# Degrees of Freedom in a Multipole Expansion of an Electromagnetic Current Over a Worldline

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A thesis submitted for the degree of
*Master of Science by Research*

March, 2022

# Declaration

I declare that the work presented in this thesis is, to the best of my knowledge and belief, original and my own work. The material has not been submitted, either in whole or in part, for a degree at this, or any other university. This thesis does not exceed the maximum permitted word length of 35,000 words including appendices and footnotes, but excluding the bibliography.
Vanya Ivanova Spasova

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# Abstract

**We are investigating the free components of an electromagnetic current, which we describe as a De Rham current over an arbitrary worldline on an arbitrary spacetime $\mathcal{M}_{ST}$. These calculations are an extension of a previous research done on dipoles and quadrupoles, in which they are described in a metric-free manner. In this thesis, we consider an adapted coordinate system to define the free components of an electromagnetic distribution over an arbitrary worldline and extend the previous work done by going up to the octupole order of the electromagnetic current expansion. Using the symmetries of the components found in the calculations, we derive a generalized expression for the number of free components of any $2^k$-tupole current, which satisfies the equation which leads to conservation of charge, $d\mathcal{J} = 0$.**

# Acknowledgements

I would like to thank my supervisor, Dr. Jonathan Gratus. His guidance, advice and insights regarding this project made it a very interesting and educative experience for me. I would also like to thank the Physics Department of Lancaster University for providing the necessary support and guidance when it was needed. Finally, I would like to thank my family and friends for giving me invaluable support during this research.

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# Notation

| | |
|---|---|
| $\mathrm{RIn}(t)$ | set of all increasing lists with repetitions |
| $\mathrm{SIn}(t)$ | set of all strictly increasing lists |
| $\mathcal{M}, \mathcal{N}$ | differentiable manifolds |
| $\dim(\mathcal{M})$ | dimension of $\mathcal{M}$ |
| $\dim(\mathcal{N})$ | dimension of $\mathcal{N}$ |
| $\tau_{\mathcal{M}}$ | topology of $\mathcal{M}$ |
| $\tau_{\mathcal{N}}$ | topology of $\mathcal{N}$ |
| $(\underline{x^{n}}) = (x^1, \ldots, x^n)$ | lists of coordinates on $\mathbb{R}^n$ |
| $T\mathcal{M}$ | tangent space of $\mathcal{M}$ |
| $T\mathcal{N}$ | tangent space of $\mathcal{N}$ |
| $\Gamma T\mathcal{M}$ | space of all tangent vectors on $\mathcal{M}$ |
| $\Gamma T^*\mathcal{M}$ | space of all covectors on $\mathcal{M}$ |
| $\Gamma\Lambda^p\mathcal{M}$ | space of all p-forms on $\mathcal{M}$ |
| $\Gamma\Lambda^p\mathcal{N}$ | space of all p-forms on $\mathcal{N}$ |
| $\Gamma_0\Lambda^q\mathcal{M}$ | space of all test forms on $\mathcal{M}$ |
| $\Gamma_0\Lambda^q\mathcal{N}$ | space of all test forms on $\mathcal{N}$ |
| $\Upsilon^{k,p}(f)$ | set of all submanifold distributions on $\mathcal{M}$ |
| $\Psi[\phi]$ | distribution acting on a test form |

# Chapter 1

# Introduction

Multipole expansions have been used extensively as a method to study the fields of sources of gravitational and electromagnetic radiation [5][7][10][18][21]. Within this method, the fields at a long distance from the source are represented by a point charge, or a point mass, followed by higher order terms called moments. The zeroth-term is called a monopole, the first-order term is called a dipole, the second-order term, quadrupole, the third-order term octupole, and so on. In this thesis, we will focus on multipole expansions of electromagnetic currents over a worldline. We are considering the multipole expansion of a charged body of size much smaller than the distance to an observer. We are relating the electromagnetic potential at a distant field point to a series of multipole moments at the point of reference, or the origin. Our aim is to generalize the expression of an electromagnetic current and also generalize the notion of moments in the language of differential geometry.

An interesting application of the multipole expansions of the electromagnetic potential is deriving the radiation fields that correspond to the source. For example, the fields resulting from a moving dipole were first evaluated by Ellis using the Liénard–Wiechert four-potential for moving point-charges [6]. Another method for evaluating the fields of arbitrary moving dipoles was proposed later by Ward, who considered a Hertzian six-vector potential and obtained the same results as Ellis [8][21]. Ellis was also the first one to derive the fields by arbitrary moving quadrupoles and multipoles with the use of invariant Green's functions. His work on multipoles has also recently been extended to an application in general relativity which introduces the stress-energy quadrupole as a source of gravitational radiation [12]. The fields that arise from a moving electric and magnetic dipole and an electric quadrupole have also been obtained using the quantum approach in [15].

There are some properties of the electromagnetic field that can be described within the dipole approximation. For example, in [14] molecular dipole moments are used to define covariant polarization. However, examples of interesting insights have been obtained from investigating higher moments. These are related to optical activity, described by electric quadrupoles and magnetic dipoles, and birefringence in certain non-magnetic crystals which requires going up to the order of electric octopoles and magnetic quadrupoles [10].

There are a number of peripheral conclusions which have stemmed from the research surrounding multipole expansions of the electromagnetic potential. As it is an ongoing topic of research, there are a number of discussions and results which concern the physical validity of the expansion and the physical meaning of the multipole terms. For example, in [10] Graham et al. claim that considering traceless multipole terms of order higher than the quadrupole, which arise from the fields $\mathbf{D}$ and $\mathbf{H}$, lead to a form of Maxwell's equations which is not translationally invariant. In the context of their work, they call this translational invariance, origin dependence. The authors consider the charge distributions as time-varying currents of volume $v$ in a vacuum.

The origin $O$ is understood by the authors to be a mathematical point anywhere inside a charge distribution in a vacuum or, where bulk matter is being considered inside a macroscopic volume element [10]. In this work the multipole expansion is obtained by the authors by relating the given vector potential at a distant field point $\mathcal{R} >> \mathbf{r}$ to a series of multipole moments at the origin $O$. To remedy the origin dependence of the multipole expansion, Graham et al. propose additional constitutive relations to ensure translational invariance of the two Maxwell equations containing $\mathbf{D}$ and $\mathbf{H}$.

The theme of origin dependence of the multipole expansion was further investigated by Raab and Lange in [13]. This piece of research proposes a multipole theory of linear constitutive relations for the response fields $\mathbf{D}$ and $\mathbf{H}$ with a new transformation which leaves the inhomogeneous Maxwell equations for the response fields unchanged. However, these results have been debated as some authors say that the definition of origin dependence is not satisfactory. Furthermore, in [20] it is shown that the "origin dependence", as defined above, is not unphysical as claimed, but only forms a part of the effect of moving the point of reference. Moreover, the author shows that the transformation proposed by Raab and Lange is unphysical as it does not leave the boundary conditions for the fields invariant. The results in [20] show that both on the macroscopic and microscopic levels the moments change with the reference point and so does the position of these moments. If both effects are taken into account then the resulting charge and current densities are independent of the reference point [20].

Previously, free components of an electromagnetic multipole expansion have been discussed by Raab and Lange in [16], where the free components are listed up to the electric octupole-magnetic quadrupole order. The calculations in [16] are obtained using tensor algebra and assumptions which are different to those presented in this thesis. The general formula for the free components of an electromagnetic $2^k$-tupole obtained in this thesis is, as far as the author is concerned, novel and has not been

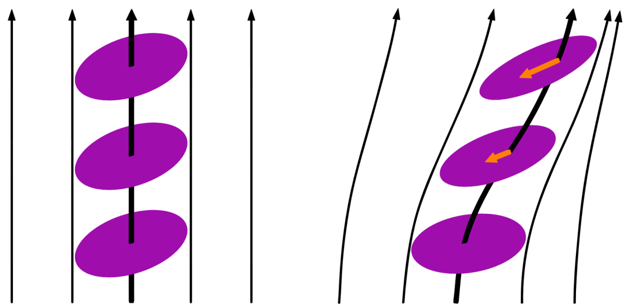

Figure 1.1: Quadrupoles represented in an adapted coordinate system (*left*), and quadrupoles in lab coordinates (*right*). The quadrupoles are represented by ellipsoids, and dipoles by arrows. We can see that dipoles appear as arrows in the lab coordinate system. It is clear that the adapted coordinate system simplifies the calculations. It should be noted that the lab coordinates do require a metric. (Figure adapted from [11]).

seen previously in the literature.

In this thesis, we will describe electromagnetic distributions as examples of De Rham currents, which we will call submanifold distributions, over an arbitrary worldline. De Rham currents are defined on a manifold in terms of the push-forward of a regular distribution. This is known as the De Rham push-forward and it will be a key technique in the calculations presented in this thesis. The De Rham push-forward maps form fields on one manifold onto distributions on another manifold. We will define a regular distribution in terms of its action on a test form of an arbitrary degree. We will endow the space of the submanifold distributions which are De Rham currents with the exterior derivative, the internal contraction and the Lie derivative [2].

All De Rham currents, defined on a closed embedding, $f : \mathcal{N} \to \mathcal{M}$, of degree $p$, and order $k$, corresponding to the number of derivatives, will be known as $\Upsilon^{k,p}(f)$. We will show that the electromagnetic distributions, the dipole, quadrupole, octupole and $2^k$-tupole, belong to the space of $\Upsilon^{k,p}(f)$ of the same degree, but different order. In order to find the number of degrees of freedom of the respective currents and generalize this to an $2^k$-tupole, we will use a representation of the De Rham currents in terms of an arbitrary number of Lie derivatives and internal contractions.

In order to calculate the degrees of freedom we will use an adapted coordinate system which, in general, makes the calculations easier as the equations of motion are adapted to the flow of the multipoles (See Fig. 1.1). Although this might be described as a disadvantage, since we are not considering a general coordinate system but a particular coordinate system which is adapted, the result which we have obtained can still be useful in continuing the discussion on searching for interesting properties of the higher order moments of the multipole expansion. The components of electromagnetic distributions as defined in this thesis are unique given they are defined on an adapted coordinate system. This has been shown in [2]. In order to extend our analysis and make conclusions on the coordinate dependence of the free components, we should transform the components to the laboratory coordinates and thus determine the direct coordinate dependence of the components.

The correct coordinate transformations for quadrupoles, for example, were first calculated in [11]. The calculations presented in this thesis are closely related to the work done by Gratus et al. in [11], where multipoles are presented in the language of differential geometry up to the quadrupole order. We will extend this by going up to the octupole order and use the mathematical tools derived in [2] in order to build the basis of our calculations. However, in these calculations we are only considering the adapted coordinate system and we are not making any conclusions about the way that the components transform from one coordinate system to another. We will extrapolate information from the octupole order and derive a general formula for the degrees of freedom, which we will call *free components*, of any $2^k$-tupole expansion of an electromagnetic current that is satisfying $d\mathcal{J} = 0$. In this thesis, we use the term degrees of freedom and free components interchangeably to mean the freedom we have in varying different components after imposing a constraint on the distributions or, in other words, the free components are the components which are left undetermined and are free to be varied after the imposed constraint.

We are interested in finding a general formula for the free components of any electromagnetic $2^k$-tupole because it can provide us with insights about the source of the electromagnetic field that we are interested in. If multipole expansions are used to describe particles knowing the number of free components can give us further information about particle interactions and their intrinsic properties. The number of free components also gives us a hint on how much information is there to find out about an electromagnetic field and its source, since the degrees of freedom show us that imposing initial conditions is not enough to find everything about the system. We would need to replace the free components with algebraic constructions and constitutive relations in order to be able to know fully how the system will behave. [12]

We will also present the calculations without a reference to a metric. This gives the advantage that the definition of multipoles can be generalized even further to higher dimensional spacetimes, and to manifolds such as phase space, or manifolds with no preferred metric. For example, we can use multipole expansions in the study of plasma and beams of particles, where moments of a probably distribution in phase space are calculated [11].

This thesis is constructed as follows:

In **Chapter 2** we will introduce the theoretical basis for our calculations. We will introduce a multi-index notation which will help us deal with the symmetries of internal contraction and Lie derivatives that appear in our calculations. We will also define an adapted coordinate system and the definition of a closed embedding in order to build the space of submanifold distribution. We will endow the space of submanifold distributions with structures such as the exterior derivative, the internal contraction and the Lie derivative. We will then define De Rham currents in terms of their actions on a test form and we will introduce their components which will be the main focus of our calculations.

In **Chapter 3** we will represent electromagnetic distributions as De Rham currents over an arbitrary worldline, which satisfy the equation for conservation of charge, $d\mathcal{J} = 0$. We will explore the independent components of the dipole, the quadrupole and the octupole. We will obtain differential equations by imposing $d\mathcal{J} = 0$ as a constraint and will make conclusion for the degrees of freedom for each of the currents.

