}^j, \tag{3.9}$$

where for $0 < \ell < N_j$, $\Psi_\ell^j = f_{n_j(\ell)+1}^{\text{ALE}} \circ f_{n_j(\ell)+2}^{\text{ALE}} \circ \cdots \circ f_{n_j(\ell+1)-1}^{\text{ALE}} = \left(\Phi_{n_j(\ell)}^{\text{ALE}}\right)^{-1} \circ \Phi_{n_j(\ell+1)-1}^{\text{ALE}}$, the map which attaches every particle landing at slits *other than* the jth between the ℓth and $(\ell+1)$th time a particle lands on the jth slit. The two special cases are $\Psi_0^j = \left(\Phi_0^{\text{ALE}}\right)^{-1} \circ \Phi_{n_j(1)-1}^{\text{ALE}}$ and $\Psi_{N_j}^j = \left(\Phi_{n_j(N_j)}^{\text{ALE}}\right)^{-1} \circ \Phi_n^{\text{ALE}}$. Note that any of these Ψ_ℓ^j maps may be the identity map on Δ. The important common feature of all these Ψ_ℓ^j maps is that their derivatives have no poles or zeroes near the jth slit.

Since the event we have conditioned on implies $n < \tau_{\text{coupling}}$, the subsequence of $(\theta_\ell^*)_{\ell=1}^n$ consisting of times when a particle was attached to the jth slit is

$$\theta_{n_j(1)}^*, \theta_{n_j(2)}^*, \dots, \theta_{n_j(N_j)}^*$$

with the same n_j as above. There is a similar decomposition to (3.9), complicated only slightly by the cluster rotation,

$$\begin{aligned}\Phi_n^* = {} & R_{\delta_1+\cdots+\delta_n} \circ \Phi_0^* \circ R_{-(\delta_1+\cdots+\delta_n)} \circ \\ & \circ \overline{\Psi}_0^j \circ f^*_{\hat{\theta}^*_{n_j(1)}+\delta_{n_j(1)}+\cdots+\delta_n} \circ \overline{\Psi}_1^j \circ \cdots \circ f^*_{\hat{\theta}^*_{n_j(N_j)}+\delta_{n_j(N_j)}+\cdots+\delta_n} \circ \overline{\Psi}_{N_j}^j,\end{aligned}$$

where $\overline{\Psi}_\ell^j = f^*_{\hat{\theta}^*_{n_j(\ell)+1}+\delta_{n_j(\ell)+1}+\cdots+\delta_n} \circ \cdots \circ f^*_{\hat{\theta}^*_{n_j(\ell+1)-1}+\delta_{n_j(\ell+1)-1}+\cdots+\delta_n}$ for $0 < \ell < N_j$ and the two end-cases are also defined similarly to Ψ_0^j and $\Psi_{N_j}^j$.

Now fix a $\theta \in \mathbb{R}$ with $|\theta| < \mathbf{c}^2$, and we will compare the densities $|(\Phi_n^{\text{ALE}})'(e^{\sigma+i\theta}e^{i\phi_n^j})|^{-\eta}$ and $|(\Phi_n^*)'(e^{\sigma+i\theta}e^{i\bar{\phi}_n^j})|^{-\eta}$ when $N_j > 0$. We omit the simpler case $N_j = 0$.

First we will bound
$$\left|\log \frac{(\Psi^j_{N_j})'(e^{\sigma+i\theta}e^{i\phi^j_n})}{(\overline{\Psi}^j_{N_j})'(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n})}\right|.$$
The two maps $\Psi^j_{N_j}$ and $\overline{\Psi}^j_{N_j}$ are solutions to Loewner's equation with driving functions
$$\xi_t = \theta^{\mathrm{ALE}}_{\lfloor t/\mathbf{c}\rfloor+n_j(N_j)+1}, \quad \overline{\xi}_t = \hat{\theta}^*_{\lfloor t/\mathbf{c}\rfloor+n_j(N_j)+1} + \delta_{\lfloor t/\mathbf{c}\rfloor+n_j(N_j)+1} + \cdots + \delta_n$$
respectively, for $t \in [0, (n-n_j(N_j))\mathbf{c})$. Remark 3.7 of [29] notes that the points $e^{i\bar{\phi}^j_n}$ are repelled from each other on the circle. We can therefore guarantee there is an $L > 0$ depending only on the initial conditions and T such that $e^{\sigma+i\theta}e^{i\phi^j_n}$ and $e^{\sigma+i\theta}e^{i\bar{\phi}^j_n}$ keep at least a distance L away from ξ_t and $\overline{\xi}_t$ respectively, so if we apply Lemma 3.2, $\Lambda_t \asymp t$, condition (3.5) is met for sufficiently small $\mathbf{c}$, and the bound on the right of (3.2) is $\asymp 1$.

We know $e^{i\phi^j_n} = \left(\Psi^j_{N_j}\right)^{-1}(\theta^{\mathrm{ALE}}_{n_j(N_j)})$ and $e^{i\bar{\phi}^j_n} = \left(\overline{\Psi}^j_{N_j}\right)^{-1}(\hat{\theta}^*_{n_j(N_j)} + \delta_{n_j(N_j)} + \cdots + \delta_1)$, and so applying Lemma 3.2 gives us, since $(n-n_j(N_j))\mathbf{c} \le T$,
$$\left|\log \frac{(\Psi^j_{N_j})'(e^{\sigma+i\theta}e^{i\phi^j_n})}{(\overline{\Psi}^j_{N_j})'(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n})}\right| \le A_L T\left(\left|\phi^j_n - \bar{\phi}^j_n\right| + |\delta_1| + \cdots + |\delta_{n-1}|\right),$$
and a further calculation using explicit estimates for f^{-1} similar to (3.25) tells us
$$|\phi^j_n - \bar{\phi}^j_n| \le A_L(|\delta_1| + \cdots + |\delta_n|).$$
Hence, with a constant A depending on A_L and T,
$$\left|\log \frac{(\Psi^j_{N_j})'(e^{\sigma+i\theta}e^{i\phi^j_n})}{(\overline{\Psi}^j_{N_j})'(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n})}\right| \le A\left(|\delta_1| + \cdots + |\delta_n|\right). \tag{3.10}$$
The next two terms in the expansions of each derivative are the two largest,
$$(f^{\mathrm{ALE}}_{n_j(N_j)})'\left(\Psi^j_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n})\right) \quad \text{and} \quad (f^*_{\hat{\theta}^*_{n_j(N_j)}+\delta_{n_j(N_j)}+\cdots+\delta_n})'\left(\overline{\Psi}^j_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n})\right). \tag{3.11}$$
For the reader's convenience, we restate (2.7), the explicit expression for f',
$$f'(z) = \frac{f(z)}{z}\frac{z-1}{(z-e^{i\beta})^{1/2}(z-e^{-i\beta})^{1/2}}, \tag{3.12}$$
so to first order the size of $|f'(z)|$ depends on $|z-1|$. This means that to find the size of the two derivatives above, we are most interested in
$$\left|\Psi^j_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}) - e^{i\theta^{\mathrm{ALE}}_{n_j(N_j)}}\right| \quad \text{and} \quad \left|\overline{\Psi}^j_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n}) - e^{i(\hat{\theta}^*_{n_j(N_j)}+\delta_{n_j(N_j)}+\cdots+\delta_1)}\right|.$$

If we apply (3.3), the first of these is

$$\begin{aligned}\left|\Psi^j_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}) - e^{i\theta^{\mathrm{ALE}}_{n_j(N_j)}}\right| &= \left|\Psi^j_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}) - \Psi^j_{N_j}(e^{i\phi^j_n})\right| \\ &= \left|(e^{\sigma+i\theta}-1)e^{i\phi^j_n}(\Psi^j_{N_j})'(e^{i\phi^j_n})\right|\left[1+O\left(|e^{\sigma+i\theta}-1|\right)\right] \\ &= |e^{\sigma+i\theta}-1|\left|(\Psi^j_{N_j})'(e^{i\phi^j_n})\right|\left[1+O\left(|e^{\sigma+i\theta}-1|\right)\right] \qquad (3.13)\end{aligned}$$

and similarly the second term is

$$|e^{\sigma+i\theta}-1|\left|(\overline{\Psi}^j_{N_j})'(e^{i\bar{\phi}^j_n})\right|\left[1+O\left(|e^{\sigma+i\theta}-1|\right)\right]. \qquad (3.14)$$

Our previous bounds on $\left|\log\frac{(\Psi^j_{N_j})'(e^{\sigma+i\theta}e^{i\phi^j_n})}{(\overline{\Psi}^j_{N_j})'(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n})}\right|$ apply equally to $\left|\log\frac{(\Psi^j_{N_j})'(e^{i\phi^j_n})}{(\overline{\Psi}^j_{N_j})'(e^{i\bar{\phi}^j_n})}\right|$, so (3.12) allows us to directly compare the derivatives in (3.11).

Firstly, a simple calculation shows that if $|z-1| < \mathbf{c}^{1/2}$ then $f(z) = 1+d+O(|z-1|)$. The next estimate $z = 1+O(|z-1|)$ is obvious. Putting this estimate into the denominator of the second fraction in (3.12), we get $|z-e^{\pm i\beta}|^{1/2} = |1-e^{i\beta}|^{1/2}(1+O\left(\frac{|z-1|}{\mathbf{c}^{1/2}}\right))$. Thus for z close to 1 the behaviour of $|f'(z)|$ mainly depends on $|z-1|$, and we can write

$$|f'(z)| = \frac{1+d(\mathbf{c})}{|1-e^{i\beta}|}|z-1|\left(1+O(\mathbf{c}^{-1/2}|z-1|)\right). \qquad (3.15)$$

Therefore

$$\begin{aligned}\left|\log\frac{(f^{\mathrm{ALE}}_{n_j(N_j)})'\left(\Psi^j_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n})\right)}{(f^*_{\hat{\theta}^*_{n_j(N_j)}+\delta_{n_j(N_j)}+\cdots+\delta_n})'\left(\overline{\Psi}^j_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n})\right)}\right| &\le \left|\log\frac{\left|(\Psi^j_{N_j})'(e^{i\phi^j_n})\right|}{\left|(\overline{\Psi}^j_{N_j})'(e^{i\bar{\phi}^j_n})\right|}\right| + A\mathbf{c}^{-1/2}|e^{\sigma+i\theta}-1| \\ &\le A\left(|\delta_1|+\cdots+|\delta_n|+\mathbf{c}^{-1/2}|e^{\sigma+i\theta}-1|\right). \\ &\qquad (3.16)\end{aligned}$$

Now as $|f^{\mathrm{ALE}}_{n_j(N_j)}(\Psi_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}))| \ge 1+\mathbf{c}^{1/2}$ and $|f^*_{n_j(N_j)}(\overline{\Psi}_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n}))| \ge 1+\mathbf{c}^{1/2}$, we can apply Lemma 3.2 to compare the two remaining derivatives

$$\left|\left(\Phi^{\mathrm{ALE}}_0 \circ \Psi^j_0 \circ f^{\mathrm{ALE}}_{n_j(1)} \circ \cdots \circ \Psi^j_{N_j-1}\right)'\left(f^{\mathrm{ALE}}_{n_j(N_j)}(\Psi_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}))\right)\right|$$

and

$$\left|\left(R_{\delta_1+\cdots+\delta_n} \circ \Phi^*_0 \circ R_{-(\delta_1+\cdots+\delta_n)} \circ \overline{\Psi}^j_0 \circ f^*_{\hat{\theta}^*_{n_j(1)}+\delta_{n_j(1)}+\cdots+\delta_n} \circ \cdots \circ \overline{\Psi}^j_{N_j-1}\right)'\left(\right.\right.$$
$$\left.\left.\left(f^*_{n_j(N_j)}(\overline{\Psi}_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n}))\right)\right|.$$

The two maps above whose derivatives we consider are generated by driving functions whose difference is bounded by $|\delta_1|+\cdots+|\delta_n|$ if we use the backward equation, so call these driving functions ξ^0 generating the first map above, and ξ^1 generating the second. Taking $z_0 = f^{\mathrm{ALE}}_{n_j(N_j)}(\Psi_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}))$, we have $|z_0| - 1 \geq \mathbf{c}^{1/2}$, so using standard estimates for conformal maps (see for example Section 1.1 of [32]), we have $\frac{\mathbf{c}^{1/2}}{A} \leq |(u^0_t)'(z_0)| \leq \frac{A}{\mathbf{c}^{1/2}}$ for all $0 < t < T$, where the constant A depends only on T and K_0.

This gives us, modifying the constant $A = A(T, K_0)$ appropriately (as we will throughout), $\frac{\mathbf{c}^{1/2}}{A}t \leq |\Lambda_t| \leq \frac{A}{\mathbf{c}^{3/2}}t$.

The right-hand side of (3.2) is bounded below by

$$\inf_{0<t\leq T}\left(\frac{\mathbf{c}^{1/2}}{A\mathbf{c}^{-1/2}} \wedge \left(t\frac{A\mathbf{c}^{-1/2}}{\mathbf{c}^{3/2}}\right)^{-1}\right) \geq A^{-1}\mathbf{c}^2, \tag{3.17}$$

so the condition (3.2) is satisfied by $z = f^*_{\hat{\theta}^*_{n_j(N_j)}+\delta_{n_j(N_j)}+\cdots+\delta_n}(\overline{\Psi}^j_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n}))$.

The resulting bounds on $\left|\log\frac{(u^0_t)'(z)}{(u^0_t)'(z_0)}\right|$ and $\left|\log\frac{(u^1_t)'(z)}{(u^0_t)'(z)}\right|$ are then, respectively,

$$\left|\log\frac{(u^0_t)'(z)}{(u^0_t)'(z_0)}\right| \leq A|z - z_0|t\mathbf{c}^{-2}$$

and

$$\left|\log\frac{(u^1_t)'(z)}{(u^0_t)'(z)}\right| \leq A(|\delta_1| + \cdots + |\delta_n|)\mathbf{c}^{-7/2}.$$

Elementary calculations show that for w close to 1,

$$f(w) = 1 + d(\mathbf{c}) + O\left(\frac{|w-1|^2}{\mathbf{c}^{1/2}}\right), \tag{3.18}$$

so using (3.13) and (3.14),

$$\begin{aligned}|z - z_0| &\leq (1 + d(\mathbf{c}))\left|\theta^{\mathrm{ALE}}_{n_j(N_j)} - (\hat{\theta}^*_{n_j(N_j)} + \delta_{n_j(N_j)} + \cdots + \delta_n)\right| \\ &\quad + A\frac{|e^{\sigma+i\theta}-1|^2}{\mathbf{c}^{1/2}}\left(\left|(\Psi^j_{N_j})'(e^{i\phi^j_n})\right| + \left|(\overline{\Psi}^j_{N_j})'(e^{i\bar{\phi}^j_n})\right|\right) \\ &\leq A(|\delta_1| + \cdots + |\delta_n| + \mathbf{c}^{-1/2}|e^{\sigma+i\theta} - 1|^2).\end{aligned}$$

Hence

$$\left|\log\frac{\left(R_{\delta_1+\cdots+\delta_n}\circ\Phi^*_0\circ R_{-(\delta_1+\cdots+\delta_n)}\circ\cdots\circ\overline{\Psi}^j_{N_j-1}\right)'\left(f^*_{n_j(N_j)}(\overline{\Psi}_{N_j}(e^{\sigma+i\theta}e^{i\bar{\phi}^j_n}))\right)}{\left(\Phi^{\mathrm{ALE}}_0\circ\Psi^j_0\circ f^{\mathrm{ALE}}_{n_j(1)}\circ\cdots\circ\Psi^j_{N_j-1}\right)'\left(f^{\mathrm{ALE}}_{n_j(N_j)}(\Psi_{N_j}(e^{\sigma+i\theta}e^{i\phi^j_n}))\right)}\right|$$

is

$$\begin{aligned}\left|\log\frac{(u^1_T)'(z)}{(u^0_T)'(z_0)}\right| &\leq \left|\log\frac{(u^1_T)'(z)}{(u^0_T)'(z)}\right| + \left|\log\frac{(u^0_T)'(z)}{(u^0_T)'(z_0)}\right| \\ &\leq A\left((|\delta_1| + \cdots + |\delta_n|)\mathbf{c}^{-7/2} + |e^{\sigma+i\theta} - 1|^2\mathbf{c}^{-5/2}\right).\end{aligned} \tag{3.19}$$

Putting together (3.10), (3.16) and (3.19) we have

$$\left|\log \frac{(\Phi_n^{\text{ALE}})'(e^{\sigma+i\theta}e^{i\phi_n^j})}{(\Phi_n^*)'(e^{\sigma+i\theta}e^{i\bar{\phi}_n^j})}\right| \leq A\left(\mathbf{c}^{-\frac{7}{2}}(|\delta_1|+\cdots+|\delta_n|)+\mathbf{c}^{-\frac{1}{2}}|e^{\sigma+i\theta}-1|+\mathbf{c}^{-\frac{5}{2}}|e^{\sigma+i\theta}-1|^2\right) \tag{3.20}$$

and we can check that this is bounded by $AT\mathbf{c}^{3/2}$. □

Corollary 3.5. *We can construct the coupling of the ALE and auxiliary models in such a way that on the event $\{n < \tau_D \wedge \tau_{\text{coupling}}\}$, then the conditional probability that $j_{n+1} \neq \bar{j}_{n+1}$ or $n+1 = \tau_D$ is bounded by*

$$ATk\mathbf{c}^{3/2} + \mathbb{P}[\delta_{n+1} > D \,|\, n < \tau_D] + \mathbb{P}[\min_j |\theta_{n+1} - \phi_n^j| > D \,|\, n < \tau_D].$$

The corollary follows immediately from the above lemma.

Next we need to show that the probability of attaching more than distance D from *any* tip is $o(\mathbf{c})$ in each model. We will demonstrate it only for the ALE, because identical proofs work for the auxiliary model, setting $D = 0$ instead of $D = \sqrt{\sigma}$.

Lemma 3.6. *There exists a constant A depending only on η, T and the initial conditions such that for any $1 \leq n < \lfloor T/\mathbf{c} \rfloor$, on the event $\{n < \tau_D\}$ we have*

$$Z_n \geq \mathbf{c}^\eta \sigma^{-(\eta-1)}$$

almost surely.

The above bound is not sharp: the $\mathbf{c}^\eta$ term can be eliminated, but the proof is substantially more complicated, and the bound in the lemma is all that we require.

Proof. For $|\theta| < \mathbf{c}$ and a fixed j, let $z = e^{\sigma + i(\phi_n^j + \theta)}$, and using (3.9) we can write

$$|(\Phi_n^{\text{ALE}})'(z)| = \left|(\Psi_{N_j}^j)'(z)\right| \left|(f_{n_j(N_j)}^{\text{ALE}})'(\Psi_{N_j}^j(z))\right| \left|(\Phi_0^{\text{ALE}} \circ \cdots \circ \Psi_{N_j-1}^j)'(f_{n_j(N_j)}^{\text{ALE}}(\Psi_{N_j}^j(z)))\right|.$$

Applying Lemma 3.2 to the first term in this decomposition with $z_0 = e^{i\phi_n^j}$, we again have a constant lower bound $L > 0$ on the distance between $u_t(z_0)$ and the driving function, so we have from (3.4)

$$|(\Psi_{N_1}^j)'(z)| = |(\Psi_{N_1}^j)'(e^{i\phi_n^j})|(1 + O(\mathbf{c})). \tag{3.21}$$

From (3.12) and (3.13) we obtain

$$\begin{aligned} |(f_{n_j(N_j)}^{\text{ALE}})'(\Psi_{N_j}^j(z))| &= \frac{1+d(\mathbf{c})}{|1-e^{i\beta_\mathbf{c}}|}(1+O(\mathbf{c}^{1/2}))\left|\Psi_{N_j}^j(z) - e^{i\theta_{n_j(N_j)}^{\text{ALE}}}\right| \\ &= \frac{1+d(\mathbf{c})}{|1-e^{i\beta_\mathbf{c}}|}|z - e^{i\phi_n^j}|\left|(\Psi_{N_j}^j)'(e^{i\phi_n^j})\right|(1+O(\mathbf{c}^{1/2})). \end{aligned} \tag{3.22}$$

Analysis of Loewner's reverse equation when z is far from the driving measure shows that there is a constant $A > 0$ with

$$A^{-1} \leq \left|(\Psi^j_{N_j})'(e^{i\phi^j_n})\right| \leq A,$$

(see, for example, equation (26) from [33]) and using (3.13) and (3.18) we know

$$|f^{\mathrm{ALE}}_{n_j(N_j)}(\Psi^j_{N_j}(z))| - 1 \geq \mathbf{c}^{1/2},$$

and so by standard conformal map estimates,

$$\frac{\mathbf{c}^{1/2}}{A} \leq \left|(\Phi^{\mathrm{ALE}}_0 \circ \cdots \circ \Psi^j_{N_j-1})'(f^{\mathrm{ALE}}_{n_j(N_j)}(\Psi^j_{N_j}(z)))\right| \leq A\mathbf{c}^{-1/2}. \tag{3.23}$$

Then combining (3.21), (3.22) and (3.23), we have

$$\begin{aligned} |(\Phi^{\mathrm{ALE}}_n)'(e^{\sigma+i(\phi^j_n+\theta)})|^{-\eta} &\geq A\mathbf{c}^{\eta/2}\left(\frac{|1-e^{i\beta_{\mathbf{c}}}|}{1+d(\mathbf{c})}\right)^{\eta}|e^{\sigma+i\theta}-1|^{-\eta} \\ &\geq A\mathbf{c}^{\eta}|e^{\sigma+i\theta}-1|^{-\eta}. \end{aligned}$$

Then by a simple calculation

$$\begin{aligned} Z_n &\geq A\mathbf{c}^{\eta}\int_{-\mathbf{c}}^{\mathbf{c}}\frac{\mathrm{d}\theta}{|e^{\sigma+i\theta}-1|^{\eta}} \\ &\geq A\mathbf{c}^{\eta}\sigma^{-(\eta-1)}\int_{-\mathbf{c}/\sigma}^{\mathbf{c}/\sigma}\frac{\mathrm{d}x}{(1+x^2)^{\eta/2}}, \end{aligned}$$

and as $\sigma \ll \mathbf{c}$ and $\int_{-\infty}^{\infty}(1+x^2)^{-\eta/2}\,\mathrm{d}x < \infty$, the integral term is absorbed into the constant. $\square$

Lemma 3.7. *For $\zeta \in \Delta$, write $f(\zeta) = e^{r+i\theta}$. For all sufficiently small $\mathbf{c}$, if $r < \mathbf{c}^{1/2}$, then $|f'(\zeta)| > 1$.*

Proof. Using similar methods as Lemma 4 in [33], we can write

$$(f^{-1})'(w) = \frac{f^{-1}(w)}{w}\frac{w-1}{\sqrt{(w+1)^2-4e^{\mathbf{c}}w}}$$

for $w \in \Delta \setminus (1, 1+d(\mathbf{c})]$.

