On the existence and evaluation of Stokes phenomena in fluid mechanics

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Abstract

A singularly perturbed, high order KdV-type model, which describes localized travelling waves ("solitons") is being considered. We focus on the *Inner solution*, and detect *Stokes phenomena* that are crucial as to whether we can obtain a suitable solution. We provide a simple proof that the corresponding *Stokes constant* is non-zero. Also, we evaluate this splitting constant numerically by using two methods that are induced by the underlying theory.

Keywords— Stokes phenomenon, Stokes constants, Exponentially small splitting of Separatrices

1 Introduction

Localized travelling waves, or simply *solitons*, have been studied for more than a hundred years now. Their peculiar nature has attracted the interest of both physicists and mathematicians, with the latter trying to understand precise conditions (regarding mainly non-linearity and dispersion) in mathematical models that govern their existence. In general, the emergence of solitons is considered a rare phenomenon, because the conditions under which they are created are specific, such as abrupt changes of certain intensity in the motion of the medium. Interestingly, in the late 90's, and early 00's it was found that solitons can exists in families, continuous or discrete. Moreover, it was discovered that solitons can resonate with typical, periodic waves in the medium. A soliton of this type is called an *embedded soliton* (*ES*), see [3],[4]. This discovery came as a surprise because it was commonly thought that a structure like this will eventually collapse by gradually loosing energy by emitting radiation in the form of periodic waves. Over the last decade, *ES* are found to exist in several systems from a variety of physical disciplines (hydrodynamics, biophysics, solid state physics etc. see [1]). Motivated by research carried out in the last decade, we consider the following model that arises from fluid dynamics, see [1] and the first reference(s) there, or in particular [9]:

$$\epsilon^2 y''''(\xi) + y''(\xi) = y(\xi) + ry^2(\xi) + y^3(\xi), \quad r > 3/\sqrt{2} \tag{1.1}$$

Looking for global solutions in the case where $\epsilon = 0$ after imposing the condition $y(\xi) \to 0$ as $\xi \to \pm\infty$, the system above can be solved explicitly and we obtain the exact solution with simple poles at points $z_k^{\pm}$:

$$y_0(\xi) = \frac{3}{\sqrt{r^2 - \frac{9}{2}} \cosh \xi + r}, \quad z_k^{\pm}(r) := \pm \tanh^{-1}\left(\frac{3}{\sqrt{2}r}\right) + k\pi i, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{1.2}$$

The singular term $\epsilon^2 y''''(\xi)$ gives rise to two complex conjugate imaginary roots of the corresponding characteristic (auxiliary) equation, i.e. the algebraic equation $\epsilon^2 x^4 + x^2 - 1 = 0$. So a question that occurs naturally is whether the *separatrix* (1.2) persists (in this case we have an ES) under this singular perturbation or whether the solution that continues from $y_0(\xi)$ tends to some

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small oscillation as $\xi \to \pm\infty$ as the linearization of (1.1) suggests, in which case we say that the separatrix splits. A complete answer to that question goes beyond the purpose of this paper, but one may start by proving existence of symmetric solutions $y_\epsilon^\pm(\xi)$ ($y_\epsilon^-(-\xi) = y_\epsilon^+(\xi)$) initially defined on $(-\infty, 0]$ and $[0, +\infty)$ respectively that decay to zero at $\pm\infty$ respectively and check whether they and their derivatives up to order three agree at the origin (then by standard uniqueness theorems they will coincide). To evaluate the respective difference(s) one may start by approximating the solutions using singular perturbation theory: finding a formal sum $\tilde{y}(\xi, \epsilon) = \sum_{j=0}^\infty y_j(\xi)\epsilon^{2k}$ assumed to satisfy: $y_\epsilon(\xi) \simeq \tilde{y}(\xi, \epsilon)$ ($y_\epsilon(\xi)$ is the assumed desired solution), that is: for every $N \in \mathbb{N} \cup \{0\}$ there is a a function $R_{N+1}(\xi, \epsilon)$ such that $y_\epsilon(\xi) = y_0(\xi) + y_1(\xi)\epsilon^2 + ... + y_N(\xi)\epsilon^{2N} + R_{N+1}(\xi, \epsilon)$ and for all ξ, $R_{N+1}(\xi, \epsilon) = O(\epsilon^{2(N+1)})$ as $\epsilon \to 0$. To find the coefficients $y_k(\xi)$ in $\tilde{y}(\xi, \epsilon)$, we formally substitute the latter in (1.1). However, any order of the approximation misses exponentially small terms, and the presence of terms like $\dfrac{\pi\Gamma e^{-1/\epsilon}}{\epsilon}\cos(\dfrac{x}{\epsilon})$ can turn the soliton into a *weakly non-local solitary wave* (see [**2**]) by spoiling its localized nature. One way to detect terms like this is to study the problem near the poles $z_k^\pm$ of $y_0(\xi)$ (i.e. the "Inner problem") and then match the "inner" and "outer" solution. In this paper, we develop a first step of this approach, in particular:

In *Section 2*, we study the inner problem, and show that the equation that is to be satisfied by the first order of the approximation of $y_\epsilon(\xi)$ in a region of a pole (after translating and re-scaling the independent variable ξ to a new variable τ and re-scaling the dependent one to $v(\tau, \epsilon) = v_0(\tau) + O(\epsilon)$), has two solutions $v_\pm(\tau)$. These solutions are obtained by *Borel re-summation* of the corresponding asymptotic formal sums defined on different (yet overlapping) regions on the complex plane that decay to zero as $Re(\tau) \to \pm\infty$ respectively. Their difference is obtained approximately by a closed formula, and is proportional to an exponentially small term. The constant of proportionality, Γ, is called the *Stokes constant*; *Subsection 2.1* contains, essentially, a proof of its existence. *Subsection 2.2* contains semi-standard results and proofs that complete the first of the main goals of this paper.

In *Section 3*, we evaluate the aforementioned constant numerically, using two different independent methods, inspired by the respective theory, completing the goal of this work.

Finally, in *Section 4*, we briefly discuss related results in the literature.

As shown in the next section and described above, the Stokes constant Γ characterizes the difference of $v_+(\tau) - v_-(\tau)$ in their common domain, which indicates that the inner solution $v_0(\tau)$ may exhibit the so-called *Stokes phenomenon*: the asymptotic formula of a function changes discontinuously in adjacent regions of the complex plane. Assuming the matching of the inner and outer solution is done properly, it is natural to expect that the outer solution $y_\epsilon^+(\xi)$, roughly speaking, behaves differently as $\epsilon \to 0$ in the regions $\xi < Re(z_k^-)$ and $\xi > Re(z_k^-)$ ($k \in \mathbb{Z}$), as it happens in a very similar problem that motivated [8] (see (1)-(2) there). However, unlike the case in [8], the solution $y_0(\xi)$ of the unperturbed problem discussed here has singularities located on two lines on the complex plane instead of one, which makes an essential difference, potentially allowing the existence of a soliton-solution as numerical experiments suggest, see [1].

Remark 1.1. A formal sum (in the form of power-series): $\tilde{\phi}(z) = \sum_{n=0}^{\infty} a_n z^{an+b}$ ($a > 0$) is an asymptotic expansion of a function $f(z)$ at 0 (or at infinity through the transformation $z \mapsto z^{-1}$), in the sense of Poincaré if for any $N \in \mathbb{N} \cup \{0\}$ there is a function $g_{N+1}(z)$ such that $f(z) = a_0 z^b + a_1 z^{a+b} + ... + a_N z^{aN+b} + g_{N+1}(z)$ and $g_{N+1}(z) = O(z^{a(N+1)+b})$ as $z \to 0$, in which case we'll write $f(z) \simeq \tilde{\phi}(z)$. In applications like in this problem, the remainder $g_{N+1}(z)$ is may be presumed analytic *a priori*, as part of an ansatz, in order for closed form operations (addition, differentiation, integration) to be applied properly while asymptotic relations are respected accordingly. Note that: i) A function can have many asymptotic expansions of different forms $\tilde{\phi}(z)$, other than its *Taylor series* (if exists) at a point, but it is easily shown by subtracting two possible (truncated) expansions that the coefficients a_n (in one form $\tilde{\phi}(z)$) are unique, ii) Formal sums need not be convergent and iii) Different functions can share the same asymptotic expansion on some sector of the complex plane (unless perhaps if it's large enough, see *Watson's uniqueness theorem* in [12]), as in the main problem in Section 2, where functions differ by a term that is asymptotically smaller than any term of the series (smaller beyond all orders). In fact given any (formal) power series there are many functions asymptotic to it: any exponentially small functions such as e^{-a/x^2}, $a > 0$ is asymptotic to the (identically) zero power series at 0. However, given a formal negative power series: $\tilde{\phi}(z) = \sum_{k=0}^{\infty} c_k z^{-k-1}$ there is a unique entire function $f(z)$ of exponential type 1, asymptotic to $\tilde{\phi}(z)$ in a half-plane, see the *Borel-Ritt Lemma* in [5]. Also, we have the following basic lemma on

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asymptotic expansion of the Laplace transform:

Lemma 1.2. Let $\hat{u}$ be an analytic function defined on an open (complex) $\delta-$neighbourhood U_δ of the half-line $\mathbb{R}_{\geq 0}$, that is, the open $U_\delta \supseteq \mathbb{R}_{\geq 0}$ with $dist(\partial U_\delta, \mathbb{R}_{\geq 0}) = \delta > 0$. If there are constants $A > 0$ and $c \in \mathbb{R}$ such that $|\hat{u}(s)| \leq Ae^{c|s|}$ on U_δ then for any (fixed) $c' > c$ the Laplace transform of $\hat{u}$ satisfies:

$$\int_0^\infty e^{-zs}\hat{u}(s)ds \simeq \sum_{k=0}^\infty \frac{\hat{u}^{(k)}(0)}{z^{k+1}} \quad as \quad |z| \to +\infty, \quad Re(z) \geq c'. \tag{1.3}$$

The proof can be done using standard tools like repeated use of integration by parts. This lemma can be viewed as a strong and special implication of the *Watson's Lemma*, see [5]. In this setting, we will deal with a generalized version of the *Laplace transform*, where the integration is on some ray (half line) $[0, \infty e^{i\theta}) \subseteq \mathbb{C}$.

Remark 1.3. The formal power series we will encounter is of *Gevrey type* (the coefficients a_k satisfy $|a_k| \leq Ak!r^{-k}$, for some $r > 0$). In this case, the *Borel transform* (inverse Laplace transform), when applied formally (term by term) to some *Gevrey type* power series, defines a unique function that is analytic at the origin. This process is called (generalized)*Borel (re-)summation*, see for example [5] and [11].

2   Main results: The Inner expansion

2.1   Existence of the Stokes constant

First, since the equation is autonomous, we consider without loss of generality $z_1^- = 0$ (simply translating this pole to the origin). We set:

$$\tau = \frac{z}{\epsilon}, \qquad v(\tau, \epsilon) = \epsilon y(z, \epsilon) \tag{2.1}$$

and equation (1.1) becomes (with $' = \partial/\partial\tau$):

$$v''''(\tau, \epsilon) + v''(\tau, \epsilon) = v^3(\tau, \epsilon) + r\epsilon v^2(\tau, \epsilon) + \epsilon^2 v(\tau, \epsilon). \tag{2.2}$$

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We follow the spirit of perturbation theory and formally substitute an asymptotic expansion of the form $\sum_{k=0}^{\infty} v_k(\tau)\epsilon^k$ to $v(\tau, \epsilon)$ to obtain a(n) (infinite) system of equations whose leading term (coefficient of ϵ^0) satisfies:

$$v_0''''(\tau) + v_0''(\tau) = v_0^3(\tau). \tag{2.3}$$

Consider the "translated and rotated" half-planes: $\mathbb{H}_{\theta,c}^{\pm} := \{\tau \in \mathbb{C} : \pm Re(e^{i\theta}\tau) > c\} \subseteq \mathbb{C}$ where $\theta \in [-\pi, \pi)$ and $c \in \mathbb{R}$ (see Figure 1 for illustration), the intersection of two such half-planes (for two different θ's) is either empty, or a sector $S_{A,\tau_0} := \{\tau \in \mathbb{C} : Arg(\tau - \tau_0) \in A\}$ for some $\tau_0 \in \mathbb{C}$ and open interval A (set of directions), or just $\mathbb{H}_\theta, S_A$ if $c, \tau_0 = 0$ respectively. The Laplace transform of a (suitable) function $\hat{v}(s)$ along a complex ray $[0, \infty e^{i\theta})$ ($\theta \in A$) will denoted as $\mathcal{L}_\theta[\hat{v}](\tau)$. In this paper, the set of positive/non-negative integers will be denoted by $\mathbb{N}$ and $\mathbb{N}_0$ respectively. Bearing these in mind, the main results of this work are the following theorem and corollary:

Theorem 2.1. *Let $\theta^+ \in [0, \pi/2)$ be fixed, and set $\theta^- = \pi - \theta^+$. Equation (2.3) has a solution $v_{\theta\pm}(\tau)$ defined on $\mathbb{H}_{\theta\pm,c}^+$ for some $c \geq 0$, uniquely determined by being asymptotic to a formal sum of Gevrey type $\tilde{v}(\tau) \simeq \sqrt{2}\tau^{-1} + \sum_{k=1}^{\infty} c_k\tau^{-2k-1}$ on $\mathbb{H}_{\theta^+,c}^+$ for unique $c_k \in \mathbb{R}$, $k \in \mathbb{N}$. Moreover, if $\theta^+ > 0$ these solutions satisfy:*

$$v_{\theta^+}(\tau) - v_{\theta^-}(\tau) = \pi C_s e^{-i\tau} + O\left(\frac{e^{-i\tau}}{\tau}\right) \text{ as } \tau \to \infty \text{ in } \mathbb{H}_{\theta^+,c}^+ \cap \mathbb{H}_{\theta^-,c}^+ \tag{2.4}$$

The proof makes use of several lemmas and propositions which are proved subsequently, and is found at the end of this section. Using almost identical arguments, the following can be proved:

Corollary 2.2. *For $\tilde{v}(\tau)$ as in Theorem 2.1, let $\theta_+ \in (-\pi/2, 0)$ and $\theta_- = \pi - \theta_+$ be fixed. Equation (2.3) has solutions $v_{\theta\pm}(\tau)$ uniquely determined by being asymptotic to $\tilde{v}(\tau)$ on half-plane $\mathbb{H}_{\theta\pm,c'}^+$ (for some $c' \geq 0$) that satisfy:*

$$v_{\theta_+}(\tau) - v_{\theta_-}(\tau) = \pi C_s e^{i\tau} + O\left(\frac{e^{i\tau}}{\tau}\right) \text{ as } \tau \to \infty \text{ in } \mathbb{H}_{\theta_+,c'}^+ \cap \mathbb{H}_{\theta_-,c'}^+ \tag{2.5}$$

We proceed heuristically. For convenience, we drop any sophisticated subscripts ($\theta^\pm$,$\theta_\pm$) for now, and consider a formal power series to which $v_0(\tau)$ is asymptotic.