In **Chapter 4** We will use the obtained results in **Chapter 3** to generalize the expression for the independent components of any electromagnetic current, $\mathcal{J}$, before imposing the constraint $d\mathcal{J} = 0$. Finally, we will present a general formula for the free components of any $2^k$-tupole satisfying the continuity equation.

# Chapter 2

# Theoretical Background

In this chapter we will introduce the building blocks for our calculations, starting with introducing our notation and building up the space of the submanifold distributions which are De Rham currents.

## 2.1 Multi-Index Notation

### 2.1.1 Sets of Increasing Lists

We will be considering an arbitrary number of Lie derivatives and internal contractions. For this reason, multi-index notation will be used for convenience.
Let the set of all increasing lists with repetitions of $m$ elements be given by:

$$\mathrm{RIn(t)} = \{[I_1, ..., I_s] \mid s, I_1, ..., I_s \in \mathbb{Z}, s \geq 0, 1 \leq I_1 \leq I_2 \leq \cdots \leq I_s \leq t\}. \tag{2.1}$$

Let the set of all strictly increasing lists of $m$ elements be given by:

$$\mathrm{SIn}(t) = \{[J_1, ..., J_s] \mid s, J_1, ..., J_s \in \mathbb{Z}, s \geq 0, 1 < J_1 < J_2 < \cdots < J_s \leq t\}, \tag{2.2}$$

where $\mid [I_1, ..., I_s] \mid = s$ is the length of the list, and $t = m, t = n,$ or $t = r$ corresponds to the range of coordinates. For antisymmetric objects, such as forms and internal contractions we will use $J, K \in \mathrm{SIn}(m)$. For symmetric objects, such as Lie derivatives, we will use $I \in \mathrm{RIn}(m)$. For abbreviation these will be written as $I \Uparrow^t$ to mean $I \in \mathrm{RIn}(t)$, and $J \uparrow^t$ to mean $J \in \mathrm{SIn}(t)$. [2]

The naturally increasing lists which include all elements in a set will be given by:

$$\underline{m} = [1, \ldots, m] \in \mathrm{SIn}(m), \quad \underline{n} = [1, \ldots, n] \in \mathrm{SIn}(n), \quad \text{and} \quad \underline{r} = [1, \ldots, r] \in \mathrm{SIn}(r). \tag{2.3}$$

For examples of unpacking the multi-index notation see Example 2 in Section A.1 of the Appendix.

### 2.1.2 Lists of Lie Derivatives, Internal Contractions and Exterior Derivatives

For the lists $I \Uparrow^t$, $J \uparrow^t$, and $K \uparrow^t$ we define the following notation for the Lie derivative, internal contraction and exterior derivative:

$$\begin{aligned} \mathcal{L}_I^{(x)} &= \mathcal{L}_{I_1}^{(x)} \mathcal{L}_{I_2}^{(x)} \cdots \mathcal{L}_{I_{|I|}}^{(x)} \\ i_J^{(x)} &= i_{J_{|J|}}^{(x)} \cdots i_{J_2}^{(x)} i_{J_1}^{(x)} \\ dx^K &= dx^{K_1} \wedge dx^{K_2} \wedge \cdots \wedge dx^{K_{|K|}}, \end{aligned} \tag{2.4}$$

where the superscript $(x)$ annotates the coordinate system, $\mathcal{L}_a^{(x)} = \mathcal{L}_{\frac{\partial}{\partial x}}^{(x)}$ and $i_a^{(x)} = i_{\frac{\partial}{\partial x}}^{(x)}$ and $a = 1, ..., m$, or $a = 1, ..., n$, or $a = 1, ..., r$.

When referring to lists of coordinates, we will write:

$$(\underline{x^m}) = (x^1, \ldots, x^m), \quad (\underline{z^r}) = (z^1, \ldots, z^r), \quad (\underline{0^r}) = (0, \ldots, 0). \tag{2.5}$$

Note that all the basic geometric operations used in the calculations in this thesis, such as the interior product, the exterior derivative and the Lie derivative, are defined in Section A.2 of the Appendix.

### 2.1.3 Summation of Increasing Lists

For compactness we will use the summation sign $\sum_{\mathrm{Rng}(I,J,K)}$. In the case where $r, p, k$ are prescribed, $\mathrm{Rng}(I, J, K)$ is defined as:

$$\mathrm{Rng}(I, J, K) := \{I \Uparrow^r, K \uparrow^r, K \uparrow^r \text{ such that } |K| - |J| = p - r, \ |I| \leq k\}. \tag{2.6}$$

## 2.2 Submanifold Distributions

Let $\mathcal{M}(M, \tau_M, \mathcal{A}_M)$ and $\mathcal{N}(N, \tau_N, \mathcal{A}_N)$ be two differentiable manifolds with $\dim(\mathcal{M}) = m$, and $\dim(\mathcal{N}) = n$, where $\tau_M$, $\tau_N$ are their respective topologies and $\mathcal{A}_\mathcal{M}$, $\mathcal{A}_\mathcal{M}$ their respective atlases.

We will first introduce a general definition of a *regular*, or *embedded* submanifold:

**Definition 2.2.1.** A subset $S$ of the manifold $\mathcal{M}$ of dimension $\dim(\mathcal{M}) = m$ is a *regular*, or an *embedded* submanifold of $\dim(S) = s$ if:

$$\forall\ p\ \in S\ \exists\ (U, \varphi) = (U, x^1, ..., x^s, x^{s+1}, ..., x^m), \tag{2.7}$$

such that $U \cap S$ is defined by the vanishing of m-s of the coordinate functions. Let the vanishing coordinate functions be given by $z^i = x^{s+1}, ..., x^m = z^1, ..., z^r$. The chart $(U, \varphi)$ will then be given by:

$$(U, \varphi) = (U, x^1, ..., x^s, z^1, ..., z^r). \tag{2.8}$$

On $U \cap S$, $\varphi$ is defined as:

$$\varphi = (x^1, ..., x^s, 0, ..., 0). \tag{2.9}$$

Such chart, $(U, \varphi)$, is called an adapted chart, relative to $S$ [19].

**Definition 2.2.2.** If $S$ is a regular submanifold of $\mathcal{M}$ and $\dim(S) = s$, and $\dim(\mathcal{M}) = m$, then $r = m - s$ is called the co-dimension of $S$ and $\mathcal{M}$.

Embedded submanifolds are the most natural and common submanifolds. Every embedded submanifold is also an immersed submanifold [19]. We are interested in *smooth* embeddings which are *injective immersions*.

### 2.2.1 Injective Immersion

**Definition 2.2.3.** A map $f : \mathcal{N} \to \mathcal{M}$ is called an immersion if:

$$\forall\ p\ \in\ \mathcal{N}\ \exists\ \ d_p \equiv f_*|_{(p)} : T_p\mathcal{N} \to T_{f(p)}\mathcal{M} \text{ is injective,} \tag{2.10}$$

where $d_p$ is the differential of $f$ at $p$ and $f_*|_{(p)}$ is the push-forward of vectors from the tangent space of $\mathcal{N}$ to the tangent space of $\mathcal{M}$.

**Remark.** The rank of the map $f$ is given by the rank of the linear map $f_*|_{(p)} : T_p\mathcal{N} \to T_{f(p)}\mathcal{M}$, so it is the rank of the matrix of partial derivatives of $f$ in any coordinate chart. Assuming that $f_*$ is injective at each point $p \in \mathcal{N}$ provides with the conclusion that $\text{rank}(f) = \dim(\mathcal{N})$ [19].

### 2.2.2 Closed Embedding

In more detail, if the manifold $\mathcal{N}$ is compact, we will have an injective immersion which is also an embedding. However, if $\mathcal{N}$ is not compact, the injective immersion is not necessarily an embedding [19]. We will look at wordlines which are examples of closed embeddings. Hence, we have the following definition of a closed embedding:

**Definition 2.2.4.** A map $f : \mathcal{N} \to \mathcal{M}$ is *proper* if the preimage of every compact set, $S$, in $\mathcal{N}$ is compact in $\mathcal{M}$:

$$\begin{aligned}&\forall \text{ compact } S\ \in \tau_M : f^{-1}(S) \text{ is compact} \in \tau_N, \text{ with}\\ &f^{-1}(S) : \{n \in N \mid f(n) \in S\}, \text{ where}\end{aligned} \tag{2.11}$$

$\tau_N$ and $\tau_M$ are the topologies of the manifolds $\mathcal{N}, \mathcal{M}$ respectfully, and $f^{-1}$ denotes the preimage of $f$.

**Definition 2.2.5.** A map $f : \mathcal{N} \to \mathcal{M}$ is a smooth closed embedding if:

$$\begin{cases} f : \mathcal{N} \to \mathcal{M} \text{ is proper.} \\ f : \mathcal{N} \to \mathcal{M} \text{ is an immersion.} \\ \text{The image } f(N) \text{ is homeomorphic to } N \text{ under } f. \end{cases} \tag{2.12}$$

$\mathcal{N}$ is called a smooth embedded submanifold of $\mathcal{M}$. Since the map $f : \mathcal{N} \to \mathcal{M}$ is an immersion it follows that $\dim(\mathcal{N}) \leq \dim(\mathcal{M})$. (Follows from the maximal rank theorem) [19].

### 2.2.3 De Rham Currents

Let $f : \mathcal{N} \to \mathcal{M}$ be a smooth closed embedding, with codimension $r = m - n$.

Let the space of all test forms on $\mathcal{M}$ be given by:

$$\Gamma_0\Lambda^q\mathcal{M} = \{\phi \in \Gamma_0\Lambda^q\mathcal{M} | \phi \text{ has compact support}\}. \tag{2.13}$$

**Definition 2.2.6.** A regular distribution can be defined by its action on a test form via:

$$\alpha^{\mathcal{D}}[\phi] = \int_{\mathcal{M}} \phi \wedge \alpha, \text{ where} \tag{2.14}$$

$\alpha \in \Gamma_0\Lambda^p\mathcal{M}$ is a smooth $p$-form.

**Definition 2.2.7.** Let $p$ be the degree of $\alpha^D$, defined by:

$$\alpha^D[\phi] = 0 \text{ for } \forall\ \phi \mid \deg(\phi) + \deg(\alpha^D) \neq m. \tag{2.15}$$

**Remark.** To ensure that $\alpha^D[\phi]$ does not vanish, it is required that $0 \leq p \leq m$. This follows from $0 \leq \deg(\phi) \leq m$ and $\deg(\phi) + \deg(\alpha^D) = m$ [2].

**Definition 2.2.8.** The De Rham currents on manifolds can be defined in terms of a push-forward of a regular distribution. [17] The De Rham push-forward, $f_\varsigma\beta$, can be defined by its action on a test form $\phi$, where $\beta \in \Gamma_0\Lambda^p\mathcal{N}$ by:

$$f_\varsigma\beta[\phi] = \int_{\mathcal{N}} f^*\phi \wedge \beta \tag{2.16}$$

with the requirement that:

$$\int_{\mathcal{N}} \omega = 0, \text{ when } \deg(\omega) \neq \deg(\mathcal{N}) = n. \tag{2.17}$$

**Remark.** To ensure that the integral in Eq.2.16 does not vanish one requires that $\deg(\phi) = n - p$ so that :

$$\deg(f_\varsigma \beta) = m - n + \deg(\beta) = r + \deg(\beta). \tag{2.18}$$

**Remark.** It is important to note that the De Rham push-forward maps fields of forms on $\mathcal{N}$ into distributions on $\mathcal{M}$ [2].

