Analogue Gravity in Laser-Driven Plasma

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February 2021

A thesis submitted to Lancaster University for the degree of Doctor of Philosophy in the Faculty of Science and Technology

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Abstract

This thesis investigates whether laser-driven plasma can be used as an analogue model of gravity in order to investigate Hawking radiation. An action describing laser-driven plasma is derived, and effective metrics are obtained in various regimes from the resulting field equations. Effective metrics exhibiting different behaviour are analysed by considering different forms of the fields. One of the effective metrics has the required properties for the analysis of Hawking radiation. It is shown that for a near-IR laser the Hawking temperature is about 4.5 K, which is small compared to typical plasma temperatures. However the waist of the laser is shown to have significant impact on the resulting Hawking temperature. As such it may be possible to obtain Hawking temperatures of several hundred Kelvin with a pulse width of a few μm. A new approach to investigating quantum fluctuations in an underdense laser-driven plasma is also presented that naturally emerges from the model underpinning the above studies. It is shown that the 1-loop effective action is expressible in terms of a massless field theory on a dilatonic curved background. Plane wave perturbations to the field equations are analysed for fields which are linear in Minkowski coordinates, and two dispersion relations are obtained. The impact on a Gaussian wave packet is calculated, suggesting it may be possible to experimentally verify this theory by utilising an x-ray laser.

Declaration

This thesis is my own work and no portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification at this or any other institute of learning. The word length of this thesis does not exceed the permitted maximum.

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Relevant publications by the author

Chapter 4

- "**Analogue Hawking temperature of a laser-driven plasma**"

  C. Fiedler and D. A. Burton, arXiv:2102.02556.

Chapter 5

- "**Quantum backreaction in laser-driven plasma**"

  A. Conroy, C. Fiedler, A. Noble, D. A. Burton, arXiv:1906.09606.

Chapter 1

Introduction

The theory of general relativity predicts that a large enough mass confined to a small enough volume will create a black hole; a region of spacetime from which nothing, not even light, can escape. The point of no return is called the event horizon. Observational evidence suggests that many galaxies have black holes at their centres [1], including our own Milky Way [2]. These objects can be completely characterised by their mass, charge and angular momentum. In 1975 Hawking [3] showed that black holes should radiate particles due to quantum effects near the horizon. The temperature of this radiation, called *Hawking temperature*, is inversely proportional to the mass of the black hole and the gravitational coupling constant. However the temperature of the radiation is very small compared to everyday temperatures. For example, for a black hole the mass of our Sun, the Hawking temperature is about 10^{-7} K. Considering that the cosmic microwave background has a temperature of about 2.7 K, detecting Hawking radiation would require precision not attainable with current technology. However, it has been discovered that black holes are not the only phenomena that can generate Hawking radiation. Certain systems can be described in terms of an effective metric, and if such a metric is equivalent to that of a black hole, the system should produce Hawking radiation. The field of research into such systems is called analogue gravity. It is thus feasible that experiments in a laboratory setting could realize the Hawking effect and potentially measure it. The aim of this thesis is to investigate whether laser-driven plasma has the necessary properties to probe this effect. Facilities such as ELI [4] and XFEL [5] are pushing the boundary in terms of attainable laser pulse intensities and power in the near-IR and x-ray spectrum respectively. Thus such systems might be promising candidates for the experimental realisation of Hawking radiation.

For the rest of this chapter, section 1.1 will give an overview of the analogue gravity topic. The rest of this thesis is organised as follows: chapter 2 will give an overview of the relevant mathematical concepts required to understand the presented work. Chapter 3 will describe the laser-driven plasma theory, deriving the two field equations used to extract effective metrics. In chapter 4 effective metrics will be derived and analysed assuming the forms of the fields. Section 4.1 will consider the field equations separately while section 4.2 will consider the full system of equations. In section 4.3 the full system will be considered again for the case of a varying spot size of the laser pulse, leading to an effective metric conformally related to the Schwarzschild metric, and it will be shown that the resulting Hawking temperature is dependant on the initial laser spot size and the initial laser intensity. Chapter 5 will present a new approach for investigating quantum effects in laser-driven plasma inspired by gravitational physics. The effects of the resulting dispersion relation will be investigated for the case of a Gaussian wave packet. Finally chapter 6 will give a summary of the presented work. Natural units will be assumed throughout, unless stated otherwise.

1.1 Analogue gravity overview

In 1981 Unruh showed that Hawking radiation can be probed experimentally through the use of an analogue model based on sound in fluid flow [16]. To do so, perturbed equations of motion for an irrotational fluid were shown to be equivalent to the equations for a massless scalar field in a spacetime that is conformal to Schwarzschild spacetime (see section 2.3.1 for mathematical detail). This publication gained little interest for several years, until this model was used to investigate whether there is a Planck scale cutoff of the Hawking radiation by taking into account the atomic nature of the fluid [17]. From that point the interest in analogue gravity grew, and various other models were discovered.

There are many physically different physical systems which allow for an analogue model of physics in curved spacetime:

- Surface waves in a shallow basin filled with liquid - the speed of the waves can be easily modified by changing the depth of the basin, and different effective metrics can be obtained by varying the shape of the basin [19]. Such models can be generalised to non-shallow water waves [20], however the analysis becomes significantly more complicated.

- Linear electrodynamics - formulating the Maxwell equations in a metric independent form and assuming a linear constitutive relationship between the electric and magnetic fields leads to an effective metric [21].

- Nonlinear electrodynamics - if the permittivity and permeability depend on the background electromagnetic field, the photon propagation is equivalent to that in curved spacetime [22].

- Dielectric media - propagation of photons in a dielectric medium characterised by permeability and permittivity tensors which are proportional to each other is equivalent to that in curved spacetime [23]. It is also possible to consider light propagating in dielectric fluids [24], requiring high refractive index to be experimentally viable.

- Bose–Einstein condensates (BEC) - perturbations of the phase of the condensate wave function leads to an effective metric [25] from a generalised nonlinear Schrödinger equation. The effective geometry depends on the state of the system, and various approximations yield different results.

- Accelerating plasma mirrors - x-ray pulses on solid plasma targets with a density gradient are analogous to the late time evolution of black hole Hawking evaporation [26].

This list is by no means exhaustive. There are many other systems that can be described by an effective geometry [18]. Since so many systems can be described by an effective metric, it is also possible to make analogies between them. For example, surface waves propagating against an external current in deep water are described by an equation that is equivalent to the Gross-Pitaevskii equation modelling the mean-field dynamics of BEC [27].

The very first experiment in the context of analogue gravity was performed in 2008. It demonstrated scattering of light waves at horizons in optical fibres [31]. In the same year, a similar experiment in which surface waves scattered at horizons in a water tank [32]. Shortly thereafter optical horizons were also realised in bulk crystals and sonic horizons in a BEC of ultra-cold atoms [33]. In 2011, the scattering of surface waves at horizons in a water tank was shown to be a thermal spectrum of emission [34].

These experiments provided evidence for the versatility of analogue gravity as well as the robustness of the Hawking effect, which describes the scattering of waves at horizons not only in gravitational physics, but also condensed matter systems. However, in these experiments, the input state was a classical probe and not the quantum vacuum, meaning the emission was stimulated and not a spontaneous scattering of quantum fluctuations. And so it was not possible to measure the entangled state of Hawking radiation and its infalling partner. In hopes of addressing this issue, atomic BECs have been studied extensively due to the low temperature of the fluid, which could lead to spontaneous emission by means of correlations between density patterns in the atomic population across the horizon [35, 36], with numerous proposed experiments [37–39]. The observation of spontaneous emission in entangled pairs has been reported in 2016 [40], and it was claimed that this emission spectrum was thermal [41], however these claims are disputed [42].

Meanwhile, optics experiments lead to observation of stimulated emission into waves of positive and negative frequency at optical horizons [43], and quantum tunnelling of waves across horizon [44]. A series of experiments based on a rotating geometry similar to the Kerr black hole were performed, observing rotational superradiance at the ergosurface [45]. This also led to the study of effects of vorticity and dispersion on fluid flows [46] as well as to the observation of classical back reaction of water waves on a vortex flow [47].

This is a small selection of experiments, and many more are planned for the future. Water, optics and BEC have been the main focus of experimental realisation of analogue gravity, however, a variety of other systems have also been proposed as potential candidates. This thesis aims to show that laser-driven plasma systems have the necessary qualities to consider it as a model for analogue gravity.

Chapter 2

Mathematical background

This chapter discusses mathematical concepts which will be utilised in subsequent chapters. The derivations presented are brief as the results are of importance, and the detail can be found in the provided references. Section 2.1 will describe the necessary definitions in general relativity, section 2.2 will discuss the relevant quantum field theory, and section 2.3 will demonstrate the derivation of two effective metrics.

2.1 Preliminary aspects

The volume form will be defined as

$$\star 1 = \sqrt{-\det g_4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{2.1}$$

throughout, where $\star$ is the Hodge star operator and $\det g_4$ is the determinant of a given 4-dimensional metric tensor. The inner product of two 1-forms α and β on a metric g is given by $\alpha \cdot \beta = g^{-1}(\alpha, \beta)$, and can also be defined for any p-forms γ and δ as $\gamma \cdot \delta = \star^{-1}(\gamma \wedge \star\delta)$. More than one metric will feature in the following calculations, and the $\alpha \cdot \beta$ notation will always be used for the background metric. A 1-form α is timelike, lightlike, or spacelike when $\alpha \cdot \alpha$ is less than zero, zero, or greater than zero respectively. It is useful to define a Hodge operator $\#$

$$\#1 = \sqrt{-\det g_2}\, dx^0 \wedge dx^1, \tag{2.2}$$

to distinguish between 2- and 4-dimensional systems. The flat 4-dimensional Minkowski metric with signature $(-,+,+,+)$ is given by

$$g_M = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz, \tag{2.3}$$

where x, y and z are Cartesian coordinates in 3-dimensional Euclidean space and t is time, and it will be assumed as the background metric throughout this thesis. For 2-dimensional systems it will be assumed as

$$g_M = -dt \otimes dt + dz \otimes dz. \tag{2.4}$$

The magnitude of the proper acceleration $|\mathcal{A}|$, calculated with respect to the metric g_M of a curve $(t(\eta), z(\eta))$, where η is a parameter, is given by

$$\begin{aligned} |\mathcal{A}|^2 =& \frac{1}{\dot{t}^2 - \dot{z}^2}\left[\left(\frac{d}{d\eta}\left(\frac{\dot{z}}{\sqrt{\dot{t}^2 - \dot{z}^2}}\right)\right)^2 - \left(\frac{d}{d\eta}\left(\frac{\dot{t}}{\sqrt{\dot{t}^2 - \dot{z}^2}}\right)\right)^2\right] \\ =& \frac{(\ddot{z}\dot{t} - \ddot{t}\dot{z})^2}{(\dot{t} - \dot{z})^3}, \end{aligned} \tag{2.5}$$

where the dots denote derivatives with respect to η. A metric is conformally flat if it can be written as $\Omega^2 g_M$, where Ω is a real function of the coordinates. The light-cone coordinates will be defined as $u = z - t$ and $v = z + t$. The Riemann curvature tensor can be written as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.6}$$

where $\Gamma^{\rho}_{\mu\nu}$ are the Christoffel symbols given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right). \tag{2.7}$$

The Ricci curvature tensor is defined as $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$, furthermore $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci curvature scalar. Another curvature scalar is the Kretschmann scalar, defined as $K = R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}$. Both scalars are zero for g_M, however they are not necessarily equal in general. A singularity in any of the scalar invariants corresponds to a physical singularity. The Schwarzschild metric is given by

$$g_S = -\left(1 - \frac{2GM}{r}\right) dt \otimes dt + \left(1 - \frac{2GM}{r}\right)^{-1} dr \otimes dr + r^2 d\Omega^2, \tag{2.8}$$

where r is the radial coordinate centred about a body of mass M and $d\Omega^2$ is the angular contribution. For this metric the Ricci tensor is zero; hence the Ricci scalar is also zero, however the Kretschmann scalar evaluates to $K_S = \frac{48G^2M^2}{r^6}$. The components of g_S in equation (2.8) exhibit two singularities at $r = r_S = 2GM$ and $r = 0$. However the singularity at r_S is a coordinate singularity, and the metric components can be made regular at this point with a different set of coordinates, for example Kruskal-Szekeres coordinates. The point $r = 0$ however is a physical singularity, as the Kretschmann scalar is also singular at that point. If the mass is contained within the Schwarzschild radius r_S, then this metric describes a static, uncharged, non-rotating black hole of mass M centred at the origin with a horizon at r_S.

2.2 Quantum field theory in curved spacetime

Consider a free scalar field ϕ satisfying the massless wave equation

$$d \star d\phi = 0. \tag{2.9}$$

A scalar product can be defined as

$$(f_1, f_2) = -i \int_\Sigma (f_1 \star df_2^* - f_2 \star df_1^*), \tag{2.10}$$

where the superscript $*$ denotes complex conjugation and Σ is a Cauchy hypersurface. For example, two solutions to the massless wave equation (2.9) in n-dimensional Minkowski spacetime with metric g_M are proportional to $e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}$ and $e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega t}$, where $\omega = |\boldsymbol{k}|$. The modes proportional to $e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}$ are called positive frequency because they are eigenfunctions of the ∂_t operator with eigenvalue $-i\omega$. Conversely the solution $e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega t}$ corresponds to negative frequency modes. Requiring $(u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}) = \delta_{\boldsymbol{k}\boldsymbol{k}'}$ leads to normalised positive frequency modes

$$u_{\boldsymbol{k}} = \frac{1}{\sqrt{2\omega(2\pi)^{n-1}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}, \tag{2.11}$$

where Σ in equation (2.10) is a hypersurface of constant t. A classical field is quantized by treating it as an operator and imposing equal time commutation relations:

$$\begin{aligned} [\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{x}')] =& 0, \\ [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{x}')] =& 0, \\ [\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{x}')] =& i\delta(\boldsymbol{x} - \boldsymbol{x}'), \end{aligned} \tag{2.12}$$

where π is the conjugate variable to ϕ. The normalised modes and their complex conjugates form a complete orthonormal basis with the scalar product (2.10). Supposing that $\boldsymbol{k}$ is discrete, ϕ can be written as

$$\phi = \sum_k (u_{\boldsymbol{k}} a_{\boldsymbol{k}} + u^*_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}}). \tag{2.13}$$

The $a^\dagger_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}$ operators are called creation and annihilation operators respectively. The equal time commutation relations given in (2.12) are equivalent to

$$\begin{aligned} [a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}] =& 0, \\ [a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] =& 0, \\ [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] =& \delta_{\boldsymbol{k}\boldsymbol{k}'}. \end{aligned} \tag{2.14}$$

A vacuum state $|0\rangle$ is defined by $a_{\boldsymbol{k}} |0\rangle = 0$ for all $\boldsymbol{k}$.

2.2.1 Unruh Effect

When an observer is accelerating in Minkowski spacetime, there will be a detectable thermal spectrum of particle excitations in the vacuum state $|0\rangle$ introduced above; this is called the Unruh effect. In this section the calculation of this effect will be performed. This calculation is compatible with 4-dimensional spacetime, however the results will be used in the context of 2-dimensional spacetime in sections 4.2.1 and 4.1.3, as such only the 2-dimensional case is presented.

Consider a 2-dimensional Minkowski space with the metric given in equation (2.4). The wave equation (2.9) has orthonormal positive frequency modes as shown in the example above given by

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ikz-i\omega t}, \tag{2.15}$$

where in this subsection the bar will indicate quantities found in Minkowski space. Now consider the two sets of coordinate transformations:

$$\begin{aligned} t =& \frac{1}{a} e^{a\xi_R} \sinh(a\eta_R) \\ z =& \frac{1}{a} e^{a\xi_R} \cosh(a\eta_R), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} t =& -\frac{1}{a} e^{a\xi_L} \sinh(a\eta_L) \\ z =& -\frac{1}{a} e^{a\xi_L} \cosh(a\eta_L), \end{aligned} \tag{2.17}$$

where a is a constant. The two pairs of coordinates (η_R, ξ_R) and (η_L, ξ_L) are known as Rindler coordinates; when ξ is constant they represent uniformly accelerating observers in Minkowski space. Both sets of Rindler coordinates cover regions of spacetime where $|z| > |t|$, known as Rindler wedges. The first pair of coordinates η_R, ξ_R covers the region where $z > 0$, whilst the second pair of coordinates η_L, ξ_L covers the region where $z < 0$. They both lead to

$$g = e^{2a\xi_{L/R}}(d\xi_{L/R} \otimes d\xi_{L/R} - d\eta_{L/R} \otimes d\eta_{L/R}). \tag{2.18}$$

Solving the wave equation (2.9) the following modes are obtained:

$$^{R}u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi_R - i\omega\eta_R} & \text{in } R \\ 0 & \text{in } L \end{cases} \tag{2.19}$$

$$^{L}u_k = \begin{cases} 0 & \text{in } R \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi_L + i\omega\eta_L} & \text{in } L \end{cases} \tag{2.20}$$

with $\omega = |k|$. Note that these are positive frequency modes with respect to ∂_{η_R} for $^{R}u_k$ and $-\partial_{\eta_L}$ for $^{L}u_k$. These modes cover Minkowski space and both sets can be used to quantize ϕ:

$$\phi = \sum_k \left(\bar{u}_k a_k + \bar{u}_k^* a_k^\dagger \right), \tag{2.21}$$

$$\phi = \sum_k \left({^{R}u_k} {^{R}b_k} + {^{R}u_k^*} {^{R}b_k^\dagger} + {^{L}u_k} {^{L}b_k} + {^{L}u_k^*} {^{L}b_k^\dagger} \right), \tag{2.22}$$

with a_k and b_k being annihilation operators. The operators a_k, $^{R}b_k$ and $^{L}b_k$ lead to two vacuum states[1] $|0_M\rangle$ and $|0_R\rangle$, corresponding to Minkowski and

[1] The subscript on $|0_R\rangle$ stands for "Rindler", whereas the superscript on $^{R}b_k$ stands for "right".