Then we know that $|f^{-1}(w)| < |w|$ for any w, and elementary calculations show that if $w = e^{r+i\theta}$, then

$$\left|\frac{w-1}{(w+1)^2-4e^{c}w}\right| < 1 \iff \cos\theta < \frac{e^{\mathbf{c}}}{\cosh r}.$$

Hence $|(f^{-1})'(w)| < 1$, and so $|f'(\zeta)| > 1$.

Since $\cosh^{-1}(e^{\mathbf{c}}) \sim \sqrt{2\mathbf{c}}$ for small $\mathbf{c}$, if $\mathbf{c}$ is sufficiently small and $r < \mathbf{c}^{1/2}$, then the condition $\cos\theta < \frac{e^{\mathbf{c}}}{\cosh r}$ is always satisfied. $\square$

Lemma 3.8. *For any sufficiently large constant A_T, there exists a constant $B > 0$ depending on T and K_0 such that on the event $\{n < \tau_{\text{coupling}}\}$, i.e. when the ALE and* *auxiliary models remain coupled so they each choose the same particle of the k choices at each of the first n steps, the following is almost surely true: for any $\theta \in \mathbb{T}$, if $k \leq n$ satisfies*

$$|(f_{l+1}^{\mathrm{ALE}} \circ f_{l+2}^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(e^{i\theta}) - e^{i\theta_l^{\mathrm{ALE}}}| \geq |e^{i(\beta_{\mathbf{c}} + \frac{\mathbf{c}^{1/2}}{A_T})} - 1| \text{ for all } k \leq l \leq n,$$

and $|e^{i\theta} - e^{i\theta_n^{\mathrm{ALE}}}| \geq |e^{i(\beta_{\mathbf{c}} + \frac{\mathbf{c}^{1/2}}{A_T})} - 1|$ *then* $\left|(f_k^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})'(e^{i\theta})\right| \leq B\mathbf{c}^{-1/2}$.

Proof. On the event $\{n < \tau_{\text{coupling}}\}$, each angle θ_l^{ALE} is within D of the tip of a previous particle. Consider the angle sequence $(\bar{\theta}_l)_{l \leq n}$ corresponding to what would happen if the lth particle is attached exactly at the tip of the particle θ_l^{ALE} landed closest to. The difference between the two angle sequences is then almost surely $\sup_{l \leq n} |\theta_l^{\mathrm{ALE}} - \bar{\theta}_l| < A\mathbf{c}^{-1}D$, for a constant $A > 0$. Then using the bound on $\left|\log \frac{(u_t^1)'(z)}{(u_t^0)'(z)}\right|$ from Lemma 11 of [33], we can establish the bound we want on the ALE process by establishing it using the angle sequence $(\bar{\theta}_l)_{l \leq n}$, since $D = \sqrt{\sigma}$ is very small. We can also think of this as first considering $D = 0$ and then perturbing the result.

Elementary calculations show that for any positive constant $L > 0$, there is a constant $A_L > 0$ such that if $e^{i\bar{\theta}_l}$ is distance at least L from $e^{i\alpha}$ and $e^{i\alpha'}$ then

$$|f'_{\mathbf{c},\bar{\theta}_l}(e^{i\alpha})| \leq e^{A_L \mathbf{c}} \tag{3.24}$$

and

$$e^{-A_L \mathbf{c}}|\alpha - \alpha'| \leq |\arg \bar{f}_{\mathbf{c},\bar{\theta}_l}(e^{i\alpha}) - \arg f_{\mathbf{c},\bar{\theta}_l}(e^{i\alpha'})| \leq e^{A_L \mathbf{c}}|\alpha - \alpha'|. \tag{3.25}$$

As the minimum separation between the preimages of tips ϕ_l^j for $1 \leq l \leq n$ is bounded below by a constant $L > 0$, for any given $\theta \in T$ there can be at most one j such that there exists an $l \leq n$ with $|\bar{f}_l(e^{i\theta}) - \phi_{l-1}^j| \leq \frac{L}{2}e^{-A_{L/2}T} =: m$. If there is no such j, or the corresponding l is less than k, then by (3.24), $\left|(\bar{f}_k \circ \cdots \circ \bar{f}_n)'(e^{i\theta})\right| \leq e^{A_m T}$.

Suppose that such a j does exist. We will split the angle sequence $(\bar{\theta}_l)_{k \leq l \leq n}$ into times when particles are attached to slit j and times when particles are attached elsewhere.

Set $n_0 = \max\{k \leq l \leq n : \bar{\theta}_l = \phi_l^j\}$, then for $i \geq 0$ set $n'_i = \max\{k \leq l < n_i : \bar{\theta}_l \neq \phi_l^j\}$ and $n_{i+1} = \max\{k \leq l < n'_i : \bar{\theta}_l = \phi_l^j\}$ until one of the sets is empty, and call the last well defined value n_{p+1} (if n'_p is the final well-defined value of the above maxima, then set $n_{p+1} = n'_p$). Define the capacities $t_0 = (n - n_0)\mathbf{c}$, and for $i \geq 0$, $t'_i = (n_i - n'_i)\mathbf{c}$ and $t_{i+1} = (n'_i - n_{i+1})\mathbf{c}$. The map $\bar{f}_k \circ \cdots \circ \bar{f}_n$ is then generated by using the backwards equation (1.6) with a driving function which first is at a distance at least $L/2$ from ϕ_n^j for time $t_0 \geq 0$, then takes the constant value $\phi_{n_0}^j$ for time $t'_0 \geq \mathbf{c}$, then is distance at least $L/2$ from tip j again for time $t_1 \geq \mathbf{c}$, and so on, terminating after a total time $(n-k+1)\mathbf{c} \leq T$.

We can therefore decompose $\bar{f}_k \circ \cdots \circ \bar{f}_n$ as

$$\Psi_{p+1} \circ f_{t'_p, \phi_{n'_p}^j} \circ \Psi_p \circ \cdots \circ f_{t'_0, \phi_{n'_0}^j} \circ \Psi_0 \tag{3.26}$$

where the Ψ maps are generated by driving functions bounded away from the jth slit.

Using (3.24), we have $|\Psi_i'(f_{t_{i-1}',\phi^j_{n_{i-1}'}} \circ \cdots \circ \Psi_0(e^{i\theta}))| \leq e^{A_{L/2}t_i}$ for each i, and so the total contribution to $\left|(\bar{f}_k \circ \cdots \circ \bar{f}_n)'(e^{i\theta})\right|$ by Ψ' terms is bounded by a constant $e^{A_{L/2}T}$.

The other terms can give a larger contribution. Let δ_i be the distance in $\mathbb{R}/2\pi\mathbb{Z}$ between $\arg[(\Psi_i \circ \cdots \circ \Psi_0)(e^{i\theta})]$ and $\phi^j_{n_i}$, and similarly let δ_i' be the distance between $\arg[f_{t_i',\phi^j_{n_i'}} \circ \Psi_i \circ \cdots \circ \Psi_0)(e^{i\theta})]$ and $\phi^j_{n_i}$. Then using the explicit form of the derivative f_t' we can compute

$$\left|f'_{t_i',\phi^j_{n_i'}}(\Psi_i \circ \cdots \circ \Psi_0)(e^{i\theta}))\right| = e^{t_i'/2}\frac{|e^{i\delta_i}-1|}{|e^{i\delta_i'}-1|.} \tag{3.27}$$

Using (3.25), we can calculate $\frac{\delta_{i+1}}{\delta_i'} \leq e^{A_{L/2}t_i}$, and so the total contribution from all of the f'_{t_i',ϕ^j_i} terms is, possibly increasing the constant $A_{L/2}$,

$$\begin{aligned} e^{\frac{1}{2}\sum_{i=0}^{p} t_i'}\frac{|e^{i\delta_0}-1|}{|e^{i\delta_p'}-1|}\prod_{i=0}^{p-1}\frac{|e^{i\delta_{i+1}}-1|}{|e^{i\delta_i'}-1|} &\leq e^{(\frac{1}{2}+A_{L/2})T}\frac{2}{\mathbf{c}^{1/2}/A_T} \\ &= B\mathbf{c}^{-1/2} \end{aligned}$$

as required. $\square$

Proposition 3.9. *For any $n < N$,*

$$\mathbb{P}[n+1 = \tau_D \,|\, n < \tau_D] \leq A\mathbf{c}^{-\eta}\sigma^{\frac{\eta-1}{2}} + A\mathbf{c}^{-2\eta}\sigma^{\eta-1} + A\mathbf{c}^{-\frac{5\eta}{4}-\frac{1}{2}}\sigma^{\eta/2}.$$

If $\sigma < \mathbf{c}^{\gamma}$ where $\gamma = \frac{2(\eta+2)}{\eta-1} \vee \frac{5\eta+10}{2\eta} \vee 8$, then this implies $\mathbb{P}[n+1 = \tau_D \,|\, n < \tau_D] \leq A\mathbf{c}^2$.

Proof. First note that on the event $\{n < \tau_D\}$, we have

$$\mathbb{P}[n+1 < \tau_D \,|\, (\theta_1, \ldots, \theta_n)] \leq \left(1 - \sum_j \int_{\phi^j_n - D}^{\phi^j_n + D} h_n(\theta)\,\mathrm{d}\theta\right) + \left(1 - \sum_j \int_{\bar{\phi}^j_n - D}^{\bar{\phi}^j_n + D} h^*_n(\theta)\,\mathrm{d}\theta\right)$$

almost surely. We will hence bound $1 - \sum_j \int_{-D}^{D} h_n(\phi^j_n + \theta)\,\mathrm{d}\theta$, and a similar bound will apply to the auxiliary term. Let $z = e^{\sigma + i\theta}$. Note that

$$1 - \sum_j \int_{-D}^{D} h_n(\phi^j_n + \theta)\mathrm{d}\theta = \sum_j \int_{D<|\theta-\phi^j_n|<\mathbf{c}^2} h_n(\theta)\mathrm{d}\theta + \int_{\{|\theta-\phi^j_n|\geq\mathbf{c}^2\,\forall j\}} h_n(\theta)\mathrm{d}\theta.$$

If $D < |\theta - \phi^j_n| < \mathbf{c}^2$ for some j, then we can use (3.21), (3.22) and (3.23) to establish the almost-sure upper bound

$$|(\Phi^{\mathrm{ALE}}_n)'(z)|^{-\eta} \leq A|z - e^{i\phi^j_n}|^{-\eta} \leq AD^{-\eta}. \tag{3.28}$$

Then using (3.28) and Lemma 3.6, almost surely

$$\int_{\phi^j_n+D}^{\phi^j_n+\mathbf{c}^2} h_n(\theta)\mathrm{d}\theta \leq A\mathbf{c}^{-\eta}D^{-(\eta-1)}\sigma^{\eta-1}.$$

Next we suppose θ is further than $\mathbf{c}^2$ from any tip ϕ_n^j.

We will classify the point z based on its projection $\hat{z} = e^{i\theta}$ and follow them both through their backwards evolutions, so let $z_k = (f_k \circ f_{k+1} \circ \cdots \circ f_n)(z)$, and also let $\hat{z}_k = (f_k \circ f_{k+1} \circ \cdots \circ f_n)(\hat{z})$. If $A_T = e^{A_{L/2}T}$ for $A_{L/2}$ as in the proof of Lemma 3.8, then there can be at most two values of k such that both $\hat{z}_k \in \mathbb{T}$ and $|\hat{z}_k - e^{i\theta_{k-1}^{\mathrm{ALE}}}| < |e^{i(\beta_{\mathbf{c}} + \frac{\mathbf{c}^{1/2}}{A_T})} - 1|$. First suppose there is no such k. Then by Lemma 3.8, $|(f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})'(\hat{z})| \leq B\mathbf{c}^{-1/2}$ almost surely, so by Lemma 11 of [33],

$$|(f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(z) - (f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(\hat{z})| \leq B\mathbf{c}^{-1/2}\sigma.$$

This gives us $|(f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(z)| - 1 \leq B\mathbf{c}^{-1/2}\sigma$, and hence by Lemma 3.7, $|(f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})'(z)| > 1$. Let j be the index of the closest ϕ_0^j to $(f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(\hat{z})$. We can decompose Φ_0 as $f_{t_j,\phi_0^j} \circ \Psi_0^j$, where Ψ_0^j is generated in the reverse equation by a driving function run for total time $c_0 - t^j$ which stays at least distance $L/4$ from $(u_s(\hat{z}))_{s \leq c_0 - t^j}$ at all times. Then using equation (26) of [33],

$$e^{-Ac_0} \leq |(\Psi_0^j)'((f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(\hat{z}))| \leq e^{Ac_0}$$

for an appropriate constant $A = A(K_0)$. If none of the n particles have been attached to the initial jth slit, then the assumption we made on θ tells us $|(\Psi_0^j \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(\hat{z}) - e^{i\phi_0^j}| \geq \mathbf{c}/A$. If any particles *were* attached to the jth slit, then we must have $|(\Psi_0^j \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(\hat{z})) - e^{i\phi_0^j}| \geq \mathbf{c}^{1/2}/A > \mathbf{c}/A$, otherwise some k as above would exist. In either case we have

$$|(\Psi_0^j \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(z)) - e^{i\phi_0^j}| \geq \mathbf{c}/2A,$$

and so

$$|f'_{t_j,\phi_0^j}((\Psi_0^j \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_n^{\mathrm{ALE}})(z))| \geq \frac{\mathbf{c}}{18A}.$$

Thus, for a modified constant A,

$$|\Psi_0'(z)| \geq \frac{\mathbf{c}}{A},$$

and so $h_n(\theta) = O(\sigma^{\eta-1}/\mathbf{c}^{2\eta})$.

Next, if there is only one such k, we must have $|\hat{z}_k - e^{i\theta_{k-1}^{\mathrm{ALE}}}| \geq \frac{\mathbf{c}^{1/2}}{2A_T^2}$, since there is an $l \geq k$ with θ_l^{ALE} attached at the same slit as $\theta_{k-1}^{\mathrm{ALE}}$ and if $\hat{z}_k$ were any closer to $e^{i\theta_{k-1}^{\mathrm{ALE}}}$ then $\hat{z}_{l+1}$ would be within $|e^{i(\beta_{\mathbf{c}} + \frac{\mathbf{c}^{1/2}}{A_T})} - 1|$ of $e^{i\theta_l^{\mathrm{ALE}}}$.

So $|(f_{k-1}^{\mathrm{ALE}})'(\hat{z}_k)| \geq \left| f'\left(e^{i\frac{\mathbf{c}^{1/2}}{2A_T^2}}\right)\right| \geq A$ for some constant $A > 0$, then using Lemma 3.2 and Lemma 3.8 we can derive

$$|(f_{k-1}^{\mathrm{ALE}})'(z_k)| \geq A^{-1} \tag{3.29}$$

provided $\sigma = o(\mathbf{c}^{3/2})$.

To compute $|(\Phi_0 \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_{k-2}^{\mathrm{ALE}})'(\hat{z}_{k-1})|$, consider two cases: (a) $|\hat{z}_k - e^{i\theta_{k-1}^{\mathrm{ALE}}}| \leq |e^{i(\beta - \frac{\mathbf{c}^{1/2}}{A_T})} - 1|$, or its negation, (b).

In case (a), $|\hat{z}_{k-1}| - 1 \geq |f(e^{i(\beta - \frac{\mathbf{c}^{1/2}}{A_T})})| - 1 \geq \frac{\mathbf{c}^{1/2}}{A}$. By an identical argument to that which established (3.23),

$$|(\Phi_0 \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_{k-2}^{\mathrm{ALE}})'(\hat{z}_{k-1})| \geq \frac{\mathbf{c}^{1/2}}{A},$$

and again this easily extends to

$$|(\Phi_0 \circ f_1^{\mathrm{ALE}} \circ \cdots \circ f_{k-2}^{\mathrm{ALE}})'(z_{k-1})| \geq \frac{\mathbf{c}^{1/2}}{A}, \tag{3.30}$$

provided $\sigma = o(\mathbf{c}^{3/2})$. Combining (3.30) with (3.29) and Lemma 3.7, we have

$$|(\Phi_n^{\mathrm{ALE}})'(z)| \geq \frac{\mathbf{c}}{A},$$

and so $h_n(\theta) = O(\sigma^{\eta-1}/\mathbf{c}^{2\eta})$.

In case (b), we have

$$\min_{\pm} |\hat{z}_k - e^{i(\theta_{k-1}^{\mathrm{ALE}} \pm \beta_{\mathbf{c}})}| < \frac{2\mathbf{c}^{1/2}}{A_T},$$

and so

$$\min_{\pm} |z_k - e^{i(\theta_{k-1}^{\mathrm{ALE}} \pm \beta_{\mathbf{c}})}| < \frac{3\mathbf{c}^{1/2}}{A_T}.$$

Without loss of generality the minimum is achieved in both cases by $e^{i(\theta_{k-1}^{\mathrm{ALE}} + \beta_{\mathbf{c}})}$. Let $\delta = |z_k - e^{i(\theta_{k-1}^{\mathrm{ALE}} + \beta_{\mathbf{c}})}|$, then by Lemma 2.4,

$$|(f_{k-1}^{\mathrm{ALE}})'(z_k)| \geq A^{-1} \frac{\mathbf{c}^{1/4}}{\delta^{1/2}} \tag{3.31}$$

and Lemma 2.5 tells us that $|z_{k-1} - e^{i\theta_{k-1}^{\mathrm{ALE}}}| \asymp \mathbf{c}^{1/4}\delta^{1/2}$. If the attachment point $\theta_{k-1}^{\mathrm{ALE}}$ is on the jth slit, then it is within D of the preimage ϕ_{k-2}^j of θ_l^{ALE} under $f_{l+1}^{\mathrm{ALE}} \circ \cdots \circ f_{k-2}^{\mathrm{ALE}}$, where $l < k-1$ was the previous time a particle was attached at the jth slit (without loss of generality such an l exists, as we can decompose the initial condition to have a slit of capacity $\mathbf{c}$ at the top of a longer slit in position j).

By (3.22), the size of $|(f_l^{\mathrm{ALE}})'(z_{l+1})|$ depends on $|z_{l+1} - e^{i\theta_l^{\mathrm{ALE}}}|$, which, similarly to (3.25), satisfies

$$e^{-AT} \leq \frac{|z_{l+1} - e^{i\theta_l^{\mathrm{ALE}}}|}{|z_{k-1} - e^{i\phi_{k-2}^j}|} \leq e^{AT}.$$

Then since

$$\begin{aligned} |z_{k-1} - e^{i\theta_{k-1}^{\mathrm{ALE}}}| &\leq |z_{k-1} - e^{i\phi_{k-2}^j}| + |e^{i\phi_{k-2}^j} - e^{i\theta_{k-1}^{\mathrm{ALE}}}| \\ &\leq |z_{k-1} - e^{i\phi_{k-2}^j}| + 2D, \end{aligned}$$

we have, for some constant A,

$$A^{-1}\mathbf{c}^{1/4}\delta^{1/2} \le |z_{k-1} - e^{i\phi^j_{k-2}}| + 2D \le e^{AT}|z_{l+1} - e^{i\theta^{\text{ALE}}_l}| + 2D,$$

and so

$$|z_{l+1} - e^{i\theta^{\text{ALE}}_l}| \ge \sigma \vee \left(\frac{\mathbf{c}^{1/4}\delta^{1/2}}{A} - 2D\right) \tag{3.32}$$

Then

$$|(f^{\text{ALE}}_l)'(z_{l+1})| \ge \frac{1}{A'}\mathbf{c}^{-1/2}|z_{l+1} - e^{i\theta^{\text{ALE}}_l}|,$$

and so combining this with Lemma 3.7 and (3.31),

$$|(f^{\text{ALE}}_l \circ \cdots \circ f^{\text{ALE}}_n)'(z)| \ge \frac{\sigma \vee (A^{-1}\mathbf{c}^{1/4}\delta^{1/2} - 2D)}{\delta^{1/2}}, \tag{3.33}$$

and since $|z_{k-1} - e^{i\phi^j_{k-2}}| \ge \sigma$, (3.33) is bounded below by

$$A^{-1}\mathbf{c}^{-1/4}\frac{\sigma}{9D^2}.$$

Then since $|z_l| - 1 \ge \mathbf{c}^{1/2}$, we have an overall lower bound on $|\Phi'_n(z)|$ of

$$A^{-1}\mathbf{c}^{1/2}\frac{\sigma \vee (A^{-1}\mathbf{c}^{1/4}\delta^{1/2} - 2D)}{\delta^{1/2}}.$$

Note that δ is proportional to $|z - (f_n \circ \cdots \circ f_{k+1})(e^{i(\theta^{\text{ALE}}_{k-1} + \beta_{\mathbf{c}})})|$, so we can integrate our bound on $|\Phi'_n(z)|^{-\eta}$ from σ to $\mathbf{c}^2$ to get $\int_{\{\text{one } k \text{ exists}\}} h_n(\theta)\,\mathrm{d}\theta \le A\sigma^{-1}D^{\eta+2}\mathbf{c}^{-\frac{5\eta}{4}-\frac{1}{2}}$.

