HOMOTOPY RELATIVE ROTA-BAXTER LIE ALGEBRAS, TRIANGULAR L_∞-BIALGEBRAS AND HIGHER DERIVED BRACKETS

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ABSTRACT. We describe L_∞-algebras governing homotopy relative Rota-Baxter Lie algebras and triangular L_∞-bialgebras, and establish a map between them. Our formulas are based on a functorial approach to Voronov's higher derived brackets construction which is of independent interest.

CONTENTS

1. INTRODUCTION

The subject of this paper is the study of two important algebraic structures: homotopy relative Rota-Baxter (RB) Lie algebras and triangular L_∞-bialgebras and their relationship.

1.1. **Higher derived brackets.** Given an inclusion of differential graded Lie algebras (dglas) $\mathfrak{i}:\mathfrak{g}\to L$ with a direct complement $\mathfrak{h}$ (so that $L\cong\mathfrak{h}\oplus\mathfrak{g}$), a homotopy fiber of $\mathfrak{i}$ is quasi-isomorphic to $\mathfrak{h}[-1]$, the desuspension of $\mathfrak{h}$. Under the additional assumption that the Lie bracket on $\mathfrak{g}$ restricted to $\mathfrak{h}$ vanishes, T. Voronov constructed L_∞-structures on the cocylinder and homotopy fiber of $\mathfrak{i}$ in [37]. The higher products of these L_∞-algebras are called *higher derived brackets*. This construction, subsequently generalized in [2, 7] proved to be extremely useful and showed up in a variety of situations such as the study of simultaneous deformations of two compatible structures in [15, 16], quantization of coisotropic submanifolds of Poisson manifolds [8] and many others. We present a functorial approach to higher derived brackets and identify explicitly Maurer-Cartan (MC) elements in the corresponding L_∞-algebras. This is our first collection of results that are subsequently applied to the two concrete cases of interest (homotopy relative Rota-Baxter Lie algebras and triangular L_∞-bialgebras); it is clear that there are many other situations where they are relevant.

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1.2. **Homotopy relative Rota-Baxter Lie algebras.** The concept of a Rota-Baxter (RB) operator was introduced by G. E. Baxter [5] with motivation from probability theory; it was later put in an abstract context as an operator on an associative algebra, satisfying a certain identity [35]. Recently, it played an important role in the Connes-Kreimer's study of renormalization in quantum field theory [12]. In the context of Lie algebras, RB operators are closely related with the classical Yang-Baxter equation [22] and thus, with the study of integrable systems, see the book [19] for more details.

It is well-known that many homotopy invariant algebraic structures are themselves MC elements in certain dglas; this point of view underlies the modern approach to algebraic deformation theory in characteristic zero (cf. for example the survey [18] explaining this). In such a case we say that an algebraic structure is *governed* by the corresponding dgla. In the previous work [26], the authors introduced the notion of a strong homotopy version of a RB Lie algebra, a so-called homotopy relative RB Lie algebra as an L_∞-algebra together with an appropriate generalization of a RB operator. See the survey article [34] for more details. In the present paper we explicitly find an L_∞-algebra governing this algebraic structure and, using our functorial approach to higher derived brackets, express the homotopy RB identities in a compact, 'synthetic' way.

1.3. **Triangular L_∞-bialgebras.** A triangular Lie bialgebra is a Lie bialgebra $\mathfrak{g}$ whose cobracket $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is the coboundary of an r-matrix, i.e. a skew-symmetric quadratic element satisfying the classical Yang-Baxter equation $[r, r] = 0$ in the Schouten Lie algebra $\Lambda^*\mathfrak{g}$ of $\mathfrak{g}$. Thus, a triangular Lie bialgebra can be viewed as a pair $(\mathfrak{g}, r)$ where $\mathfrak{g}$ is a Lie algebra and $r \in \Lambda^2\mathfrak{g}$ is an r-matrix. Triangular Lie bialgebras play an important role in deformation quantization [11, 32, 33]; in particular, it is known that they can be quantized [14], in the sense that their universal enveloping algebras admit formal deformations as *triangular* Hopf algebras, an important notion that goes beyond the scope of this paper and will not be discussed here.

Just as a Lie algebra has a strong homotopy generalization, called an L_∞-algebra, Lie bialgebras have a strong homotopy version called L_∞-bialgebras, cf. [3, 21]. More recently, a strong homotopy generalization of a triangular Lie bialgebra was introduced in [4], together with an appropriate infinity analogue of an r-matrix, the so-called r_∞-matrix. A triangular L_∞-bialgebra can then be defined as a pair $(\mathfrak{g}, r)$ where $\mathfrak{g}$ is an L_∞-algebra and r is a (suitably defined) r_∞-matrix. One of the motivations behind studying triangular L_∞-bialgebras is the problem of quantizing them and, in particular, making sense of the notion of a *quantum* r_∞-matrix (or, in other words, A_∞ quantum Yang-Baxter equation).

It is well-known that L_∞-bialgebras with a fixed underlying graded vector space are governed by a certain dgla. This point of view, for example, allows interpreting deformation quantization of Lie bialgebras as an L_∞-quasi-isomorphism between dglas (or L_∞-algebras) governing L_∞-bialgebras and strong homotopy associative bialgebras, cf. [32] regarding this approach. By contrast, triangular L_∞-bialgebras are not governed by any dgla but rather, by a certain L_∞-algebra that we explicitly identify. This suggests that the perspective of [32] may be applicable to quantization of triangular L_∞-bialgebras.

Finally, we show that a triangular L_∞-bialgebra gives rise to an associated homotopy relative RB Lie algebra; this correspondence comes from a map between the L_∞-algebras governing the corresponding structures. This substantially strengthens a result in the authors' previous paper [26] where it was proved in the ungraded case and only on the level of cohomology. It is interesting to observe that, while the construction of homotopy RB Lie algebras carries over easily to the associative (or A_∞) context, the corresponding analogue of triangular L_∞-bialgebras (that we can putatively call *triangular A_∞-Hopf algebras*) is rather less straightforward to construct. We hope to return to this problem in a future work.

1.4. **Notation and conventions.** We work in the category DGVect of differential graded (dg) vector spaces over a field $\mathbf{k}$ of characteristic zero; the grading is always cohomological. The *n-fold suspension* of a graded vector space $\mathfrak{g}$ is defined by the convention $\mathfrak{g}[n]^i = \mathfrak{g}^{i+n}$; the differential is therefore a map $d : \mathfrak{g} \to \mathfrak{g}[1]$. Given an element $x \in \mathfrak{g}^k$, the corresponding element in $\mathfrak{g}[n]^{k-n}$ will be denoted by $x[n]$. There is an isomorphism $(\mathfrak{g}[n])^* \cong \mathfrak{g}^*[-n]$.

The category DGVect is symmetric monoidal, and monoids (respectively commutative monoids) in it are called dg algebras (dgas) and commutative dg algebras (cdgas). We also need to work with *pseudocompact* dg vector spaces, or projective limits of finite-dimensional vector spaces; thus a pseudocompact dg vector space V can be written as $V = \varprojlim_\alpha V_\alpha$ for a projective system $\{V_\alpha\}$ of finite dimensional dg vector spaces. The category of pseudocompact dg vector spaces is equivalent to the opposite category of DGVect with anti-equivalence established by the $\mathbf{k}$-linear duality functor. This category also admits a symmetric monoidal structure which we will denote simply by $\otimes$ and monoids (respectively commutative monoids) in it are called pseudocompact dgas (respectively pseudocompact cdgas). We will call local pseudocompact dgas (or cdgas) *complete*, another name for a complete (c)dga is a local pro-Artinian (c)dga. Occasionally we need to consider the tensor product of a pseudocompact dg vector space $V = \varprojlim_\alpha V_\alpha$ and a discrete one U; in this situation we will always write $V \otimes U$ for $\varprojlim_\alpha V_\alpha \otimes U$; such a tensor product is in general neither discrete nor pseudocompact. The category of complete cdgas and continuous multiplicative maps will be denoted by $\mathrm{CDGA}^\wedge_{\mathbf{k}}$. The maximal ideal of a complete cdga A will be denoted by $A_{\geq 1}$; thus $A = \mathbf{k} \oplus A_{\geq 1}$; there is a filtration $A =: A_{\geq 0} \supset A_{\geq 1} \supset \ldots \supset A_{\geq n} \supset \ldots$ where $A_{\geq n}$ stands for the n-th power of the maximal ideal.

We will also need the notion of a dgla; this is a dg vector space $(\mathfrak{g}, d)$ with an anti-symmetric product $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying the graded Jacobi identity. A Maurer-Cartan (MC) element in a dgla $\mathfrak{g}$ is an element $x \in \mathfrak{g}^1$ such that $dx + \frac{1}{2}[x, x] = 0$; the set of MC elements in a dgla $\mathfrak{g}$ will be denoted by $\mathrm{MC}(\mathfrak{g})$.

For a graded Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$ we denote by ad_x the *right* adjoint action of x defined by $\mathrm{ad}_x y = [y, x]$ for $y \in \mathfrak{g}$.

2. L_∞-algebras and their extensions

Let $\mathfrak{g}$ be a graded vector space. We denote by $\mathrm{Der}\, \hat{S}\mathfrak{g}^*[-1]$ the graded Lie algebra consisting of continuous derivations of the complete symmetric algebra $\hat{S}\mathfrak{g}^*[-1]$ and by $\overline{\mathrm{Der}}\, \hat{S}\mathfrak{g}^*[-1]$ its graded Lie subalgebra of derivations vanishing at zero. We will briefly recall the definition of an L_∞-algebra following e.g. [25, 27]. See also [23, 24] for more details and a different point of view.

Definition 2.1. An *L_∞-algebra* structure on $\mathfrak{g}$ is a continuous degree 1 derivation m of the complete cdga $\hat{S}\mathfrak{g}^*[-1]$, such that $m \circ m = 0$ and m has no constant term. The pair $(\mathfrak{g}, m)$ is called an L_∞-algebra, and $(\hat{S}\mathfrak{g}^*[-1], m)$ is called its representing complete cdga. Sometimes we will refer to $\mathfrak{g}$ as an L_∞-algebra leaving m understood.

Remark 2.2. Thus, an L_∞-algebra structure is an MC element m in the graded Lie algebra $\overline{\mathrm{Der}}\, \hat{S}\mathfrak{g}^*[-1]$. If the full graded Lie algebra $\mathrm{Der}\, \hat{S}\mathfrak{g}^*[-1]$ is taken in place of $\overline{\mathrm{Der}}\, \hat{S}\mathfrak{g}^*[-1]$ we get the definition of a *curved* L_∞-algebra. Many of our results hold, with appropriate modifications, for curved L_∞-algebras.