We notice that $\sqrt{2}\tau^{-1}$ solves the "unperturbed" system $v''(\tau) = v^3(\tau)$, so adding the "perturbation" term $v''''(\tau)$ and noticing that $(\sqrt{2}\tau^{-1})'' = 2\sqrt{2}\tau^{-3} = O(\tau^{-3})$ we understand that a fruitful educated guess is to assume *a priori* that $v_0(\tau)$ is asymptotic to a formal sum in the form of (negative) power series: $\tilde{v}_0(\tau) = \sqrt{2}\tau^{-1} + \sum_{k=1}^{\infty} c_k\tau^{-2k-1}$. In other words, we assume that for any $N \in \mathbb{N}_0$ we can write: $v_0(\tau) = \sqrt{2}\tau^{-1} + c_1\tau^{-3} + ... + c_N\tau^{-2N-1} + g_{N+1}(\tau)$ with $g_{N+1}(\tau) = O(\tau^{-2N-3})$ for some $g_{N+1}(\tau)$ analytic on the domain of interest. The coefficients c_k are found to satisfy:

$$c_{k+1} = \frac{1}{4k^2 + 14k + 6}\Big(\sum_{S_{3,k}} c_i c_j c_l - \big(\prod_{i=1}^{4}(2k+i)\big)c_k\Big), \quad c_0 = \sqrt{2}, c_1 = -4\sqrt{2} \tag{2.6}$$

$$\text{where} \quad S_{3,k} = \{(i,j,l) \in \mathbb{N}_0^3 : i+j+l = k+1 | i,j,l \le k\}.$$

Remark 2.3. The reason why we consider this ansatz is because of the form the asymptotic expansion the Laplace transform possesses (Lemma 1.2). Assuming that the equation admits a solution in the form of a Laplace transform, we can consider any $N \in \mathbb{N}$ arbitrary, yet fixed, and studying the limiting behaviour of the emerging sequence, we can obtain a nice recipe for constructing the (unique) *Borel transform* (inverse Laplace) $\hat{v}_0(s)$, of $v_0(\tau)$.

To show that the formal series $\tilde{v}_0(\tau) := \sum_{k=0}^{\infty} c_k\tau^{-2k-1}$, to which $v_0(\tau)$ is presumably asymptotic (at ∞), is of *Gevrey* type, we use the two following Lemmas (2.4 and 2.5) that lead to the proof of Proposition 2.6 (which essentially says that $\tilde{v}_0(\tau)$ is of *Gevrey* type):

Lemma 2.4. The sequence $C_k := \dfrac{(-1)^k c_k}{(2k)!}$ is positive and strictly increasing. Moreover, if convergent to a $\Gamma \in \mathbb{R}$, it satisfies:

$$C_k = \Gamma - \frac{3\Gamma}{2k} + O\Big(\frac{1}{k^2}\Big), \quad \text{as} \quad k \to +\infty \tag{2.7}$$

Proof. We divide the recurrence relation of c_{k+1} (eq. (2.6)) by $(2k+2)!$ and we obtain the following equivalent (and normalized) relation for Γ:

$$C_{k+1} = \frac{1}{4k^2 + 14k + 6}\Big(F(k) + \big(\prod_{i=1}^{4}(2k+i)\big)\frac{C_k}{(4k^2 + 6k + 2)}\Big) \tag{2.8}$$

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where $F(k) := \dfrac{1}{(2k+2)!} \displaystyle\sum_{S_{3,k}} (2i)! C_i (2j)! C_j (2l)! C_l$.

The positiveness of C_k is easily shown inductively, we have:

$C_0 = \sqrt{2} > 0$ and $C_1 = 2\sqrt{2} > 0$. Assume that $C_n > 0$, $\forall n \leq k$ for some $k \in \mathbb{N}$, then $F(n) > 0$ for such n (straightforward) and so $C_{n+1} > 0$ since the RHS of (2.8) is positive. Also:

$$\frac{1}{(4k^2 + 14k + 6)(4k^2 + 6k + 2)} \Big(\prod_{i=1}^{4} (2k+i) \Big) = \frac{(k+1)(k+2)(2k+1)(2k+3)}{(2k^2 + 7k + 3)(2k^2 + 3k + 1)} = 1 + \frac{3}{2k^2 + 7k + 3} > 1.$$

We set $G(k) = \dfrac{3}{2k^2 + 7k + 3} = \dfrac{3}{2k^2} - \dfrac{21}{4k^3} + O\Big(\dfrac{1}{k^4}\Big)$ (at infinity) and write:

$$C_{k+1} = \frac{F(k)}{4k^2 + 14k + 6} + (1 + G(k)) C_k. \tag{2.9}$$

Thus $C_{k+1} > C_k$, as $F(k) > 0$ (since $C_m > 0$, $\forall m \leq k$) by the induction assumptions. The proof of the first part of Lemma 2.4 is complete.

Now, in order to prove the second part, i.e. $n^2 \Big(C_n - \Gamma + \dfrac{3\Gamma}{2n} \Big) \leq M$, (where Γ is the limit of C_n) for some $M > 0$, we re-write formula (2.8) as follows:

$$C_{n+1} = \frac{1}{4n^2 + 14n + 6} \Big(\sum_{S'_{3,n}} \frac{(2i)! C_i (2j)! C_j (2l)! C_l}{(2n+2)!} + \Big((2n+3)(2n+4) + g_2(n) \Big) C_n \Big) \tag{2.10}$$

where $S'_{3,n} = S_{3,n} \cap \{i, j, l \leq n-1\}$ and $g_2(n) = \dfrac{12}{(n+1)(2n+1)} = O(n^{-2})$ at infinity. We have:

$$\frac{1}{4n^2 + 14n + 6} \Big(\sum_{S'_{3,n}} \frac{(2i)! C_i (2j)! C_j (2j)! (C_l)}{(2n+2)!} \Big) \leq \frac{\Gamma^3}{4n^2} \sum_{S'_{3,n}} \frac{(2i)! (2j)! (2l)!}{(2n+2)!}$$

$$\leq \frac{\Gamma^3}{4n^2} \sum_{S'_{3,n}} \frac{6(2)!(2)!(2n-2)!}{(2n+2)!} \leq \frac{\Gamma^3}{n^6} |S'_{3,n}| = \frac{\Gamma^3}{n^6} \Big| \frac{(n-1)^2 + 5(n-1)}{2} \Big| \leq \frac{\Gamma^3 n^2}{n^6} = O\Big(\frac{1}{n^4} \Big).$$

We notice: $\displaystyle\sum_{k=n}^{\infty} \frac{1}{k^3} \leq \sum_{k=n}^{\infty} \frac{1}{k^2(k-1)} = \sum_{k=n}^{\infty} \frac{1}{k} \Big(\frac{1}{k-1} - \frac{1}{k} \Big) \leq \frac{1}{n} \sum_{k=n}^{\infty} \Big(\frac{1}{k-1} - \frac{1}{k} \Big) \leq \frac{2}{n^2}$

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$$\implies \sum_{k=n}^{\infty} \frac{1}{k^4} \le \frac{1}{n} \sum_{k=n}^{\infty} \frac{1}{k^3} \le \frac{2}{n^3},$$

$$\sum_{k=n}^{\infty} \frac{1}{4k^2 + 14k + 6} \left(\sum_{S'_{3,k}} \frac{(2i)! C_i (2j)! C_j (2j)! C_l}{(2k+2)!} \right) \le 3\Gamma^3 \sum_{k=n}^{\infty} \frac{1}{k^4} = O\left(\frac{1}{n^3}\right) \tag{2.11}$$

$$\text{Also:} \quad \sum_{k=n}^{\infty} \frac{12}{(4k^2 + 14k + 6)(k+1)(2k+1)} \le \frac{3}{2} \sum_{k=n}^{\infty} \frac{1}{k^4} = O\left(\frac{1}{n^3}\right) \tag{2.12}$$

$$\text{and} \quad C_{n+1} - C_n = \frac{3C_n}{2n^2 + 7n + 3} + O\left(\frac{1}{n^4}\right) \implies \tag{2.13}$$

$$C_n = (C_n - C_{n+1}) + \dots + (C_{n+N-1} - C_{n+N}) + C_{n+N}, \quad N \in \mathbb{N} \text{ (arbitrary - let } N \to \infty)$$

$$\text{I.e. } C_n = \Gamma - 3 \sum_{k=n}^{\infty} \frac{C_k}{2k^2 + 7k + 3} + O\left(\frac{1}{n^3}\right) = \Gamma - \frac{3}{2} \sum_{k=n}^{\infty} \left(\frac{C_k}{k^2} + O\left(\frac{1}{k^3}\right) \right) + O\left(\frac{1}{n^3}\right) \implies$$

$$C_n = \Gamma - \frac{3}{2} \left(\sum_{k=n+1}^{\infty} \frac{C_k}{k^2} \right) + O\left(\frac{1}{n^2}\right). \tag{2.14}$$

The last equation holds for every $n \in \mathbb{N}$, so, we can replace C_k in the sum in the RHS with the ("same") expression (2.14) of C_n for $n = k$ and write:

$$C_n = \Gamma - \frac{3}{2} \left(\sum_{k=n+1}^{\infty} \frac{\Gamma}{k^2} \right) + \frac{9}{4} \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \sum_{m=k+1}^{\infty} \frac{C_m}{m^2} \right) + O\left(\frac{1}{n^2}\right), \quad \text{but}$$

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} \sum_{m=k+1}^{\infty} \frac{C_m}{m^2} = \sum_{m=n+1}^{\infty} \frac{C_m}{m^2} \sum_{k=n+1}^{m} \frac{1}{k^2} \le \sum_{m=n+1}^{\infty} \frac{\Gamma}{m^2} \sum_{k=n+1}^{m} \frac{1}{(k-1)k} \le \frac{\Gamma}{n^2}$$

$$\implies \quad C_n - \Gamma + \frac{3\Gamma}{2n} = O\left(\frac{1}{n^2}\right).$$

$\square$

Lemma 2.5. Take a (positive) natural k_0. For all $a \in (0, 1/2)$ there exists $A(a) \in \mathbb{R}_{>0}$ and $n' \in \mathbb{N}$ such that for all naturals $n \ge n'$ the sequence $D_{n+1} := \frac{D_n^3}{2(n+k_0)^2} + \left(1 + \frac{3}{2(n+k_0)^2}\right) D_n, \; D_0 = C_{k_0}$ satisfies: $D_n \le A(a) n^a$.

Proof. In a way similar to the proof of the fact that C_k is positive previously, we immediately see that $D_n > 0$, for each $n \in \mathbb{N}$ and is increasing.

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Also, we see from the RHS of (2.9) that $D_n \geq C_{n+k_0}$ for $n \in \mathbb{N}$, this follows from standard arguments and simple calculations. Take an $a \in (0, 1/2)$ (arbitrary). We will see that for $n' \in \mathbb{N}$ sufficiently large, we are able to prove the statement inductively.

We take an $n' \in \mathbb{N}$ and an $A(a)$ such that $D_{n'} \leq A(a)(n')^a$, we'll see that for n' large enough we can apply induction and prove the statement. Let an $n \geq n'$ such that the result holds, then for D_{n+1} we have: $D_{n+1} = \dfrac{D_n^3}{2(n+k_0)^2} + \left(1 + \dfrac{3}{2(n+k_0)^2}\right) D_n \leq A(a) n^a \left(1 + \dfrac{A^2(a) n^{2a} + 3}{2(n+k_0)^2}\right)$

So, we get $D_{n+1} \leq A(a) n^a \left(1 + \dfrac{A^2(a) n^{2a-2}}{2} + \dfrac{3n^{-2}}{2}\right)$.