If there are two values of k, $k_2 < k_1$, we can find a bound of the same order using the same argument as above, replacing k by k_1 and l by k_2. □

3.3.2 Auxiliary to multinomial model

Next we will define the *multinomial model*, in which the probability of attaching a particle at each tip depends on the second derivative of the relevant map. This essentially corresponds to taking $\sigma \to 0$, and we will show that replacing the auxiliary ALE model by the multinomial model does not affect the limiting behaviour as $\mathbf{c} \to 0$.

We explained the significance of the second derivative of the map for the Laplacian path model in Section 3.1, so the multinomial model can be viewed as a halfway point between the ALE and LPM.

Definition. Begin with the same initial condition as the other models, $\Phi^{\text{multi}}_0 = \Phi_0$. Let the preimages of the k tips under Φ^{multi}_n be ϕ^j_n for $j = 1, 2, \ldots, k$. Choose θ_{n+1} from $\{\phi^1_n, \ldots, \phi^k_n\}$, with probabilities

$$\mathbb{P}(\theta_{n+1} = \phi^j_n) = \frac{|(\Phi^{\text{multi}}_n)''(e^{i\phi^j_n})|^{-\eta}}{Z_n},$$

where $Z_n = \sum_j |(\Phi_n^{\mathrm{multi}})''(e^{i\phi_n^j})|^{-\eta}$. Define inductively the maps

$$\Phi_{n+1}^{\mathrm{multi}} = \Phi_n^{\mathrm{multi}} \circ f_{\theta_{n+1}}.$$

If they choose the same tips, the auxiliary model (re-rotated to fix the basepoints) and the multinomial model coincide exactly. It is therefore fairly simple to prove a coupling result between the two.

Proposition 3.10. *Let $(\theta_l^*)_{l\le n}$ be the angle sequence for the auxiliary model, without the rotation used in Section 3.3.1, and let $(\theta_l)_{l\le n}$ be the angle sequence for the multinomial model with the same initial conditions. Define the stopping time $\tau_{\neq} = \inf\{l : \theta_l^* \neq \theta_l\}$. On the event $\{l < \tau_{\neq} \wedge n\}$, the conditional distributions of θ_{l+1}^* and θ_{l+1} are almost surely supported on the same set $\{\phi_l^1, \dots, \phi_l^k\}$, and*

$$\max_{1\le j\le k} |\mathbb{P}(\theta_{l+1}^* = \phi_l^j) - \mathbb{P}(\theta_{l+1} = \phi_l^j)| \le A\mathbf{c}^{-1}D$$

almost surely for a deterministic constant A, when σ and D are as in the previous section.

Corollary 3.11. *We can construct the coupling of $(\theta_l^*)_{l\le n}$ and $(\theta_l)_{l\le n}$ in such a way that $\mathbb{P}[\tau_{\neq} < \lfloor T/\mathbf{c}\rfloor] \le AT\mathbf{c}^{-2}D$.*

The proof of Proposition 3.10 is relatively simple: since $(\Phi_n^*)'(e^{i\phi_l^j}) = 0$, the value of $\mathbb{P}(\theta_{l+1}^* = \phi_l^j)$ is asymptotically proportional to $|(\Phi_n^*)''(e^{i\phi_l^j})|^{-\eta}$, and hence approximates $\mathbb{P}(\theta_{l+1} = \phi_l^j)$.

Proof of Proposition 3.10. We need to show $\int_{-D}^{D} |(\Phi_l^*)'(e^{\sigma+i(\phi_l^j+\theta)})|^{-\eta}\,\mathrm{d}\theta$ is proportional to $|(\Phi_l^*)''(e^{i\phi_l^j})|^{-\eta}$. For $|\theta| < D$, let γ be the line segment in $\overline{\Delta}$ from $e^{i\phi_l^j}$ to $e^{\sigma+i(\phi_l^j+\theta)}$. Then by the fundamental theorem of calculus,

$$(\Phi_l^*)'(e^{\sigma+i(\phi_l^j+\theta)}) = (e^{\sigma+i\theta}-1)e^{i\phi_l^j}(\Phi_l^*)''(e^{i\phi_l^j}) + \int_\gamma (e^{\sigma+i(\phi_l^j+\theta)} - z)(\Phi_l^*)^{(3)}(z)\,\mathrm{d}z,$$

and so

$$\left|\log \frac{|(\Phi_l^*)'(e^{\sigma+i(\phi_l^j+\theta)})|}{|e^{\sigma+i\theta}-1| \times |(\Phi_l^*)''(e^{i\phi_l^j})|}\right| \le \frac{|e^{\sigma+i\theta}-1| \times \sup_{z\in\gamma}|(\Phi_l^*)^{(3)}(z)|}{|(\Phi_l^*)''(e^{i\phi_l^j})|}. \tag{3.34}$$

We can decompose Φ_l^* as $\Phi_{k-1}^* \circ f_k \circ \Psi_k$, where $1 \le k \le l$ is the last time a particle was attached to slit j (if no particle has been attached we can regard the top part of the initial slit as a particle of capacity $\mathbf{c}$, so without loss of generality we can assume at least one particle has been added to each slit). For u satisfying the backwards equation (1.6) with driving function ξ on $[0,T]$, equation (26) of [33] gives an expression for the spatial derivative

$$u_t'(z) = \exp\left(t - \int_0^t \frac{2\,\mathrm{d}s}{(u_s(z)e^{-i\xi_{T-s}}-1)^2}\right).$$

Using this expression and its higher spatial derivatives, we find that for a constant A depending only on η, T and the initial conditions, we have bounds $A^{-1} \le |\Psi_k'(z)| \le A$,

$|\Psi_k''(z)| \le A$, $|\Psi_k^{(3)}(z)| \le A$ for all $z \in \gamma$. Then by the chain rule, and as $f_k'(\Psi_k(e^{i\phi_l^j})) = 0$,

$$\begin{aligned}|(\Phi_l^*)''(e^{i\phi_l^j})| &= \left|\Psi_k''(e^{i\phi_l^j})(\Phi_{k-1}^* \circ f_k)'(\Psi_k(e^{i\phi_l^j})) + (\Psi_k'(e^{i\phi_l^j}))^2(\Phi_{k-1}^* \circ f_k)''(\Psi_k(e^{i\phi_l^j}))\right| \\ &\ge A^{-2}\left|(\Phi_{k-1}^* \circ f_k)''(\Psi_k(e^{i\phi_l^j}))\right| \\ &= A^{-2}|f_k''(\Psi_k(e^{i\phi_l^j}))| \times |(\Phi_{k-1}^*)'(f_k(\Psi_k(e^{i\phi_l^j})))|.\end{aligned}$$

Since $\Psi_k(e^{i\phi_l^j}) = e^{i\theta_k^*}$,

$$|f_k''(\Psi_k(e^{i\phi_l^j}))| = f''(1) = \frac{1+d(\mathbf{c})}{2\sqrt{1-e^{-\mathbf{c}}}} \ge \frac{\mathbf{c}^{-1/2}}{4},$$

and $|f_k(\Psi_k(e^{i\phi_l^j}))| - 1 \ge \mathbf{c}^{1/2}$, so $|(\Phi_{k-1}^*)'(f_k(\Psi_k(e^{i\phi_l^j})))| \ge \frac{\mathbf{c}^{1/2}}{A}$. Hence we have a constant lower bound on $|(\Phi_l^*)''(e^{i\phi_l^j})|$, and only need an upper bound on $\sup_{z\in\gamma}|(\Phi_l^*)^{(3)}(z)|$. By repeated application of the chain rule with the same decomposition of Φ_l^*, similarly to the above we find

$$|(\Phi_l^*)^{(3)}(z)| \le A\mathbf{c}^{-1}$$

for all $z \in \gamma$. Hence the right-hand side of (3.34) is almost surely bounded by $A\mathbf{c}^{-1}D$, and so $\mathbb{P}(\theta_{l+1}^* = \phi_l^j) = e^{O(D/\mathbf{c})}\mathbb{P}(\theta_{l+1} = \phi_l^j)$. □

3.3.3 Multinomial to Laplacian path model

As the Laplacian path model is generated by a driving *measure* rather than a function, we will first specify what we mean by convergence of driving measures.

Definition. Given a metric space (X, d), and two measures μ_1, μ_2 on X, the bounded Wasserstein distance $d_{\mathrm{BW}}(\mu_1, \mu_2)$ is defined

$$d_{\mathrm{BW}}(\mu_1, \mu_2) = \sup_{\varphi\in\mathcal{H}}\left|\int_X \varphi\,\mathrm{d}\mu_1 - \int_X \varphi\,\mathrm{d}\mu_2\right|,$$

where $\mathcal{H} = \{\varphi \in C(X) : \|\varphi\|_{\mathrm{Lip}} + \|\varphi\|_\infty \le 1\}$, for

$$\|\varphi\|_{\mathrm{Lip}} = \sup_{x\ne y}\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}, \quad \|\varphi\|_\infty = \sup_x |\varphi(x)|.$$

Proposition 3.12. *Let (X, d) be a separable metric space, then d_{BW} metrises weak convergence of finite measures on X, i.e. if μ, μ_n for $n \in \mathbb{N}$ are finite measures on X, then $\mu_n \Rightarrow \mu$ if and only if $d_{\mathrm{BW}}(\mu_n, \mu) \to 0$ as $n \to \infty$.*

Proof. See Theorem 11.3.3 of [8]. □

Remark. Weak convergence $\mu_n \Rightarrow \mu$ of measures on X can also be implied by convergence $\int_X \varphi\,\mathrm{d}\mu_n \to \int_X \varphi\,\mathrm{d}\mu$ for bounded continuous functions φ. We use the smaller space of test functions $\varphi \in \mathcal{H}$ in this section because it is substantially easier to prove convergence of the integrals for our two measures in this case, as we make use of the bound on $\|\varphi\|_{\mathrm{Lip}}$ in the proof of Corollary 3.17, and this still suffices to imply weak convergence of the measures.

Remark. We can view each driving measure $(\mu_t)_{t\in[0,T]}$ as a single probability measure μ on the cylinder $S = \mathbb{T} \times [0,T]$ given by $\mu_t \otimes m_{[0,T]}$ where $m_{[0,T]}$ is normalised Lebesgue measure on $[0,T]$. Then by Proposition 1 of [16], weak convergence of these measures on the cylinder implies convergence of the corresponding clusters in the Carathéodory topology.

To show that the multinomial model and Laplacian path model are close, we will use the fact that in the measures of each model, consisting of k atoms, the weights of the atoms as well as their locations on $\mathbb{T}$ are Lipschitz in t.

Lemma 3.13. *Let Φ_t be the solution to Loewner's equation for driving function ξ_t corresponding to the ALE as above. Let $q_t(z) = \frac{\partial}{\partial t}\Phi_t(z) = \Phi_t'(z) z\frac{z+\xi_t}{z-\xi_t}$ for $z \in \Delta$. Note that we can write Loewner's equation in the integral form $\Phi_t(z) = \Phi_0(z) + \int_0^t q_s(z)\,\mathrm{d}s$ and for $m \geq 1$, $\Phi_t^{(m)}(z) = \Phi_0^{(m)}(z) + \int_0^t q_s^{(m)}(z)\,\mathrm{d}s$. Then for $m \geq 0$, at times t when $\xi_t = e^{i\phi_t^j}$ for some j, we have*

$$q_t^{(m)}(e^{i\phi_t^j}) := \lim_{z\to e^{i\phi_t^j}} q_t^{(m)}(z) = m\Phi_t^{(m)}(\xi_t) + 3\xi_t\Phi_t^{(m+1)}(\xi_t) + \frac{2\xi_t^2}{m+1}\Phi_t^{(m+2)}(\xi_t). \tag{3.35}$$

Proof. Let $h(z) = z\frac{z+\xi_t}{z-\xi_t} = z + 2\xi_t + \frac{2\xi_t^2}{z-\xi_t}$. Then $h'(z) = 1 - \frac{2\xi_t^2}{(z-\xi_t)^2}$, and for $p \geq 2$, $h^{(p)}(z) = 2\xi_t^2\frac{(-1)^p p!}{(z-\xi_t)^{p+1}}$.

Using the product rule,

$$\begin{aligned} q_t^{(m)}(z) &= \sum_{p=0}^{m}\binom{m}{p} h^{(m-p)}(z)\Phi_t^{(p+1)}(z) \qquad (3.36)\\ &= 2\xi_t^2\sum_{p=0}^{m-2}\binom{m}{p}\frac{(-1)^{m-p}(m-p)!}{(z-\xi_t)^{m-p+1}}\Phi^{(p+1)}(z) + mh'(z)\Phi_t^{(m)}(z) + h(z)\Phi_t^{(m+1)}(z). \end{aligned}$$

Consider the Taylor expansion of each derivative of Φ_t about $\xi_t = e^{i\phi_t^j}$,

$$\begin{aligned} \Phi_t^{(p+1)}(z) = \Phi_t^{(p+1)}(\xi_t) + (z-\xi_t)\Phi_t^{(p+2)}(\xi_t) + \cdots + \frac{(z-\xi_t)^{n-m+1}}{(n-m+1)!}\Phi_t^{(n+2)}(\xi_t)\\ + O\left((z-\xi_t)^{n-m+2}\sup_{|w-\xi_t|<|z-\xi_t|}|\Phi_t^{(n+3)}(w)|\right). \end{aligned}$$

Substituting each of these into (3.36), for $1 \leq r \leq m-1$, the coefficient of $\Phi_t^{(r)}(\xi_t)$ is

$$\begin{aligned} 2\xi_t^2\sum_{p=0}^{r-1}\binom{m}{p}\frac{(-1)^{m-p}(m-p)!}{(z-\xi_t)^{m-p+1}}\frac{(z-\xi_t)^{r-p-1}}{(r-p-1)!} &= \frac{2\xi_t^2}{(z-\xi_t)^{m-r+2}}\sum_{p=0}^{r-1}\frac{m!}{p!(r-1-p)!}(-1)^{m-p}\\ &= \frac{2\xi_t^2\, m(m-1)\dots r}{(z-\xi_t)^{m-r+2}}\sum_{p=0}^{r-1}\binom{r-1}{p}(-1)^{m-p}. \end{aligned}$$

If $r > 1$ this is just a multiple of the binomial expansion of $(1-1)^{r-1} = 0$, so there is no $\Phi_t^{(r)}(\xi_t)$ term in $q_t^{(m)}(\xi_t)$. If $r = 1$ the coefficient is non-zero, but $\Phi_t'(\xi_t) = 0$. Next, the coefficient of $\Phi_t^{(m)}(\xi_t)$ is, comparing the with the binomial expansion of $(1-1)^{m-1}$ again,

$$\begin{aligned}
&2\xi_t^2 \sum_{p=0}^{m-2} \binom{m}{p} \frac{(-1)^{m-p}(m-p)!}{(z-\xi_t)^{m-p+1}} \frac{(z-\xi_t)^{m-p-1}}{(m-p-1)!} + m(1-\frac{2\xi_t^2}{(z-\xi_t)^2}) \\
&= \frac{2m\xi_t^2}{(z-\xi_t)^2} \sum_{p=0}^{m-2} \binom{m-1}{p} (-1)^{m-p} + m(1-\frac{2\xi_t^2}{(z-\xi_t)^2}) \\
&= \frac{2m\xi_t^2}{(z-\xi_t)^2}(-(1-1)^{m-1}+1) + m(1-\frac{2\xi_t^2}{(z-\xi_t)^2}) \\
&= m
\end{aligned}$$

if $m \geq 2$. Then writing $\sum_{p=0}^{m-2} \binom{m}{p}(-1)^{m-p} = (1-1)^m - \binom{m}{m-1}(-1)^1 - \binom{m}{m}(-1)^0 = m-1$, we find that the coefficient of $\Phi_t^{(m+1)}(\xi_t)$ is

$$\begin{aligned}
&2\xi_t^2 \sum_{p=0}^{m-2} \binom{m}{p} \frac{(-1)^{m-p}(m-p)!}{(z-\xi_t)^{m-p+1}} \frac{(z-\xi_t)^{m-p}}{(m-p)!} + m(1-\frac{2\xi_t^2}{(z-\xi_t)^2})(z-\xi) + z + 2\xi_t + \frac{2\xi_t^2}{z-\xi_t} \\
&= \frac{2\xi_t^2}{z-\xi_t}(0 - \binom{m}{m-1}(-1)^1 - \binom{m}{m}(-1)^0) + m(z-\xi_t) - \frac{2(m-1)\xi_t^2}{z-\xi_t} + z + 2\xi_t \\
&= \frac{2\xi_t^2}{z-\xi_t}(m-1) + m(z-\xi_t) - \frac{2(m-1)\xi_t^2}{z-\xi_t} + z - \xi_t + 3\xi_t \\
&= 3\xi_t + (m+1)(z-\xi_t).
\end{aligned}$$

Finally, by a similar trick, the coefficient of $\Phi_t^{(m+2)}(\xi_t)$ is

$$\begin{aligned}
&2\xi_t^2 \sum_{p=0}^{m-2} \binom{m}{p} \frac{(-1)^{m-p}(m-p)!}{(z-\xi_t)^{m-p+1}} \frac{(z-\xi_t)^{m-p+1}}{(m-p+1)!} \\
&\quad + \frac{m}{2}(1-\frac{2\xi_t^2}{(z-\xi_t)^2})(z-\xi_t)^2 + (z+2\xi_t+\frac{2\xi_t^2}{z-\xi_t})(z-\xi_t) \\
&= 2\xi_t^2 \sum_{p=0}^{m-2} \frac{m!}{p!(m-p+1)!}(-1)^{m-p} - m\xi_t^2 + 2\xi_t^2 + O(z-\xi_t) \\
&= \xi_t^2 \left(\frac{-2}{m+1} \sum_{p=0}^{m-2} \binom{m+1}{p} (-1)^{m+1-p} - m + 2 \right) + O(z-\xi_t) \\
&= \xi_t^2 \left(\frac{-2}{m+1}(0 - \frac{(m+1)m}{2} + m + 1 - 1) - m + 2 \right) + O(z-\xi_t) \\
&= \frac{2\xi_t^2}{m+1} + O(z-\xi_t),
\end{aligned}$$

and so taking the limit $z \to \xi_t$, we have

$$q_t^{(m)}(\xi_t) = m\Phi_t^{(m)}(\xi_t) + 3\xi_t\Phi_t^{(m+1)}(\xi_t) + \frac{2\xi_t^2}{m+1}\Phi_t^{(m+2)}(\xi_t). \qquad \square$$

Corollary 3.14. *When $\xi_t \neq e^{i\phi_t^j}$, we have $q_t(e^{i\phi_t^j}) = \Phi_t'(e^{i\phi_t^j})e^{i\phi_t^j}\frac{e^{i\phi_t^j}+\xi_t}{e^{i\phi_t^j}-\xi_t} = 0$, and so for* *all t,*

$$q_t(e^{i\phi_t^j}) = 1\{\xi_t = e^{i\phi_t^j}\}2e^{2i\phi_t^j}\Phi_t''(e^{i\phi_t^j}).$$

Also, by a near-identical argument, if $\bar{q}_t(z) = \bar{\Phi}_t'(z)z\sum_{j=1}^k \bar{p}_t^j \frac{z+e^{i\bar{\phi}_t^j}}{z-e^{i\bar{\phi}_t^j}}$, then

$$\bar{q}_t(e^{i\bar{\phi}_t^j}) = 2\bar{p}_t^j e^{2i\bar{\phi}_t^j}\bar{\Phi}_t''(e^{i\bar{\phi}_t^j}).$$

Remark. Applying Lemma 3.13 with $m = 1$ gives $q_t'(e^{i\phi_t^j}) = 3e^{i\phi_t^j}\Phi_t''(e^{i\phi_t^j}) + e^{2i\phi_t^j}\Phi_t^{(3)}(e^{i\phi_t^j})$. But since $\Phi_t'(e^{i\phi_t^j}) = 0$ for all time, this tells us that $\Phi_t^{(3)}(e^{i\phi_t^j}) = -3e^{-i\phi_t^j}\Phi_t''(e^{i\phi_t^j})$. Repeatedly taking the derivative of this relationship with respect to time would let us establish expressions for every odd power of Φ_t in terms of lower powers.

Remark. Note that if $t \in (n\mathbf{c}, (n+1)\mathbf{c})$, the conditional expectation of $q_t(e^{i\phi_t^j})$ given $(\theta_1, \ldots, \theta_n)$ is $2p_{n\mathbf{c}}^j e^{2i\phi_t^j}\Phi_t''(e^{i\phi_t^j})$, which is of a similar form to the expression for $\bar{q}_t(e^{i\bar{\phi}_t^j})$ above. We will rely on this in the martingale methods used in the proof of Theorem 3.16.

