Equal sums, sums of squares and sums of cubes

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Sums of squares; triples

Consider the problem of finding triples of numbers (x_1, x_2, x_3) and (y_1, y_2, y_3) satisfying

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 \tag{1}$$

and

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2. \tag{2}$$

The variables x_j, y_j are taken to be real numbers (not excluding negative numbers), but we shall be particularly interested in integer solutions.

It will help to use vector notation. We write $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. If (1) and (2) hold, we write $\mathbf{x} \sim \mathbf{y}$, and say that $\mathbf{x}$ and $\mathbf{y}$ are *associates*.

Of course, the problem is not really restricted to pairs of vectors. For given S_1 and S_2, all solutions of the pair of equations

$$x_1 + x_2 + x_3 = S_1, \qquad x_1^2 + x_2^2 + x_3^2 = S_2$$

are associates of each other. Geometrically, this is the intersection of a plane and a sphere in three-dimensional space. However, it is not at all a pleasant exercise to find solutions (still less, integer solutions) of this pair of equations for given S_1 and S_2. Instead, we will outline a method that generates associate pairs without effort.

Some elementary observations will help to pave the way.

(i) If $\mathbf{x} \sim \mathbf{y}$, then $\lambda\mathbf{x} \sim \lambda\mathbf{y}$ for any λ.

(ii) If $\mathbf{x} \sim \mathbf{y}$, then $\mathbf{x}' \sim \mathbf{y}$ for any permutation $\mathbf{x}'$ of $\mathbf{x}$, for example (x_2, x_1, x_3). For example, we can re-order $\mathbf{x}$ so that $x_1 \le x_2 \le x_3$ (we will say that $\mathbf{x}$ is *aligned* if this holds).

(iii) If $\mathbf{x} \sim \mathbf{y}$, then $(x_1 + c, x_2 + c, x_3 + c) \sim (y_1 + c, y_2 + c, y_3 + c)$ for any c, since

$$\sum_{j=1}^{3}(y_j + c)^2 - \sum_{j=1}^{3}(x_j + c)^2 = \sum_{j=1}^{3}(y_j^2 - x_j^2) + 2c\sum_{j=1}^{3}(y_j - x_j).$$

So it is enough to present a pair normalised so that (for example) $x_1 = 0$. Also, any pair of vectors containing negative numbers can be converted in this way to a pair with all numbers non-negative.

Next, we note that the problem for 2-vectors is trivial. Suppose that $x_1 + x_2 = y_1 + y_2$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Then $y_1 - x_1 = x_2 - y_2$ and $y_1^2 - x_1^2 = x_2^2 - y_2^2$, so that $(y_1 - x_1)(y_1 + x_1) =$

$(x_2 - y_2)(x_2 + y_2)$. So either we have $x_1 = y_1$ and $x_2 = y_2$, or $y_1 + x_1 = x_2 + y_2$, in which case $x_1 = y_2$ and $x_2 = y_1$.

Consequently, for 3-vectors, if $\mathbf{x} \sim \mathbf{y}$ and any x_i equals any y_j, then $\mathbf{y}$ is simply a permutation of $\mathbf{x}$.

A given integer triple can have at most a finite number of integer-valued associates, because condition (2) sets a bound on $|y_j|$. There are plenty of triples that have no integer-valued associates other than permutations, for example $(0, 1, 2)$ and $(0, 1, 3)$ (this is easily checked, with the help of the previous remark).

A neat observation (supplied to me by Nick Lord) is that if $\mathbf{x} \sim \mathbf{y}$, then $(p\mathbf{x} + q\mathbf{y}) \sim (q\mathbf{x} + p\mathbf{y})$ for any p, q.

