A GEOMETRIC APPROACH TO QUILLEN'S CONJECTURE

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Abstract. We introduce *admissible collections* for a finite group G and use them to prove that most of the finite classical groups in non-defining characteristic satisfy the *Quillen dimension at p property*, a strong version of Quillen's conjecture, at a given odd prime divisor p of $|G|$. Compared to the methods in [2], our techniques are simpler.

1. Introduction

Let G be a finite group and let p be a prime number dividing the order of G. Let $\mathcal{A}_p(G)$ be the poset of all non-trivial elementary abelian p-subgroups of G ordered by inclusion and let $|\mathcal{A}_p(G)|$ be its realization as a topological space. This space is a simplicial complex which has as n-simplices the chains of length n in $\mathcal{A}_p(G)$:

$$P_0 < P_1 < \ldots < P_n, \text{ with } P_i \in \mathcal{A}_p(G).$$

We denote by $O_p(G)$ be the largest normal p-subgroup of G, and by $\mathrm{rk}_p(G)$ the p-rank of G, that is, the maximum of the ranks of the elementary abelian p-subgroups of G. A finite elementary abelian p-group E has rank r if $|E| = p^r$, or equivalently, $r = \dim_{\mathbb{F}_p} E$, when we regard E as an $\mathbb{F}_p$-vector space.

Conjecture 1.1 (Quillen's conjecture [11])**.** *If $|\mathcal{A}_p(G)|$ is contractible then $O_p(G) \neq 1$.*

Quillen stated the conjecture in his seminal paper [11], in which he also proved the conjecture for finite groups of p-rank at most 2 [11, Proposition 2.10], for finite groups of Lie type in defining characteristic [11, Theorem 3.1] and for solvable finite groups [11, Theorem 12.1]. Recently, Piterman et al. proved Quillen's conjecture for finite groups of p-rank at most 3 [10]. Quillen's results were extended to p-solvable finite groups by several people using different methods, notably by Alperin, using results on coprime actions and the Classification of the Finite Simple Groups [13, Theorem 8.2.12], and by Hawkes and Isaacs in the particular case of p-solvable finite groups with abelian Sylow p-subgroups, via a combinatorial method and without resorting to the Classification [7, Theorem A]. In [2], Aschbacher and Smith tackled Quillen's conjecture via reduction theorems based on the assumption of the existence of a counter-example to the conjecture of minimal order. Thus, they showed that Quillen's conjecture holds for p-solvable finite groups in another way, and their overarching results are the more general ones to date: if $p > 5$ and G does not have a unitary component $U_n(q)$ with $q \equiv -1 \pmod{p}$ and q odd, then G satisfies Quillen's conjecture [2, Main Theorem, p. 474]. In their work,

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the authors introduced the following property involving reduced homology with rational coefficients. Here, we employ integral coefficients.

Definition 1.2 ($\mathcal{QD}_p$)**.** The finite group G with $r = \mathrm{rk}_p(G)$ has the *Quillen dimension at p property*, written $\mathcal{QD}_p$, if

$$\widetilde{H}_{r-1}(|\mathcal{A}_p(G)|) \neq 0.$$

By $\widetilde{H}_*(|\mathcal{A}_p(G)|)$ we mean reduced integral simplicial homology (cf. [12, Ch. 7]), i.e., $\widetilde{H}_*(|\mathcal{A}_p(G)|) = H_*(|\mathcal{A}_p(G)|;\mathbb{Z})$ for $* > 0$ and $H_0(|\mathcal{A}_p(G)|;\mathbb{Z}) = \widetilde{H}_0(|\mathcal{A}_p(G)|) \oplus \mathbb{Z}$. Note that, as $|\mathcal{A}_p(G)|$ is an $(r-1)$-dimensional complex, $r-1$ is the top dimension for which reduced homology can possibly be non-zero, and that, in this dimension, the reduced homology group is necessarily a free abelian group. Hence, the rational and integral versions of this property are equivalent. In addition, by [9, Theorem 2], Quillen's conjecture is equivalent to the acyclicity condition that $\widetilde{H}_*(|\mathcal{A}_p(G)|) \neq 0$.

In [2, Theorem 3.1], they consider p-extensions of finite simple groups, that is, almost-simple groups with an elementary abelian p-group inducing outer automorphisms. Aschbacher and Smith prove that most of these p-extensions satisfy $\mathcal{QD}_p$, and they list those which do not. In [3], the first author proves that Quillen's conjecture holds for solvable and p-solvable finite groups via new geometrical methods. In the present paper, we elaborate on these methods with the objective to find shorter and easier proofs of the results in [2]. We deal with the alternating, symmetric and classical finite groups in non-defining characteristic and for an odd prime p. By a finite *classical group*, we mean a finite linear, unitary, symplectic or orthogonal group. With these methods, we prove the following result.

Theorem 1.3. *Let p be an odd prime, and let G be one of the following groups:*

(i) G is an alternating or symmetric group of degree n for $p \geq 5$, and for $p = 3$, $n = 4, 5, 8$.
(ii) G is a linear, unitary, symplectic or orthogonal group defined over a field of characteristic different from p, unless $p|(q+1)$ and G is a unitary group defined over $\mathbb{F}_{q^2}$ or G is a symplectic or orthogonal group defined over $\mathbb{F}_2$ or $G = \mathrm{PSL}_3(q)$ with $q \equiv 1 \pmod 3$.

Then G has the Quillen dimension at p property, and therefore Quillen's conjecture holds for G.

The structure of the paper is as follows. In Section 2, we define *faithful collections* and discuss some of their properties. Such a collection for a group G consists of an arrangement of elementary abelian p-subgroups of G together with certain elements of G that centralize/normalize these p-subgroups. In Section 3, we find further conditions on faithful collections which imply that a finite group has the Quillen dimension at p property $\mathcal{QD}_p$, and we call such faithful collections *admissible*. In Section 4, we briefly study when the Quillen dimension at p property of a given finite group is inherited by its quotient groups. Then we show the Quillen dimension at p property for the symmetric and alternating groups, and for the finite classical groups in non-defining characteristic, excluding certain cases when $p = 3$. Thus Theorem 1.3 summarizes these results. We also present some limitations of our method and open questions.

2. Faithful collections

In this section, we define faithful collections for a finite group G. Given $E \in \mathcal{A}_p(G)$, we regard E as an $\mathbb{F}_p$-vector space (generally written additively) and a faithful collection for G, if it exists, is a certain arrangement of hyperplanes of E and elements of G subject to certain constraints.

Definition 2.1. Let r be a positive integer. For any integer l with $0 \leq l \leq r$, we define an *l-tuple for r* to be an ordered sequence of integers $\mathbf{i} = [i_1, \ldots, i_l]$ with $1 \leq i_j \leq r$ and no repetition. By S^r_l we denote the set of all l-tuples for r for a given l, and by $S^r = \cup_{l=0,\ldots,r} S^r_l$ the set of all l-tuples for r for all $0 \leq l \leq r$.

For $l = r$ the unique r-tuple is $[1, 2, \ldots, r]$, and for $l = 0$ the unique 0-tuple corresponds to the empty sequence and we denote it by $[\,]$. Thus, $S^r_r = \{[1, 2, \ldots, r]\}$ and $S^r_0 = \{[\,]\}$.

Definition 2.2. Let $E = \langle e_1, \ldots, e_r\rangle$ be an elementary abelian p-subgroup of G of rank $r \leq \mathrm{rk}_p(G)$. For each l-tuple $\mathbf{i} = [i_1, \ldots, i_l] \in S^r_l$, set:

$$E_{\mathbf{i}} = E_{[i_1,\ldots,i_l]} = \langle e_1, \ldots, \widehat{e_{i_1}}, \ldots, \widehat{e_{i_l}}, \ldots, e_r\rangle,$$

for the subgroup of E generated by all the e_i, except $e_{i_1}, \ldots, e_{i_l}$.