Proposition 3.15. *There exists a constant $A = A(k, K_0, T, \eta)$ such that, almost surely, for $0 \le t, s \le T$, $|e^{i\phi_t^j} - e^{i\phi_s^j}| \le A|t-s|$ and $|p_t^j - p_s^j| \le A|t-s|$, for all $j = 1, \ldots, k$ in the multinomial model. For the Laplacian path model, for all j and for $0 \le t, s \le T$, we also have $|e^{i\bar{\phi}_t^j} - e^{i\bar{\phi}_s^j}| \le A|t-s|$ and $|\bar{p}_t^j - \bar{p}_s^j| \le A|t-s|$.*

Proof. Equation (2.7) of [4] gives a useful expression for the evolution of $e^{i\phi_t^j}$ over time:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{i\phi_t^j} = \begin{cases} 0 & \text{if } \xi_t = e^{i\phi_t^j}, \\ -e^{i\phi_t^j}\frac{e^{i\phi_t^j}+e^{i\phi_t^l}}{e^{i\phi_t^j}-e^{i\phi_t^l}} & \text{if } \xi_t = e^{i\phi_t^l} \text{ for } l \neq j \end{cases}, \tag{3.37}$$

and so as the denominator is bounded by $L = L(k, K_0, T, \eta)$,

$$|e^{i\phi_t^j} - e^{i\phi_s^j}| \le \frac{2}{L}|t-s|.$$

Next, by the chain rule and by definition of $q_t(z)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t''(e^{i\phi_t^j}) = q_t''(e^{i\phi_t^j}) + \Phi_t^{(3)}(e^{i\phi_t^j})\frac{\mathrm{d}}{\mathrm{d}t}e^{i\phi_t^j}.$$

When $\xi_t \neq e^{i\phi_t^j}$, using (3.37) we can calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t''(e^{i\phi_t^j}) = 2\left(1 - \frac{2\xi_t^2}{(e^{i\phi_t^j} - \xi_t)^2}\right)\Phi_t''(e^{i\phi_t^j}),$$

so $\frac{\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t''(e^{i\phi_t^j})}{\Phi_t''(e^{i\phi_t^j})} = \exp(O(1))$. When $\xi_t = e^{i\phi_t^j}$, we will compute $(\Phi_t \circ f_{\phi_t^j,c})''(e^{i\phi_t^j})$ for $0 < c < \mathbf{c}$. Since $f'_{\phi_t^j,c}(e^{i\phi_t^j}) = 0$, the chain rule gives

$$(\Phi_t \circ f_{\phi_t^j,c})''(e^{i\phi_t^j}) = f''_{\phi_t^j,c}(e^{i\phi_t^j})\Phi_t'(f_{\phi_t^j,c}(e^{i\phi_t^j})). \tag{3.38}$$

By differentiating the expression for f' in (3.12), we find that

$$f''_{\phi_t^j,c}(e^{i\phi_t^j}) = e^{-i\phi_t^j}\frac{1+d(c)}{2\sqrt{1-e^{-c}}}. \tag{3.39}$$

As $\Phi_t'(e^{i\phi_t^j}) = 0$ and $f_{\phi_t^j,c}(e^{i\phi_t^j}) - e^{i\phi_t^j} = e^{i\phi_t^j}d(c)$, we also have

$$\Phi_t'(f_{\phi_t^j,c}(e^{i\phi_t^j})) = e^{i\phi_t^j}d(c)\Phi_t''(e^{i\phi_t^j}) + \frac{e^{2i\phi_t^j}d(c)^2}{2}\Phi_t^{(3)}(e^{i\phi_t^j}) + O(d(c)^3).$$

Since $\Phi_t^{(3)}(e^{i\phi_t^j}) = -3e^{-i\phi_t^j}\Phi_t''(e^{i\phi_t^j})$, and $d(c) = O(c^{1/2})$, we have

$$\begin{aligned}\Phi_t'(f_{\phi_t^j,c}(e^{i\phi_t^j})) &= e^{i\phi_t^j}d(c)(1 - \frac{3d(c)}{2} + O(c))\Phi_t''(e^{i\phi_t^j}) \\ &= e^{i\phi_t^j}d(c)(1 - 3c^{1/2} + O(c))\Phi_t''(e^{i\phi_t^j}).\end{aligned} \tag{3.40}$$

Hence, substituting (3.39) and (3.40) back into (3.38), we obtain

$$(\Phi_t \circ f_{\phi_t^j,c})''(e^{i\phi_t^j}) = \frac{d(c)(1+d(c))}{2\sqrt{1-e^{-c}}}(1 - 3c^{1/2} + O(c))\Phi_t''(e^{i\phi_t^j}).$$

More computations give us $\frac{d(c)(1+d(c))}{2\sqrt{1-e^{-c}}} = 1 + 3c^{1/2} + O(c)$, and so

$$(\Phi_t \circ f_{\phi_t^j,c})''(e^{i\phi_t^j}) = (1 + O(c))\Phi_t''(e^{i\phi_t^j}).$$

Hence while attaching a slit at j, $\frac{\mathrm{d}}{\mathrm{d}t}\log\Phi_t''(e^{i\phi_t^j}) = \exp(O(1))$. Together with the previous case, this shows $p_t^j = \frac{|\Phi_t''(e^{i\phi_t^j})|^{-\eta}}{\sum_l |\Phi_t''(e^{i\phi_t^l})|^{-\eta}}$ is Lipschitz in t. Similar arguments apply to the Laplacian path model. □

Now we will be able to show that the quantities governing the growth of the multinomial model, ϕ_t^j and p_t^j, are with high probability close to their counterparts in the Laplacian path model, $\bar{\phi}_t^j$ and $\bar{p}_t^j$.

Theorem 3.16. *There exists a universal constant R and a constant $A = A(T, L, k, K_0, \eta)$ such that*

$$\mathbb{P}\left[\sup_{t\le T}\sum_{j=1}^{k}\left(|e^{i\phi_t^j} - e^{i\bar{\phi}_t^j}| + |p_t^j - \bar{p}_t^j|\right) \le A\mathbf{c}^{1/2R}\right] \to 1$$

as $\mathbf{c} \to 0$.

Corollary 3.17. *If $\mu_t = \delta_{\theta_{\lfloor t/\mathbf{c}\rfloor+1}}$ for the multinomial model and $\bar{\mu}_t = \sum_{j=1}^k \bar{p}_t^j \delta_{\bar{\phi}_t^j}$ for the Laplacian path model, then*

$$d_{\mathrm{BW}}(\mu_t \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}) \to 0$$

in probability as $\mathbf{c} \to 0$. Hence $\mu_t \otimes m_{[0,T]}$ converges in distribution, as a random element of the space of measures on $S = \mathbb{T} \times [0,T]$, to $\bar{\mu}_t \otimes m_{[0,T]}$.

Proof of Theorem 3.16. Throughout the proof, A represents a constant which may change from line to line, but all occurrences have a common upper bound depending only on T, L, k, K_0 and η.

Let $x_j(t) = e^{i\phi_t^j}$ and $\bar{x}_j(t) = e^{i\bar{\phi}_t^j}$. Write $\delta_{\mathrm{x}}(t) = \sup_{s\le t}\sum_{j=1}^k |x_j(s) - \bar{x}_j(s)|$ and $\delta_{\mathrm{p}}(t) = \sup_{s\le t}\sum_{j=1}^k \left|\frac{1}{\Phi_s''(x_j(s))} - \frac{1}{\bar{\Phi}_s''(\bar{x}_j(s))}\right|$. Since $p_s^j = |\Phi_s''(x_j(s))|^{-\eta}/\sum_{l=1}^k |\Phi_s''(x_l(s))|^{-\eta}$ and $\bar{p}_s^j$ is similarly expressed in terms of $(|\Phi_s''(\bar{x}_l(s))|^{-\eta})_{l=1}^k$, and all these second derivatives stay away from 0 and ∞, to obtain the bound on $\sup_{t\le T}|p_t^j - \bar{p}_t^j|$ from the theorem statement it will suffice to bound $\delta_{\mathrm{p}}(T)$. Hence we aim to find an inequality of the form $\delta_{\mathrm{x}}(t) + \delta_{\mathrm{p}}(t) \le \alpha(s) + \int_0^t \beta(s)(\delta_{\mathrm{x}}(s) + \delta_{\mathrm{p}}(s))\,\mathrm{d}s$ for all $t \le T$ with high probability, for some suitable functions α and β, which will give us the claimed upper bound on $\delta_{\mathrm{x}}(T) + \delta_{\mathrm{p}}(T)$.

Equation (2.6) in [4] shows that the movement of the preimages $\bar{\phi}_t^j$ in the Laplacian path model is determined by

$$\frac{\partial}{\partial t}\bar{x}_j(t) = -\sum_{l\neq j}\bar{p}_t^j \bar{x}_j(t)\frac{\bar{x}_j(t) + \bar{x}_l(t)}{\bar{x}_j(t) - \bar{x}_l(t)} \tag{3.41}$$

and similarly for the multinomial model,

$$\frac{\partial}{\partial t}x_j(t) = \begin{cases} 0 & \text{if } \xi_t = x_j(t), \\ -x_j(t)\frac{x_j(t)+x_l(t)}{x_j(t)-x_l(t)} & \text{if } \xi_t = x_l(t) \text{ for } l \neq j \end{cases}. \tag{3.42}$$

Define

$$\lambda_t^{j,l} = \begin{cases} 0 & \text{if } j = l \\ -x_j(t)\frac{x_j(t)+x_l(t)}{x_j(t)-x_l(t)} & \text{otherwise} \end{cases},$$

and likewise

$$\bar{\lambda}_t^{j,l} = \begin{cases} 0 & \text{if } j = l \\ -\bar{x}_j(t)\frac{\bar{x}_j(t)+\bar{x}_l(t)}{\bar{x}_j(t)-\bar{x}_l(t)} & \text{otherwise} \end{cases},$$

so for $t \in [0, T]$ we can write, integrating (3.41) and (3.42) with respect to time,

$$\bar{x}_j(t) = \bar{x}_j(0) + \int_0^t \sum_{l\neq j}\bar{\lambda}_s^{j,l}\bar{p}_s^l\,\mathrm{d}s, \tag{3.43}$$

$$x_j(t) = x_j(0) + \sum_{n=0}^{\lfloor t/\mathbf{c}\rfloor}\int_{n\mathbf{c}}^{(n+1)\mathbf{c}\wedge t}\sum_{l\neq j}\lambda_s^{j,l}I_s^l\,\mathrm{d}s, \tag{3.44}$$

where $I_s^l := 1\{\xi_s = e^{i\phi_s^j}\} = 1\{\theta_{\lfloor s/\mathbf{c}\rfloor+1} = \phi^l_{\lfloor s/\mathbf{c}\rfloor\mathbf{c}}\} = I_{n\mathbf{c}}^l$ in the line above. Write $X_{n+1}^j = \sum_{l\neq j}\lambda_{n\mathbf{c}}^{j,l}(I_{n\mathbf{c}}^l - p_{n\mathbf{c}}^l)$, and note that X_{n+1}^j is a bounded martingale increment,

$\mathbb{E}[X^j_{n+1}|\theta_1,\dots,\theta_n]=0$. Then using Lemma 3.15, we have

$$\left|\sum_{l\neq j}\lambda^{j,l}_s I^l_s-\left(\sum_{l\neq j}\lambda^{j,l}_s p^l_s+X^j_{n+1}\right)\right|\le A(\mathbf{c}\wedge s),$$

for any $s\le T$, and so for $t\le T$, we can expand (3.44):

$$x_j(t)=x_j(0)+\int_0^t\sum_{l\neq j}p^l_s\lambda^{j,l}_s\,\mathrm{d}s+\mathbf{c}\sum_{n=1}^{\lfloor t/\mathbf{c}\rfloor}X^j_n+(t-\left\lfloor\frac{t}{\mathbf{c}}\right\rfloor\mathbf{c})X^j_{\lfloor t/\mathbf{c}\rfloor+1}+E^j_t \tag{3.45}$$

where $|E^j_t|\le A(\mathbf{c}t\wedge t^2)$. Let $M^j_t=\mathbf{c}\sum_{n=1}^{\lfloor t/\mathbf{c}\rfloor}X^j_n+(t-\lfloor t/\mathbf{c}\rfloor\mathbf{c})X^j_{\lfloor t/\mathbf{c}\rfloor+1}$, and note that now (3.45) is the sum of something of the same form as the right-hand side of (3.43) with a martingale-type term M^j_t and a small error E^j_t. Since $\bar{x}_j(0)=x_j(0)$, we can use (3.43) and (3.44) to obtain

$$\begin{aligned}|x_j(t)-\bar{x}_j(t)|&\le\int_0^t\sum_{l\neq j}|p^l_s\lambda^{j,l}_s-\bar{p}^l_s\bar{\lambda}^{j,l}_s|\,\mathrm{d}s+|M^j_t|+|E^j_t|\\&\le\int_0^t\sum_{l\neq j}|\lambda^{j,l}_s-\bar{\lambda}^{j,l}_s|\,\mathrm{d}s+A\int_0^t\sum_{l\neq j}|p^l_s-\bar{p}^l_s|\,\mathrm{d}s+|M^j_t|+|E^j_t|.\end{aligned}$$

As the denominators in the above definitions of $\lambda^{j,l}_t$ and $\bar{\lambda}^{j,l}_t$ are bounded below, it is a simple calculation to linearise $|\lambda^{j,l}_s-\bar{\lambda}^{j,l}_s|\le A\left(|x_j(t)-\bar{x}_j(t)|+|x_l(t)-\bar{x}_l(t)|\right)$, and so

$$\begin{aligned}|x_j(t)-\bar{x}_j(t)|\le A\int_0^t&\left((k-1)|x_j(s)-\bar{x}_j(s)|+\sum_{l\neq j}|x_l(s)-\bar{x}_l(s)|+\sum_{l\neq j}|p^l_s-\bar{p}^l_s|\right)\mathrm{d}s\\&+|M^j_t|+|E^j_t|.\end{aligned} \tag{3.46}$$

Recall $\delta_{\mathrm{x}}(t)=\sup_{s\le t}\sum_{j=1}^k|x_j(s)-\bar{x}_j(s)|$. Let $M_t=\sum_{j=1}^k|M^j_t|$ and $E_t=\sum_{j=1}^k|E^j_t|$, so by (3.46),

$$\delta_{\mathrm{x}}(t)\le\int_0^t A\delta_{\mathrm{x}}(s)\,\mathrm{d}s+A\int_0^t\sum_{j=1}^k|p^j_s-\bar{p}^j_s|\,\mathrm{d}s+M_t+E_t \tag{3.47}$$

for $t\in[0,T]$.

Note that $(M^j_{n\mathbf{c}})_{n\ge0}$ is a martingale, and so $(M_{n\mathbf{c}})_{n\ge0}$ is a submartingale. For every $t\le T$, $|M^j_t-M^j_{\lfloor t/\mathbf{c}\rfloor\mathbf{c}}|\le A\mathbf{c}$, and so

$$\sup_{t\le T}M_t\le\max_{n\le T/\mathbf{c}}M_{n\mathbf{c}}+A\mathbf{c},$$

and by Doob's submartingale inequality

$$\mathbb{P}\left[\max_{n\le T/\mathbf{c}}M_{n\mathbf{c}}\ge\mathbf{c}^{1/4}\right]\le\frac{\mathbb{E}M^2_{\lfloor T/\mathbf{c}\rfloor}}{\mathbf{c}^{1/2}}\le A\mathbf{c}^{1/2}.$$

So on an event of probability at least $1 - A\mathbf{c}^{1/2}$, which we will call E_{x}, for all $t \leq T$ we have $M_t \leq 2\mathbf{c}^{1/4}$. On this event,

$$\delta_{\mathrm{x}}(t) \leq \int_0^t A\delta_{\mathrm{x}}(s)\,\mathrm{d}s + A\int_0^t \sum_{j=1}^k |p_s^j - \bar{p}_s^j|\,\mathrm{d}s + 2\mathbf{c}^{1/4} + A\mathbf{c}. \tag{3.48}$$

We will establish a similar integral bound for $\sum_{j=1}^k |p_t^j - \bar{p}_t^j|$, and then use a Grönwall-type argument to bound $\delta_x(t) + \sum_{j=1}^k |p_t^j - \bar{p}_t^j|$.

To establish a bound on $|p_t^j - \bar{p}_t^j|$ the argument is similar to the previous, but substantially more complicated. Following ideas from Section 4.2 of [4], we will be able to establish precise expressions for $1/\Phi_t''(e^{i\phi_t^j})$ and $1/\bar{\Phi}_t''(e^{i\bar{\phi}_t^j})$. Recall

$$\delta_{\mathrm{p}}(t) = \sup_{s\leq t}\sum_{j=1}^k \left|\frac{1}{\Phi_s''(x_j(s))} - \frac{1}{\bar{\Phi}_s''(\bar{x}_j(s))}\right|.$$

Fix $t \in [0,T]$, and for $s \leq t$ define the transition map $h_s := \Phi_s^{-1}\circ\Phi_t$, and similarly define $\bar{h}_s = \bar{\Phi}_s^{-1}\circ\bar{\Phi}_t$. As t is fixed, $\dot{h}_s$ will now denote $\frac{\mathrm{d}}{\mathrm{d}s}h_s$, *etc.*. Note that h_s evolves from $h_0 = \Phi_0^{-1}\circ\Phi_t$ (which is *not* Φ_t since the initial condition is non-trivial) to $h_t = \mathrm{Id}_\Delta$, the identity map. Also note that h_s satisfies the inverse Loewner equation

$$\frac{\partial}{\partial s}h_s(z) = -h_s(z)\sum_l I_s^l \frac{h_s(z) + x_l(s)}{h_s(z) - x_l(s)}, \tag{3.49}$$

and similarly

$$\frac{\partial}{\partial s}\bar{h}_s(z) = -\bar{h}_s(z)\sum_l \bar{p}_s^l \frac{\bar{h}_s(z) + \bar{x}_l(s)}{\bar{h}_s(z) - \bar{x}_l(s)}. \tag{3.50}$$

We will first establish an expression involving $1/\Phi_t''(e^{i\phi_t^j})$, and a near-identical argument gives an equivalent expression (replacing I_s^l by $\bar{p}_s^l$) for $1/\bar{\Phi}_t''(e^{i\bar{\phi}_t^l})$.

Adopting the notation used in [4], write $w_j(s) = h_s(x_j(t))$, and where it is unambiguous we will write $x_j = x_j(s)$, $w_j = w_j(s)$, *etc.*. Then (3.49) becomes

$$\dot{w}_j = -\sum_l I_s^l w_j \frac{w_j + x_l}{w_j - x_l} = -\sum_l I_s^l\left(w_j + 2x_l + \frac{2x_l^2}{w_j - x_l}\right). \tag{3.51}$$

Note that $w_j(0) = \Phi_0^{-1}(\Phi_t(x_j(t)))$, which is approximately $x_j(t)(1 + 2\sqrt{p_t^j}\sqrt{t})$ for small t, and $w_j(t) = x_j(t)$.

Let $\kappa(s) = h_s''(x_j(t))$. Since $h_s'(x_j(t)) = 0$ for $s < t$, differentiating the right-hand side of (3.50) twice with respect to z and evaluating at $x_j(t)$ gives us

$$\frac{\dot{\kappa}(s)}{\kappa(s)} = -\sum_l I_s^l\left(1 - \frac{2x_l(s)^2}{(w_j(s) - x_l(s))^2}\right).$$

Since $w_j(t) = x_j(t)$, the $l = j$ term in the above sum is singular as $s \to t$. So before we use it to determine $\log \kappa$, we would like to subtract off this singularity. Subtracting (3.42) from (3.51) gives

$$\begin{aligned}\dot{w}_j - \dot{x}_j &= -\sum_l I_s^l \left(w_j + 2x_l + \frac{2x_l^2}{w_j - x_l} \right) + \sum_{l \neq j} I_s^l x_j \frac{x_j + x_l}{x_j - x_l} \\ &= -I_s^j \left(w_j + 2x_j + \frac{2x_j^2}{w_j - x_j} \right) + \sum_{l \neq j} I_s^l \left(\frac{2x_l^2}{(x_j - x_l)(w_j - x_l)} - 1 \right) (w_j - x_j),\end{aligned} \tag{3.52}$$

and so

$$\frac{\dot{w}_j - \dot{x}_j}{w_j - x_j} = -I_s^j \left(\frac{w_j + 2x_j}{w_j - x_j} + \frac{2x_j^2}{(w_j - x_j)^2} \right) + \sum_{l \neq j} I_s^l \left(\frac{2x_l^2}{(x_j - x_l)(w_j - x_l)} - 1 \right).$$

Hence

$$\frac{\dot{\kappa}}{\kappa} + \frac{\dot{w}_j - \dot{x}_j}{w_j - x_j} = -I_s^j \left(2 + \frac{3x_j}{w_j - x_j} \right) + \sum_{l \neq j} I_s^l \left(\frac{2x_l^2}{(w_j - x_l)^2} + \frac{2x_l^2}{(w_j - x_l)(x_j - x_l)} - 2 \right). \tag{3.53}$$

We have reduced the order of the singularity by one, and now since $|w_j(s) - x_j(s)| \approx \sqrt{t - s}$ as $s \uparrow t$, the remaining singularity is integrable. Note that we can collect some of the terms as $-2I_s^j - 2\sum_{l \neq j} I_s^l = -2$. We will give names to the remaining terms:

$$Q_s^{j,l} = \frac{2x_l(s)^2}{(w_j(s) - x_l(s))^2} + \frac{2x_l(s)^2}{(w_j(s) - x_l(s))(x_j(s) - x_l(s))} \tag{3.54}$$

for $l \neq j$, and

$$Q_s^{j,j} = \frac{3x_j(s)}{w_j(s) - x_j(s)}. \tag{3.55}$$

Note that for $l \neq j$, $|Q_s^{j,l}| \leq A$, for a constant A proportional to L^{-2}. Integrating the equation (3.53) over $s \in [0, t)$, we obtain

$$\log \frac{\lim_{s \uparrow t} \kappa(s)(w_j(s) - x_j(s))}{\kappa(0)(w_j(0) - x_j(0))} = -2t + \sum_{l=1}^{k} \int_0^t I_s^l Q_s^{j,l} \, \mathrm{d}s. \tag{3.56}$$

We have an analogous expression

$$\log \frac{\lim_{s \uparrow t} \bar{\kappa}(s)(\bar{w}_j(s) - \bar{x}_j(s))}{\bar{\kappa}(0)(\bar{w}_j(0) - \bar{x}_j(0))} = -2t + \sum_{l=1}^{k} \int_0^t \bar{p}_s^l \bar{Q}_s^{j,l} \, \mathrm{d}s, \tag{3.57}$$

for $\bar{\kappa}$ and $\bar{Q}_s^{j,l}$ defined in the obvious way.