Rindler spaces respectively. The states $|0_M\rangle$ and $|0_R\rangle$ satisfy $a_k\,|0_M\rangle = 0$ and ${}^R b_k\,|0_R\rangle = {}^L b_k\,|0_R\rangle = 0$ respectively. Bogolubov transformations between the two sets of modes are commonly used to show that the vacuum state $|0_M\rangle$ represents a thermal bath of particles according to the accelerated observer. However a more elegant method due to Unruh [6] is to use

$$ {}^1U_k = {}^R u_k + e^{-\frac{\pi\omega}{a}}\,{}^L u^*_{-k}, \tag{2.23}$$

$$ {}^2U_k = {}^R u^*_{-k} + e^{\frac{\pi\omega}{a}}\,{}^L u_k. \tag{2.24}$$

It can be shown that ${}^1U_k \propto (z-t)^{\frac{i\omega}{a}}$ and ${}^2U_k \propto (z+t)^{\frac{i\omega}{a}}$ when $k > 0$. Likewise ${}^1U_k \propto (z+t)^{-\frac{i\omega}{a}}$ and ${}^2U_k \propto (z-t)^{-\frac{i\omega}{a}}$ when $k < 0$. The constants of proportionality are $\frac{a^{\pm\frac{i\omega}{a}}}{\sqrt{4\pi\omega}}$, $\pm$ corresponding to the same sign as the exponent of $z+t$ or $z-t$. Both ${}^1U_k$ and ${}^2U_k$ are analytic and share the positive frequency properties of the Minkowski modes. They must also share a common vacuum state, namely $|0_M\rangle$. The normalization factor of ${}^1U_k$ is $\big(2e^{-\frac{\pi\omega}{a}}\sinh\big(\frac{\pi\omega}{a}\big)\big)^{-\frac{1}{2}}$, and for ${}^2U_k$ it is $\big(2e^{\frac{\pi\omega}{a}}\sinh\big(\frac{\pi\omega}{a}\big)\big)^{-\frac{1}{2}}$, hence the field ϕ can be expressed as

$$ \phi = \sum_k \left(2\sinh\left(\frac{\pi\omega}{a}\right)\right)^{-\frac{1}{2}} \left[{}^1c_k\big(e^{\frac{\pi\omega}{2a}}\,{}^R u_k + e^{-\frac{\pi\omega}{2a}}\,{}^L u^*_{-k}\big) + {}^2c_k\big(e^{\frac{-\pi\omega}{2a}}\,{}^R u^*_{-k} + e^{\frac{\pi\omega}{2a}}\,{}^L u_k\big)\right] + \text{h.c.} \tag{2.25}$$

where now ${}^1c_k\,|0_M\rangle = 0$ and ${}^2c_k\,|0_M\rangle = 0$. Taking inner products $(\phi, {}^R u_k)$ and $(\phi, {}^L u_k)$ with ϕ given by (2.22) and (2.25) results in

$$ {}^R b_k = \left(2\sinh\left(\frac{\pi\omega}{a}\right)\right)^{-\frac{1}{2}} \left(e^{\frac{\pi\omega}{2a}}\,{}^1c_k + e^{-\frac{\pi\omega}{2a}}\,{}^2c^\dagger_k\right), \tag{2.26}$$

$$
{}^{L}b_k = \left(2\sinh\left(\frac{\pi\omega}{a}\right)\right)^{-\frac{1}{2}} \left(e^{\frac{\pi\omega}{2a}}\,{}^{2}c_k + e^{-\frac{\pi\omega}{2a}}\,{}^{1}c_k^{\dagger}\right). \tag{2.27}
$$

Thus it may be deduced that a Rindler observer will detect

$$
\langle 0_M|^{L}\, b_k^{\dagger}\,{}^{L}b_k\,|0_M\rangle = \langle 0_M|^{R}\, b_k^{\dagger}\,{}^{R}b_k\,|0_M\rangle = \frac{e^{-\frac{\pi\omega}{a}}}{2\sinh\left(\frac{\pi\omega}{a}\right)} = \frac{1}{e^{\frac{2\pi\omega}{a}}-1} \tag{2.28}
$$

particles in mode k. This is the Planck spectrum for radiation at temperature

$$
T = \frac{a}{2\pi}. \tag{2.29}
$$

2.2.2 Hawking radiation

Hawking radiation [3] is the thermal radiation predicted to be spontaneously emitted by black holes, thereby reducing their mass. Intuitively this radiation arises when a pair of virtual photons, one with positive energy and one with negative energy, are created due to quantum vacuum fluctuations just outside the event horizon. The negative energy photon then crosses the event horizon, while the positive one escapes to infinity, constituting a part of the Hawking radiation. This effect is widely discussed in literature [7, 9], as such the derivation will be omitted. The main result is that the temperature of this radiation is given by

$$
T_H = \frac{\kappa}{2\pi}, \tag{2.30}
$$

where κ is the surface gravity of the black hole, which for a Schwarzschild black hole (metric in equation (2.8)) is $\kappa = (4GM)^{-1}$. This temperature is conformally invariant [10], which will be exploited in section 4.3.

2.2.3 Effective action for a dilaton in two dimensions

This section describes the renormalization and variation of a one-loop effective action arising from a massless scalar field theory on a background 2-dimensional dilatonic spacetime, following the calculation performed in Ref. [11]. The results will be exploited in chapter 5. The one-loop effective action W given by the path integral [12]

$$\exp\{iW[g,\phi,\psi]\} = \int \mathcal{D}\tilde{f} \exp\left\{\frac{1}{2}i\int_{\mathcal{M}} e^{-2\phi} d\tilde{f}\cdot d\tilde{f}\#1\right\} \tag{2.31}$$

describes the coupling of a dilaton ϕ to some Lorentzian metric $g_{\mu\nu}$ and their self-couplings, due to the vacuum fluctuations of a massless scalar field $\tilde{f}$. The properties of the measure $\mathcal{D}\tilde{f}$ are fixed [11] by introducing $\tilde{f} = e^{\psi}f$, for some scalar field ψ, and taking $\mathcal{D}\tilde{f}$ to be the standard measure for f, not $\tilde{f}$ [11]. This is equivalent to requiring the quantity $\int \mathcal{D}\tilde{f} e^{i\langle\tilde{f},\tilde{f}\rangle}$ to be a field-independent constant, where the inner product $\langle\cdot,\cdot\rangle$ is given by $\langle\mathfrak{a},\mathfrak{b}\rangle = \int_{\mathcal{M}} e^{-2\psi}\mathfrak{a}^*\mathfrak{b}\#1$ and $*$ denotes complex conjugate. The integral in the exponent on the right-hand side of equation (2.31) can be written as

$$\int_{\mathcal{M}} e^{-2\phi} d\tilde{f}\wedge\#d\tilde{f} = -\int_{\mathcal{M}} \tilde{f} d(e^{-2\phi}\#d\tilde{f}) = -\int_{\mathcal{M}} \tilde{f}\#^{-1}d(e^{-2\phi}\#d\tilde{f})\#1. \tag{2.32}$$

This integrand can be written as fAf, where the operator A is given by

$$\begin{aligned} Af =& -e^{\psi}\#^{-1}d[e^{-2\phi}\#d(e^{\psi}f)] \\ =& -e^{2\psi-2\phi}[\#^{-1}d\#df + 2(d\psi - d\phi)\cdot df \\ &+ (d\psi\cdot d\psi - 2d\psi\cdot d\phi + \#^{-1}d\#d\psi)f], \end{aligned} \tag{2.33}$$

or equivalently

$$Af = -e^{2\psi-2\phi}g^{\mu\nu}(\nabla_\mu\nabla_\nu f+2(\nabla_\mu\psi-\nabla_\mu\phi)\nabla_\nu f+f\nabla_\mu\nabla_\nu\psi-2f\nabla_\mu\phi\nabla_\nu\psi+f\nabla_\mu\psi\nabla_\nu\psi), \tag{2.34}$$

where ∇_μ is the covariant derivative[2]. Introducing subscript E to denote quantities defined on a Riemannian metric, by definition $e^{-W_E} = \mathrm{Det}(A_E)^{-\frac{1}{2}}$, where Det denotes a functional determinant, hence

$$W_E = \frac{1}{2}\,\mathrm{Tr}_E \ln A_E, \tag{2.35}$$

where Tr_E is the functional trace [7, 8]. In zeta-function regularization W_E can be expressed as

$$W_E = -\frac{1}{2}\zeta'_{A_E}(0), \tag{2.36}$$

where $\zeta_{A_E}(s) = \mathrm{Tr}_E(A_E^{-s})$ and prime denotes derivative with respect to s. Mapping (2.36) to the Lorentzian domain gives

$$W = -\frac{1}{2}\zeta'_A(0). \tag{2.37}$$

By introducing an infinitesimal Weyl transformation $\delta g_{\mu\nu} = \delta k\, g_{\mu\nu}$ (therefore $\delta g^{\mu\nu} = -\delta k\, g^{\mu\nu}$), for some infinitesimal scalar field δk, the energy-momentum tensor $T_{\mu\nu}$ by definition satisfies [13]

$$\delta_g W = \frac{1}{2}\int_{\mathcal{M}} \delta g^{\mu\nu} T_{\mu\nu} \#1 = -\frac{1}{2}\int_{\mathcal{M}} \delta k\, T^\mu_\mu \#1. \tag{2.38}$$

[2]Since $\#^{-1}d\#d\alpha = g^{\mu\nu}\nabla_\mu\nabla_\nu\alpha$ and $d\alpha\cdot d\beta = g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\beta$ for any 0-forms α and β.

Noting that $\delta_g \Gamma^\alpha_{\mu\nu} = \frac{1}{2}(\partial_\mu \delta k \, \delta^\alpha_\nu + \partial_\nu \delta k \, \delta^\alpha_\mu - g^{\alpha\beta} g_{\mu\nu} \partial_\beta \delta k)$ gives $g^{\mu\nu} \delta_g \Gamma^\alpha_{\mu\nu} = 0$, and hence inspecting (2.34) yields $\delta_g(Af) = -\delta k(Af)$. Furthermore defining $\zeta(s|\delta k, A) = \mathrm{Tr}(\delta k \, A^{-s})$ results in

$$\zeta(0|\delta k, A) = \int_{\mathcal{M}} \delta k T^\mu_\mu \# 1. \tag{2.39}$$

It can be shown that $\zeta(0|\delta k, A) = a_2(\delta k, A)$ up to a divergent part [14], where a_2 is a coefficient in the asymptotic expansion of the heat kernel $\mathrm{Tr}\big(\delta k \, e^{-At}\big) = \sum_{n=0} t^{n-2} a_n$. It is not straightforward to compute this expansion for an arbitrary A, however Ref. [14] provides the result for operators of the form $P = -(\hat{g}^{\mu\nu} D_\mu D_\nu + E)$, for some metric $\hat{g}_{\mu\nu}$, connection D_μ and scalar field E:

$$a_2(\delta k, A) = \frac{1}{24\pi} \,\mathrm{tr} \int \delta k (\hat{R} + 6E) \# 1, \tag{2.40}$$

where $\hat{R}$ is the Ricci scalar obtained from $\hat{g}_{\mu\nu}$. Identifying $\hat{g}^{\mu\nu}$ as $\hat{g}^{\mu\nu} = e^{2\psi - 2\phi} g^{\mu\nu}$ and introducing the connection $D_\mu = \hat{\nabla}_\mu + \omega_\mu$ with $\omega_\mu = \hat{\nabla}_\mu \psi - \hat{\nabla}_\mu \phi$, where $\hat{\nabla}_\mu$ is the covariant derivative with respect to $\hat{g}_{\mu\nu}$, leads to the conclusion that the operator A is of the required form, where $E = \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \phi - \hat{g}^{\mu\nu} \hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi$. Written in the original metric in conjunction with (2.39) this yields [15]

$$T^\mu_\mu = \frac{1}{24\pi}(R - 6(\nabla\phi)^2 + 4\Box\phi + 2\Box\psi), \tag{2.41}$$

where R is the Ricci scalar, $\Box$ is the d'Alembert operator and $(\nabla\phi)^2 = g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$. The effective action can be computed from its functional derivatives with respect to ρ, ϕ and ψ, where ρ is introduced by $g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu}$, where

$\eta_{\mu\nu}$ is a flat metric. All metrics on 2-dimensional manifolds are conformally flat, so the previous expressions for $g_{\mu\nu}$ is general. The variation of the metric becomes $\delta g_{\mu\nu} = 2\delta\rho\, g_{\mu\nu}$, and thus $\delta_\rho W = -\int_{\mathcal{M}} \delta\rho T^\mu_\mu \#1$, leading to

$$\frac{\delta W}{\delta\rho} = -\frac{1}{12\pi}\sqrt{-\eta}(-\Delta\rho + \Delta\psi + 2\Delta\phi - 3(\partial\phi)^2), \tag{2.42}$$

where Δ and ∂_μ are the d'Alembert operator and covariant derivative with respect to $\eta_{\mu\nu}$ respectively, and $(\partial\phi)^2 = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. To obtain the remaining functional derivatives, firstly note that

$$\frac{\delta^2 W}{\delta\phi(x)\delta\rho(x')} = -\frac{1}{6\pi}\sqrt{-\eta}\big(\Delta\delta(x-x') + 3\eta^{\mu\nu}\partial_\nu(\delta(x-x')\partial_\mu\phi(x))\big), \tag{2.43}$$

follows from (2.42). By definition $\delta_\rho \frac{\delta W}{\delta\phi(x)} = \int_{\mathcal{M}} d^2x' \sqrt{-\eta}\frac{\delta^2 W}{\delta\phi(x)\delta\rho(x')}\delta\rho(x')$, and utilising integration by parts and Stokes' theorem gives

$$\delta_\rho \frac{\delta W}{\delta\phi} = -\frac{1}{6\pi}\sqrt{-\eta}\left(\Delta\delta\rho + 3\eta^{\mu\nu}\partial_\nu(\delta\rho\partial_\mu\phi)\right), \tag{2.44}$$

thus

$$\frac{\delta W}{\delta\phi} = -\frac{1}{6\pi}\sqrt{-\eta}(\Delta\rho + 3\eta^{\mu\nu}\partial_\nu(\rho\partial_\mu\phi)) + \left.\frac{\delta W}{\delta\phi}\right|_{\rho=0}. \tag{2.45}$$

With a similar approach,

$$\frac{\delta W}{\delta\psi} = -\frac{1}{12\pi}\sqrt{-\eta}\Delta\rho + \left.\frac{\delta W}{\delta\psi}\right|_{\rho=0} \tag{2.46}$$

can be obtained. Calculating $\left.\frac{\delta W}{\delta\psi}\right|_{\rho=0}$ and $\left.\frac{\delta W}{\delta\phi}\right|_{\rho=0}$ is not trivial, since neither $\delta_\phi A$ or $\delta_\psi A$ are proportional to A. To evaluate these, Ref. [11] proposes upgrading f to a two-component spinor $\vec{f}$ by

$$e^{iW|_{\rho=0}} = \frac{1}{4}\int \mathcal{D}\vec{f}\exp\{i\int_{\mathcal{M}} \vec{f}^{\dagger}A\vec{f}\#1\}. \tag{2.47}$$

Introducing $D\vec{f} = i\gamma^{\mu}e^{\psi}\partial_{\mu}(e^{-\phi}\vec{f})$ and $D^{\dagger}\vec{f} = i\gamma^{\mu}e^{-\phi}\partial_{\mu}(e^{\psi}\vec{f})$, where γ^{μ} are the 2-dimensional Dirac matrices with respect to $\eta^{\mu\nu}$, it can be shown that in flat spacetime

$$\int_{\mathcal{M}} \vec{f}^{\dagger}A\vec{f}\#1 = \int_{\mathcal{M}} \vec{f}^{\dagger}DD^{\dagger}\vec{f}\#1. \tag{2.48}$$

The zeta-function corresponding to $DD^{\dagger}$ can be written as [14]

$$\zeta_{DD^{\dagger}}(s) = \mathrm{Tr}\big((DD^{\dagger})^{-s}\big) = \frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1}\,\mathrm{Tr}\exp\big(-tDD^{\dagger}\big), \tag{2.49}$$

where $\Gamma(s) = \int_0^{\infty} dt\, t^{s-1}e^{-t}$, and the variations with respect to ϕ and ψ become [11]

$$\delta\zeta_{DD^{\dagger}}(s) = -2s\,\mathrm{Tr}\big((DD^{\dagger})^{-s}\delta\psi - (D^{\dagger}D)^{-s}\delta\phi\big). \tag{2.50}$$

Noting that $W|_{\rho=0} = -\frac{1}{4}\zeta'_{DD^{\dagger}}(0)$ yields

$$\begin{aligned}\delta(W|_{\rho=0}) =& \frac{1}{2}(\zeta(0|\delta\psi, DD^{\dagger}) - \zeta(0|\delta\phi, D^{\dagger}D)) \\ =& \frac{1}{2}(a_2(\delta\psi, DD^{\dagger}) - a_2(\delta\phi, D^{\dagger}D)).\end{aligned} \tag{2.51}$$