We abuse notation and write $E_i = E_{[i]} = \langle e_1, \ldots, \widehat{e_i}, \ldots, e_r\rangle$. In Definition 2.2, the subgroups E_i are subspaces of E of codimension 1. Note that we have the following:

$$E_{[i_1,\ldots,i_l]} = E_{i_1} \cap \ldots \cap E_{i_l} \quad , \quad E_{[\,]} = E = \langle E_i, e_i\rangle \quad , \quad E_{[1,2,\ldots,r]} = \{0\},$$

and, in particular:

$$\langle e_i\rangle = E_{[1,\ldots,\hat{i},\ldots,r]} = E_1 \cap \ldots \cap \widehat{E_i} \cap \ldots \cap E_r. \tag{1}$$

Remark 2.3. Note that

$$E_{[i_1,\ldots,i_l]} = E_{[i'_1,\ldots,i'_{l'}]} \Leftrightarrow l = l' \quad \text{and} \quad \{i_1, \ldots, i_l\} = \{i'_1, \ldots, i'_{l'}\}.$$

Definition 2.4 (Faithful collection)**.** Let $E = \langle e_1, \ldots, e_r\rangle$ be an elementary abelian p-subgroup of G of rank $r \leq \mathrm{rk}_p(G)$, and let $\mathbf{c} = (c_1, \ldots, c_r)$ be an ordered r-tuple of elements of G. Suppose that for any $\mathbf{i} \in S^r$, and for any ordered r-tuple $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r)$ of integers $\epsilon_i \in \{-1, 0, 1\}$, we have

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} \leq E \Rightarrow {}^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} = E_{\mathbf{i}},$$

where

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} = {}^{c_1^{\epsilon_1}\ldots c_r^{\epsilon_r}}E_{\mathbf{i}}.$$

Then we say that

$$\{E_i, c_j\} \quad \text{is a } \textit{faithful collection.}$$

Note that the subgroups $E_i = E_{[i]} = \langle e_1, \ldots, \widehat{e_i}, \ldots, e_r\rangle$ of E in a given faithful collection determine the subgroups $\langle e_i\rangle$ by Equation (1), and hence also the subgroups $E_{\mathbf{i}}$ for any $\mathbf{i} \in S^r$. So, $\{E_i, c_j\}$ is a faithful collection if the subspaces $E_{\mathbf{i}}$ of E in Definition 2.2 can be conjugated by the elements $\mathbf{c}^{\boldsymbol{\epsilon}}$ of G to a subspace of E if and only if $\mathbf{c}^{\boldsymbol{\epsilon}}$ normalizes $E_{\mathbf{i}}$. This mimics a property of coprime actions, see [3, Proposition 2.2(1)]. Note that if $\mathbf{i} \in S^r_0$ or S^r_r, the implication

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} \leq E \Rightarrow {}^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} = E_{\mathbf{i}}.$$

is vacuous.

The next result is a characterization of faithful collections in terms of the elements e_i generating E in Definition 2.2. By Equation (1), these elements are determined by the hyperplanes E_i up to a power.

Lemma 2.5. *Let $E = \langle e_1, \ldots, e_r\rangle$ be an elementary abelian p-subgroup of G of rank r. Let $\mathbf{c} = (c_1, \ldots, c_r) \in G^r$. Then $\{E_i, c_j\}$ is a faithful collection if and only if for any $\boldsymbol{\epsilon} \in \{-1, 0, 1\}^r$ and any $1 \le i \le r$ we have*

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}\langle e_i\rangle \le E \Rightarrow {}^{\mathbf{c}^{\boldsymbol{\epsilon}}}\langle e_i\rangle = \langle e_i\rangle.$$

Proof. That a faithful collection satisfies the given conditions is clear by considering the $(r-1)$-tuples $[1, \ldots, \hat{i}, \ldots, r]$. Conversely, let $\mathbf{i} \in S^r_{r-l}$ for some $0 \le l \le r$, and let $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in \{-1, 0, 1\}^r$ such that

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} \le E,$$

where $E_{\mathbf{i}} = \langle e_{j_1}, \ldots, e_{j_l}\rangle$ has rank l. In particular, we have

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}\langle e_{j_k}\rangle \le E$$

for $1 \le k \le l$, and so, by hypothesis, $^{\mathbf{c}^{\boldsymbol{\epsilon}}}\langle e_{j_k}\rangle = \langle e_{j_k}\rangle$ for $1 \le k \le l$. It follows that $^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}} = {}^{\mathbf{c}^{\boldsymbol{\epsilon}}}\langle e_{j_1}, \ldots, e_{j_l}\rangle = \langle {}^{\mathbf{c}^{\boldsymbol{\epsilon}}}e_{j_1}, \ldots, {}^{\mathbf{c}^{\boldsymbol{\epsilon}}}e_{j_l}\rangle = \langle e_{j_1}, \ldots, e_{j_l}\rangle = E_{\mathbf{i}}$. □

The next result gives sufficient conditions for the existence of a faithful collection.

Lemma 2.6. *Let $E = \langle e_1, \ldots, e_r\rangle$ be an elementary abelian p-subgroup of G of rank r and let $\mathbf{c} = (c_1, \ldots, c_r) \in G^r$. Suppose $c_i \in C_G(E_i) \setminus C_G(e_i)$ and $[c_i, c_j] = 1$ for all i, j. Then $\{E_i, c_j\}$ is a faithful collection.*

Proof. We use Lemma 2.5. Consider $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r)$ with $\epsilon_i \in \{-1, 0, 1\}$ and let $1 \le i_0 \le r$ such that

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}\langle e_{i_0}\rangle \le E.$$

Then we have

$$^{\mathbf{c}^{\boldsymbol{\epsilon}}}e_{i_0} = {}^{c_1^{\epsilon_1}\cdots c_r^{\epsilon_r}}e_{i_0} = {}^{c_{i_0}^{\epsilon_{i_0}}}e_{i_0} = \lambda_1 e_1 + \ldots + \lambda_{i_0} e_{i_0} + \ldots + \lambda_r e_r,$$

where $\lambda_i \in \mathbb{F}_p$ (not all zero), and in the second equality we have used that $c_i \in C_G(E_i)$, that $[c_i, c_{i_0}] = 1$ and that $e_{i_0} \in E_i$ for $i \neq i_0$. Assume $\lambda_{i_1} \neq 0$ for some $i_1 \neq i_0$. Then,

$$E = \langle E_{i_1}, {}^{c_{i_0}^{\epsilon_{i_0}}}e_{i_0}\rangle.$$

Now, by hypothesis c_{i_1} centralizes $e_{i_0} \in E_{i_1}$, and we calculate

$$^{c_{i_1}c_{i_0}^{\epsilon_{i_0}}}e_{i_0} = {}^{c_{i_0}^{\epsilon_{i_0}}c_{i_1}}e_{i_0} = {}^{c_{i_0}^{\epsilon_{i_0}}}e_{i_0},$$

using that $[c_{i_1}, c_{i_0}] = 1$, that $c_{i_1} \in C_G(E_{i_1})$ and that $e_{i_0} \in E_{i_1}$ (as $i_0 \neq i_1$). This is a contradiction with $c_{i_1} \notin C_G(E)$, and hence $\lambda_i = 0$ for all $i \neq i_0$ and $^{\mathbf{c}^{\boldsymbol{\epsilon}}}e_{i_0} = \lambda_{i_0} e_{i_0} \in \langle e_{i_0}\rangle$. □

In the next section, we will use the existence of faithful collections with E of rank $r = \mathrm{rk}_p(G)$ subject to certain constraints to prove that $\mathcal{QD}_p$ holds for G, see Theorem 3.5.

By constrast, we prove that, under certain assumptions on G, there cannot be a faithful collection subject to the conditions in Lemma 2.6. Recall that a *maximal elementary abelian p-subgroup of G* is a maximal element in the poset $\mathcal{A}_p(G)$, i.e., an elementary abelian p-subgroup E of G which is not properly contained in any other elementary abelian p-subgroup of G. But E need not have maximal rank $\mathrm{rk}_p(G)$.

Lemma 2.7. *Suppose that $O_p(G) \cap Z(G) > 1$ for some odd prime p. Let E be a maximal elementary abelian p-subgroup of G, not necessarily of maximal order. Then there exists no collection $\{E_i, c_j\}$ for E subject to $c_i \in C_G(E_i) \setminus C_G(e_i)$ for all i.*

Proof. Let $E = \langle e_1, \dots, e_r\rangle \in \mathcal{A}_p(P)$ be a maximal element, and suppose that $\{E_i, c_j\}$ is a collection subject to $c_i \in C_G(E_i)\setminus C_G(e_i)$ for all i. Let $V = \Omega_1\big(O_p(G)\cap Z(G)\big)$. Note that $E = VE \geq V$ by maximality of E, and the fact that V lies in the center of every Sylow p-subgroup of G.

Let $1 \neq v = \sum_{i=1}^{r} \lambda_i e_i \in V$, and without loss, suppose $\lambda_1 = 1$. (Here we use the additive notation of E seen as vector space.) So $v = e_1 + v'$ with $v' \in E_1$. Because $v \in V \leq Z(G)$ and $c_1 \in C_G(E_1)$, we have $v = {}^{c_1}v = {}^{c_1}(e_1 + v') = ({}^{c_1}e_1) + v'$. Therefore $e_1 = v - v' = {}^{c_1}e_1$, saying that $c_1 \in C_G(e_1)$, a contradiction. □

3. Generalization

We want to find sufficient conditions which imply that G has the Quillen dimension at p property, or more generally, which imply that Quillen's conjecture holds for G. Theorem 3.5 below is a generalization of [3, Theorem 5.3].

Definition 3.1. For an l-tuple $\mathbf{i} = [i_1, \dots, i_l] \in S^r_l$, we define the *signature*

$$\mathrm{sgn}(\mathbf{i}) = (-1)^{n+m} \quad \text{of } \mathbf{i}, \text{ where}$$

- n is the number of transpositions we need to apply to the l-tuple $\mathbf{i}$ to rearrange it in increasing order $[j_1, \dots, j_l]$, and
- m is the number of positions in which $[j_1, \dots, j_l]$ differ from $[1, \dots, l]$.