Next, we will carefully analyse the left-hand sides of (3.56) and (3.57), whose difference will be useful in bounding the difference of $1/\Phi_t''(x_j(t))$ and $1/\bar{\Phi}_t''(\bar{x}_j(t))$.

In the multinomial model, the left-hand side of (3.56) is only finite if the previous particle attached by time t was at slit j, otherwise $\kappa(t)(w_j(t) - x_j(t)) = 0$. Fix a large positive $R > 0$, which we will determine later, and pick a sequence of times $0 = T^j(0) < T^j(1) < T^j(2) < \cdots < T^j(N_j) \leq T$ such that each $T^j(n)$ is an integer multiple of $\mathbf{c}$, and for all $n \geq 1$, $\mathbf{c}^{1/R} \leq T^j(n) - T^j(n-1) \leq \mathbf{c}^{1/2R}$ and $\xi_t = x_j(t)$ for all $t \in (T^j(n) - \mathbf{c}, T^j(n))$. It will suffice to have an estimate like (3.48) only for each time $t := \{T^j(1), \dots, T^j(N_j)\}$, as we will show later. Note also that $N_j \leq T\mathbf{c}^{-1/R}$. First we need to show that with high probability we can choose such a sequence.

Note that there exists $p > 0$ depending on T, η, L and K_0 such that $\min_j \inf_{s\in[0,T]} p_s^j \geq p > 0$, and so we can couple the multinomial model with a sequence of $\lfloor T/\mathbf{c} \rfloor$ independent trials of success probability p, so that on a success a particle is attached at slit j, and if X is the longest run of consecutive failures, $\max_{n\geq 1}(T^j(n) - T^j(n-1)) \leq \mathbf{c}X$. Let F be the number of runs of consecutive failures of length $\lfloor \mathbf{c}^{-(1-1/R)} \rfloor$, i.e. $F = \sum_{n=1}^{\lfloor T\mathbf{c}^{-1} - \mathbf{c}^{-(1-1/R)} \rfloor} 1\{\text{trials } n, n+1, ..., n + \lfloor \mathbf{c}^{-(1-1/R)} \rfloor \text{ all fail}\}$. Then by Markov's inequality,

$$\mathbb{P}[X \geq \mathbf{c}^{-(1-1/R)}] = \mathbb{P}[F \geq 1] \leq \mathbb{E}F \leq \frac{T}{\mathbf{c}}(1-p)^{\lfloor \mathbf{c}^{-(1-1/R)} \rfloor}.$$

So the event

$$E_{\mathrm{T}} := \{\text{for all } t \in [\mathbf{c}^{1/2R}, T] \text{ and all } j,\ \xi_s = x_j(s) \text{ for some } s \in (t - \mathbf{c}^{1/2R}, t]\} \tag{3.58}$$

has probability at least $1 - \frac{kT}{\mathbf{c}}(1-p)^{\lfloor \mathbf{c}^{-(1-1/R)} \rfloor}$, which is very close to 1.

So if we take (3.56) with $t \in \{T^j(1), \dots, T^j(N_j)\}$ for a given j, we have

$$\log \frac{\lim_{s\uparrow t} \kappa(s)(w_j(s) - x_j(s))}{\kappa(0)(w_j(0) - x_j(0))} = -2t + \sum_{l=1}^{k} \int_0^t I_s^l Q_s^{j,l}\, \mathrm{d}s. \tag{3.59}$$

If $t - \mathbf{c} < s < t$, then h_s is exactly the slit map $f_{x_j(t), t-s}$, and so differentiating (3.12) to obtain an expression for $\kappa(s) = \frac{f''_{t-s}(1)}{x_j(t)}$, we find

$$\begin{aligned}
\kappa(s)(w_j(s) - x_j(s)) &= \frac{f''_{t-s}(1)}{x_j(t)}(x_j(t) f_{t-s}(1) - x_j(t)) \\
&= \frac{1 + d(t-s)}{2\sqrt{1 - e^{-(t-s)}}} d(t-s) \\
&\sim \frac{1}{2\sqrt{t-s}} 2\sqrt{t-s} = 1
\end{aligned}$$

as $s \uparrow t$, and as $\bar{\Phi}_s^{-1} \circ \bar{\Phi}_t$ is locally a slit map of capacity $\bar{p}_t^j(t-s)$ for s close to t, we also have

$$\bar{\kappa}(s)(\bar{w}_j(s) - \bar{x}_j(s)) \sim \frac{1}{2\sqrt{\bar{p}_t^j(t-s)}} 2\sqrt{\bar{p}_t^j(t-s)} = 1$$

as $s \uparrow t$. So the numerators of both (3.56) and (3.57) are equal to 1.

Next we use the difference of the denominators to bound the difference of $1/\Phi''_t(x_j(t))$ and $1/\bar{\Phi}''_t(\bar{x}_j(t))$.

The remaining calculations in this proof repeatedly make use of expansions of the form $ab - \bar{a}\bar{b} = (a - \bar{a})b + \bar{a}(b - \bar{b}) = a(b - \bar{b}) + (a - \bar{a})\bar{b}$ to linearise the difference of expressions which are the product of two (or more) terms. It will be much easier to follow the computations if the reader keeps this trick in mind.

We have $1/\kappa(0) = \frac{\Phi_0'(w_j(0))}{\Phi_t''(x_j(t))}$, and so

$$\frac{1}{\Phi_t''(x_j(t))} = \frac{w_j(0) - x_j(0)}{\Phi_0'(w_j(0))} \frac{1}{\kappa(0)(w_j(0) - x_j(0))}.$$

Then expanding $\frac{1}{\Phi_t''(x_j(t))} - \frac{1}{\bar{\Phi}_t''(\bar{x}_j(t))}$ into two terms using the above expression and our linearisation trick, then applying the triangle inequality, we have

$$\begin{aligned}\left|\frac{1}{\Phi_t''(x_j(t))} - \frac{1}{\bar{\Phi}_t''(\bar{x}_j(t))}\right| \leq & \left|\frac{w_j(0) - x_j(0)}{\Phi_0'(w_j(0))} - \frac{\bar{w}_j(0) - \bar{x}_j(0)}{\Phi_0'(\bar{w}_j(0))}\right| \frac{1}{|\kappa(0)(w_j(0) - x_j(0))|} \\ & + \left|\frac{\bar{w}_j(0) - \bar{x}_j(0)}{\Phi_0'(\bar{w}_j(0))}\right| \left|\frac{1}{\kappa(0)(w_j(0) - x_j(0))} - \frac{1}{\bar{\kappa}(0)(\bar{w}_j(0) - \bar{x}_j(0))}\right|.\end{aligned} \tag{3.60}$$

We claim that the coefficient $\left|\frac{\bar{w}_j(0) - \bar{x}_j(0)}{\Phi_0'(\bar{w}_j(0))}\right|$ is bounded above and below by constants. Both $\Phi_0'(\bar{w}_j(0)) \to 0$ and $\bar{w}_j(0) - \bar{x}_j(0) \to 0$ as $t \to 0$. More specifically, as $|\bar{w}_j(0) - \bar{x}_j(t)| = |(\Phi_0^{-1} \circ \bar{\Phi}_t)(\bar{x}_j(t)) - \bar{x}_j(t)|$ is proportional to $\sqrt{\bar{p}_t^j t}$ when t is small, and Φ_0 is locally a slit map, we can use (3.12) to estimate

$$A^{-1}\sqrt{\bar{p}_t^j t} \leq |\Phi_0'(\bar{w}_j(0))| \leq A\sqrt{\bar{p}_t^j t}. \tag{3.61}$$

For the numerator, $|\bar{w}_j(0) - \bar{x}_j(0)| \sim d(\bar{p}_t^j t) \sim 2\sqrt{\bar{p}_t^j t}$ as $t \to 0$, so $A^{-1} \leq \left|\frac{\bar{w}_j(0) - \bar{x}_j(0)}{\Phi_0'(\bar{w}_j(0))}\right| \leq A$. A similar argument gives us $A^{-1} \leq \frac{1}{|\kappa(0)(w_j(0) - x_j(0))|} \leq A$.

For the first increment in (3.60), the bounds above on $\left|\frac{\bar{w}_j(0) - \bar{x}_j(0)}{\Phi_0'(\bar{w}_j(0))}\right|$ and the analogous bounds for the multinomial model allow us to write

$$\left|\frac{w_j(0) - x_j(0)}{\Phi_0'(w_j(0))} - \frac{\bar{w}_j(0) - \bar{x}_j(0)}{\Phi_0'(\bar{w}_j(0))}\right| \leq A\left|\frac{\Phi_0'(w_j(0))}{w_j(0) - x_j(0)} - \frac{\Phi_0'(\bar{w}_j(0))}{\bar{w}_j(0) - \bar{x}_j(0)}\right|. \tag{3.62}$$

Since $\Phi_0'(x_j(0)) = 0$, we can use the fundamental theorem of calculus to write

$$\Phi_0'(w_j(0)) = (w_j(0) - x_j(0)) \int_0^1 \Phi_0''(\alpha w_j(0) + (1 - \alpha)x_j(0))\,\mathrm{d}\alpha,$$

and similarly for $\Phi_0'(\bar{w}_j(0))$. Then, using the fact that $x_j(0) = \bar{x}_j(0)$ and the bound $|\Phi_0^{(3)}(z)| \leq A$ for z in the convex hull of $\{w_j(0), \bar{w}_j(0), x_j(0)\}$, we can bound the right-hand side of (3.62) by

$$A\int_0^1 \left|\Phi_0''(pw_j(0) + (1 - p)x_j(0)) - \Phi_0''(p\bar{w}_j(0) + (1 - p)\bar{x}_j(0))\right| \mathrm{d}p \leq A|w_j(0) - \bar{w}_j(0)|.$$

Applying the fundamental theorem of calculus again with the bound (3.61) and its analogue for the multinomial model, we have

$$\begin{aligned}|w_j(0) - \bar{w}_j(0)| &= |\Phi_0^{-1}(\Phi_t(x_j(t))) - \Phi_0^{-1}(\bar{\Phi}_t(\bar{x}_j(t)))| \\ &\le A\frac{|\Phi_t(x_j(t)) - \bar{\Phi}_t(\bar{x}_j(t))|}{\sqrt{t}}.\end{aligned} \tag{3.63}$$

Then we can write, using Lemma 3.13 and writing $I_s^j = p_s^j + (I_s^j - p_s^j)$,

$$\begin{aligned}|\Phi_t(x_j(t)) - \bar{\Phi}_t(\bar{x}_j(t))| &= \left|\int_0^t (q_s(x_j(s)) - \bar{q}_s(\bar{x}_j(s)))\,\mathrm{d}s\right| \\ &= \left|2\int_0^t (I_s^j x_j(s)^2\Phi_s''(x_j(s)) - \bar{p}_s^j\bar{x}_j(s)^2\bar{\Phi}_s''(\bar{x}_j(s)))\,\mathrm{d}s\right| \\ &\le 2\int_0^t |p_s^j x_j(s)^2\Phi_s''(x_j(s)) - \bar{p}_s^j\bar{x}_j(s)^2\bar{\Phi}_s''(\bar{x}_j(s))|\,\mathrm{d}s \\ &\quad + \left|2\int_0^t (I_s^j - p_s^j)x_j(s)^2\Phi_s''(x_j(s))\,\mathrm{d}s\right|.\end{aligned}$$

Our linearisation trick applied twice shows the first term is bounded by

$$\begin{aligned}&2\left|\Phi_s''(x_j(s)) - \bar{\Phi}_s''(\bar{x}_j(s))\right| + A\left(|p_s^j - \bar{p}_s^j| + |x_j(s) - \bar{x}_j(s)|\right) \\ &\le A\left(\left|\frac{1}{\Phi_s''(x_j(s))} - \frac{1}{\bar{\Phi}_s''(\bar{x}_j(s))}\right| + |p_s^j - \bar{p}_s^j| + |x_j(s) - \bar{x}_j(s)|\right),\end{aligned}$$

and further routine calculations show that

$$|p_s^j - \bar{p}_s^j| \le A\sum_l \left|\frac{1}{\Phi_s''(x_l(s))} - \frac{1}{\bar{\Phi}_s''(\bar{x}_l(s))}\right|,$$

so we have

$$|\Phi_t(x_j(t)) - \bar{\Phi}_t(\bar{x}_j(t))| \le A\int_0^t (\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s))\,\mathrm{d}s + 2\left|\int_0^t (I_s^j - p_s^j)x_j(s)^2\Phi_s''(x_j(s))\,\mathrm{d}s\right|.$$

The second term can be decomposed into a martingale and small error, as we did in (3.44),

$$2\int_0^t (I_s^j - p_s^j)x_j(s)^2\Phi_s''(x_j(s))\,\mathrm{d}s = \sum_{n=0}^{\lfloor t/\mathbf{c}\rfloor}\int_{n\mathbf{c}}^{(n+1)\mathbf{c}\wedge t} X_{n+1}^j\,\mathrm{d}s + E_t^j,$$

where using Lemma 3.15, $|E_t^j| \le A(\mathbf{c}t \wedge t^2)$, and $X_{n+1}^j := 2(I_{n\mathbf{c}}^j - p_{n\mathbf{c}}^j)x_j(n\mathbf{c})^2\Phi_{n\mathbf{c}}''(x_j(n\mathbf{c}))$ is a bounded martingale increment with respect to $\mathcal{F}_n = \sigma(\theta_1, \ldots, \theta_n)$. Again define

$$M_t^j = \mathbf{c}\sum_{n=1}^{\lfloor t/\mathbf{c}\rfloor} X_n^j + (t - \left\lfloor\frac{t}{\mathbf{c}}\right\rfloor\mathbf{c})X_{\lfloor t/\mathbf{c}\rfloor+1}^j. \tag{3.64}$$

For a given t, define the event

$$E^t_{\mathrm{p},1} = \left\{ \sum_{j=1}^k |M^j_t| \le \mathbf{c}^{\frac{1}{2}-\frac{2}{R}} \sqrt{t} \right\},$$

where R is the large positive constant we defined earlier and will determine the value of later. Then by Markov's inequality

$$\mathbb{P}(E^t_{\mathrm{p},1}) \ge 1 - A\mathbf{c}^{4/R}. \tag{3.65}$$

So overall, on the event $E^t_{\mathrm{p},1}$, the first term in (3.60) is bounded by

$$\frac{A}{\sqrt{t}} \int_0^t (\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s))\,\mathrm{d}s + \mathbf{c}^{\frac{1}{2}-\frac{2}{R}} + A\mathbf{c}\sqrt{t}, \tag{3.66}$$

and as $t \le T$ the final term is bounded by a multiple of the second, and so on the event $E^t_{\mathrm{p},1}$, (3.60) becomes

$$\begin{aligned} \left| \frac{1}{\Phi''_t(x_j(t))} - \frac{1}{\bar{\Phi}''_t(\bar{x}_j(t))} \right| &\le \frac{A}{\sqrt{t}} \int_0^t (\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s))\,\mathrm{d}s + A\mathbf{c}^{\frac{1}{2}-\frac{2}{R}} \\ &\quad + A \left| \frac{1}{\kappa(0)(w_j(0) - x_j(0))} - \frac{1}{\bar{\kappa}(0)(\bar{w}_j(0) - \bar{x}_j(0))} \right|. \end{aligned} \tag{3.67}$$

For the second line in (3.67), note that as $A^{-1} \le \left| \frac{1}{\kappa(0)(w_j(0)-x_j(0))} \right| \le A$, and similarly for the LPM version of the same term, we have

$$\begin{aligned} &\left| \frac{1}{\kappa(0)(w_j(0) - x_j(0))} - \frac{1}{\bar{\kappa}(0)(\bar{w}_j(0) - \bar{x}_j(0))} \right| \\ &\le A \left| \log \frac{1}{\kappa(0)(w_j(0) - x_j(0))} - \log \frac{1}{\bar{\kappa}(0)(\bar{w}_j(0) - \bar{x}_j(0))} \right| \\ &= A \left| \sum_{l=1}^k \int_0^t (I^l_s Q^{j,l}_s - \bar{p}^l_s \bar{Q}^{j,l}_s)\,\mathrm{d}s \right|. \end{aligned} \tag{3.68}$$

We decompose the integrands into three terms using our linearisation trick,

$$I^l_s Q^{j,l}_s - \bar{p}^l_s \bar{Q}^{j,l}_s = (I^l_s - p^l_s) Q^{j,l}_s + (p^l_s - \bar{p}^l_s) Q^{j,l}_s + \bar{p}^l_s (Q^{j,l}_s - \bar{Q}^{j,l}_s),$$

and so (3.68) is bounded by a constant multiple of

$$\sum_{l=1}^k \left(\left| \int_0^t (I^l_s - p^l_s) Q^{j,l}_s\,\mathrm{d}s \right| + \int_0^t |p^l_s - \bar{p}^l_s||Q^{j,l}_s|\,\mathrm{d}s + \int_0^t |Q^{j,l}_s - \bar{Q}^{j,l}_s|\,\mathrm{d}s \right). \tag{3.69}$$

For $l \ne j$, by the linearisation trick applied to (3.54) (and linearising the denominators as they all stay away from 0 since $l \ne j$),

$$|Q^{j,l}_s - \bar{Q}^{j,l}_s| \le A \left(|(w_j(s) - x_j(s)) - (\bar{w}_j(s) - \bar{x}_j(s))| + |x_j(s) - \bar{x}_j(s)| + |x_l(s) - \bar{x}_l(s)| \right),$$

$|Q_s^{j,l}| \leq A$, and $|Q_s^{j,j}| = \frac{3}{|w_j(s) - x_j(s)|} \asymp \frac{1}{\sqrt{t-s}}$. The first term in (3.69) can be bounded, as the other martingale terms were, by something which has second moment less than $A\mathbf{c}\log\frac{t}{\mathbf{c}}$, and so if we define the event

$$E_{\mathrm{p},2} = \left\{ \sup_{t \leq T} \sum_{l=1}^{k} \left| \int_0^t (I_s^l - p_s^l) Q_s^{j,l} \, \mathrm{d}s \right| \leq 2\mathbf{c}^{\frac{1}{2} - \frac{2}{R}} \right\}, \tag{3.70}$$

then by Doob's martingale inequality, $\mathbb{P}(E_{\mathrm{p},2}) \geq 1 - A\mathbf{c}^{4/R}\log(T/\mathbf{c})$.

We can bound the second term in (3.69) by

$$A \int_0^t \frac{\delta_{\mathrm{p}}(s)}{\sqrt{t-s}} \, \mathrm{d}s. \tag{3.71}$$

Then to bound the final term, consider the $l = j$ case:

$$\begin{aligned} |Q_s^{j,j} - \bar{Q}_s^{j,j}| &\leq \frac{3|x_j(s) - \bar{x}_j(s)|}{|\bar{w}_j(s) - \bar{x}_j(s)|} + \frac{|(w_j(s) - x_j(s)) - (\bar{w}_j(s) - \bar{x}_j(s))|}{|w_j(s) - x_j(s)||\bar{w}_j(s) - \bar{x}_j(s)|} \\ &\leq A\frac{\delta_{\mathrm{x}}(s)}{\sqrt{t-s}} + A\frac{|(w_j(s) - x_j(s)) - (\bar{w}_j(s) - \bar{x}_j(s))|}{t-s}. \end{aligned} \tag{3.72}$$

So on the event $E_{\mathrm{p},1}^t \cap E_{\mathrm{p},2}$, which has probability at least $1 - A\log(T/\mathbf{c})\mathbf{c}^{4/R}$, (3.67) becomes

$$\begin{aligned} \left| \frac{1}{\Phi_t''(x_j(t))} - \frac{1}{\bar{\Phi}_t''(\bar{x}_j(t))} \right| \leq \frac{A}{\sqrt{t}} &\int_0^t (\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s)) \, \mathrm{d}s + A\mathbf{c}^{\frac{1}{2} - \frac{2}{R}} \\ &+ A \int_0^t \frac{\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s)}{\sqrt{t-s}} \, \mathrm{d}s \\ &+ A \int_0^t \frac{|(w_j(s) - x_j(s)) - (\bar{w}_j(s) - \bar{x}_j(s))|}{t-s} \, \mathrm{d}s, \end{aligned} \tag{3.73}$$

where the $l = j$ summand in the final term of (3.69) gives the final line of (3.73), which, together with the terms involving δ_{x} also provide an upper bound on the $l \neq j$ summands.