Introducing $\mathcal{D}_{\mu} = \partial_{\mu} + \partial_{\mu}\psi - \partial_{\mu}\phi - \gamma^5\eta_{\mu\omega}\epsilon^{\nu\omega}\partial_{\nu}\phi$ and $E = \hat{g}^{\mu\nu}\hat{\nabla}_{\mu}\hat{\nabla}_{\nu}\phi$, where $\gamma^5\epsilon^{\mu\nu} = \frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}]$, allows $DD^{\dagger}$ to be written in the form of $-(\hat{g}^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu} + E)$. Thus both terms can be evaluated by noting that $D^{\dagger}D$ is obtained from $DD^{\dagger}$

by $\phi \mapsto -\psi$ and $\psi \mapsto -\phi$, hence [11]

$$\delta(W|_{\rho=0}) = \frac{1}{12\pi} \int_{\mathcal{M}} ((\Delta\phi + 2\Delta\psi)\delta\phi + (\Delta\psi + 2\Delta\phi)\delta\psi)\#1. \tag{2.52}$$

The variations of W are now obtained:

$$\begin{aligned} \frac{\delta W}{\delta \phi} &= -\frac{1}{12\pi}\sqrt{-\eta}(2\Delta\rho + 6\partial^{\mu}(\rho\partial_{\mu}\phi) - \Delta\phi - 2\Delta\psi), \\ \frac{\delta W}{\delta \psi} &= -\frac{1}{12\pi}\sqrt{-\eta}(\Delta\rho - \Delta\psi - 2\Delta\phi). \end{aligned} \tag{2.53}$$

These together with (2.42) result in

$$\begin{aligned} W = -\frac{1}{24\pi} \int_{\mathcal{M}} \big(& -\rho\Delta\rho + 2\psi\Delta\rho - \psi\Delta\psi - 6\rho(\partial\phi)^2 + 4\phi\Delta\rho \\ & - 4\phi\Delta\psi - \phi\Delta\phi\big)\#1, \end{aligned} \tag{2.54}$$

which written in terms of the original metric $g_{\mu\nu}$ becomes

$$\begin{aligned} W = -\frac{1}{24\pi} \int_{\mathcal{M}} \Big(& -\frac{1}{4}R\Box^{-1}R + 3(\nabla\phi)^2\Box^{-1}R - R(\psi + 2\phi) \\ & + (\nabla\psi)^2 + (\nabla\phi)^2 + 4(\nabla^{\mu}\psi)(\nabla_{\mu}\phi)\Big)\#1, \end{aligned} \tag{2.55}$$

after integration by parts has been used. The renormalization term $\mathcal{W}(\mu, \mu') = \zeta_A(0)\ln\mu + \frac{1}{2}\zeta_{DD^{\dagger}}(0)\ln\mu'$ associated with the zeta-function regularization methodology must be added [11] to (2.55). It can be shown [11] that $\zeta_{DD^{\dagger}}(0)$ is a boundary term and as such does not contribute to the field equations. Thus a renormalized effective action for a theory in two dimensions is given by

$$
\begin{aligned}
W = &- \frac{1}{24\pi} \int_{\mathcal{M}} \Big(- \frac{1}{4} R \Box^{-1} R + 3(\nabla\phi)^2 \Box^{-1} R - R(\psi + 2\phi) \\
&+ (\nabla\psi)^2 + (\nabla\phi)^2 + 4(\nabla^\mu \psi)(\nabla_\mu \phi) \Big) \# 1 - \frac{1}{4\pi} \ln(\mu) \int_{\mathcal{M}} (\nabla\phi)^2 \# 1,
\end{aligned}
\tag{2.56}
$$

where the boundary terms have been dropped. This result will be used in chapter 5, in which the natural inner product is given by $\int d^2x \sqrt{-\eta}\, \mathfrak{a}^* \mathfrak{b}$. As such ψ can be set to zero, allowing for a definition of $w = W|_{\psi=0}$:

$$
\begin{aligned}
w[g, \phi, \mu] =& \frac{1}{24\pi} \int_{\mathcal{M}} \Big(\frac{1}{4} R \Box^{-1} R + 2R\phi - 3(\nabla\phi)^2 \Box^{-1} R) \# 1 \\
&- \frac{1}{24\pi} (1 + 6 \ln \mu) \int_{\mathcal{M}} (\nabla\phi)^2 \# 1.
\end{aligned}
\tag{2.57}
$$

2.3 Analogue gravity

The notion of an effective metric is a key component of analogue gravity. Section 2.3.1 will show the derivation of an effective metric from equations governing a perfect fluid, in similar fashion to the first effective metric found by Unruh [16]. Section 2.3.2 will show that effective metrics arise naturally from the linearisation process for any scalar field governed by a Lagrangian which depends only on the field and its first derivatives.

2.3.1 Acoustic metric

To derive an effective acoustic metric g_{eff} for a perfect fluid of energy density $\varrho(n)$ and unit-normalized 4-velocity $\tilde{V}$, n being the proper density, local balance of energy-momentum and particle number conservation can be used:

$$\iota_V d\left(\frac{d\varrho}{dn}\tilde{V}\right) = 0, \tag{2.58}$$

$$d \star (n\tilde{V}) = 0. \tag{2.59}$$

Here tilde denotes metric dual with respect to the spacetime metric g. Note that equation (2.58) is satisfied when $\frac{d\varrho}{dn}\tilde{V} = d\phi$, leading to the relations

$$d\phi \cdot d\phi = -\left(\frac{d\varrho}{dn}\right)^2, \tag{2.60}$$

$$d \star \left(n\frac{1}{\frac{d\varrho}{dn}}d\phi\right) = 0, \tag{2.61}$$

where $\cdot$ denotes metric product given by g. Next, consider a perturbation to the system such that $\phi = \phi_0 + \epsilon\phi_1 + \mathcal{O}(\epsilon^2)$, $n = n_0 + \epsilon n_1 + \mathcal{O}(\epsilon^2)$ and

$$\frac{n}{\frac{d\varrho}{dn}} = \left(\frac{n}{\frac{d\varrho}{dn}}\right)\Bigg|_{n=n_0} + \epsilon\frac{d}{dn}\left(\frac{n}{\frac{d\varrho}{dn}}\right)\Bigg|_{n=n_0} n_1 + \mathcal{O}(\epsilon^2). \tag{2.62}$$

The zeroth order in ϵ returns unperturbed equations (2.60) and (2.61), and the first order from equation (2.60) gives

$$d\phi_0 \cdot d\phi_1 = -\left.\frac{d\varrho}{dn}\frac{d^2\varrho}{dn^2}\right|_{n=n_0} n_1, \tag{2.63}$$

from which an equation for n_1 can be obtained. This can be combined with the first order in ϵ of equation (2.61), which results in

$$d \star (\alpha d\phi_0 \cdot d\phi_1 \, d\phi_0 + \beta d\phi_1) = 0, \tag{2.64}$$

where

$$\alpha = -\left.\frac{1}{\frac{d\varrho}{dn}\frac{d^2\varrho}{dn^2}}\frac{d}{dn}\left(\frac{n}{\frac{d\varrho}{dn}}\right)\right|_{n=n_0}, \tag{2.65}$$

$$\beta = \left.\left(\frac{n}{\frac{d\varrho}{dn}}\right)\right|_{n=n_0}. \tag{2.66}$$

Requiring

$$\kappa_0 \star_{\text{eff}} d\phi_1 = \star(\alpha d\phi_0 \cdot d\phi_1\, d\phi_0 + \beta d\phi_1), \tag{2.67}$$

where κ_0 is a dimensionful constant such that the dimensions of the spacetime metric and the effective metric are the same, and $\star_{\text{eff}}$ is the Hodge map resulting from the effective metric g_{eff}, leads to an expression for the effective metric g_{eff}. Firstly introduce a 0-form γ such that

$$\star_{\text{eff}} 1 = \gamma \star 1, \tag{2.68}$$

$$\star_{\text{eff}} d\phi_1 = \iota_{g_{\text{eff}}^{-1}(d\phi_1,-)}\gamma \star 1 = \gamma \star [g_{\text{eff}}^{-1}(d\phi_1,-)]^{\flat}, \tag{2.69}$$

where $\flat$ denotes metric dual with respect to g. With this equation (2.67) yields

$$\begin{aligned} \kappa_0\gamma g_{\text{eff}}^{-1}(d\phi_1,-) &= \alpha d\phi_0 \cdot d\phi_1 \widetilde{d\phi}_0 + \beta\widetilde{d\phi}_1 \\ &= (\alpha\widetilde{d\phi}_0 \otimes \widetilde{d\phi}_0 + \beta g^{-1})(d\phi_1,-), \end{aligned} \tag{2.70}$$

thus obtaining the expression

$$\kappa_0\gamma g_{\text{eff}}^{-1} = \alpha\widetilde{d\phi}_0 \otimes \widetilde{d\phi}_0 + \beta g^{-1}, \tag{2.71}$$

for the inverse g_{eff}^{-1} of the effective metric This effective metric can exhibit sonic horizons with the right choice of ϕ_0 and the results have been studied in the non-relativistic limit in Ref. [18].

2.3.2 Effective metric from a scalar field

It is straightforward to derive an effective metric for any scalar field ϕ whose dynamics are given by a Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ [28], which depends only on the field and its first derivatives. Perturbing the field $\phi = \phi_0 + \epsilon\phi_1 + \frac{1}{2}\epsilon^2\phi_2 + \mathcal{O}(\epsilon^3)$, the Lagrangian can be written as

$$\begin{aligned}\mathcal{L}(\phi, \partial_\mu \phi) =& \mathcal{L}(\phi_0, \partial_\mu \phi_0) + \epsilon \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \phi_1 + \frac{\partial \mathcal{L}}{\partial \phi} \phi_1 \right) + \frac{\epsilon^2}{2} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \phi_2 + \frac{\partial \mathcal{L}}{\partial \phi} \phi_2 \right) \\ &+ \frac{\epsilon^2}{2} \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \partial_\mu \phi_1 \partial_\nu \phi_1 + 2 \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial \phi} \phi_1 \partial_\mu \phi_1 + \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi} \phi_1^2 \right) \\ &+ \mathcal{O}(\epsilon^3),\end{aligned} \tag{2.72}$$

where the derivatives of the Lagrangian are evaluated at $\epsilon = 0$. Consider the action

$$S[\phi] = \int d^n x \mathcal{L}(\phi, \partial_\mu \phi). \tag{2.73}$$

Utilising integration by parts and Euler-Lagrange equations for ϕ yields [28]

$$\begin{aligned} S[\phi] = S[\phi_0] + \frac{\epsilon^2}{2} \int d^n x \Bigg[& \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \partial_\mu \phi_1 \partial_\nu \phi_1 \\ & + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi} - \partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial \phi} \right) \right) \phi_1^2 \Bigg]. \end{aligned} \tag{2.74}$$

Thus the equation of motion for the linearised perturbation is

$$\partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \partial_\nu \phi_1 \right) - \left(\frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi} - \partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial \phi} \right) \right) \phi_1 = 0. \tag{2.75}$$

By defining

$$\sqrt{-g_{\text{eff}}}\, g_{\text{eff}}^{\mu\nu} \equiv \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right) \Bigg|_{\phi=\phi_0}, \tag{2.76}$$

equation (2.75) can be written as

$$(\Delta - \mathcal{U})\phi_1 = 0, \tag{2.77}$$

where Δ is the d'Alembert operator associated with the effective metric g_{eff}, and $\mathcal{U}$ is a potential given by

$$\mathcal{U} = \frac{1}{\sqrt{-g_{\text{eff}}}} \left(\frac{\partial^2 \mathcal{L}}{\partial\phi\partial\phi} - \partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial\phi} \right) \right) \Bigg|_{\phi=\phi_0}. \tag{2.78}$$

This result shows that an effective metric arises naturally from the linearisation process. The signature and properties of such effective metric will depend on the details of $\mathcal{L}$. However, it is also possible to derive effective metrics with multiple fields [29] (a fact that will be exploited in subsequent chapters), and even systems that cannot be described by Lagrangians [30]. This availability of models and their robustness lends itself to experimental searches for analogue Hawking radiation, as well as analogies of other effects arising from general relativity.

Chapter 3

Underdense laser-driven plasma theory

A laser-driven plasma is said to be underdense if the laser frequency is greater than the plasma frequency. The plasma acts as a non-linear optical medium for the laser. The field equations for an underdense laser-driven plasma will be derived in this chapter. Firstly, in section 3.1 the ponderomotive force equation will be derived, which describes the bulk behaviour of the plasma electrons, then the continuity equation will be derived in section 3.2 by considering conservation of particle number current. Section 3.3 will introduce an action for the theory of electron-ion plasma and discuss what form the energy density should take, as well as extracting the laser pulse contribution to the electromagnetic field. Section 3.3.1 will present the field equations resulting from stationary variations of the action for a general energy density and the cold plasma model will be assumed. Section 3.3.2 will describe the system obtained from considering an energy density that leads to field equations that are commonly used in laser-driven plasma physics and discuss an approximation regime in which it reduces to one spatial dimension. Finally section 3.4 will show that the results of section 3.3.2 can be obtained by considering scalar quantum electrodynamics (QED).

3.1 Relativistic ponderomotive force

A heuristic derivation of a commonly used method for modelling the effects of an intense laser pulse on charged particles will be presented. Consider a timelike unit normalised vector field $\mathcal{V}$. The relativistic equation of motion for a collection of particles with mass m and charge q moving in an electric field represented by $\mathcal{V}$ can be written as

$$\iota_{\mathcal{V}} d\tilde{\mathcal{V}} = \frac{q}{m}\iota_{\mathcal{V}}\mathcal{F}, \tag{3.1}$$

where $\mathcal{F}$ is the Maxwell tensor. This equation can be simplified to

$$\iota_{\mathcal{V}} d\tilde{\mathcal{V}} = \frac{q}{m}\iota_{\mathcal{V}}\mathcal{F} \implies d\tilde{\mathcal{V}} = \frac{q}{m}\mathcal{F} + \Omega, \tag{3.2}$$

where $\iota_{\mathcal{V}}\Omega = 0$ is required, and also $d\Omega = 0$ since $d\mathcal{F} = 0$. For simplicity, the 2-form Ω will be set to zero, meaning the motion is irrotational when $\mathcal{F} = 0$. Let $\langle\alpha\rangle$ denote the differential p-form that results from averaging the differential p-form α over the period of the fast oscillations of the laser pulse. The precise details of the averaging process are unimportant[1], but the

[1] Typically a time-averaging process is used [48, 49], however a different process can be used, such as phase-averaging [50].

key properties of $\langle\cdot\rangle$ include

$$\begin{aligned}\langle\alpha+\beta\rangle &= \langle\alpha\rangle+\langle\beta\rangle,\\ \langle\langle\alpha\rangle\rangle &= \langle\alpha\rangle,\\ \langle d\alpha\rangle &= d\langle\alpha\rangle.\end{aligned} \tag{3.3}$$

Let $\mathcal{V}$ and $\mathcal{F}$ be a sum of slowly varying parts from the background plasma and fast varying parts from the laser pulse;

$$\begin{aligned}\tilde{\mathcal{V}} &= \langle\tilde{\mathcal{V}}\rangle+\tilde{\mathcal{V}}_{\text{fast}},\\ \mathcal{F} &= \langle\mathcal{F}\rangle+\mathcal{F}_{\text{fast}},\end{aligned} \tag{3.4}$$

where $\langle\tilde{\mathcal{V}}_{\text{fast}}\rangle = 0$ and $\langle\mathcal{F}_{\text{fast}}\rangle = 0$. Using (3.2) and (3.3) (with $\Omega = 0$) leads to

$$d\langle\tilde{\mathcal{V}}\rangle = \frac{q}{m}\langle\mathcal{F}\rangle, \tag{3.5}$$

$$d\tilde{\mathcal{V}}_{\text{fast}} = \frac{q}{m}\mathcal{F}_{\text{fast}}. \tag{3.6}$$

Now introducing $\tilde{V} = \frac{\langle\tilde{\mathcal{V}}\rangle}{|\langle\tilde{\mathcal{V}}\rangle|}$ and using the fact that V is timelike and unit normalised, the following equality can be derived:

$$\langle\tilde{\mathcal{V}}\rangle = \tilde{V}\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}. \tag{3.7}$$

With this equation (3.5) becomes

$$\begin{aligned}\frac{q}{m}\langle\mathcal{F}\rangle =& d\left(\tilde{V}\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\right)\\ =& d\left(\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\right)\tilde{V}+\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\,d\tilde{V}.\end{aligned} \tag{3.8}$$

Dividing both sides by $\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}$ and applying ι_V yields

$$\begin{aligned}\frac{q}{m}\iota_V\langle\mathcal{F}\rangle =&\iota_V\left[d\left(\ln\left(\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\right)\right)\tilde{V}\right]+\iota_V d\tilde{V}\\ =&\iota_V d\ln\left(\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\right)\tilde{V}-d\ln\left(\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\right)\iota_V\tilde{V}+\iota_V d\tilde{V}.\end{aligned} \tag{3.9}$$

Let Π_V be the V-orthogonal projection operator defined by

$$\Pi_V\alpha\equiv\alpha+\tilde{V}\wedge\iota_V\alpha, \tag{3.10}$$

where α is any p-form. Utilising the fact that $\iota_V\tilde{V}=-1$, equation (3.9) can be rearranged as

$$m\iota_V d\tilde{V}=-m\Pi_V d\ln\left(\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}\right)+\frac{q}{\sqrt{1+\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle}}\iota_V\langle\mathcal{F}\rangle. \tag{3.11}$$

The first term on the right-hand side is the ponderomotive force term, and the extra factor in the second term can be interpreted as a multiplicative correction to the relativistic mass of particles undergoing the averaged motion.