Now, it suffices to prove that for all $n \geq n'$ with n' sufficiently large we get:

$$(*) \quad A(a) n^a \left(1 + an^{-1} + \dfrac{a(a-1)}{2} n^{-2}\right) \geq A(a) n^a \left(1 + \dfrac{A^2(a) n^{2a-2}}{2} + \dfrac{3n^{-2}}{2}\right).$$

This inequality will be enough, since from *Taylor's theorem* we get (for large n):

$$\left(1 + \dfrac{1}{n}\right)^a = 1 + \dfrac{a}{n} + \dfrac{a(a-1)}{2n^2} + \dfrac{a(a-1)(a-2)}{6n^3}(1+\xi_n)^a, \text{ for some } \xi_n \in \left(0, \dfrac{1}{n}\right).$$

So $A(a)(1+n)^a = A(a) n^a \left(1 + \dfrac{1}{n}\right)^a \geq A(a) n^a \left(1 + an^{-1} + \dfrac{a(a-1)}{2} n^{-2}\right)$ since $a < 1/2$.

We have $2a - 2 < -1$, and so for all $n > n'$ for some $n' \in \mathbb{N}$ large enough we have (since $a \in (0, 1/2) \implies 2 - 2a > 1$):

$$\dfrac{a}{n} > \dfrac{A^2(a)}{2n^{2-2a}} + \dfrac{a(1-a)}{2n^2} + \dfrac{3}{2n^2} \implies \dfrac{a}{n} + \dfrac{a(a-1)}{2n^2} > \dfrac{A^2(a)}{2n^{2-2a}} + \dfrac{3}{2n^2}.$$

We multiply both sides of the latest inequality with $A(a) n^a$ and we obtain inequality $(*)$. Moreover, $n^{2-2a}/n = n^{1-2a} \to +\infty$ as $n \to +\infty$.

So (after picking n' sufficiently large) the inequality $D_n \leq A(a) n^a$ at the top of this page holds, and so does the induction step. Lastly, we have that such a large n' exists, and we are free to choose it as large as we need. The proof of Lemma 2.5 is complete. $\qquad\square$

Proposition 2.6. There is a positive constant Γ such that $\dfrac{(-1)^k c_k}{(2k)!} = \Gamma - \dfrac{3\Gamma}{2k} + O(\dfrac{1}{k^2})$ (at infinity).

Proof. Having proved Lemma 2.4, it suffices to prove that there is a constant $C > 0$ such that $C_k \leq C$, $\forall k \in \mathbb{N}$, this implies convergence since C_k increases. Following it we have: $G(k) \leq \dfrac{3}{2k^2}$, $\forall k \in \mathbb{N}$ and for $m \in \{2, 3, ..., k_0\}$ for some $k_0 \in \mathbb{N}$ we get:

$$F(m) = \sum_{S_{3,m}} \frac{(2i)!C_i(2j)!C_j(2l)!C_l}{(2m+2)!} \leq \frac{C_{k_0}^3}{2m^2} \sum_{S_{3,m}} 1 = \frac{C_{k_0}^3}{2m^2} \frac{m^2 + 5m}{2} = \frac{C_{k_0}^3}{4}\left(1 + \frac{5}{m}\right)$$

because $C_i, C_j, C_l \leq C_{k_0}$ and so $C_i C_j C_l \leq C_{k_0}^3$. Also: $|S_{3,m}| = \dfrac{m^2 + 5m}{2}$ $(= \#S_{3,m})$ and for the multinomial coefficients with $(i, j, l) \in S_{3,m}$ we have:

$$\frac{(2m+2)!}{(2i)!(2j)!(2l)!} = \binom{2m+2}{2i, 2j, 2l} \geq (m+1)(2m+1) \geq 2m^2 \implies \frac{(2i)!C_i(2j)!C_j(2l)!C_l}{(2m+2)!} \leq \frac{C_{k_0}^3}{2m^2}$$

and so

$$F(m) \leq \frac{C_{k_0}^3}{4}\left(1 + \frac{5}{m}\right). \text{ Therefore, } \frac{F(m)}{4m^2 + 14m + 6} \leq \frac{C_{k_0}^3}{16m^2}\left(1 + \frac{5}{m}\right) \leq \frac{C_{k_0}^3}{2m^2} \text{ for } m \geq 2.$$

So, since k_0 was arbitrary, we get for $m \in \{2, 3, ...\}$ the inequality:

$$C_{m+1} \leq \frac{C_{k_0}^3}{16m^2} + \frac{5C_{k_0}^3}{64m^3} + (1 + G(m))C_m \leq \frac{C_m^3}{2m^2} + \left(1 + \frac{3}{2m^2}\right)C_m. \tag{2.15}$$

Now, we take a $k_0 \geq 2 \in \mathbb{N}$ and consider $M(k_0) := \dfrac{\tilde{C}_{k_0}^2}{2} + \dfrac{3}{2} \equiv M_0$ where:

$\tilde{C}_{k_0} := \max\{C_{2k_0}, 2C_{k_0}\}(:= \tilde{C}_0$ for simplicity). We have: $C_{m+1} \leq \left(1 + \dfrac{M_0}{m^2}\right)C_m$ $\forall m \in \{2, ..., 2k_0\}$

This happens for such m since $\dfrac{C_m^2}{2m^2} + \dfrac{3}{2m^2} \leq \dfrac{M_0}{m^2}$ $\forall m \in \{2, ..., 2k_0\}$. So we have:

$$C_{2k_0+1} \leq \left(1 + \frac{M_0}{(2k_0)^2}\right)C_{2k_0}. \tag{2.16}$$

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Iterating (2.16) backwards (in C_{2k_0}) $k_0 - 1$ times we get:

$$C_{2k_0+1} \leq \Big(\prod_{i=k_0}^{2k_0} \big(1 + \frac{M_0}{i^2}\big)\Big)C_{k_0} \leq e^{\sum_{i=k_0}^{2k_0} \frac{M_0}{i^2}} C_{k_0}, \text{ since } e^x \geq 1+x, \, \forall x \in \mathbb{R}.$$

We notice:

$$\sum_{i=k_0}^{2k_0} \frac{1}{i^2} \leq \sum_{i=k_0}^{\infty} \frac{1}{i^2} \leq \frac{1}{k_0} + \sum_{i=k_0+1}^{\infty} \frac{1}{i(i+1)} = \frac{1}{k_0} + \sum_{i=k_0+1}^{\infty} \Big(\frac{1}{i} - \frac{1}{i+1}\Big) \leq \frac{2}{k_0}.$$

So, we obtain the inequality:

$$C_{2k_0+1} \leq \Big(\prod_{i=k_0}^{2k_0} \big(1 + \frac{M_0}{i^2}\big)\Big)C_{k_0} \leq e^{\sum_{i=k_0}^{2k_0} \frac{M_0}{i^2}} C_{k_0} \leq e^{2\frac{M_0}{k_0}} C_{k_0}.$$

Now, in order to prove a uniform bound, if suffices to show that $\dfrac{M_0}{k_0} \to 0$ as $k_0 \to +\infty$ (recall $M_0 \equiv M(k_0)$), as in this case we will get that:

There is a $k_0' \in \mathbb{N}$ such that $2M(k_0')/k_0' \leq \ln 2$, and so: $C_{2k_0'+1} \leq 2C_{k_0'} \leq \tilde{C}_0$.

Then we'll be able to iterate the (forward) inequality (2.16) as many times as we want and maintaining the bound $\tilde{C}_0$ since the same bound will hold for $C_{2k_0'+1}$ with k_0' arbitrary, albeit large enough. Afterwards, we do the same for $C_{(2k_0'+1)+1}$ with $M(k_0') \equiv M_0'$ (staying the same as in the case of $C_{2k_0'+1} \leq ...$) and obtain the same bound:

$$C_{2k_0'+2} \leq ... \leq \Big(\prod_{i=k_0'}^{2k_0'} \big(1 + \frac{M_0'}{i^2}\big)\Big)C_{k_0'} \leq e^{\sum_{i=k_0'}^{2k_0'} \frac{M_0'}{i^2}} C_{k_0'} \leq e^{2\frac{M_0'}{k_0'}} C_{k_0'}$$

and proceeding inductively in this manner, we get: $C_k \leq \max\{C_{2k_0'}, 2C_{k_0'}\} < \infty$, $\forall k \in \mathbb{N}$.

To show $M_0/k_0 \to 0$ as $k_0 \to +\infty$ we proceed by considering the sequence D_n from Lemma 2.5 and following it, we take an $a \in (0, 1/2)$ and we have that $C_{n+k_0} \leq D_n$, and so $C_{n+k_0} \leq A(a)n^a \Rightarrow C_{n+k_0}^2 \leq A^2(a)n^{2a} \leq A^2(a)n^{1-\epsilon} < A^2(a)n$ for some $A(a) > 0$ and $\forall \epsilon < 1 - 2a$, with $a \in (0, 1/2)$ arbitrary.

Eventually, letting $k_0 \to +\infty$ we get $\dfrac{C_{k_0}^2}{k_0} \to 0$ and now we can easily deduce that for $C'(k_0) = \max\{C_{2k_0}, 2C_{k_0}\}$ we have $\dfrac{(C'(k_0))^2}{k_0} \to 0$ in the limit $k_0 \to +\infty$.

Thus, for $M(k_0) = \dfrac{3}{2} + \dfrac{\tilde{C}^2(k_0)}{2}$ we obtain $\dfrac{2M(k_0)}{k_0} \to 0$ as $k_0 \to +\infty$. Proposition 2.6 is proved. $\square$

2.2 The analytic continuation of the Borel transform

Define $\Gamma := \lim_{n\to\infty} C_n$ (i.e. $\lim_{n\to\infty} (-1)^n c_n/(2n)!$). It is clear now, since $(-1)^k c_k \sim \Gamma(2k)!$ as $k \to +\infty$, that the formal power series $\tilde{v}_0(\tau) = \sum_{n=0}^{\infty} c_n \tau^{-2n-1}$ is of *Gevrey* type. So, we can obtain a function $\hat{v}_0(s)$ that is analytic on the unit disk $\mathbb{D}(0,1) \equiv \mathbb{D}_0$ by defining:

$$\hat{v}_0(s) := \sum_{n=0}^{\infty} \frac{c_n}{(2n)!} s^{2n} \tag{2.17}$$

In fact, $\hat{v}_0(s)$ can be analytically extended to a function of exponential type in a suitable sector (united with the unit disk), see Proposition 2.9.

Remark 2.7. The RHS of (2.17) solves the equation: $s^2(s^2+1)\hat{v}(s) = (\hat{v}*\hat{v}*\hat{v})(s)$ which is obtained by applying the inverse Laplace (Borel) transform in (2.3). Letting $\hat{v}_0(s) = \sum_{n=0}^{N-1} \frac{c_n}{(2n)!} s^{2n} + \hat{g}_N(s)$ we can easily see that the remainder satisfies: $\hat{g}_N(s) = O(s^{2N+2})$ while being analytic at zero, also:

$$v_0(\tau) = \mathcal{L}[\hat{v}_0(s)] = \int_0^{+\infty} e^{-\tau s}\hat{v}_0(s)ds = \sum_{n=0}^{N} c_n \tau^{-2n-1} + g_{N+1}(\tau),$$

where the remainder satisfies $g_{N+1}(\tau) = O(\tau^{-2N-3})$ and is analytic at a region of interest. Indeed, if $g_{N+1}(\tau) = O(\tau^{-2N-3})$ (at infinity), then its Borel transform satisfies $\hat{g}_{N+1}(\tau) = O(s^{2N+2})$ (at zero) and vice-versa.

The asymptotic behaviour of $\hat{v}_0(s)$ can be easily found by studying the limiting behaviour of its series expression as $s \to \pm i$ along the segment $\{i\psi : \psi \in (-1,1)\}$ (or along any line segment that belongs to the unit disk and has i or resp. $-i$ as one of its limit points). However, we obtain a much more informative description of $\hat{v}_0(s)$:

Lemma 2.8. There exists a function $r(s)$ analytic and bounded on the unit disk $\mathbb{D}_0$ such that the function $\hat{v}_0(s)$ (eq. (2.17)) has the form:

$$\hat{v}_0(s) = \frac{\Gamma}{s^2+1} + \frac{3\Gamma}{2}\ln(1+s^2) + r(s), \quad s \in \mathbb{D}_0 \tag{2.18}$$

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Proof. We set $r_k = C_k - \Gamma$, $k \in \mathbb{N}$ and we have:

$$\hat{v}_0(s) = \sum_{k=0}^{\infty} \frac{c_k}{(2k)!} s^{2k} = \sum_{k=0}^{\infty} (-1)^k C_k s^{2k} = \sum_{k=0}^{\infty} C_k(-s^2)^k =$$

$$\Gamma \sum_{k=0}^{\infty} (-s^2)^k - \sum_{k=0}^{\infty} (\Gamma - C_k)(-s^2)^k = \frac{\Gamma}{1+s^2} - \frac{(-3\Gamma)}{2} \sum_{k=1}^{\infty} \frac{(-s^2)^k}{k} + \sum_{k=0}^{\infty} r_k s^{2k} =$$

$$\frac{\Gamma}{1+s^2} + \frac{3\Gamma}{2} \ln(1+s^2) + \sum_{k=0}^{\infty} r_k s^{2k} \Rightarrow \sum_{k=0}^{\infty} r_k s^{2k} = \hat{v}_0(s) - \frac{\Gamma}{1+s^2} - \frac{3\Gamma}{2} \ln(1+s^2)$$

where $r_k = O(k^{-2})$ at infinity (Proposition 2.6).