Remark 2.3. Let $(\mathfrak{g}, m)$ be an L_∞-algebra. The element m can be written as a sum

$$m = m_1 + \cdots + m_n + \cdots$$

where m_n is the order n part of m so we can write $m_n : \mathfrak{g}^*[-1] \to \hat{S}^n \mathfrak{g}^*[-1]$. Consider the dual map of m_n, thus we have the degree 1 map $\check{m}_n : S^n\mathfrak{g}[1] \to \mathfrak{g}[1]$ for $n = 1, 2, \cdots$. Namely an L_∞-algebra on a graded vector space $\mathfrak{g}$ is a sequence of linear maps of degree 1:

$$\check{m}_n : S^n\mathfrak{g}[1] \to \mathfrak{g}[1], \quad n \geq 1$$

which satisfy the following relations for any homogeneous elements $x_1, \cdots, x_n \in \mathfrak{g}[1]$:

$$(2.1)\quad \sum_{i=1}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma; x_1, \cdots, x_n) \check{m}_{n-i+1}(\check{m}_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$

Definition 2.4. Let $(\mathfrak{g}, m)$ and $(\mathfrak{h}, m')$ be two L_∞-algebras. An L_∞*-map* f from $\mathfrak{g}$ to $\mathfrak{h}$ is, by definition, a continuous map of degree 0 between the corresponding representing complete cdgas so that $f : \hat{S}\mathfrak{h}^*[-1] \to \hat{S}\mathfrak{g}^*[-1]$.

Remark 2.5. An L_∞-map $f : \mathfrak{g} \to \mathfrak{h}$ can be represented as $f = f_1 + \cdots + f_n + \cdots$ where f_n is the order n part of f so that $f_n : \mathfrak{h}^*[-1] \to \hat{S}^n\mathfrak{g}^*[-1]$. If $f_n = 0$ for $n \neq 1$ then f is called a *strict* L_∞*-map*. Furthermore, dualizing, an L_∞-map $f : \mathfrak{g} \to \mathfrak{h}$ is a sequence of linear maps of degree 0:

$$\check{f}_n : S^n\mathfrak{g}[1] \to \mathfrak{h}[1], \quad n \geq 1$$

such that the following relation holds for any homogeneous elements $x_1, \cdots, x_n \in \mathfrak{g}[1]$:

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma; x_1, \cdots, x_n)\check{f}_{n-i+1}(\check{m}_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) \\ = &\sum_{j=1}^{n} \sum_{k_1+\cdots+k_j=n} \sum_{\sigma \in \mathbb{S}_{(k_1,\cdots,k_j)}} \varepsilon(\sigma; x_1, \cdots, x_n) \\ &\frac{1}{j!}\check{m}'_j(\check{f}_{k_1}(x_{\sigma(1)}, \cdots, x_{\sigma(k_1)}), \cdots, \check{f}_{k_j}(x_{\sigma(k_1+\cdots+k_{j-1}+1)}, \cdots, x_{\sigma(n)})). \end{aligned}$$

Definition 2.6. Let $(\mathfrak{g}, m)$ be an L_∞-algebra. Then an element $\xi \in (\mathfrak{g}[1])^0$ is an MC *element* if it satisfies the MC equation

$$\sum_{i=1}^{\infty} \frac{1}{i!}\check{m}_i(\xi, \cdots, \xi) = 0. \tag{2.2}$$

The set of MC elements in an L_∞-algebra $(\mathfrak{g}, m)$ will be denoted by $\mathrm{MC}(\mathfrak{g})$.

Remark 2.7. The definition of an MC element in an L_∞-algebra assumes that the left hand side of (2.2) converges. If this is the case for all $\xi \in \mathfrak{g}^1$, we will say that $\mathfrak{g}$ contains MC elements. For example, if $m_n = 0$ for $n > 2$ (e.g. $\mathfrak{g}$ is essentially a dgla) then $\mathfrak{g}$ contains MC elements. We now formulate some other sufficient conditions for an L_∞-algebra to contain MC elements.

Definition 2.8. Let $\mathfrak{g}$ be an L_∞-algebra and $\mathcal{F}_\bullet\mathfrak{g}$ be a descending filtration of the graded vector space $\mathfrak{g}$ such that $\mathfrak{g} = \mathcal{F}_1\mathfrak{g} \supset \cdots \supset \mathcal{F}_n\mathfrak{g} \supset \cdots$ and $\mathfrak{g}$ is complete with respect to this filtration, i.e. there is an isomorphism of graded vector spaces $\mathfrak{g} \cong \varprojlim \mathfrak{g}/\mathcal{F}_n\mathfrak{g}$.

(1) If for all $k, n_1, \ldots, n_k \geq 1$ it holds that
$$\check{m}_k(\mathcal{F}_{n_1}\mathfrak{g}[1], \cdots, \mathcal{F}_{n_k}\mathfrak{g}[1]) \subseteq \mathcal{F}_{n_1+\cdots+n_k}\mathfrak{g}[1],$$
we say that the pair $(\mathfrak{g}, \mathcal{F}_\bullet\mathfrak{g})$ is a *filtered* L_∞-algebra.

(2) If there exists $l \geq 0$ such that for all $k > l$ it holds that
$$\check{m}_k(\mathfrak{g}[1], \cdots, \mathfrak{g}[1]) \subseteq \mathcal{F}_k\mathfrak{g}[1],$$
we say that the pair $(\mathfrak{g}, \mathcal{F}_\bullet\mathfrak{g})$ is a *weakly filtered* L_∞-algebra.

Remark 2.9. The definition of a filtered L_∞-algebra belongs to Dolgushev and Rogers [13]. It is the *weak* notion that is relevant to our immediate purposes while Dolgushev-Rogers's notion is given for comparison. Taking in the definition of a filtered L_∞-algebra $n_1 = n_2 = \cdots = n_k = 1$, we obtain the condition of being weakly filtered with $l = 0$, i.e. one notion is indeed stronger than the other. Furthermore, when $\mathfrak{g}$ is finite-dimensional, it is easy to see that the condition of being weakly filtered is equivalent to requiring that the differential m on the representing complete cdga $\hat{S}\mathfrak{g}^*[-1]$ of $\mathfrak{g}$ restricts to the (uncompleted) symmetric algebra $S\mathfrak{g}^*[-1]$ whereas the stronger Dolgushev-Rogers condition means that the cdga $(S\mathfrak{g}^*[-1], m)$ is cofibrant in the model category of cdgas. Thus, any dgla $\mathfrak{g}$ is weakly filtered with the filtration $\mathfrak{g} \supset 0$ and $l = 2$, however not every dgla has a cofibrant representing cdga. For example, the representing cdga of the dgla spanned by two elements $x, [x, x]$ with $d(x) = \frac{1}{2}[x, x]$ is not cofibrant and so, it cannot be filtered in the sense of Dolgushev and Rogers.

Furthermore, it is clear that a weakly filtered L_∞-algebra contains MC elements. Indeed, being weakly filtered is essentially a minimal condition ensuring the existence of MC elements. Later we will see that the L_∞-algebras constructed from admissible V-structures are all weakly filtered (Lemma 3.10) though they are not filtered.

Given a complete cdga A and an L_∞-algebra $(\mathfrak{g}, m)$, consider the tensor product $A_{\geq 1}\otimes\mathfrak{g}$ and extend the L_∞-structure maps $\check{m}_n : \mathfrak{g}[1]^{\otimes n}\to\mathfrak{g}[1]$ by A-linearity to maps

$$\check{m}_n^A : (A_{\geq 1}\otimes\mathfrak{g}[1])^{\otimes n}\to A_{\geq 1}\otimes\mathfrak{g}[1]$$

so that

$$\check{m}_n^A(a_1\otimes x_1,\cdots,a_n\otimes x_n)=\begin{cases} d_A(a_1)\otimes x_1+(-1)^{|a_1|}a_1\otimes\check{m}_1(x_1), & n=1,\\ (-1)^{\sum_{i=1}^n |a_i|(|x_1|+\cdots+|x_{i-1}|+1)}(a_1\cdots a_n)\otimes\check{m}_n(x_1,\cdots,x_n), & n\geq 2,\end{cases}$$

where $a_1,\cdots,a_n\in A_{\geq 1}$ and $x_1,\cdots,x_n\in\mathfrak{g}[1]$. With this, $A_{\geq 1}\otimes\mathfrak{g}$ becomes an L_∞-algebra.

Proposition 2.10. *Given an L_∞-algebra $\mathfrak{g}$ and a complete cdga A, the L_∞-algebra $A_{\geq 1}\otimes\mathfrak{g}$ is filtered.*

Proof. We define a filtration on $A_{\geq 1}\otimes\mathfrak{g}$ as follows:

$$\mathcal{F}_n(A_{\geq 1}\otimes\mathfrak{g})=A_{\geq n}\otimes\mathfrak{g}.$$

The corresponding conditions of Definition 2.8 are trivial to check. □

Given a complete cdga A and an L_∞-map $f:\mathfrak{g}\to\mathfrak{h}$ with components $\check{f}_n$, there is an L_∞-map $f^A : A_{\geq 1}\otimes\mathfrak{g}\to A_{\geq 1}\otimes\mathfrak{h}$ with components $\check{f}_n^A$ defined by the formulas:

$$\check{f}_n^A(a_1\otimes x_1,\cdots,a_n\otimes x_n)=(-1)^{\sum_{i=1}^n |a_i|(|x_1|+\cdots+|x_{i-1}|)}(a_1\cdots a_n)\otimes\check{f}_n(x_1,\cdots,x_n),$$

where $a_1,\cdots,a_n\in A_{\geq 1}$ and $x_1,\cdots,x_n\in\mathfrak{g}[1]$.

Proposition 2.11. *Let $(\mathfrak{g},m)$ be an L_∞-algebra and $f:A\to B$ be a morphism of complete cdgas. Then $f\otimes\mathrm{Id}_\mathfrak{g} : A_{\geq 1}\otimes\mathfrak{g}\to B_{\geq 1}\otimes\mathfrak{g}$ is a strict L_∞-map.*

Proof. Since f is a morphism of complete cdgas, it follows that $f(A_{\geq 1})\subset B_{\geq 1}$. For any $a_1\otimes x_1,\cdots,a_n\otimes x_n\in A_{\geq 1}\otimes\mathfrak{g}[1]$, we have

$$\begin{aligned} & (f\otimes\mathrm{Id}_\mathfrak{g})(\check{m}_n^A(a_1\otimes x_1,\cdots,a_n\otimes x_n))\\ = & \begin{cases} f(d_A(a_1))\otimes x_1+(-1)^{|a_1|}f(a_1)\otimes\check{m}_1(x_1), & n=1,\\ (-1)^{\sum_{i=1}^n |a_i|(|x_1|+\cdots+|x_{i-1}|+1)}f(a_1\cdots a_n)\otimes\check{m}_n(x_1,\cdots,x_n), & n\geq 2,\end{cases}\\ = & \check{m}_n^B((f\otimes\mathrm{Id}_\mathfrak{g})(a_1\otimes x_1),\cdots,(f\otimes\mathrm{Id}_\mathfrak{g})(a_n\otimes x_n)).\end{aligned}$$

Thus, we obtain that $f\otimes\mathrm{Id}_\mathfrak{g}$ is a strict L_∞-map. □

Moreover, given an L_∞-algebra $(\mathfrak{g},m)$, there is a set-valued functor $\mathrm{MC}_\mathfrak{g}$ on the category $\mathrm{CDGA}_\mathbf{k}^\wedge$, which is defined on the set of objects and on the set of morphisms respectively by:

$$\mathrm{MC}_\mathfrak{g}(A) = \mathrm{MC}(\mathfrak{g},A)=\mathrm{MC}(A_{\geq 1}\otimes\mathfrak{g}), \tag{2.3}$$

$$\mathrm{MC}_\mathfrak{g}(A\xrightarrow{f}B) = \mathrm{MC}(\mathfrak{g},A)\xrightarrow{f\otimes\mathrm{Id}_\mathfrak{g}}\mathrm{MC}(\mathfrak{g},B), \tag{2.4}$$

for $A,B\in\mathrm{CDGA}_\mathbf{k}^\wedge$ and $f\in\mathrm{Hom}_{\mathrm{CDGA}_\mathbf{k}^\wedge}(A,B)$. Since $f\otimes\mathrm{Id}_\mathfrak{g}$ is a strict L_∞-map, we conclude that $f\otimes\mathrm{Id}_\mathfrak{g}$ takes elements in $\mathrm{MC}(\mathfrak{g},A)$ to elements in $\mathrm{MC}(\mathfrak{g},B)$. So the functor $\mathrm{MC}_\mathfrak{g}$ is well-defined. Moreover, it is representable.