We are now ready to describe our method. Suppose that $\mathbf{x}$ and $\mathbf{y}$ satisfy (1). Let $z_j = \frac{1}{2}(x_j + y_j)$. Then for some a and b, we have

$$x_1 = z_1 - a, \quad x_2 = z_2 + a + b, \quad x_3 = z_3 - b, \tag{3}$$

$$y_1 = z_1 + a, \quad y_2 = z_2 - a - b, \quad y_3 = z_3 + b. \tag{4}$$

For non-trivial examples, a and b must be non-zero. Now

$$\sum_{j=1}^{3} y_j^2 - \sum_{j=1}^{3} x_j^2 = 4az_1 - 4(a + b)z_2 + 4bz_3,$$

so for (2) to hold, we must have $g(\mathbf{z}) = 0$, where

$$g(\mathbf{z}) = az_1 - (a + b)z_2 + bz_3 = a(z_1 - z_2) + b(z_3 - z_2). \tag{5}$$

For a chosen a and b, the set $\{\mathbf{z} : g(\mathbf{z}) = 0\}$ is a two-dimensional linear subspace E of $\mathbb{R}^3$. Two obvious members are $(0, b, a + b)$ and $(1, 1, 1)$. It is easily verified that all elements of E are linear combinations of these two, in other words, of the form

$$\mathbf{z} = \lambda(0, b, a + b) + \mu(1, 1, 1). \tag{6}$$

For each such $\mathbf{z}$, an associate pair $\mathbf{x}$, $\mathbf{y}$ is then defined by (3) and (4). All associate pairs are obtained by allowing λ, μ, a and b to vary freely. However, a good deal of duplication occurs, in ways which will emerge below.

To normalise with $x_1 = 0$, we take $\mu = a$, with the effect that the scheme becomes

$$\mathbf{x} = \lambda(0, b, a + b) + (0, 2a + b, a - b), \tag{7}$$

$$\mathbf{y} = \lambda(0, b, a + b) + (2a, -b, a + b). \tag{8}$$

Any choice of a, b and λ delivers an associate pair. However, the choice $\lambda = 1$ is not productive: it gives $\mathbf{x} = (0, 2a + 2b, 2a)$ and $\mathbf{y} = (2a, 0, 2a + 2b)$, a permutation of $\mathbf{x}$. Similarly for $\lambda = -1$.

The x_j and y_j will be integers if a, b and λ are integers, but more exactly, it is easily checked that necessary and sufficient conditions are that $2a$, $2b$, $(\lambda - 1)a$ and $(\lambda - 1)b$ are all integers. Since $\mathbf{y} - \mathbf{x} = (2a, -2a - 2b, 2b)$, it is entirely natural to consider cases where a or b is a half integer.

Example 1. Take $a = b = \frac{1}{2}$. By (7) and (8), we have $\mathbf{x} = (0, \frac{1}{2}(\lambda + 3), \lambda)$ and $\mathbf{y} = (1, \frac{1}{2}(\lambda - 1), \lambda + 1)$. For integer values, we need λ to be an odd integer. Note that $\mathbf{x}$ and $\mathbf{y}$ will be aligned if $\lambda \geq 3$. We record a few such cases:

λ	$\mathbf{x}$	$\mathbf{y}$
3	$(0, 3, 3)$	$(1, 1, 4)$
5	$(0, 4, 5)$	$(1, 2, 6)$
7	$(0, 5, 7)$	$(1, 3, 8)$
9	$(0, 6, 9)$	$(1, 4, 10)$
11	$(0, 7, 11)$	$(1, 5, 12)$

Meanwhile, $\lambda = 4$ gives $\mathbf{x} = (0, \frac{7}{2}, 4)$ and $\mathbf{y} = (1, \frac{3}{2}, 5)$, which we can double to give the integer-valued pair $(0, 7, 8)$ and $(2, 3, 10)$. Also, $\lambda = 2$, after doubling, gives $\mathbf{x} = (0, 5, 4)$ and $\mathbf{y} = (2, 1, 6)$, permutations of the pair derived from $\lambda = 5$. This is actually an instance of a more general fact: one can check that if $\lambda > 3$ generates the pair (x_1, x_2, x_3), (y_1, y_2, y_3) and $\mu = (\lambda + 3)/(\lambda - 1)$, then $\mu < 3$ and μ generates the pair $\alpha(x_1, x_3, x_2)$ and $\alpha(y_2, y_1, y_3)$, where $\alpha = 2/(\lambda - 1)$.