So, there is an $M > 0$ such that $r_k \leq Mk^{-2}$ for all $n \in \mathbb{N}$ and the sum:

$$|\sum_{k=0}^{\infty} r_k s^{2k}| \leq \sum_{k=0}^{\infty} |r_k||s|^{2k} \leq r_0 + \sum_{k=1}^{\infty} \frac{M}{k^2} |s|^{2k} \leq r_0 + \sum_{k=1}^{\infty} \frac{M}{k^2} = r_0 + \frac{M\pi^2}{6}$$

converges absolutely on $\mathbb{D}_0$, and so $r(s)$ is analytic and bounded on $\mathbb{D}_0$. $\qquad\square$

If $\hat{v}_0(s)$ can be analytically extended on a δ-neighbourhood of some ray $[0, \infty e^{i\theta})$ to a function $\hat{v}(s)$ of some exponential type c, we can then apply the *Laplace* transform on $\hat{v}(s)$ and obtain a function which is analytic, solves eq. (2.3) and is asymptotic to the formal sum $\tilde{v}(\tau)$ on $\mathbb{H}_{\theta,c}^+ = \{z \in \mathbb{C} : Re(e^{i\theta}z) > c\}$. That is exactly what we get by the next Proposition (2.9). After that, we will study the differences of functions obtained by *Laplace*-transforming $\hat{v}(s)$ along different rays, essentially proving Theorem 2.1, which is the main results of that paper. The aforementioned difference is exponentially small yet non-zero, indicating the presence of a *Stokes phenomenon* (the asymptotic formula of a function changes discontinuously in adjacent regions of the complex plane) on the *inner* (and consequently outer) solution. The reader may consult Figure 2 for an illustration of the domain $\mathbb{D}_0 \cup S_{A^+} \cup S_{A^-}$ which we see in the following proposition.

Proposition 2.9. Consider $a \in (0, \pi/2)$. The function $\hat{v}_0(s) : \mathbb{D}_0 \mapsto \mathbb{C}$ can be analytically extended on $\mathbb{D}_0 \cup S_{A^+} \cup S_{A^-}$, where $A^+ = (-a, a)$ and $A^- = (\pi - a, \pi + a)$ the sets of directions and $S_{A^\pm}$ the respective sectors. Let $\hat{v}(s)$ be this continuation. Moreover there exist $M_a, L_a > 0$ such that $|\hat{v}(s)| \leq M_a e^{L_a|s|}$ on $\mathbb{D}_0 \cup S_{A^+} \cup S_{A^-}$.

Proof. We consider the functional equation:

$$s^4 \hat{v}_0(s) + s^2 \hat{v}_0(s) = (\hat{v}_0 * \hat{v}_0 * \hat{v}_0)(s) \tag{2.19}$$

for $s \in \mathbb{D}_0$, which is derived after applying the transform $\mathcal{L}^{-1}$ to (2.3).

We will denote the triple convolution on the RHS of (2.19) as $\hat{v}_0^{*3}$ through the straight line segment connecting $[0, s]$ for s such that the segment does not pass through singularities of $\hat{v}_0(s)$.

We let $r \in (0, 1)$ and for $s \in \mathbb{D}(0, r) \subseteq \mathbb{D}_0$ we write:

$$\hat{v}_0(s) = \frac{\hat{v}_0^{*3}}{s^2(s^2 + 1)}, \quad s \in \mathbb{D}(0, r).$$

We see that even though s^2 appears in the denominator in the latest formula, $s = 0$ is not a singularity, because $\hat{v}_0(s) = \sqrt{2} + O(s^2)$ as $s \to 0$. However, $\hat{v}_0(s)$ has singularities at $s = \pm i$ as proven in Lemma 2.8.

Now, we set $S_{1,r}(A) = \mathbb{D}(0, 2r) \cap S_A$ (where $A = A^{\pm}$) and for $s \in S_{1,r}(A) \setminus \mathbb{D}(0, r)$ we consider the integral equation:

$$\hat{v}_1(s) = \frac{1}{s^2(s^2 + 1)} \hat{v}_1^{*3}. \tag{2.20}$$

We now set $\underline{S}_{1,r}(A) = S_{1,r}(A) \setminus \mathbb{D}(0, r)$ and study whether equation (2.20) has a solution in $\underline{S}_{1,r}(A)$.

To do that we set $\underline{\hat{v}}_0(s) = \hat{v}_0(s)$ for $s \in S_{1,r}(A)$ and 0 otherwise, and consider an auxiliary function/unknown $\underline{\hat{v}}_1(s) : S_{1,r}(A) \mapsto \mathbb{C}$ such that $\underline{\hat{v}}_1(s) = 0$ on $\mathbb{D}(0, r)$, and $\hat{v}_1(s)$ otherwise, the unknown in eq. (2.20). We will prove existence of such solution $\underline{\hat{v}}_1(s)$, after that we will be able to define an analytic continuation $\hat{v}_1 : S_{1,r}(A) \mapsto \mathbb{C}$ of $\hat{v}_0$ to $S_{1,r}(A)$ by setting $\hat{v}_1(s) = \hat{v}_0(s)$ on $\mathbb{D}(0, r)$ and $\underline{\hat{v}}_1$ on $\underline{S}_{1,r}(A)$.

We re-write (2.20) as:

$$\underline{\hat{v}}_1 = \frac{1}{s^2(s^2 + 1)} (\underline{\hat{v}}_0^{*3} + 3\underline{\hat{v}}_0^{*2} * \underline{\hat{v}}_1). \tag{2.21}$$

Equation (2.20) is equivalent to (2.21) because we have:

$$\underline{\hat{v}}_1^{*2}(s) = \int_0^s \underline{\hat{v}}_1(\tau)\underline{\hat{v}}_1(s - \tau)d\tau = 0, \quad for \quad s \in \underline{S}_{1,r}(A)$$

because for each $\tau \in [0, s)$ and $\tau' \in [0, s - \tau)$ we have that either $\underline{\hat{v}}_1(\tau) = 0$, or $\underline{\hat{v}}_1(s - \tau) = 0$, in other words, there are no "non-linear" terms of $\underline{\hat{v}}_1$, bearing in mind the associative property of convolution. We fix an $a \in (0, \pi/2)$, we set $s_r = re^{ia} \in [0, s] \cap \partial \mathbb{D}(0, r)$ and we have:

$$\underline{\hat{v}}_1(s) = \frac{C_r(s)}{s^2(s^2 + 1)} + \frac{3}{s^2(s^2 + 1)}(\hat{\underline{v}}_0^{*2} * \hat{\underline{v}}_1) \tag{2.22}$$

where $C_r(s) = \int_{s - s_r}^{s_r} \hat{\underline{v}}_0(\tau) \int_0^{s_r - \tau} \hat{\underline{v}}_0(\tau') \underline{\hat{v}}_0(s - \tau - \tau') d\tau' d\tau.$

The integral form of (2.21-22) can be written:

$$\underline{\hat{v}}_1(s) = f_r(s) + \frac{3}{s^2(s^2 + 1)} \int_{s_r}^s \underline{\hat{v}}_1(\tau) \int_0^{s - \tau} \hat{\underline{v}}_0(\tau') \underline{\hat{v}}_0(s - \tau - \tau') d\tau' d\tau \tag{2.23}$$

with $f_r(s) = \frac{C_r(s)}{s^2(s^2 + 1)} = \frac{1}{s^2(s^2 + 1)} \int_0^{s_r} \hat{\underline{v}}_0(\tau) \int_{s - s_r}^{s - \tau} \hat{\underline{v}}_0(\tau') \underline{\hat{v}}_0(s - \tau - \tau') d\tau' d\tau.$ \hfill (2.24)

Equation (2.23) is a *Volterra equation of 2nd kind* with *Integral Kernel*:

$$K(s, \tau) = K(s - \tau) = \frac{3}{s^2(s^2 + 1)} \int_0^{s - \tau} \hat{\underline{v}}_0(\tau') \underline{\hat{v}}_0(s - \tau - \tau') dt', \quad \forall \tau \in [0, s). \tag{2.25}$$

and 0 otherwise. We have:

$Arg(s), Arg(\tau) = a \in (0, \pi/2)$ fixed, so $\left| \frac{1}{s^2 + 1} \right| = \left| \frac{1}{(s + i)(s - i)} \right|$ is bounded in $\mathbb{D}(0, r) \cup S_{1,r}(A)$. Similarly, for some $M_f, M_K > 0$ we have bounds: $|f_r(s)| \leq M_f$ on $\underline{S}_{1,r}(A)$ and:

$|K(s, \tau)| = |(\hat{\underline{v}}_0 * \hat{\underline{v}}_0)(s - \tau)| = \left| \int_0^{s - \tau} \hat{\underline{v}}_0(\tau') \underline{\hat{v}}_0(s - \tau - \tau') d\tau' \right| \leq M_K, \ s \in \underline{S}_{1,r}(A), \ \tau \in [s_r, s].$

Consider $\mathcal{X} := \mathcal{A}(\underline{S}_{1,r}(A))$, the space of analytic functions on $\underline{S}_{1,r}(A)$ and $\mathcal{X}_\epsilon$ the space of analytic functions on an ϵ-neighbourhood of $\underline{S}_{1,r}(A)$, for ϵ small enough, and the operator:

$T_K : \mathcal{X} \mapsto \mathcal{X}$ defined by: $T_K(v(s)) := f_r(s) + \int_{s_r}^s v(\tau)K(s, \tau)d\tau.$

The problem of proving existence of solution to (2.23) with certain properties reduces to a problem of proving existence of a fixed point of T_K on a suitable subspace of $\mathcal{X}$ (obviously $\mathcal{X}_\epsilon \subseteq \mathcal{X}$). For $x, y \in \mathcal{X}_\epsilon$ we have:

$$|T_K(x) - T_K(y)| = \left| \int_{s_r}^s (x - y)K(s, \tau)d\tau \right| \leq \left| \int_{s_r}^s |x - y||K(s, \tau)|d\tau \right| \implies$$

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$$|T_K(x(s)) - T_K(y(s))| \leq M_K \left| \int_{s_r}^{s} |x(\tau) - y(\tau)| d\tau \right| \leq M_K |s - s_r| \cdot ||x - y||_\infty. \qquad (2.26)$$

(the supremum norm is finite since $x, y \in \mathcal{X}_\epsilon$).

By induction, we can similarly show that:

$$|T_K^n(x(s)) - T_K^n(y(s))| \leq \frac{M_K^n |s - s_r|^n}{n!} ||x(s) - y(s)||_\infty. \qquad (2.27)$$

Since $||x(s) - y(s)||_\infty < \infty$, the RHS of (2.27) tends to 0 as $n \to +\infty$, consequently, for all n large enough, i.e. T_K^n is a contraction mapping. We set $x_0 := f_r(s) \in \mathcal{A}(\underline{S}_{1,r}(A))$ and $x_k = T(x_{k-1})$ $(\ldots = T^k(x_0))$ and have:

$$x_n = x_0 + \sum_{k=1}^{n} (x_k - x_{k-1}). \qquad (2.28)$$

From (2.27) $\big($after setting $y = x_0, x = T(y) = x_1\big)$, we have that the RHS of (2.28) converges absolutely as $n \to +\infty$, in particular:

$$|x_n| \leq |x_0| + |x_1 - x_0| \sum_{j=0}^{\infty} \frac{M_K^j}{j!} |s - s_r|^j = |x_0| + |x_1 - x_0| e^{M_K |s - s_r|} < \infty \qquad (2.29)$$

uniformly $\forall n \in \mathbb{N}$. Let $M_\infty := ||x_0||_\infty + ||x_1 - x_0||_\infty e^{M_K}$, therefore, there is a subsequence $x_{k_n} \in \mathcal{X}_\epsilon \cap \{||x||_\infty \leq M_\infty\}$ of x_n, that converges to a function $x_\infty \in \mathcal{X}_\epsilon \cap \{||x|| \leq M_\infty\}$. So, $T^n(x_0) \to x_\infty$ and by (2.27) we have

$$\left| T^{n+1}(x_0(s)) - T^n(x_0(s)) \right| = \left| T(T^n(x_0)) - T^n(x_0) \right| \to 0$$

as $n \to \infty$, meaning that $T(x_\infty) = x_\infty$, analyticity is preserved through uniform convergence. The terms T^n are the partial sums of the *Neumann series*. To avoid confusion regarding the notation, we set this $x_\infty(\tau) = \underline{\hat{v}}_1(\tau) \in \mathcal{X}$. We apply the same procedure infinitely many times, each time extending the domain as $S_{2^k}(A) = \mathbb{D}(0, 2^{k+1}r) \cap S_A$ (each time we "double" the domain of definition), similarly: $\underline{S}_{2^k}(A) = S_{2^k}(A) \setminus \mathbb{D}(0, 2^k r)$ and obtain a solution $\hat{v}_{2^k}(\tau)$. Consequently we have extended $\hat{v}_0(s)$ to an analytic function $\hat{v}_{2^k}(s)$ on $S_{2^k}(A)$.

Thereafter, we can inductively (in k) show that that $\hat{v}(s)$ can be extended to an analytic function on $\mathbb{D}_0 \cup S_A$ ($r < 1$ was arbitrary, and $\hat{v}_0(\tau)$ is analytic on $\mathbb{D}_0$).

Now, we prove that this analytic continuation is of exponential type: We will study the problem inductively for $s \in S_n(A) = \mathbb{D}(0, (n+1)r) \cap S_A$, $\underline{S}_n(A) = S_n(A) \setminus \mathbb{D}(0, nr)$ and proceed with induction in n. To avoid confusion, notice that in the part where we showed existence we doubled the radius $r \mapsto 2r$.