Theorem 2.12. *Let $(\mathfrak{g},m)$ be an L_∞-algebra. Then the functor $\mathrm{MC}_\mathfrak{g}$ is represented by the complete cdga $(\hat{S}\mathfrak{g}^*[-1],m)$. In other words, for any complete cdga A there is an isomorphism*

$$\mathrm{MC}_\mathfrak{g}(A)\cong\mathrm{Hom}_{\mathrm{CDGA}^\wedge}\left((\hat{S}\mathfrak{g}^*[-1],m),A\right),$$

functorial in A.

Proof. See, e.g. [9, Proposition 2.2 (1)] where this result is proved in the $\mathbb{Z}/2$-graded setting but the arguments are the same in the $\mathbb{Z}$-graded case. ☐

By the Yoneda embedding theorem, the functor $\mathrm{MC}_{\mathfrak{g}}$ determines the L_∞-algebra $\mathfrak{g}$ up to a canonical L_∞-isomorphism. Conversely, given a functor $\mathcal{F}$ on $\mathrm{CDGA}^\wedge_{\mathbf{k}}$, we will often be interested in whether it is isomorphic to $\mathrm{MC}_{\mathfrak{g}}$ for a suitable L_∞-algebra $(\mathfrak{g}, m)$.

Furthermore, for a fixed complete cdga A, the correspondence $\mathfrak{g} \mapsto \mathrm{MC}(\mathfrak{g}, A)$ is functorial with respect to L_∞-maps. Namely, the following result holds.

Proposition 2.13. *Let $f : \mathfrak{g} \to \mathfrak{h}$ with components $(\check{f}_1, \cdots, \check{f}_n, \cdots)$. Then for any complete cdga A it induces a map $f_* : \mathrm{MC}(\mathfrak{g}, A) \to \mathrm{MC}(\mathfrak{h}, A)$ according to the formula*

$$\xi \mapsto f_*(\xi) = \sum_{k=1}^{\infty} \frac{1}{k!} \check{f}_k^A(\xi, \cdots, \xi), \quad \forall \xi \in \mathrm{MC}(\mathfrak{g}, A). \tag{2.5}$$

Proof. Since $f : \hat{S}\mathfrak{h}^*[-1] \to \hat{S}\mathfrak{g}^*[-1]$ is a map of complete cdgas, it clearly induces a map of sets

$$\mathrm{MC}(\mathfrak{g}, A) \cong \mathrm{Hom}_{\mathrm{CDGA}^\wedge}(\hat{S}\mathfrak{g}^*[-1], A) \to \mathrm{Hom}_{\mathrm{CDGA}^\wedge}(\hat{S}\mathfrak{h}^*[-1], A) \cong \mathrm{MC}(\mathfrak{h}, A),$$

and a straightforward inspection shows that it is given in components by the stated formula. ☐

Remark 2.14. Proposition 2.13 is well-known in formal deformation theory. It was formulated explicitly in [20, Section 4.2] for dglas and in [31, Section 2.5.5] in general.

Remark 2.15. All told, the set $\mathrm{MC}(-,-)$ can be viewed as a functor of two arguments. It is natural with respect to L_∞-maps in the first argument and maps of complete cdgas in the second argument. In order to determine an L_∞-map $\mathfrak{g} \to \mathfrak{h}$ it suffices to specify, for any complete cdga A, a map $\mathrm{MC}(\mathfrak{g}, A) \to \mathrm{MC}(\mathfrak{h}, A)$, functorial in A (by Yoneda's lemma). Moreover, it is clear that if $f = (\check{f}_1, \cdots, \check{f}_n, \cdots)$ is such that for any $\xi \in \mathrm{MC}(\mathfrak{g}, A)$ the element $f_*(\xi) \in \mathrm{MC}(\mathfrak{h}, A)$ given by formula (2.5) is an MC element, then f is an L_∞-map. This can sometimes be used for explicit constructions of L_∞-maps out of MC elements.

Recall that given a Lie algebra $\mathfrak{g}$, its *representation* in a vector space V is a Lie algebra map from $\mathfrak{g}$ to $\mathfrak{gl}(V)$. This generalizes in a straightforward way to the L_∞-case [24].

Definition 2.16. Let $(\mathfrak{g}, m)$ be an L_∞-algebra with the representing cdga $(\hat{S}\mathfrak{g}^*[-1], m)$ and V be a graded vector space. Then a *representation* of $\mathfrak{g}$ in V is an L_∞-map f from $\mathfrak{g}$ to $\mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the graded Lie algebra of endomorphisms of V.

Remark 2.17. Let ρ be a representation of an L_∞-algebra $(\mathfrak{g}, m)$ in a graded vector space V. By Definition 2.4, we deduce that $\rho \in \mathrm{Hom}_{\mathrm{CDGA}^\wedge_{\mathbf{k}}}(\hat{S}\mathfrak{gl}(V)^*[-1], \hat{S}\mathfrak{g}^*[-1])$. Then, by Theorem 2.12, ρ can be viewed as an MC element of the dgla $\hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V)$.

Given a complete cdga A, we will need the notion of an A-linear L_∞-algebra; this notion, with a slight modification, was used in [9].

Definition 2.18. Let A be a complete cdga and $\mathfrak{g}$ be a graded vector space. Then an *A-linear L_∞-algebra* structure on $A \otimes \mathfrak{g}$ is an MC element of the dgla $A_{\geq 1} \otimes \overline{\mathrm{Der}}\, \hat{S}\mathfrak{g}^*[-1]$. If B is another complete cdga and $A \to B$ is a map, then an A-linear L_∞-algebra on $A \otimes \mathfrak{g}$ obviously determines a B-linear L_∞-algebra on $B \otimes \mathfrak{g}$ that will be referred to as obtained from $A \otimes \mathfrak{g}$ by change of scalars.

Remark 2.19. Note that an A-linear L_∞-algebra structure on $A \otimes \mathfrak{g}$ is a deformation of the trivial L_∞-algebra structure on $\mathfrak{g}$ with a dg base A. Alternatively, we could have called an A-linear L_∞-algebra structure on $A \otimes \mathfrak{g}$ an MC element of $A \otimes \overline{\mathrm{Der}}\, \hat{S}\mathfrak{g}^*[-1]$. Our notion, slightly more restrictive, means that it is *not* a generalization of an ordinary L_∞-algebra over $\mathbf{k}$ (because $\mathbf{k}_{\geq 1} = 0$).

We now have the following result.

Proposition 2.20. *Let $\mathfrak{g}$ be a graded vector space and $\mathcal{F}_{\mathrm{HL}}$ be the functor associating to a complete cdga A the set of A-linear L_∞-algebra structures on $A \otimes \mathfrak{g}$. Then $\mathcal{F}_{\mathrm{HL}}$ is represented by the complete cdga $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big)$.*

Proof. By Definition 2.18, we deduce that $\mathcal{F}_{\mathrm{HL}}(A) = \mathrm{MC}(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1], A)$. Moreover, by Theorem 2.12, it follows that $\mathcal{F}_{\mathrm{HL}}(A) \cong \mathrm{Hom}_{\mathrm{CDGA}^\wedge_{\mathbf{k}}}(\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big), A)$ and we are done. □

Corollary 2.21. *Given a graded vector space $\mathfrak{g}$, there exists a 'universal' $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big)$-linear L_∞-algebra structure on $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big) \otimes \mathfrak{g}$ such that any other A-linear L_∞-structure on $A \otimes \mathfrak{g}$ is obtained by change of scalars from a unique map of complete cdgas $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big) \to A$.*

Proof. By Proposition 2.20, there is a natural isomorphism α as following:

$$\alpha : \mathrm{Hom}_{\mathrm{CDGA}^\wedge_{\mathbf{k}}}(\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big), \cdot) \to \mathcal{F}_{\mathrm{HL}}.$$

By the universality of the representing cdga, we deduce that for any complete cdga A and any $y \in \mathcal{F}_{\mathrm{HL}}(A)$, there exists a unique map of complete cdgas $f : \hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big) \to A$ such that $y = \mathcal{F}_{\mathrm{HL}}(f)(\mathcal{X})$. Here $\mathcal{X} \in \mathcal{F}_{\mathrm{HL}}(\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big))$ is given by

$$\mathcal{X} = \alpha(\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big))(\mathrm{Id}).$$

This completes the proof. □

Remark 2.22. The universal L_∞-algebra $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big) \otimes \mathfrak{g}$ can be viewed as a universal deformation of the trivial L_∞-algebra on $\mathfrak{g}$; its universal properties persist upon passing to the homotopy category of complete cdgas cf. [18, 27] regarding this approach to deformation theory. In the present paper we will not be concerned with this aspect of the theory.

Suppose that we have an A-linear L_∞-algebra structure m_A on $A \otimes \mathfrak{g}$, where $A = (\hat{S}U^*[-1], m_U)$ is itself the representing complete cdga of an L_∞-algebra U. The element m_A is an A-linear derivation of

$$A \otimes \hat{S}\mathfrak{g}^*[-1] = \hat{S}U^*[-1] \otimes \hat{S}\mathfrak{g}^*[-1] \cong \hat{S}(U \oplus \mathfrak{g})^*[-1].$$

Forgetting that m_A is A-linear, we can view it as an MC element in $\overline{\mathrm{Der}}\,\hat{S}(U \oplus \mathfrak{g})^*[-1]$, i.e. an L_∞-structure on $U \oplus \mathfrak{g}$ with the representing cdga $(\hat{S}(U \oplus \mathfrak{g})^*[-1], m_A)$. Moreover, $A = (\hat{S}U^*[-1], m_U)$ is a sub-cdga of $(\hat{S}(U \oplus \mathfrak{g})^*[-1], m_A)$ and we can form an L_∞-structure on $\mathfrak{g}$ with the representing cdga $\hat{S}\mathfrak{g}^*[-1]$. All told, we have a sequence of L_∞-algebras and strict L_∞-maps:

$$U \to U \oplus \mathfrak{g} \to \mathfrak{g}. \tag{2.6}$$

This leads naturally to the notion of an *extension* of L_∞-algebras, cf. [10, 27, 29].

Definition 2.23. The sequence of L_∞-algebras and strict L_∞-maps of the form (2.6) is called an *extension* of $\mathfrak{g}$ by U.

