# An equality underlying Hardy's inequality

**G. J. O. Jameson**

**Abstract.** A classical inequality of G. H. Hardy states that $\|Cx\| \le 2\|x\|$ for $x$ in $\ell_2$, where $C$ is the Cesàro (alias averaging) operator. This inequality has been strengthened to $\|(C - I)x\| \le \|x\|$. It has also been shown that $\|C^T x\| \le \|Cx\|$ for $x$ in $\ell_2$. We present equalities that imply these inequalities, together with the reverse inequalities $\|(C - I)x\| \ge (1/\sqrt{2})\|x\|$ and $\|Cx\| \le \sqrt{2}\|C^T x\|$. We also present companion results involving the shift operator.

**1. INTRODUCTION.** For real sequences $x = (x_n)$, the Cesàro (or averaging) operator is defined by $Cx = y$, where

$$y_j = \frac{1}{j}(x_1 + x_2 + \cdots + x_j).$$

Here $C$ is the lower-triangular matrix defined by:

$$c_{j,k} = \begin{cases} \frac{1}{j} & \text{for } k \le j, \\ 0 & \text{for } k > j. \end{cases}$$

The transposed operator $C^T$ is defined by $C^T x = z$, where $z_j = \sum_{k=j}^{\infty} \frac{x_k}{k}$.

Recall that $\ell_2$ denotes the space of all real sequences $x = (x_n)$ such that $\sum_{n=1}^{\infty} x_n^2$ is convergent, with norm defined by $\|x\| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$. For a matrix $A$, we denote by $\|A\|$ the norm of $A$ as an operator on $\ell_2$, in other words the least $M$ such that $\|Ax\| \le M\|x\|$ for all $x$ in $\ell_2$. Also, the inner product $\sum_{n=1}^{\infty} x_n y_n$ is denoted by $\langle x, y\rangle$, so $\langle x, x\rangle = \|x\|^2$. The $n$th unit vector will be denoted by $e_n$.

The English mathematician G. H. Hardy (1877–1947) established a number of significant inequalities, each known, in its own context, simply as "Hardy's inequality". One of them states, in this notation, that $\|C\| = 2$. However, a stronger statement is actually true: $\|C - I\| = 1$. This can be proved by a minor adjustment to the proof in [**4**, p. 239–241]. A very neat alternative proof, given in [**2**] and [**3**], uses basic Hilbert space theory, as follows. It is quite easy to show that $CC^T$ is the matrix having $1/\max(j, k)$ in place $(j, k)$. Hence $CC^T = C + C^T - \Delta_1$, where $\Delta_1$ is the diagonal matrix with entries $\frac{1}{n}$. Equivalently,

$$(C - I)(C^T - I) = I - \Delta_1. \tag{1}$$

Now for any operator $A$ on $\ell_2$, we have $\|AA^T\| = \|A\|^2 = \|A^T\|^2$. Clearly, $\|I - \Delta_1\| = 1$, so $\|C^T - I\| = \|C - I\| = 1$, as stated. But we can deduce more than this from (1). In general, $\langle AA^T x, x\rangle = \langle A^T x, A^T x\rangle = \|A^T x\|^2$, so

$$\|(C^T - I)x\|^2 = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n^2. \tag{2}$$

This is an *equality* that implies the inequality $\|(C^T - I)x\| \le \|x\|$. It is of the form $\sum_{n=1}^{\infty} z_n^2 = \sum_{n=1}^{\infty} \delta_n x_n^2$, where $(x_n)$ and $(z_n)$ are two sequences we wish to compare, and $\delta_n \to 1$ as $n \to \infty$. We will present a number of further equalities of this type, including one that implies $\|(C - I)x\| \le \|x\|$, and hence Hardy's inequality. We start with one relating $\|Cx\|$ and $\|C^T x\|$.

**2. COMPARISON BETWEEN $\|Cx\|$ AND $\|C^Tx\|$.** The reasoning in [**1**, p. 47], together with $\|C - I\| = 1$, implies that $\|Cx\| \geq \|C^Tx\|$ for all $x$ in $\ell_2$. Here we present an equality which implies this inequality, together with the reverse inequality $\|Cx\| \leq \sqrt{2}\|C^Tx\|$. We use another identity for $CC^T$. It would make things easy if we had $C^TC = CC^T$, but this is not true! However, by inserting a suitable diagonal matrix, we arrive at a correct statement that is nearly as good.