### 2.2.4 The Space of All Submanifold Distributions $\Upsilon^{k,p}(f)$

**Definition 2.2.9.** Let $\Upsilon^{k,p}(f)$ be the set of all submanifold distributions on $\mathcal{M}$ of degree $p$, and order $k$. We say that $f_\varsigma(\alpha)$ is of order 0, so that:

$$f_\varsigma(\alpha) \in \Upsilon^{0,p}(f) \ \text{ where } \ p = r - \deg(\alpha) \ [2]. \tag{2.19}$$

**Definition 2.2.10.** Let $\Phi$, $\Psi \in \Upsilon^{k,p}(f)$. Let $v$ be a field $\in T\mathcal{M}$ and $\phi \in \Gamma_0 \Lambda^q \mathcal{M}$ as before. We can define the following operations on the submanifold distributions:

$$\Phi + \Psi \in \Upsilon^{k,p}(f) \ \text{ with } \ (\Psi + \Phi)[\phi] = \Psi[\phi] + \Phi[\phi] \tag{2.20}$$

$$i_v \Psi \in \Upsilon^{k,p-1}(f) \ \text{ with } \ i_v \Psi[\phi] = -(-1)^{deg(\phi)} \Psi[i_v \phi] \tag{2.21}$$

$$d\Psi \in \Upsilon^{k+1,p+1}(f) \ \text{ with } \ d\Psi[\phi] = -(-1)^{deg(\phi)} \Psi[d\phi]. \tag{2.22}$$

Using Cartan's identity, $\mathcal{L}_v = d i_v + i_v d$, we can define the Lie derivative on the submanifold distributions as:

$$\mathcal{L}_v \Psi \in \Upsilon^{k+1,p}(f) \ \text{ with } \ \mathcal{L}_v \Psi[\phi] = -\Psi[\mathcal{L}_v \phi]. \tag{2.23}$$

Let $\beta \in \Gamma \Lambda^q \mathcal{M}$. We can define the wedge product on the submanifold distributions as:

$$\beta \wedge \Psi \in \Upsilon^{k,p+q}(f) \ \text{ with } \ \beta \wedge \Psi[\phi] = \Psi[\phi \wedge \beta] = (-1)^{(deg\varphi)(deg\beta)} \Psi[\beta \wedge \phi]. \tag{2.24}$$

**Definition 2.2.11.** The order of $\Psi \in \Upsilon^{k,p}$ is given by $k$, where $k$ is the maximum number of derivatives of $\Psi$. Additionally the order of the submanifold distribution can also be expressed via:

$$\Psi[\lambda^{k+1} \phi] = 0, \quad \forall \ \phi \in \Gamma_0 \Lambda^{m-p} \mathcal{M}, \ \text{ and } \ \lambda \in \Gamma \Lambda^0 \mathcal{M} \text{ such that } f^* \lambda = 0. \tag{2.25}$$

**Remark.** As stated, $\Psi \in \Upsilon^{k,p}$ has no more than $k$ derivatives. We can add two submanifold distributions of order $k$ and obtain a submanifold distribution of lower order. For example, given $u, v \in \Gamma T\mathcal{M}$, and $\mathcal{L}_u\mathcal{L}_v f_\varsigma(1) \in \Upsilon^{2,0}, \mathcal{L}_v\mathcal{L}_u f_\varsigma(1) \in \Upsilon^{2,0}$ we have that:

$$\mathcal{L}_v\mathcal{L}_u f_\varsigma(1) - \mathcal{L}_u\mathcal{L}_v f_\varsigma(1) = \mathcal{L}_{[v,u]} f_\varsigma(1) \in \Upsilon^{1,0}. \tag{2.26}$$

Therefore each $\Upsilon^{k,p}(f)$ is a subspace of the space of higher order submanifold distributions:

$$\Upsilon^{k,p}(f) \subset \Upsilon^{j,p} \text{ for } k \leq j. \tag{2.27}$$

**Lemma 2.2.1.** If the order of $\Psi \in \Upsilon^{k,p}(f)$ is defined by Eq.2.25, then the order of $f_\varsigma(\alpha)$ is zero as stated in Eq.2.19. Moreover, the addition, defined in 2.20, the internal contraction, defined in 2.21, and the wedge, defined in 2.24 do not change the order of the submanifold distribution. However, the external derivative, defined in 2.22, and the Lie derivative, defined in 2.23 increase the order of the submanifold distribution by one [2].

**Definition 2.2.12.** To define the support of $\Psi$ one takes the complement of the largest open set where $\Psi$ vanishes for all test forms $\phi$ with compact support:

$$\mathcal{M} \setminus U \text{ such that } U \subset \mathcal{M}, \text{ and } \Psi[\phi] = 0, \quad \forall\, \phi \in \Gamma_o \Lambda^{p-m}\mathcal{M} \text{ such that } \mathrm{supp}(\phi) \subset U. \tag{2.28}$$

**Remark.** It is therefore implied that $\mathrm{supp}(\Psi) \subset f(\mathcal{N})$ [2].

### 2.2.5 Adapted Coordinates and Components of $\Upsilon^{k,p}(f)$

Let $(U, \varphi)$ be an adapted chart on $\mathcal{M}$ such that:

$$\varphi(x^1, ..., x^n, z^1, ..., z^r) = (x^1, ..., x^n, 0, ..., 0), \tag{2.29}$$

as defined in Eq.2.9.

**Theorem 2.2.2.** The elements of $\Psi \in \Upsilon^{k,p}(f)$ can be expressed locally in terms of an adapted chart as:

$$\Psi[\phi] = \sum_{\mathrm{Rng}(I,J,K)} \int_{\mathcal{N}} \Psi_K^{I,J} dx^K \wedge f^*(i_J^{(z)} \mathcal{L}_I^{(z)} \phi), \text{ [2]} \tag{2.30}$$

where

$$\mathrm{supp}(\phi) \in f(\mathcal{N}), \text{ and } \Psi_K^{I,J} \in \Gamma\Lambda^0\mathcal{N}. \tag{2.31}$$

**Lemma 2.2.3.** The elements of $\Psi \in \Upsilon^{k,p}(f)$ can be also written without a reference to a test form in terms of the action of the internal contraction and Lie derivatives on the De Rham push-forward, $f_\varsigma(\Psi_K^{I,J} dx^K)$:

$$\Psi = \sum_{\mathrm{Rng}(I,J,K)} (-1)^{|I|+(p-r)(m-p+|J|)} i_J^{(z)} \mathcal{L}_I^{(z)} f_\varsigma(\Psi_K^{I,J} dx^K). \tag{2.32}$$

The latter definition is used in conducting the calculations in this work. For the rest of this paper the following notation will be used for compactness:

$$^*\Psi_K^{I,J} = (-1)^{|I|+(p-r)(m-p+|J|)} \Psi_K^{I,J}, \tag{2.33}$$

so that Eq.2.32 now reads:

$$\Psi = \sum_{\mathrm{Rng}(I,J,K)} i_J^{(z)} \mathcal{L}_I^{(z)} f_\varsigma(^*\Psi_K^{I,J} dx^K). \tag{2.34}$$

Expressing $\Psi$ locally with the help of adapted coordinates guarantees that the components $\Psi_K^{I,J}$ are unique [2].

# Chapter 3

# Electromagnetic Distributions

We will consider electromagnetic $2^k$-tupoles as distributions over an arbitrary moving worldline, $C$.

**Definition 3.0.1.** Let $C : \mathcal{I} \to \mathcal{M}_{ST}$ be a closed embedding onto a four-dimensional differentiable manifold, $\mathcal{M}_{ST}$, which will be referred to as *spacetime*, where $\mathcal{I} : \{\tau \mid \tau_{min} < \tau < \tau_{max}\} \subseteq \mathbb{R}$ is the domain of $C$. It is important to note that here there is no prescribed metric to the manifold $\mathcal{M}_{ST}$, so the parameter $\tau$ will not be considered as proper time.

**Remark.** Since $C$ is a closed embedding, there exists an adapted chart $(U, \varphi)$, such that:

$$\begin{cases} \phi : U \to \mathbb{R}^4 \\ \tau : U \to \mathbb{R} \\ z^\mu : U \to \mathbb{R} \\ \tau|_{C(\tau')} = \tau' \\ z^\mu|_{C(\tau')} = 0. \end{cases} \tag{3.1}$$

The worldline $C$ will then be represented by the coordinate functions $C^0(\tau) = \tau$, and $C^\mu = 0$ with $\mu = 1, 2, 3$.

**Definition 3.0.2.** Let $\mathcal{J} \in \Upsilon^{k,3}(C)$ be the current 3-form which is a source of Maxwell's equations given by:

$$d\mathcal{F} = 0 \tag{3.2}$$

$$d\mathcal{H} = \mathcal{J}, \tag{3.3}$$

where $\mathcal{F} \in \Gamma\Lambda^2\mathcal{M}_{ST}$ embodies the electric field $\mathbf{E}$ and the magnetic flux density $\mathbf{B}$, and $\mathcal{H} \in \Gamma\Lambda^2\mathcal{M}_{ST}$ embodies the displacement field $\mathbf{D}$ and the magnetic field intensity

**H** [2].
The fields $\mathcal{F}$ and $\mathcal{H}$ are related by constitutive relations given by:

$$\mathcal{H} = \star\mathcal{F}, \tag{3.4}$$

where $\star$ is the Hodge dual map defined in Section A.4 of the Appendix.

It should be noted here that the Hodge dual map requires further structures on the manifold, namely an orientation and a metric to be defined. In this case, we are not considering the further constraints which will be imposed from $\mathcal{H} = \star\mathcal{F}$ and we are restricting our analysis to the conservation of charge, given by $d\mathcal{J} = 0$.
Taking the exterior derivative of Eq.3.3 we have the following equation for the conservation of charge:

$$d\mathcal{J} = 0. \tag{3.5}$$

We also point out that any $\mathcal{J}$ satisfying Eq.3.5 is closed.

**Remark.** Alternatively, we can state that the closeness of $\mathcal{J}$ in Eq.3.5 is also guaranteed by its exactness, as every exact form is by definition also closed. (See Theorem A.2.4 in the Appendix).

### 3.0.1 General Current

Consider $\mathcal{J} \in \Upsilon^{k,3}(C)$ such that $d\mathcal{J} = 0$ in an adapted coordinate system such as the one above:

$$\mathcal{J} = \sum_{I\Uparrow^3,|I|\leq k,J\uparrow^3,K\uparrow^1} (-1)^{(|I|+(p-3)(4-p)+|J|)} i_J^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\mathcal{J}_K^{I,J} dx^K), \tag{3.6}$$

with $|J| + |K| = 1$.
It can be seen that this is equivalent to Eq. 2.32 but here we have specified that $r = 3$,as the co-dimension of $\mathcal{I}$ and $\mathcal{M}_{ST}$, and $m = 4$ as the dimension of $\mathcal{M}_{ST}$.
Let $(-1)^{(|I|+(p-3)(4-p)+|J|)}\mathcal{J} = \hat{\mathcal{J}}$, then $d\mathcal{J} = 0$ gives:

$$d\mathcal{J} = \sum_{I\Uparrow^3,|I|\leq k} \mathcal{L}_I^{(z)} C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{I,\varnothing}) + \sum_{I\Uparrow^3,J\uparrow^3,|I|\leq k,|J|=1} i_J^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}_o^{I,J} d\tau) \tag{3.7}$$

The sum is split into two where $K = [\varnothing]$ and $K = [0]$ with $dx^1 = d\tau$. The general expression for $d\mathcal{J} = 0$ will help extract the free components for each of the distributions.

After establishing the equation for the general electromagnetic current, we will now give specific examples of currents of different orders, starting with the zeroth order, which is the monopole current, and going up to the third order, which is the octupole current.

### 3.0.2 Monopole Current

**Lemma 3.0.1.** Given $\mathcal{J}_M \in \Upsilon^{0,3}(C)$ such that $d\mathcal{J}_M = 0$, we can define the elementary charge as:

$$\mathcal{J}_M = qC_\varsigma(1), \text{ where } q = \text{constant}. \tag{3.8}$$

*Proof.* Using Eq.3.6 the monopole current is given by:

$$\mathcal{J} = C_\varsigma(\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\nu=1}^{3} i_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau).$$

Setting $d\mathcal{J} = 0$, we obtain:

$$0 = d\mathcal{J} = C_\varsigma(d\mathcal{J}_\varnothing^{\hat{\varnothing},\varnothing}) + \sum_{\nu=1}^{3} d\{i_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau)\}.$$

Applying Cartan's identity to the second term gives:

$$0 = d\mathcal{J} = C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\nu=1}^{3} (\mathcal{L}_\nu^{(z)}(C_\varsigma \hat{\mathcal{J}}_0^{\varnothing,\nu}) - i_\nu^{(z)}(C_\varsigma d\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau)).$$

The last term vanishes so finally $d\mathcal{J} = 0$ leads to two expressions:

$$\begin{aligned} \partial_0^{(\tau)} \hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing} &= 0 \\ \hat{\mathcal{J}}_0^{\varnothing,\nu} &= 0. \end{aligned}$$

From the first equation it follows that $\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}$ is a constant, or a conserved quantity. One can set $\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing} = q$. This represents the total monopole charge of the current [2]. □

### 3.0.3 Dipole Current

Consider $\mathcal{J}_D \in \Upsilon^{1,3}(C)$ such that $d\mathcal{J}_D = 0$. This is the dipole current given by:

$$\mathcal{J}_D = C_\varsigma(\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3} \mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing}) + \sum_{\nu=1}^{3} i_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau) + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} i_\nu^{(z)} \mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu} d\tau). \tag{3.9}$$

Using the symmetries of the Lie derivatives and internal contractions, we can count the independent components of the dipole current. These are presented in Table 3.1 below.