Example 2.24. The universal $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big)$-linear L_∞-algebra

$$\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big) \otimes \mathfrak{g}$$

gives rise to an L_∞-extension

$$\mathfrak{g} \to \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{g} \to \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1].$$

Here $\mathfrak{g}$ is given the trivial L_∞-structure and the L_∞-structure on $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{g}$ can be read off the $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big)$-linear L_∞-structure on $\hat{S}\big(\big(\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]\big)^*[-1]\big) \otimes \mathfrak{g}$. Specifically (cf. [10, Example 3.8]), for $\phi[1] \in \mathrm{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}) \subset \mathrm{Der}\,\hat{S}\mathfrak{g}^*[-1]$ and $v_1, \cdots, v_n \in \mathfrak{g}$ we have

$$\check{m}_n(\phi[1], v_1, \cdots, v_n) = \phi(v_1, \cdots, v_n).$$

Note that the L_∞-algebra $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{g}$ represents the functor associating to a complete cdga A an A-linear L_∞-algebra on $A \otimes \mathfrak{g}$ together with an MC element in it. Later on, we will consider a higher version of this construction with an MC element replaced with the so-called r_∞*-matrix*, cf. Definition 5.7 below associated to an L_∞-algebra and see how that leads to triangular L_∞-bialgebras.

Our next task is to describe a dgla controlling the pair of an L_∞-algebra and its representation in a graded vector space. We arrange the set of such pairs as a functor on complete cdgas.

Definition 2.25. An L_∞Rep *pair* consists of an L_∞-algebra $(\mathfrak{g}, m)$ and a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$ in a graded vector space V.

There is a natural action of the graded Lie algebra $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]$ on $\hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V)$ given by

$$[\phi, x \otimes y] = \phi(x) \otimes y, \quad \forall \phi \in \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1],\ x \otimes y \in \hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V).$$

Let $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V) = \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \ltimes (\hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V))$ be the corresponding semidirect product graded Lie algebra. Note that an L_∞Rep pair $((\mathfrak{g}, m), \rho)$ is nothing but an MC element in the graded Lie algebra $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$.

Definition 2.26. Let A be a complete cdga. Then an A*-linear* L_∞Rep *pair* with the underlying graded vector spaces $\mathfrak{g}$ and V is an element in $\mathrm{MC}(\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V), A)$.

Let $\mathcal{F}_{L_\infty\mathrm{Rep}}$ be the functor associating to a complete cdga A the set of A-linear L_∞Rep pairs with the underlying graded vector spaces $\mathfrak{g}$ and V. Then we have the following result.

Proposition 2.27. *The functor* $\mathcal{F}_{L_\infty\mathrm{Rep}}$ *is represented by the complete cdga* $\hat{S}\mathcal{L}^*_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)[-1]$.

Proof. Let $(m, x \otimes y)$ be a degree 1 element in $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$. We have

$$[(m, x \otimes y), (m, x \otimes y)] = ([m, m], 2m(x) \otimes y + x^2 \otimes [y, y]).$$

Thus, $(m, x \otimes y)$ is an MC element of $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$ if and only if

$$[m, m] = 0, \quad m(x) \otimes y + \frac{1}{2}x^2 \otimes [y, y] = 0.$$

By Remark 2.2 and Remark 2.17, we deduce that $(m, x \otimes y)$ is an MC element of $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$ if and only if $(\mathfrak{g}, m)$ is an L_∞-algebra and $x \otimes y$ is a representation of the L_∞-algebra $(\mathfrak{g}, m)$ in a graded vector space V. Thus, we obtain that $\mathcal{F}_{L_\infty\mathrm{Rep}}(A) = \mathrm{MC}(\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V), A)$. Moreover, by Theorem 2.12, $\mathcal{F}_{L_\infty\mathrm{Rep}}$ is represented by the complete cdga $\hat{S}\mathcal{L}^*_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)[-1]$. ☐

Remark 2.28. There is an inclusion of graded Lie algebras $i : \mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V) \subset \overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1]$ where $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \subset \overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1]$ in an obvious way and $\hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V) \subset \overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1]$ via the isomorphism $\hat{S}^n\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V) \cong \mathrm{Hom}(V^*[-1], \hat{S}^n\mathfrak{g}^*[-1] \otimes V^*[-1])$; a simple check shows that the Lie bracket is preserved under this inclusion. By the proof of Proposition 2.27, the structure of an L_∞-algebra on $\mathfrak{g}$ together with a representation of $\mathfrak{g}$ in a graded vector space V is equivalent to an MC element $(m, x \otimes y) \in \mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$ and thus, $i(m, x \otimes y) \in \mathrm{MC}\left(\overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1]\right)$. The graded Lie algebra $\overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1]$ supplied with the differential $d = [i(m, x \otimes y), \cdot]$ can be identified with the Chevalley-Eilenberg complex of the L_∞-algebra $(\mathfrak{g}, m)$ with coefficients in the representation V.

3. Voronov's higher derived brackets and MC elements

In this section we review Voronov's constructions [37] of higher derived brackets from the point of view of MC elements and L_∞-extensions. Related results are contained in [15].

Definition 3.1. Let L be a dgla, $x \in \mathrm{MC}(L)$ and $h \in L^0$. The *right gauge transformation* by h on x is given by the following formula:

$$x \mapsto x * h := x + \sum_{n=1}^{\infty} \frac{1}{n!}(\mathrm{ad}_h^n(x) + \mathrm{ad}_h^{n-1}(d(h))).$$

Remark 3.2. In the above definition it is assumed that the $e^{\mathrm{ad}_h} := \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_h)^n}{n!}$ is a well-defined operator on L. This is the case, e.g. when L is a pronilpotent dgla. The right gauge transformation $x * h$ agrees with the ordinary (left) gauge action on x by the element $-h$ given in [17, Section 1.3]. It was also proved in [17, Section 1.3] that these transformation preserve Maurer-Cartan elements, i.e. $x * h \in \mathrm{MC}(L)$.

Definition 3.3. We say that a dgla L is supplied with a V*-structure* if there is given an operator $P : L \to L$ with $P^2 = P$ (so that P is a projector) such that

(1) The subspace $\ker P$ is a sub-dgla of L,
(2) The image of P (denoted hereafter by $\mathfrak{h}$) is an abelian graded Lie subalgebra of L.

From now on, we will denote a V-structure by a pair (L, P). If, for a given V-structure, there is the following filtration on L:

$$L \supset P[L, \mathfrak{h}] \supset P[[L, \mathfrak{h}], \mathfrak{h}] \supset \cdots \supset P[\cdots [[L, \underbrace{\mathfrak{h}], \mathfrak{h}] \cdots, \mathfrak{h}]}_{n-1} \supset \cdots$$

which is complete (e.g. if the adjoint action of $\mathfrak{h}$ on L is pronilpotent), the corresponding V-structure is called *admissible.*

Remark 3.4. Note that for an admissible V-structure on L, the operator $e^{\mathrm{ad}_h} : L \to L$ makes sense for any $h \in \mathfrak{h}^0$.

Associated to an admissible V-structure on a dgla L is the notion of a VMC functor.

Definition 3.5. Let (L, P) be an admissible V-structure. A VMC *element* associated to it is a pair (x, h) where $x \in \mathrm{MC}(L)$ and $h \in \mathfrak{h}^0$ such that $P(x * h) = 0$. The set of VMC elements associated to (L, P) will be denoted by $\mathrm{VMC}(L)$ (leaving P understood).

Just as the ordinary MC set in a dgla, the VMC set can be made into a functor of two arguments. Let L be a dgla with a V-structure and A be a complete cdga. Then $A_{\geq 1} \otimes L$ is pronilpotent and has an induced admissible V-structure given by the projector $\mathrm{id} \otimes P$.

Definition 3.6. The VMC *set* of L with values in A is defined as $\mathrm{VMC}(A_{\geq 1} \otimes L)$ and denoted by $\mathrm{VMC}(L, A)$.

It is clear that $\mathrm{VMC}(-,-)$ is a functor in the second variable. We will show that it is represented by a complete cdga that is the representing cdga of a certain L_∞-algebra. Recall the following result by Voronov [37].

Theorem 3.7. *Let (L, P) be a V-structure. Then the graded vector space $L \oplus \mathfrak{h}[-1]$ is an L_∞-algebra where*

$$\begin{aligned}
\check{m}_1(x[1], h) &= (-d(x)[1], P(x + d(h))), \\
\check{m}_2(x[1], y[1]) &= (-1)^{|x|}[x, y][1], \\
\check{m}_k(x[1], h_1, h_2, \cdots, h_{k-1}) &= P[\cdots [[x, h_1], h_2] \cdots, h_{k-1}], \quad k \geq 2, \\
\check{m}_k(h_1, h_2, \cdots, h_k) &= P[\cdots [d(h_1), h_2] \cdots, h_k], \quad k \geq 2.
\end{aligned}$$

Here $h, h_1, \cdots, h_k$ are homogeneous elements of $\mathfrak{h}$ and x, y are homogeneous elements of L. All the other L_∞-algebra products that are not obtained from the ones written above by permutations of arguments, will vanish. □

Remark 3.8. The L_∞-products on $L \oplus \mathfrak{h}[-1]$ restrict to $\mathfrak{h}[-1]$ making the latter into an L_∞-algebra given by

(3.1) $$\check{m}_k(h_1, \cdots, h_k) = P[\cdots [d(h_1), h_2] \cdots, h_k], \quad \text{for homogeneous } h_1, \cdots, h_k \in \mathfrak{h}.$$

It is included into an L_∞-extension

(3.2) $$\mathfrak{h}[-1] \to \mathfrak{h}[-1] \oplus L \to L,$$

where the second arrow is the natural projection and L is viewed as a dgla (hence an L_∞-algebra).

For later use we record the following obvious observation.

Remark 3.9. Let L' be a graded Lie subalgebra of L that satisfies $d(L') \subset L'$. Then $L' \oplus \mathfrak{h}[-1]$ is an L_∞-subalgebra of the above L_∞-algebra $(L \oplus \mathfrak{h}[-1], \{\check{m}_k\}_{k=1}^{\infty})$.

The following key lemma interprets a VMC element of an admissible V-structure as an MC element.

Lemma 3.10. *Let L be a dgla with an admissible V-structure. Then*

1. *The L_∞-algebra $L \oplus \mathfrak{h}[-1]$ is weakly filtered (so it contains* MC *elements).*
2. *The following isomorphism of sets holds:*

$$\mathrm{MC}(L \oplus \mathfrak{h}[-1]) \cong \mathrm{VMC}(L).$$

Proof. For (1) consider the following filtration on $L \oplus \mathfrak{h}[-1]$:

$$\mathcal{F}_1 := L \oplus \mathfrak{h}[-1] \supset \mathcal{F}_2 := P([L, \mathfrak{h}])[-1] \supset \cdots \supset \mathcal{F}_n := P([\cdots [[L, \underbrace{\mathfrak{h}], \mathfrak{h}] \cdots, \mathfrak{h}}_{n-1}])[-1] \supset \cdots .$$

By the definition of an admissible V-structure, the above filtration is complete; moreover it clearly satisfies condition (2) of Definition 2.8 with $l = 3$.