3.2 Continuity equation

Consider a 4-dimensional spacetime $\mathcal{M}$ and a 3-dimensional manifold $\mathcal{B}$. Let f be a function that maps worldlines in $\mathcal{M}$ to points in $\mathcal{B}$, such that $f:(\check{\xi}^0,\check{\xi}^A)\mapsto(\xi^A)$, where A, B and C run over 1, 2 and 3. The point (ξ^A) in $\mathcal{B}$ corresponds to the particle with world line $\check{\xi}^0\mapsto(\check{\xi}^0,\check{\xi}^A)$ in $\mathcal{M}$. A general

3-form ς on $\mathcal{B}$ is given by

$$\varsigma = \frac{1}{3!}\varsigma_{ABC}\, d\xi^A \wedge d\xi^B \wedge d\xi^C. \tag{3.12}$$

Letting $f^X = \xi^X \circ f$ leads to

$$f^*\varsigma = \frac{1}{3!}\varsigma_{ABC} \circ f\, df^A \wedge df^B \wedge df^C, \tag{3.13}$$

where f^* is a pullback map induced from f. For any 3-form j on manifold $\mathcal{M}$ given by $j = f^*\varsigma$,

$$dj = 0, \tag{3.14}$$

since $dj = df^*\varsigma = f^*d\varsigma = 0$ as ς is a 3-form on a 3-dimensional manifold. Let j be the particle number 3-form of f; note that particle number conservation follows immediately from equation (3.14). Introducing proper density n and 4-velocity of matter V, the particle number 3-form j can be written as

$$j = n \star \tilde{V}. \tag{3.15}$$

Requiring V to be unit normalised results in $n = \sqrt{j \cdot j}$. Note that both n and V are dependent on f.

3.3 Action

An action describing a laser-driven plasma whose plasma electrons are represented by $\mathfrak{f} : \mathcal{B} \to \mathcal{M}$ can be written as

$$\mathcal{S}[\mathfrak{f}, \mathcal{A}] = \int_{\mathcal{M}} \left(\rho \star 1 + q\mathcal{A} \wedge \mathcal{J} + \frac{1}{2}\mathcal{F} \wedge \star\mathcal{F} - q\mathcal{A} \wedge \mathcal{J}_{\text{ext}} \right), \tag{3.16}$$

where ρ is the proper energy density of the plasma and a function of the proper density of the plasma electrons $\mathcal{N}$, $\mathcal{A}$ is the electromagnetic potential 1-form, $\mathcal{F} = d\mathcal{A}$ is the Maxwell tensor, and $\mathcal{J} = \mathfrak{f}^*\varsigma = \mathcal{N} \star \widetilde{\mathcal{V}}$. The 3-form $\mathcal{J}_{\text{ext}}$ describes the external number current of the plasma ions. The proper energy density of a cold plasma is given by $m\mathcal{N}$. The discussion in section 3.1 suggests the action

$$\begin{aligned} S[f, A] &= \int_{\mathcal{M}} \langle \left(\rho \star 1 + q\mathcal{A} \wedge \mathcal{J} + \frac{1}{2}\mathcal{F} \wedge \star\mathcal{F} - q\mathcal{A} \wedge \mathcal{J}_{\text{ext}} \right) \rangle \\ &= \int_{\mathcal{M}} \left(\varrho \star 1 + qA \wedge j + \frac{1}{2}\langle \mathcal{F} \wedge \star\mathcal{F} \rangle - qA \wedge j_{\text{ext}} \right), \end{aligned} \tag{3.17}$$

where ϱ is the proper energy density of the averaged motion, and a function of the averaged number density $n = \langle \mathcal{N} \rangle$. The 3-form $j_{\text{ext}} = \langle \mathcal{J}_{\text{ext}} \rangle$ is the averaged external number current. Here $\langle \mathcal{A} \wedge \mathcal{J} \rangle = A \wedge j$ and $\langle \mathcal{A} \wedge \mathcal{J}_{\text{ext}} \rangle = A \wedge j_{\text{ext}}$ are assumed rather than shown directly, as the resulting field equations agree with the commonly used ones [48–53]. Also note that the statement $\langle \star P \rangle = \star \langle P \rangle$ for any p-form P requires the metric components to also be slowly varying, however the results will be applied to Minkowski space and as such it is a valid statement.

Keeping the background plasma and the laser pulse contributions to the

Maxwell tensor together would lead to a complicated system. Thus following the same approach as in section 3.1, let $\mathcal{F}$ be split into fast varying part, $\mathcal{F}_{\text{fast}}$ due to the laser pulse and a slow varying part, $F = \langle \mathcal{F} \rangle$, due to the plasma, such that $\mathcal{F} = F + \mathcal{F}_{\text{fast}}$. Assuming a solution for the laser pulse with a rapidly changing $e^{i\Phi}$ compared to the amplitude α_0 gives

$$\mathcal{F}_{\text{fast}} = d\left(\text{Re}\left(\alpha_0 e^{i\Phi}\right)\right) \approx \text{Re}\left(i d\Phi \wedge \alpha_0 e^{i\Phi}\right). \tag{3.18}$$

Equation (3.18) leads to

$$\begin{aligned}\langle \mathcal{F} \wedge \star\mathcal{F} \rangle =& \langle F \wedge \star F + F \wedge \star\mathcal{F}_{\text{fast}} + \mathcal{F}_{\text{fast}} \wedge \star F + \mathcal{F}_{\text{fast}} \wedge \star\mathcal{F}_{\text{fast}} \rangle \\ =& F \wedge \star F + \langle \mathcal{F}_{\text{fast}} \wedge \star\mathcal{F}_{\text{fast}} \rangle,\end{aligned} \tag{3.19}$$

since $\langle F \wedge \star\mathcal{F}_{\text{fast}} \rangle = F \wedge \star\langle \mathcal{F}_{\text{fast}} \rangle$. The last term written in terms of α_0 and Φ is

$$\begin{aligned}\langle \mathcal{F}_{\text{fast}} \wedge \star\mathcal{F}_{\text{fast}} \rangle =& \langle \frac{1}{4} i (d\Phi \wedge \alpha_0 e^{i\Phi} - d\Phi \wedge \bar{\alpha}_0 e^{-i\Phi}) \wedge \star i (d\Phi \wedge \alpha_0 e^{i\Phi} - d\Phi \wedge \bar{\alpha}_0 e^{-i\Phi}) \rangle \\ =& \frac{1}{2} \langle (d\Phi \wedge \bar{\alpha}_0 e^{-i\Phi}) \wedge \star(d\Phi \wedge \alpha_0 e^{i\Phi}) \rangle \\ =& \frac{1}{2} (d\Phi \wedge \bar{\alpha}_0) \wedge \star(d\Phi \wedge \alpha_0),\end{aligned} \tag{3.20}$$

since $\langle \Phi \rangle = \Phi$, $\langle \alpha_0 \rangle = \alpha_0$ and $\langle e^{i\Phi} \rangle = 0$. Here bar denotes complex conjugate. Incorporating this into the action (3.17) yields

$$S[\alpha_0, \Phi, f, A] = \int_{\mathcal{M}} \left(\varrho \star 1 + qA \wedge (j - j_{\text{ext}}) + \frac{1}{2} F \wedge \star F + \frac{1}{4} d\Phi \wedge \bar{\alpha}_0 \wedge \star(d\Phi \wedge \alpha_0) \right), \tag{3.21}$$

where now $F = dA$. Compared to the action in (3.17), the contribution of the plasma and the laser pulse to the electromagnetic field has been split into the last two terms respectively. The final issue is the lack of interaction between the laser pulse and plasma. This can be added by modifying $\varrho(n) \mapsto \lambda(n, \mu)$, where μ is a dimensionless intensity parameter relating the back-reaction of the laser pulse and the plasma, but that leaves the question of what forms should λ and μ have. One possible choice is $\mu = \mu_{\mathrm{I}} = \frac{\alpha_0 \cdot \bar{\alpha}_0}{2}$. This comes from requiring the form of equation (3.31) in the following section to match that of equation (3.11) for λ that is chosen appropriately (see section 3.3.2). However the residual gauge invariance of (3.20) under the transformation $\alpha_0 \mapsto \alpha_0 + \xi d\Phi$ for some 0-form ξ is then not respected by the entire action (3.21). To restore it, another choice for μ can be made:

$$\mu = \mu_{\mathrm{II}} = \frac{\iota_V(d\Phi \wedge \alpha_0) \cdot \iota_V(d\Phi \wedge \bar{\alpha}_0)}{2(\iota_V d\Phi)^2}, \tag{3.22}$$

which can be written as

$$\mu_{\mathrm{II}} = \frac{\alpha_0 \cdot \bar{\alpha}_0}{2} + \frac{\iota_V \alpha_0 \iota_V \bar{\alpha}_0}{2(\iota_V d\Phi)^2} d\Phi \cdot d\Phi - \frac{\iota_V \bar{\alpha}_0(\alpha_0 \cdot d\Phi) + \iota_V \alpha_0(\bar{\alpha}_0 \cdot d\Phi)}{2\iota_V d\Phi}. \tag{3.23}$$

When $\alpha_0 \cdot d\Phi = 0$ and $\alpha_0 \cdot \tilde{V} = 0$, μ_{II} reduces to μ_{I}. However, the form $\mu = \mu_{\mathrm{II}}$ will be assumed for the purpose of varying the action, to respect the residual gauge invariance. As for λ, there is a form that corresponds to standard modelling techniques in laser-driven plasma literature. The particular form will be discussed in section 3.3.2, but will be kept general for now, thus finally arriving at the action

$$S[\alpha_0,\Phi,f,A]=\int_{\mathcal{M}}\left(\lambda\star 1+qA\wedge(j-j_{\text{ext}})+\frac{1}{2}F\wedge\star F+\frac{1}{4}d\Phi\wedge\bar{\alpha}_0\wedge\star(d\Phi\wedge\alpha_0)\right). \tag{3.24}$$

3.3.1 Variation of the action

Firstly consider variation with respect to A:

$$\begin{aligned}\delta_A S&=\int_{\mathcal{M}}\left(q\delta A\wedge(j-j_{\text{ext}})+d\delta A\wedge\star F\right)\\&=\int_{\mathcal{M}}\left(q\delta A\wedge(j-j_{\text{ext}})+d(\delta A\wedge\star F)+\delta A\wedge d\star F\right)\\&=\int_{\mathcal{M}}\delta A\wedge(q(j-j_{\text{ext}})+d\star F).\end{aligned} \tag{3.25}$$

Requiring $\delta_A S=0$ recovers the Maxwell equation $d\star F=-q(j-j_{\text{ext}})$. Next, variation with respect to f requires more care. First let $f_\epsilon:\mathcal{M}\to\mathcal{B}$ be a 1-parameter family of maps with $f_0=f$ and consider a 0-form h:

$$\begin{aligned}\delta_f(f^*h)&=\frac{d}{d\epsilon}(h\circ f_\epsilon)\bigg|_{\epsilon=0}=\frac{\partial h}{\partial\xi^A}\frac{df^A_\epsilon}{d\epsilon}\bigg|_{\epsilon=0}=\frac{\partial f^*h}{\partial\check{\xi}^A}\frac{df^A_\epsilon}{d\epsilon}\bigg|_{\epsilon=0}\\&=\frac{df^A_\epsilon}{d\epsilon}\bigg|_{\epsilon=0}\frac{\partial}{\partial\check{\xi}^A}(f^*h)=\mathcal{L}_X(f^*h),\end{aligned} \tag{3.26}$$

where $X=\frac{df^A_\epsilon}{d\epsilon}\Big|_{\epsilon=0}\frac{\partial}{\partial\check{\xi}^A}$. Now looking at a 1-form:

$$\begin{aligned}\delta_f(f^*(d\xi^A))&=d\delta_f f^A=d\frac{df^A}{d\epsilon}\bigg|_{\epsilon=0}=d\iota_X d\check{\xi}^A=(d\iota_X+\iota_X d)d\check{\xi}^A=\mathcal{L}_X d\check{\xi}^A\\&=\mathcal{L}_X(f^*d\xi^A).\end{aligned} \tag{3.27}$$

This can be extended to any p-form β obtained by pulling back from $\mathcal{B}$ using f^*, thus $\delta_f(\beta) = \mathcal{L}_X\beta$ for any such p-form β. Variation of $n\star 1$ and $\mu\star 1$ with respect to f is as follows:

$$
\begin{aligned}
\delta_f\sqrt{j\cdot j}\star 1 &= \frac{1}{2\sqrt{j\cdot j}}(\delta_f j\cdot j + j\cdot\delta_f j)\star 1 = \frac{1}{n}\star(\delta_f j\cdot j) = \frac{1}{n}\delta_f j\wedge\star j \\
&= \delta_f j\wedge\tilde{V},
\end{aligned}
\tag{3.28}
$$

since $\star\star\tilde{V} = \tilde{V}$, and using (3.22)

$$
\begin{aligned}
\delta_f\mu\star 1 =& \frac{\partial\mu}{\partial\tilde{V}}\wedge\star\delta_f\tilde{V} = \frac{\partial\mu}{\partial\tilde{V}}\wedge\delta_f\frac{j}{n} = \frac{\partial\mu}{\partial\tilde{V}}\wedge\left(\frac{1}{n}\mathcal{L}_X j + j\delta_f\frac{1}{n}\right) \\
=& \frac{\partial\mu}{\partial\tilde{V}}\wedge\left(\frac{1}{n}d\iota_X j - \frac{j}{n^3}\delta_f j\cdot j\right) = \frac{\partial\mu}{\partial\tilde{V}}\wedge\left(\frac{1}{n}d\iota_X j - \frac{1}{n}(\delta_f j\cdot\star\tilde{V})\star\tilde{V}\right) \\
=& \frac{\partial\mu}{\partial\tilde{V}}\wedge\left(\frac{1}{n}d\iota_X j - \frac{1}{n}\iota_V(d\iota_X j\wedge\tilde{V})\right) = -\frac{\partial\mu}{\partial\tilde{V}}\wedge\left(\frac{1}{n}\iota_V d\iota_X j\wedge\tilde{V}\right) \\
=& -\frac{1}{n}\iota_V\left(\frac{\partial\mu}{\partial\tilde{V}}\wedge\tilde{V}\wedge d\iota_X j\right) + \frac{1}{n}\iota_V\left(\frac{\partial\mu}{\partial\tilde{V}}\wedge\tilde{V}\right)\wedge d\iota_X j \\
=& \frac{1}{n}\left(\Pi_V\frac{\partial\mu}{\partial\tilde{V}}\right)\wedge d\iota_X j,
\end{aligned}
\tag{3.29}
$$

where $\frac{\partial\mu}{\partial\tilde{V}}$ is a 1-form and Π_V denotes the V-orthogonal projection operator on forms. Thus varying the action gives

$$\begin{aligned}
\delta_f S &= \int_{\mathcal{M}} \left(\frac{\partial\lambda}{\partial n}\delta_f n \star 1 + \frac{\partial\lambda}{\partial\mu}\delta_f\mu \star 1 + qA \wedge \delta_f j \right) \\
&= \int_{\mathcal{M}} d\iota_X j \wedge \left(\frac{\partial\lambda}{\partial n}\tilde{V} - qA - \frac{1}{n}\frac{\partial\lambda}{\partial\mu}\Pi_V \frac{\partial\mu}{\partial\tilde{V}} \right) \\
&= \int_{\mathcal{M}} \iota_X j \wedge d \left(\frac{\partial\lambda}{\partial n}\tilde{V} - qA - \frac{1}{n}\frac{\partial\lambda}{\partial\mu}\Pi_V \frac{\partial\mu}{\partial\tilde{V}} \right) \\
&= \int_{\mathcal{M}} \iota_X \iota_V (n \star 1) \wedge d \left(\frac{\partial\lambda}{\partial n}\tilde{V} - qA - \frac{1}{n}\frac{\partial\lambda}{\partial\mu}\Pi_V \frac{\partial\mu}{\partial\tilde{V}} \right) \\
&= \int_{\mathcal{M}} (n \star 1) \wedge \iota_X \iota_V d \left(\frac{\partial\lambda}{\partial n}\tilde{V} - qA - \frac{1}{n}\frac{\partial\lambda}{\partial\mu}\Pi_V \frac{\partial\mu}{\partial\tilde{V}} \right).
\end{aligned} \tag{3.30}$$