Note that in Definition 3.1, the number n of transpositions is not uniquely defined, but its parity is.

For instance, if $\mathbf{i} = [1, 4, 2]$, then $n = 1$, since we need to apply $(2, 4)$ to reorder $\mathbf{i}$ as $[1, 2, 4]$, and $m = 1$, since $[1, 2, 4]$ differs from $[1, 2, 3]$ only in one place. Thus $\mathrm{sgn}(\mathbf{i}) = 1$.

Let $E = \langle e_1, \dots, e_r\rangle$ be an elementary abelian p-subgroup of rank r of the group G. We now generalize the chains introduced in [3, Section 3] to the case when E need not have maximal rank $\mathrm{rk}_p(G)$ and G need not be a semi-direct product. We consider the poset $\mathcal{A}_p(E)$ and its order complex $\Delta(\mathcal{A}_p(E))$. We define an element of the integral simplicial chains of dimension $r-1$, $C_{r-1}(\Delta(\mathcal{A}_p(E)))$. For $\mathbf{i} = [i_1, \dots, i_{r-1}] \in S^r_{r-1}$, we define the $(r-1)$-simplex

$$\sigma_{\mathbf{i}} = \big(E_{[i_1,\dots,i_{r-1}]} < E_{[i_1,\dots,i_{r-2}]} < \cdots < E_{i_1} < E\big) \in \Delta(\mathcal{A}_p(E)),$$

and for $a \in \mathbb{Z}$, the chain

$$Z_{E,a} = a \sum_{\mathbf{i} \in S^r_{r-1}} \mathrm{sgn}(\mathbf{i}) \sigma_{\mathbf{i}} \quad \text{in} \quad C_{r-1}(\Delta(\mathcal{A}_p(E))).$$

By definition of the differential,

$$d(Z_{E,a}) = a \sum_{0 \leq j < r} \sum_{\mathbf{i} \in S^r_{r-1}} (-1)^j \,\mathrm{sgn}(\mathbf{i}) d_j(\sigma_{\mathbf{i}}) \quad \text{where}$$

$$d_j(\sigma_{\mathbf{i}}) = \big(E_{[i_1,\dots,i_{r-1}]} < \cdots < E_{[i_1,\dots,i_{r-j}]} < E_{[i_1,\dots,i_{r-j-2}]} < \cdots < E_{i_1} < E\big)$$

removes the $(j+1)$-st term from the left in the chain for $0 \leq j < r$. So, in particular,

$$d_0(\sigma_{\mathbf{i}}) = \big(E_{[i_1,\ldots,i_{r-2}]} < \cdots < E\big) \quad \text{and} \quad d_{r-1}(\sigma_{\mathbf{i}}) = \big(E_{[i_1,\ldots,i_{r-1}]} < \cdots < E_{i_1}\big).$$

Hence, define the $(r-2)$-simplex

$$\tau_{\mathbf{i}} = d_{r-1}(\sigma_{\mathbf{i}}) = \big(E_{[i_1,\ldots,i_{r-1}]} < \cdots < E_{i_1}\big).$$

Let us recall the following useful property ([3, Proposition 3.2]).

Proposition 3.2. *With the above notation,*

$$d(Z_{E,a}) = (-1)^{r-1} a \sum_{\mathbf{i}\in S^r_{r-1}} \operatorname{sgn}(\mathbf{i})\tau_{\mathbf{i}}.$$

Proof. Suppose that $d_k(\sigma_{\mathbf{i}}) = d_l(\sigma_{\mathbf{j}})$ for some $\mathbf{i},\mathbf{j} \in S^r_{r-1}$ and $0 \leq k,l \leq r-1$, that is,

$$\big(E_{[i_1,\ldots,i_{r-1}]} < \cdots < E_{[i_1,\ldots,i_{r-k}]} < E_{[i_1,\ldots,i_{r-k-2}]} < \cdots < E_{i_1} < E\big) = \\ = \big(E_{[j_1,\ldots,j_{r-1}]} < \cdots < E_{[j_1,\ldots,i_{r-l}]} < E_{[j_1,\ldots,j_{r-l-2}]} < \cdots < E_{j_1} < E\big).$$

For both chains to be equal, the jump by an index p^2 must occur in the same place, saying that $k = l$. If $0 < k = l < r-1$, then, by Remark 2.3, the tuples $\mathbf{i}$ and $\mathbf{j}$ are identical but for $\{i_{r-k-1}, i_{r-k}\} = \{j_{r-k-1}, j_{r-k}\}$. So either $\mathbf{i} = \mathbf{j}$, or $\mathbf{i}$ and $\mathbf{j}$ differ by one transposition and hence $\operatorname{sgn}(\mathbf{i}) = -\operatorname{sgn}(\mathbf{j})$. In the latter case, the corresponding summands $\operatorname{sgn}(\mathbf{i})d_k(\sigma_{\mathbf{i}})$ and $\operatorname{sgn}(\mathbf{j})d_l(\sigma_{\mathbf{j}})$ add up to zero. Assume now that $k = l = 0$. Then, again by Remark 2.3, $[j_1,\ldots,j_{r-2}] = [i_1,\ldots,i_{r-2}]$ and either $\mathbf{i} = \mathbf{j}$ or $j_{r-1} \neq i_{r-1}$. In the latter case, by [3, Lemma 2.4], $\operatorname{sgn}(\mathbf{i}) = -\operatorname{sgn}(\mathbf{j})$ and again the two terms cancel each other out. Finally, if $k = l = r-1$, the tuples $\mathbf{i}$ and $\mathbf{j}$ are identical and the terms contribute to the sum in the statement of the proposition. □

Consider the order complex $\Delta(\mathcal{A}_p(G))$. Then the group G acts by conjugation on $C_*(\Delta(\mathcal{A}_p(G)))$. For $x \in G$, the element

$${}^xZ_{E,a} = a \sum_{\mathbf{i}\in S^r_{r-1}} \operatorname{sgn}(\mathbf{i})\,{}^x\sigma_{\mathbf{i}} = a \sum_{\mathbf{i}\in S^r_{r-1}} \operatorname{sgn}(\mathbf{i})\big({}^xE_{[i_1,\ldots,i_{r-1}]} < \cdots < {}^xE_{i_1} < {}^xE\big)$$

belongs to $\Delta(\mathcal{A}_p({}^xE)) \subseteq \Delta(\mathcal{A}_p(G))$.

Let J be a non-empty finite indexing set and consider subsets $\mathbf{x} = \{x_j\}_{j\in J} \subseteq G$ and $\mathbf{a} = \{a_j\}_{j\in J} \subseteq \mathbb{Z}$. Define the chain in $C_{r-1}(\Delta(\mathcal{A}_p(G)))$:

$$Z_{G,\mathbf{x},\mathbf{a}} = \sum_{j\in J} {}^{x_j}Z_{E,a_j} = \sum_{j\in J} a_j \sum_{\mathbf{i}\in S^r_{r-1}} \operatorname{sgn}(\mathbf{i})\,{}^{x_j}\sigma_{\mathbf{i}}. \tag{2}$$

Then, by Proposition 3.2,

$$d(Z_{G,\mathbf{x},\mathbf{a}}) = (-1)^{r-1} \sum_{j\in J} a_j \sum_{\mathbf{i}\in S^r_{r-1}} \operatorname{sgn}(\mathbf{i})\,{}^{x_j}\tau_{\mathbf{i}}.$$

So, given $j \in J$ and $\mathbf{i} \in S^r_{r-1}$, the coefficients of ${}^{x_j}\sigma_{\mathbf{i}}$ and ${}^{x_j}\tau_{\mathbf{i}}$ are respectively:

$$C_{j,\mathbf{i}} := \sum_{(l,\mathbf{k})\in\mathcal{C}(j,\mathbf{i})} a_l \operatorname{sgn}(\mathbf{k}) \quad \text{and} \tag{3}$$

$$D_{j,\mathbf{i}} := (-1)^{r-1} \sum_{(l,\mathbf{k})\in\mathcal{D}(j,\mathbf{i})} a_l \operatorname{sgn}(\mathbf{k}), \tag{4}$$

where

$$\mathcal{C}(j,\mathbf{i}) = \{(l,\mathbf{k}) \in J \times S^r_{r-1} \mid {}^{x_l}E_{[k_1,\dots,k_t]} = {}^{x_j}E_{[i_1,\dots,i_t]}\ ,\ \forall\, 0 \le t < r\} \quad \text{and}$$
$$\mathcal{D}(j,\mathbf{i}) = \{(l,\mathbf{k}) \in J \times S^r_{r-1} \mid {}^{x_l}E_{[k_1,\dots,k_t]} = {}^{x_j}E_{[i_1,\dots,i_t]}\ ,\ \forall\, 1 \le t < r\}.$$

Note that $\mathcal{C}(j,\mathbf{i}) \subseteq \mathcal{D}(j,\mathbf{i})$ (and recall that $E_{[\,]} = E$).