For (2), let $(x[1], h) \in \mathrm{MC}(L \oplus \mathfrak{h}[-1])$ so that $x \in L^1$ and $h \in \mathfrak{h}^0$. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n!} \check{m}_n\Big((x[1], h), \ldots, (x[1], h)\Big) \\
= \;& \check{m}_1(x[1], h) + \frac{1}{2}\check{m}_2\Big((x[1], h), (x[1], h)\Big) + \sum_{n=3}^{\infty} \frac{1}{n!} \check{m}_n\Big((x[1], h), \ldots, (x[1], h)\Big) \\
= \;& \Big(-d(x)[1], P(x + d(h))\Big) + \Big(-\frac{1}{2}[x, x][1], P\,\mathrm{ad}_h(x) + \frac{1}{2} P\,\mathrm{ad}_h(d(h))\Big) \\
& + \Big(0, P \sum_{n=3}^{\infty} \frac{1}{(n-1)!}\,\mathrm{ad}_h^{n-1}(x)\Big) + \Big(0, P \sum_{n=3}^{\infty} \frac{1}{n!}\,\mathrm{ad}_h^{n-1}(d(h))\Big)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
d(x) + \frac{1}{2}[x, x] &= 0, \\
P(x) + \sum_{n=1}^{\infty} \frac{1}{n!} P\Big((\mathrm{ad}_h)^n(x) + \mathrm{ad}_h^{n-1}(dh)\Big) &= 0.
\end{aligned}$$

Therefore, $x \in \mathrm{MC}(L)$ and $P(x * h) = 0$. All told, we obtain that $(x, h) \in \mathrm{VMC}(L)$. The same calculation performed in the reverse order, shows that, conversely, if $(x, h) \in \mathrm{VMC}(L)$, then $(x[1], h) \in \mathrm{MC}(L \oplus \mathfrak{h}[-1])$. □

Furthermore, the following result holds.

Proposition 3.11. *Let L be a dgla with a V-structure. Then the functor* $\mathrm{VMC}(L, -)$ *is representable. The complete cdga representing it is the representing cdga of the L_∞-algebra $L \oplus \mathfrak{h}[-1]$ constructed in Theorem 3.7.*

Proof. Let A be a complete cdga. Then $(A_{\geq 1} \otimes L, \mathrm{id} \otimes P)$ is a pronilpotent dgla with a V-structure (which is then admissible) and by Lemma 3.10 we have:

$$\begin{aligned}
\mathrm{VMC}(L, A) &= \mathrm{VMC}(A_{\geq 1} \otimes L) \\
&\cong \mathrm{MC}(A_{\geq 1} \otimes (L \oplus \mathfrak{h}[-1])) \\
&= \mathrm{MC}(L \oplus \mathfrak{h}[-1], A)
\end{aligned}$$

as required. □

It is clear that the L_∞-algebra $L\oplus\mathfrak{h}[-1]$ defined above, is quasi-isomorphic to the dgla $\ker P$ and thus, there is an L_∞-map $j : L\oplus\mathfrak{h}[-1]\to\ker P$ that is homotopy inverse to the inclusion $\ker P \hookrightarrow L \hookrightarrow L\oplus\mathfrak{h}[-1]$. It follows from (3.2) that there is a homotopy fibre sequence of L_∞-algebras and maps:

$$\mathfrak{h}[-1]\to\ker P\to L. \tag{3.3}$$

Denote by $i : \mathfrak{h}[-1]\to\ker P$ the corresponding L_∞-map in the above homotopy fibre sequence. If the given V-structure is admissible, there is an induced map $\mathrm{MC}(i) : \mathrm{MC}(\mathfrak{h}[-1])\to\mathrm{MC}(\ker P)$. We will find this map explicitly.

Proposition 3.12. *Let L be a pronilpotent dgla with a V-structure. Then:*

(1) *The L_∞-map $j : L\oplus\mathfrak{h}[-1]\to\ker P$ induces the map*

$$\mathrm{MC}(j) : \mathrm{MC}(L\oplus\mathfrak{h}[-1])\to\mathrm{MC}(\ker P),$$

so that for $(x[1], h)\in\mathrm{MC}(L\oplus\mathfrak{h}[-1])$ it holds that

$$\mathrm{MC}(j)(x[1], h) = x * h.$$

(2) *The L_∞-map $i : \mathfrak{h}[-1]\to\ker P$ induces the map*

$$\mathrm{MC}(i) : \mathrm{MC}(\mathfrak{h}[-1])\to\mathrm{MC}(\ker P),$$

so that for $h\in\mathrm{MC}(\mathfrak{h}[-1])$ it holds that

$$\mathrm{MC}(i)(h) = 0 * h := \sum_{n=1}^{\infty}\frac{1}{n!}\operatorname{ad}_h^{n-1} d(h).$$

Proof. It is clear that (2) follows from (1). For (1) let $(x[1], h)\in\mathrm{MC}(L\oplus\mathfrak{h}[-1])$, i.e. a VMC element where $x\in L^1$ and $h\in\mathfrak{h}^0$. Then the element $x * h$ is an MC element in L (gauge equivalent to x) and it belongs to $\ker P$ by definition of a VMC element. To finish the proof it suffices to observe that this morphism splits the canonical map $\mathrm{MC}(\ker P)\to\mathrm{MC}(L\oplus\mathfrak{h}[-1])$ induced by the inclusion $\ker P\hookrightarrow L\oplus\mathfrak{h}[-1]$. ☐

Corollary 3.13. *Let L be as in Proposition 3.12. Then:*

(1) *The L_∞-map $j : L\oplus\mathfrak{h}[-1]\to\ker P$ has the form:*

$$\begin{aligned}\check{j}_1(x[1], h) =&(\mathrm{id}-P)(x)[1],\\ \check{j}_k(x[1], h_1,\cdots, h_{k-1}) =&(\mathrm{id}-P)[\cdots[[x, h_1], h_2]\cdots, h_{k-1}][1],\\ \check{j}_k(h_1,\dots, h_k) =&(\mathrm{id}-P)[\cdots[d(h_1), h_2]\cdots, h_k][1],\quad k\geq 2.\end{aligned}$$

(2) *The L_∞-map $i : \mathfrak{h}[-1]\to\ker P$ from (3.3) has the following form:*

$$\check{i}_k(h_1,\dots, h_k) =(\mathrm{id}-P)[\cdots[d(h_1), h_2]\cdots, h_k][1],\quad k\geq 2.$$

Proof. It suffices to show that the map $f := (f_1,\dots, f_k\dots)$ as defined above induces the correct map on MC elements with values in arbitrary complete cdgas (i.e. it agrees with the formula of Proposition 3.12 (2)). This is a straightforward calculation similar to that of Lemma 3.10. ☐

Example 3.14. Here is one of the simplest yet nontrivial examples of the above construction. Let (V, m) be an L_∞-algebra. Set $L := \operatorname{Der}\hat{S}V^*[-1]$ and P be the natural projection $\operatorname{Der}\hat{S}V^*[-1]\to V[1]$ and $\Delta := m\in\overline{\operatorname{Der}}\,\hat{S}V^*[-1]$. Note that $(L, [\Delta, -])$ is nothing but the Chevalley-Eilenberg complex of the L_∞-algebra V whereas the dg Lie subalgebra $\overline{\operatorname{Der}}\,\hat{S}V^*[-1]$ is its truncation. Then $\mathfrak{h}\cong V[1]$ and the L_∞-structure on V coming from higher derived brackets is just the original L_∞-structure. This result, obtained by a different method, as well as an explicit L_∞-map $V\to\overline{\operatorname{Der}}\,\hat{S}V^*[-1]$ is contained in [10].

Remark 3.15. The L_∞-algebra $\mathfrak{h}[-1]$ is a model for a homotopy fiber of the inclusion of dglas $\ker P\to L$. Our methods can be extended to the case when $\mathfrak{h} = \mathrm{Im}(P)$ is not abelian, but we will not pursue this route as our applications are only concerned with the abelian case. The non-abelian case was treated in [2] using different methods.

The functor VMC constructed above is analogous to the functor Def_χ associated to an arbitrary morphism (not necessarily an inclusion) of dglas $\chi : M \to L$ introduced in [28] in the sense that both can be interpreted as MC functors of homotopy fibers of the corresponding maps.

4. Homotopy relative Rota-Baxter Lie algebras

Let $(\mathfrak{g}, m)$ be an L_∞-algebra and V be a representation of $\mathfrak{g}$, i.e. a graded vector space together with an L_∞-map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Recall that this data can be represented as an MC element

$$\Phi := (m, \rho) \in \mathcal{L}_{L_\infty \mathrm{Rep}}(\mathfrak{g}, V) := \overline{\mathrm{Der}}\, \hat{S}\mathfrak{g}^*[-1] \ltimes (\hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V)).$$

Via the natural inclusion

$$\mathcal{L}_{L_\infty \mathrm{Rep}}(\mathfrak{g}, V) \subset \overline{\mathrm{Der}}\, \hat{S}(\mathfrak{g} \oplus V)^*[-1]$$

the element Φ can be regarded as an MC element of the graded Lie algebra $\overline{\mathrm{Der}}\, \hat{S}(\mathfrak{g} \oplus V)^*[-1]$, cf. Remark 2.28. We set

$$\mathfrak{h} = \mathrm{Hom}(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}V^*[-1]).$$

Then $\mathfrak{h}$ is the abelian Lie subalgebra in $\overline{\mathrm{Der}}\, \hat{S}(\mathfrak{g} \oplus V)^*[-1]$ consisting of continuous derivations vanishing on $V^*[-1]$ and mapping $\mathfrak{g}^*[-1]$ to $\hat{S}_{\geq 1}V^*[-1]$. Note that the natural direct complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\overline{\mathrm{Der}}\, \hat{S}(\mathfrak{g} \oplus V)^*[-1]$ is closed with respect to the commutator bracket and contains Φ. We will define P to be the projector in $\overline{\mathrm{Der}}\, \hat{S}(\mathfrak{g} \oplus V)^*[-1]$ onto $\mathfrak{h}$. Thus, we obtain a V-structure as following:

Proposition 4.1. *Let $(\mathfrak{g}, m)$ be an L_∞-algebra and (V, ρ) be a representation of $\mathfrak{g}$. Then the projection onto the subspace $\mathfrak{h}$ determines an admissible V-structure on the dgla $\overline{\mathrm{Der}}\, \hat{S}(\mathfrak{g} \oplus V)^*[-1]$ supplied with the commutator bracket and the differential $d = [\Phi, \cdot]$.*

Consequently, $(\mathfrak{h}[-1], \{\check{m}_k\}_{k=1}^\infty)$ is a weakly filtered L_∞-algebra, where higher products $\check{m}_k$ are given by formulas (3.1).

Proof. We only need to show that the given V-structure is admissible. To that end let $T \in \mathfrak{h} \cong \mathrm{Hom}(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}V^*[-1])$. The element T can be written as a sum

$$T = T_1 + T_2 + \cdots , \tag{4.1}$$

where T_n is the order n part of T so we can write $T_n : \mathfrak{g}^*[-1] \to \hat{S}^n V^*[-1]$. The element T is a derivation of $\hat{S}(\mathfrak{g} \oplus V)^*[-1]$ but we will view it merely as an endomorphism of $\hat{S}(\mathfrak{g} \oplus V)^*[-1]$. It is clear that T_1 is a nilpotent endomorphism (in fact $T_1^2 = 0$) and it follows that the adjoint action of T is pronilpotent. □

Remark 4.2. In fact, the L_∞-algebra $\mathfrak{h}$ is even *filtered*, cf. [26, Remark 5.9]

We can now give a compact definition of a homotopy relative RB operator.

Definition 4.3. A *homotopy relative* RB *operator* on an L_∞-algebra $(\mathfrak{g}, m)$ supplied with a representation (V, ρ) is an element $T \in \mathfrak{h}^0$ such that $P(e^{\mathrm{ad}_T}(m + \rho)) = 0$ (i.e. T is an MC element of the L_∞-algebra $(\mathfrak{h}[-1], \{\check{m}_k\}_{k=1}^\infty)$).