If X and V are linearly dependent then $\delta_f S = 0$ is satisfied trivially, otherwise this gives the Lorentz force associated with the averaged motion and an extra contribution:

$$\iota_V d \left(\frac{\partial\lambda}{\partial n}\tilde{V} - \frac{1}{n}\frac{\partial\lambda}{\partial\mu}\Pi_V \frac{\partial\mu}{\partial\tilde{V}} \right) = q\iota_V F, \tag{3.31}$$

where

$$\begin{aligned}
\frac{\partial\mu}{\partial\tilde{V}} =& \frac{((\iota_V\alpha_0)\bar{\alpha}_0 + (\iota_V\bar{\alpha}_0)\alpha_0)(d\Phi\cdot d\Phi) + ((\iota_V\alpha_0)(\bar{\alpha}_0\cdot d\Phi) + (\iota_V\bar{\alpha}_0)(\alpha_0\cdot d\Phi))d\Phi}{2(\iota_V d\Phi)^2} \\
&- \frac{(\alpha_0\cdot d\Phi)\bar{\alpha}_0 + (\bar{\alpha}_0\cdot d\Phi)\alpha_0}{2\iota_V d\Phi} - \frac{(\iota_V\alpha_0)(\iota_V\bar{\alpha}_0)(d\Phi\cdot d\Phi)d\Phi}{(\iota_V d\Phi)^3}.
\end{aligned} \tag{3.32}$$

Varying (3.24) with respect to α_0 and Φ gives

$$(d\Phi\cdot d\Phi)\bar{\alpha}_0 - (d\Phi\cdot\bar{\alpha}_0)d\Phi = -2\frac{\partial\lambda}{\partial\mu}\left(\bar{\alpha}_0 + \frac{\iota_V\bar{\alpha}_0}{(\iota_V d\Phi)^2}(d\Phi\cdot d\Phi)\tilde{V} - \frac{\bar{\alpha}_0\cdot d\Phi}{\iota_V d\Phi}\tilde{V} - \frac{\iota_V\bar{\alpha}_0}{\iota_V d\Phi}d\Phi \right), \tag{3.33}$$

and

$$
\begin{aligned}
d\star\Bigg(&\frac{\bar{\alpha}_0\cdot d\Phi}{4}\alpha_0+\frac{\alpha_0\cdot d\Phi}{4}\bar{\alpha}_0-\frac{\bar{\alpha}_0\cdot\alpha_0}{2}d\Phi+\frac{\partial\lambda}{\partial\mu}\Bigg(\frac{\iota_V\alpha_0\iota_V\bar{\alpha}_0}{(\iota_V d\Phi)^2}d\Phi\\
&-\frac{\iota_V\bar{\alpha}_0}{2\iota_V d\Phi}\alpha_0-\frac{\iota_V\alpha_0}{2\iota_V d\Phi}\bar{\alpha}_0-\frac{\iota_V\bar{\alpha}_0\iota_V\alpha_0}{(\iota_V d\Phi)^3}(d\Phi\cdot d\Phi)\tilde{V}\\
&-\frac{\iota_V\bar{\alpha}_0}{2(\iota_V d\Phi)^2}(\alpha_0\cdot d\Phi)\tilde{V}-\frac{\iota_V\alpha_0}{2(\iota_V d\Phi)^2}(\bar{\alpha}_0\cdot d\Phi)\tilde{V}\Bigg)\Bigg)=0,
\end{aligned}
\tag{3.34}
$$

respectively. As presented, these equations are very complicated and unmanageable. They become much simpler when using μ_{I}, or seeking solutions where $\alpha_0\cdot d\Phi=0$ and $\alpha_0\cdot\tilde{V}=0$ (i.e. the polarisation vector of the laser pulse is orthogonal to the motion of the plasma electrons). The former abandons residual gauge invariance but leads to simpler field equations, while the latter is applicable when the pointwise behaviour of the system is 2-dimensional on spacetime. The form of μ_{I} leads to field equations that are commonly used in laser-driven plasma physics, but from a physical standpoint it is a more natural choice to keep residual gauge invariance. Both choices reduce the field equations in the same way. However from now on it will be assumed that one of these choices was made, thus the field equations from varying the action with respect to α_0 and Φ respectively become

$$
d\Phi\cdot d\Phi=-2\frac{\partial\lambda}{\partial\mu},
\tag{3.35}
$$

$$
d\star(\mu d\Phi)=0.
\tag{3.36}
$$

Furthermore the force from the laser will be assumed to dominate over the force from the electromagnetic field, thus equation (3.31) obtained from varying the action with respect to f becomes

$$\iota_V d\left(\frac{\partial\lambda}{\partial n}\tilde{V}\right) = 0. \tag{3.37}$$

The flow of electrons can be chosen to be irrotational [53], thus this equation can be solved by introducing a momentum potential ψ such that

$$d\psi = \frac{\partial\lambda}{\partial n}\tilde{V}. \tag{3.38}$$

The following relations are also satisfied:

$$d\psi \cdot d\psi = -\left(\frac{\partial\lambda}{\partial n}\right)^2, \tag{3.39}$$

$$d\star\left(n\frac{1}{\frac{\partial\lambda}{\partial n}}d\psi\right) = 0, \tag{3.40}$$

where (3.40) follows from (3.14) and (3.15).

3.3.2 Minimal energy density

The typical form of λ [48–53], which will be referred to as minimal energy density, that leads to field equations commonly used in laser-driven plasma physics is

$$\lambda = nm\sqrt{1+\frac{q^2}{m^2}\mu}. \tag{3.41}$$

With the form of μ discussed previously, the field equation (3.31) is that of the standard relativistic ponderomotive force found in section 3.1, with μ being proportional to $\langle\tilde{\mathcal{V}}^2_{\text{fast}}\rangle$. In this regime the effective mass of the electron is given by $\sqrt{m^2 + q^2\mu}$. This form of λ lends itself in specifying the field equations further; now equations (3.35) and (3.39) become

$$d\Phi \cdot d\Phi = -\frac{nq^2}{m\sqrt{1 + \frac{q^2}{m^2}\mu}}, \tag{3.42}$$

$$d\psi \cdot d\psi = -m^2 - q^2\mu, \tag{3.43}$$

respectively. The field equations (3.36) and (3.40) can be written as

$$d((d\psi \cdot d\psi + m^2) \star d\Phi) = 0, \tag{3.44}$$

$$d((d\Phi \cdot d\Phi) \star d\psi) = 0. \tag{3.45}$$

These in turn suggest the action

$$S[\Phi, \psi] = \frac{1}{2}\int_{\mathcal{M}} (d\Phi \cdot d\Phi)(d\psi \cdot d\psi + m^2) \star 1. \tag{3.46}$$

The considerations in section 3.3.1 leading to requiring $\alpha_0 \cdot d\Phi = 0$ and $\alpha_0 \cdot \tilde{V} = 0$ suggest that both Φ and ψ will vary more quickly in the direction of propagation of the laser pulse than transverse to it. Furthermore supposing that it is possible to approximate $(d\Phi \cdot d\Phi)(d\psi \cdot d\psi + m^2)$ as a product of two functions of (t, z) and (x, y), where the laser pulse propagates in the z-direction, and using the approximation $\int \mathfrak{h}(x, y) dxdy \approx \max(\mathfrak{h}) \int_{\mathrm{supp}(\mathfrak{h})} dxdy$, for a sufficiently well behaved function $\mathfrak{h}$, allows the introduction of

$$\Lambda = \frac{\max[(d\Phi \cdot d\Phi)(d\psi \cdot d\psi + m^2)]_{x,y} \int_{\mathcal{S}} dxdy}{[(d\Phi \cdot d\Phi)(d\psi \cdot d\psi + m^2)]|_{x=y=0}}, \tag{3.47}$$

where $\mathcal{S} = \text{supp}((d\Phi \cdot d\Phi)(d\psi \cdot d\psi + m^2))$. The quantity $\int_{\mathcal{S}} dxdy$ represents the cross-sectional area of the laser pulse (spot size), and thus Λ will depend on (z, t), unless the spot size is assumed to be constant. Thus the action becomes

$$S[\Phi, \psi] = \frac{1}{2} \int_{\mathcal{M}} \Lambda[(d\Phi \cdot d\Phi)(d\psi \cdot d\psi + m^2)]|_{x=y=0} \# 1, \tag{3.48}$$

and the domain of the field equations is reduced to 2-dimensional spacetime:

$$d(\Lambda(d\psi \cdot d\psi + m^2) \# d\Phi) = 0, \tag{3.49}$$

$$d(\Lambda(d\Phi \cdot d\Phi) \# d\psi) = 0. \tag{3.50}$$

Analysis of a system with non-varying cross sectional area can be done with the field equations (3.44) and (3.45), and one or three spatial dimensions can be considered. The field equations (3.49) and (3.50) describe a system in one spatial dimension, and involve a varying spot size which is encoded in Λ.

3.4 Effective field theory of laser-driven plasma motivated from scalar QED

Although the action (3.48) may seem mysterious, it will be shown that this action can be motivated through scalar quantum electrodynamics. To show this, consider the action

$$S = -\int_{\mathcal{M}} \left[\frac{1}{2} \overline{D\varphi} \wedge \star D\varphi + \frac{1}{2} m^2 |\varphi|^2 \star 1 + \frac{1}{2} F \wedge \star F - A \wedge \mathrm{j}_{\mathrm{ext}} \right], \tag{3.51}$$

on a flat 4-dimensional spacetime, where φ is a complex scalar field, $F = dA$, $D\varphi = d\varphi + ieA\varphi$, $\mathfrak{j}_{\text{ext}}$ is the background electric current 3-form, A is the electromagnetic potential 1-form and e is the elementary charge. Let φ be a function of (t, z) only, $A = \underline{A} + A_x dx + A_y dy$, where the components of the 1-form $\underline{A}$ are functions of (t, z) only and underline indicates projection into the $t - z$ plane. The first term in the integrand of (3.51) becomes

$$\frac{1}{2}\overline{D\varphi} \wedge \star D\varphi = \frac{1}{2}(\overline{\underline{D}\varphi} \wedge \star \underline{D}\varphi + e^2|\mathcal{A}|^2|\varphi|^2\#1) \wedge \#_\perp 1, \tag{3.52}$$

while the third term becomes

$$\begin{aligned}\frac{1}{2}F \wedge \star F =& \frac{1}{2}\underline{F} \wedge \star \underline{F} \wedge \#_\perp 1 + \frac{1}{2}\underline{d}A_x \wedge dx \wedge \star(\underline{d}A_x \wedge dx) \\ &+ \frac{1}{2}\underline{d}A_y \wedge dy \wedge \star(\underline{d}A_y \wedge dy) \\ =& \frac{1}{2}(\underline{F} \wedge \star \underline{F} + \underline{d}\bar{\mathcal{A}} \wedge \#\underline{d}\mathcal{A}) \wedge \#_\perp 1,\end{aligned} \tag{3.53}$$

where $\mathcal{A} = A_x + iA_y$ and $\#_\perp 1 = dx \wedge dy$. Assuming that the background current has no components in the x and y directions, the following action is motivated from (3.51):

$$\begin{aligned}S = -\int_{\mathcal{M}} \Lambda\Big[&\frac{1}{2}(\overline{\underline{D}\varphi} \wedge \#\underline{D}\varphi + e^2|\mathcal{A}|^2|\varphi|^2\#1 + m^2|\varphi|^2\#1) \\ &+ \frac{1}{2}(\underline{F} \wedge \#\underline{F} + \underline{d}\bar{\mathcal{A}} \wedge \#\underline{d}\mathcal{A}) - \underline{A} \wedge \underline{j}\Big],\end{aligned} \tag{3.54}$$

where $\mathfrak{j}_{\text{ext}} = \underline{j} \wedge dx \wedge dy$ for some 1-form $\underline{j}$. Here Λ is the cross-sectional area of the domain in the $x - y$ plane of the action, assumed to be a function of (t, z) . The field equations arising from stationary variations in $\varphi, \underline{A}$ and $\mathcal{A}$

respectively are

$$\underline{D}(\Lambda\#\underline{D}\varphi) - \Lambda(e^2|\mathcal{A}|^2 + m^2)\varphi\#1 = 0, \tag{3.55}$$

$$\frac{1}{2}ie\Lambda(\varphi\#\overline{\underline{D}\varphi} - \bar{\varphi}\#\underline{D}\varphi) + \underline{d}(\Lambda\#\underline{F}) - \Lambda\underline{j} = 0, \tag{3.56}$$

$$e^2\Lambda\mathcal{A}|\varphi|^2\#1 - \underline{d}(\Lambda\#\underline{d}\mathcal{A}) = 0. \tag{3.57}$$

By introducing $\varphi = |\varphi|e^{i\psi}$ and $\mathcal{A} = |\mathcal{A}|e^{i\Phi}$, the field equations result in

$$\Lambda(\underline{d}\psi + e\underline{A})\cdot(\underline{d}\psi + e\underline{A}) - \frac{1}{|\varphi|}\#^{-1}\underline{d}(\Lambda\#\underline{d}|\varphi|) + e^2\Lambda|\mathcal{A}|^2 + \Lambda m^2 = 0, \tag{3.58}$$

$$\underline{d}\#(\Lambda|\varphi|^2(\underline{d}\psi + e\underline{A})) = 0, \tag{3.59}$$

$$e^2\Lambda|\varphi|^2\#(\underline{d}\psi + e\underline{A}) + \underline{d}(\Lambda\#\underline{F}) - \Lambda\underline{j} = 0. \tag{3.60}$$

$$\Lambda\underline{d}\Phi\cdot\underline{d}\Phi + e^2\Lambda|\varphi|^2 - \frac{1}{|\mathcal{A}|}\#^{-1}\underline{d}(\Lambda\#\underline{d}|\mathcal{A}|) = 0, \tag{3.61}$$

$$\Lambda\underline{d}\#(\Lambda|\mathcal{A}|^2\underline{d}\Phi) = 0. \tag{3.62}$$

Equations (3.58) and (3.59) arise from the real and imaginary parts of (3.55), similarly (3.61) and (3.62) arise from the real and imaginary parts of (3.57). Inspection of equations (3.58), (3.59), (3.61) and (3.62) when $\underline{d}\#\underline{d}|\varphi|$, $\underline{d}\#d\underline{|}\mathcal{A}|$ and $\underline{A}$ are negligible leads to equations (3.49) and (3.50). Applying the above amplitude-phase decomposition to the action (3.54) gives

$$
\begin{aligned}
S = -\int_{\mathcal{M}} \Lambda\big[&\frac{1}{2}(\underline{d}\psi + e\underline{A}) \wedge \#(\underline{d}\psi + e\underline{A})|\varphi|^2 + \frac{1}{2}\underline{d}|\varphi| \wedge \#\underline{d}|\varphi| \\
&+ \frac{1}{2}e^2|\mathcal{A}|^2|\varphi|^2\#1 + \frac{1}{2}m^2|\varphi|^2 + \frac{1}{2}\underline{F} \wedge \#\underline{F} \\
&+ \frac{1}{2}|\mathcal{A}|^2\underline{d}\Phi \wedge \#\underline{d}\Phi + \frac{1}{2}\underline{d}|\mathcal{A}| \wedge \#\underline{d}|\mathcal{A}| - \underline{A} \wedge \underline{j}\big].
\end{aligned}
\tag{3.63}
$$

Let $|\mathrm{I}\rangle = |\underline{A}_{\mathrm{I}}, \mathcal{A}_{\mathrm{I}}, \varphi_{\mathrm{I}}; t_{\mathrm{I}}\rangle$ and $|\mathrm{II}\rangle = |\underline{A}_{\mathrm{II}}, \mathcal{A}_{\mathrm{II}}, \varphi_{\mathrm{II}}; t_{\mathrm{II}}\rangle$ be the eigenstates of the field operators at times t_{I} and t_{II} respectively. The transition amplitude is given by [8]

$$
\begin{aligned}
\langle \mathrm{II}|\mathrm{I}\rangle &= \int \mathcal{D}\underline{A}\mathcal{D}\mathcal{A}\mathcal{D}\mathcal{A}^*\mathcal{D}\varphi\mathcal{D}\varphi^* e^{iS} \\
&= \int \mathcal{D}\underline{A}\mathcal{D}\Phi\mathcal{D}\psi\mathcal{D}(|\mathcal{A}|^2)\mathcal{D}(|\varphi|^2)e^{iS},
\end{aligned}
\tag{3.64}
$$

where the lower and upper limits in the action integral are t_{I} and t_{II} respectively.