If we further assume that E is a *maximal* elementary abelian p-subgroup of G, we want to find sufficient conditions for the existence of a non-zero cycle in $\widetilde{H}_{r-1}(|\mathcal{A}_p(G)|)$.

Theorem 3.3. *Let $E = \langle e_1,\dots,e_r\rangle$ be a* maximal *elementary abelian p-subgroup of rank r of the group G. If the subsets $\mathbf{x} = \{x_j\}_{j\in J} \subseteq G$ and $\mathbf{a} = \{a_j\}_{j\in J} \subseteq \mathbb{Z}$ satisfy that $C_{j,\mathbf{i}} \neq 0$ for some $j \in J$ and some $\mathbf{i} \in S^r_{r-1}$ and that $D_{j,\mathbf{i}} = 0$ for all $j \in J$ and all $\mathbf{i} \in S^r_{r-1}$, then*

$$0 \neq [Z_{G,\mathbf{x},\mathbf{a}}] \in \widetilde{H}_{r-1}(|\mathcal{A}_p(G)|).$$

In particular, $|\mathcal{A}_p(G)|$ is not contractible and Quillen's conjecture holds for G. If furthermore $r = \mathrm{rk}_p(G)$ then $\mathcal{QD}_p$ holds for G.

Proof. Consider the chain $Z_{G,\mathbf{x},\mathbf{a}} \in C_{r-1}(\Delta(\mathcal{A}_p(G)))$ defined in Equation (2). The condition in the statement for the coefficients $C_{j,\mathbf{i}}$ is clearly equivalent to $Z_{G,\mathbf{x},\mathbf{a}} \neq 0$. The condition on the coefficients $D_{j,\mathbf{i}}$ is equivalent to $d(Z_{G,\mathbf{x},\mathbf{a}}) = 0$ (cf. also [3, Proposition 4.2]). By the maximality of E , this cycle cannot be a boundary and hence it gives rise to a non-zero homology class in $\widetilde{H}_{r-1}(|\mathcal{A}_p(G)|)$. □

The assumptions of Theorem 3.3 are fulfilled when we can find a faithful collection subject to certain constraints, as described in the next theorem.

Theorem 3.4. *Let $E = \langle e_1,\dots,e_r\rangle$ be a maximal elementary abelian p-subgroup of G of rank r and let $\{E_i, c_j\}$ be a faithful collection such that $[c_i, c_j] = 1$ for all i, j. Set $J = \{\boldsymbol{\delta} = (\delta_1,\dots,\delta_r)\ ,\ \delta_i \in \{0,1\}\}$, let $\mathbf{a} = \{a_{\boldsymbol{\delta}}\}_{\boldsymbol{\delta}\in J} \subseteq \mathbb{Z}$ and consider the following subset of G:*

$$\mathbf{x} = \{\mathbf{c}^{\boldsymbol{\delta}} = c_1^{\delta_1}\cdots c_r^{\delta_r} | \boldsymbol{\delta} \in J\}.$$

Then, for each such $\boldsymbol{\delta} \in J$ and each $\mathbf{i} = [i_1,\dots,i_{r-1}] \in S^r_{r-1}$:

1. *$C_{\boldsymbol{\delta},\mathbf{i}} = \mathrm{sgn}(\mathbf{i})\big(\sum_{\boldsymbol{\delta}'} a_{\boldsymbol{\delta}'}\big)$, where the sum is over all $\boldsymbol{\delta}' \in J$ such that $\mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'} \in N_G(E)$.*
2. *$D_{\boldsymbol{\delta},\mathbf{i}} = (-1)^{r-1}\,\mathrm{sgn}(\mathbf{i})\big(\sum_{\boldsymbol{\delta}'} a_{\boldsymbol{\delta}'}\big)$, where the sum is over all $\boldsymbol{\delta}' \in J$ such that $\mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'} \in N_G(E_{i_1})$.*
3. *If $c_i \in C_G(E_i) \setminus N_G(\langle e_i\rangle)$ for all $1 \le i \le r$, then $C_{\boldsymbol{\delta},\mathbf{i}} = \mathrm{sgn}(\mathbf{i})a_{\boldsymbol{\delta}}$.*
4. *If $c_i \in N_G(E_i)$ for all $1 \le i \le r$, then*

$$D_{\boldsymbol{\delta},\mathbf{i}} = (-1)^{r-1}\,\mathrm{sgn}(\mathbf{i})\big(\sum (a_{\boldsymbol{\delta}'} + a_{\boldsymbol{\delta}''})\big),$$

where the sum runs through the pairs $\boldsymbol{\delta}', \boldsymbol{\delta}'' \in J$ such that $\mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'}, \mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}''} \in N_G(E_{i_1})$, $\delta''_j = \delta'_j$ for all $j \neq i_1$ and $\delta'_{i_1} + \delta''_{i_1} = 1$.

Proof. Using the faithfulness condition in Definition 2.4 and Remark 2.3, a straightforward computation shows that $(\boldsymbol{\delta}', \mathbf{k}) \in \mathcal{C}(\boldsymbol{\delta}, \mathbf{i})$ (cf. definition of $\mathcal{C}(\boldsymbol{\delta}, \mathbf{i})$ in Equation (3) above) if and only if $\mathbf{k} = \mathbf{i}$ and $\mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'} \in N_G(E)$, and similarly, $(\boldsymbol{\delta}', \mathbf{k}) \in$

$\mathcal{D}(\boldsymbol{\delta},\mathbf{i})$ (cf. definition of $\mathcal{D}(\boldsymbol{\delta},\mathbf{i})$ in Equation (4) above) if and only if $\mathbf{k}=\mathbf{i}$ and $\mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'}\in N_G(E_{i_1})$. In other words,

$$\mathcal{C}(\boldsymbol{\delta},\mathbf{i})=\{(\boldsymbol{\delta}',\mathbf{i})\in J\times S^r_{r-1}\mid \mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'}\in N_G(E)\}\quad\text{and}$$
$$\mathcal{D}(\boldsymbol{\delta},\mathbf{i})=\{(\boldsymbol{\delta}',\mathbf{i})\in J\times S^r_{r-1}\mid \mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'}\in N_G(E_{i_1})\}.$$

From here, we immediately get the assertions (1) and (2).

Suppose the hypotheses in point (3) hold. Again, since ${}^{\mathbf{c}^{\boldsymbol{\delta}}}E={}^{\mathbf{c}^{\boldsymbol{\delta}'}}E\iff {}^{\mathbf{c}^{\boldsymbol{\delta}-\boldsymbol{\delta}'}}E=E$, where each $\boldsymbol{\delta}_j-\boldsymbol{\delta}'_j\in\{-1,0,+1\}$, the faithfulness condition implies that

$${}^{c^{\boldsymbol{\delta}-\boldsymbol{\delta}'}}\langle e_i\rangle={}^{c_i^{\boldsymbol{\delta}_i-\boldsymbol{\delta}'_i}}\langle e_i\rangle=\langle e_i\rangle,$$

where we have used that $[c_i,c_j]=1$, that $c_j\in C_G(E_j)$ and that $e_i\in E_j$ for $i\neq j$. By assumption $c_i\notin N_G(\langle e_i\rangle)$, which forces $\delta_i=\delta'_i$, and this holds for all $1\leq i\leq r$.

Finally, for point (4), let $\boldsymbol{\delta}'$ be such that ${}^{\mathbf{c}^{\boldsymbol{\delta}}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}'}}E_{i_1}$. If $\delta'_{i_1}=0$ then $\boldsymbol{\delta}'=(\delta'_1,\ldots,0,\ldots,\delta'_r)$ and we conjugate by c_{i_1} to obtain

$${}^{c_{i_1}\mathbf{c}^{\boldsymbol{\delta}'}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}'}c_{i_1}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}'}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}}}E_{i_1},$$

where we have used that $[c_i,c_j]=1$ and that $c_{i_1}\in N_G(E_{i_1})$. So we deduce that $\boldsymbol{\delta}''=(\delta'_1,\ldots,1,\ldots,\delta'_r)$ also appears in the sum in (2) of this theorem. If $\delta'_{i_1}=1$ then $\boldsymbol{\delta}'=(\delta'_1,\ldots,1,\ldots,\delta'_r)$ and we conjugate by $c_{i_1}^{-1}$ to obtain

$${}^{c_{i_1}^{-1}\mathbf{c}^{\boldsymbol{\delta}'}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}'}c_{i_1}^{-1}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}'}}E_{i_1}={}^{\mathbf{c}^{\boldsymbol{\delta}}}E_{i_1},$$

where we have used that $[c_i,c_j]=1$ and that $c_{i_1}\in N_G(E_{i_1})$. So we deduce that $\boldsymbol{\delta}''=(\delta'_1,\ldots,0,\ldots,\delta'_r)$ also appears in the sum. □