Now, we just increase it by r: $nr \mapsto (n+1)r$. Without loss of generality we consider the case with $Re(s) > 0$ (the case where $Re(s) < 0$ is similar). For $n = 1$ we have:

$$|\hat{v}_1(s)| \leq |f_0(s)| + \left| \int_{s_0}^s |\hat{v}_1(\tau)| |K(s,\tau)| d\tau \right| \leq M_0 + L_0 \int_1^x |\hat{v}_1(e^{ia}y)| dy \tag{2.30}$$

where, $x = se^{-ia} \in \mathbb{R}$, $M_0 = |\hat{v}_0^{*3}|$, $L_0 = |K(s,\tau)|$, both considered on the line segment: $\{|s_0| \leq |s'| \leq |s| : Arg(s) = a\}$ and $y = \tau e^{-ia}$ (and so $d\tau = e^{ia}dy$).

We set $V_1(x) = \int_1^x |\hat{v}_1(e^{ia}y)| dy$, $x \in [1,2]$ and equivalently we have (for $' = \frac{d}{dx}$):

$$V_1'(x) \leq M_0 + L_0 V_1(x), \quad V_1(x_0) = 0. \tag{2.31}$$

We consider the initial value problem (for $x \in [1,2]$):

$$\bar{V}_1'(x) = M_0 + L_0 \bar{V}_1(x), \quad \bar{V}_1(x_0) = 0. \tag{2.32}$$

For which the (unique) solution is: $\bar{V}_1(x) = -\dfrac{M_0}{L_0} + \left(\dfrac{M_0}{L_0}\right) e^{L_0(x-x_0)}$.

By standard theory of ODEs (Grönwall's inequality) we get:

$\bar{V}_1(x) \geq V_1(x)$, $\implies M_0 e^{-1} e^{L_0|s|} = M_0 e^{L_0(|s|-|s_0|)} = M_0 + L_0 \bar{V}_1(x) \geq M_0 + L_0 V_1(x) \geq |\hat{v}_1(s)|$. i.e. $|\hat{v}_1(s)|$ is of exponential order L_0. Also, we have $|s_0| = |e^{ia}| = 1$ and s has the same direction ($Arg(s) = a$) and continues from $s_0 = e^{ia}$, which is why we get: $|s \pm s_0| = |s| \pm |s_0|$.

Now, we take $n \in \mathbb{N}$ and $L, M > 0$ such that $|\hat{v}_k(s)| \leq M e^{L|s|}$ for $s \in \underline{S}_n(A)$, $\forall k \in \{1, 2, ..., n\}$ and search for possible exponential bounds of $|\underline{\hat{v}}_{n+1}(s)|$.

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We set $\hat{\underline{v}}_k(s) = \hat{v}_k(s)\mathbf{1}_{\underline{S}_k(A)}$ for $k = 1, 2, .., n$ and $\hat{\underline{v}}_0(s) = \hat{v}_0 \mathbf{1}_{\mathbb{D}_0}$ we have:

$$\hat{\underline{v}}_{n+1}(s) = \frac{\hat{\underline{v}}_{n+1}^{*3}(s)}{s^2(s^2+1)} = \frac{1}{s^2(s^2+1)}\Big(\sum_{k=0}^{n+1} \hat{\underline{v}}_k\Big)^{*3}(s). \tag{2.33}$$

Because trivially, we have: $\hat{\underline{v}}_{n+1}(s) = \sum_{k=0}^{n+1} \hat{\underline{v}}_k(s)$ for $s \in \underline{S}_{n+1}(A)$, and this sum equals $\hat{v}_{n+1}(s)$ for all $s \in S_{n+1}(A)$. However, we have: $\hat{\underline{v}}_{n+1}^{*j}(s) = 0$ for $j = 2, 3$ and $\hat{\underline{v}}_{n+1} * \hat{\underline{v}}_k(s)$ for $k = 2, 3, ..., n-1$.

$$\hat{\underline{v}}_{n+1}(s) = \frac{1}{s^2(s^2+1)}\Big(\sum_{k=0}^{n} \hat{\underline{v}}_k\Big)^{*3}(s) + \frac{3}{s^2(s^2+1)}\big(\hat{\underline{v}}_0^{*2} * \hat{\underline{v}}_{n+1}\big)(s) \tag{2.34}$$

$$\implies |\hat{v}_{n+1}(s)| \leq \frac{1}{|s^2+1||s|^2} \sum_{S_{3,n}} |\hat{\underline{v}}_{i_1} * \hat{\underline{v}}_{i_2} * \hat{\underline{v}}_{i_3}| + \frac{3}{|s^2+1||s|^2}|\hat{\underline{v}}_0^{*2} * \hat{\underline{v}}_{n+1}(s)|. \tag{2.35}$$

As before, we set $s = e^{ia}x$ for $x \in (n, n+1)$ we have (the n^2 term comes from the number of terms in the sum in (2.35) - $|S_{3,n}| = (n^2 + 5n)/2$):

$$|\hat{\underline{v}}_{n+1}(s)| \leq \frac{M^3 n^2}{2|s^2+1||s|^2}e^{L_0|s_n|} + L_s \int_n^x |\hat{\underline{v}}_{n+1}(e^{ia}y)|dy \tag{2.36}$$

where $L_s = \frac{3}{|s|^2|s^2+1|}|\hat{\underline{v}}_0 * \hat{\underline{v}}_0| \leq \frac{3}{|s_0|^2|s_0^2+1|}|\hat{\underline{v}}_0 * \hat{\underline{v}}_0| \leq L_0$ for $s \in [ns_0, (n+1)s_0]$ with $s_0 = e^{ia}$.

So, we get:

$$|\hat{\underline{v}}_{n+1}(s)| \leq \frac{M^3 e^{L_0|s_n|}}{2|s^2+1|} + L_0 \int_n^x |\hat{\underline{v}}_{n+1}(e^{ia}y)|dy \leq \frac{M^3 e^{nL_0}}{2|n^2-1|} + L_0 \int_n^x |\hat{\underline{v}}_{n+1}(e^{ia}y)|dy \tag{2.37}$$

we apply the steps presented in the case $n = 1$ and we get (for $n \in \mathbb{N} \setminus \{1\}$):

$$|\hat{\underline{v}}_{n+1}(s)| \leq \frac{M^3 e^{L_0|s_n|}}{2|n^2-1|}e^{L_0(|s|-|s_n|)} = \frac{M^3}{2(n^2-1)}e^{L_0|s|}. \tag{2.38}$$

So, if $\bar{M}_n$ is the infimum of the upper bound of $\hat{\underline{v}}_n(s)e^{-L_0|s|}$, on $[0, s_n]$, then the infimum of the upper bound of $|\hat{v}_{n+1}(s)|e^{-L_0|s|}$ on $[s_n, s_{n+1}]$ is bounded above by $\frac{\bar{M}_n^3}{2(n^2-1)} \leq \frac{\bar{M}_n^3}{n^2}$, since $n \geq 2$.

If $n \geq \sqrt{\frac{\bar{M}_n^2}{2} + 1}$, then $\bar{M}_n \geq \frac{\bar{M}_n^3}{2(n^2-1)}$ and proceed inductively.

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If not true, we continue, and without loss of generality we set: $M_{n+1} \equiv \dfrac{\bar{M}_n^3}{n^2}$ and apply the procedure for $n+2$ to get an upper bound for $|\hat{v}_{n+2}(s)|e^{-L_0|s|}$ in $[s_{n+1}, s_{n+2}]$:

$$M_{n+2} \equiv \frac{M_{n+1}^3}{(n+1)^2} = \frac{\bar{M}_n^9}{n^6(n+1)^2}$$

We apply the same procedure k times and we obtain:

$$M_{n+k} \equiv \frac{M_{n+k-1}^3}{(n+k-1)^2} = ... = \frac{\bar{M}_n^{3^k}}{n^{2(3)^{k-1}}(n+1)^{2(3)^{k-2}}...(n+k-1)^2}.$$

We have 3^k terms equal to $\bar{M}_n$ in the nominator and: $2 \cdot (1 + 3 + ... + 3^{k-1}) = 3^k - 1$ terms varying from n to $n+k-1$, where each $n+i$ appears (at least) twice for $i = 0, ..., k-1$, in the denominator. We have n and M_n fixed, so for k sufficiently large, we have $n+i \geq \bar{M}_n$ for all $i \geq k_0$ for some $k_0 \in \{0, 1, ..., k-1\}$. That means: $1 \geq \dfrac{\bar{M}_n}{n+i}$ for all $i \geq k_0$, moreover: $1 \geq \dfrac{\bar{M}_n}{n+k_0} \geq \dfrac{\bar{M}_n}{n+k_0+1} \geq ... \geq \dfrac{M_n}{n+k-1} > 0$. which means that M_{n+m} has a maximum at $m = k_0 - 1$ or k_0, and for $m \geq k_0 + 1$ it start decreasing. In other words, we have that $M_{n+k} \geq 0$ is bounded as a sequence of $k \in \mathbb{N}$ ($n \in \mathbb{N}$ is fixed). Therefore, there is an $M_a > 0$ such that $\bar{M}_m \leq M_a$ for all $m \in \mathbb{N}$, so $|\hat{v}_m(s)| \leq M_a e^{L_0|s|}$ for $s \in [0, s_m] = [0, me^{ia}]$, for all $m \in \mathbb{N}$. We set $L_0 = L_a$ and since this estimate is uniform in $m \in \mathbb{N}$, Proposition 2.9 is proved. $\qquad\square$

Now, we can apply the *Laplace transform* along an arbitrary (non-singular: with direction other than $\pm\pi/2$) ray. Set $a \in (0, \pi/2)$ and consider a set of directions $A^+ = (-a, a)$ and its reflection with respect to the imaginary axis $A^- = (\pi - a, \pi + a)$. For $\theta^+ \in A^+$ we have $\theta^- = \pi - \theta^+ \in A^-$, and:

$$v_\theta(\tau) = \mathcal{L}_\theta[\hat{v}(s)](\tau) := \int_0^{\infty e^{i\theta}} e^{-\tau s}\hat{v}(s)ds \quad \text{where } \theta = \theta^\pm \tag{2.39}$$

Figure 3 provides a helpful illustration of these paths of integration. Note that for each $\theta_1, \theta_2 \in A^+$ (equiv. A^-), the domain of definition of $v_{\theta_1}(\tau)$ and $v_{\theta_2}(\tau)$ have non-empty intersection that consists of one connected component. Also, for such $\theta_{1,2}$, since $\hat{v}_{\theta_{1,2}}(s)$ is analytic (and of exponential type) for each $\theta \in A^+$, $v_{\theta_{1,2}}(\tau)$ agree on their common subset of definition, by the Identity theorem from standard Complex Analysis. Since $\theta_{1,2} \in A^+$ (resp. A^-) are arbitrary we can define $v_{A^\pm}(\tau)$ the extension of $v_0(\tau) = \mathcal{L}_0[\hat{v}(s)](\tau)$ where $\theta = 0$ in (2.40) as θ runs all over A^+ (resp. A^-).

The domain of definition ($\subseteq \mathbb{C}$, with analytic properties) of $v_{A\pm}(z)$ is the union of half-planes: $\mathbb{H}^+_{A\pm,c} = \bigcup_{\theta \in A\pm} \mathbb{H}^+_{\theta,c}$ where c is the exponential type of $\hat{v}(s)$. Notice that the domains of definitions of $v_{A\pm}(\tau)$ have two totally disconnected components, one in the upper and one in the lower half-plane. In fact, these are the sectors, S_U, S_L such that:

$$\mathbb{H}^+_{A+,c} \cap \mathbb{H}^+_{A-,c} = S_U \cup S_L \tag{2.40}$$

with $S_U \subseteq \mathbb{H}_U$ and $S_L \subseteq \mathbb{H}_L$ ("upper" and "lower" half-plane respectively). However, $v_{A\pm}(\tau)$ don't agree on $S_U \cup S_L$. In fact we are now ready to prove Theorem 2.1 which is our main result:

Proof of Theorem 2.1. Let $\theta^+ \in A^+ \cap (0,\pi)$, $\theta^- = \pi - \theta^+$ and $v_\pm(\tau)$ be as defined in (2.39) for $\tau \in S_L$. We have:

$$v_{\theta^+}(\tau) - v_{\theta^-}(\tau) = \int_0^{\infty e^{i\theta^+}} e^{-\tau s} \hat{v}(s) ds - \int_0^{\infty e^{i\theta^-}} e^{-\tau s} \hat{v}(s) ds = \int_{C^+} e^{-\tau s} \hat{v}(s) ds \tag{2.41}$$

where the contour of integration is $C^+ := (-\infty e^{i\theta^-}, 0] \cup [0, \infty e^{i\theta^+})$ (and s is moving counterclockwise, i.e. "positive" direction). By utilizing Lemma 2.8 and Proposition 2.9 we get:

$$\int_{C^+} e^{-\tau s} \hat{v}(s) ds = \frac{C_s}{2i} \int_{C^+} \frac{e^{-\tau s}}{s^2 + 1} ds + \frac{3C_s}{2} \int_{C^+} \left(\ln(s^2 + 1) + r(s) \right) e^{-\tau s} ds \tag{2.42}$$

The term $\ln(s^2 + 1)$ is analytic on the cut-plane $(\mathbb{C} \setminus \{i\mathbb{R}\}) \cup \mathbb{D}_0 \ (\supseteq C^+)$ and can be decomposed as $\ln(s^2 + 1) = \ln(s - i) + \ln(s + i)$ there and the term $r(s)$ is defined as:

$$r(s) = \hat{v}(s) - \frac{C_s}{s^2 + 1} - \frac{3C_s}{2} \ln(s^2 + 1) \text{ for } s \in \left(S_{A^+} \cup S_{A^-} \right) \cup \mathbb{D}_0$$

By standard analysis and the aforementioned results we have that $\ln(s \pm i)$ and $r(s)$ are of exponential type, i.e. $|\ln(s \pm i)|, |r(s)| \leq M e^{c|s|} \ \forall s \in \left(S_{A^+} \cup S_{A^-} \right) \setminus \mathbb{D}_0$ for some $M, c > 0$. They are also bounded in $\mathbb{D}_0$:

$$\left| \int_0^{\infty e^{i\theta^\pm}} e^{-\tau s} \left(\ln(s^2 + 1) + r(s) \right) ds \right| = O\left(\frac{1}{\tau} \right) \text{ as } \tau \to \infty \tag{2.43}$$

Also: $\left| \int_{C^+} e^{-\tau s} \left(\ln(s^2 + 1) + r(s) \right) ds \right| = O\left(\frac{e^{-\tau i}}{\tau} \right)$ as $\tau \to \infty$.