Remark 4.4. Let T be a homotopy relative RB operator compatible with an L_∞-algebra $(\mathfrak{g}, m)$ and its representation (V, ρ) and $T_n, n = 1, 2, \cdots$ be its components as in (4.1). Consider the dual map of T_n, thus we have the degree 0 map $\check{T}_n : \hat{S}^n V[1] \to \mathfrak{g}[1]$ for $n = 1, 2, \cdots$. Moreover, $\check{T}_n$ is graded symmetric and satisfy the homotopy relative RB relation. See [26] for more details about homotopy relative RB operators.

Remark 4.5. By the well-known formula for the exponential of the adjoint representation, the homotopy relative RB condition $P(e^{\mathrm{ad}_T}(m + \rho)) = 0$ can be rewritten as $P(e^{-T}(m + \rho)e^T) = 0$.

Remark 4.6. In the classical case for $(\mathfrak{g}, [-, -]_\mathfrak{g})$ is an ordinary Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on V. For any $T \in \mathrm{Hom}(V, \mathfrak{g})$, we have

$$e^{\mathrm{ad}_T}(m + \rho) = e^{-T} \circ (m + \rho) \circ (e^T \otimes e^T).$$

For all $u, v \in V$, we have

$$\begin{aligned}(e^{-T} \circ (m+\rho) \circ (e^T \otimes e^T))(u,v) &= (1-T)(m+\rho)(u+Tu, v+Tv)\\ &= (1-T)([Tu,Tv]_{\mathfrak{g}} + \rho(Tu)v - \rho(Tv)u)\\ &= \rho(Tu)v - \rho(Tv)u + [Tu,Tv]_{\mathfrak{g}} - T(\rho(Tu)v - \rho(Tv)u).\end{aligned}$$

Therefore, the condition $P(e^{\mathrm{ad}_T}(m+\rho)) = 0$ will give us

$$[Tu, Tv]_{\mathfrak{g}} = T(\rho(Tu)v - \rho(Tv)u),$$

which implies that T is a *relative* RB *operator* (also called an $\mathcal{O}$-operator) on the Lie algebra $(\mathfrak{g}, m)$ with respect to the representation (V, ρ). See [1] for more details.

Definition 4.7. Let $\mathfrak{g}$ and V be graded vector spaces. A *homotopy relative* RB *Lie algebra* on $\mathfrak{g}$ and V is a triple (m, ρ, T), where m is an L_∞-algebra structure on $\mathfrak{g}$, ρ is a representation of $(\mathfrak{g}, m)$ on V and T is a homotopy relative RB operator compatible the L_∞-algebra $(\mathfrak{g}, m)$ and its representation (V, ρ).

Let $\mathfrak{g}$ and V be graded vector spaces. Since $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$ is a graded Lie subalgebra of $\overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1]$, we have the following corollary:

Corollary 4.8. *With the above notation, $(\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V) \oplus \mathfrak{h}[-1], \{\check{m}_i\}_{i=1}^\infty)$ is an L_∞-algebra, where $\check{m}_i$ are given by*

$$\begin{aligned}\check{m}_2(Q[1], Q'[1]) &= (-1)^{|Q|}[Q, Q'][1],\\ \check{m}_k(Q[1], \theta_1, \cdots, \theta_{k-1}) &= P[\cdots[Q, \theta_1], \cdots, \theta_{k-1}],\end{aligned}$$

for homogeneous elements $\theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$, homogeneous elements $Q, Q' \in \mathcal{L}$, and all the other higher products vanish, unless they can be obtained from the ones listed by permutations of arguments. We denote the L_∞-algebra $(\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V) \oplus \mathfrak{h}[-1], \{\check{m}_i\}_{i=1}^\infty)$ by $\mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V)$.

Proof. It follows from Theorem 3.7, Remark 3.9 and Proposition 4.1. □

Now we show that the L_∞-algebra $\mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V)$ governs homotopy relative RB Lie algebras.

Theorem 4.9. *Let $\mathfrak{g}$ and V be graded vector spaces, $m \in \overline{\mathrm{Der}}^1\,\hat{S}\mathfrak{g}^*[-1]$, $\rho \in (\hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(V))^1$ and $T \in \mathrm{Hom}^0(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}V^*[-1])$. Then the triple (m, ρ, T) is a homotopy relative* RB *Lie algebra structure on $\mathfrak{g}$ and V if and only if $(\Phi[1], T)$ is an* MC *element of the L_∞-algebra $\mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V)$ given in Corollary 4.8, where $\Phi = (m, \rho)$.*

Proof. Let $(\Phi[1], T)$ be an MC element of the L_∞-algebra $\mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V)$. Using the inclusion of L_∞-algebras

$$(\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V) \oplus \mathfrak{h}[-1], \{\check{m}_i\}_{i=1}^\infty) \subset \overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1] \oplus \mathfrak{h}[-1], \{\check{m}_k\}_{k=1}^\infty),$$

the pair $(\Phi[1], T)$ can be viewed as an MC element in $\overline{\mathrm{Der}}\,\hat{S}(\mathfrak{g} \oplus V)^*[-1] \oplus \mathfrak{h}[-1], \{\check{m}_k\}_{k=1}^\infty)$. This implies, by Lemma 3.10, that $P(e^{-T}(m+\rho)e^T) = 0$, i.e. T is a homotopy relative RB operator on $\mathfrak{g}$ with respect to the representation ρ. Conversely, given a homotopy relative RB operator T, the same argument traced in the reverse order, shows that the pair $(\Phi[1], T)$ is an MC element of the L_∞-algebra $\mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V)$. □

Remark 4.10. It is clear that the L_∞-algebra $\mathcal{L}_{\mathsf{HRB}}$ is included in an L_∞-extension

$$\mathfrak{h}[-1] \to \mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V) \to \mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, V)$$

where $\mathfrak{h}[-1]$ has the L_∞-structure which is given in Proposition 4.1.

Definition 4.11. Let A be a complete cdga, $\mathfrak{g}$ and V be graded vector spaces. Then an *A-linear homotopy relative* RB *Lie algebra* structure on $A \otimes \mathfrak{g}$ and $A \otimes V$ is a pair $(\Phi_A[1], T_A)$, which is an MC element of the L_∞-algebra $A_{\geq 1} \otimes \mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V)$, where $\Phi_A = (m_A, \rho_A)$.

Let $\mathcal{F}_{\mathrm{HRB}}$ be the functor associating to a complete cdga A the set of A-linear homotopy relative RB Lie algebra structures on $A \otimes \mathfrak{g}$ and $A \otimes V$. Then we have the following result.

Theorem 4.12. *The functor $\mathcal{F}_{\mathrm{HRB}}$ is represented by the complete cdga $\hat{S}\mathcal{L}^*_{\mathsf{HRB}}(\mathfrak{g}, V)[-1]$.*

Proof. By the definition of an A-linear homotopy relative RB Lie algebra and Theorem 4.9, we deduce that $\mathcal{F}_{\mathrm{HRB}}(A) = \mathrm{MC}(\mathcal{L}_{\mathsf{HRB}}(\mathfrak{g}, V), A)$. Therefore, by Theorem 2.12, we obtain that the functor $\mathcal{F}_{\mathrm{HRB}}$ is represented by the complete cdga $\hat{S}\mathcal{L}^*_{\mathsf{HRB}}(\mathfrak{g}, V)[-1]$. □

5. Shifted Poisson algebras, r_∞-matrices and triangular L_∞-bialgebras

In this section, we describe a certain doubling construction for shifted Poisson algebras. Our exposition is a straightforward modification of the corresponding $\mathbb{Z}/2$-graded construction in [6]. The construction of higher derived brackets applied to the shifted Poisson algebra, gives rise to the higher Schouten Lie algebra $(\hat{S}\mathfrak{g}[1-n])[n-1], \{\check{m}_k\}_{k=1}^{\infty})$. We define an n-shifted r_∞-matrix to be a Maurer-Cartan element of the L_∞-algebra $(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$, and show that r_∞-matrices give rise to triangular L_∞-bialgebras. It is possible to define an r_∞-matrix using the L_∞-algebra $(\hat{S}\mathfrak{g}[1-n])[n-1]$ as opposed to the smaller L_∞-algebra $(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$; however since the former does not contain MC elements, it is then necessary to introduce an auxiliary complete cdga. Such a definition in the 0-shifted case and taking $\mathbf{k}[[\lambda]]$ as a complete cdga, was given in [4] using a different method.

5.1. Doubling construction for shifted Poisson algebras.

Let $\mathfrak{g}$ be a graded vector space, here and later on assumed to be finite-dimensional. Then the n-shifted cotangent bundle $T^*[n]\mathfrak{g}[1]$ is a graded symplectic manifold equipped with a degree n symplectic structure. Consequently, the pairing of degree $-n$

$$\mathfrak{g}^*[-1] \otimes \mathfrak{g}[1-n] \to \mathbf{k}$$

determines an n-shifted Poisson algebra structure on the complete pseudocompact algebra $\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])$. The corresponding Poisson bracket $\{-,-\}$ can be viewed as a graded Lie algebra structure on $(\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$, in other words, it is a degree zero map

$$\{-,-\} : (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n] \otimes (\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n] \to (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$$

satisfying the graded Jacobi identity.

Later on we will also need to work with the graded vector space

$$\hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]) := \hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \hat{S}_{\geq 1}(\mathfrak{g}[1-n]),$$

which is clearly a (shifted) Poisson subalgebra of $\hat{S}\mathfrak{g}^*[-1] \otimes \hat{S}(\mathfrak{g}[1-n])$.

Consider the graded Lie algebra $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]$ with respect to the commutator bracket. Since any derivation is determined by its value on $\mathfrak{g}^*[-1]$, we can identify $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]$ with the graded vector space

$$\mathrm{Hom}(\mathfrak{g}^*[-1], \hat{S}\mathfrak{g}^*[-1]) \cong \hat{S}\mathfrak{g}^*[-1] \otimes \mathfrak{g}[1].$$

Definition 5.1. The *n-shifted double* is the map

$$D_n : \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \cong \hat{S}\mathfrak{g}^*[-1] \otimes \mathfrak{g}[1] \to (\hat{S}\mathfrak{g}^*[-1])[n] \otimes \hat{S}_{\geq 1}(\mathfrak{g}[1-n]) \subset (\hat{S}_{\geq 1}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$$

given by the formula

$$\hat{S}\mathfrak{g}^*[-1] \otimes \mathfrak{g}[1] \supset f \otimes w \mapsto (-1)^{n|f|} f[n] \otimes (w[-n]) \in (\hat{S}\mathfrak{g}^*[-1])[n] \otimes \hat{S}_{\geq 1}(\mathfrak{g}[1-n]).$$

Then we have the following result.

Proposition 5.2. *The map D_n is a map of graded Lie algebras.*

Proof. The proof of [6, Theorem 3.2] carries over with only notational modifications. The cases when n is odd or even are slightly different. □

5.2. **r_∞-matrices and triangular L_∞-bialgebras.** Let $(\mathfrak{g}, m)$ be an L_∞-algebra. In this subsection we define, using the doubling construction and Voronov's higher derived brackets, the higher shifted Schouten Lie algebra and an n-shifted r_∞-matrix for $\mathfrak{g}$. This leads naturally to the notion of an n-shifted triangular L_∞-bialgebra. Ordinary (or 0-shifted in our terminology) L_∞-bialgebras were introduced in [21] and its shifted version was considered in [3, 30].