Theorem 3.5. *Let $E=\langle e_1,\ldots,e_r\rangle$ be a maximal elementary abelian p-subgroup of G of rank r and assume that there are elements $c_i\in C_G(E_i)\setminus N_G(\langle e_i\rangle)$ with $[c_i,c_j]=1$ for all $1\leq i,j\leq r$. Then $\widetilde{H}_{r-1}(|\mathcal{A}_p(G)|)\neq 0$ and hence Quillen's conjecture holds for G. If furthermore $r=\mathrm{rk}_p(G)$, then $\mathcal{QD}_p$ holds for G.*

Proof. Note that the collection $\{E_i,c_j\}$ is faithful by Lemma 2.6. Now apply Theorem 3.3 with $A=\mathbb{Z}$, and Theorem 3.4(3) and (4) with $a_{\boldsymbol{\delta}}=(-1)^{\boldsymbol{\delta}}=(-1)^{\delta_1+\cdots+\delta_r}$ for $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_r)$, $\delta_i\in\{0,1\}$. □

Definition 3.6. A collection satisfying the assumptions of Theorem 3.5 is called *admissible*. That is, given a maximal elementary abelian p-subgroup $E=\langle e_1,\ldots,e_r\rangle$ of G with $\mathrm{rk}_p(E)=r$, an admissible collection for G is a collection $\{E_i,c_j\}$ of subgroups $E_i=\langle e_1,\ldots,\widehat{e_i},\ldots,e_r\rangle$ of G and elements c_j of G, such that $c_i\in C_G(E_i)\setminus N_G(\langle e_i\rangle)$ and $[c_i,c_j]=1$ for all $1\leq i,j\leq r$. Note that such a collection is faithful by Lemma 2.6.

4. Applications to symmetric and classical groups

We start this section with an observation which is useful when investigating Quillen's conjecture and $\mathcal{QD}_p$ for p'-central quotients of finite groups.

Lemma 4.1. *Let $E=\langle e_1,\ldots,e_r\rangle$ be a maximal elementary abelian p-subgroup of G of rank $r\leq\mathrm{rk}_p(G)$ and let $c_1,\ldots,c_r\in G$. Let N be a normal p'-subgroup of G satisfying that for each $\mathbf{i}\in S^r_l$ and for all $\boldsymbol{\epsilon}=(\epsilon_1,\ldots,\epsilon_r)$ with $\epsilon_i\in\{-1,0,1\}$ we have ${}^{\mathbf{c}^{\boldsymbol{\epsilon}}}E_{\mathbf{i}}\cap EN\leq E$. In particular this holds whenever N is a normal p'-subgroup of G which also centralizes E (for instance, if N is a central p'-subgroup of G). Set*

$\overline{G} = G/N$, and denote by an overline $\overline{(*)}$ the image of elements or subgroups of G in $\overline{G}$. The following hold.

(1) $\{E_i, c_j\}$ is faithful if and only if $\{\overline{E_i}, \overline{c_j}\}$ is faithful.
(2) $c_i \in C_G(E_i) \setminus N_G(\langle e_i \rangle)$ if and only if $\overline{c_i} \in C_{\overline{G}}(\overline{E}_i) \setminus N_{\overline{G}}(\langle \overline{e_i} \rangle)$.

Proof. For part (1), assume first that $\{E_i, c_j\}$ is faithful and that ${}^{\overline{\mathbf{c}}^\epsilon}\overline{e}_i \leq \overline{E}$, i.e., that ${}^{\mathbf{c}^\epsilon}e_i \in EN$. Then, by assumption we have ${}^{\mathbf{c}^\epsilon}e_i \in E$ and, as $\{E_i, c_j\}$ is faithful, we get that $\mathbf{c}^\epsilon \in N_G(\langle e_i \rangle)$. Hence we also get $\overline{\mathbf{c}}^\epsilon \in N_{\overline{G}}(\langle \overline{e}_i \rangle)$. Conversely, suppose that $\{\overline{E_i}, \overline{c_j}\}$ is faithful. Suppose that ${}^{\mathbf{c}^\epsilon}e_i \in E$. Then ${}^{\overline{\mathbf{c}}^\epsilon}\overline{e}_i \in \overline{E}$ and by assumption $\overline{\mathbf{c}}^\epsilon \in N_{\overline{G}}(\langle \overline{e}_i \rangle)$. It follows that ${}^{\mathbf{c}^\epsilon}e_i \in \langle e_i \rangle N \leq E$. By Dedekind's modular law, $\langle e_i \rangle N \cap E = \langle e_i \rangle (N \cap E) = \langle e_i \rangle$. Thus $\{E_i, c_j\}$ is faithful.

For part (2), assume that $c_i \in C_G(E_i) \setminus N_G(\langle e_i \rangle)$ for all i. Then $\overline{c_i} \in C_{\overline{G}}(\overline{E_i})$. We claim that $\overline{c_i} \notin N_{\overline{G}}(\langle \overline{e_i} \rangle)$. Indeed, by assumption, if ${}^{c_i}e_i = e_i^{a_i} n$ for some $n \in N$ and integer $0 < a_i < p$, then ${}^{c_i}e_i \in {}^{c_i}\langle e_i \rangle \cap EN \leq E$ and since $E \cap N = \{1\}$, we must have $n = 1$ and $c_i \in N_G(\langle e_i \rangle)$.

Conversely, if $\overline{c_i} \notin N_{\overline{G}}(\langle \overline{e_i} \rangle)$, then c_i cannot normalize $\langle e_i \rangle$ for any $1 \leq i \leq r$. It remains to see that if $\overline{c_i} \in C_{\overline{G}}(\overline{E}_i)$, then c_i centralizes E_i. For any $j \neq i$, if ${}^{c_i}e_j \in e_j N$, then by assumption, ${}^{c_i}e_j \in {}^{c_i}\langle e_j \rangle \cap EN \leq E$, and so we must have ${}^{c_i}e_j = e_j$ since $E \cap N = \{1\}$. $\square$

In the remainder of this section, we exhibit admissible collections for many alternating groups and finite classical groups of Lie type in non-defining characteristic, thereby proving that these groups possess the Quillen dimension at p property (cf. Definition 3.6).

We write Sym_X for the full permutation group on a finite set X. If $X = \{1, \ldots, n\}$, then we write Sym_n instead. Accordingly, we write A_X and A_n for the corresponding alternating groups.

Theorem 4.2. *Suppose that $p \geq 5$ and let $n \geq p$. Let $G = A_n$ or $G = \mathrm{Sym}_n$. Then there exists an admissible collection for G.*

Proof. We refer the reader to [1, Ch. IV] for the p-local structure of the alternating and symmetric groups. The p-rank of G is $r = \mathrm{rk}_p(A_n) = \lfloor \frac{n}{p} \rfloor$. Consider the maximal elementary abelian p-subgroup E of rank r generated by

$$e_1 = (1, \ldots, p), \ldots,\ e_i = ((i-1)p+1, \ldots, ip), \ldots,\ e_r = ((r-1)p+1, \ldots, rp).$$

Define hyperplanes of E by $E_i = \langle e_1, e_2, \ldots, \hat{e}_i, \ldots, e_r \rangle$. Write $n = rp + b$ with $0 \leq b < p$. Recall that $|N_{\mathrm{Sym}_n}(\langle e_i \rangle) : N_{A_n}(\langle e_i \rangle)| = |C_{\mathrm{Sym}_n}(E_i) : C_{A_n}(E_i)| = 2$, and that, since $p \geq 5$, the groups A_n are simple for $n \geq p$; in particular, its Sylow p-subgroups are not normal (A_p contains $(p-2)!$ Sylow p-subgroups). We have

$$N_G(\langle e_i \rangle) = \big(\langle e_i, t_i \rangle \times \mathrm{Sym}_{\{1, \ldots, (i-1)p, ip+1, \ldots n\}} \big) \cap G \quad \text{and} \tag{5}$$

$$C_G(E_i) = \big(E_i \times \mathrm{Sym}_{\{(i-1)p+1, \ldots, ip, rp+1, \ldots, n\}} \big) \cap G, \tag{6}$$

where $\langle e_i, t_i \rangle \cong C_p \rtimes C_{p-1}$ is the normalizer of $\langle e_i \rangle$ in $\mathrm{Sym}_{\{(i-1)p+1, \ldots, ip\}}$. Now, by assumption, $r \geq 1$ and $p \geq 5$, and, for each $1 \leq i \leq r$, we choose $c_i = ((i-1)p+1\ ,\ (i-1)p+2\ ,\ (i-1)p+3\ )$. This element clearly belongs to $C_G(E_i)$, for $G = A_n$ and for $G = \mathrm{Sym}_n$, and a short computation shows that it does not normalize $\langle e_i \rangle$. The collection $\{c_i\ ,\ 1 \leq i \leq r\}$ satisfies the hypothesis of Theorem 3.5. Indeed, by definition $[c_i, c_j] = 1$ for all i, j because the c_i are all disjoint cycles.