**Theorem 1.** *Let $\Delta_2$ be the diagonal matrix with $n$th component $n/(n+1)$. Then $C^T\Delta_2 C = CC^T$. Hence if $Cx = y$ and $C^Tx = z$, then*

$$\sum_{n=1}^{\infty} \frac{n}{n+1} y_n^2 = \sum_{n=1}^{\infty} z_n^2. \tag{3}$$

*Proof.* The matrix $\Delta_2 C$ is obtained from $C$ by multiplying row $j$ by $j/(j+1)$:

$$(\Delta_2 C)_{j,k} = \begin{cases} \frac{1}{j+1} & \text{for } k \leq j, \\ 0 & \text{for } k > j. \end{cases}$$

Meanwhile $(C^T)_{j,k}$ is $1/k$ for $j \leq k$ and 0 for $j > k$. So if $k \leq j$, then element $(j,k)$ of $C^T\Delta_2 C$ is

$$\sum_{r=j}^{\infty} \frac{1}{r(r+1)} = \frac{1}{j},$$

agreeing with $CC^T$. Both products are symmetric, so their entries also coincide for $k \geq j$.

Now $\langle CC^Tx, x\rangle = \|C^Tx\|^2 = \sum_{n=1}^{\infty} z_n^2$, while

$$\langle C^T\Delta_2 Cx, x\rangle = \langle \Delta_2 Cx, Cx\rangle = \langle \Delta_2 y, y\rangle = \sum_{n=1}^{\infty} \frac{n}{n+1} y_n^2. \qquad \blacksquare$$

Since $n/(n+1) \geq \frac{1}{2}$ for all $n \geq 1$, we deduce:

**Corollary 2.** *For all $x$ in $\ell_2$, we have $\|C^Tx\| \leq \|Cx\| \leq \sqrt{2}\|C^Tx\|$.*

Both constants here are optimal. For the right-hand inequality, note that if $x = (1, -1, 0, \ldots)$, then $Cx = (1, 0, 0, \ldots)$ while $C^Tx = (\frac{1}{2}, -\frac{1}{2}, 0, \ldots)$. For the left-hand inequality, it is easily checked that $\|Ce_n\|/\|C^Te_n\|$ tends to 1 as $n \to \infty$

**3. AN IDENTITY RELATING TO $(C - I)x$.** We now present the promised equality underlying the inequality $\|(C - I)x\| \leq \|x\|$.

**Theorem 3.** *Let $x \in \ell_2$ and $(C - I)x = z$. Then*

$$\sum_{n=2}^{\infty} \frac{n}{n-1}\, z_n^2 = \sum_{n=1}^{\infty} x_n^2. \tag{4}$$

We will prove this by direct algebra rather than matrix identities. Some preliminary remarks will clear the way. We write $X_n = \sum_{j=1}^{n} x_j$ and $y = Cx$, so that $y_n = X_n/n$ and $z_n = y_n - x_n$. Note that $z_1 = 0$. It is essential to recognize that (4) applies strictly to *infinite* sequences. In fact, if $x_j = 1$ for $1 \leq j \leq n$, then $z_j = 0$ for $1 \leq j \leq n$. Let us clarify what (4) actually says for $x$ of the form $(x_1, x_2, \ldots, x_n, 0, \ldots)$. For such $x$, we have $z_j = y_j = X_n/j$ for $j > n$, hence

$$\sum_{j=n+1}^{\infty} \frac{j}{j-1} z_j^2 = X_n^2 \sum_{j=n+1}^{\infty} \frac{1}{j(j-1)} = \frac{X_n^2}{n},$$

so (4) becomes

$$\sum_{j=2}^{n} \frac{j}{j-1}\, z_j^2 + \frac{X_n^2}{n} = \sum_{j=1}^{n} x_j^2 \tag{5}$$

We will prove that (5) holds for all $x$ in $\ell_2$ (not just $x$ with finitely many non-zero terms). To deduce (4), we then need the following lemma.

**Lemma 4.** *For $x \in \ell_2$, we have $X_n^2/n \to 0$ as $n \to \infty$.*

*Proof.* Choose $\varepsilon > 0$. There exists $n_0$ such that $\sum_{n=n_0+1}^{\infty} x_n^2 \le \varepsilon$. For $n > n_0$, we have $X_n = X_{n_0} + S_n$, where $S_n = \sum_{j=n_0+1}^{n} x_j$. By the Cauchy-Schwarz inequality,

$$S_n^2 \le (n - n_0) \sum_{j=n_0+1}^{n} x_j^2 < \varepsilon n.$$

For sufficiently large $n$, we have $X_{n_0}^2 \le \varepsilon n$. By the elementary inequality $(a+b)^2 \le 2(a^2 + b^2)$, we now have

$$X_n^2 \le 2X_{n_0}^2 + 2S_n^2 \le 4\varepsilon n. \qquad \blacksquare$$

*Proof of Theorem 3.* For a given $x$ in $\ell_2$, we prove (5) by induction. For $n = 2$, the left-hand side is

$$2z_2^2 + \tfrac{1}{2}X_2^2 = \tfrac{1}{2}(x_1 - x_2)^2 + \tfrac{1}{2}(x_1 + x_2)^2 = x_1^2 + x_2^2.$$