Consider the graded Lie algebra

$$(\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$$

where the graded Lie bracket was defined in the previous subsection. The L_∞-algebra structure m is an MC element in $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]$ and so, $D_n(m)$ is an MC element in the graded Lie algebra $(\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$ making it a dgla. Note that $(\hat{S}\mathfrak{g}[1-n])[n] \subset (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$ is an abelian Lie subalgebra whose direct complement is a dg Lie subalgebra. Then, Voronov's derived brackets construction implies the following result.

Theorem 5.3. *Let $(\mathfrak{g}, m)$ be an L_∞-algebra. Then the differential graded Lie algebra*

$$\Big(L := (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n], \{-,-\}, d = \{D_n(m), \cdot\}\Big)$$

is endowed with an admissible V-structure $P : L \to L$, which is the projection to

$$\mathfrak{h} := (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n].$$

Consequently, $(\mathfrak{h}[-1], \{\check{m}_k\}_{k=1}^\infty)$ is an L_∞-algebra, where $\check{m}_k$ is given by (3.1).

Moreover, there is an L_∞-algebra structure on $L \oplus \mathfrak{h}[-1]$ given by

$$(5.1)\qquad \begin{cases} \check{m}_1(x[1], h) &= (-d(x)[1], P(x + d(h))), \\ \check{m}_2(x[1], y[1]) &= (-1)^{|x|}\{x, y\}[1], \\ \check{m}_k(x[1], h_1, \cdots, h_{k-1}) &= P\{\cdots\{\{x, h_1\}, h_2\}, \cdots, h_{k-1}\}, \quad k \geq 2, \\ \check{m}_k(h_1, \cdots, h_{k-1}, h_k) &= P\{\cdots\{d(h_1), h_2\}, \cdots, h_k\}, \quad k \geq 2, \end{cases}$$

for all $x, y \in L$ and $h, h_1, \ldots, h_k \in \mathfrak{h}$. The remaining L_∞-products vanish, unless they are obtained from those above by permutations of arguments. Moreover there exists the following L_∞-extension:

$$(5.2)\qquad \mathfrak{h}[-1] \to (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n] \oplus (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1] \to L.$$

Proof. Arguing as in the proof of Proposition 4.1, let $h \in \mathfrak{h}$; then $h = h_1 + h_2 + \cdots$ where $h_n \in (S_{\geq n+1}\mathfrak{g}[1-n])[n-1]$. In particular, the element $h_1 \in (S_{\geq 2}\mathfrak{g}[1-n])[n-1]$ viewed as an endomorphism of L is nilpotent (even has square zero) and we conclude that the specified V-structure is admissible. The stated formulas for the L_∞-products follow from Theorem 3.7. □

Remark 5.4. If we choose $(\hat{S}\mathfrak{g}[1-n])[n]$ as the abelian subalgebra in L, then the corresponding V-structure will not be admissible. Nevertheless, the derived brackets construction is applicable and $(\hat{S}\mathfrak{g}[1-n])[n-1]$ becomes an L_∞-algebra where formulas (5.1) still hold. Clearly, $\mathfrak{h} = (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$ is an L_∞-subalgebra in $(\hat{S}\mathfrak{g}[1-n])[n-1]$.

Definition 5.5. The L_∞-algebra $(\hat{S}\mathfrak{g}[1-n])[n-1], \{\check{m}_k\}_{k=1}^\infty)$ is called the *higher Schouten Lie algebra* of the L_∞-algebra $(\mathfrak{g}, m)$.

Remark 5.6. It is well known that for an ordinary Lie algebra $\mathfrak{g}$, the exterior algebra $\Lambda(\mathfrak{g}) := \oplus_{k=0}^\infty \wedge^{k+1} \mathfrak{g}$ is a graded Lie algebra ($\wedge^{k+1}\mathfrak{g}$ is of degree k), which is also called the Schouten Lie algebra. Now the L_∞-algebra structure on $(\hat{S}\mathfrak{g}[1-n])[n-1]$ given in Theorem 5.3 only contains m_2. If, furthermore, $n = 2$, it reduces to the Schouten Lie algebra, i.e.

$$(\hat{S}\mathfrak{g}[-1])[1] = \Lambda(\mathfrak{g}).$$

If, on the other hand, $n = 1$, we obtain a graded Lie algebra structure on $\hat{S}\mathfrak{g}$; this is the well-known Poisson bracket on the completion of $\mathrm{gr}\, U\mathfrak{g}$, the associated graded to the universal enveloping algebra of $\mathfrak{g}$ (which is isomorphic, by the Poincare-Birkhoff-Witt theorem, to $S\mathfrak{g}$).

We can now give the definition of an n-shifted r_∞-matrix.

Definition 5.7. Let $(\mathfrak{g}, m)$ be an L_∞-algebra.

(1) An *n-shifted r_∞-matrix for* $\mathfrak{g}$ is a degree 0 element $r \in (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$ such that $P(e^{\mathrm{ad}_r} D_n(m)) = 0$, i.e. r is an MC element in the L_∞-algebra $(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$.

(2) Let A be a complete cdga. Then an *n-shifted r_∞-matrix for* $\mathfrak{g}$ *with values in* A is a degree 0 element $r_A \in A_{\geq 1} \otimes (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$ such that $P(e^{\mathrm{ad}_{r_A}} D_n(m)) = 0$, i.e. r_A is an MC element in the L_∞-algebra $(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$ with values in A.

Remark 5.8. As far as we know, n-shifted r_∞-matrices for an odd number n have not been considered, even in the case of ordinary (graded) Lie algebras. Our definition is closer to the classical notion of an r-matrix [36] and specializes to this notion in the case when the formal power series $r \in (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$ contains only the quadratic term.

We will now define a shifted L_∞-bialgebra. Note that the notion of an $L_\infty[l,k]$-bialgebra was introduced in [3, Definition 2.5], which is a degree $3+l+k$ element $t_{p,q} \in S^{p,q}(V^*[-1-l] \oplus V[-1-k])$, for $p, q \in \{1, 2, \ldots, \}$, such that $\langle\!\langle \sum_{p=1,q=1}^\infty t_{p,q}, \sum_{p=1,q=1}^\infty t_{p,q} \rangle\!\rangle = 0$, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is a graded version of the big bracket on the symmetric algebra. The following definition is a special case of an $L_\infty[l,k]$-bialgebra for $l = 0$ and $k = n-2$.

Definition 5.9. The structure of an *n-shifted L_∞-bialgebra* on a graded vector space $\mathfrak{g}$ is an MC element in $\hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$, i.e. an element $h \in \hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$ of degree 1 such that $\{h, h\} = 0$.

Remark 5.10. The projections

$$\hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n] \to \mathfrak{g}[1] \otimes \hat{S}\mathfrak{g}^*[-1] \cong \mathrm{Der}\, \hat{S}\mathfrak{g}^*[-1]$$

and

$$\hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n] \to (\hat{S}\mathfrak{g}[1-n])[n] \otimes \mathfrak{g}^*[-1] \cong \mathrm{Der}\, \hat{S}(\mathfrak{g}[1-n]) \cong \mathrm{Der}\, \hat{S}(\mathfrak{g}^*[n-2])^*[-1]$$

are graded Lie algebra maps. It follows that an n-shifted L_∞-bialgebra structure on $\mathfrak{g}$ determines

- an L_∞-algebra structure on $\mathfrak{g}$ and
- an L_∞-algebra structure on $\mathfrak{g}^*[n-2]$ (equivalently, an L_∞-*coalgebra* structure on $\mathfrak{g}[2-n]$).

Additionally, there are appropriate compatibility conditions between these structures. This explains the terminology 'shifted L_∞-bialgebra'.

Note also that the Poisson algebra $\hat{S}_{\geq 1}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$ (as opposed to its Poisson subalgebra $\hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$) leads to the notion of an L_∞-quasi-bialgebra.

Classically, an r-matrix in a Lie algebra $\mathfrak{g}$ gives rise to a Lie bialgebra structure on $\mathfrak{g}$, called *triangular*. We will now formulate a higher version of this result.

Theorem 5.11. *Let* $(\mathfrak{g}, m)$ *be an* L_∞*-algebra and* $r \in (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$ *be an* n*-shifted* r_∞*-matrix. Define* $r(m) \in (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$ *by*

$$r(m) := e^{\mathrm{ad}_r} D_n(m). \tag{5.3}$$

Then $r(m)$ *is an* MC *element in* $(\hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$ *and determines the structure of an* n*-shifted* L_∞*-bialgebra on* $\mathfrak{g}$.

Proof. Note that $D_n(m)$ is an MC element in the graded Lie algebra $(\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$ as the image of the MC element m under the gla map D_n, cf. Proposition 5.2. It follows that $e^{\mathrm{ad}_r} D_n(m)$ is likewise an MC element as obtained from $D_n(m)$ by a gauge transformation. Next, since r is an r_∞-matrix, we have $P(e^{\mathrm{ad}_r} D_n(m)) = 0$ and it follows that $e^{\mathrm{ad}_r} D_n(m) \in \hat{S}'(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$ and we are done. □

Definition 5.12. Given an L_∞-algebra $(\mathfrak{g}, m)$ and an n-shifted r_∞-matrix r, the n-shifted L_∞-bialgebra $r(m)$ defined by formula (5.3) will be called a *triangular n-shifted L_∞-bialgebra.* By abuse of terminology, we will also refer to the triple $(\mathfrak{g}, m, r)$ as a *triangular n-shifted L_∞-bialgebra.*

Remark 5.13. (1) The statement of Theorem 5.11 for $n = 2$ and in the context of $\mathbf{k}[[\lambda]]$-linear r_∞-matrices is essentially the main result of [4], cf. Theorem 2 and the discussion following it in op. cit.

(2) For an ordinary Lie algebra, Theorem 5.11 reduces to the standard construction of a triangular Lie bialgebra.

Let $\mathfrak{g}$ be a vector space. Recall that $\mathfrak{h} = (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$. Since $\overline{\operatorname{Der}}\,\hat{S}\mathfrak{g}^*[-1]$ is a graded Lie subalgebra of $(\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]$, we have the following corollary:

Corollary 5.14. *With the above notation,* $(\overline{\operatorname{Der}}\,\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{h}[-1], \{\check{m}_i\}_{i=1}^{\infty})$ *is an* L_∞*-algebra, where* $\check{m}_i$ *are given by*

$$\begin{aligned} \check{m}_2(Q[1], Q'[1]) &= (-1)^{|Q|}[Q, Q'][1], \\ \check{m}_k(Q[1], \theta_1, \cdots, \theta_{k-1}) &= P[\cdots[Q, \theta_1], \cdots, \theta_{k-1}], \end{aligned}$$

for homogeneous elements $\theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$*, homogeneous elements* $Q, Q' \in \overline{\operatorname{Der}}\,\hat{S}\mathfrak{g}^*[-1]$*, and all the other possible combinations vanish.*

Proof. It follows from Remark 3.9 and Theorem 5.3. □

We denote the above L_∞-algebra $(\overline{\operatorname{Der}}\,\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{h}[-1], \{\check{m}_i\}_{i=1}^{\infty})$ by $\mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})$. We will see that it governs triangular L_∞-bialgebras; the proof is formally analogous to that of Theorem 4.9.