To bound the final line of (3.73), let $u_s = (w_j(s) - x_j(s)) - (\bar{w}_j(s) - \bar{x}_j(s))$. We will bound $|u_s|$ using Grönwall's lemma. Since the s-derivative of u_s is singular as $s \uparrow t$, we will carefully analyse and control the singularity. Subtract (3.52) from its LPM equivalent to obtain

$$\begin{aligned} \dot{u}_s = &- ((w_j(s) - x_j(s)) - (\bar{w}_j(s) - \bar{x}_j(s))) \\ &+ \sum_{l \neq j} \left(- I_s^l \left(\frac{2x_l(s)^2}{(x_j(s) - x_l(s))(w_j(s) - x_l(s))} \right) (w_j(s) - x_j(s)) \right. \\ &\qquad \left. + \bar{p}_s^l \left(\frac{2\bar{x}_l(s)^2}{(\bar{x}_j(s) - \bar{x}_l(s))(\bar{w}_j(s) - \bar{x}_l(s))} \right) (\bar{w}_j(s) - \bar{x}_j(s)) \right) \\ &- I_s^j \left(3x_j(s) + \frac{2x_j(s)^2}{w_j(s) - x_j(s)} \right) + \bar{p}_s^j \left(3\bar{x}_j(s) + \frac{2\bar{x}_j(s)^2}{\bar{w}_j(s) - \bar{x}_j(s)} \right). \end{aligned} \tag{3.74}$$

The first line on the right-hand side is clearly $-u_s$, and repeating the linearisation trick for the lth summand in the next term we find it is the sum of three things:

$$(I^l_s - p^l_s)\left(\frac{2x_l(s)^2}{(x_j(s)-x_l(s))(w_j(s)-x_l(s))}\right)(w_j(s)-x_j(s)),$$

and something with size bounded by

$$A\left(|p^l_s - \bar{p}^l_s| + |x_l(s)-\bar{x}_l(s)| + |x_j(s)-\bar{x}_j(s)|\right),$$

where the constant A is proportional to L^{-3}, and a bounded multiple of u_s. The final term in the expansion of $\dot{u}_s$, on the last line of (3.74), is the sum of

$$(I^j_s - p^j_s)\left(3x_j(s) + \frac{2x_j(s)^2}{w_j(s)-x_j(s)}\right), \tag{3.75}$$

another term which can be simplified to

$$\frac{2\bar{p}^j_s x_j(s)\bar{x}_j(s)}{(w_j(s)-x_j(s))(\bar{w}_j(s)-\bar{x}_j(s))}u_s, \tag{3.76}$$

and something bounded by

$$A\frac{|p^j_s - \bar{p}^j_s| + |x_j(s)-\bar{x}_j(s)|}{|\bar{w}_j(s)-\bar{x}_j(s)|}. \tag{3.77}$$

For s close to t, $\bar{w}_j(s)-\bar{x}_j(s) = 2\bar{x}_j(s)\sqrt{\bar{p}^j_s}\sqrt{t-s}+O(t-s)$, and we said earlier that $\bar{p}^j_s$ is bounded below by a positive constant, so we can bound (3.77) by

$$A\frac{|p^j_s - \bar{p}^j_s| + |x_j(s)-\bar{x}_j(s)|}{\sqrt{t-s}}.$$

Similarly $w_j(s)-x_j(s) = 2x_j(s)\sqrt{t-s}+O(t-s)$, and so the coefficient of u_s in (3.76) is

$$\frac{\sqrt{\bar{p}^j_s}}{2(t-s)} + O\left(\frac{1}{\sqrt{t-s}}\right),$$

and note that the leading order term is a positive real number, and the error term can absorb the $-u_s$ from the first line of (3.74) and the bounded multiple of u_s from each summand of the second and third lines. Hence (3.74) can be written as

$$\dot{u}_s = \left(\frac{\sqrt{\bar{p}^j_s}}{2(t-s)} + O\left(\frac{1}{\sqrt{t-s}}\right)\right)u_s + H^{\text{regular}}_s + H^{\text{singular}}_s, \tag{3.78}$$

where the first forcing term is

$$H^{\text{regular}}_s = \sum_{l\neq j}(I^l_s - p^l_s)\left(\frac{2x_l(s)^2(w_j(s)-x_j(s))}{(x_j(s)-x_l(s))(w_j(s)-x_l(s))}\right) + B^{\text{regular}}_s, \tag{3.79}$$

where

$$|B_s^{\text{regular}}| \leq A \sum_l (|p_s^l - \bar{p}_s^l| + |x_l(s) - \bar{x}_l(s)|). \tag{3.80}$$

The other forcing term is

$$H_s^{\text{singular}} = (I_s^j - p_s^j)\left(3x_j(s) + \frac{2x_j(s)^2}{w_j(s) - x_j(s)}\right) + B_s^{\text{singular}}, \tag{3.81}$$

where

$$|B_s^{\text{singular}}| \leq A \frac{|p_s^j - \bar{p}_s^j| + |x_j(s) - \bar{x}_j(s)|}{\sqrt{t-s}}. \tag{3.82}$$

For some $\alpha > 0$, let $v_s = s^\alpha u_{t-s}$, then as $\dot{v}_s = \alpha s^{\alpha-1} u_{t-s} - s^\alpha \dot{u}_{t-s}$, the sum of terms involving u_{t-s} in $\dot{v}_s$ is

$$\alpha s^{\alpha-1} u_{t-s} - s^\alpha \left(\frac{\sqrt{\bar{p}_{t-s}^j}}{2s} + O\left(s^{-1/2}\right)\right) u_{t-s} = \left(\alpha - \frac{\sqrt{\bar{p}_{t-s}^j}}{2} + O(s^{1/2})\right) \frac{v_s}{s}.$$

Hence (3.74) becomes

$$\dot{v}_s = \left(\alpha - \frac{\sqrt{\bar{p}_{t-s}^j}}{2} + O(s^{1/2})\right) \frac{v_s}{s} - s^\alpha H_{t-s}^{\text{regular}} - s^\alpha H_{t-s}^{\text{singular}}. \tag{3.83}$$

We noted before that there is a constant $p = p(T, K_0, k, \eta) \in (0, 1/k)$ such that $\bar{p}_s^j \geq p$ for all j and s, so the leading order term is a real number less than $\alpha - \sqrt{p}/2$.

Hence if $\alpha > \sqrt{p}/2$, which is the only requirement on the constant α, $|\alpha - \sqrt{\bar{p}_{t-s}^j}/2 + O(s^{1/2})| \leq \alpha - \sqrt{p}/2 + O(s^{1/2})$. Then integrating (3.83) and applying the triangle inequality, we have

$$|v_s| \leq \int_0^s \left(\alpha - \frac{\sqrt{p}}{2} + O(r^{\frac{1}{2}})\right) \frac{|v_r|}{r} \, \mathrm{d}r + \left|\int_0^s r^\alpha H_{t-r}^{\text{regular}} \, \mathrm{d}r\right| + \left|\int_0^s r^\alpha H_{t-r}^{\text{singular}} \, \mathrm{d}r\right|. \tag{3.84}$$

Substituting in (3.79) and (3.80), the second term on the right-hand side in (3.84) is bounded by

$$\begin{aligned}
&\sum_{l \neq j} \left| \int_0^s (I_{t-r}^l - p_{t-r}^l) r^\alpha \left(\frac{2x_l(t-r)^2 (w_j(t-r) - x_j(t-r))}{(x_j(t-r) - x_l(t-r))(w_j(t-r) - x_l(t-r))}\right) \mathrm{d}r \right| \\
&+ A \sum_l \int_0^s r^\alpha |p_{t-r}^l - \bar{p}_{t-r}^l| \, \mathrm{d}r + A \sum_l \int_0^s r^\alpha |x_l(t-r) - \bar{x}_l(t-r)| \, \mathrm{d}r.
\end{aligned} \tag{3.85}$$

Let the integral on the first line be $M_{t,s}^{\mathrm{w},j,l}$. Note that we have not used the triangle inequality to take the absolute value of the integrand in the first line of (3.85), unlike the second line. This is because we will later use martingale methods to establish a good bound on $|M_{t,s}^{\mathrm{w},j,l}|$ which holds with high probability. Note that the absolute value of the integrand is bounded by $Ar^{\alpha+1/2}$, since $|I^l_{t-r} - p^l_{t-r}| \le 2$, $|w_j(t-r) - x_j(t-r)| \asymp r^{1/2}$, and the other terms are bounded. The resulting almost-sure bound $|M^{\mathrm{w},j,l}_{t,s}| \le \frac{A}{\alpha+3/2} s^{\alpha+3/2}$ will be useful as a kind of bootstrap for the later improved bound.

The final term in (3.84) is similarly bounded by

$$\begin{aligned}
&\left| \int_0^s (I^j_{t-r} - p^j_{t-r}) r^\alpha \left(3x_j(t-r) + \frac{2x_j(t-r)}{w_j(t-r) - x_j(t-r)} \right) \mathrm{d}r \right| \\
&+ A \int_0^s r^{\alpha-1/2} |p^j_{t-r} - \bar{p}^j_{t-r}| \,\mathrm{d}r + A \int_0^s r^{\alpha-1/2} |x_j(t-r) - \bar{x}_j(t-r)| \,\mathrm{d}r.
\end{aligned} \tag{3.86}$$

Again let the integral on the first line be $M^{\mathrm{w},j,j}_{t,s}$. As the absolute value of the integrand is bounded by $Ar^{\alpha-1/2}$, we have an almost-sure bound $|M^{\mathrm{w},j,j}_{t,s}| \le \frac{A}{\alpha+1/2} s^{\alpha+1/2}$, and we will also use martingale methods to improve this later.

It will be convenient to define

$$\chi(r) := \sum_{l=1}^k (|p^l_r - \bar{p}^l_r| + |x_l(r) - \bar{x}_l(r)|). \tag{3.87}$$

Note that the sum of the second lines of (3.85) and (3.86) is bounded by

$$A \int_0^s r^{\alpha-1/2} \chi(t-r) \,\mathrm{d}r \tag{3.88}$$

which will be more convenient to work with.

As $|p^l_{t-r} - \bar{p}^l_{t-r}| \le 1$ and $|x_l(t-r) - \bar{x}_l(t-r)| \le 2$ for all l, t and r, $\chi(r)$ is uniformly bounded by a constant, and so (3.88) is bounded by $\frac{A}{\alpha+1/2} s^{\alpha+1/2}$.

Overall, combining the second lines of (3.85) and (3.86) we have

$$|v_s| \le \int_0^s \left(\alpha - \frac{\sqrt{p}}{2} + O(r^{1/2}) \right) \frac{|v_r|}{r} \,\mathrm{d}r + \sum_l |M^{\mathrm{w},j,l}_{t,s}| + A \int_0^s r^{\alpha-1/2} \chi(t-r) \,\mathrm{d}r.$$

If we let $a_t(s) = \sum_l |M^{\mathrm{w},j,l}_{t,s}| + A \int_0^s r^{\alpha-1/2} \chi(t-r) \,\mathrm{d}r$ be the sum of the "forcing terms" on the right-hand side above, then by Grönwall's lemma

$$|v_s| \le a_t(s) + A\alpha s^{\alpha - \frac{\sqrt{p}}{2}} \int_0^s \frac{a_t(r)}{r^{\alpha+1-\sqrt{p}/2}} \,\mathrm{d}r$$

for all $0 < s < t$. Note that since $a_t(r) \le \frac{A}{\alpha+1/2} r^{\alpha+1/2}$, the integral above converges.

Passing from v back to u, this bound becomes

$$|u_s| = (t-s)^{-\alpha} |v_{t-s}| \le (t-s)^{-\alpha} a_t(t-s) + A\alpha (t-s)^{-\frac{\sqrt{p}}{2}} \int_0^{t-s} \frac{a_t(r)}{r^{\alpha+1-\sqrt{p}/2}} \,\mathrm{d}r.$$

Substituting this back into the final term of (3.73), we get

$$\int_0^t \frac{|u_s|}{t-s} \,\mathrm{d}s \le A \int_0^t \frac{a_t(t-s)}{(t-s)^{\alpha+1}} \,\mathrm{d}s + A\alpha \int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}} \int_0^{t-s} \frac{a_t(r)}{r^{\alpha+1-\sqrt{p}/2}} \,\mathrm{d}r \,\mathrm{d}s. \tag{3.89}$$

Our crude bound $a_t(r) \le \frac{A}{\alpha+1/2} r^{\alpha+1/2}$ shows that each of the above integrals converges. For the first term on the right-hand side of (3.89) we have

$$\int_0^t \frac{a_t(t-s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s = \sum_l \int_0^t \frac{|M^{\mathrm{w},j,l}_{t,t-s}|}{(t-s)^{\alpha+1}}\,\mathrm{d}s + A\int_0^t \frac{1}{(t-s)^{\alpha+1}}\int_0^{t-s} r^{\alpha-1/2}\chi(t-r)\,\mathrm{d}r\,\mathrm{d}s.$$

We will come back to the term involving $|M^{\mathrm{w},j,l}|$ later. First, for the double integral, we can apply Fubini's theorem to change the order of integration:

$$\begin{aligned}
&\int_0^t \frac{1}{(t-s)^{\alpha+1}}\int_0^{t-s} r^{\alpha-1/2}\chi(t-r)\,\mathrm{d}r\,\mathrm{d}s\\
&= \int_0^t r^{\alpha-1/2}\chi(t-r)\int_0^{t-r}(t-s)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}r\\
&= \frac{1}{\alpha}\left(\int_0^t r^{-1/2}\chi(t-r)\,\mathrm{d}r - t^{-\alpha}\int_0^t r^{\alpha-1/2}\chi(t-r)\,\mathrm{d}r\right)\\
&\le \frac{1}{\alpha}\int_0^t r^{-1/2}\chi(t-r)\,\mathrm{d}r.
\end{aligned}$$

Substituting the two summands in the definition of $a_t(r)$ back into the final term of (3.89), we expand it to

$$\begin{aligned}
&A\alpha\sum_l \int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\int_0^{t-s}\frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1-\frac{\sqrt{p}}{2}}}\,\mathrm{d}r\,\mathrm{d}s\\
&+A\alpha\int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\int_0^{t-s}\frac{1}{r^{\alpha+1-\frac{\sqrt{p}}{2}}}\int_0^r x^{\alpha-1/2}\chi(t-x)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}$$

We will leave the first term for later. To simplify the triple integral, we can apply Fubini's theorem to the innermost two integrals:

$$\begin{aligned}
&\int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\int_0^{t-s}\frac{1}{r^{\alpha+1-\frac{\sqrt{p}}{2}}}\int_0^r x^{\alpha-1/2}\chi(t-x)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s\\
&= \int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\int_0^{t-s} x^{\alpha-1/2}\chi(t-x)\int_x^{t-s} r^{-\alpha-1+\sqrt{p}/2}\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\int_0^{t-s} x^{\alpha-1/2}\chi(t-x)\int_x^{\infty} r^{-\alpha-1+\sqrt{p}/2}\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}s\\
&= \frac{1}{\alpha-\sqrt{p}/2}\int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\int_0^{t-s} x^{\frac{\sqrt{p}}{2}-1/2}\chi(t-x)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

By a second application of Fubini's theorem followed by the substitution $y = t-s$ we can simplify this further:

$$
\begin{aligned}
&\int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}} \int_0^{t-s} x^{\frac{\sqrt{p}}{2}-1/2}\chi(t-x)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t x^{\frac{\sqrt{p}}{2}-1/2}\chi(t-x) \int_0^{t-x} \frac{\mathrm{d}s}{(t-s)^{1+\frac{\sqrt{p}}{2}}}\,\mathrm{d}x \\
&= \int_0^t x^{\frac{\sqrt{p}}{2}-1/2}\chi(t-x) \int_x^t y^{-1-\sqrt{p}/2}\,\mathrm{d}y\,\mathrm{d}x \\
&\le \int_0^t x^{\frac{\sqrt{p}}{2}-1/2}\chi(t-x) \int_x^\infty y^{-1-\sqrt{p}/2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{2}{\sqrt{p}} \int_0^t x^{-1/2}\chi(t-x)\,\mathrm{d}x.
\end{aligned}
$$

Now note that

$$
\int_0^t x^{-1/2}\chi(t-x)\,\mathrm{d}x = \int_0^t (t-x)^{-1/2}\chi(x)\,dx \le \int_0^t (t-x)^{-1/2}(\delta_{\mathrm{x}}(x)+A\delta_{\mathrm{p}}(x))\,\mathrm{d}x,
$$

and so (3.89) is bounded by

$$
\begin{aligned}
A\int_0^t \frac{\delta_{\mathrm{x}}(s)+\delta_{\mathrm{p}}(s)}{\sqrt{t-s}}\,\mathrm{d}s + A\sum_l \int_0^t \frac{|M^{\mathrm{w},j,l}_{t,t-s}|}{(t-s)^{\alpha+1}}\,\mathrm{d}s& \\
+ A\alpha \sum_l \int_0^t \frac{1}{(t-s)^{1+\frac{\sqrt{p}}{2}}} \int_0^{t-s} \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1-\frac{\sqrt{p}}{2}}}\,\mathrm{d}r\,\mathrm{d}s.& \qquad (3.90)
\end{aligned}
$$

Making the substitution $x = t-s$ and exchanging the integrals, the lth summand on the second line of (3.90) is equal to

$$
\begin{aligned}
\int_0^t \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1-\frac{\sqrt{p}}{2}}} \int_r^t \frac{\mathrm{d}x}{x^{1+\frac{\sqrt{p}}{2}}}\,\mathrm{d}r &\le \int_0^t \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1-\frac{\sqrt{p}}{2}}} \int_r^\infty \frac{\mathrm{d}x}{x^{1+\frac{\sqrt{p}}{2}}}\,\mathrm{d}r \\
&= \frac{2}{\sqrt{p}} \int_0^t \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1}}\,\mathrm{d}r,
\end{aligned}
$$

so the final term in (3.90) is bounded by a multiple of the second term. Hence we only need to bound $\sum_{l=1}^k \int_0^t \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1}}\,\mathrm{d}r$.

First we will look at the case $l = j$. Recall the definition of $M^{\mathrm{w},j,j}_{t,s}$ via (3.86), and let $C_x = x^\alpha\left(3x_j(t-x) + \frac{2x_j(t-x)}{w_j(t-x)-x_j(t-x)}\right)$, and note $|C_x| \le Ax^{\alpha-1/2}$.

If $r < \mathbf{c}$, then

$$
|M^{\mathrm{w},j,j}_{t,r}| = \left|\int_0^r (I^j_{t-x} - p^j_{t-x})C_x\,\mathrm{d}x\right| \le 2\int_0^r |C_x|\,\mathrm{d}x \le Ar^{\alpha+1/2}.
$$

If $r \ge \mathbf{c}$, note that for $x \in (n\mathbf{c}, (n+1)\mathbf{c})$, by Proposition 3.15, $|p_{t-x} - p_{t-(n+1)\mathbf{c}}| \le \mathbf{c}$ and $I_{t-x} = I_{t-(n+1)\mathbf{c}}$, so substituting $p_{t-x} = p_{t-(n+1)\mathbf{c}} + (p_{t-x} - p_{t-(n+1)\mathbf{c}})$ and using the triangle inequality,

$$|M^{\mathrm{w},j,j}_{t,r}| \leq \left| \sum_{n=0}^{\lfloor r/\mathbf{c} \rfloor} (I^j_{t-(n+1)\mathbf{c}} - p^j_{t-(n+1)\mathbf{c}}) \int_{n\mathbf{c}}^{(n+1)\mathbf{c}\wedge r} C_x \, \mathrm{d}x \right| + A\mathbf{c} \int_0^r |C_x| \, \mathrm{d}x. \tag{3.91}$$

The second term is less than $A\mathbf{c}r^{\alpha+1/2}$. Using the same types of martingale arguments as we did following (3.44) and also (3.64), the second moment of the first term is bounded by

$$4 \sum_{n=0}^{\lfloor r/\mathbf{c} \rfloor} \left(\int_{n\mathbf{c}}^{(n+1)\mathbf{c}\wedge r} |C_x| \, \mathrm{d}x \right)^2 \leq A\mathbf{c}r^{2\alpha}.$$

Define the event $E^{t,r}_{\mathrm{w},j,j} = \{|M^{\mathrm{w},j,j}_{t,r}| \leq \mathbf{c}^{\frac{1}{2}-\frac{4}{R}} r^{\alpha+\frac{1}{2R}}\}$. By Markov's inquality, for any $r \geq \mathbf{c}$,

$$1 - \mathbb{P}(E^{t,r}_{\mathrm{w},j,j}) = \mathbb{P}\left(|M^{\mathrm{w},j,j}_{t,r}|^2 > (\mathbf{c}^{\frac{1}{2}-\frac{4}{R}} r^{\alpha+\frac{1}{2R}})^2 \right) \leq \frac{A\mathbf{c}r^{2\alpha}}{\mathbf{c}^{1-\frac{8}{R}} r^{2\alpha+\frac{1}{R}}} = A\mathbf{c}^{\frac{8}{R}} r^{-\frac{1}{R}} \leq A\mathbf{c}^{\frac{7}{R}}.$$

We will find a good upper bound on $\int_0^t \frac{|M^{\mathrm{w},j,j}_{t,r}|}{r^{\alpha+1}}$ by showing that $\bigcap_{m=1}^{N_t} E^{t,r_m}_{\mathrm{w},j,j}$ occurs with high probability, where $(r_m)_{1\leq m\leq N_t}$ is a finite sequence of points which are "sufficiently dense" in $[0,t]$. Choose this sequence of times inductively: set $r_1 = \mathbf{c}$, and for $m \geq 1$ set $r_{m+1} = r_m + \mathbf{c}^{\frac{1}{R}}\sqrt{r_m}$. Then $r_{m+1} - r_m \geq \mathbf{c}^{\frac{1}{R}}\sqrt{r_m - r_{m-1}}$, and so $r_{m+1} - r_m \geq \mathbf{c}^{\frac{1}{2^m}+\frac{1}{R}(2-\frac{1}{2^m})} \geq \mathbf{c}^{\frac{4}{R}}$ for m greater than some constant m_R. Hence $N_t := \min\{m : r_m \geq t\} \leq A\mathbf{c}^{-\frac{4}{R}}$.