There are total of 16 independent components in the dipole distribution before setting any constraints on $\mathcal{J}_D$. These are consistent with the number of independent components in the dipole distribution previously calculated in [11]. To find the number of constraints we will set $d\mathcal{J}_D$ to zero:

$$\begin{aligned}0 = d\mathcal{J}_D = C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3} \mathcal{L}_\mu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing}) + \sum_{\nu=1}^{3} d\{i_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau)\} + \\ + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} d\{i_\nu^{(z)} \mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu} d\tau)\}.\end{aligned} \tag{3.10}$$

Using Cartan's identity on the last two terms gives:

$$\begin{aligned}0 = d\mathcal{J}_D = C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3} \mathcal{L}_\mu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing}) + \sum_{\nu=1}^{3}(\mathcal{L}_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau) - \sum_{\nu=1}^{3} i_\nu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_0^{\varnothing,\nu} \wedge d\tau)) \\ + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3}(\mathcal{L}_\nu^{(z)}\mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu} d\tau) - \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} i_\nu \mathcal{L}_\mu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_0^{\mu,\nu} \wedge d\tau)),\end{aligned} \tag{3.11}$$

with the terms $\sum_{\nu=1}^{3} i_\nu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_0^{\varnothing,\nu} \wedge d\tau)$ and $\sum_{\nu=1}^{3}\sum_{\mu=1}^{3} i_\nu \mathcal{L}_\mu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_0^{\mu,\nu} \wedge d\tau)$ vanishing, we obtain:

$$0 = d\mathcal{J}_D = C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3} \mathcal{L}_\mu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing}) + \sum_{\nu=1}^{3} \mathcal{L}_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu} d\tau) + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} \mathcal{L}_\nu^{(z)}\mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu} d\tau). \tag{3.12}$$

From that we can deduce the following equations for the components:

$$\begin{aligned}&\partial_0^{(\tau)} \hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing} = 0 \\ &\partial_0^{(\tau)} \hat{\mathcal{J}}_\varnothing^{\mu,\varnothing} + \hat{\mathcal{J}}_0^{\varnothing,\mu} = 0 \\ &\hat{\mathcal{J}}_0^{\mu,\nu} + \hat{\mathcal{J}}_0^{\nu,\mu} = 0.\end{aligned} \tag{3.13}$$

Here we are using the symmetries of the indices as above to deduce the number of differential equations. The resulting numbers of constraints is given in Table 3.2.

There are 10 constraints that are imposed on the dipole current. This means that we obtain 6 free components of the current after the imposed constraints.

Table 3.1: Number of Independent Components of the Dipole Distribution

| *Component* | *Number of Configurations* |
|---|---|
| $\hat{\mathcal{J}}_{\varnothing}^{\varnothing,\varnothing}$ | 1 |
| $\hat{\mathcal{J}}_{\varnothing}^{\mu,\varnothing}$ | 3 |
| $\hat{\mathcal{J}}_{0}^{\varnothing,\nu}$ | 3 |
| $\hat{\mathcal{J}}_{0}^{\mu,\nu}$ | 9 |

Table 3.2: Number of Constraints for the Dipole Distribution

| *Constraints* | *Number of Constraints* |
|---|---|
| $\hat{\mathcal{J}}_{\varnothing}^{\varnothing,\varnothing} = q$ | 1 |
| $\partial_0^{(\tau)} \hat{\mathcal{J}}_{\varnothing}^{\mu,\varnothing} + \hat{\mathcal{J}}_{0}^{\varnothing,\mu} = 0$ | 3 |
| $\hat{\mathcal{J}}_{0}^{(\mu,\nu)} = 0$ | 6 |

### 3.0.4 Quadrupole Current

Consider $\mathcal{J}_Q \in \Upsilon^{2,3}(C)$ such that $d\mathcal{J}_Q = 0$. This is the quadrupole current given by:

$$\mathcal{J}_Q = C_\varsigma(\hat{\mathcal{J}}_{\varnothing}^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3} \mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_{\varnothing}^{\mu,\varnothing}) + \sum_{\nu=1}^{3} i_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_{0}^{\varnothing,\nu} d\tau) + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} i_\nu^{(z)} \mathcal{L}_\mu^{(z)} C_\varsigma(\hat{\mathcal{J}}_{0}^{\mu,\nu} d\tau)$$
$$+ \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \mathcal{L}_\mu^{(z)} \mathcal{L}_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_{\varnothing}^{\mu\nu,\varnothing}) + \sum_{\delta=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} i_\delta^{(z)} \mathcal{L}_\mu^{(z)} \mathcal{L}_\nu^{(z)} C_\varsigma(\hat{\mathcal{J}}_{0}^{\mu\nu,\delta} d\tau). \tag{3.14}$$

Using the same method as above, we find the independent components of the quadrupole current, presented in Table 3.4 below.

To illustrate the independent components, we will use $\hat{\mathcal{J}}_{0}^{\mu\nu,\delta}$ as an example. Without considering any symmetries we will naturally have 27 components coming from $\hat{\mathcal{J}}_0^{\mu\nu,\delta}$:

$$
\begin{aligned}
&\hat{\mathcal{J}}_0^{11,1}, \hat{\mathcal{J}}_0^{12,1}, \hat{\mathcal{J}}_0^{13,1},\\
&\hat{\mathcal{J}}_0^{21,1}, \hat{\mathcal{J}}_0^{22,1}, \hat{\mathcal{J}}_0^{23,1}\\
&\hat{\mathcal{J}}_0^{31,1}, \hat{\mathcal{J}}_0^{32,1}, \hat{\mathcal{J}}_0^{33,1}\\
&\hat{\mathcal{J}}_0^{11,2}, \hat{\mathcal{J}}_0^{12,2}, \hat{\mathcal{J}}_0^{13,2}\\
&\hat{\mathcal{J}}_0^{21,2}, \hat{\mathcal{J}}_0^{22,2}, \hat{\mathcal{J}}_0^{23,2}\\
&\hat{\mathcal{J}}_0^{31,2}, \hat{\mathcal{J}}_0^{32,2}, \hat{\mathcal{J}}_0^{33,2}\\
&\hat{\mathcal{J}}_0^{11,3}, \hat{\mathcal{J}}_0^{12,3}, \hat{\mathcal{J}}_0^{13,3},\\
&\hat{\mathcal{J}}_0^{21,3}, \hat{\mathcal{J}}_0^{22,3}, \hat{\mathcal{J}}_0^{23,3}\\
&\hat{\mathcal{J}}_0^{31,3}, \hat{\mathcal{J}}_0^{32,3}, \hat{\mathcal{J}}_0^{33,3}.
\end{aligned}
\tag{3.15}
$$

Considering the symmetries of the Lie derivatives, namely $\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)} = \mathcal{L}_\nu^{(z)}\mathcal{L}_\mu^{(z)}$, we observe that:

$$\hat{\mathcal{J}}_0^{\mu\nu,\rho} = \hat{\mathcal{J}}_0^{(\mu\nu),\rho}, \tag{3.16}$$

where

$$\hat{\mathcal{J}}_0^{(\mu\nu),\rho} = \frac{1}{2}\big(\hat{\mathcal{J}}_0^{\mu\nu,\rho} + \hat{\mathcal{J}}_0^{\nu\mu,\rho}\big). \tag{3.17}$$

Or alternatively:

$$\hat{\mathcal{J}}_0^{[\mu\nu],\rho} = 0, \tag{3.18}$$

where

$$\hat{\mathcal{J}}_0^{[\mu\nu],\rho} = \frac{1}{2}\big(\hat{\mathcal{J}}_0^{\mu\nu,\rho} - \hat{\mathcal{J}}_0^{\nu\mu,\rho}\big). \tag{3.19}$$

Finally, we have 18 independent component coming from $\hat{\mathcal{J}}_0^{\mu\nu,\delta}$, given by:

$$
\begin{aligned}
&\hat{\mathcal{J}}_0^{11,1}, \hat{\mathcal{J}}_0^{12,1}, \hat{\mathcal{J}}_0^{13,1}\\
&\hat{\mathcal{J}}_0^{22,1}, \hat{\mathcal{J}}_0^{23,1}, \hat{\mathcal{J}}_0^{33,1}\\
&\hat{\mathcal{J}}_0^{11,2}, \hat{\mathcal{J}}_0^{12,2}, \hat{\mathcal{J}}_0^{13,2}\\
&\hat{\mathcal{J}}_0^{23,2}, \hat{\mathcal{J}}_0^{22,2}, \hat{\mathcal{J}}_0^{33,2}\\
&\hat{\mathcal{J}}_0^{11,3}, \hat{\mathcal{J}}_0^{12,3}, \hat{\mathcal{J}}_0^{13,3}\\
&\hat{\mathcal{J}}_0^{22,3}, \hat{\mathcal{J}}_0^{23,3}, \hat{\mathcal{J}}_0^{33,3}.
\end{aligned}
\tag{3.20}
$$

To find the number of constraints we again set $d\mathcal{J}_Q = 0$ to obtain:

$$\begin{aligned}0 = d\mathcal{J}_Q &= C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3}\mathcal{L}_\mu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing}) + \sum_{\nu=1}^{3} d\{i_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu}d\tau)\}\\ &+ \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} d\{i_\nu^{(z)}\mathcal{L}_\mu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu}d\tau)\} + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu\nu,\varnothing})\\ &+ \sum_{\delta=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} d\{i_\delta^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu\nu,\delta}d\tau)\}.\end{aligned} \tag{3.21}$$

Applying Cartan's identity to a few terms we obtain the following relations:

$$\begin{aligned}&\sum_{\nu=0}^{3} d\{i_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu}d\tau)\} = \sum_{\nu=0}^{3}(\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu}d\tau) - \sum_{\nu=0}^{3} i_\nu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\nu}\wedge d\tau)),\\ &\sum_{\nu=0}^{3}\sum_{\mu=0}^{3} d\{i_\nu^{(z)}\mathcal{L}_\mu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu}d\tau)\} = \sum_{\nu=0}^{3}\sum_{\mu=0}^{3}\mathcal{L}_\nu^{(z)}\mathcal{L}_\mu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu}d\tau) - \sum_{\nu=0}^{3} i_\nu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_0^{\varnothing,\nu}\wedge d\tau) \text{ and}\\ &\sum_{\delta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} d\{i_\delta^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu\nu,\delta}d\tau)\} = \sum_{\delta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\mathcal{L}_\delta^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu\nu,\delta}d\tau)\\ &- \sum_{\delta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} i_\delta^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_0^{\mu\nu,\delta}\wedge d\tau).\end{aligned} \tag{3.22}$$

Simplifying using the expressions above, we obtain:

$$\begin{aligned}0 = d\mathcal{J}_Q &= C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing}) + \sum_{\mu=1}^{3}\mathcal{L}_\mu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing}) + \sum_{\nu=1}^{3}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu}d\tau) + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3}\mathcal{L}_\nu^{(z)}\mathcal{L}_\mu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu}d\tau)\\ &+ \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{\mu\nu,\varnothing}) + \sum_{\delta=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\mathcal{L}_\delta^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu\nu,\delta}d\tau).\end{aligned} \tag{3.23}$$

This gives rise to the following equations:

$$\begin{aligned}&\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing} = q\\ &\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing} + \hat{\mathcal{J}}_0^{\varnothing,\mu} = 0\\ &\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu\nu,\varnothing} + \hat{\mathcal{J}}_0^{(\mu,\nu)} = 0\\ &\hat{\mathcal{J}}_0^{(\mu\nu,\rho)} = 0.\end{aligned} \tag{3.24}$$

The number of all constraints on the quadrupole current is presented in Table 3.3 below:

We will explicitly show the equations given in Table 3.3 in order to illustrate how we have applied the increasing lists:

- There are 3 equations coming from $\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{\mu,\varnothing} + \hat{\mathcal{J}}_0^{\varnothing,\mu} = 0$:

$$\begin{aligned}\hat{\mathcal{J}}_0^{\varnothing,1} &= -\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{1,\varnothing}\\ \hat{\mathcal{J}}_0^{\varnothing,2} &= -\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{2,\varnothing}\\ \hat{\mathcal{J}}_0^{\varnothing,3} &= -\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{3,\varnothing}.\end{aligned} \tag{3.25}$$

- There are 6 equations coming from $\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{\mu\nu,\varnothing} + \hat{\mathcal{J}}_0^{(\mu,\nu)} = 0$:

$$\begin{aligned}\hat{\mathcal{J}}_0^{(1,1)} + \partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{11,\varnothing} &= 0\\ \hat{\mathcal{J}}_0^{(2,2)} + \partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{22,\varnothing} &= 0\\ \hat{\mathcal{J}}_0^{(3,3)} + \partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{33,\varnothing} &= 0\\ \hat{\mathcal{J}}_0^{(1,2)} + 2\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{12,\varnothing} &= 0\\ \hat{\mathcal{J}}_0^{(1,3)} + 2\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{13,\varnothing} &= 0\\ \hat{\mathcal{J}}_0^{(3,2)} + 2\partial_0^{(\tau)}\hat{\mathcal{J}}_{\varnothing}^{32,\varnothing} &= 0.\end{aligned} \tag{3.26}$$