□

Remark 4.3. Computational evidence using GAP [4] shows that there is no admissible collection for $p = 3$ for the alternating and symmetric groups of degree n unless $n = 4, 5$ or 8. For $n = 4, 5$, these groups have 3-rank 1 and no non-trivial normal 3-subgroup. Then it is immediate that there exists an admissible collection. For A_8, we can take $E = \langle e_1, e_2 \rangle$ with $e_1 = (1, 2, 3)$ and $e_2 = (4, 5, 6)$, and put $c_1 = (1, 7)(2, 3)$ and $c_2 = (4, 8)(5, 6)$. Then, a direct check shows that $\{E_i, c_j\}$ is an admissible collection for A_8 (and so for Sym_8 too). Note that these cases are known via other methods [11, Proposition 2.10], since A_n has 3-rank at most 2 for $n = 4, 5, 8$ and $p = 3$.

We will now adapt the above argument to find admissible collections in classical finite groups with trivial p-core in non-defining characteristic, and therefore show that they have the Quillen dimension at p property. By a *classical group*, we mean G a finite group of Lie type A_n, ${}^2\mathrm{A}_n$, B_n, C_n, D_n or ${}^2\mathrm{D}_n$, and we refer the reader to [8, Table 22.1] for the different isogeny types and the standard notation that we will use. In particular, if G is of type A_{n-1} defined over a field with q elements, then G is a central quotient of a group G^* such that $\mathrm{SL}_n(q) \leq G^* \leq \mathrm{GL}_n(q)$, and $G = G^*/Z$ for a central subgroup $Z \leq Z(G^*)$. If $Z = Z(G^*)$, then we write $PG^* = G^*/Z(G^*)$. We also assume that p is coprime to the characteristic of the field of definition of G.

Notation 4.4. For any natural number n, we write $\mathrm{GL}_n^+(q) = \mathrm{GL}_n(q)$, $\mathrm{GL}_n^-(q) = \mathrm{GU}_n(q)$, $\mathrm{SL}_n^+(q) = \mathrm{SL}_n(q)$ and $\mathrm{SL}_n^-(q) = \mathrm{SU}_n(q)$ for linear, unitary, special linear and special unitary groups respectively. So a superscript "+" means "linear" and "−" means "unitary". We let d be the multiplicative order of q modulo p, that is, the smallest positive integer such that p divides $q^d - 1$. Finally, if n is an integer, we write $(n)_p$ for the p-part of n.

For the central quotient $\mathrm{P}G = G/Z(G)$ of a classical group G, recall that $O_p(G) \leq Z(G)$ except for some small linear and unitary groups. Moreover, we have isomorphisms $Z(\mathrm{GL}_n^\epsilon(q)) \cong C_{q-\epsilon}$ and $Z(\mathrm{SL}_n^\epsilon(q)) \cong C_{(n,q-\epsilon)}$, for $\epsilon = \pm 1$, and the centers of symplectic and orthogonal groups are 2-groups (cf. [8, Table 24.2]). Hence, $\mathrm{P}G$ is a p'-quotient of G if and only if $O_p(G) = 1$. If so, $\mathrm{P}G$ and G have the same p-subgroup structure [2, Lemma 0.11], in particular, $\mathcal{A}_p(G) \cong \mathcal{A}_p(PG)$. In this case, we show that $\mathcal{QD}_p$ holds for both groups by exhibiting an admissible collection for G, and then applying Lemma 4.1, we obtain that PG possesses an admissible collection too.

In the remaining case, i.e., if $O_p(G) > 1$, we show that there is no admissible collection for G, using Lemma 2.7, and we prove that $\mathcal{QD}_p$ holds for $\mathrm{P}G$ by exhibiting an admissible collection. The next table summarizes the different cases, indicating the conditions and the results to be applied in each case. For conciseness, we omitted from the table the exceptions $(p, q) = (3, 2)$, which apply whenever G is not a linear group, and the cases $\mathrm{PSL}_3(q)$ with $q \equiv 1 \pmod 3$ at the prime 3.

Group G	$O_p(G)=1$		$O_p(G)>1$
	$d>1$		$d=1$
$\mathrm{GL}_n(q)$	$\mathcal{QD}_p$ by 4.7, 4.10		$\nexists$ admissible coll. by 4.5
$\mathrm{PGL}_n(q)$	$\mathcal{QD}_p$ by 4.1		$\mathcal{QD}_p$ by 4.9
	$d>1$	$d=1,(p,n)=1$	$d=1,(p,n)=p$
$\mathrm{SL}_n(q)$	$\mathcal{QD}_p$ by 4.7, 4.10	$\mathcal{QD}_p$ by 4.8	$\nexists$ admissible coll. by 4.5
$\mathrm{PSL}_n(q)$	$\mathcal{QD}_p$ by 4.1	$\mathcal{QD}_p$ by 4.1	$\mathcal{QD}_p$ by 4.9
	$d\neq 2$		$d=2$
$\mathrm{GU}_n(q)$	$\mathcal{QD}_p$ by 4.10		$\nexists$ admissible coll. by 4.5
$\mathrm{PGU}_n(q)$	$\mathcal{QD}_p$ by 4.1		
	$d\neq 2$		$d=2,(p,n)=p$
$\mathrm{SU}_n(q)$	$\mathcal{QD}_p$ by 4.10		$\nexists$ admissible coll. by 4.5
$\mathrm{PSU}_n(q)$	$\mathcal{QD}_p$ by 4.1		
symplectic	$\mathcal{QD}_p$ for G by 4.10		
orthogonal	$\mathcal{QD}_p$ for $\mathrm{P}G$ by 4.1		–

Note that $d=1$ if and only if $q\equiv 1 \pmod p$, and that $d=2$ if and only if $q\equiv -1 \pmod p$, which determines when a linear or unitary group has a nontrivial p-core. The last two rows denote any symplectic or orthogonal finite group of Lie type, as described in [8, Table 22.1].

Proposition 4.5. *Suppose that p is odd and that $n\geq 1$. Let $G=\mathrm{GL}^{\epsilon}_n(q)$, or $G=\mathrm{SL}^{\epsilon}_n(q)$, and assume that $q-\epsilon\equiv 0 \pmod p$, or $(q-\epsilon,n)\equiv 0 \pmod p$, respectively. Then, there exists no admissible collection for G.*

Proof. Recall that $Z(\mathrm{GL}^{\epsilon}_n(q))$ is cyclic of order $q-\epsilon$ and $Z(\mathrm{SL}^{\epsilon}_n(q))$ is cyclic of order $(q-\epsilon,n)$. So, the assumptions say that $O_p(G)\cap Z(G)>1$ and the assertion follows from Lemma 2.7. □

We now consider the linear groups with $d>1$.

Notation 4.6. We denote by I_n the $n\times n$ identity matrix in $\mathcal{M}_{n\times n}(\mathbb{F}_q)$ and by M_{ab} the standard basis matrix in $\mathcal{M}_{n\times n}(\mathbb{F}_q)$, i.e., the $n\times n$ matrix with coefficients in $\mathbb{F}_q$ whose entry at row c and column d is given by

$$(M_{ab})_{cd}=\begin{cases}1 & \text{if } a=c \text{ and } b=d\text{, and}\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 4.7. *Suppose that p is odd and let $G=\mathrm{GL}_n(q)$ or $G=\mathrm{SL}_n(q)$ with $d>1$ except $\mathrm{SL}_2(2)$ for $p=3$. Then there exists an admissible collection for G.*

Proof. Write $n=rd+f$, with $0\leq f<d$. Then the unique (up to G-conjugacy) maximal elementary abelian p-subgroup E of G has rank r. We split the proof into two cases. First, suppose that $(p,q)\neq(3,2)$. Using [6, Section 4.10, 4.8], and the fact that $\mathrm{GL}_1(q^d)\hookrightarrow\mathrm{GL}_d(q)$, we can choose $E\leq G$ as the set of block diagonal matrices of the form

$$E=\langle\theta_1(u),\dots,\theta_r(u)\rangle\quad\text{where } u\in\mathrm{GL}_1(q^d) \text{ has order } p\text{, and where}$$

$$\begin{array}{rcccccl}\theta_i & : & \mathrm{GL}_1(q^d) & \to & \mathrm{GL}_d(q) & \to & \mathrm{GL}_n(q)\\ & & x & \mapsto & X & \mapsto & \mathrm{diag}(I_{(i-1)d},X,I_{n-id})\end{array}.$$

Put $e_j = \theta_i(u)$ for $1 \leq i \leq r$. Note that $\det(e_i) = 1$ because $\det(e_i)$ must be a p-th root of 1 in $\mathbb{F}_q$ and $(p, q-1) = 1$. Then

$$C_G(E) = G \cap \Big(\operatorname{diag}\big(\operatorname{Im}(\theta_1), \ldots, \operatorname{Im}(\theta_r)\big) \times \operatorname{GL}_f(q) \Big)$$

with $\operatorname{Im}(\theta_i) \cong \operatorname{GL}_1(q^d)$ for all $1 \leq i \leq r$. We have

$$C_G(E_i) \cong G \cap \big(C_{\operatorname{GL}_n(q)}(E)\, \operatorname{GL}_{d+f}(q) \big) \quad \text{and}$$

$$N_G(\langle e_i \rangle) \cong G \cap \big((\operatorname{GL}_1(q^d) \rtimes C_d) \times \operatorname{GL}_{n-d}(q) \big)$$

where the factors $\operatorname{GL}_{d+f}(q)$ and $\operatorname{GL}_{n-d}(q)$ intersect in a subgroup isomorphic to $\operatorname{GL}_f(q)$. So we can choose $c_i \in C_G(E_i) \setminus N_G(\langle e_i \rangle)$ in the factor $\operatorname{GL}_d(q)$ properly containing the subgroup $(\operatorname{GL}_1(q^d) \rtimes C_d)$ in the i-th diagonal block for all i. Such choice is possible whenever p is odd, unless $q = 2$, $d = 2$ and $p = 3$, because $\operatorname{GL}_1(2^2) \rtimes C_2 \cong \operatorname{GL}_2(2) \cong \operatorname{Sym}_3$. Now, our choice satisfies $c_i \in C_G(E_i) \setminus N_G(\langle e_i \rangle)$ and $[c_j, c_i] = 1$ for all i, j. Thus $\{E_i, c_j\}$ is an admissible collection for G.