The reader may care to write out some examples delivered by other choices of a and b.

We mention some further consequences of our reasoning. First, if $x_1 = x_2 = x_3$, then $g(\mathbf{x}) = 0$. However, by (3) and the fact that $g(\mathbf{z}) = 0$, we have $g(\mathbf{x}) = -a^2 - (a + b)^2 - b^2$. Hence $a = b = 0$, so there are no associates other than $\mathbf{x}$ itself.

Second, a fact about the possible interweaving of x_j and y_j. Suppose that associates $\mathbf{x}$ and $\mathbf{y}$ are aligned and $x_3 < y_3$. We show that $x_1 < y_1$, hence also $x_2 > y_2$. Now $b > 0$, and since $\mathbf{y}$ is not a permutation of $\mathbf{x}$, we have $a \neq 0$. We have to show that $a > 0$. By (5), $a(z_2 - z_1) = b(z_3 - z_2)$. If $z_1 = z_2 = z_3$, then, since $\mathbf{x}$ and $\mathbf{y}$ are aligned, $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$: this is not possible with $x_3 < y_3$. So $z_1 < z_2 < z_3$, hence $a > 0$.

Now let us address the different problem of finding associates of a given vector $\mathbf{x}$. For this, we vary the previous method slightly. Any $\mathbf{y}$ satisfying (1) can be expressed as follows:

$$y_1 = x_1 + a, \quad y_2 = x_2 - a - b, \quad y_3 = x_3 + b \tag{9}$$

for some a, b (note that a replaces the previous $2a$). Then $\sum_{j=1}^{3}(y_j^2 - x_j^2) = 2R + 2S$, where

$$R = a^2 + ab + b^2,$$

$$S = ax_1 - (a+b)x_2 + bx_3 = a(x_1 - x_2) + b(x_3 - x_2).$$

We have to choose a and b so that $R + S = 0$. Write $b = qa$. Then $R = Qa^2$, where $Q = 1 + q + q^2$, and the condition $R + S = 0$ equates to

$$(x_1 - x_2) + q(x_3 - x_2) + Qa = 0,$$

so

$$a = \frac{1}{Q}[(x_2 - x_1) + q(x_2 - x_3)]. \tag{10}$$

To obtain associates, we choose q freely, then define a by (10) and $\mathbf{y}$ by (9), with $b = qa$. If the x_j are integers and a and q are rational, then the y_j may or may not be integers, but they will certainly be rational. Integer-valued associate pairs can then be derived by multiplying through by the denominator.

Example 2: Associates of $(0, 3, 3)$. We have seen the associate $(1, 1, 4)$ in Example 1. Using the principle that the y_j must be distinct from the x_j, it is easily checked that (apart from permutations), this is the only integer-valued associate. We record some rational associates. By (10), we have $a = 3/Q$.

q	Q	$\mathbf{y}$
1	3	$(1, 1, 4)$
2	7	$(\frac{3}{7}, \frac{12}{7}, \frac{27}{7})$
3	13	$(\frac{3}{13}, \frac{27}{13}, \frac{48}{13})$
4	21	$(\frac{1}{7}, \frac{16}{7}, \frac{25}{7})$

Multiplying through by 7, we can exhibit the following example of multiple integer-valued associates:

$$(0, 21, 21) \sim (7, 7, 28) \sim (3, 12, 27) \sim (1, 16, 25).$$

The reader might care to verify the following fact: if (y_1, y_2, y_3) is derived from q in this way, then the associate derived from $1/q$ is (y_2, y_1, y_3), and the associate derived from $-q - 1$ is (y_1, y_3, y_2).