- Finally, there are 10 equations coming from: $\hat{\mathcal{J}}_0^{(\mu\nu,\rho)} = 0$:

$$\begin{aligned}&\hat{\mathcal{J}}_0^{11,1} = \hat{\mathcal{J}}_0^{22,2} = \hat{\mathcal{J}}_0^{33,3} = 0\\ &\hat{\mathcal{J}}_0^{12,3} + \hat{\mathcal{J}}_0^{13,2} + \hat{\mathcal{J}}_0^{23,1} = 0\\ &\hat{\mathcal{J}}_0^{11,2} + 2\hat{\mathcal{J}}_0^{12,1} = 0\\ &\hat{\mathcal{J}}_0^{11,3} + 2\hat{\mathcal{J}}_0^{13,1} = 0\\ &\hat{\mathcal{J}}_0^{22,1} + 2\hat{\mathcal{J}}_0^{21,2} = 0\\ &\hat{\mathcal{J}}_0^{22,3} + 2\hat{\mathcal{J}}_0^{23,2} = 0\\ &\hat{\mathcal{J}}_0^{33,1} + 2\hat{\mathcal{J}}_0^{31,3} = 0\\ &\hat{\mathcal{J}}_0^{33,2} + 2\hat{\mathcal{J}}_0^{32,3} = 0.\end{aligned} \tag{3.27}$$

**Remark.** Here we can see the usefulness of considering increasing lists, as we have to take the symmetries of the Lie derivatives into account. For example, $\hat{\mathcal{J}}_0^{12,2} = \hat{\mathcal{J}}_0^{21,2}$, and $\hat{\mathcal{J}}_{\varnothing}^{23,\varnothing} = \hat{\mathcal{J}}_{\varnothing}^{32,\varnothing}$, which follows directly from $\mathcal{L}_1^{(z)}\mathcal{L}_2^{(z)} = \mathcal{L}_2^{(z)}\mathcal{L}_1^{(z)}$, and $\mathcal{L}_2^{(z)}\mathcal{L}_3^{(z)} = \mathcal{L}_3^{(z)}\mathcal{L}_2^{(z)}$.

The obtained number of independent components and constraints here are consistent with the calculations made previously by Gratus *et al.* in [2] and [11].

Table 3.3: Number of Constraints for the Quadrupole Distribution

| *Constraints* | *Number of Constraints* |
|---|---|
| $\hat{\mathcal{J}}^{\varnothing,\varnothing}_{\varnothing} = q$ | 1 |
| $\partial^{(\tau)}_0 \hat{\mathcal{J}}^{\mu,\varnothing}_{\varnothing} + \hat{\mathcal{J}}^{\varnothing,\mu}_0 = 0$ | 3 |
| $\partial^{(\tau)}_0 \hat{\mathcal{J}}^{\mu\nu,\varnothing}_{\varnothing} + \hat{\mathcal{J}}^{(\mu,\nu)}_0 = 0$ | 6 |
| $\hat{\mathcal{J}}^{(\mu\nu,\rho)}_0=0$ | 10 |

Table 3.4: Number of Independent Components of the Quadrupole Distribution

| *Component* | *Number of Configurations* |
|---|---|
| $\hat{\mathcal{J}}^{\varnothing,\varnothing}_{\varnothing}$ | 1 |
| $\hat{\mathcal{J}}^{\mu,\varnothing}_{\varnothing}$ | 3 |
| $\hat{\mathcal{J}}^{\varnothing,\nu}_0$ | 3 |
| $\hat{\mathcal{J}}^{\mu,\nu}_0$ | 9 |
| $\hat{\mathcal{J}}^{\mu\nu,\varnothing}_{\varnothing}$ | 6 |
| $\hat{\mathcal{J}}^{\mu\nu,\delta}_{\varnothing}$ | 18 |

### 3.0.5 Octupole Current

Consider $\mathcal{J}_O \in \Upsilon^{3,3}$ such that $d\mathcal{J}_O = 0$. This is the octupole current given by:

$$\mathcal{J}_O = \sum_{I\Uparrow 3, |I|\leq 3} \mathcal{L}^{(z)}_I C_\varsigma(\hat{\mathcal{J}}^{I,\varnothing}_{\varnothing}) + \sum_{I\Uparrow 3, J\uparrow 3, |I|\leq 3, |J|=1} i^{(z)}_J \mathcal{L}^{(z)}_I C_\varsigma(\hat{\mathcal{J}}^{I,J}_0 d\tau). \tag{3.28}$$

Writing the terms explicitly, we obtain:

$$\begin{aligned}\mathcal{J}_O = {} & C_\varsigma(\hat{\mathcal{J}}^{\varnothing,\varnothing}_{\varnothing}) + \sum_{\mu=1}^{3} \mathcal{L}^{(z)}_\mu C_\varsigma(\hat{\mathcal{J}}^{\mu,\varnothing}_{\varnothing}) + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \mathcal{L}^{(z)}_\mu \mathcal{L}^{(z)}_\nu C_\varsigma(\hat{\mathcal{J}}^{\mu\nu,\varnothing}_{\varnothing}) \\ & + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\sum_{\delta=1}^{3} \mathcal{L}^{(z)}_\mu \mathcal{L}^{(z)}_\nu \mathcal{L}^{(z)}_\delta C_\varsigma(\hat{\mathcal{J}}^{\mu\nu\delta,\varnothing}_{\varnothing}) + \sum_{\nu=1}^{3} i^{(z)}_\nu C_\varsigma(\hat{\mathcal{J}}^{\varnothing,\nu}_0 d\tau) + \sum_{\nu=1}^{3}\sum_{\mu=1}^{3} i^{(z)}_\nu \mathcal{L}^{(z)}_\mu C_\varsigma(\hat{\mathcal{J}}^{\mu,\nu}_0 d\tau) \\ & + \sum_{\rho=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} i^{(z)}_\rho \mathcal{L}^{(z)}_\mu \mathcal{L}^{(z)}_\nu C_\varsigma(\hat{\mathcal{J}}^{\mu\nu,\rho}_0 d\tau) + \sum_{\rho=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\sum_{\delta=1}^{3} i^{(z)}_\rho \mathcal{L}^{(z)}_\mu \mathcal{L}^{(z)}_\nu \mathcal{L}^{(z)}_\delta C_\varsigma(\hat{\mathcal{J}}^{\mu\nu\delta,\rho}_0 d\tau).\end{aligned} \tag{3.29}$$

The independent components of the octupole current are give in Table 3.5 below. Setting $d\mathcal{J}_O = 0$ to find the constraints gives:

$$
\begin{aligned}
0 = d\mathcal{J}_O = {} & C_\varsigma((\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing})d\tau) + \sum_{\mu=1}^{3}\mathcal{L}_\mu^{(z)}C_\varsigma((\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing})d\tau) \\
& + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma((\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu\nu,\varnothing})d\tau) + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\sum_{\delta=1}^{3}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}\mathcal{L}_\delta^{(z)}C_\varsigma((\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu\nu\delta,\varnothing})d\tau) \\
& + \sum_{\nu=1}^{3}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\varnothing,\nu}d\tau) + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu,\nu}d\tau) \\
& + \sum_{\rho=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\mathcal{L}_\rho^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu\nu,\rho}d\tau) + \sum_{\rho=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\sum_{\delta=1}^{3}\mathcal{L}_\rho^{(z)}\mathcal{L}_\rho^{(z)}\mathcal{L}_\mu^{(z)}\mathcal{L}_\nu^{(z)}\mathcal{L}_\delta^{(z)}C_\varsigma(\hat{\mathcal{J}}_0^{\mu\nu\delta,\rho}d\tau).
\end{aligned}
\tag{3.30}
$$

This leads to the following equations:

$$
\begin{aligned}
& \hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing} = q \\
& \partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing} + \hat{\mathcal{J}}_0^{\varnothing,\mu} = 0 \\
& \partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu\nu,\varnothing} + \hat{\mathcal{J}}_0^{(\mu,\nu)} = 0 \\
& \partial_0^{(\tau)}(\hat{\mathcal{J}}_\varnothing^{\mu\nu\rho,\varnothing}) + \hat{\mathcal{J}}_0^{(\mu\nu,\rho)} = 0 \\
& \hat{\mathcal{J}}_0^{\mu\nu,\varnothing} = 0.
\end{aligned}
\tag{3.31}
$$

Now we have the following number of constraints on the octupole distribution, presented in Table 3.6 below:

Table 3.5: Number of Constraints for the Octupole Distribution

| *Constraints* | *Number of Constraints* |
|---|---|
| $\hat{\mathcal{J}}_\varnothing^{\varnothing,\varnothing} = q$ | 1 |
| $\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu,\varnothing} + \hat{\mathcal{J}}_0^{\varnothing,\mu} = 0$ | 3 |
| $\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu\nu,\varnothing} + \hat{\hat{\mathcal{J}}}_0^{(\mu,\nu)} = 0$ | 6 |
| $\partial_0^{(\tau)}\hat{\mathcal{J}}_\varnothing^{\mu\nu\rho,\varnothing} + \hat{\mathcal{J}}_0^{(\mu\nu,\rho)} = 0$ | 10 |
| $\hat{\mathcal{J}}_0^{(\mu\nu\delta,\rho)} = 0$ | 15 |

For clarity and illustration of the increasing lists of indices, the 30 independent

Table 3.6: Number of Independent Components of the Octupole Distribution

| *Component* | *Number of Configurations* |
|---|---|
| $\hat{\mathcal{J}}_{\varnothing}^{\varnothing,\varnothing}$ | 1 |
| $\hat{\mathcal{J}}_{\varnothing}^{\mu,\varnothing}$ | 3 |
| $\hat{\mathcal{J}}_{0}^{\varnothing,\nu}$ | 3 |
| $\hat{\mathcal{J}}_{0}^{\mu,\nu}$ | 9 |
| $\hat{\mathcal{J}}_{\varnothing}^{\mu\nu,\varnothing}$ | 6 |
| $\hat{\mathcal{J}}_{\varnothing}^{\mu\nu,\delta}$ | 18 |
| $\hat{\mathcal{J}}_{\varnothing}^{\mu\nu\delta}$ | 10 |
| $\hat{\mathcal{J}}_{0}^{\mu\nu\delta,\rho}$ | 30 |

components coming from $\hat{\mathcal{J}}_0^{\mu\nu\delta,\rho}$ are listed explicitly below:

$$\begin{gathered}
\hat{\mathcal{J}}_0^{111,1}, \hat{\mathcal{J}}_0^{111,2}, \hat{\mathcal{J}}_0^{111,3} \\
\hat{\mathcal{J}}_0^{112,1}, \hat{\mathcal{J}}_0^{112,2}, \hat{\mathcal{J}}_0^{112,3} \\
\hat{\mathcal{J}}_0^{113,1}, \hat{\mathcal{J}}_0^{113,2}, \hat{\mathcal{J}}_0^{113,3} \\
\hat{\mathcal{J}}_0^{122,1}, \hat{\mathcal{J}}_0^{122,2}, \hat{\mathcal{J}}_0^{122,3} \\
\hat{\mathcal{J}}_0^{123,1}, \hat{\mathcal{J}}_0^{123,2}, \hat{\mathcal{J}}_0^{123,3} \\
\hat{\mathcal{J}}_0^{133,1}, \hat{\mathcal{J}}_0^{133,2}, \hat{\mathcal{J}}_0^{133,3} \\
\hat{\mathcal{J}}_0^{222,1}, \hat{\mathcal{J}}_0^{222,2}, \hat{\mathcal{J}}_0^{222,3} \\
\hat{\mathcal{J}}_0^{223,1}, \hat{\mathcal{J}}_0^{223,2}, \hat{\mathcal{J}}_0^{223,3} \\
\hat{\mathcal{J}}_0^{233,1}, \hat{\mathcal{J}}_0^{233,2}, \hat{\mathcal{J}}_0^{233,3} \\
\hat{\mathcal{J}}_0^{333,1}, \hat{\mathcal{J}}_0^{333,2}, \hat{\mathcal{J}}_0^{333,3}.
\end{gathered} \tag{3.32}$$

# Chapter 4

# Number of Free Components for $\mathcal{J}$

We will obtain the *free components* of a general electromagnetic current, $\mathcal{J}$, over a worldline by subtracting the number of differential equations, which arise from the condition $d\mathcal{J} = 0$ which we will call *constraints*, from the number of independent components in the general expressions for the distributional current $\mathcal{J}$.

**Definition 4.0.1.** Let $N_{in}$ be the number of independent components of the distribution $\mathcal{J}$.

**Remark.** It is important to note that to obtain this number we are taking into consideration only the symmetries of the Lie derivatives and internal contractions before any further conditions are imposed.