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The Residue theorem and other standard tools from Complex Analysis give:

$$\int_{C^+} \frac{C_s e^{-\tau s}}{s^2 + 1} ds = Res\Big(\frac{C_s e^{-\tau s}}{s^2 + 1}\Big)\Big|_{s=i} = \pi C_s e^{-\tau i} \tag{2.44}$$

Combining equations (2.41) to (2.44) we get eq. (2.4). By formula (2.39) and Lemma 1.2 the asymptotic condition in Theorem 2.1 is guaranteed. Uniqueness follows by the fact that the coefficients c_k of the formal sum $\tilde{v}(\tau)$ are unique, and they are determined by the recurrence relation in (2.6), which defined (a unique) $\hat{v}_0(s)$ by (2.17) which is uniquely extended to an analytic function on a suitable set (Proposition 2.9). $\qquad \square$

The proof of Corollary 2.1 follows by an almost identical manner.

Remark 2.10 (On uniqueness of solutions satisfying asymptotic conditions). Clearly, equation (2.3) has no unique solution on $S_U \cup S_L$ (defined in (2.40)), even with the asymptotic condition $v_0(\tau) \simeq \tilde{v}_0(\tau)$. Moreover, equation (2.3) is autonomous, which means that if $v_0(\tau)$ is a solution on a domain D_0 then $v_0(\tau+a)$ too will solve (2.3) on a domain D_a (typically other that D_0 unless for it is invariant under a set of certain translations - such as when D_0 is the right half-plane and a is purely imaginary). Making the substitution $\tau \mapsto \tau + a$ we see that the (formal) sum $\tilde{v}(\tau)$ "solves" (2.3) formally while having the same leading term as $\tilde{\tau} \to \infty$. Therefore we do not expect uniqueness of solutions induced by the leading term of the asymptotic expansion. Moreover, we can still obtain a solution under the sign change $\sqrt{2} \to -\sqrt{2}$ (the choice of sign might be determined by a necessary condition for the Inner and Outer solution to match) in the leading order coefficient (c_0 in (2.6)) of the asymptotic expansion (this forces the rest c_k to change too). However, by predetermining the sign of the leading order coefficient and imposing $v_0(\tau) \simeq \sum_{k=0}^{\infty} c_k \tau^{-2k-1}$ (for c_k determined accordingly) we eliminate both of the aforementioned freedoms and obtain a unique solution.

Moreover, using ideas similar to the ones on Proposition 3.1 (see below in Section 3) one may be able prove that the condition $v_0(\tau) = \sqrt{2}\tau^{-1} + g_1(\tau)$ with $g_1(\tau) = O(\tau^{-3})$ at infinity, g_1 analytic on a suitable half-plane $P \subseteq \mathbb{C}$ is enough to guarantee uniqueness: Considering the difference between two solution and studying a "linearized" version of (2.3) and yielding two trigonometric & two exponential non-trivial solutions, none of which vanishing along all directions of the half-plane P.

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3 Numerics

In Section 2, we saw that the limit $\lim_{n\to\infty} C_n$ $(= \lim_{k\to+\infty} (-1)^k (c_k/(2k)!))$ (provided that it exists) defines a constant, Γ, that characterizes the "exponentially small" differences of functions that are asymptotic to the same (*Gevrey*) formal power series (and the potential "splitting of separatrices" as explained in the end of the previous section). In this chapter we evaluate the constant Γ using two independent computational methods, based on the theory we presented before.

The <u>first method</u> consists of evaluating C_n for some large values of n and extrapolating based on the formula (2.7). The coding is done in Mathematica 12.0 (see the end of this section - first 6 lines of code). This method gives for example: $C_{200}(= c_{200}/400!) \approx 6.31463$.

Considering Proposition 2.6 we have that the rate of convergence is "$O(1/n)$", so it is instructive to extrapolate by best-fitting the functions $1/n^k, k = 0, 1, .., m$ for some $m \in \mathbb{N}$. That is:

Choose integers $n_{max} \gg n_0 \gg 1$ and fit $\{1, 1/n, .., 1/n^m\}$ to the data $\{(n_0, C_{n_0}), .., (n_{max}, C_{n_{max}})\}$ (n_i equidistant) for some $1 \leq m < 2\sqrt{n_{max}}$ using the Mathematica built-in function "Fit", which finds the best-fitting combinations of functions to data by the method of least squares. We set $m = 1$ and compute:

```
Fit[Table[{i, N[C[i]]}, {i, 50, 100}],Table[1/n^k, {k, 0, 1}],n]
```

Such a computation yields a function: $f(n) = a + b/n$, with $a \approx 6.35$, and $b \approx -9.01$.

Intuitively, this is a function that describes the limiting behaviour of C_n. This result is in agreement with Lemma 2.2. Using standard tools from Linear Algebra (such as Cramer's rule) carefully we can show(after slightly abusing notation) that for the resulting $f(n) = a_n + b_n/n$, we have $a_n = \Gamma + O(1/n^2)$ and $b_n = -(3/2)\Gamma + O(1/n)$, which means that this extrapolation method provides a better approximation by design. Similarly we can show that directly extrapolating the data by solving the following system:

$$\begin{pmatrix} 1 & \frac{1}{n} \\ 1 & \frac{1}{n-1} \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} C[n] \\ C[n-1] \end{pmatrix} \tag{3.1}$$

yields a similar approximation.

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Regarding extrapolation to a higher order ($m > 1$), in both of those extrapolation tricks, we naively expect to obtain an approximation like: $a_n = \Gamma + O(n^{-m-1})$, just as happens when $m = 1$. The results strongly agree with this expectation (the higher the order - the more and more first digits stabilize, see Figure 4) indicating that $C_n = \Gamma - (3/2)\Gamma n^{-1} + \sum_{k=2}^{m+2} C_{s,k} n^{-k} + O(n^{-m-3})$ for some constants $C_{s,k}$ and large $n \in \mathbb{N}$. A similar expansion evidently holds in the case of the case of the *Second method* (see the next page) as well. Having computed the first 100 coefficients C_n, we fix $n_{min} = 50$ and $n_{max} = 100$, and experimenting with the order m, we find that the first 10 decimal digits stabilize pretty quickly as m grows (not indefinitely), see the table at the end of the text. Surprisingly, the first 10-decimals approximation persists for m even larger than the bound $RU_{ph} := 2\sqrt{n_{max} - n_{min} + 1}$ $(= 2\sqrt{\#data})$ set to avoid the *Runge's phenomenon*, if $m > RU_{ph}$ the coefficients of $1/n^k$ for k large start to diverge but the 10-12 decimals approximation persists. The same approximation is obtained even as n_{min} increases, and remains the same even when the number of data $\#data$ is as small as 10. After several such experiments and keeping attention to some more decimals, we find that Γ, with 12 decimal digits precision, is: **6.361878187125**.

Lastly we calculate the first $n = 365$ coefficients with precision of at least 800 decimal digits and start extrapolating with the method of *least squares* (on a sample $\{C[n'], C[201], ..., C[n]\}$, with $n' = 200$ for example) as well as directly, first at order 1. We notice (in both methods) that as the order of extrapolation increases the data change "in a continuous manner" stabilizing more and more first decimal digits, until it reaches a critical value that depends on the number of data (the higher the number the higher the critical value) where the pattern breaks and the results become inconsistent. However, this comes as no surprise to us due to *Runge's phenomenon*. Slightly varying the parameters of the problem, mainly the number n of terms, the number n' in the sample in the *least squares fitting* and experimenting the order of extrapolation we see these two extrapolation tricks (with order $m = 10 \pm 0, \pm 1, \pm 2...$) tend to agree in the first 20 digits (unless we get very loose with the parameters). Also, for fixed m and fixed $n - n'$, the approximation obtained by the *least squares fitting* looks like it converges when n' increases, similarly when m', n' are fixed and n increases, taking these in mind we've obtained an indication that: $\mathbf{\Gamma \approx 6.36187818712587330176}$.

Now we move on to the second method and cross-check the obtained results.

Recall $v_{\theta\pm}(\tau)$ from eq. (2.39). For simplicity, since the values of θ^+ and $a > 0$ do not really matter we drop $\theta^\pm \in A^\pm$ from the subscript of $v_\pm(\tau)$ and simply denote them by $v_\pm(\tau)$.

The <u>second method</u> consists of directly approximating the difference of solutions $v_{\theta\pm}(\tau)$, with "initial conditions" governed by the formal sum $\tilde{v}_0(\tau)$:

<u>Step 1</u>: Pick a $z_0 = -i\psi \in \{z \in \mathbb{C} : Re(z) = 0, Im(z) < -10\}$[1]

<u>Step 2</u>: Optimally truncate the series with respect to z $\left(\text{in this case } N(\psi) = \left\lfloor \frac{\psi}{2} \right\rfloor \right)$

<u>Step 3</u>: For $t \in [0, \psi]$, solve the ψ-parametric initial value problems ($\psi > 10$):

$$u_\pm''''(t) + u_\pm''(t) = u_\pm^3(t) \tag{3.2}$$

$$u_\pm^{(k)}(0) = \frac{d^k}{dz^k}\left(\sum_{n=0}^{N(\psi)} \frac{c_n}{z^{2n+1}} \right)\Big|_{z=\pm\psi-\psi i}, k = 0, 1, 2, 3 \tag{3.3}$$

where the contour of integration is: $\{\pm(\psi - t) - i\psi \in \mathbb{C} : t \in [0, \psi]\}$ respectively, i.e. the line segment that connects $\pm\psi - i\psi$ and $-i\psi \in i\mathbb{R}$.

<u>Step 4</u>: Calculate:

$$\lim_{\psi \to +\infty} e^\psi (u_+(i\psi) - u_-(i\psi))/\pi \tag{3.4}$$

However, unlike the first method which directly computes Γ by its definition, it's not very clear the limit in (3.4) equals Γ, even though it's intuitive. The initial conditions in (3.3) are not equal to $v_{\theta\pm}(\pm\psi - i\psi)$ (the conditions that uniquely define $v_{\theta\pm}(\tau)$) but provide an approximation. In fact they are "exponentially close" to them since they are the optimally truncated *Gevrey series* to which the $v_\pm(\tau)$ are asymptotic, see the *Exponential accuracy remark* in [**5**]. The reason why this method works is that the error of these approximations is smaller than the exponential quantity (eq. (2.42)) we try to evaluate.

We can now show the following proposition:

---

[1]The lower bound $Im(z) < -10$ is arbitrary, and we have chosen it to be 10 just for simplicity.

Proposition 3.1. *Let $v_{\pm}^{[N]}(\tau) : \{\pm(\psi - t) + i\psi | t \in [0, \psi]\} \subseteq \mathbb{C} \to \mathbb{C}$ be the numerical solutions obtained by the second method (i.e. the ones that solve the Initial Value problem (3.2)-(3.3) numerically) and assume they satisfy: $v_{\pm}^{[N]}(\tau) = c_0\tau^{-1} + o(\tau^{-2})$ for large $|\tau|$ (as in Proposition 2.7)[2], then we have:*

$$v_{+}^{[N]}(-i\psi) - v_{-}^{[N]}(-i\psi) = v_{+}(-i\psi) - v_{-}(-i\psi) + O\left(\psi^{3/2}e^{-\sqrt{2}\psi}\right) \tag{3.5}$$

Proof. It suffices to study the difference $v_{-}(-i\psi) - v_{-}^{[N]}(-i\psi)$, since by symmetry, we obtain information on the difference $v_{+}(-i\psi) - v_{+}^{[N]}(-i\psi)$ too. For simplicity, we drop the subscripts $\pm$ from $v_{\pm}, v_{\pm}^{[N]}$ for now. We set $\vec{x}(z) = (x_1, x_2, x_3, x_4)^T$ where $x_1 = v$, $x_2 = v'$, $x_3 = v''$ and $x_4 = v'''$ and considering ψ as a parameter we define $\vec{x}_{\psi}(t) := \vec{x}(-\psi + t - i\psi)$ and consider the equation:

$$\frac{d}{dt}\vec{x}_{\psi}(t) = A_0 \cdot \vec{x}_{\psi}(t) + f(\vec{x}_{\psi}(t)) \tag{3.6}$$

(see below - (3.10) - for A_0 and f) and equip it with the initial condition:

$$\vec{x}_{\psi}(0) := \vec{x}(-\psi - i\psi). \tag{3.7}$$

Similarly we define the "truncated" system:

$$\frac{d}{dt}\vec{x}_{\psi}^{[N]}(t) = A_0 \cdot \vec{x}_{\psi}^{[N]}(t) + f\left(\vec{x}_{\psi}^{[N]}(t)\right) \tag{3.8}$$

with the initial condition:

$$\vec{x}_{\psi}^{[N]}(0) = \vec{x}^{[N]}(-\psi - i\psi) \tag{3.9}$$

$$\text{for} \quad f(\vec{x}) = f(x_1, x_2, x_3, x_4) = (0, 0, 0, x_1^3)^T \ , \ A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.10}$$

---

[2]Numerical experiments very strongly support the previous assumption (as well as that: $v_{+}^{[N]}(\tau) - v_{-}^{[N]}(\tau) = \pi\Gamma e^{-i\tau}(1 + 3\tau^{-1} + O(\tau^{-2}))$, which is necessary in order to ensure that the obtained numerical solutions follow the respective actual solutions to (3.2)-(3.3) closely enough.