It remains to handle the case $(p, q) = (3, 2)$. If $n = 2$, then $O_3(\operatorname{SL}_2(2)) \neq 1$ and this case is excluded by assumption. If $n = 3$, recall that a Sylow 3-subgroup of $G = \operatorname{SL}_3(2)$ has order 3 and it is not normal in G, saying that, trivially, there exists an admissible collection for G. If $n \geq 4$, we observe that it is enough to consider the groups $G = \operatorname{SL}_{2k}(2)$, for $k \geq 2$, as the index of $\operatorname{SL}_{2k}(2)$ in $\operatorname{SL}_{2k+1}(2)$ is not divisible by 3, so that an admissible collection for $\operatorname{SL}_{2k}(2)$ induces an admissible collection for $\operatorname{SL}_{2k+1}(2)$. As above, we can choose an elementary abelian 3-subgroup E of G of maximal rank embedded in the subgroup formed by the diagonal 2×2 blocks in $\operatorname{SL}_{2k}(2)$. Moreover, we see that it suffices to find admissible collections with respect to such an E for $G = \operatorname{SL}_4(2)$ and for $G = \operatorname{SL}_6(2)$, because from them we can obtain admissible collections for $\operatorname{SL}_{2n}(2)$, for all $n \geq 2$. Using GAP [4], we find the following admissible collections. Let $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If $G = \operatorname{SL}_4(2)$, put

$$e_1 = \begin{pmatrix} X^2 & \\ & X^2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} X^2 & \\ & X \end{pmatrix}, \quad c_1 = \begin{pmatrix} X & \\ Y & X \end{pmatrix} \text{ and } c_2 = \begin{pmatrix} X & \\ X & X^2 \end{pmatrix},$$

where a blank entry means the zero matrix of appropriate size. A direct calculation shows that $E = \langle e_1, e_2 \rangle = C_3 \times C_3$ is an elementary abelian 3-subgroup of G of maximal rank, with $c_j \in C_G(e_{3-j})$ and $c_j \notin N_G(\langle e_j \rangle)$ for $j = 1, 2$, and also that $[c_1, c_2] = 1$. Thus these elements form an admissible collection for $\operatorname{SL}_4(2)$. If $G = \operatorname{SL}_6(2)$, put

$$e_1 = \begin{pmatrix} X^2 & & \\ & I_2 & \\ & & I_2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} X & & \\ & X & \\ & & X \end{pmatrix} \text{ and } e_3 = \begin{pmatrix} X^2 & & \\ & X^2 & \\ & & X \end{pmatrix},$$

and

$$c_1 = \begin{pmatrix} X & & \\ X & I_2 & \\ & & I_2 \end{pmatrix}, \; c_2 = \begin{pmatrix} X & & \\ & X & Y \\ & & X \end{pmatrix} \text{ and } c_2 = \begin{pmatrix} X & & \\ & X & X \\ & & X^2 \end{pmatrix}.$$

A direct computation shows that $E = \langle e_1, e_2, e_3 \rangle = C_3 \times C_3 \times C_3$ is an elementary abelian 3-subgroup of G of maximal rank, with $c_i \in C_G(\langle e_j, e_k \rangle)$ and $c_i \notin N_G(\langle e_i \rangle)$ for any $\{i, j, k\} = \{1, 2, 3\}$, and also that $[c_i, c_j] = 1$ for any $1 \leq i, j \leq 3$. Thus these elements form an admissible collection for $\operatorname{SL}_6(2)$. $\square$

The cases $G = \mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$, and $\mathrm{PGL}_n(q)$ when $d = 1$ are more subtle. In this case, $O_p(\mathrm{SL}_n(q)) = 1$ if and only if $(n, p) = 1$, i.e. when $\mathrm{SL}_n(q)$ has no scalar matrix ζI_n where $\zeta^n = \zeta^p = 1 \neq \zeta$. From [6, Section 4.10], the p-ranks of $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$ and $\mathrm{PGL}_n(q)$ may be found, and we use this information in the proofs of the next two propositions.

Proposition 4.8. *Let $G = \mathrm{SL}_n(q)$ with p an odd prime dividing $q-1$ and coprime to n. Then there exists an admissible collection for G.*

Proof. By assumption $O_p(G) = 1$ because p does not divide n. Let $u \in \mathrm{GL}_1(q)$ be an element of order p and put, for all $1 \leq i \leq n-1$,

$$e_i = \mathrm{diag}(uI_{i-1}, u^{1-n}, uI_{n-i}).$$

Let $E = \langle e_1, \ldots, e_{n-1} \rangle$, so that $E \in \mathcal{A}_p(G)$ has maximal rank $n - 1 = \mathrm{rk}(G)$. Then $C_G(E) = T = C_G(T) \cong \mathrm{GL}_1(q)^{n-1}$ is the subgroup of G formed by all diagonal matrices with determinant 1. We have

$$C_G(E_i) \cong G \cap \big(T\,\mathrm{GL}_2(q)\big) \quad \text{and} \quad N_G(\langle e_i \rangle) = C_G(e_i) \cong \mathrm{GL}_{n-1}(q),$$

where such $\mathrm{GL}_{n-1}(q)$ contains T, and where

$$C_G(E_i) \cap N_G(\langle e_i \rangle) = T.$$

Choose $c_j = I_n + M_{jn}$ (see Notation 4.6). A routine calculation shows that $c_j \in C_G(E_j) \setminus N_G(\langle e_j \rangle)$ has order q, and that $[c_j, c_i] = 1$ for all i, j. Thus $\{E_i, c_j\}$ is an admissible collection for G. □

Proposition 4.9. *Let $G = \mathrm{PSL}_n(q)$ or $\mathrm{PGL}_n(q)$ with p an odd prime prime dividing $(n, q-1)$ or $q-1$ respectively, excluding the case $\mathrm{PSL}_3(q)$ for $p = n = 3$. Then there exists an admissible collection for G.*

Proof. First consider $G = \mathrm{PSL}_n(q)$. If $p = n = 3$ then the p-rank is 2 and we exclude this case by assumption. Otherwise, from [5, Theorem 10.6(1)] we have

$$\mathrm{rk}_p(G) = \begin{cases} n-2, \text{ if } (n)_p \geq (q-1)_p, \\ n-1, \text{ if } (n)_p < (q-1)_p. \end{cases}$$

For the former case, we embed $\mathrm{SL}_{n-1}(q)$ in $\mathrm{PSL}_n(q)$ as the subgroup

$$\left(\begin{array}{c|c} & 0 \\ \mathrm{SL}_{n-1}(q) & \vdots \\ & 0 \\ \hline 0 \;\ldots\; 0 & 1 \end{array} \right),$$

and, since p does not divide $n-1$, we invoke Proposition 4.8 to construct an admissible collection with the maximal elementary abelian p-subgroup E of G of rank $n-2$ and $c_1, \ldots, c_{n-2}$ all sitting in $\mathrm{SL}_{n-1}(q)$.

For the case $G = \mathrm{PSL}_n(q)$ and $(n)_p < (q-1)_p$, let $z, u \in \mathbb{F}_q^*$ be elements such that z has order $(n)_p$ and $u^p = z$. Set

$$e = \mathrm{diag}(z, \ldots, z) \in Z(\mathrm{SL}_n(q))$$

and, as in Proposition 4.8, set for all $1 \leq i \leq n-1$,

$$e_i = \mathrm{diag}(uI_{i-1}, u^{1-n}, uI_{n-i}) \in \mathrm{SL}_n(q).$$

Then $e_i^p = e$ and in the quotient $G = \mathrm{SL}_n(q)/Z(\mathrm{SL}_n(q))$ the classes of the elements $e_1, \ldots, e_{n-1}$ generate a subgroup $E \in \mathcal{A}_p(G)$ of rank $n - 1 = \mathrm{rk}_p(G)$. Choose c_j

to be the class of the element $I_n + M_{jn} \in \mathrm{SL}_n(q)$ (see Notation 4.6). A routine calculation shows that we have built an admissible collection for G.