**Definition 4.0.2.** Let $N_{free}$ be the number of free components of a distribution $\mathcal{J} \in \Upsilon^{k,3}(C)$ after the condition $d\mathcal{J} = 0$ is imposed.

**Definition 4.0.3.** Let $N_c$ be the number of the resulting differential equations after we have imposed $d\mathcal{J} = 0$.

**Definition 4.0.4.** The number of free components, $N_{free}$, of the distribution $\mathcal{J}$ will then be given by:

$$N_{free} = N_{\text{in}} - N_c. \tag{4.1}$$

## 4.1 Stars and Bars

To obtain the number of independent components in a general electromagnetic distribution, $\mathcal{J}$, we will use the counting method known as *Stars and Bars*.
This counting method will allow us to determine how many ways there are of allocating $s$ number of stars into $s'$ number of bins. The stars are indistinguishable but the bins are not. We can distinguish the bins by how many stars are present in it. We only

need to know when we go from one bin to another. This is achieved by separating each bin with $s' - 1$ bars which tell us when we are moving to the next bin.
The number of ways to allocate $s$ objects into $s'$ number of bins is given by the Stars and Bars theorem and it is equal to:

$$N = \binom{s + s' - 1}{s' - 1}. \tag{4.2}$$

**Example 1.** Let us say that we would like to know in how many ways we could distribute 5 identical objects between 3 people. Using the Stars and Bars counting method, we will have $s = 5$ stars, allocated in 3 bins, separated by $s' - 1 = 2$ bars. The number of ways will then be:

$$\binom{s + s' - 1}{s' - 1} = \binom{5 + 2}{2} = \binom{7}{2} = 21. \tag{4.3}$$

To illustrate pictorially one way of distributing these objects between the 3 people will be given by:

$$|\bigstar\bigstar|\bigstar\bigstar\bigstar. \tag{4.4}$$

Here we have given 0 objects to the first person, 2 objects to the second one, and three objects to the third one. It can be clearly seen that the bars help us know when to stop giving objects to one person and move to the other.
Another option will be, for example:

$$\bigstar|\bigstar\bigstar\bigstar|\bigstar. \tag{4.5}$$

Here we have given 1 object to the first person, 3 objects to the second one, and one to the third person.

We can now introduce the core theorem of this thesis:

**Theorem 4.1.1.** For $\mathcal{J} \in \Upsilon^{k,3}$ such that $d\mathcal{J} = 0$ the number of free components, $N_{free}$, as defined in Def. 4.0.2, is given by:

$$N_{free} = \frac{k(k+2)(k+3)}{2}. \tag{4.6}$$

Before we prove Theorem 4.1.1 we will introduce a few definitions and lemmas.

## 4.2 Independent components of $\mathcal{J}$

To find the general expressions for both $\hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}$, and $\hat{\mathcal{J}}_{0}^{I,\nu}$, we will first consider a general distribution $\Upsilon^{k,3}(C)$ over a worldline $C$ as before. Let $\mathcal{J} \in \Upsilon^{k,3}(C)$ represent a $2^k$-tupole distribution of order k given by:

$$\mathcal{J} = \sum_{I\Uparrow^3,|I|\leq k,J\uparrow^3,|J|=1,K\uparrow^1} (-1)^{(|I|+1)} i_J^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\mathcal{J}_K^{I,J} dx^K), \tag{4.7}$$

where $|J| + |K| = 1$.
Let $(-1)^{(|I|+1)}\mathcal{J} = \hat{\mathcal{J}}$ as before. Then the equation for $\mathcal{J}$ becomes:

$$\mathcal{J} = \sum_{I\Uparrow^3,|I|\leq k,J\uparrow^3,|J|=1,K\uparrow^1} i_J^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}_K^{I,J} dx^K). \tag{4.8}$$

For more clarity we will now split the sum into two parts: one part where $K = [\varnothing]$, and one where $K = [0]$ with $dx^1 = d\tau$. Since we are working with three spatial components we will have $J = [\nu_1, \nu_2, \nu_3]$ with $\nu = 1, 2, 3$:

$$\mathcal{J} = \sum_{I\Uparrow^3,|I|\leq k} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}) + \sum_{\nu=1}^{3} \sum_{I\Uparrow^3,|I|\leq k,} i_\nu^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{I,\nu} d\tau). \tag{4.9}$$

Since we are interested in the number of configurations, $N_{in}$, we want to include independent components from both $\hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}$, and $\hat{\mathcal{J}}_{0}^{I,\nu}$.

**Definition 4.2.1.** Let the number of independent components of $\mathcal{J}$ be given by:

$$N_{in} = N_1 + N_2, \tag{4.10}$$

where $N_1$ will contain components from $\hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}$, and $N_2$ components from $\hat{\mathcal{J}}_{0}^{I,\nu}$ respectively.

**Remark.** $N_{in}$ will represent only those components which are non-repeating and thus we will have to exclude all repetitions due to symmetries.

## 4.3 Symmetry of Indices

To identify the number of independent components for both $\hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}$, and $\hat{\mathcal{J}}_{0}^{I,\nu}$ considering the symmetry we will use the Stars and Bars counting method. In this particular case, $s$ represents the number of symmetric indices, and $s'$ represents the number of values each index can take (e.g. $\mu = 1, 2, 3$). Since each index can take only three values $s' = 3$ always.

In order to take into account that we are only considering increasing lists, we will have $s$ number of symmetric indices (or stars) separated by $s'-1=2$ bars, where the bars will represent going up by 1 integer with each bin.
Thus, in order to find the number of independent components where symmetries have been considered, we will use the following binomial coefficient:

$$\binom{s+s'-1}{s'-1} = \binom{s+2}{2}. \tag{4.11}$$

To illustrate the application of the Stars and Bars counting method in calculating the symmetric indices, we will look at Example 1 again but this time using the number of the value each of the index can take. Thus, we will have:

$$[22333], \tag{4.12}$$

representing Eq.4.4 where the 2 objects given to person number 2 are now representing our dummy indices taking a value '2', and 3 objects given to person 3 represent the dummy indices taking a value '3'. And:

$$[12223], \tag{4.13}$$

representing Eq.4.5 where the object given to person 1 now represents the dummy index taking value '1', 3 objects given to person 2 now represent dummy indices taking value '2', and 1 object, given to person 3 now represents 1 index taking the value '3', recalling that here the bars represent going up with one integer.

To showcase this with an example of components, Eq.4.12 will correspond to $\hat{\mathcal{J}}^{22333,1}$ and Eq. 4.13 will correspond to $\hat{\mathcal{J}}^{12223,1}$ where we are paying attention to the indices before the comma, as they are the ones with the corresponding symmetries. (See the independent components of the octupole current, listed in Eq. 3.32, for comparison).

Here it is useful to remind that, since we are working in an adapted chart, we have the following symmetries of the Lie derivatives and internal contractions:

$$\begin{aligned} \mathcal{L}_I^{(z)}\mathcal{L}_J^{(z)} &= \mathcal{L}_J^{(z)}\mathcal{L}_I^{(z)} \\ i_I^{(z)}\mathcal{L}_J^{(z)} &= \mathcal{L}_J^{(z)}i_I^{(z)}. \end{aligned} \tag{4.14}$$

**Lemma 4.3.1.** The number of independent components obtained from $\hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}$ is given by:

$$N_1 = \sum_{s=0}^{k} \binom{s+2}{2}. \tag{4.15}$$

*Proof.* Using the Stars and Bars counting method and the arguments above we know that there are:

$$\binom{s+s'-1}{s'-1} = \binom{s+2}{2} \tag{4.16}$$

number of ways of allocating $s$ symmetric indices into $s' = 3$ number of slots, where the slots represent the number of values each index can take as explained above. Taking the symmetry of the Lie derivative and the fact that $|I| \leq k$ we find that to find the total number of independent components coming from $\hat{\mathcal{J}}^{I,\varnothing}_{\varnothing}$ we need to sum Eq. 4.16 up to $k$ to take into account all the lenght of the list $I$. □

**Lemma 4.3.2.** The number of independent components obtained from $\hat{\mathcal{J}}^{I,\nu}_0$ is given by

$$N_2 = \frac{3!}{2} \sum_{s=0}^{k} \binom{s+2}{2}. \tag{4.17}$$

*Proof.* Similar to 4.3.1 we find that when we take the symmetry of the Lie derivatives into account we will have:

$$\binom{s+s'-1}{s'-1} = \binom{s+2}{2}$$

number of ways to allocate $s$ symmetric indices into the three possible values. When we take the length of the list $I$ into account, we are again summing up to $k$, as $|I| \leq k$. However, in this case we also have to take into account the anti-symmetries of the internal contractions. To achieve that, we have multiplied the sum by a factor of 3 to take into account the number of possible permutations that arise from $\nu = 1, 2, 3$ in the expression: $\sum_{\nu=1}^{3} \sum_{I \Uparrow^3, |I| \leq k,} i_\nu^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}^{I,\nu}_0 d\tau)$. □

Hence, for the total number of independent components of the distribution $\mathcal{J}$, we obtain:

$$N_{in} = \sum_{s=0}^{k} \binom{s+2}{2} + 3 \sum_{s=0}^{k} \binom{s+2}{2}. \tag{4.18}$$

## 4.4 Number of Constraints $N_c$

Here we obtain a general expression for the differential equations arising from the imposed condition: $d\mathcal{J} = 0$. Letting $d\mathcal{J} = 0$ gives:

$$d\mathcal{J} = \sum_{I \Uparrow^3, |I| \leq k} \mathcal{L}_I^{(z)} C_\varsigma(d\hat{\mathcal{J}}^{I,\varnothing}_{\varnothing}) + \sum_{\nu=1}^{3} \sum_{I \Uparrow^3, |I| \leq k,} d(i_\nu^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}^{I,\nu}_0 d\tau)) = 0. \tag{4.19}$$

Using Cartan's identity $di_X\alpha = \mathcal{L}_X\alpha - i_X d\alpha$, the expression for $d\mathcal{J}$ becomes:

$$d\mathcal{J} = \sum_{I \Uparrow^3, |I| \leq k} \mathcal{L}_I^{(z)} C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{I,\varnothing}) + \sum_{\nu=1}^{3} \sum_{I \Uparrow^3, |I| \leq k,} \left( \mathcal{L}_\nu^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{I,\nu} d\tau) - i_\nu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_0^{I,\nu} \wedge d\tau) \right) = 0, \tag{4.20}$$

where the last terms: $i_\nu^{(z)} C_\varsigma(d\hat{\mathcal{J}}_0^{I,\nu} \wedge d\tau)$, vanish since $d\hat{\mathcal{J}}_0^{I,\nu} \wedge d\tau = 0$.
Finally we obtain:

$$d\mathcal{J} = \sum_{I \Uparrow^3, |I| \leq k} \mathcal{L}_I^{(z)} C_\varsigma(d\hat{\mathcal{J}}_\varnothing^{I,\varnothing}) + \sum_{\nu=1}^{3} \sum_{I \Uparrow^3, |I| \leq k,} \mathcal{L}_\nu^{(z)} \mathcal{L}_I^{(z)} C_\varsigma(\hat{\mathcal{J}}_0^{I,\nu} d\tau) = 0. \tag{4.21}$$

This gives rise to the following differential equations:

$$\partial_0^{(\tau)} \hat{\mathcal{J}}_\varnothing^{I,\varnothing} + \hat{\mathcal{J}}_0^{(I,\nu)} = 0, \tag{4.22}$$

which can be written as:

$$\hat{\mathcal{J}}_0^{(I,\nu)} = -\partial_0^{(\tau)} \hat{\mathcal{J}}_\varnothing^{I,\varnothing}, \tag{4.23}$$

where $\hat{\mathcal{J}}_0^{(I,\nu)}$ is the totally symmetric part of $\hat{\mathcal{J}}_0^{I,\nu}$.

**Remark.** Note that here we have $(k+2)$ number of ordinary differential equations, including the first one which shows that $\Psi_\varnothing^{\varnothing,\varnothing}$ = constant, with $k$ being the order of the distribution. These correspond to $k$ unknown functions. In order to discuss the independence of the solutions of these equations, we require more information about the functions on the left hand side. For example, no specific dependence on the variable with respect to which we differentiate, $\tau$, would make this system autonomous. More analysis is required in order to get a clearer idea of the independence of the solutions of these constraints. If we, however, assume that the functions on the left hand side are smooth, then there should exist a unique solution in some neighbourhood of the point which correspond to the initial conditions that we may impose [9].