Suppose that the derivatives of $|\varphi|$ and $|\mathcal{A}|$ are negligible compared to the derivatives of Φ and ψ. Furthermore suppose $\underline{d}|\varphi| \cdot \underline{d}|\varphi|$ and $\underline{d}|\mathcal{A}| \cdot \underline{d}|\mathcal{A}|$ are negligible compared to other terms in (3.63). Now the functional integrals over $|\varphi|^2$ and $|\mathcal{A}|^2$ are infinite dimensional analogues of

$$
I = \int_0^\infty dx \int_0^\infty dy\, e^{i(ax+by+cxy)}, \tag{3.65}
$$

where $\mathrm{Im}\{a\} = 0^+$ and $\mathrm{Im}\{b\} = 0^+$, and $c < 0$ is a real constant. The notation 0^+ indicates replacing 0^+ with ϵ and taking the limit $\epsilon \to 0$ of the whole expression. The variables corresponding to a b, c, x and y are $-\frac{1}{2}\Lambda(\underline{d}\psi + e\underline{A})^2 - \frac{1}{2}\Lambda m^2$, $-\frac{1}{2}\Lambda(\underline{d}\Phi)^2$, $-\frac{1}{2}\Lambda e^2$, $|\varphi|^2$ and $|\mathcal{A}|^2$ respectively. Note that $a \mapsto a + i0^+$ and $b \mapsto b + i0^+$ corresponds to the Feynman prescription

$M^2 \mapsto M^2 - i0^+$, with $M = m$ for φ and $M = 0$ for $\mathcal{A}$. Either integral evaluates to

$$I = \frac{1}{c} e^{-i\frac{ab}{c}} \lim_{\epsilon \to 0^+} \int_{\mathrm{Re}\{\frac{a}{c}\}}^{\infty} dz \frac{e^{ibz}}{z - i\epsilon} = \frac{1}{c} e^{-i\frac{ab}{c}} \lim_{\epsilon \to 0^+} \int_{\mathrm{Re}\{\frac{b}{c}\}}^{\infty} dz \frac{e^{iaz}}{z - i\epsilon}. \tag{3.66}$$

Numerical integration reveals that

$$I \approx -\frac{2\pi}{c} \theta(a)\theta(b) e^{-i\frac{ab}{c}}, \tag{3.67}$$

where θ is the Heaviside function, is a reasonable approximation when $|\frac{a}{c}| > 1$ and $|\frac{b}{c}| > 1$. The right-hand side of equation (3.67) arises from considering the results of (3.66) when the lower integration limits are replaced by $\pm\infty$. Hence with these approximations equation (3.67) suggests that the functional integrals in (3.64) give

$$\langle \mathrm{II}|\mathrm{I}\rangle \approx \int \mathcal{D}\underline{A}\mathcal{D}\Phi\mathcal{D}\psi\theta[U_\Phi]\theta[U_\psi]e^{iS'}, \tag{3.68}$$

up to a numerical factor, where $U_\Phi = -\underline{d}\Phi \cdot \underline{d}\Phi$, $U_\psi = -(\underline{d}\psi + e\underline{A}) \cdot (\underline{d}\psi + e\underline{A}) - m^2$ and

$$\begin{aligned} S' = \int_{\mathcal{M}} \Lambda \bigg(& \frac{1}{2e^2}(\underline{d}\psi + e\underline{A}) \wedge \#(\underline{d}\psi + e\underline{A})\underline{d}\Phi \cdot \underline{d}\Phi - \frac{1}{2}\underline{F} \wedge \#\underline{F} + \underline{A} \wedge \underline{j} \\ & + \frac{1}{2e^2} m^2 \underline{d}\Phi \cdot \underline{d}\Phi \# 1 \bigg). \end{aligned} \tag{3.69}$$

The action S' is that of equation (3.63) with the stated assumptions. This approach is valid for

$$\left|\frac{\underline{d}\sqrt{U_\Phi}\cdot\underline{d}\sqrt{U_\Phi}}{U_\Phi U_\psi}\right| \ll 1, \tag{3.70}$$

$$\left|\frac{\underline{d}\sqrt{U_\psi}\cdot\underline{d}\sqrt{U_\psi}}{U_\Phi U_\psi}\right| \ll 1, \tag{3.71}$$

which can be deduced from equations (3.58) and (3.61) respectively since by assumption the kinetic terms of $|\varphi|$ and $|\mathcal{A}|$ in (3.63) are negligible. When $\underline{A}$ is negligible, the action (3.69) is that of (3.48). There is a point worth noting: Φ and ψ in (3.69) have finite ranges as they are the angles in polar decomposition of φ and $\mathcal{A}$. However when U_Φ and U_ψ are large, small variations in Φ and ψ will make significant contributions to (3.69), enhancing the phase interference in the functional integral over configurations in (3.68). Thus Φ and ψ having finite ranges lessens in significance when U_Φ and U_ψ are large, and in such case there is no obstacle in extending their ranges to the real line. Thus Φ and ψ can be regarded as scalar fields. Finally note that this approach only captures the effects of quantum fluctuations of Φ and ψ and not those of $|\varphi|$ or $|\mathcal{A}|$. The above considerations support that if the quantum fluctuations of $|\varphi|$ and $|\mathcal{A}|$ are negligible then the physics of the bi-scalar field theory (3.48) should capture a flavour of the physics of (3.51).

Chapter 4

Effective metric derivation

There are several ways of deriving an effective metric from the field equations found in section 3.3, and different functions can be ascribed to the fields giving rise to distinct metrics. Firstly section 4.1 will consider metrics derived from individual field equations separately, with one of the solutions leading to the Unruh effect. Section 4.2 will consider effective metrics derived from perturbing the general field equations (3.36) and (3.40). In each of these, subsections will follow with specified forms of the fields Φ and ψ. While there will be no solutions analogous to the Schwarzschild metric, the Unruh effect is attainable again. Finally section 4.3 will consider the system with varying laser spot size as discussed in section 3.3.2 and show that it is possible to get an effective metric conformal to the Schwarzschild metric.

4.1 Individual field equations

Before turning to the bi-scalar field theory of Φ and ψ, it is instructive to explore the case of a single field, as is typically the focus of other analogue gravity systems. To do so, an effective metric can be derived from equation (3.40) or (3.36) individually. Note that they both resemble the field equation

$$d(\mathfrak{f}'(\nu) \star d\varphi) = 0, \tag{4.1}$$

for some function $\mathfrak{f}$, where $\nu = d\varphi \cdot d\varphi$, φ is some field, and prime indicates derivative with respect to ν. The effective metric will be derived for this general field and its properties dependant on Φ and ψ will be discussed in the immediately following sections.

Consider the action

$$S[\varphi] = \int_{\mathcal{M}} \mathfrak{f}(\nu) \star 1, \tag{4.2}$$

of which stationary variation gives equation (4.1). Perturbing the field such that $\varphi = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2 + \mathcal{O}(\epsilon^3)$ and Taylor expanding $\mathfrak{f}$ yields

$$\mathfrak{f}(\nu) = \mathfrak{f}_0 + 2\epsilon\mathfrak{f}_0' d\varphi_0 \cdot d\varphi_1 + \epsilon^2(\mathfrak{f}_0'(d\varphi_0 \cdot d\varphi_2 + d\varphi_1 \cdot d\varphi_1) + 2\mathfrak{f}_0''(d\varphi_0 \cdot d\varphi_1)^2) + \mathcal{O}(\epsilon^3), \tag{4.3}$$

where the zero subscript denotes functions evaluated at the solution φ_0 to equation (4.1). Noting that for any χ

$$\int_{\mathcal{M}} d\varphi_0 \cdot d\varphi_\chi \mathfrak{f}_0' \star 1 = \int_{\mathcal{M}} d\varphi_\chi \wedge (\mathfrak{f}_0' \star d\varphi_0) = \int_{\mathcal{M}} d(\varphi_\chi \mathfrak{f}_0' \star d\varphi_0) - \int_{\mathcal{M}} \varphi_\chi d(\mathfrak{f}_0' \star d\varphi_0) = 0, \tag{4.4}$$

results in

$$S[\varphi] = \epsilon^2 \int (\mathfrak{f}_0' d\varphi_1 \cdot d\varphi_1 + 2\mathfrak{f}_0''(d\varphi_0 \cdot d\varphi_1)^2) \star 1. \tag{4.5}$$

From this, an effective metric is given by

$$\sigma g_{\text{eff}}^{-1} = \mathfrak{f}_0' g^{-1} + 2\mathfrak{f}_0'' \widetilde{d\varphi}_0 \otimes \widetilde{d\varphi}_0, \tag{4.6}$$

where $\star_{\text{eff}} 1 = \sigma \star 1$, so that $g_{\text{eff}}^{-1}(d\varphi_1, d\varphi_1)$ is the integrand of equation (4.5). The function σ can be obtained by expressing the effective metric in terms of an orthonormal frame. To see this let $U = \frac{\widetilde{d\varphi_0}}{|d\varphi_0|}$, then

$$\begin{aligned} \sigma g_{\text{eff}}^{-1} =& \mathfrak{f}_0'(-U \otimes U + \Pi_U g^{-1}) + 2\mathfrak{f}_0''|d\varphi_0|^2 U \otimes U \\ =& -(\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\varphi_0|^2) U \otimes U + \mathfrak{f}_0' \Pi_U g^{-1}. \end{aligned} \tag{4.7}$$

Assuming g_{eff} is Lorentzian suggests the frame

$$\begin{aligned} X_0^{\text{eff}} =& \sqrt{\frac{\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\varphi_0|^2}{\sigma}} X_0, \quad X_j^{\text{eff}} = \sqrt{\frac{\mathfrak{f}_0'}{\sigma}} X_j \\ \implies e_{\text{eff}}^0 =& \sqrt{\frac{\sigma}{\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\varphi_0|^2}} e^0, \quad e_{\text{eff}}^j = \sqrt{\frac{\sigma}{\mathfrak{f}_0'}} e^j, \end{aligned} \tag{4.8}$$

where $e^0 = \tilde{U}$ and $j = 1, 2, 3$. Thus $\star_{\text{eff}} 1 = \sigma \star 1$ gives

$$\begin{aligned} \frac{\sigma^2}{\sqrt{(\mathfrak{f}_0')^4 - 2(\mathfrak{f}_0')^3 \mathfrak{f}_0''|d\varphi_0|^2}} =& \sigma \\ \implies \sigma = (\mathfrak{f}_0')^{\frac{3}{2}} \sqrt{\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\varphi_0|^2}&, \end{aligned} \tag{4.9}$$

and the effective metric becomes

$$g_{\text{eff}}^{-1} = \frac{\mathfrak{f}_0'}{\sqrt{(\mathfrak{f}_0')^3(\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\varphi_0|^2)}} g^{-1} + \frac{2\mathfrak{f}_0''}{\sqrt{(\mathfrak{f}_0')^3(\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\varphi_0|^2)}} \widetilde{d\varphi_0} \otimes \widetilde{d\varphi_0}. \quad (4.10)$$

Section 4.1.1 will consider the field equation for ψ, showing that it will never lead to a Lorentzian effective metric. Section 4.1.2 will consider the field equation for Φ using minimal energy density, showing it leads to a complex metric. Section 4.1.3 will consider the field equation for Φ with a general energy density, however the time dependence of the field will be assumed to be linear in order to obtain a solution. It will be shown to lead to a peculiar energy density, but also an effective metric that is conformally flat and that of a homogeneous plane wave, and a calculation of the Unruh effect will be presented. Similarly section 4.1.4 will consider the same system, but in radial coordinates, showing that the resulting equations are too difficult to solve analytically.

4.1.1 Field equation for ψ

Consider the case when $\varphi = \psi$ and the laser strength μ is not an independent variable. Equation (3.39) gives $|d\psi| = \frac{d\lambda}{dn}$, with which comparing equations (3.40) and (4.1) results in

$$\mathfrak{f}_0'(\nu) = \frac{\omega_p^2}{\sqrt{-\nu}}, \quad (4.11)$$

where $\omega_p^2 = \frac{nq^2}{m}$, and thus $\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\psi_0|^2 = 0$. Hence trivially there is no Lorentzian effective metric corresponding to (3.39) and (3.40) alone.

4.1.2 Field equation for Φ with minimal energy density

Consider the Φ field when the minimal energy density (3.41) is used and n is not an independent variable. Equation (3.35) becomes

$$d\Phi_0 \cdot d\Phi_0 = -\frac{\omega_p^2}{\sqrt{1+\frac{q^2}{m^2}\mu}}, \tag{4.12}$$

where $\omega_p^2 = \frac{nq^2}{m}$, which in conjunction with equation (3.36) results in $\mathfrak{f}_0' = \frac{\omega_p^4}{\nu^2} - 1$. Furthermore $\mathfrak{f}_0' - 2\mathfrak{f}_0''|d\Phi_0|^2 = -\frac{\omega_p^4}{\nu^2} - 1$, so

$$\sigma^2 = -\frac{1}{\nu^8}(\omega_p^4+\nu^2)(\omega_p^4-\nu^2)^3 = -\frac{1}{\nu^8}(\omega_p^8-\nu^4)^3(\omega_p^4-\nu^2)^2. \tag{4.13}$$

A consequence of equation (4.12) is that $\omega_p^8 > \nu^4$, making σ imaginary, and thus there is no Lorentzian effective metric corresponding to (3.35) and (3.36) alone for the minimal energy density case.

4.1.3 Field equation for Φ with general energy density

Equations (3.35) and (3.36), when n is not an independent variable, can describe a laser pulse propagating through a dielectric medium, not just a laser-driven plasma system. Instead of requiring minimal energy density given in (3.41), a form of $\lambda(\mu)$ can be found that is compatible with the existence of a Lorentzian effective metric. Let Φ_0 be of the form

$$\Phi_0 = \gamma t + h(z), \tag{4.14}$$

where γ is a constant and h is a function. Furthermore the timelike condition $d\Phi_0 \cdot d\Phi_0 < 0$ gives the constraint

$$\left(\frac{dh}{dz}\right)^2 < \gamma^2. \tag{4.15}$$

Looking at equation (4.1) now results in

$$\frac{d}{dz}\left[\frac{dh}{dz}\mathfrak{f}_0'(\nu)\right] = 0, \implies \mathfrak{f}_0'(\nu) = \frac{c_\mathfrak{f}}{\frac{dh}{dz}}, \tag{4.16}$$

for some constant $c_\mathfrak{f}$. At this point it is possible to deduce the form of λ from the field equation in terms of ν: $\mathfrak{f}_0'$ has to equal $\pm k\mu$ for some constant k due to equation (3.36), furthermore $\frac{dh}{dz} = \sqrt{\nu + \gamma^2}$, thus with equation (3.35):

$$\lambda = \frac{\gamma^2}{2}\mu + \frac{c_\mathfrak{f}^2}{2k^2\mu}. \tag{4.17}$$

The last term in the energy density increases for sufficiently small μ, but in the large μ limit it disappears. Noting that $\frac{d}{d\nu} = \frac{dz}{d\nu}\frac{d}{dz}$ and $\frac{dz}{d\nu} = \frac{1}{\frac{d\nu}{dz}} = \frac{1}{2\frac{d^2h}{dz^2}\frac{dh}{dz}}$ yields

$$\begin{aligned} \mathfrak{f}_0''(\nu) =& \frac{1}{2\frac{d^2h}{dz^2}\frac{dh}{dz}}\frac{d}{dz}\left(\frac{c_\mathfrak{f}}{\frac{dh}{dz}}\right) \\ =& -\frac{c_\mathfrak{f}}{2\left(\frac{dh}{dz}\right)^3}. \end{aligned} \tag{4.18}$$

A change of sign in $\frac{dh}{dz}$ will change the overall sign of the effective metric, so without any loss of generality the numerators can be brought into the square roots in equation (4.10), thus with $dz = \frac{1}{\frac{dh}{dz}} dh$,

$$g_{\text{eff}} = \frac{|c_{\text{f}}\gamma|}{\left(\frac{dh}{dz}\right)^2} \left[\frac{1}{\gamma}(dt \otimes dh + dh \otimes dt) + \left(\frac{1}{\gamma^2} + \frac{1}{\left(\frac{dh}{dz}\right)^2} \right) dh \otimes dh + dx \otimes dx + dy \otimes dy \right]. \tag{4.19}$$

Introducing a coordinate change

$$t = v - \Xi(h), \quad \zeta = \frac{h}{\gamma}, \tag{4.20}$$

where $2\frac{1}{\gamma}\frac{d\Xi}{dh} = \frac{1}{\gamma^2} + \frac{1}{\left(\frac{dh}{dz}\right)^2}$ and letting $\Omega(h) = \frac{|c_{\text{f}}\gamma|}{\left(\frac{dh}{dz}\right)^2}$, shows that the effective metric is conformally flat:

$$g_{\text{eff}} = \Omega(\gamma\zeta) \left[dv \otimes d\zeta + d\zeta \otimes dv + dx \otimes dx + dy \otimes dy \right]. \tag{4.21}$$

With a further substitution of $du = \Omega d\zeta$, this metric becomes that of a homogeneous plane wave in Rosen coordinates. While there is no particle creation in such a spacetime [54], it might be of interest in pursuing analogue models of string theory [55], but that is beyond the scope of this discussion. However, if considered as a 2-dimensional effective metric, the Unruh effect calculation applies, hence the motion of a detector in the laboratory frame corresponding to a uniformly accelerated observer in the effective metric can be deduced. By using the coordinate transformation

$$u = \frac{Z - T}{\sqrt{2}} \quad v = \frac{Z + T}{\sqrt{2}} \tag{4.22}$$

the metric becomes

$$g_{\text{eff}} = -dT \otimes dT + dZ \otimes dZ. \tag{4.23}$$

A uniformly accelerated observer in the right Rindler wedge of this metric is described by the parametrization

$$Z = \frac{1}{a} e^{a\xi} \cosh(a\eta) \quad T = \frac{1}{a} e^{a\xi} \sinh(a\eta), \tag{4.24}$$

for some constants a and ξ. Firstly note that

$$\begin{aligned} \xi =& \frac{1}{2a} \ln\big(a^2(Z^2 - T^2)\big) = \frac{1}{2a} \ln\big(2a^2 uv\big), \\ \eta =& \frac{1}{2a} \ln\left(\frac{Z+T}{Z-T}\right) = \frac{1}{2a} \ln\left(\frac{v}{u}\right), \end{aligned} \tag{4.25}$$

thus

$$v = \frac{1}{a\sqrt{2}} e^{a(\xi+\eta)}, \quad u = \frac{1}{a\sqrt{2}} e^{a(\xi-\eta)}. \tag{4.26}$$

By definition u satisfies

$$\frac{du}{dz}\frac{dh}{dz} = \frac{|c_{\mathfrak{f}}\gamma|}{\gamma}, \tag{4.27}$$

furthermore $\Xi = \frac{h}{2\gamma} + \frac{\gamma}{2}\int (\frac{dh}{dz})^{-1} dz$, hence

$$t(\eta) = \frac{1}{a\sqrt{2}} e^{a(\xi+\eta)} - \frac{h(z(\eta))}{2\gamma} - \frac{\gamma}{2}\left[\int \frac{1}{\frac{dh}{dz}} dz\right]\Bigg|_{z=z(\eta)}, \tag{4.28}$$

Now $z(\eta)$ and $t(\eta)$ can be found once $h(z)$ is specified. Figures 4.1 and 4.2 show the motion and proper acceleration respectively in the laboratory frame for a simple choice of $h(z) = z$, with arbitrarily chosen constants as an example. The plots show the motion starts with an approximately constant velocity in the past followed by a period of acceleration near $t = 0$ and tending to $z = 0$ for large t, and the acceleration is prominent around $t = 0$ and tends to zero for large $|t|$.