On the event $E^{t,r_m}_{\mathrm{w},j,j}$ for some $m \geq 1$, let $r \in (r_m, r_{m+1})$, then by the mean value theorem

$$\begin{aligned}
|M^{\mathrm{w},j,j}_{t,r} - M^{\mathrm{w},j,j}_{t,r_m}| = \left| \int_{r_m}^r (I^j_{t-x} - p^j_{t-x}) C_x \, \mathrm{d}x \right| &\leq 2 \int_{r_m}^r |C_x| \, \mathrm{d}x \\
&\leq \frac{A}{\alpha+1/2}(r^{\alpha+1/2} - r_m^{\alpha+1/2}) \\
&\leq A(r - r_m) r^{\alpha-1/2} \\
&\leq A\mathbf{c}^{\frac{1}{R}} \sqrt{r_m} r^{\alpha-1/2} \\
&\leq A\mathbf{c}^{\frac{1}{2R}} r^{\alpha+\frac{1}{2R}}.
\end{aligned}$$

Hence for any given t, as the $\mathbf{c}^{\frac{1}{2R}} r^{\alpha+\frac{1}{2R}}$ above absorbs the $\mathbf{c}^{\frac{1}{2}-\frac{4}{R}} r^{\alpha+\frac{1}{2R}}$ bound on $|M^{\mathrm{w},j,j}_{t,r_m}|$,

$$\mathbb{P}\left(|M^{\mathrm{w},j,j}_{t,r}| \leq A\mathbf{c}^{\frac{1}{2R}} r^{\alpha+\frac{1}{2R}} \text{ for all } \mathbf{c} \leq r \leq t \right) \geq \mathbb{P}\left(\bigcap_{m=1}^{N_t} E^{t,r_m}_{\mathrm{w}} \right) \geq 1 - A\mathbf{c}^{\frac{3}{R}}, \tag{3.92}$$

where we have bounded $\mathbb{P}\left(\bigcup_{m=1}^{N_t} (E^{t,r_m}_{\mathrm{w},j,j})^c \right)$ by a union bound and using $N_t \leq A\mathbf{c}^{-4/R}$. Note that the bound $|M^{\mathrm{w},j,j}_{t,r}| \leq Ar^{\alpha+1/2}$ for $r < \mathbf{c}$ is almost sure. So on the event

$E^t_{\mathrm{w},j,j} := \bigcap_{m=1}^{N_t} E^{t,r_m}_{\mathrm{w},j,j}$, we can use the bound on $|M^{\mathrm{w},j,j}_{t,r}|$ in (3.92) to find

$$\int_0^t \frac{|M^{\mathrm{w},j,j}_{t,r}|}{r^{\alpha+1}}\,\mathrm{d}r \le \int_0^{\mathbf{c}} \frac{Ar^{\alpha+\frac{1}{2}}}{r^{\alpha+1}}\,\mathrm{d}r + \int_{\mathbf{c}}^t \frac{A\mathbf{c}^{\frac{1}{2R}}r^{\alpha+\frac{1}{2R}}}{r^{\alpha+1}}\,\mathrm{d}r$$
$$\le A\mathbf{c}^{\frac{1}{2}} + AR\mathbf{c}^{\frac{1}{2R}}t^{\frac{1}{2R}} \tag{3.93}$$

if $R \ge 9$.

For $l \ne j$, a very similar argument (which we omit) shows that if we define events $E^{t,r}_{\mathrm{w},j,l} = \{|M^{\mathrm{w},j,l}_{t,r}| \le \mathbf{c}^{\frac{1}{2}-\frac{4}{R}}r^{\alpha+1+\frac{1}{2R}}\}$ and $E^t_{\mathrm{w},j,l} = \bigcap_{m=1}^{N_t} E^{t,r_m}_{\mathrm{w},j,l}$ for the same sequence $(r_m)_{m=1}^{N_r}$, we have $\mathbb{P}(\bigcap_{m=1}^{N_t} E^{t,r_m}_{\mathrm{w},j,l}) \ge 1 - A\mathbf{c}^{3/R}$ and on the event $E^t_{\mathrm{w},j,l}$ we have the bound

$$\int_0^t \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1}}\,\mathrm{d}r \le A\mathbf{c}^{\frac{3}{2}} + AR\mathbf{c}^{\frac{1}{2R}}t^{1+\frac{1}{2R}}.$$

Define the combined event $E^t_{\mathrm{w},j} = \bigcap_{l=1}^k E^t_{\mathrm{w},j,l}$. Since k is constant, we still have the lower bound $\mathbb{P}(E^t_{\mathrm{w},j}) \ge 1 - A\mathbf{c}^{\frac{3}{R}}$, and on $E^t_{\mathrm{w},j}$ the latter two terms in (3.90) are bounded by $A\mathbf{c}^{\frac{1}{2}} + AR\mathbf{c}^{\frac{1}{2R}}t^{\frac{1}{2R}}(t \vee 1)$. As $t \le T$ and R is a constant, we can just use the upper bound

$$\sum_{l=1}^k \int_0^t \frac{|M^{\mathrm{w},j,l}_{t,r}|}{r^{\alpha+1}}\,\mathrm{d}r \le A\mathbf{c}^{\frac{1}{2R}}. \tag{3.94}$$

Therefore, on the event $E_{\mathrm{T}} \cap E^t_{\mathrm{p},1} \cap E_{\mathrm{p},2} \cap E^t_{\mathrm{w},j}$, using (3.89), (3.90) and (3.94), we can update the bound (3.73) to

$$\left|\frac{1}{\Phi''_t(x_j(t))} - \frac{1}{\bar\Phi''_t(\bar x_j(t))}\right| \le \frac{A}{\sqrt{t}}\int_0^t (\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s))\,\mathrm{d}s + A\mathbf{c}^{\frac{1}{2}-\frac{2}{R}}$$
$$+ A\int_0^t \frac{\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s)}{\sqrt{t-s}}\,\mathrm{d}s$$
$$+ A\mathbf{c}^{\frac{1}{2R}}. \tag{3.95}$$

Note that $\mathbf{c}^{\frac{1}{2}-\frac{2}{R}} < \mathbf{c}^{\frac{1}{2R}}$, and as $\frac{1}{\sqrt{t}} < \frac{1}{\sqrt{t-s}}$ for all $0 < s < t$, we can simplify (3.95) to

$$\left|\frac{1}{\Phi''_t(x_j(t))} - \frac{1}{\bar\Phi''_t(\bar x_j(t))}\right| \le A\int_0^t \frac{\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s)}{\sqrt{t-s}}\,\mathrm{d}s + A\mathbf{c}^{\frac{1}{2R}}. \tag{3.96}$$

The above event holds with probability at least $1 - A\frac{T}{\mathbf{c}}(1-p)^{\lfloor c^{-(1-1/R)}\rfloor} - A\mathbf{c}^{3/R} \ge 1 - 2A\mathbf{c}^{3/R}$. So on an event of probability at least $1 - 2AT\mathbf{c}^{2/R}$, which we call E^j, the above inequality holds for all $t \in \{0, T^j(1), \dots, T^j(N_j)\}$.

Then using Lemma 3.13, for any $T^j(m) < t \le T^j(m+1)$, including $T^j(0) = 0$, since $T^j(m+1) - T^j(m) \le \mathbf{c}^{1/2R}$ on the event E_{T} by its definition, (3.58), we have

$$\left|\frac{1}{\Phi''_t(x_j(t))} - \frac{1}{\bar\Phi''_t(\bar x_j(t))}\right| \le A\int_0^t \frac{\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s)}{\sqrt{t-s}}\,\mathrm{d}s + A\mathbf{c}^{\frac{1}{2R}} + A(T^j(m+1) - T^j(m))$$
$$\le A\int_0^t \frac{\delta_{\mathrm{p}}(s) + \delta_{\mathrm{x}}(s)}{\sqrt{t-s}}\,\mathrm{d}s + 2A\mathbf{c}^{\frac{1}{2R}}, \tag{3.97}$$

and the final inequality holds for all $t \in (0, T]$, almost surely on the event $E^j \cap E_{\mathrm{T}}$. Let $\varepsilon = 2A\mathbf{c}^{\frac{1}{2R}}$. Define the event $E := E_{\mathrm{T}} \cap E_{\mathrm{x}} \cap \bigcap_{j=1}^k E^j$, which has probability at least $1 - 3AkT\mathbf{c}^{2/R}$. Then on E, summing (3.97) over j, and then adding the above inequality to (3.48), for the total error $\delta_{\mathrm{total}} := \delta_{\mathrm{p}} + \delta_{\mathrm{x}}$ we get

$$\delta_{\mathrm{total}}(t) \leq A \int_0^t \frac{\delta_{\mathrm{total}}(s)}{\sqrt{t-s}} \, \mathrm{d}s + \varepsilon$$

for all $0 < t \leq T$ almost surely.

By the theorem on page 375 of [24], δ_{total} is bounded by any solution to

$$y(t) = A \int_0^t \frac{y(s)}{\sqrt{t-s}} \, \mathrm{d}s + \varepsilon. \tag{3.98}$$

This is Abel's integral equation of the second kind, with standard solution given in [27] (page 136) as

$$\begin{aligned} y(t) &= \varepsilon + 2A\varepsilon\sqrt{t} + \pi A^2 \int_0^t e^{t-s}(\varepsilon + 2A\varepsilon\sqrt{s}) \, \mathrm{d}s \\ &\leq A\varepsilon, \end{aligned}$$

and so

$$\sup_{t \leq T} \sum_{j=1}^k \Big(|x_j(t) - \bar{x}_j(t)| + |p_t^j - \bar{p}_t^j| \Big) \leq A\mathbf{c}^{1/2R}$$

on an event of probability $1 - o(1)$, as required. □

Proof of Corollary 3.17. Let $\varphi \in \mathcal{H}$. For simplicity assume T is an integer multiple of $\mathbf{c}$. Then

$$\begin{aligned} \int_0^T \int_{\mathbb{T}} \varphi \, \mathrm{d}\mu_t \, \mathrm{d}t - \int_0^T \int_{\mathbb{T}} \varphi \, \mathrm{d}\bar{\mu}_t \, \mathrm{d}t &= \sum_{j=1}^k \int_0^T \Big(I_t^j \varphi(x_j(t)) - \bar{p}_t^j \varphi(\bar{x}_j(t)) \Big) \, \mathrm{d}t \\ = \sum_{j=1}^k \int_0^T \Big((I_t^j - p_t^j)\varphi(x_j(t)) &+ (p_t^j - \bar{p}_t^j)\varphi(x_j(t)) + \bar{p}_t^j(\varphi(x_j(t)) - \varphi(\bar{x}_j(t))) \Big) \, \mathrm{d}t. \end{aligned}$$

The latter two terms are small with high probability using Theorem 3.16, so we only need to bound the martingale terms $\int_0^T (I_t^j - p_t^j)\varphi(x_j(t)) \, \mathrm{d}t$ for each j. We can write this as

$$\sum_{n=0}^{\frac{T}{\mathbf{c}}-1} \left[(I_{n\mathbf{c}}^j - p_{n\mathbf{c}}^j) \int_{n\mathbf{c}}^{(n+1)\mathbf{c}} \varphi(x_j(t)) \, \mathrm{d}t + \int_{n\mathbf{c}}^{(n+1)\mathbf{c}} (p_{n\mathbf{c}}^j - p_t^j)\varphi(x_j(t)) \, \mathrm{d}t \right].$$

Since $|p_{n\mathbf{c}}^j - p_t^j| \leq A\mathbf{c}$, we have $|\sum_{n=0}^{\frac{T}{\mathbf{c}}-1} \int_{n\mathbf{c}}^{(n+1)\mathbf{c}} (p_{n\mathbf{c}}^j - p_t^j)\varphi(x_j(t)) \, \mathrm{d}t| \leq AT\mathbf{c}$. For the remaining term, we would like to simply compute second moments and so show the term is small, but need some way of doing this uniformly in φ. Choose some large N, and for simplicity assume it is a factor of $T/\mathbf{c}$. Then write

$$\begin{aligned}
&\sum_{m=1}^{N} \sum_{n=\frac{(m-1)T}{\mathbf{c}N}}^{\frac{mT}{\mathbf{c}N}-1} (I^j_{n\mathbf{c}} - p^j_{n\mathbf{c}}) \int_{n\mathbf{c}}^{(n+1)\mathbf{c}} \varphi(x_j(t))\,\mathrm{d}t \\
&= \sum_{m=1}^{N} \mathbf{c}\varphi(x_j(mT/N)) \sum_{n=\frac{(m-1)T}{\mathbf{c}N}-1}^{\frac{mT}{\mathbf{c}N}} (I^j_{n\mathbf{c}} - p^j_{n\mathbf{c}}) + \delta,
\end{aligned}$$

where $|\delta| \leq \frac{AT^2}{N}$ using $\|\varphi\|_{\mathrm{Lip}} \leq 1$. Now we will be able to find a bound independent of φ, as

$$\mathbb{E}\left(\mathbf{c} \sum_{n=\frac{(m-1)T}{\mathbf{c}N}}^{\frac{mT}{\mathbf{c}N}-1} (I^j_{n\mathbf{c}} - p^j_{n\mathbf{c}})\right)^2 \leq 4\mathbf{c}^2 \frac{T}{\mathbf{c}N} = \frac{4T\mathbf{c}}{N}$$

by conditional independence of the increments $I^j_{n\mathbf{c}} - p^j_{n\mathbf{c}}$. Define the event

$$E_m = \left\{ \left| \mathbf{c} \sum_{n=\frac{(m-1)T}{\mathbf{c}N}-1}^{\frac{mT}{\mathbf{c}N}} (I^j_{n\mathbf{c}} - p^j_{n\mathbf{c}}) \right| \geq \frac{2\sqrt{T}\mathbf{c}^{1/4}}{N} \right\},$$

then $\mathbb{P}(E_m) \leq \mathbf{c}^{1/2} N$, and so

$$\begin{aligned}
\mathbb{P}\left(\sup_{\varphi\in\mathcal{H}} \left| \sum_{m=1}^{N} \mathbf{c}\varphi(x_j(mT/N)) \sum_{n=\frac{(m-1)T}{\mathbf{c}N}-1}^{\frac{mT}{\mathbf{c}N}} (I^j_{n\mathbf{c}} - p^j_{n\mathbf{c}}) \right| \geq 2\sqrt{T}\mathbf{c}^{1/4} \right) &\leq \mathbb{P}\left(\bigcup_{m=1}^{N} E_m \right) \\
&\leq \mathbf{c}^{1/2} N^2.
\end{aligned}$$

So if $1 \ll N \ll \mathbf{c}^{-1/4}$, we have

$$\sup_{\varphi\in\mathcal{H}} \left| \int_0^T \int_{\mathbb{T}} \varphi\,\mathrm{d}\mu_t\,\mathrm{d}t - \int_0^T \int_{\mathbb{T}} \varphi\,\mathrm{d}\bar{\mu}_t\,\mathrm{d}t \right| \to 0$$

in probability, as required. □

We can now bring together each of the steps to show convergence of the ALE to the LPM.

Proof of Theorem 3.1. Let ξ_t be the driving function of the ALE, and $\bar{\mu}_t$ the driving measure of the LPM. To show that the two converge in distribution, we will show that $d_{\mathrm{BW}}(\delta_{\xi_t} \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}) \to 0$ in probability as $\mathbf{c} \to 0$.

Note that another way of writing (3.1) for the auxiliary process is

$$\Phi^*_n = R_{\delta_1+\cdots+\delta_n} \circ \left(\Phi_0 \circ f_{\theta^*_1-\delta_1} \circ f_{\theta^*_2-(\delta_1+\delta_2)} \circ \cdots \circ f_{\theta^*_n-(\delta_1+\cdots+\delta_n)} \right) \circ R_{-(\delta_1+\cdots+\delta_n)}$$

So let ξ^*_t be the driving measure for the angle sequence $(\theta^*_n - (\delta_1 + \cdots + \delta_n))_{n\leq\lfloor T/\mathbf{c}\rfloor}$. By

Proposition 3.3 we have a coupling between ξ^* and ξ, such that if we define the event $E_1 = \{\tau_D \wedge \tau_{\text{coupling}} > \lfloor T/\mathbf{c} \rfloor\}$, then $\mathbb{P}(E_1) \geq 1 - A\mathbf{c}$. On E_1 note that

$$\sup_{t\in[0,T]} |\xi_t - \xi_t^*| \leq \left(\frac{T}{\mathbf{c}} + 2\right) D.$$

Next, to pass from the auxiliary model to the multinomial model, let ξ_t^{multi} be the driving measure of the multinomial model, define the event $E_2 = \{\tau_{\neq} > \lfloor T/\mathbf{c} \rfloor\}$, and note that on E_2, $\xi_t^{\text{multi}} = \xi_t^*$ for all $t \in [0,T]$. By Corollary 3.11, $\mathbb{P}(E_2) \geq 1 - AT\mathbf{c}^{-2}D$.

Finally, by Corollary 3.17

$$d_{\text{BW}}(\delta_{\xi_t^{\text{multi}}} \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}) \to 0 \tag{3.99}$$

in probability as $\mathbf{c} \to 0$.

Then by the triangle inequality,

$$\begin{aligned} d_{\text{BW}}(\delta_{\xi_t} \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}) \leq\ & d_{\text{BW}}(\delta_{\xi_t} \otimes m_{[0,T]}, \delta_{\xi_t^*} \otimes m_{[0,T]}) \\ & + d_{\text{BW}}(\delta_{\xi_t^*} \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}), \end{aligned}$$

and on $E_1 \cap E_2$ this is bounded by

$$T\left(\frac{T}{\mathbf{c}} + 2\right) D + d_{\text{BW}}(\delta_{\xi_t^{\text{multi}}} \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}).$$

Hence as $\mathbf{c}^{-1}D = o(1)$, this upper bound tends to zero in probability as $\mathbf{c} \to 0$ and $\mathbb{P}(E_1 \cap E_2) \to 1$, giving us $d_{\text{BW}}(\delta_{\xi_t} \otimes m_{[0,T]}, \bar{\mu}_t \otimes m_{[0,T]}) \xrightarrow{p} 0$ as required. □

Chapter 4

Stability of the ALE for $\eta > 1$

We showed in the previous chapter that for $\eta > 1$ the ALE$(\eta, 0)$ model started from an initial configuration with k slits of a constant size converges as $\mathbf{c} \to 0$ to the Laplacian path model (LPM) started from the same configuration.

A natural question which arises is the stability of the ALE in this regime.

In [4], Carleson and Makarov established several stability results for variants of the Laplacian path model. For example, their Theorem 3 states (for the *chordal* geodesic LPM, where all slits grow from 0 in the domain $\mathbb{C} \setminus [0, \infty)$) that for an initial configuration with two arms, both arms survive if $\eta < 2$, and if $\eta > 2$ this is not always the case. This should be interpreted as increasing stability as η increases: they show that for $\eta > 2$ a one-arm solution is a local attractor for the dynamics, and introducing a small perturbation by adding a sufficiently small second arm does not affect the long-term behaviour: the ratio of the lengths of the arms is $o(1)$ as the cluster grows.

We will also show a phase transition in the stability of the ALE$(0, \eta)$ model as η varies in the "LPM-like" region $\eta > 1$.

To simplify matters, we will consider an initial configuration with only two arms, attached at opposite sides of the circle. Let $a_0, b_0 > 0$, and $K_0 = (1, 1+b_0] \cup [-1-a_0, -1)$. Exploiting the symmetry of this configuration, if Φ_0 is the conformal map $\Delta \to \Delta \setminus K_0$ with $\Phi_0(z) = e^{c_0}z + O(1)$ near ∞, we can decompose Φ_0 in two ways:

$$f_{a_0,\pi} \circ f_{\widetilde{b}_0,0} = \Phi_0 = f_{b_0,0} \circ f_{\widetilde{a}_0,\pi}, \tag{4.1}$$

where $\widetilde{a}_0 = -f^{-1}_{b_0,0}(-1 - a_0) - 1 < a_0$ and $\widetilde{b}_0 = f^{-1}_{a_0,\pi}(1 + b_0) - 1 < b_0$, and $f_{d,\theta}$ attaches a slit of length d at $e^{i\theta}$. This is shown in Figure 4.1.

As we have already shown it has the same limit as the ALE, we will use the multinomial model from Chapter 3. So when we have attached n particles, let $p_n^a = \frac{|\Phi_n''(1)|^{-\eta}}{|\Phi_n''(1)|^{-\eta} + |\Phi_n''(-1)|^{-\eta}}$ and $p_n^b = 1 - p_n^a$. With probability p_n^a, let $\theta_{n+1} = 1$, and with probability p_n^b let $\theta_{n+1} = -1$. Then let $\Phi_{n+1} = \Phi_n \circ f_{\theta_{n+1}}$.

The nth cluster K_n still consists of two straight slits attached at ± 1, so let the length

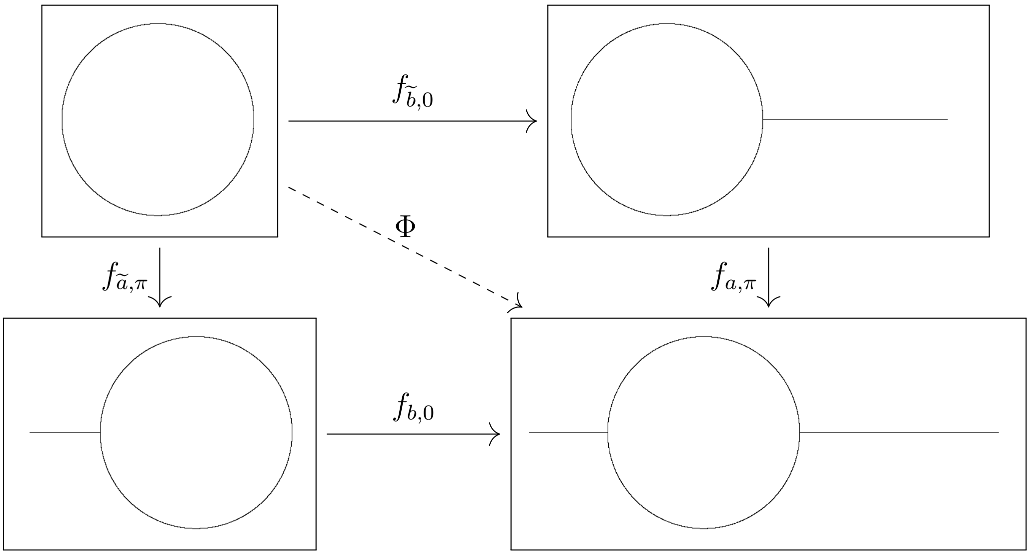

Figure 4.1: The two ways we can compose maps to construct the two-slit cluster.

of the left slit at -1 be $a(n)$, and the length of the right slit at $+1$ be $b(n)$.