**Lemma 4.4.1.** The number of constraints is equal to the number of partial differential equations given in Eq.4.22 and is given by:

$$N_c = \sum_{s=0}^{k+1} \binom{s+2}{2}. \tag{4.24}$$

*Proof.* To find the number of constraints we have utilized the Stars and Bars method as above. Considering the constraints again:

$$\hat{\mathcal{J}}_0^{(I,\nu)} = -\partial_0^{(\tau)} \hat{\mathcal{J}}_\varnothing^{I,\varnothing}. \tag{4.25}$$

Since we are only considering the symmetric part of $\hat{\mathcal{J}}_0^{I,\nu}$ we will have $s$ indices to allocate to three values. As above, we are symmetrizing over all lengths $|I| \leq k+1$ and hence the sum goes up to $k+1$. Furthermore:

$$\hat{\mathcal{J}}_0^{(I,\nu)} = \begin{cases} 0, \quad \text{when } |I| = s \\ -\partial_0^{(\tau)} \hat{\mathcal{J}}_{\varnothing}^{I,\varnothing}, \quad \text{otherwise,} \end{cases}$$

where $s$ is the maximal length of the list $I$. □

Now given the definitions and lemmas above and their respective proofs, we can prove Theorem 4.1.1 as follows:

*Proof.* The number of free components, $N_{free}$, as defined in Def. 4.0.2, after the imposed constraints on the distribution is given by:

$$\begin{aligned} N_{free} = N_{\text{config}} - N_c &= \sum_{s=0}^{k} \frac{(s+2)(s+1)}{2} + 3\sum_{s=0}^{k} \frac{(s+2)(s+1)}{2} - \sum_{s=0}^{k+1} \frac{(s+2)(s+1)}{2} \\ &= \frac{4}{6}(k+1)(k+2)(k+3) - \frac{1}{6}(k+2)(k+3)(k+4) \\ &= \frac{1}{6}\{(k+2)(k+3)\big(4(k+1) - (k-4)\big)\} \\ &= \frac{1}{6}(k+2)(k+3)3k \\ &= \frac{3k(k+2)(k+3)}{6}, \end{aligned}$$

which gives:

$$N_{free} = \frac{k(k+2)(k+3)}{2}. \tag{4.26}$$

Finally, the degrees of freedom of the different $2^k$-tupoles are given in Table 4.1. □

Table 4.1: Degrees of Freedom of a $2^k$-tupole

| *Charge Distribution* | *Notation* | *Order* (k) | *Free Components* ($N_{\text{free}}$) |
|---|---|---|---|
| **monopole** | $\mathcal{J}_M \in \Upsilon^{0,3}$ | 0 | 1 |
| **dipole** | $\mathcal{J}_D \in \Upsilon^{1,3}$ | 1 | 6 |
| **quadrupole** | $\mathcal{J}_Q \in \Upsilon^{2,3}$ | 2 | 20 |
| **octupole** | $\mathcal{J}_O \in \Upsilon^{3,3}$ | 3 | 45 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $2^k$**-tupole** | $\mathcal{J}_k \in \Upsilon^{k,3}$ | k | $\dfrac{k(k+2)(k+3)}{2}$ |

# Chapter 5

# Conclusion and Disscusion

In this thesis we have calculated the number free components for a $2^k$-pole distribution satisfying the equation for conservation of charge, $d\mathcal{J} = 0$. We achieved our results by considering a distribution on a worldline, defined on a arbitrary spacetime without specifying a metric. Although we know that a metric, a connection and an orientation are natural structures on spacetime, we are still interested in knowing which objects can be described without a metric or a connection. There is also an interest in finding how the metric and the connection of spacetime is affecting objects such as the sources of electromagnetic fields or gravitational fields. This is why it is very useful to define multipoles without a reference of a metric or a connection, but only by referring to the following structures on a manifold, such as the Lie derivatives, the interior product, the exterior derivative and the tangent and cotangent bundles. [12]

The results we have obtained do not directly depend on the metric or a connection of a manifold and this provides with the freedom of modeling multipoles on higher dimensional manifolds such as phase space and to manifolds with no preferred metric [12]. These calculations are also useful particularly in situations where we have to vary a metric: it is useful to know how much the expansions depend on the metric itself. Knowing the number of free components can be useful in providing insight into different sources of electromagnetic fields. An interesting application of modeling multipoles on higher dimensional manifolds is found in accelerator physics where high energy electrons can be represented by a multipoe expansion in 7 dimensions (phase space and time) [11]. The free components show us that the sources of electromagnetic radiation depend not only on the initial conditions that we impose but also on constitutive relations that should be established in order to understand the system that we are investigating in detail.

In the future, research can be done on finding the appropriate constitutive relations which will provide with further information about the sources of fields and also intrinsic properties of materials and particles which are also investigated with the help of multipole expansions. Some work has already been done on the free components of the quadrupole stress-energy source of gravity in [12]. There is now a wide interest in exotic compact objects in Gravitational Wave Astronomy [1][3][4]. These objects are alternative models of black holes characterized by Planckian corrections at the event horizon. A lot of analysis is done on the ringdown signal of black hole and neutron star mergers in order to find the footprint of such objects, if they are physical. A calculation of the free components of a multipole expansion, similar to that presented in this thesis, of the stress-energy distribution might give us an insight on the physical properties of exotic compact objects and how we would be able to detect their footprint in the gravitational wave data. In this way, in the future our analysis could be possibly extended to gravity where we will be interested in looking at the number of free components of a stress-energy distribution which is a source of a gravitational field.

# Appendix A

# Appendix

## A.1 Multi-Index Notation Examples

In our calculations we work with components of distributions, which are expressed as $\Psi^{I,J}_{K}$ (See Eq.2.34), where $I$ will show the lists of indices that directly depend on the symmetries endowed by the Lie derivatives, $J$ will show the anti-symmetry endowed by the internal contractions, and $K$ is related to whether the components are multiplied by the base $dx^K$.
To help illustrate the unpacking of increasing and strictly increasing lists, let us consider the following example:

**Example 2.** Let $I = [\mu_0, \mu_1, \mu_2]$, with $\mu_0 = \varnothing$ so that $I = [\varnothing, \mu_1, \mu_2]$. Let $J = [\varnothing, \nu]$ and $K = [\varnothing, 0]$.

First, let us consider the case where $J = \varnothing$ and $K = \varnothing$ and $I \Uparrow^3, |I| \leq 2$. We will have the following list of components compactly written as:

$$\Psi^{I,\varnothing}_{\varnothing} \tag{A.1}$$

Here we specify that the values of the elements of $I$ that are not $\varnothing$ take up values $\mu = 1, 2, 3$ so that $I \Uparrow^t$ means $I \in \mathrm{RIn}(t)$ with $t = 3$. We also specify for the length of the list of $I$ the following $|I| \leq 2$. If we now unpack the components, we will have:

$$\Psi^{I,\varnothing}_{\varnothing} \rightarrow \Psi^{\varnothing,\varnothing}_{\varnothing}, \Psi^{\mu_1,\varnothing}_{\varnothing}, \Psi^{\mu_1\mu_2,\varnothing}_{\varnothing}. \tag{A.2}$$

Now taking into account that $\mu$ can take values $\mu = 1, 2, 3$ we will have the following components considering the increasing lists:

$$\begin{aligned} &\Psi^{1,\varnothing}_{\varnothing}, \Psi^{2,\varnothing}_{\varnothing}, \Psi^{3,\varnothing}_{\varnothing} \quad \text{from} \quad \Psi^{\mu_1,\varnothing}_{\varnothing} \\ &\Psi^{11,\varnothing}_{\varnothing}, \Psi^{12,\varnothing}_{\varnothing}, \Psi^{13,\varnothing}_{\varnothing}, \Psi^{22,\varnothing}_{\varnothing}, \Psi^{23,\varnothing}_{\varnothing}, \Psi^{33,\varnothing}_{\varnothing}, \quad \text{from} \quad \Psi^{\mu_1\mu_2,\varnothing}_{\varnothing}. \end{aligned} \tag{A.3}$$

Now, let us consider the case when $K = 0$ and $I \Uparrow^3, |I| \leq 2, J \uparrow^3, |J| = 1$. This can be compactly written as:

$$\Psi_0^{I,J} \tag{A.4}$$

Here we note that $I \Uparrow^t$ means $I \in \mathrm{RIn}(t)$ with $t = 3$ as above, and $|J| = 1$ shows us that the list $J$ contains one element, $J = [\nu]$. Moreover, $J \uparrow^t$ means $J \in \mathrm{SIn}(t)$ with $t = 3$ which gives us $\nu = 1, 2, 3$. Unpacking the lists we obtain:

$$\Psi_0^{I,J} \to \Psi_0^{\varnothing,\nu} + \Psi_0^{\mu,\nu} + \Psi_0^{\mu_1\mu_2,\nu} \tag{A.5}$$

As above, taking into account that $\mu$ can take values $\mu = 1, 2, 3$, and also $\nu = 1, 2, 3$ we will have the following components considering the increasing lists:

$$\Psi_0^{\varnothing,1}, \Psi_0^{\varnothing,2}, \Psi_0^{\varnothing,3} \quad \text{from} \quad \Psi_0^{\varnothing,\nu} \tag{A.6}$$

$$\Psi_0^{1,1}, \Psi_0^{2,1}, \Psi_0^{3,1}, \Psi_0^{1,2}, \Psi_0^{2,2}, \Psi_0^{3,2}, \Psi_0^{1,3}, \Psi_0^{2,3}, \Psi_0^{3,3} \quad \text{from} \quad \Psi_0^{\mu,\nu} \tag{A.7}$$

and finally:

$$\begin{aligned} &\Psi^{11,1}, \Psi^{12,1}, \Psi^{13,1}, \Psi^{11,2}, \Psi^{12,2}, \Psi^{13,2}, \Psi^{11,3}, \Psi^{12,3}, \Psi^{13,3}, \\ &\Psi^{22,1}, \Psi^{2,2}, \Psi^{22,3}, \Psi^{23,1}, \Psi^{23,2}, \Psi^{23,3}, \Psi^{33,1}, \Psi^{33,2}, \Psi^{33,3} \end{aligned} \tag{A.8}$$

all coming from $\Psi_0^{\mu_1\mu_2,\nu}$.

Note that here we have not considered any additional symmetries and also note that the list of $I$ is always strictly increasing and we do not have an order of indices such as 32 or 21, for example.

## A.2 Operations on forms

### A.2.1 Interior product

We will consider two differentiable manifolds $\mathcal{M}(M, \tau_M, \mathcal{A}_M)$ and $\mathcal{N}(N, \tau_N, \mathcal{A}_N)$ with $\dim(\mathcal{M}) = m$, and $\dim(\mathcal{N}) = n$, as before.

**Definition A.2.1.** The interior product or internal contraction takes a vector field $V \in \Gamma T\mathcal{M}$ and a p-form, $\alpha \in \Gamma\Lambda^p\mathcal{M}$ resulting in a $(p-1)$ form field annotated as $i_V\alpha \in \Gamma\Lambda^{p-1}\mathcal{M}$:

$$i : \Gamma T\mathcal{M} \times \Gamma\Lambda^p\mathcal{M} \to \Gamma\Lambda^{p-1}\mathcal{M}. \tag{A.9}$$

It satisfies the following properties:

$$\begin{aligned} &i_{(U+V)}\alpha = i_U\alpha + i_V\alpha, \quad \text{for} \quad U, V \in \Gamma T\mathcal{M} \\ &i_{(fV)}\alpha = f i_V\alpha, \quad \text{for} \quad f \in \Gamma\Lambda^0\mathcal{M} \\ &i_V f = 0. \end{aligned} \tag{A.10}$$

### A.2.2 Exterior Differential Operator

**Definition A.2.2.** The exterior differential operator is defined via the map:

$$d : \Gamma\Lambda^p\mathcal{M} \to \Gamma\Lambda^{p+1}\mathcal{M}. \tag{A.11}$$

It satisfies the following properties:

$$\begin{aligned}
&d(f)V = V(f), \quad \text{for} \;\; f \in \Gamma\Lambda^0\mathcal{M}, \;\; V \in \Gamma T\mathcal{M} \\
&d(\alpha+\beta) = d(\alpha) + d(\beta), \quad \text{for} \;\; \alpha, \beta \in \Gamma\Lambda^p\mathcal{M} \\
&d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^p(\alpha \wedge d\beta), \quad \text{where} \;\; \alpha \in \Gamma\Lambda^p\mathcal{M}, \beta \in \Gamma\Lambda^q\mathcal{M} \\
&d^2 = 0 \;\; \text{(see Theorem } A.2.1).
\end{aligned} \tag{A.12}$$

#### A.2.2.1 Lie Derivatives

We can define the Lie derivative, $\mathcal{L}$, as an operator on tensors and forms. We do not need extra structure on the manifold in order to define the Lie derivative, as it is already available for smooth manifolds such as $\mathcal{M}(M, \tau_M, \mathcal{A}_M)$.