We define $\vec{w}(t) := \vec{x}_\psi(t) - \vec{x}_\psi^{[N]}(t) = (w_1(t), w_2(t), w_3(t), w_4(t))$ and by equations (3.8-9) we get that $\vec{w}(t)$ satisfies:

$$\vec{w'}(t) = A_\psi(t) \cdot \vec{w}(t) \tag{3.11}$$

with $\vec{w}(0) := \vec{x}_0 - \vec{x}_0^{[N]}$ where $A_\psi(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g_\psi(t) & 0 & -1 & 0 \end{pmatrix}$, and:

$$g_\psi(t) = v^2 + v\tilde{v}^{[N]} + (\tilde{v}^{[N]})^2 = \frac{6}{(t - \psi - i\psi)^2} + O\left(\frac{1}{\psi^4}\right) = -\frac{6}{\psi^2} + O\left(\frac{1}{\psi^3}\right) \text{ (as } \psi \to \infty, \text{ for } t \in [0, \psi])$$

defined as the coefficient of $w_1(t)$ (fourth dimension) by:

$$f(\vec{x}(t)) - f(\vec{x}^{[N]}(t)) = (0,0,0,v^3)^T - (0,0,0,(\tilde{v}^{[N]})^3)^T =$$
$$(0,0,0,(v^2 + v\tilde{v}^{[N]} + (\tilde{v}^{[N]})^2)(v - \tilde{v}^{[N]}))^T = \left(0,0,0,g_\psi(t)w_1(t)\right)$$

By standard theory of ODEs we can ignore the "$O(\psi^{-3})$"-correction terms in the aforementioned asymptotic expansion of $g_\psi(t)$ as they don't have any qualitative importance. So, we treat the later as $-6/\psi^2$ (the leading term of the asymptotic behaviour). With this in mind, we slightly abuse notation and so from standard theory, the system (3.11) has a unique solution given by:

$$\vec{w}(t) = \exp\left(\int_0^t A_\psi(s)ds\right)\vec{w}(0) \tag{3.12}$$

Also: $\vec{x}_\psi(0) - \vec{x}_\psi^{[N]}(0) = \vec{x}(z_0) - \vec{x}^{[N]}(z_0) = c_N z_0^{-2N-1} + O(z_0^{-2N-3})$, (at ∞) for $z_0 = -\psi + i\psi$. We fix z_0, and have (Lemma 2.2): $c_N = (-1)^N (2N)!\left(\Gamma - (\frac{3}{2}\Gamma)N^{-1} + O(N^{-2})\right)$, so for N in a neighbourhood of infinity we have:

$$\vec{x}_\psi(0) - \vec{x}^{[N]}(0) = \frac{(-1)^N (2N)!\Gamma}{z_0^{2N+1}}\left(1 + O\left(\frac{(2N)!}{N}\right)\right)$$

and for some $C \geq \Gamma$ we have:

$$|\vec{w}(0)| = |\vec{x}_0 - \vec{x}_0^{[N]}| \leq C\frac{(2N)!}{|z_0|^{2N+1}} \tag{3.13}$$

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From *Stirling's formula* we have: $N! = \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N} \left(1 + \frac{1}{12N} + O\left(\frac{1}{N^2}\right)\right)$

$$\Rightarrow (2N)! = \frac{\sqrt{2\pi}(2N)^{2N+\frac{1}{2}}}{e^{2N}}\left(1 + O\left(\frac{1}{N}\right)\right) = \frac{2^{2N}}{\sqrt{N\pi}}\left(\frac{\sqrt{2\pi}N^{N+\frac{1}{2}}}{e^N}\right)^2\left(1 + O\left(\frac{1}{N}\right)\right)$$

and $\dfrac{(2N)!}{|z_0|^{2N+1}} = \dfrac{\pi 2^{2N+1}}{\sqrt{N\pi}}\left(\dfrac{N^{2N+1}e^{-2N}}{|z_0|^{2N+1}}\right)\left(1 + O\left(\frac{1}{N}\right)\right) \Rightarrow \dfrac{(2N)!}{|z_0|^{2N+1}} \leq \dfrac{M}{\sqrt{N}}e^{-2N}$

for some $M > 0$ and $N = \left\lfloor \dfrac{|z_0|}{2} \right\rfloor \left(\leq \dfrac{|z_0|}{2}\right)$ (optimal truncation rule), and since $z_0 = -\psi - i\psi \Rightarrow$

$|z_0| = \sqrt{2}\psi$ we have: $\dfrac{(2N)!}{|z_0|^{2N+1}} \leq \dfrac{M}{\sqrt{N}}e^{-2N} \leq \dfrac{M\sqrt{2}e^2}{\sqrt{|z_0|-2}}e^{-|z_0|} = O\left(\dfrac{e^{-|z_0|}}{\sqrt{|z_0|}}\right) = O\left(\dfrac{e^{-\sqrt{2}\psi}}{\sqrt{\psi}}\right)$.

So, by (3.13) we have:

$$|\vec{w}(0)| \equiv ||\vec{w}(0)||_2 \leq \frac{\bar{M}e^{-\sqrt{2}\psi}}{\sqrt{\psi}} \tag{3.14}$$

for some $\bar{M} > 0$. In order to proceed with estimating $\vec{w}(\psi)$, we return to (3.12) and analyse the matrix $\int_0^\psi A_\psi(s)ds$, we have:

$$\int_0^\psi A_\psi(s)ds = \begin{pmatrix} 0 & \psi & 0 & 0 \\ 0 & 0 & \psi & 0 \\ 0 & 0 & 0 & \psi \\ G(\psi) & 0 & -\psi & 0 \end{pmatrix} \equiv \tilde{A}, \text{ where } G(\psi) = \frac{6}{\psi} + O\left(\frac{1}{\psi^2}\right).$$

For simplicity, we will ignore the $O(\psi^{-2})$ terms from $G(\psi)$ and from $\tilde{A}$, as in the limit $\psi \to +\infty$ they become so small compared to any other terms, that they do not have a qualitative effect on our results (no qualitative effect on the eigenvalues). We integrate eq. (3.11) from 0 to t, and obtain a first-order linear integral equation whose characteristic polynomial of $\tilde{A}$ evaluates as: $\chi_{\tilde{A}}(x) = x^4 + \psi^2 x^2 - 6\psi^2$ which gives us 4 distinct eigenvalues, two real and two (conjugate) purely imaginary: $\lambda_{1,2} = \pm\lambda_R = \pm\sqrt{6} + O(\psi^{-2})$ and $\lambda_{3,4} = \pm i\lambda_I = \pm i\left(\psi + O(1)\right)$ (as $\psi \to \infty$).

The corresponding eigenvectors are:

$\vec{v}_1 = (1, \lambda_R\psi^{-1}, \lambda_R^2\psi^{-2}, \lambda_R\psi^{-3})^T$, $\vec{v}_2 = (1, -\lambda_R\psi^{-1}, \lambda_R^2\psi^{-2}, -\lambda_R\psi^{-3})^T$,

$\vec{v}_3 = (1, -i\lambda_I\psi^{-1}, -\lambda_I^2\psi^{-2}, i\lambda_I^2\psi^{-3})^T$ and $\vec{v}_4 = \bar{\vec{v}}_3$

Therefore, from standard theory of *ODE's* and *Linear Algebra* (*Jordan Canonical Form*) we can "factorize" (decompose) the matrix $\tilde{A}$ as follows: $\tilde{A} = PJP^{-1}$ where:

$$J = \begin{pmatrix} \lambda_R & 0 & 0 & 0 \\ 0 & -\lambda_R & 0 & 0 \\ 0 & 0 & 0 & -\lambda_I \\ 0 & 0 & \lambda_I & 0 \end{pmatrix} \tag{3.15}$$

and $P := col(\vec{v}_1, \vec{v}_2, Re(\vec{v}_3), Im(\vec{v}_3))$ the (invertible) matrix whose columns consist of the eigenvectors $\vec{v}_{1,2}$ that correspond to the eigenvalues $\lambda_{1,2}$ respectively (first two columns) and the real and imaginary part of the (complex) eigenvector $\vec{v}_3 = \bar{\vec{v}}_4$ that corresponds to the eigenvalue $\lambda_3 = \bar{\lambda}_4$ as third and fourth column respectively. Again, we proceed with standard theory and we have:

$$e^{\tilde{A}} = e^{PJP^{-1}} = Pe^J P^{-1} = P \begin{pmatrix} e^{\lambda_R} & 0 & 0 & 0 \\ 0 & e^{-\lambda_R} & 0 & 0 \\ 0 & 0 & cos(\lambda_I) & -sin(\lambda_I) \\ 0 & 0 & sin(\lambda_I) & cos(\lambda_I) \end{pmatrix} P^{-1} = P \Big(\sum_{k=0}^{\infty} \frac{J^k}{k!} \Big) P^{-1} \tag{3.16}$$

Returning to eq. (3.12), we get: $|\vec{w}(\psi)| = |e^{\tilde{A}} \vec{w}(0)| \leq ||e^{\tilde{A}}||_2 ||\vec{w}(0)||_2 \leq ||P||_2 ||P^{-1}||_2 ||e^J||_2 ||\vec{w}_0||_2$. However, $||P||_2 = O(1)$ and $||P^{-1}||_2 = O(\psi^2)$ as $\psi \to \infty$, and that is because: $|\det(P)| = 4\sqrt{6}\psi^{-2} + O(\psi^{-3})$ and since $P^{-1} = \det(A)^{-1} adj(P)$ and the elements of $adj(P)$ (i.e. the sub-determinants of P) are bounded (they are "at most $O(1)$"). In other words, we have:

$$v_-(-i\psi) - v_-^{[N]}(-i\psi) = O\Big(\psi^{3/2} e^{-\sqrt{2}\psi} \Big) \tag{3.17}$$

Similarly for $v_-(-i\psi) - v_-^{[N]}(-i\psi)$ and by subtracting these and rearranging a few terms we get (3.5). $\square$

Corollary 3.2. The *first* and *second* method are consistent with each other in computing the same constant.

Now that we know that the second method agrees with the first, we implement it in Mathematica 12.0. The code is at the end of this section and evaluates $e^{200}(u_+(200i) - u_-(200i))/\pi \approx 6.26830$, the first 6 lines calculate the first 200 terms of the sequence c_n with practically arbitrary precision. Notice the symmetry $v_-(-\tau) = -\overline{v_+(-\overline{\tau})}$, and so when $\tau = -i\psi \in i\mathbb{R}_{<0}$ we have that $v_-(-i\psi) = -\overline{v_+(-i\psi)}$ and so $e^{\psi}(v_+(-i\psi) - v_-(-i\psi)) = -2e^{\psi}Re(v_-(-i\psi))$, i.e. the solution $u_-(\tau)$ is enough to approximate the *Stokes constant*. This trick halves the computation time. Another approximation is $e^{300}(u_+(300i) - u_-(300i))/\pi \approx 6.2991$

Proposition 2.7 indicates that extrapolating by best-fitting functions of type $1/n^m$ to the data (obtained by the *second method*) is fruitful (strongly indicates it for $m = 1$). We apply:

Step 1: Calculate: $C_k^* = e^{\psi_k}(v_+(-i\psi_k) - v_-(-i\psi_k))$ when y_k, $k = 1, 2, ..., k_{max}$ are equidistant integers such that $y_{k+1} - y_k = L$, for each $k = 1, 2, ...k_{max} - 1$, where y_1 is a "large enough" value, let $y_1 \geq 25$, and let $y_{k_{max}} \leq max\{2y_1, 200\}$ for some $L \in \mathbb{N}^*$ and $k_{max} \geq k_0$ a sufficiently large amount of calculations.

Step 2: Find the best-fit of the functions $\{1, n^{-1}, ..., n^{-m}\}$ for $1 \leq m < 2\sqrt{k_{max}}$ ($m \in \mathbb{N}$) to the data $\{(y_1, C_1^*), ..., (y_k, C_k^*), ..., (y_{k_{max}}, C_{k_{max}}^*)\}$, where the last inequality is imposed in order for us to avoid *Runge's Phenomenon*.