Theorem 5.15. *Let* $\mathfrak{g}$ *be a graded vector space,* $m \in \overline{\operatorname{Der}}^1\,\hat{S}\mathfrak{g}^*[-1]$ *and* $r \in (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$*. Then the pair* $(m[1], r)$ *is a triangular* L_∞*-bialgebra structure on* $\mathfrak{g}$ *if and only if* $(m[1], r)$ *is an* MC *element of the* L_∞*-algebra* $\mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})$ *given in Corollary 5.14.*

Proof. Let $(m[1], r)$ be an MC element of the L_∞-algebra $\mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})$. Using the inclusion of L_∞-algebras

$$\overline{\operatorname{Der}}\,\hat{S}\mathfrak{g}^*[-1] \oplus \mathfrak{h}[-1] \subset (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n] \oplus \mathfrak{h}[-1]$$

the pair $(m[1], r)$ can be viewed as an MC element in $(\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n] \oplus \mathfrak{h}[-1]$. This implies, by Lemma 3.10, that $P(e^{\operatorname{ad}_r} D_n(m)) = 0$, i.e. r is an r_∞-matrix for $\mathfrak{g}$. Conversely, given an r_∞-matrix r, the same argument traced in the reverse order, shows that the pair $(m[1], r)$ is an MC element of the L_∞-algebra $\mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})$. □

Remark 5.16. It follows from the above that there is a commutative diagram where horizontal arrows are L_∞-extensions and vertical arrows are strict L_∞-maps or graded Lie algebra maps:

$$\begin{array}{ccccc} (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1] & \longrightarrow & \mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g}) & \longrightarrow & \overline{\operatorname{Der}}\,\hat{S}\mathfrak{g}^*[-1] \\ \| & & \downarrow & & \downarrow{\scriptstyle D_n} \\ (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1] & \longrightarrow & K & \longrightarrow & (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n]. \end{array}$$

and $K = (\hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]))[n] \oplus (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$.

Definition 5.17. Let A be a complete cdga and $\mathfrak{g}$ be a graded vector space. Then an *A-linear triangular n-shifted L_∞-bialgebra* structure on $A \otimes \mathfrak{g}$ is a pair $(m_A[1], r_A)$, which is an MC element in the L_∞-algebra $A_{\geq 1} \otimes \mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})$.

Let $\mathfrak{g}$ be a graded vector space and $\mathcal{F}_{\mathsf{TL}_\infty\mathrm{Bi}}$ be the functor associating to a complete cdga A the set of A-linear triangular n-shifted L_∞-bialgebra structures on $A \otimes \mathfrak{g}$. Then we have the following result.

Theorem 5.18. *The functor* $\mathcal{F}_{\mathsf{TL}_\infty\mathrm{Bi}}$ *is represented by the complete cdga* $\hat{S}\mathcal{L}^*_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})[-1]$.

Proof. By the definition of an A-linear triangular n-shifted L_∞-bialgebra and Theorem 5.15, we deduce that $\mathcal{F}_{\mathsf{TL}_\infty\mathrm{Bi}}(A) = \mathrm{MC}(\mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g}), A)$. Therefore, by Theorem 2.12, we obtain that the functor $\mathcal{F}_{\mathsf{TL}_\infty\mathrm{Bi}}$ is represented by the complete cdga $\hat{S}\mathcal{L}^*_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g})[-1]$. □

6. From triangular L_∞-bialgebras to homotopy relative Rota-Baxter Lie algebras

In this section, we establish the relation between the L_∞-algebra governing triangular L_∞-bialgebras and the L_∞-algebra governing homotopy relative RB Lie algebras.

Consider again the graded Lie algebra

$$\hat{S}\mathfrak{g}^*[-1] \otimes (\hat{S}\mathfrak{g}[1-n])[n] \cong \hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$$

together with its Poisson bracket $\{-,-\}$. An element in it (a hamiltonian) gives rise to a hamiltonian (formal) vector field on $\mathfrak{g}[1] \oplus \mathfrak{g}^*[n-1]$, and Poisson brackets correspond to commutators of vector fields. Thus, we have an inclusion of graded Lie algebras

$$H : \hat{S}_{\geq 1}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n] \hookrightarrow \operatorname{Der} \hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n]) \cong \operatorname{Der} \hat{S}\big((\mathfrak{g} \oplus \mathfrak{g}^*[n-2])^*[-1]\big).$$

MC elements in $\hat{S}_{\geq 2}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$ correspond, under this map, to elements in the graded Lie algebra

$$\overline{\operatorname{Der}}\, \hat{S}\big((\mathfrak{g} \oplus \mathfrak{g}^*[n-2])^*[-1]\big) \subset \operatorname{Der} \hat{S}\big((\mathfrak{g} \oplus \mathfrak{g}^*[n-2])^*[-1]\big)$$

that are cyclic L_∞-algebra structures on $\mathfrak{g} \oplus \mathfrak{g}^*[n-2]$.

What is important for us, is that the image of the abelian Lie subalgebra

$$(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n] \subset \hat{S}_{\geq 2}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n]$$

under the map H is inside the abelian Lie subalgebra in $\overline{\operatorname{Der}}\, \hat{S}\big((\mathfrak{g} \oplus \mathfrak{g}^*[n-2])^*[-1]\big)$ having the form

$$\operatorname{Hom}(\mathfrak{g}^*[-1], \hat{S}(\mathfrak{g}^*[n-1])^*) \cong \operatorname{Hom}(\mathfrak{g}^*[-1], \hat{S}\mathfrak{g}[1-n]) \cong \operatorname{Hom}(\hat{S}\mathfrak{g}^*[n-1], \mathfrak{g}[1]).$$

This is precisely the abelian subalgebra that was used to construct (using derived brackets) the L_∞-algebra controlling homotopy relative RB Lie algebras given in Theorem 4.9. Therefore, for any $r \in (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$, we have $H(r) \in \operatorname{Hom}(\hat{S}_{\geq 1}\mathfrak{g}^*[n-1], \mathfrak{g}[1])$.

Theorem 6.1. *Let $r \in (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n]$ be an n-shifted r_∞-matrix in an L_∞-algebra $(\mathfrak{g}, m)$, i.e. the pair $(\mathfrak{g}, m, r)$ is an n-shifted triangular L_∞-bialgebra. Then $H(r)$ is a homotopy relative* RB *operator on $\mathfrak{g}$ with respect to the coadjoint representation* ad^* *of $\mathfrak{g}$ on $\mathfrak{g}^*[n-2]$. The resulting correspondence*

$$(m, r) \mapsto (m, \operatorname{ad}^*, H(r))$$

between triangular L_∞-bialgebras and homotopy relative RB *Lie algebras is realized as a strict L_∞-map*

$$\mathcal{L}_{\mathrm{TL}_\infty \mathrm{Bi}}(\mathfrak{g}) \to \mathcal{L}_{\mathrm{HRB}}(\mathfrak{g}, \mathfrak{g}^*[n-2])$$

between L_∞-algebras governing the corresponding structures. □

Proof. Taking into account Remark 5.16, we obtain the following commutative diagram whose rows are L_∞-extensions:

$$\begin{array}{ccccc}
(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1] & \longrightarrow & \mathcal{L}_{\mathrm{TL}_\infty \mathrm{Bi}}(\mathfrak{g}) & \longrightarrow & \overline{\operatorname{Der}}\, \hat{S}\mathfrak{g}^*[-1] \\
\Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle D_n} \\
(\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1] & \longrightarrow & K & \longrightarrow & \hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n] \\
\Big\downarrow{\scriptstyle H} & & \Big\downarrow & & \Big\downarrow{\scriptstyle H} \\
\operatorname{Hom}(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}\mathfrak{g}[1-n]) & \longrightarrow & X & \longrightarrow & \overline{\operatorname{Der}}\, \hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])
\end{array}$$

where $K = \hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])[n] \oplus (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1]$ and X is the L_∞-extension obtained from the graded Lie algebra $\overline{\operatorname{Der}}\, \hat{S}(\mathfrak{g}^*[-1] \oplus \mathfrak{g}[1-n])$ and its abelian subalgebra $\operatorname{Hom}(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}\mathfrak{g}[1-n])$ by the derived brackets construction. The vertical maps in the above diagram are dgla maps or strict L_∞-maps.

Recall from Remark 4.10 that there is an L_∞-extension

$$\mathrm{Hom}(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}\mathfrak{g}[1-n]) \to \mathcal{L}_{\mathrm{HRB}}(\mathfrak{g}, \mathfrak{g}^*[n-2]) \to \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \ltimes \hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(\mathfrak{g}^*)$$

where the L_∞-algebra $\mathcal{L}_{\mathrm{HRB}}(\mathfrak{g}, \mathfrak{g}^*[n-2])$ controls homotopy relative RB Lie algebras on the graded vector space $\mathfrak{g}$ with a representation on $\mathfrak{g}^*[n-2]$, cf. Theorem 4.9 (note that $\mathfrak{gl}(\mathfrak{g}^*[n-2])$ is canonically isomorphic to $\mathfrak{gl}(\mathfrak{g}^*)$).

Noting that the image of $\overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1]$ inside $\overline{\mathrm{Der}}\,\hat{S}((\mathfrak{g}\oplus\mathfrak{g}^*[n-2])^*[-1])$ under the map $H\circ D_n$ is contained in the Lie subalgebra $\mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, \mathfrak{g}^*[n-2]) := \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \ltimes \hat{S}_{\geq 1}\mathfrak{g}^*[-1] \otimes \mathfrak{gl}(\mathfrak{g}^*)$, we obtain the following commutative diagram where, as above, the rows are L_∞-extensions and vertical arrows are graded Lie algebra maps or strict L_∞-maps:

$$\begin{array}{ccccc} (\hat{S}_{\geq 2}\mathfrak{g}[1-n])[n-1] & \longrightarrow & \mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g}) & \longrightarrow & \overline{\mathrm{Der}}\,\hat{S}\mathfrak{g}^*[-1] \\ \downarrow H & & \downarrow & & \downarrow H\circ D_n \\ \mathrm{Hom}(\mathfrak{g}^*[-1], \hat{S}_{\geq 1}\mathfrak{g}[1-n]) & \longrightarrow & \mathcal{L}_{\mathrm{HRB}}(\mathfrak{g}, \mathfrak{g}^*[n-2]) & \longrightarrow & \mathcal{L}_{L_\infty\mathrm{Rep}}(\mathfrak{g}, \mathfrak{g}^*[n-2]). \end{array} \tag{6.1}$$

Since L_∞-maps preserve Maurer-Cartan elements, it follows that if (m, r) is an n-shifted triangular L_∞-bialgebra structure, then $(m, \mathrm{ad}^*, H(r))$ is a homotopy relative RB Lie algebra. □

Remark 6.2. When we consider the ungraded case (i.e. when $\mathfrak{g}$ is an ordinary Lie algebra), the strict L_∞-map $\mathcal{L}_{\mathsf{TL}_\infty\mathrm{Bi}}(\mathfrak{g}) \to \mathcal{L}_{\mathrm{HRB}}(\mathfrak{g}, \mathfrak{g}^*[n-2])$ constructed above, recovers the map $\mathfrak{i}$ given in [26, Proposition 4.19]; note that $\mathfrak{i}$ was only constructed in op. cit. as a chain map rather than a strict L_∞-map.

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