**Definition A.2.3.** Let $X \in \Gamma T\mathcal{M}$ be a vector field. The Lie derivative, $\mathcal{L}_X$ sends a pair of vector field, $X$, and a $(p, q)$-tensor field $\mathcal{T}^p_q(\mathcal{M})$ to a $(p, q)$-tensor field:

$$\begin{aligned}
&\mathcal{L}_X : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) \;\; \text{where} \;\; \mathcal{F}(\mathcal{M}) \;\; \text{is a scalar field on } \mathcal{M} \\
&\mathcal{L}_X : \Gamma T\mathcal{M} \to \Gamma T\mathcal{M} \\
&\mathcal{L}_X : \Gamma T^*\mathcal{M} \to \Gamma T^*\mathcal{M} \\
&\mathcal{L}_X : \Gamma\Lambda^p\mathcal{M} \to \Gamma\Lambda^p\mathcal{M}.
\end{aligned} \tag{A.13}$$

It satisfies the following properties:

$$\mathcal{L}_X f = Xf, \quad \text{for } f \in \mathcal{F}(\mathcal{M}) \tag{A.14}$$

$$\mathcal{L}_X(Y) = [X, Y] \;\; \text{for} \;\; X, Y \in \Gamma T\mathcal{M} \tag{A.15}$$

$$(\mathcal{L}_X\zeta) \cdot Y = \mathcal{L}_X(\zeta \cdot Y) - \zeta \cdot \mathcal{L}_X(Y), \quad \text{for} \;\; Y \in \Gamma T\mathcal{M}, \zeta \in \Gamma T^*\mathcal{M} \tag{A.16}$$

$$\mathcal{L}(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta). \tag{A.17}$$

The Lie derivative of forms also satisfies Cartan's magic formula:

$$\mathcal{L}_X(\zeta) = i_X(d\zeta) + d(i_X\zeta), \tag{A.18}$$

for $X, \zeta$ defined as above.

Note that in the definition above:

- scalar fields are considered to be $(0,0)$ tensor fields, $\mathcal{T}^0_0$,
- tangent vector fields belonging to $\Gamma T\mathcal{M}$ are regarded as $(1,0)$ tensor fields, $\mathcal{T}^1_0$,
- linear form fields belonging to $\Gamma T^*\mathcal{M}$ are regarded as $(0,1)$ tensor fields,$\mathcal{T}^0_1$,
- and finally p-forms belonging to $\Gamma\Lambda^p\mathcal{M}$ are regarded as tensor fields which are completely antisymmetric [9].

#### A.2.2.2 Examples of calculating Lie Derivatives

Let $(U,\phi)$ be a chart on $\mathcal{M}(M,\tau_M,\mathcal{A}_M)$ and $X, Y \in \Gamma T\mathcal{M}$ be vector fields on $\mathcal{M}$ with the components of $X, Y$ annotated by $Y^i, X^j$. Let $\mathcal{T}^1_1$ be a $(1,1)$ tensor field on $\mathcal{M}$ with its components annotated by $T^i_j$.
The Lie derivative of $Y$ with respect to the field $X$, $\mathcal{L}_X Y$ is then given by:

$$\begin{aligned}(\mathcal{L}_X Y)^i &= X(Y^i) - Y(X^i)\\ &= X^m\frac{\partial}{\partial x^m}(Y^i) - Y^s\frac{\partial}{\partial x^s}(X^i)\\ &= X^m\frac{\partial}{\partial x^m}(Y^i) - \frac{\partial}{\partial x^s}(X^i)Y^s.\end{aligned} \tag{A.19}$$

The Lie Derivative of $\mathcal{T}^1_1$ with respect to the field $X$ is given by:

$$(\mathcal{L}_X T)^i_j = X^m\frac{\partial T^i_j}{\partial x^m} - \frac{\partial X^i}{\partial x^s}T^s_j + \frac{\partial X^s}{\partial x^j}T^i_j. \tag{A.20}$$

### A.2.3 Exact Forms

We will now show that all exact forms are also closed.

**Definition A.2.4.** Let $\mathcal{M}$ be a smooth manifold and let $\alpha \in \Gamma\Lambda^p\mathcal{M}$.
We say that $\alpha$ is:

$$\begin{cases} \text{closed} \;\text{ if }\; d\alpha = 0 \\ \text{exact} \;\text{ if }\; \exists\, \beta \in \Gamma\Lambda^{n-1}\mathcal{M} \;\text{ such that }\; \alpha = d\beta. \end{cases} \tag{A.21}$$

**Theorem A.2.1.** The operator

$$d^2 \equiv d \circ d : \Gamma\Lambda^p\mathcal{M} \to \Gamma\Lambda^{n+2}\mathcal{M} \tag{A.22}$$

is identically zero, $d^2 = 0$.

Following the statement of Theorem A.2.1 we take the exterior derivative of $\alpha$ to obtain:

$$d\alpha = d(d\beta) = 0. \tag{A.23}$$

Thus we can conclude that a form which is exact is also closed by definition [9].

## A.3 Metric on a Manifold

**Definition A.3.1.** A metric, $g$, on a smooth manifold, $\mathcal{M}$, is a$(0, 2)$ tensor field satisfying:

- Symmetry:

$$g(X, Y) = g(Y, X), \text{ for } X, Y \in \Gamma T\mathcal{M} \tag{A.24}$$

- Non-Degeneracy:

$$\flat : \Gamma T\mathcal{M} \to \Gamma T^*\mathcal{M}, \text{ with } \flat(X) = g(X, \cdot). \tag{A.25}$$

## A.4 Hodge Dual Map

The Hodge dual map takes a $p$-form, $\alpha \in \Gamma\Lambda^p\mathcal{M}$, and gives a $(m-p)$-form, $\star\alpha \in \Gamma\Lambda^{m-p}\mathcal{M}$:

$$\star : \Gamma\Lambda^p\mathcal{M} \to \Gamma\Lambda^{m-p}\mathcal{M}. \tag{A.26}$$

It has the following properties:

- '$f$'-linearity:

$$\star(f\alpha) = f \star \alpha, \text{ for } f \in \mathcal{F}(\mathcal{M}). \tag{A.27}$$

- '+'-linearity:

$$\star(\alpha + \beta) = \star\alpha + \star\beta \text{ for } \alpha, \beta \in \Gamma\Lambda^p\mathcal{M}. \tag{A.28}$$

Let $\tilde{V} = g(V, \cdot)$, where $g$ is the metric on $\mathcal{M}$, and $V \in \Gamma T\mathcal{M}$.
Let $\star 1$ be the orientation on $\mathcal{M}$, defined by:

$$\star 1 = e^1 \wedge \cdots \wedge e^n \in \Gamma\Lambda^n\mathcal{M}. \tag{A.29}$$

We can now define $\star\alpha$ via:

$$i_V \star \alpha = \star(\alpha \wedge \tilde{V}). \tag{A.30}$$

## A.5 Operations on $\Upsilon^{k,p}(f)$

Let $f : \mathcal{N} \to \mathcal{M}$ be a closed embedding.
Let $f_* : T\mathcal{N} \to T\mathcal{M}$ be the push-forward map of vector fields on the tangent bundle of $\mathcal{N}$, $T\mathcal{N}$, onto the tangent bundle of $\mathcal{M}$, $T\mathcal{M}$.
Let $V$ and $W \in \Gamma(T\mathcal{M})$ be vector fields on $\mathcal{M}$, with $V = f_*(W)|_p$.
Let $\alpha \in \Gamma\Lambda^p\mathcal{M}$.

Let $\Gamma_0\Lambda^q\mathcal{M}$ be the space of test forms on $\mathcal{M}$, defined by:

$$\Gamma_0\Lambda^q\mathcal{M} := \big\{\phi \in \Gamma\Lambda^q\mathcal{M} \text{ such that } \phi \text{ has compact support}\big\}, \tag{A.31}$$

as before. We will show that:

$$i_v\alpha^D[\phi] = -(-1)^{deg(\phi)}\alpha^D[i_v\phi] \tag{A.32}$$

$$d\alpha^D[\phi] = -(-1)^{deg(\phi)}\alpha^D[d\phi]. \tag{A.33}$$

*Proof.* By the definition of a regular distribution given in Eq.2.14, we have:

$$\int_{\mathcal{M}}(i_v\phi)\wedge\alpha = \int_{\mathcal{M}} i_v(\phi\wedge\alpha) - (-1)^{deg(\phi)}\int_{\mathcal{M}}\phi\wedge(i_v\alpha), \tag{A.34}$$

with the first integral on the RHS vanishing as $\varphi\wedge\alpha = 0$, we obtain:

$$\int_{\mathcal{M}}(i_v\phi)\wedge\alpha = -(-1)^{deg(\phi)}\int_{\mathcal{M}}\phi\wedge(i_v\alpha). \tag{A.35}$$

Similarly,

$$\int_{\mathcal{M}}(d\phi)\wedge\alpha = \int_{\mathcal{M}} d(\phi\wedge\alpha) - (-1)^{deg(\phi)}\int_{\mathcal{M}}\phi\wedge(d\alpha), \tag{A.36}$$

with $\int_{\mathcal{M}} d(\varphi\wedge\alpha)$ vanishing as well, we have:

$$\int_{\mathcal{M}}(d\phi)\wedge\alpha = -(-1)^{deg(\phi)}\int_{\mathcal{M}}\phi\wedge(d\alpha). \tag{A.37}$$

□

## A.6 Representing $\Psi \in \Upsilon^{k,p}(f)$ locally

As $\Psi$ is defined by its action on a test form, we will show that we can write it in terms of the De Rham push-forward without a reference to a test form $\phi$, as stated in Lemma 2.2.3:

$$\Psi = \sum_{Rng(I,J,K)}(-1)^{|I|+(p-r)(m-p+|J|)} i_J^{(z)}\mathcal{L}_I^{(z)} f_\varsigma(\Psi_K^{I,J}dx^K). \tag{A.38}$$

*Proof.*

$$\begin{aligned} &\Psi[i^{(z)}_J \phi] = \Psi[i^{(z)}_{J_s} \cdots i^{(z)}_{J_1}] = \\ &= (-1)^{m-p-s} i^{(z)}_{J_s} \Psi[i^{(z)}_{J_{s-1}} \cdots i_{J_1} \phi] = \\ &= (-1)^{2(m-p-s)} i^{(z)}_{J_s} i^{(z)}_{J_{s-1}} \Psi[i^{(z)}_{J_{s-2}} \cdots i_{J_1} \phi] = \\ &= (-1)^{s(m-p)-s^2} i^{(z)}_{J_s} \cdots i_{J_1} \Psi[\phi] = \\ &= (-1)^{s(m-p)+s} i^{(z)}_J \Psi[\phi], \end{aligned} \tag{A.39}$$

with $|J| = s$.
Recalling the statement of Theorem 2.2.2:

$$\Psi[\phi] = \sum_{Rng(I,J,K)} \int_{\mathcal{N}} \Psi^{I,J}_K dx^K \wedge f^*(i^{(z)}_J \mathcal{L}^{(z)}_I \phi), \tag{A.40}$$

we have:

$$\begin{aligned} \Psi[\phi] &= \sum_{Rng(I,J,K)} (-1)^{|K|(m-p-|J|} \int_{\mathcal{N}} f^*(i^{(z)}_J \mathcal{L}^{(z)}_I \phi) \wedge \Psi^{I,J}_K dx^K = \\ &= \sum_{Rng(I,J,K)} (-1)^{|K|(m-p-|J|} f_\varsigma(\Psi^{I,J}_K dx^K)[i^{(z)}_J \mathcal{L}^{(z)}_I \phi] = \\ &= \sum_{Rng(I,J,K)} (-1)^{|K|(m-p-|J|+|J|(m-p+1)} i^{(z)}_J f_\varsigma(\Psi^{I,J}_K dx^K[\mathcal{L}^{(z)}_I \phi] \\ &= \sum_{Rng(I,J,K)} (-1)^{|K|(m-p-|J|+|J|(m-p+1)+|I|} i^{(z)}_J \mathcal{L}^{(z)}_I f_\varsigma(\Psi^{I,J}_K dx^K)[\phi], \end{aligned} \tag{A.41}$$

with

$$\begin{aligned} &(-1)^{|K|(m-p-|J|)+|J|(m-p+1)} = (-1)^{(n-m+p+|J|)(m-p-|J|)+|J|(m-p+|J|)} = \\ &= (-1)^{(p-r)(m-p+|J|)}. \end{aligned} \tag{A.42}$$

Using the same notation as before for compactness:

$$^*\Psi^{I,J}_K = (-1)^{|I|+(p-r)(m-p+|J|)} \Psi^{I,J}_K, \tag{A.43}$$

so that locally we now have:

$$\Psi = \sum_{Rng(I,J,K)} i^{(z)}_J \mathcal{L}^{(z)}_I f_\varsigma(^*\Psi^{I,J}_K dx^K). \tag{A.44}$$

□

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