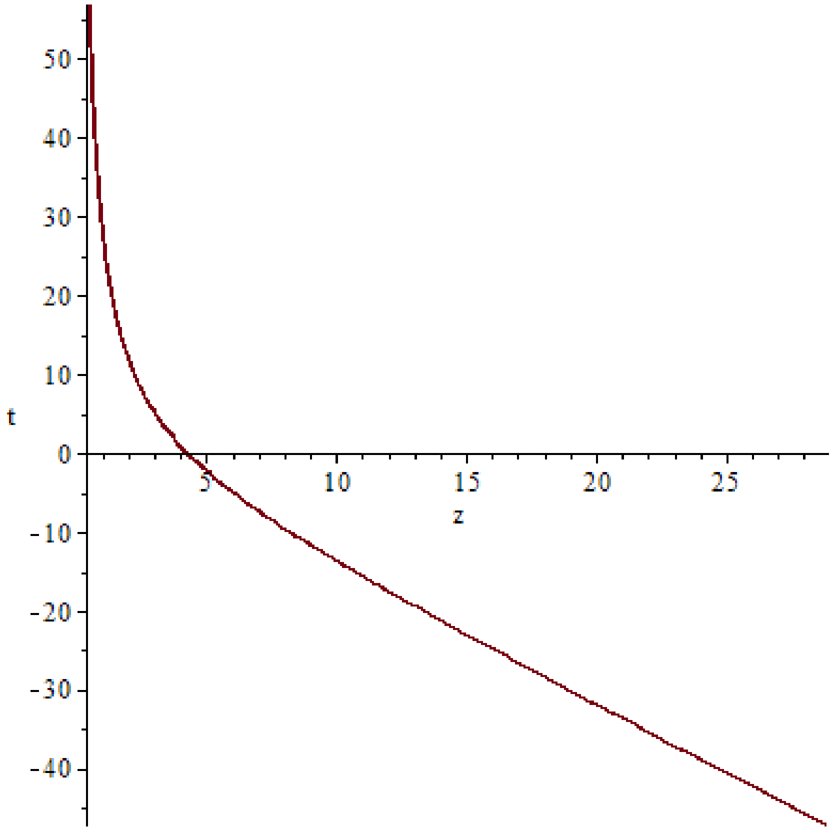

Figure 4.1: Plot of the motion described by (4.28) with $h(z) = \ln(\cosh(z))$ for arbitrarily chosen constants $a = 1$, $\xi = 2$, $c_{\mathfrak{f}} = 2$ and $\gamma = 3$.

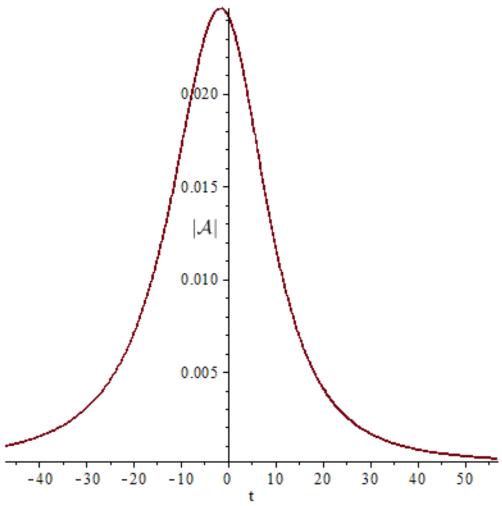

Figure 4.2: Plot of the magnitude of the acceleration $|\mathcal{A}|$ (see equation (2.5)) described by (4.28) with $h(z) = \ln(\cosh(z))$ for arbitrarily chosen constants $a = 0.1$, $\xi = 1$, $c_{\mathfrak{f}} = 2$ and $\gamma = 3$.

4.1.4 Field equation for Φ with general energy density in spherical coordinates

Let Φ_0 be of the form

$$\Phi_0 = \gamma t + h(r), \tag{4.29}$$

where γ is a constant and h is a function of the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$ in spherical coordinates. Looking at equation (4.1) now results in

$$\begin{aligned} &\frac{d}{dr}\left[r^2 \frac{dh}{dr}\mathfrak{f}'\left(-\gamma^2 + \left(\frac{dh}{dr}\right)^2\right)\right] = 0, \\ &\implies \mathfrak{f}_0'(\xi) = \frac{c}{r^2 \frac{dh}{dr}}, \end{aligned} \tag{4.30}$$

for some constant c. With $\frac{d}{d\xi} = \frac{dr}{d\xi}\frac{d}{dr}$ and $\frac{dr}{d\xi} = \frac{1}{\frac{d\xi}{dr}} = \frac{1}{2\frac{d^2h}{dr^2}\frac{dh}{dr}}$,

$$\begin{aligned} f_0''(\xi) =& \frac{1}{2\frac{d^2h}{dr^2}\frac{dh}{dr}}\frac{d}{dr}\left(\frac{c}{r^2\frac{dh}{dr}}\right) \\ =& -c\frac{r\frac{d^2h}{dr^2}+2\frac{dh}{dr}}{2r^3\frac{d^2h}{dr^2}\left(\frac{dh}{dr}\right)^3}. \end{aligned} \tag{4.31}$$

Ricci and Kretschmann scalars can be obtained with the use of algebraic software, however there are no simple algebraic solutions when the former is equal to zero or the latter is proportional to $\frac{k}{r}$ for some constant k. Thus this approach is not fruitful for obtaining static effective metrics that are of interest in analogue gravity.

4.2 High frequency approach and the prominence of the minimal energy density

The full field system (3.35), (3.36), (3.39) and (3.40) for Φ and ψ will now be considered. Unfortunately, simply perturbing the field equations does not immediately yield an obvious way to extract an effective metric as was the case in section 4.1. However as will be shown, it is possible with the assumption that the perturbations have high frequency. This method also requires the minimal energy density to be used, as will be demonstrated. Consider the field equations (3.36) and (3.40) using a general λ. By perturbing μ and n such that $\mu = \mu_0 + \epsilon\mu_1 + \mathcal{O}(\epsilon^2)$, $n = n_0 + \epsilon n_1 + \mathcal{O}(\epsilon^2)$ and defining

$\mathfrak{k}(\mu,n)\big|_0 \equiv \mathfrak{k}(\mu_0,n_0)$ for any function $\mathfrak{k}$, the following relations are acquired:

$$-2\frac{\partial\lambda}{\partial\mu} = -2\left.\frac{\partial\lambda}{\partial\mu}\right|_0 - 2\epsilon\left(\left.\frac{\partial}{\partial n}\frac{\partial\lambda}{\partial\mu}\right|_0 n_1 + \left.\frac{\partial}{\partial\mu}\frac{\partial\lambda}{\partial\mu}\right|_0 \mu_1\right), \tag{4.32}$$

$$-\left(\frac{\partial\lambda}{\partial n}\right)^2 = -\left.\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 - \epsilon\left(\left.\frac{\partial}{\partial n}\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 n_1 + \left.\frac{\partial}{\partial\mu}\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 \mu_1\right), \tag{4.33}$$

$$\frac{n}{\frac{\partial\lambda}{\partial n}} = \left.\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0 + \epsilon\left(\left.\frac{\partial}{\partial n}\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0 n_1 + \left.\frac{\partial}{\partial\mu}\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0 \mu_1\right). \tag{4.34}$$

Perturbing Φ and ψ in a similar fashion, equations (3.35) and (3.39) in first order of ϵ give

$$d\Phi_0\cdot d\Phi_1 = -\left.\frac{\partial^2\lambda}{\partial n\partial\mu}\right|_0 n_1 - \left.\frac{\partial^2\lambda}{\partial\mu^2}\right|_0 \mu_1, \tag{4.35}$$

$$2d\psi_0\cdot d\psi_1 = -\left.\frac{\partial}{\partial n}\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 n_1 - \left.\frac{\partial}{\partial\mu}\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 \mu_1 \tag{4.36}$$

respectively. These can be solved as a system of linear equations for n_1 and μ_1, giving

$$n_1 = \frac{2\left.\frac{\partial^2\lambda}{\partial\mu^2}\right|_0 (d\psi_0\cdot d\psi_1) - \left.\frac{\partial}{\partial\mu}\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 (d\Phi_0\cdot d\Phi_1)}{\left.\left(\frac{\partial^2\lambda}{\partial n\partial\mu}\frac{\partial}{\partial\mu}\left(\frac{\partial\lambda}{\partial n}\right)^2 - \frac{\partial^2\lambda}{\partial\mu^2}\frac{\partial}{\partial n}\left(\frac{\partial\lambda}{\partial n}\right)^2\right)\right|_0}, \tag{4.37}$$

$$\mu_1 = \frac{\left.\frac{\partial}{\partial n}\left(\frac{\partial\lambda}{\partial n}\right)^2\right|_0 (d\Phi_0\cdot d\Phi_1) - 2\left.\frac{\partial^2\lambda}{\partial n\partial\mu}\right|_0 (d\psi_0\cdot d\psi_1)}{\left.\left(\frac{\partial^2\lambda}{\partial n\partial\mu}\frac{\partial}{\partial\mu}\left(\frac{\partial\lambda}{\partial n}\right)^2 - \frac{\partial^2\lambda}{\partial\mu^2}\frac{\partial}{\partial n}\left(\frac{\partial\lambda}{\partial n}\right)^2\right)\right|_0}. \tag{4.38}$$

The field equations (3.36) and (3.40) in first order of ϵ read

$$d(\mu_0\star d\Phi_1 + \mu_1\star d\Phi_0) = 0, \tag{4.39}$$

$$d\left(\left.\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0 \star d\psi_1 + \left(\left.\frac{\partial}{\partial n}\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0 n_1 + \left.\frac{\partial}{\partial\mu}\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0 \mu_1\right)\star d\psi_0\right) = 0. \tag{4.40}$$

Substituting the expressions for n_1 and μ_1 and introducing

$$p_0 = \left.\frac{n}{\frac{\partial\lambda}{\partial n}}\right|_0, \tag{4.41}$$

$$q_0 = \left.\frac{\frac{\partial^2\lambda}{\partial n^2}}{\left(\left(\frac{\partial^2\lambda}{\partial n\partial\mu}\right)^2 - \frac{\partial^2\lambda}{\partial n^2}\frac{\partial^2\lambda}{\partial\mu^2}\right)}\right|_0, \tag{4.42}$$

$$r_0 = -\left.\frac{\frac{\partial^2\lambda}{\partial n\partial\mu}}{\frac{\partial\lambda}{\partial n}\left(\left(\frac{\partial^2\lambda}{\partial n\partial\mu}\right)^2 - \frac{\partial^2\lambda}{\partial n^2}\frac{\partial^2\lambda}{\partial\mu^2}\right)}\right|_0, \tag{4.43}$$

$$s_0 = 2\left.\frac{\left(\frac{\partial^2\lambda}{\partial\mu^2}\frac{\partial}{\partial n}\frac{n}{\frac{\partial\lambda}{\partial n}} - \frac{\partial^2\lambda}{\partial n\partial\mu}\frac{\partial}{\partial\mu}\frac{n}{\frac{\partial\lambda}{\partial n}}\right)}{\left(\frac{\partial^2\lambda}{\partial n\partial\mu}\frac{\partial}{\partial\mu}\left(\frac{\partial\lambda}{\partial n}\right)^2 - \frac{\partial^2\lambda}{\partial\mu^2}\frac{\partial}{\partial n}\left(\frac{\partial\lambda}{\partial n}\right)^2\right)}\right|_0, \tag{4.44}$$

the field equations (4.39) and (4.40) become

$$d(\mu_0 \star d\Phi_1 + q_0(d\Phi_0\cdot d\Phi_1)\star d\Phi_0 + r_0(d\psi_0\cdot d\psi_1)\star d\Phi_0,) = 0, \tag{4.45}$$

$$d(p_0 \star d\psi_1 + r_0(d\Phi_0\cdot d\Phi_1)\star d\psi_0 + s_0(d\psi_0\cdot d\psi_1)\star d\psi_0) = 0. \tag{4.46}$$

To extract a metric, the perturbations will be assumed to have high frequency, such that $\Phi_1 = \mathrm{Re}\left(a_\eta e^{i\frac{K}{\eta}}\right)$ and $\psi_1 = \mathrm{Re}\left(b_\eta e^{i\frac{K}{\eta}}\right)$, for some parameter η. Only the lowest order of η is required, where $a_\eta = a_0 + \mathcal{O}(\eta)$ and $b_\eta = b_0 + \mathcal{O}(\eta)$. Using $d\Phi_1 = \mathrm{Re}\left(\frac{i}{\eta}a_0 e^{i\frac{K}{\eta}}dK\right)$ and $d\psi_1 = \mathrm{Re}\left(\frac{i}{\eta}b_0 e^{i\frac{K}{\eta}}dK\right)$, the field equations (4.45) and (4.46) in the lowest order of η lead to

$$(q_0(d\Phi_0 \cdot dK)a_0 + r_0(d\psi_0 \cdot dK)b_0)(d\Phi_0 \cdot dK) + \mu_0(dK \cdot dK)a_0 = 0, \quad (4.47)$$

$$(r_0(d\Phi_0 \cdot dK)a_0 + s_0(d\psi_0 \cdot dK)b_0)(d\psi_0 \cdot dK) + p_0(dK \cdot dK)b_0 = 0 \quad (4.48)$$

respectively. These in turn can be written as a matrix multiplied by a vector:

$$\begin{pmatrix} q_0(d\Phi_0 \cdot dK)^2 + \mu_0(dK \cdot dK) & r_0(d\Phi_0 \cdot dK)(d\psi_0 \cdot dK) \\ r_0(d\Phi_0 \cdot dK)(d\psi_0 \cdot dK) & s_0(d\psi_0 \cdot dK)^2 + p_0(dK \cdot dK) \end{pmatrix} \begin{pmatrix} b_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.49)$$

The determinant of this matrix must be zero for the vector on the left-hand side of (4.49) to be non-zero. Thus

$$\begin{aligned} (q_0(d\Phi_0 \cdot dK)^2 + \mu_0(dK \cdot dK))(s_0(d\psi_0 \cdot dK)^2 + p_0(dK \cdot dK)) & \\ -r_0^2(d\Phi_0 \cdot dK)^2(d\psi_0 \cdot dK)^2 &= 0, \end{aligned} \quad (4.50)$$

which can be rearranged to give

$$\begin{aligned} \mu_0 p_0(dK \cdot dK)^2 + (q_0 s_0 - r_0^2)(d\Phi_0 \cdot dK)^2(d\psi_0 \cdot dK)^2 & \\ +(dK \cdot dK)(p_0 q_0(d\Phi_0 \cdot dK)^2 + s_0\mu_0(d\psi_0 \cdot dK)^2) = 0. & \end{aligned} \quad (4.51)$$

The simplest way to extract one or more effective metrics from (4.50) is to demand that it can be factorised. Introducing two symmetric rank two tensors

$$g_{\mathrm{I}}^{-1} = a_{\mathrm{I}} g^{-1} + b_{\mathrm{I}} \widetilde{d\Phi}_0 \otimes \widetilde{d\Phi}_0 + c_{\mathrm{I}} \widetilde{d\psi}_0 \otimes \widetilde{d\psi}_0 + d_{\mathrm{I}}(\widetilde{d\Phi}_0 \otimes \widetilde{d\psi}_0 + \widetilde{d\psi}_0 \otimes \widetilde{d\Phi}_0), \quad (4.52)$$

$$g_{\mathrm{II}}^{-1} = a_{\mathrm{II}} g^{-1} + b_{\mathrm{II}} \widetilde{d\Phi_0} \otimes \widetilde{d\Phi_0} + c_{\mathrm{II}} \widetilde{d\psi_0} \otimes \widetilde{d\psi_0} + d_{\mathrm{II}} (\widetilde{d\Phi_0} \otimes \widetilde{d\psi_0} + \widetilde{d\psi_0} \otimes \widetilde{d\Phi_0}), \quad (4.53)$$

the coefficients need to be matched such that $g_{\mathrm{I}}^{-1}(dK, dK) g_{\mathrm{II}}^{-1}(dK, dK)$ gives the determinant of the matrix. Here g^{-1} is the inverse of the background metric, which will be set to the Minkowski metric in the following sections. These two tensors are then the inverses of two effective metrics. This procedure gives the following relations:

$$a_{\mathrm{I}} a_{\mathrm{II}} = \mu_0 p_0, \quad (4.54)$$

$$4 d_{\mathrm{I}} d_{\mathrm{II}} + b_{\mathrm{I}} c_{\mathrm{II}} + b_{\mathrm{II}} c_{\mathrm{I}} = q_0 s_0 - r_0^2, \quad (4.55)$$

$$a_{\mathrm{I}} b_{\mathrm{II}} + a_{\mathrm{II}} b_{\mathrm{I}} = p_0 q_0, \quad (4.56)$$

$$a_{\mathrm{I}} c_{\mathrm{II}} + a_{\mathrm{II}} c_{\mathrm{I}} = s_0 \mu_0, \quad (4.57)$$

$$b_{\mathrm{I}} b_{\mathrm{II}} = 0, \quad (4.58)$$

$$c_{\mathrm{I}} c_{\mathrm{II}} = 0, \quad (4.59)$$

$$a_{\mathrm{I}} d_{\mathrm{II}} + a_{\mathrm{II}} d_{\mathrm{I}} = 0, \quad (4.60)$$

$$b_{\mathrm{I}} d_{\mathrm{II}} + b_{\mathrm{II}} d_{\mathrm{I}} = 0, \quad (4.61)$$

$$c_{\mathrm{I}} d_{\mathrm{II}} + c_{\mathrm{II}} d_{\mathrm{I}} = 0. \quad (4.62)$$