Equal sums and products

We digress briefly to consider the problem of finding triples $\mathbf{x}$ and $\mathbf{y}$ such that $\sum_{j=1}^{3} x_j = \sum_{j=1}^{3} y_j$ and $x_1x_2x_3 = y_1y_2y_3$. To be of any interest, the x_j and y_j must be non-zero. This can be done very simply. Having chosen x_1, x_2, y_1 and y_2 however we like, we require x_3 and y_3 to satisfy $y_3 - x_3 = b$, where $b = x_1 + x_2 - y_1 - y_2$, and $y_3 = cx_3$, where $c = (x_1x_2)/(y_1y_2)$.

If $c = 1$, then $x_3 = y_3$, and as before, we see that x_1 and x_2 coincide with y_1 and y_2 in either order. So assume that $c \neq 1$. Then $(c-1)x_3 = b$, so

$$x_3 = \frac{b}{c-1}, \qquad y_3 = \frac{bc}{c-1}.$$

If the chosen numbers are integers, then a sufficient (but not necessary) condition for x_3 and y_3 to be integers is $c = 1 + \frac{1}{k}$, where k is an integer. Also, x_3 and y_3 will be positive if $x_1 + x_2 > y_1 + y_2$ and $x_1 x_2 > y_1 y_2$, or if the opposite inequalities hold.

Example 3. Let $x_1 = 10$, $x_2 = 8$, $y_1 = 12$, $y_2 = 5$. Then $b = 1$ and $c = \frac{80}{60} = \frac{4}{3}$, so $x_3 = 3$ and $y_3 = 4$. The triples are $(10, 8, 3)$ and $(12, 5, 4)$.

Is it possible for such triples also to satisfy (2)? We can use our earlier work to show that is is not. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are given by (3) and (4). With a bit of algebra, we find that the condition $x_1 x_2 x_3 = y_1 y_2 y_3$ is equivalent to

$$a z_3 (z_2 - z_1) + b z_1 (z_2 - z_3) = ab(a+b).$$

If (2) is satisfied, then $\mathbf{z}$ is given by (6). With these values substituted, the left-hand side becomes

$$\lambda ab[\lambda(a+b) + \mu] - \lambda ab\mu = \lambda^2 ab(a+b).$$

Hence $\lambda = \pm 1$. As mentioned earlier, this implies that $\mathbf{y}$ is a permutation of $\mathbf{x}$.

Sums of cubes: 4-vectors

What happens if we demand that $\sum_{j=1}^{3} x_j^3 = \sum_{j=1}^{3} y_j^3$ in addition to (1) and (2)? With 3-vectors, there are no non-trivial solutions. This fact is not obvious, but it is a case of the following result proved in [1]: if $\mathbf{x}$ and $\mathbf{y}$ are aligned associates with $x_3 < y_3$, then $\sum_{j+1}^{3} f(x_j) < \sum_{j=1}^{3} f(y_j)$ for all functions f with strictly convex derivative f', so in particular for $f(x) = x^3$.

So we will try our luck with 4-vectors. We wish them to satisfy

$$\sum_{j=1}^{4} x_j = \sum_{j=1}^{4} y_j, \qquad \sum_{j=1}^{4} x_j^2 = \sum_{j=1}^{4} y_j^2, \qquad \sum_{j=1}^{4} x_j^3 = \sum_{j=1}^{4} y_j^3. \tag{11}$$

As before, the property is preserved if all x_j and y_j are mutliplied by λ or increased by c.

We will not attempt anything like a general solution. Instead, we will describe solutions that satisfy the extra condition $x_1 + x_2 = y_1 + y_2$ (hence also $x_3 + x_4 = y_3 + y_4$). This will be enough to deliver a plentiful supply of examples.