Assume now that (5) holds for $n-1$, where $n \ge 3$. To deduce that it holds for $n$, we require

$$\frac{n}{n-1} z_n^2 + \frac{X_n^2}{n} - \frac{X_{n-1}^2}{n-1} = x_n^2. \tag{6}$$

Since $z_n = \frac{1}{n}(X_n - nx_n)$ and $X_{n-1} = X_n - x_n$, the left-hand side of (6) equals

$$\frac{1}{n(n-1)}(X_n - nx_n)^2 + \frac{X_n^2}{n} - \frac{(X_n - x_n)^2}{n-1} = \left(\frac{n}{n-1} - \frac{1}{n-1}\right) x_n^2 = x_n^2$$

(the $X_n^2$ and $X_n x_n$ terms cancel to 0). $\blacksquare$

Since $n/(n-1) \le 2$ for all $n \ge 2$, we deduce at once:

**Corollary 5.** *For $x \in \ell_2$, we have $\|(C - I)x\| \ge (1/\sqrt{2})\|x\|$.*

Equality in Corollary 5 occurs in the case $(C-I)(e_1-e_2)=e_2$.

Can (4) be deduced from a matrix identity similar to (1)? The statement is $\langle \Delta_3 z, z\rangle = \langle x, x\rangle$, where $z=(C-I)x$ and $\Delta_3$ is the diagonal matrix with entries $(0, \frac{2}{1}, \frac{3}{2}, \ldots)$. This would follow from the matrix identity $(C^T-I)\Delta_3(C-I)=I$, equivalently $C^T\Delta_3 C - \Delta_3 C - C^T\Delta_3 = I - \Delta_3$. This is indeed true, but the proof is rather tricky and involves at least as much work as the proof we have given.

*Note on the continuous case*. The continuous analogue of $C$ is the operator on $L_2(0,\infty)$ defined by $(Af)(x) = \frac{1}{x}\int_0^x f(t)\,dt$, with dual $(A^T f)(x) = \int_x^\infty [f(t)/t]\,dt$. It is shown in [**6**] that $(A^T-I)(A-I)=I$, so that $A-I$ is actually *isometric*: $\|(A-I)f\| = \|f\|$ for all $f$. It is a general fact that operator norms often behave more smoothly in the continuous case than in the discrete case.

**4. COMPANION IDENTITIES FOR THE SHIFT OPERATOR.** The *shift operator* $S$ is defined by $Sx=(0,x_1,x_2,\ldots)$. Clearly, $\|Sx\|=\|x\|$. Also, $S^Tx = (x_2, x_3, \ldots)$, hence $S^TS=I$. There are simple identities, exploited in [**1**], relating $S$, $C$ and their transposes. Our results for $C-I$ have neat companion results with $I$ replaced by $S$ or $S^T$. The analogues of (1) and (2) are as follows.

**Theorem 6.** *We have*

$$(C-S^T)(C^T-S) = I + \Delta_1, \tag{7}$$

*hence* $\|C-S^T\| = \|C^T-S\| = \sqrt{2}$ *and for* $x\in\ell_2$,

$$\|(C^T-S)x\|^2 = \sum_{n=1}^\infty \frac{n+1}{n} x_n^2. \tag{8}$$

*Proof.* This time, instead of describing matrix products explicitly, we consider the action on a vector. We use the notation $(Cx)_n$ for component $n$ of $Cx$. First,

$$(CSx)_n = \frac{1}{n}(x_1+x_2+\cdots+x_{n-1}) = (Cx)_n - \frac{x_n}{n},$$

so $CS = C-\Delta_1$, hence also $S^TC^T = C^T-\Delta_1$. Recall that $CC^T = C + C^T - \Delta_1$. We deduce

$$(C-S^T)(C^T-S) = CC^T - (C-\Delta_1) - (C^T-\Delta_1) + I = I+\Delta_1.$$

Clearly, $\|I+\Delta_1\| = 2$, so the further statements follow. ■

**Corollary 7.** *We have* $\|x\| \le \|(C^T-S)x\| \le \sqrt{2}\|x\|$ *for all* $x$ *in* $\ell_2$.

We now prove the analogue of (4), by a suitable variation of the previous method.