Next let $G = \mathrm{PGL}_n(q)$. Then $\mathrm{rk}_p(G) = n-1$. We choose elements $e_1, \ldots, e_{n-1}$ as follows: Let $u \in \mathrm{GL}_1(q)$ be an element of order p and and set $e_i = \mathrm{diag}(uI_{i-1}, 1, uI_{n-i})$ for $1 \leq i \leq n-1$. Let $E = \langle e_1, \ldots, e_{n-1}\rangle$, and define $c_j = I_n + M_{jn}$ (see Notation 4.6). It is clear that $c_j \in C_G(E_j)$ and that $[c_i, c_j] = 1$ for all $1 \leq j \leq n-1$. Moreover, a routine calculation shows that the matrices ${}^{c_j}e_j$ are not diagonal, and hence $c_j \notin N_G(\langle e_j\rangle)$ and $\{E_i, c_j\}$ is an admissible collection for G. □

Machine computations indicate that the cases excluded in Proposition 4.9 are probably genuine exceptions to the existence of admissible collections.

We are now ready to handle the remaining classical groups (cf. the paragraph above Notation 4.4 for our conventions). We follow the methods in [5, Section 8], and so let G act on a vector space V, defined over $\mathbb{F}_q$, unless G is unitary, in which case V is defined over $\mathbb{F}_{q^2}$, and V comes equipped with a hermitian form. Let $\mathrm{Isom}(V)$ denote the full isometry group of V. So, for instance, if G has Lie type $\mathrm{A}_{n-1}(q)$, then $\mathrm{Isom}(V) = \mathrm{GL}_n(q)$, and G is a central quotient of a group G^* such that $\mathrm{SL}_n(q) \leq G^* \leq \mathrm{GL}_n(q)$.

Theorem 4.10. *Let G be a finite linear, unitary, symplectic or orthogonal group defined over a field with q elements (respectively q^2 elements if G is unitary), and let p be an odd prime with $p \nmid q$. Suppose that the following conditions hold:*

(i) *p divides $|G|$ and $O_p(G) = 1$.*
(ii) *$(p, q) \neq (3, 2)$ unless G is linear.*
(iii) *$p \nmid q + 1$ if G is unitary.*

Then there exists an admissible collection for G.

Remark 4.11. Some well known properties of small finite groups of Lie type (cf. [8, Remark 24.18] for instance), and computational evidence using GAP [4] show that there are no admissible collections for small dimensional unitary, symplectic and orthogonal groups for $p = 3$ and $q = 2$. So condition (ii) in the statement is necessary. The case when G is linear and $(p, q) = (3, 2)$ is dealt with in Proposition 4.7.

Proof. From Propositions 4.8 and 4.9, we can assume that if G is linear, then $d > 1$. Condition (iii) is equivalent to saying that if G is unitary, then $d \neq 2$.

Let V be the underlying vector space of G, and $\mathrm{Isom}(V)$ its full isometry group, as explained above. From [8, Tables 22.1 and 24.2], our assumptions allow us to suppose that G is simply connected. Indeed, the index of the group of simply connected type in $\mathrm{Isom}(V)$ is coprime to p, so that both have have isomorphic Sylow p-subgroups. Thus, if there is an admissible collection for the group of simply connected type, then its inclusion in $\mathrm{Isom}(V)$ yields an admissible collection for $\mathrm{Isom}(V)$. Hence, if G is a central quotient of G^*, the image of such admissible collection is an admissible collection for G, by Lemma 4.1. For instance, if G has Lie type $\mathrm{A}_{n-1}(q)$, then we can suppose that $G = \mathrm{SL}_n(q)$. Indeed, if $d > 1$ (i.e. if $q \not\equiv 1 \pmod p$), then an admissible collection for $\mathrm{SL}_n(q)$ gives admissible collections for any group G^*/Z, where $\mathrm{SL}_n(q) \leq G^* \leq \mathrm{GL}_n(q)$ and $Z \leq Z(G^*)$.

Let E be a maximal elementary abelian p-subgroup of G of maximal order. Choose generators $\{e_1, \ldots, e_r\}$ of E and a basis $\{v_1, \ldots, v_n\}$ of V such that e_i acts as the identity everywhere except on $V_i = \langle v_{(i-1)d+1}, \ldots, v_{id}\rangle$ for all $1 \leq i \leq r$. Put

$V_0 = \langle v_{rd+1}, \dots, v_n \rangle$, possibly $V_0 = \{0\}$. Thus $V = \oplus_{0 \le i \le r} V_i$. We refer the reader to [5, Table 10:1] for the values of r depending on the type of G, and the values of n and d. (Recall that under our assumptions, G and $\mathrm{Isom}(V)$ have isomorphic Sylow p-subgroups.)

From [5, 8-1], we have, for all $1 \le i \le r$,

$$C_G(e_i) = G \cap \big(\mathrm{Isom}(V/V_i) \times \mathrm{GL}_1^{\epsilon}(q^e)\big).$$

It follows that

$$C_G(E_i) = \bigcap_{j \neq i} C_G(e_j) =$$

$$G \cap \Bigg(\mathrm{GL}_1^{\epsilon}(q^e)^{(1)} \times \cdots \times \mathrm{GL}_1^{\epsilon}(q^e)^{(i-1)} \times \mathrm{Isom}(V_i \oplus V_0) \times \mathrm{GL}_1^{\epsilon}(q^e)^{(i+1)} \times \cdots \times \mathrm{GL}_1^{\epsilon}(q^e)^{(r)} \Bigg)$$

for $1 \le i \le r$, and

$$C_G(E) = G \cap \big(\mathrm{GL}_1^{\epsilon}(q^e)^{(1)} \times \cdots \times \mathrm{GL}_1^{\epsilon}(q^e)^{(r)} \times \mathrm{Isom}(V_0)\big),$$

where $\mathrm{GL}_1^{\epsilon}(q^e)^{(i)} = C_{\mathrm{Isom}(V_i)}(e_{i|_{V_i}})$ is the centralizer of the action of e_i restricted to V_i, where the sign $\epsilon = \pm 1$ and the value of e depend on the parity of d and the type of G, and where $\mathrm{Isom}(V_0)$ has the same type as G. Explicitly,

- If G is linear: $\epsilon = +$ and $d = e$.
- If G is unitary: if $d \equiv 2 \pmod 4$, then $\epsilon = -$, and $\epsilon = +$ otherwise. We put $e = 2d$ if d is odd, $e = \frac{d}{2}$ if $d \equiv 1 \pmod 4$ and $e = d$ if $d \equiv 0 \pmod 4$.
- If G is symplectic or orthogonal, put $f = \frac{1}{2}\,\mathrm{lcm}(2,d)$ and hence $\epsilon = +$ if f is odd and $-$ otherwise, and put $e = 2f$. (We refer the reader to [5, Section 8] for the type of the orthogonal space, as it does not impact on our argument.)

By definition, $\dim(V_0) < d$ (resp $2d$ if G is symplectic or orthogonal), saying that $p \nmid |\mathrm{Isom}(V_0)|$.

Also,

$$N_G(\langle e_i \rangle) = G \cap \Bigg(\big(\mathrm{GL}_1^{\epsilon}(q^e)^{(i)} \rtimes C_e^{(i)} \big) \times \mathrm{Isom}(\oplus_{j \neq i} V_j) \Bigg),$$

where e is as above.

Conditions (i)-(iii) in the statement ensure that $\big(\mathrm{GL}_1^{\epsilon}(q^d)^{(j)} \rtimes C_e^{(j)}\big) \lneq \mathrm{Isom}(V_j)$, so that we can pick $c_j \in G$ such that c_j acts as the identity on V_i for all $i \neq j$ and the restriction $c_{j|_{V_j}} \in \mathrm{Isom}(V_j) \setminus \big(\mathrm{GL}_1^{\epsilon}(q^d)^{(j)} \rtimes C_e^{(j)}\big)$.

Since $V = \oplus_{0 \le i \le r} V_i$, we have $[\mathrm{Isom}(V_i), \mathrm{Isom}(V_j)] = 1$ for all $i \neq j$, and therefore the elements c_j commute pairwise, i.e. $[c_i, c_j] = 1$ for any $1 \le i, j \le r$. We conclude that $\{E_i, c_j\}$ is an admissible collection for G. □

Our objective in this work was to devise a simpler argument to show the Quillen dimension at p property for the symmetric and alternating groups, and for the finite classical groups in non-defining characteristic. Although our results do not fully meet our initial objective, we believe that our methods can be further generalized to tackle the cases excluded from the present paper, and also p-extensions, i.e., almost-simple groups with an elementary abelian p-group inducing outer automorphisms. The first author will pursue this aim in a subsequent work.

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