Let $z_j = \frac{1}{2}(x_j + y_j)$. Any pair $\mathbf{x}$, $\mathbf{y}$ satisfying the conditions can be expressed as follows:

$$x_1 = z_1 - a, \quad x_2 = z_2 + a, \quad x_3 = z_3 - b, \quad x_4 = z_4 + b, \tag{12}$$

$$y_1 = z_1 + a, \quad y_2 = z_2 - a, \quad y_3 = z_3 + b, \quad y_4 = z_4 - b \tag{13}$$

for some a, b (both non-zero for non-trivial solutions). Then

$$\sum_{j=1}^{4} y_j^2 - \sum_{j=1}^{4} x_j^2 = 4a(z_1 - z_2) + 4b(z_3 - z_4),$$

so

$$a(z_1 - z_2) = b(z_4 - z_3). \tag{14}$$

So if $z_1 = z_2$, then $z_3 = z_4$ and $\mathbf{y}$ is a permutation of $\mathbf{x}$. Assume that $z_1 \neq z_2$. Now $y_1^3 - x_1^3 = 6az_1^2 + 2a^3$, hence

$$y_1^3 - x_1^3 + y_2^3 - x_2^3 = 6a(z_1^2 - z_2^2),$$

$$y_3^3 - x_3^3 + y_4^3 - x_4^3 = 6b(z_3^2 - z_4^2).$$

So

$$a(z_1^2 - z_2^2) = b(z_4^2 - z_3^2).$$

With (14), this implies

$$z_1 + z_2 = z_3 + z_4. \tag{15}$$

So $\mathbf{z}$ has to satisfy (14) and (15). One could solve this pair of equations following the rules, but it is very easy to spot two solutions: $(-b, b, a, -a)$ and $(1, 1, 1, 1)$. Other solutions are linear combinations of these two: $\mathbf{z} = \lambda(-b, b, a, -a) + \mu(1, 1, 1, 1)$. Corresponding to each such $\mathbf{z}$, a pair $\mathbf{x}$, $\mathbf{y}$ is delivered by (12) and (13), However, it is easily checked that if $a = b$, or if $\lambda = 1$, then $\mathbf{y}$ is just a permutation of $\mathbf{x}$.

We illustrate this by working through the case $a = 1$, $b = 3$ (the reader might like to investigate the case $a = 1$, $b = 2$).

Example 4. Let $a = 1$, $b = 3$. Then $\mathbf{z} = \lambda'(-3, 3, 1, -1) + \mu'(1, 1, 1, 1)$ for some λ' and μ'. To arrange for non-negative $\mathbf{x}$ and $\mathbf{y}$, we modify this to $\lambda(0, 6, 4, 2) + \mu(1, 1, 1, 1)$, and choose μ to make the smallest x_j or y_j zero. (Alternatively, one could do without these modifications and adjust $\mathbf{x}$ and $\mathbf{y}$ afterwards.) Also, it is now natural to take half-integer values for λ. The results are set out in the following table.

λ	μ	$\mathbf{z}$	$\mathbf{x}$	$\mathbf{y}$
$\frac{1}{2}$	2	$(2, 5, 4, 3)$	$(1, 6, 1, 6)$	$(3, 4, 7, 0)$
$\frac{3}{2}$	1	$(1, 10, 7, 4)$	$(0, 11, 4, 7)$	$(2, 9, 10, 1)$
$\frac{5}{2}$	1	$(1, 16, 11, 6)$	$(0, 17, 8, 9)$	$(2, 15, 4, 3)$

One might now choose to rewrite these vectors in increasing order. Also, recall that any multiple of $(1, 1, 1, 1)$ can be added to $\mathbf{x}$ and $\mathbf{y}$. To reassure ourselves that the process has worked, note that in the first example $\sum_{j=1}^{4} x_j = 14$, $\sum_{j=1}^{4} x_j^2 = 74$ and $\sum_{j=1}^{4} x_j^3 = 434$, with the same values for $\mathbf{y}$.

A different perspective on these examples is given by considering $\sum_{j=1}^4 x_j^p - \sum_{j=1}^4 y_j^p$ as a function of p: denote it by $F(p)$ (now assuming that the x_j and y_j are all positive). We have ensured that $F(1) = F(2) = F(3) = 0$. Clearly, also $F(0) = 0$. It is shown in [2], by a generalisation of Descartes' rule of signs, that a function $F(p)$ of this kind can have at most four zeros. So $F(p)$ is non-zero for all other values of p, and alternates signs on the intervals between 0, 1, 2 and 3.

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References

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