Now we can state our stability result, which is that there is a phase transition in the stability of the multinomial model (and hence the ALE) at $\eta = 2$.

Theorem 4.1. *Let Φ_n be the multinomial model as above with $\eta > 1$. Fix a constant T and $b_0 > 0$, and let $a_0 = d(\mathbf{c})$. Then as $\mathbf{c} \to 0$,*

$$a(\lfloor T/\mathbf{c} \rfloor) \to 0 \text{ in probability} \iff \eta \geq 2.$$

Proof. See Proposition 4.2 and Proposition 4.3. □

4.1 The stable region: $\eta \geq 2$

We claim that for $\eta \geq 2$ the multinomial model with one arm is stable, in the sense that if we add an extra small arm then the small arm only grows to size $o(1)$ for small $\mathbf{c}$ in time T.

Proposition 4.2. *Consider the multinomial model as above. Then if $\eta \geq 2$, $a(\lfloor T/\mathbf{c} \rfloor) \to 0$ in probability as $\mathbf{c} \to 0$.*

Proof. Let C_n be the capacity of the map $f_{\widetilde{a}(n),\pi}$, which is given by $e^{C_n} = \frac{(\widetilde{a}(n)+2)^2}{4(\widetilde{a}(n)+1)}$, and satisfies $C_n \asymp \widetilde{a}(n)^2$. Then $a(\lfloor T/\mathbf{c} \rfloor) \to 0$ if and only if $C_{\lfloor T/\mathbf{c} \rfloor} \to 0$.

Using the first decompositions of Φ_n in (4.1), we can find

$$\begin{aligned} \left|\Phi_n''(1)\right| &= \left|f''_{\widetilde{b}(n),0}(1)\right| \times \left|f'_{a(n),\pi}(1+\widetilde{b}(n))\right| \\ &= \frac{1+\widetilde{b}(n)}{2\sqrt{1-e^{-c(\widetilde{b}(n))}}} \frac{(1+b(n))(2+\widetilde{b}(n))}{1+\widetilde{b}(n)} \frac{1}{\sqrt{\widetilde{b}(n)^2 + 4e^{-c(a(n))}(1+\widetilde{b})}}, \end{aligned}$$

where $c(\widetilde{b}(n))$ is the capacity of the map $f_{\widetilde{b}(n),0}$ and $c(a(n))$ is the capacity of the map $f_{a(n),\pi}$. Since $T < \infty$, there is a constant $A > 0$ depending on T and b_0 such that all of $b(n)$, $\widetilde{b}(n)$ and $c(\widetilde{b}(n))$ are bounded below by A^{-1} and above by A. Hence, possibly modifying A, we have

$$A^{-1} \leq |\Phi_n''(1)| \leq A.$$

Using the second decomposition in (4.1) and similar arguments, we have

$$\frac{A^{-1}}{\sqrt{C_n}} \leq |\Phi_n''(-1)| \leq \frac{A}{\sqrt{C_n}}. \tag{4.2}$$

Hence we can find $p_n^a \leq AC_n^{\eta/2}$.

Consider the change in C_n when we add the $(n+1)$th particle. If $\theta_{n+1} = -1$ then $C_{n+1} = C_n + \mathbf{c}$. If $\theta_{n+1} = +1$ then $C_{n+1} \leq C_n$. Hence the process $(C_n)_{n \leq \lfloor T/\mathbf{c} \rfloor}$ is dominated by the Markov process $(\widehat{C}_n)_{n \leq \lfloor T/\mathbf{c} \rfloor}$ with $\widehat{C}_0 = \mathbf{c}$ and

$$\widehat{C}_{n+1} = \begin{cases} \widehat{C}_n + \mathbf{c} & \text{with probability } A\widehat{C}_n^{\eta/2} \wedge 1 \\ \widehat{C}_n & \text{with probability } (1 - A\widehat{C}_n^{\eta/2}) \vee 0 \end{cases}.$$

To see that $\widehat{C}_{\lfloor T/\mathbf{c} \rfloor} \to 0$ in probability as $\mathbf{c} \to 0$, fix $\varepsilon > 0$ sufficiently small so that $A\varepsilon^{\eta/2} < 1$.

Let T_j be time of the $(j-1)$th increase of $\widehat{C}$, i.e. $T_j = \min\{n : \widehat{C}_n = j\mathbf{c}\}$, and let $S_j = T_{j+1} - T_j$ be the jth holding time.

Then $S_j \sim \text{Geometric}\left(A(j\mathbf{c})^{\eta/2}\right)$, and $T_{\lceil \varepsilon/\mathbf{c} \rceil} = \sum_{j=1}^{\lceil \varepsilon/\mathbf{c} \rceil - 1} S_j$, and the random variables S_j are independent. If $\varepsilon < 1$, $\mathbb{P}[\widehat{C}_n \geq \varepsilon]$ is decreasing in η, so we only need to consider $\eta = 2$. Since $\mathbb{E}S_j = A^{-1}(j\mathbf{c})^{-\eta/2}$, we can calculate

$$\mathbb{E}T_{\lceil \varepsilon/\mathbf{c} \rceil} \geq A^{-1}\mathbf{c}^{-1} \log \frac{\varepsilon}{\mathbf{c}}$$

for sufficiently small $\mathbf{c}$, and since $\text{Var}(S_j) \leq A^{-2}(j\mathbf{c})^{-\eta}$, we have $\text{Var}(T_{\lceil \varepsilon/\mathbf{c} \rceil}) \leq A^{-2}\mathbf{c}^{-1}$ and so by Chebyshev's inequality

$$\mathbb{P}[\widehat{C}_{\lfloor T/\mathbf{c} \rfloor} \geq \varepsilon] = \mathbb{P}[T_{\lceil \varepsilon/\mathbf{c} \rceil} \leq \lfloor T/\mathbf{c} \rfloor] \lesssim \frac{1}{(\log \frac{\varepsilon}{\mathbf{c}})^2} \to 0$$

as $\mathbf{c} \to 0$, hence $\widehat{C}_n \to 0$ in probability. □

4.2 The unstable region: $1 < \eta < 2$

In this section we will prove the more difficult half of Theorem 4.1, that when $1 < \eta < 2$, a particle of size $o(1)$ grows to something of order 1 in a finite time T.

As we did above, let C_n be the capacity of a single slit of length $\widetilde{a}(n)$. Then for a fixed T, $a(\lfloor T/\mathbf{c} \rfloor) \to 0$ as $\mathbf{c} \to 0$ if and only if $C_{\lfloor T/\mathbf{c} \rfloor} \to 0$.

Hence, to show that $a(\lfloor T/\mathbf{c} \rfloor) \not\to 0$ as $\mathbf{c} \to 0$, we will show that C_n reaches some positive size within time $\lfloor T/\mathbf{c}\rfloor$. We will do this by defining a stopping time which indicates when C_n has reached this size, and then showing this stopping time is not too large.

Proposition 4.3. *Let $0 < \varepsilon < \varepsilon_0$ for some constant ε_0. Let $\tau_\varepsilon = \inf\{n : C_n \geq \varepsilon\}$. Then if ε_0 is sufficiently small, we have*

$$\liminf_{\mathbf{c}\to 0} \mathbb{P}[\tau_\varepsilon \leq \lfloor T/\mathbf{c}\rfloor] > 0,$$

and moreover

$$\lim_{\varepsilon\to 0} \liminf_{\mathbf{c}\to 0} \mathbb{P}[\tau_\varepsilon \leq \lfloor T/\mathbf{c}\rfloor] = 1.$$

As we did in the stable regime $\eta \geq 2$, we will prove this result by comparing C_n with a Markov chain.

Recall that if $\theta_{n+1} = -1$, then $C_{n+1} = C_n + \mathbf{c}$. To show $C_{\lfloor T/\mathbf{c}\rfloor}$ is not too small, we will need to show C_n does not decrease by too much when $\theta_{n+1} = +1$.

Lemma 4.4. *If $\theta_{n+1} = +1$, then*

$$C_{n+1} - C_n \geq \mathbf{c} - e^{C_n}(e^{\mathbf{c}} - 1).$$

If $C_n < 1/2$ and $\mathbf{c}$ is sufficiently small, this implies

$$C_{n+1} \geq e^{-3\mathbf{c}} C_n.$$

Together with the lower bound in (4.2), this allows us to find a lower bound on C_n using another process.

Definition. Let $(X(n))_{n\geq 1}$ be a Markov chain with $X(0) = \mathbf{c}$ and

$$X(n+1) = \begin{cases} X(n) + \mathbf{c} & \text{with probability } A(X(n))^{\eta/2} \wedge 1 \\ e^{-3\mathbf{c}} X(n) & \text{with probability } (1 - A(X(n))^{\eta/2}) \vee 0. \end{cases}$$

It is easy to see from Lemma 4.4 that $(C_n)_{n\geq 1}$ stochastically dominates $(X(n))_{n\geq 1}$ up until the first time that $C_n \geq 1/2$.

Proof of Lemma 4.4. Consider Loewner's equation on the time interval $(n\mathbf{c}, (n+1)\mathbf{c})$, i.e. the transition from Φ_n to Φ_{n+1}. If $\theta_{n+1} = +1$, then the driving function $\xi_t = +1$ for all $t \in (n\mathbf{c}, (n+1)\mathbf{c})$. Let C_t be the capacity of a slit of length $\widetilde{a}_t$, and C'_t the capacity of the right slit, of length b_t. Since capacity is additive, $C_t + C'_t = c_0 + t$. It is a general fact about slit maps that $e^{C'_t} = \frac{(b_t+2)^2}{4(b_t+1)}$, so $\frac{\mathrm{d}}{\mathrm{d}t} e^{C'_t} = \dot{b}_t \frac{b_t(b_t+2)}{4(b_t+1)^2}$. Using

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(z) = \Phi'_t(z) z \frac{z+1}{z-1}$$

and taking the limit $z \to 1$, we have

$$\dot{b}_t = 2\Phi''_t(1).$$

Using the second decomposition in (4.1) and the fact that $f'_{b_t,0}(1) = 0$, we have

$$\Phi''_t(1) = f''_{b_t,0}(1) \times (f'_{\tilde{a}_t,\pi}(1))^2.$$

Using the explicit forms of both and again using $e^{C'_t} = \frac{(b_t+2)^2}{4(b_t+1)}$, we can find

$$\begin{aligned}(f'_{\tilde{a}_t,\pi}(1))^2 &= e^{C_t/2} = e^{c_0+t}\frac{4(b_t+1)}{(b_t+2)^2},\\ f''_{b_t,0}(1) &= \frac{1+b_t}{2\sqrt{1-e^{-C'_t}}} = \frac{(b_t+1)(b_t+2)}{2b_t},\end{aligned}$$

and so

$$\Phi''_t(1) = 2e^{c_0+t}\frac{(b_t+1)^2}{b_t(b_t+2)}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{C'_t} = e^{c_0+t},$$

and so $e^{C'_{n+1}} - e^{C'_n} = e^{c_0+n}(e^{\mathbf{c}}-1)$. Then $C_{n+1} - C_n = \mathbf{c} - (C'_{n+1} - C'_n)$, and by the mean value theorem

$$C'_{n+1} - C'_n \leq \frac{e^{c_0+n}(e^{\mathbf{c}}-1)}{e^{C_{n'}}} = e^{C_n}(e^{\mathbf{c}}-1),$$

which gives the claimed bound. □

Corollary 4.5. *For $n < \inf\{n' \geq 0 : C_{n'} \geq 1/2\}$, C_n dominates $X(n)$.*

So to show $C_{\lfloor T/\mathbf{c}\rfloor} \not\to 0$ in probability as $\mathbf{c} \to 0$, we will show the same for X. More precisely, we will show that there is a constant $\varepsilon_0 = \varepsilon_0(T,\eta) \in (0,1/2)$ such that $\mathbb{P}[X(\lfloor T/\mathbf{c}\rfloor) \geq \varepsilon_0] \not\to 0$ as $\mathbf{c} \to 0$.

The change of scale from $\mathbf{c}$ to ε_0 makes the process X difficult to analyse, as larger values of $X(n)$ decrease the probability of taking a downward step, but increase the size of the downward steps when they do occur. Rather than examining X itself, we will find it more convenient to keep track of the number of upward steps of X.

Definition. For $n \geq 1$, let $K(n)$ be the number of upward steps taken by the process X. For $k \geq 1$, let T_k be the time of the kth increase of X, i.e. $T_k = \min\{n : K(n) = k\}$, and let S_k be the kth holding time, $S_1 = T_1$, $S_k = T_k - T_{k-1}$ for $k \geq 2$, the time taken for K to jump from $k-1$ to k.

Remark. If $K(n) = k$ then $X(n)$ is minimised among all possible orderings of the n steps if the first k steps are upward and the remaining $n-k$ steps are downward, giving the almost-sure bound $X(n) \geq e^{-3(n-K(n))\mathbf{c}}(K(n)+1)\mathbf{c}$. If $n \leq \lfloor T/\mathbf{c}\rfloor$ then $3(n-K(n))\mathbf{c} \leq T$

almost surely, and so we have an almost-sure bound

$$X(n) \geq e^{-3T}(K(n)+1)\mathbf{c} \tag{4.3}$$

for all $n \leq \lfloor T/\mathbf{c} \rfloor$.

Hence to show $X(\lfloor T/\mathbf{c} \rfloor) \not\to 0$ as $\mathbf{c} \to 0$, it will suffice to show the same for $\mathbf{c}K(\lfloor T/\mathbf{c} \rfloor)$.

Lemma 4.6. *For $k \geq 0$ let $p_k = Ae^{-3\eta T/2}(k+1)^{\eta/2}\mathbf{c}^{\eta/2} \wedge 1$, and let $(\widehat{S}_k)_{k\geq 1}$ be a family of independent random variables with $\widehat{S}_k \sim$ Geometric(p_{k-1}). Then for any k,*

$$\mathbb{P}\left[T_k \leq \lfloor T/\mathbf{c} \rfloor\right] \geq \mathbb{P}\left[\sum_{j=1}^{k} \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor\right]. \tag{4.4}$$

Proof. If $K(n) = k$ then $X(n)$ is minimised if the first k steps are upward and the remaining $n-k$ steps were downward. So, almost surely on the event $\{K(n) = k\}$, we have $X(n) \geq e^{-3(n-k)\mathbf{c}}(k+1)\mathbf{c}$. But if $n \leq \lfloor T/\mathbf{c} \rfloor$, then $3(n-k) \leq T$, and so $X(n) \geq e^{-3T}(k+1)\mathbf{c}$. We can then find a lower-bound on the probability of an upward step,

$$\mathbb{P}(X(n+1) = X(n) + \mathbf{c} \,|\, K(n) = k) \geq p_k \tag{4.5}$$

for all $n \leq \lfloor T/\mathbf{c} \rfloor$. Hence $\mathbb{P}(K(n+1) = k+1 \,|\, K(n) = k) \geq p_k$ for $n \leq \lfloor T/\mathbf{c} \rfloor$.

We can couple X and the family $(\widehat{S}_k)_{k\geq 1}$ as follows: let $(U_{k,s})_{k\geq 0, s\geq 1}$ be independent random variables uniformly distributed on $[0,1)$. Let $\widehat{S}_k = \min\{s \geq 1 : U_{k-1,s} < p_{k-1}\}$. Then $(\widehat{S}_k)_{k\geq 1}$ are independent random variables with $\widehat{S}_k \sim$ Geometric(p_{k-1}).

Informally, X uses the kth row of $(U_{k,s})_{k\geq 0,s\geq 1}$ while $K(n) = k$ to decide whether to jump down or up, starting from the beginning of each row. More formally, for $n \geq 1$, we set $X(n+1) = X(n) + \mathbf{c}$ if $U_{K(n),n-T_{K(n)}+1} < A(X(n))^{\eta/2} \wedge 1$, and otherwise set $X(n+1) = e^{-3\mathbf{c}}X(n)$. Under this construction, $S_k = \min\{s \geq 1 : U_{k-1,s} < A(X(T_{k-1}+s))^{\eta/2} \wedge 1\}$, with $T_0 := 0$.

On the event $\{\sum_{j=1}^{k} \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor\}$, for all $1 \leq j \leq k$, we claim $S_j \leq \widehat{S}_j$ almost surely. We prove this claim by induction. Since on the given event we have $\widehat{S}_1 \leq \lfloor T/\mathbf{c} \rfloor$, we have $A(X(s))^{\eta/2} \wedge 1 \geq p_0$ almost surely for all $1 \leq s \leq \widehat{S}_1$, and so $U_{0,s} < p_0$ implies $U_{0,s} < A(X(s))^{\eta/2} \wedge 1$ if $s \leq \widehat{S}_j$. This implies $S_1 \leq \widehat{S}_1$ almost surely.

Suppose for $k' \leq k$ that $S_j \leq \widehat{S}_j$ for all $1 \leq j \leq k'-1$ almost surely. Then the given event implies that $\sum_{j=1}^{k'} \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor$, which, together with the induction hypothesis, implies that $T_{k'-1} + \widehat{S}_{k'} \leq \sum_{j=1}^{k'} \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor$ almost surely. Hence $T_{k'-1} + s \leq \lfloor T/\mathbf{c} \rfloor$ for all $1 \leq s \leq \widehat{S}_{k'}$, and so $U_{k'-1,s} < p_{k'-1}$ implies $U_{k'-1,s} < A(X(T_{k-1}+s))^{\eta/2} \wedge 1$ for s in this range. As above, this implies $S_{k'} \leq \widehat{S}_{k'}$ almost surely.

Hence the claim holds by induction. It follows that on the event $\{\sum_{j=1}^{k} \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor\}$, we have $T_k = \sum_{j=1}^k S_j \leq \sum_{j=1}^k \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor$ almost surely. Therefore

$$\mathbb{P}\left[\sum_{j=1}^{k} \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor\right] \leq \mathbb{P}\left[T_k \leq \lfloor T/\mathbf{c} \rfloor\right]$$

as required. □

Since $\sum_{j=1}^k \widehat{S}_j$ is a sum of independent geometric random variables, we can use standard methods to show $\mathbb{P}\left[\sum_{j=1}^k \widehat{S}_j \leq \lfloor T/\mathbf{c} \rfloor\right] \not\to 0$ as $\mathbf{c} \to 0$ for an appropriately chosen $k = k(\mathbf{c})$.

Proposition 4.7. *Suppose $1 < \eta < 2$. There exists an $\varepsilon_0 > 0$ depending only on η and T such that if $0 < \varepsilon < \varepsilon_0$ then, for $k_{\varepsilon,\mathbf{c}} = \lceil \varepsilon/\mathbf{c} \rceil$,*

$$\liminf_{\mathbf{c} \to 0} \mathbb{P}\left[\sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \widehat{S}_j \leq \frac{T}{\mathbf{c}}\right] > 0,$$

and moreover

$$\lim_{\varepsilon \to 0} \liminf_{\mathbf{c} \to 0} \mathbb{P}\left[\sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \widehat{S}_j \leq \frac{T}{\mathbf{c}}\right] = 1.$$

Proof. First, note that for any ε,

$$\mathbb{E}\sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \widehat{S}_j = \sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \frac{1}{p_{j-1}},$$

and recall the definition $p_{j-1} = Ae^{-3\eta T/2} j^{\eta/2} \mathbf{c}^{\eta/2} \wedge 1$. For a sufficiently small ε_0 depending only on η and T, if $\varepsilon < \varepsilon_0$ then $p_{j-1} = Ae^{-3\eta T/2} j^{\eta/2} \mathbf{c}^{\eta/2}$ for all $1 \leq j \leq k_{\varepsilon,\mathbf{c}}$. Therefore

$$\begin{aligned}
\mathbb{E}\sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \widehat{S}_j &\leq \frac{e^{3\eta T/2}}{A} \mathbf{c}^{-\eta/2} \sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} j^{-\eta/2} \\
&\leq \frac{e^{3\eta T/2}}{A} \mathbf{c}^{-\eta/2} \int_0^{k_{\varepsilon,\mathbf{c}}} x^{-\eta/2} \, \mathrm{d}x \\
&= \frac{e^{3\eta T/2}}{A(1-\eta/2)} \mathbf{c}^{-\eta/2} k_{\varepsilon,\mathbf{c}}^{1-\eta/2},
\end{aligned}$$

and note that $1 - \eta/2 > 0$. Then $\mathbf{c}^{-\eta/2} k_{\varepsilon,\mathbf{c}}^{1-\eta/2} = \mathbf{c}^{-\eta/2} (\lceil \varepsilon/\mathbf{c} \rceil)^{1-\eta/2} \leq 2\mathbf{c}^{-\eta/2} (\varepsilon/\mathbf{c})^{1-\eta/2}$ for sufficiently small $\mathbf{c}$. This simplifies to give us

$$\mathbb{E}\sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \widehat{S}_j \leq \frac{2e^{3\eta T/2}}{A(1-\eta/2)} \times \frac{\varepsilon^{1-\eta/2}}{\mathbf{c}}.$$

So by Markov's inequality,

$$\mathbb{P}\left[\sum_{j=1}^{k_{\varepsilon,\mathbf{c}}} \widehat{S}_j > \frac{T}{\mathbf{c}}\right] \leq \frac{2e^{3\eta T/2}}{AT(1-\eta/2)} \varepsilon^{1-\eta/2},$$

which is strictly less than 1 if ε_0 is sufficiently small, and tends to 0 as $\varepsilon \to 0$. $\square$

Proof of Proposition 4.3. Using Corollary 4.5 and Lemma 4.6, the result of Proposition 4.7 implies Proposition 4.3. $\square$

Hence we have shown both parts of the phase transition result for stability stated in Theorem 4.1.

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