**Theorem 8.** *Let* $x\in\ell_2$ *and* $(C-S^T)x=z$. *Then*

$$\sum_{n=1}^\infty \frac{n}{n+1}\, z_n^2 = \sum_{n=1}^\infty x_n^2. \tag{9}$$

*Proof.* Note that $z_n = X_n/n - x_{n+1}$. Again guided by the what the statement says for $(x_1, x_2, \ldots, x_n, 0, \ldots)$, we show by induction that

$$\sum_{j=1}^n \frac{j}{j+1} z_j^2 + \frac{X_{n+1}^2}{n+1} = \sum_{j=1}^{n+1} x_j^2. \tag{10}$$

Lemma 4 then gives (9). For $n = 1$, the left-hand side is

$$\tfrac{1}{2}(x_1 - x_2)^2 + \tfrac{1}{2}(x_1 + x_2)^2 = x_1^2 + x_2^2.$$

Assume (10) for $n - 1$. To deduce it for $n$, we require

$$\frac{n}{n+1} z_n^2 + \frac{X_{n+1}^2}{n+1} - \frac{X_n^2}{n} = x_{n+1}^2.$$

The left-hand side is

$$\frac{1}{n(n+1)}(X_n - nx_{n+1})^2 + \frac{(X_n + x_{n+1})^2}{n+1} - \frac{X_n^2}{n},$$

which indeed equates to $x_{n+1}^2$. ∎

**Corollary 9.** *We have* $\|x\| \le \|(C - S^T)x\| \le \sqrt{2}\|x\|$ *for all* $x$ *in* $\ell_2$.

**Remark.** By (7) and (1), we also have $C = (C - S^T)C^T$. With this identity, (9) gives an alternative proof of (3).

**5. EXTENSION TO COMPLEX NUMBERS AND VECTORS.** We have presented our results exclusively for real numbers $x_k$. However, a very simple trick from [**5**] extends them not only to complex numbers, but also to vectors $x_k$.

Our results are of the folllowing form: $A$ and $B$ are matrices such that for all $x$ in $\ell_2$, we have $\sum_{j=1}^{\infty} \delta_j y_j^2 = \sum_{j=1}^{\infty} z_j^2$ for a certain sequence $(\delta_j)$, where $y = Ax$ and $z = Bx$. Now let $(x_k)_{k\ge1}$ be a sequence of vectors (i.e. elements of $\ell_2$) instead of scalars. We now write $x_k(r)$ for term $r$ of the sequence $x_k$. Let $y_j = \sum_{k=1}^{\infty} a_{j,k} x_k$ and $z_j = \sum_{k=1}^{\infty} b_{j,k} x_k$. Then for each $r$, we have $y_j(r) = \sum_{k=1}^{\infty} a_{j,k} x_k(r)$, similarly for $z_j(r)$. Summing over $r$, we deduce

$$\sum_{j=1}^{\infty} \delta_j \|y_j\|^2 = \sum_{j=1}^{\infty} \|z_j\|^2.$$

So all the results can be extended in this way. The extension applies, in particular, to complex numbers (with $y_j$ replaced by $|y_j|$), since they equate to 2-dimensional vectors.

REFERENCES


1. Bennett, G. (1996). Factorizing the Classical Inequalities. *Mem. Amer. Math. Soc.* 576, Providence: American Math. Soc.
2. Brown, A., Halmos, P.R., Shields, A.L. (1965). Cesàro operators. *Acta Sci. Math.* 26(1–2): 125–137.
3. Choi, M.D. (1983). Tricks or treats with the Hilbert matrix. *Amer. Math. Monthly* 90(5): 301–312.
4. Hardy, G.H., Littlewood, J., Pólya, G. (1967). *Inequalities*, 2nd ed. Cambridge: Cambridge Univ. Press.
5. Jameson, G.J.O. (2019). Operator-valued extensions of matrix norm inequalities. *Amer. Math. Monthly* 126(9): 809–815.
6. Kaiblinger, N., Maligranda, L., Persson, L.-E. (2000). Norms in weighted $L^2$-spaces and Hardy operators. *Function spaces, the fifth conference, Poznań 1998, Lecture Notes in Pure Appl. Math.* (213): 205–216. New York: Marcel Dekker.



*Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, United Kingdom*
*g.jameson@lancaster.ac.uk*