As in the case of the *first method*, we apply the procedure several times, each time with setting different values at the parameters ($k_{max}, y_1, y_{k_{max}}, L$ and m): In particular we repeat the experiment after applying small changes to these parameters and study how do they affect the result, by observing whether the final result changes "continuously" (small changes to parameters $\mapsto$ small changes to the result) and how consistent it is with the results obtained by the *first method*. After several repetitions, we can safely say that the final results depend continuously on the values of the aforementioned parameters. However, this method is a lot more "unstable" than the *first method* (small changes to parameters result in larger and larger changes of the result), but this comes as no surprise to us since almost every common method of numerical integration will produce more errors, as they require multiple steps to be executed. Keeping this in mind, we observe that the extrapolated results tend to agree in the first 5-10 decimal digits, unless we get very loose with parameters (e.g. let y_1, k_{max} become very small) which means that at least the first 5 decimal

digits these two results have in common are precisely the decimal digits of the Stokes constant. The reason is that since these two methods work independently, it is impossible of these two being both wrong while agreeing to the same ("wrong") result, especially when they keep agreeing when small changes on their parameters are applied to them on each of the many repetitions.

Remark 3.3. The second method does not work if $Re(z_0)$ is chosen too large or small compared to $Im(z_0)$. For the first case the reason is that the numerical errors increase as $Re(z_0)$ increases because the integration takes place on a larger interval (more steps implies more errors). On the other hand, if we choose $Re(z_0)$ to be small enough, then the error on the initial conditions $v_0 - v_0^{[N]} = O(|z_0|^{-N-1})$ will be "comparable" to $e^{Im(z_0)}$, ($Im(z_0) < 0$ and $|Im(z_0)|$ large) and the term $e^{-Im(z_0)}$ which will be multiplied by: $v(-i\psi) - v^{[N]}(-i\psi)$ for the computation of the Stokes constant will magnify the errors. So, as $\psi = |Im(z_0)|$ increases, a choice of $Re(z_0)$, like $Re(z_0) = \psi = Im(z_0)$ is what we need to obtain an accurate approximation of Γ.

The code:

```
c[0] = Sqrt[2];
d14[n_] := (2n-1)(2n)(2n+1)(2n+2)c[n-1]
d12[n_] := (2n+1)(2n+2)c[n]
m13[n_] := Sum[Sum[c[n-k]c[k-m]c[m],{m,0,k}], {k,0,n}]
EQ1[n_] := Evaluate[d14[n] + d12[n] - m13[n]]
Do[c[nn] = c[nn] /. Solve[EQ1[nn] == 0][[1]];,{nn,1,200}]
y = 200; d = Floor[y/2];
xl1[z_] = Sum[c[n]/(z-(y)I)^(2n+1),{n,0,d}];
xl2[z_] = Sum[(-(2n+1)c[n])/(z-(y)I)^(2n+2),{n,0,d}];
xl3[z_] = Sum[((2n+1)(2n+2)c[n])/(z-(y)I)^(2n+3),{n,0,d}];
xl4[z_] = Sum[-(2n+1)(2n+2)(2n+3)c[n]/(z-(y)I)^(2n+4),{n,0,d}];
soll = NDSolve[{xl''''[t]+xl''[t] == (xl[t])^3, xl[-y] == xl1[-y], xl'[-y] == xl2[-y],
xl''[-y] == xl3[-y], xl'''[-y] == xl4[-y]}, xl,{t,-y,0}, Method -> "StiffnessSwitching",
AccuracyGoal -> 100, PrecisionGoal -> 200, WorkingPrecision -> 200];
x1 = xl[0] /. soll; -2Re[x1*Exp[y0]]/Pi
```

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4   Discussion

Problems similar to the one that motivated our study are studied in [**6**],[**7**],[**8**] and [**9**] as well as in [**10**] where the author points out some critical mistakes in the literature. The case of the *Discrete Klein-Gordon* model with bistable nonlinearity as presented in [**1**] (example 3) may be treated in same way too, but a proof of that diverges from the purpose of this paper, which is to present a simplified approach to these problems with computational methods that lead to strong numerical results regarding the *Stokes phenomenon*.

The same procedure can easily be applied to problems similar to our main problem but with nonlinearities higher than cubic. The corresponding formulae for C_k will involve stronger nonlinearities yet they will be able to be tackled by the tricks in Proposition 2.6 (which can also be used to obtain bounds for the *Stokes constant*). The formal series that emerges will have a algebraic branch point but we will still be able to extract a solution in a form of a Laplace transform. Formally, we may obtain a *transseries* solution to these problems in the form $\sum_{k=0}^{\infty} e^{\pm ik\tau} \tilde{v}_k(\tau)$, where $\tilde{v}_k(\tau)$ is a formal negative power series and this analysis is expected to give insights to higher order correction terms, but rigorous theory is still developing. One way to proceed rigorously, involves the implementation of *Resurgent Theory*, which loosely speaking, studies formal expansions and generalized *Borel-Laplace* transforms of complex functions with isolated singularities. This theory is a very powerful mathematical tool as it can be applied to a wide class of dynamical systems, see for example [**11**] for an introduction.

Lastly, it is worth mentioning that in 2015, P. H. Trinh and S. J. Chapman ([**13**]) published a paper where they studied certain classes of water wave problems, in which the terms of the corresponding asymptotic expansion are characterized by "exponential over power" rate of divergence.

Acknowledgements

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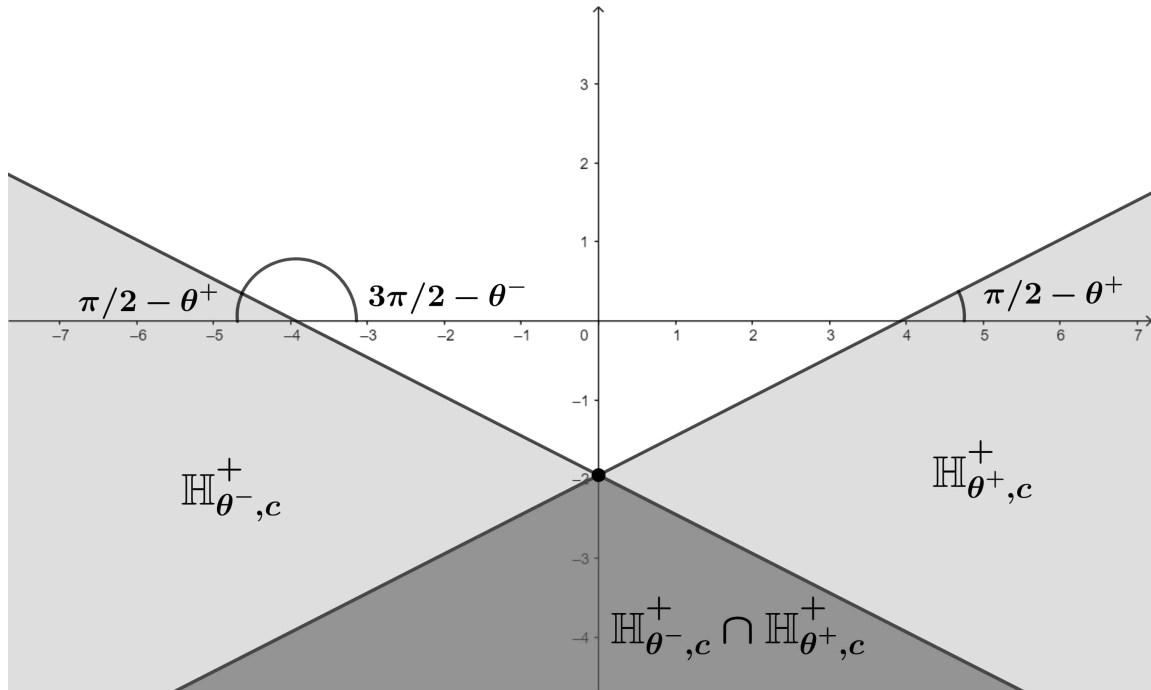
Figure 1: The half-planes $\mathbb{H}^+_{\theta^\pm,c}$ are where $v_{\theta^\pm}(\tau)$ (2.39) are defined respectively. The point of intersection $\tau_0 \approx -2i$ of $\partial\mathbb{H}^+_{\theta^\pm,c}$ and c (the exponential type of $\hat{v}(s)$) satisfy $c = Re(e^{i\theta^\pm}\tau_0)$.
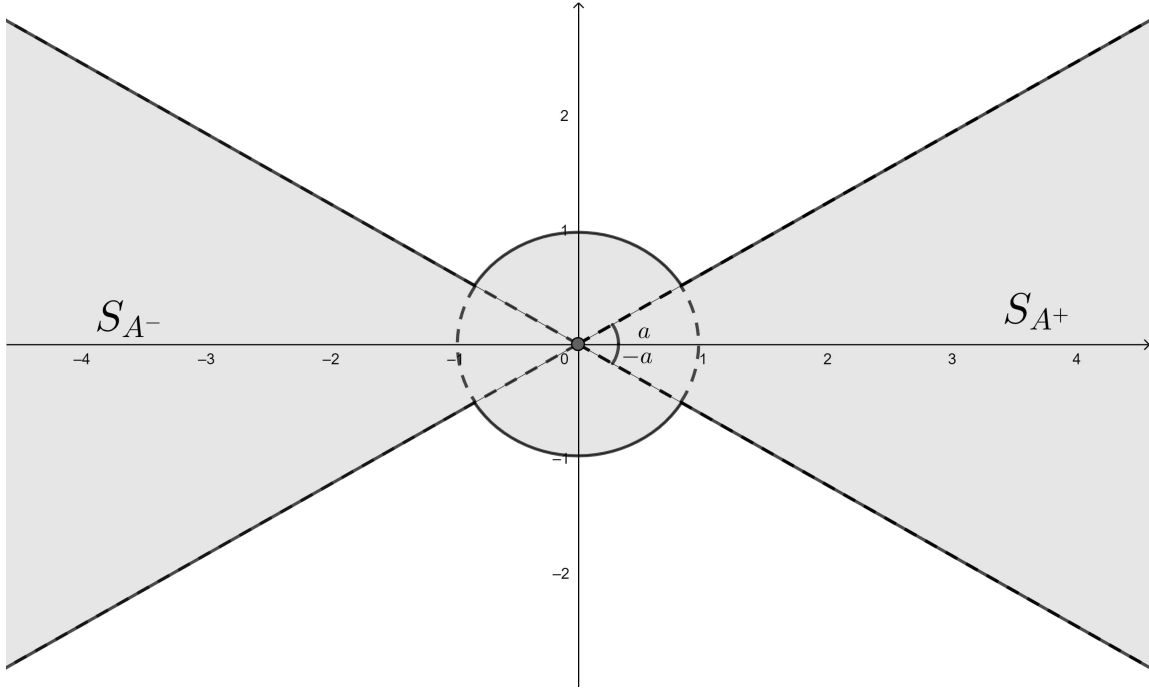
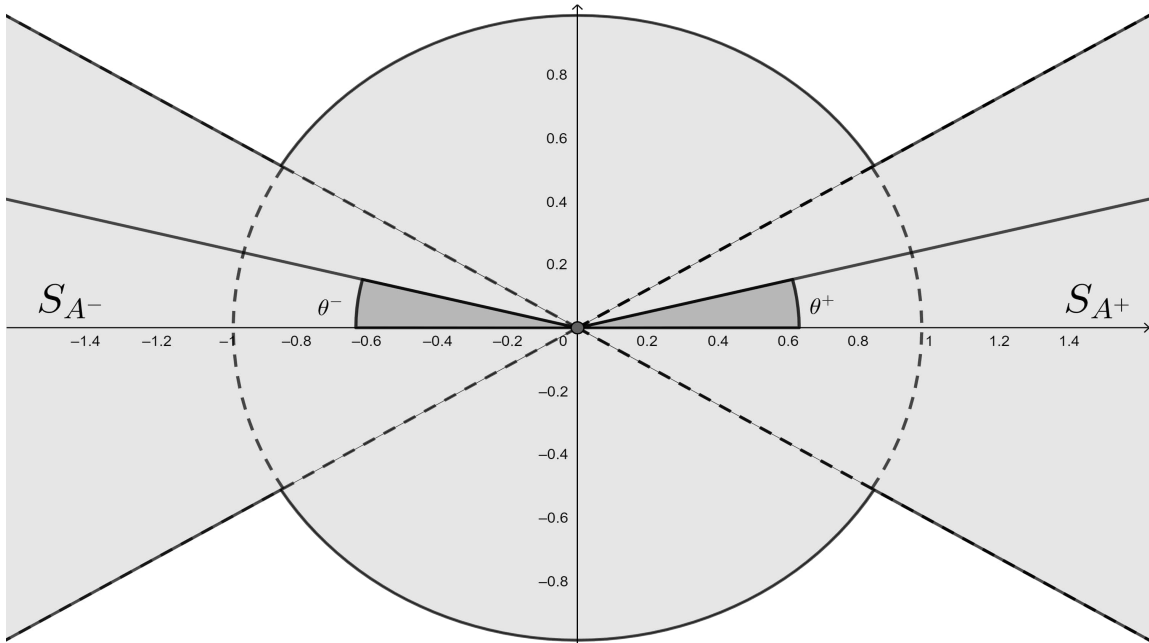Figure 2: The domain $\mathbb{D}_0 \cup S_{A^+} \cup S_{A^-}$.

Figure 3: The domain $\mathbb{D}_0 \cup S_{A^+} \cup S_{A^-}$ with the rays of integration $e^{i\theta^\pm}$.

m	Approximate Stokes constant
2	6.36002275906475562231
3	6.36183244858689510936
4	6.36187764510770954622
5	6.36187816610065157225
6	6.36187818751520525975
7	6.36187818708544022430
8	6.36187818712914981043
9	6.36187818712554608116
10	6.36187818712591062235

Figure 4: Approximations of the Stokes constant via extrapolation. The extrapolation is obtained by the method of least squares fitting to data. The latter are obtained from the first method.