Since $\mu_0 p_0 \neq 0$, inspection of (4.54) shows $a_{\mathrm{I}} \neq 0$ and $a_{\mathrm{II}} \neq 0$. If $d_{\mathrm{I}} \neq 0$ and $d_{\mathrm{II}} \neq 0$, then equations (4.60), (4.61) and (4.62) can be re-written as $a_{\mathrm{I}} b_{\mathrm{II}} - a_{\mathrm{II}} b_{\mathrm{I}} = 0$, $a_{\mathrm{I}} c_{\mathrm{II}} - a_{\mathrm{II}} c_{\mathrm{I}} = 0$ and $b_{\mathrm{I}} c_{\mathrm{II}} - b_{\mathrm{II}} c_{\mathrm{I}} = 0$ which together with

(4.58) and (4.59) yield

$$b_{\mathrm{I}} = b_{\mathrm{II}} = c_{\mathrm{I}} = c_{\mathrm{II}} = 0. \tag{4.63}$$

This in turn gives $q_0 = 0$ and $s_0 = 0$ by (4.56) and (4.57), leading to two partial differential equations:

$$\frac{\partial^2 \lambda}{\partial n^2} = 0, \tag{4.64}$$

$$n\left(\left(\frac{\partial^2 \lambda}{\partial \mu \partial n}\right)^2 - \frac{\partial^2 \lambda}{\partial \mu^2}\frac{\partial^2 \lambda}{\partial n^2}\right) + \frac{\partial^2 \lambda}{\partial \mu^2}\frac{\partial \lambda}{\partial n} = 0, \tag{4.65}$$

from equations (4.42) and (4.44) respectively. Equation (4.64) follows from (4.42), and (4.65) follows from (4.44). Equation (4.64) yields $\lambda = F_1(\mu)n + F_2(\mu)$, which in conjunction with (4.65) results in

$$n\left(\frac{dF_1}{d\mu}\right)^2 + \left(\frac{d^2 F_1}{d\mu^2}n + \frac{d^2 F_2}{d\mu^2}\right) F_1 = 0. \tag{4.66}$$

Note that $F_1 = 0$ trivially satisfies this equation, but this is not a physical choice for λ as it is independent of the averaged number density n. Solving (4.66) for F_2 yields

$$F_2 = -n \int\int \frac{\left(\frac{dF_1}{d\mu}\right)^2 + \frac{d^2 F_1}{d\mu^2} F_1}{F_1} d\mu d\mu, \tag{4.67}$$

however no matter what F_1 is, there is a factor of n thus contradicting the fact that F_2 is just a function of μ. Hence $F_2 = 0$. Finally (4.66) becomes

$$\left(\frac{dF_1}{d\mu}\right)^2 + \frac{d^2F_1}{d\mu^2}F_1 = 0, \tag{4.68}$$

which solves to $F_1 = \pm\sqrt{C_1 + C_2\mu}$, thus

$$\lambda = \pm n\sqrt{C_1 + C_2\mu}. \tag{4.69}$$

Note that $d_{\mathrm{I}} = d_{\mathrm{II}} = 0$ could have been chosen instead of (4.63). However this choice does not lead to choices of λ that include (4.69). The form of λ in (4.69) recovers the standard relativistic ponderomotive force, and as such $d_{\mathrm{I}} = d_{\mathrm{II}} = 0$ is not a valid physical choice. In summary, starting from the full field equations (3.35), (3.36), (3.39) and (3.40) for Φ and ψ with a general energy density λ, the perturbations are assumed to be of high frequency in order to derive two effective metrics. When the coefficients of the metrics are matched to the underlying field equations, it appears that the only physically sensible choice for λ is the minimal energy density.

The remaining coefficients in (4.52) and (4.53) are

$$\begin{aligned} a_{\mathrm{I}} &= \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)}d_{\mathrm{I}},\ d_{\mathrm{I}} = d_{\mathrm{I}}, \\ a_{\mathrm{II}} &= -\sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)}d_{\mathrm{II}},\ d_{\mathrm{II}} = \frac{1}{d_{\mathrm{I}}}. \end{aligned} \tag{4.70}$$

Choosing $d_{\mathrm{I}} = 1$ yields two effective metrics that are given by

$$g_{\mathrm{eff}}^{-1} = \widetilde{d\Phi_0} \otimes \widetilde{d\psi_0} + \widetilde{d\psi_0} \otimes \widetilde{d\Phi_0} \pm \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)}g^{-1}. \tag{4.71}$$

Note that at this stage it is not important to distinguish between the $+$ and $-$ sign choice. It is also required that the effective metrics are Lorentzian, which leads to a constraint on $d\Phi_0$ and $d\psi_0$. For example, suppose that the fields are functions of the light-cone coordinates only, then the $\partial_x \otimes \partial_x$ and $\partial_y \otimes \partial_y$ components of the metric are both either positive or negative. The $\partial_u \otimes \partial_u$ and $\partial_v \otimes \partial_v$ components are $8\partial_v \Phi_0 \partial_v \psi_0$ and $8\partial_u \Phi_0 \partial_u \psi_0$, the $\partial_u \otimes \partial_v$ and $\partial_v \otimes \partial_u$ components are $4(\partial_u \Phi_0 \partial_v \psi_0 + \partial_v \Phi_0 \partial_u \psi_0) \pm 2\sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)}$ which is equivalent to $2(d\Phi_0 \cdot d\psi_0 \pm \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)})$, and the remaining components are zero. A Lorentzian signature is obtained when $\det\left(g_{\text{eff}}^{-1}\right) < 0$, which gives

$$4[(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0) - (d\Phi_0 \cdot d\psi_0 \pm \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)})^2] < 0. \tag{4.72}$$

A little rearranging and taking the square root yields the constraint

$$-d\Phi_0 \cdot d\psi_0 > \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0)} + \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)}. \tag{4.73}$$

Although both metrics are Lorentzian, it can be inferred that the effective metric (4.71) with positive sign choice will have the same signature as the background metric, while the negative sign choice gives the opposite signature. To proceed further, the forms of Φ_0 and ψ_0 are required. The field equations are too complicated for a general solution, but there are several regimes which lead to a manageable system which require the fields to be functions of two variables (t, z), or equivalently the light-cone coordinates (u, v). Section 4.2.1 will consider the fields as linear functions of t and z, leading to metrics with constants coefficients which are conformally flat, and the Unruh effect calculation will be discussed. It is also worth noting that the field equations result in linear fields when requiring them to be of the form $\gamma t + h(z)$. Section 4.2.2 will consider fields as solutions to the 2-dimensional wave equation, leading to effective metrics with geometric singularities, however it will be shown that there are no horizons; thus they are naked singularities. Section 4.2.3 will consider the system in spherical coordinates with a linear time component, showing they lead to systems which are not physical. However, it will be shown later that introducing an extra degree of freedom (the spot size of the laser) leads to physically interesting effective metrics.

4.2.1 Fields as linear functions

The simplest model is given by Φ_0 and ψ_0 being linear functions of t and z, or equivalently the light-cone coordinates. The background metric will be considered as 2-dimensional for brevity, but the result easily translates to three spatial dimensions. Introduce $\Phi_0 = c_1 u + c_2 v$, $\psi_0 = c_3 u + c_4 v$, for some constants c_1, c_2, c_3, c_4, where u and v are the light-cone coordinates. In order to keep $d\Phi_0$ and $d\psi_0$ timelike, either c_1 is positive with c_2 being negative or the other way around, and similarly either c_3 is negative and c_4 is positive or the converse is true. For this analysis it will be assumed that c_2 and c_4 are negative, but it is straightforward to change that by introducing minus signs in relevant places. The effective metrics now become

$$g_{\text{eff}} = \Omega^2 \Big[-4c_1c_3 du \otimes du - 4c_2c_4 dv \otimes dv \\ + \Big(2c_1c_4 + 2c_2c_3 \pm \sqrt{4c_1c_2(4c_3c_4 + m^2)}\Big) (du \otimes dv + dv \otimes du) \Big], \tag{4.74}$$

where $\Omega^2 = -\frac{2}{\det\left(g_{\text{eff}}^{-1}\right)}$. With the transformation $\mathfrak{z} - \mathfrak{t} = \sqrt{4c_1c_3}u$, $\mathfrak{z} + \mathfrak{t} = \sqrt{4c_2c_4}v$, the metrics become

$$g_{\text{eff}} = 2\Omega^2(-(C_{uv} + 1)d\mathfrak{t} \otimes d\mathfrak{t} + (C_{uv} - 1)d\mathfrak{z} \otimes d\mathfrak{z}), \tag{4.75}$$

where

$$\begin{aligned} C_{uv} &= -\frac{2c_1c_4 + 2c_2c_3 \pm \sqrt{4c_1c_2(4c_3c_4 + m^2)}}{\sqrt{16c_1c_2c_3c_4}} \\ &= -\frac{d\Phi_0 \cdot d\psi_0 \pm \sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)}}{\sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0)}}. \end{aligned} \tag{4.76}$$

Finally letting $T = \Omega\sqrt{2}\sqrt{C_{uv} + 1}\mathfrak{t}$ and $Z = \Omega\sqrt{2}\sqrt{C_{uv} - 1}\mathfrak{z}$ puts the metrics into the form:

$$g_{\text{eff}} = -dT \otimes dT + dZ \otimes dZ. \tag{4.77}$$

Requiring $C_{uv} - 1 > 0$ is equivalent to the condition given in (4.73). Now an observer that is uniformly accelerated in the effective spacetime, in the right Rindler wedge is described by the parametrization

$$Z = \frac{1}{a}e^{a\xi}\cosh(a\eta), \quad T = \frac{1}{a}e^{a\xi}\sinh(a\eta). \tag{4.78}$$

Thus $\mathfrak{z} = \frac{1}{a\Omega\sqrt{2}\sqrt{C_{uv}-1}} e^{a\xi}\cosh(a\eta)$ and $\mathfrak{t} = \frac{1}{a\Omega\sqrt{2}\sqrt{C_{uv}+1}} e^{a\xi}\sinh(a\eta)$. In the light-cone coordinates adapted to the background metric this becomes

$$
\begin{aligned}
u =& \frac{1}{2a\Omega\sqrt{2}\sqrt{C_{uv}-1}\sqrt{c_1c_3}} e^{a\xi}\cosh(a\eta) - \frac{1}{2a\Omega\sqrt{2}\sqrt{C_{uv}+1}\sqrt{c_1c_3}} e^{a\xi}\sinh(a\eta),\\
v =& \frac{1}{2a\Omega\sqrt{2}\sqrt{C_{uv}-1}\sqrt{c_2c_4}} e^{a\xi}\cosh(a\eta) + \frac{1}{2a\Omega\sqrt{2}\sqrt{C_{uv}+1}\sqrt{c_2c_4}} e^{a\xi}\sinh(a\eta),
\end{aligned}
\tag{4.79}
$$

yielding

$$
\begin{aligned}
z(\eta) =& \frac{e^{a\xi}}{4a\Omega\sqrt{2}}\left[\frac{\cosh(a\eta)}{\sqrt{C_{uv}-1}}\left(\frac{1}{\sqrt{c_2c_4}} + \frac{1}{\sqrt{c_1c_3}}\right) + \frac{\sinh(a\eta)}{\sqrt{C_{uv}+1}}\left(\frac{1}{\sqrt{c_2c_4}} - \frac{1}{\sqrt{c_1c_3}}\right)\right],\\
t(\eta) =& \frac{e^{a\xi}}{4a\Omega\sqrt{2}}\left[\frac{\cosh(a\eta)}{\sqrt{C_{uv}-1}}\left(\frac{1}{\sqrt{c_2c_4}} - \frac{1}{\sqrt{c_1c_3}}\right) + \frac{\sinh(a\eta)}{\sqrt{C_{uv}+1}}\left(\frac{1}{\sqrt{c_2c_4}} + \frac{1}{\sqrt{c_1c_3}}\right)\right].
\end{aligned}
\tag{4.80}
$$

Noting that

$$
\left(\frac{dt}{d\eta}\right)^2 - \left(\frac{dz}{d\eta}\right)^2 = \frac{e^{2a\xi}}{4\Omega^2\sqrt{2}}\frac{1 + C_{uv} - 2\cosh^2(a\eta)}{\sqrt{c_1c_2c_3c_4}(C_{uv}^2 - 1)}, \tag{4.81}
$$

reveals that the observer can only exist in a small portion of the background spacetime given by $1 + C_{uv} - 2\cosh^2(a\eta) > 0$, because the tangent to the curve $(t(\eta), z(\eta))$ becomes spacelike beyond it. The proper acceleration also becomes divergent at the transition point. The Unruh effect is strictly defined for an observer that is accelerating for all time; however this is experimentally unfeasible, and instead a finite time interval needs to be chosen. Thus the timelike region may be extended, which can be achieved by tweaking the parameters associated with Φ_0 and ψ_0 to maximise C_{uv}, such that it is sufficiently large. Figure 4.3 shows the motion of such an observer in the laboratory frame, which resembles a uniformly accelerated motion, but figure 4.4 shows non-constant acceleration, which diverges as expected at the points where $1 + C_{uv} - 2\cosh^2(a\eta) = 0$. The results for the minus sign choice in the effective metric are similar, with only minor changes in the numerical values, and as such their discussion will be omitted.

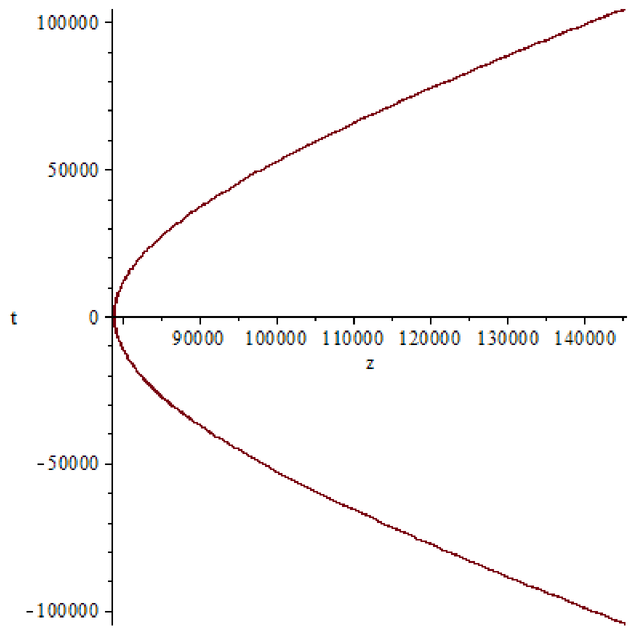

Figure 4.3: Plot of the motion in the lab frame of the corresponding uniformly accelerated observer in the effective metric with plus sign chosen, in the range $1 + C_{uv} - 2\cosh^2(a\eta) > 0$. The constants were chosen to be $c_1 = -c_4 = 15$ and $-c_2 = c_3 = a = \xi = m = 1$. These values are for illustrative purposes only.

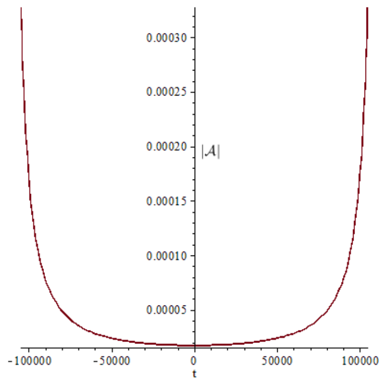

Figure 4.4: Plot of the magnitude of the acceleration $|\mathcal{A}|$ (see equation (2.5)) in the lab frame of the corresponding uniformly accelerated observer in the effective metric with plus sign chosen, in the range $1 + C_{uv} - 2\cosh^2(a\eta) > 0$. The constants were chosen to be $c_1 = -c_4 = 15$ and $-c_2 = c_3 = a = \xi = m = 1$. These values are for illustrative purposes only.

4.2.2 Fields as solutions of 2-dimensional wave equation

Suppose that Φ_0 and ψ_0 are functions of (t, z) only, and $d \star d\Phi_0 = 0$ and $d \star d\psi_0 = 0$. Utilising light-cone coordinates gives the general solutions $\Phi_0 = \Phi_-(u) + \Phi_+(v)$ and $\psi_0 = \psi_-(u) + \psi_+(v)$, with which the field equations (3.44) and (3.45) reduce to

$$\frac{d^2\Phi_-}{du^2}\frac{d\psi_+}{dv}\frac{d\Phi_+}{dv} = -\frac{d^2\Phi_+}{dv^2}\frac{d\psi_-}{dv}\frac{d\Phi_-}{dv}, \tag{4.82}$$

$$\frac{d^2\psi_-}{du^2}\frac{d\psi_+}{dv}\frac{d\Phi_+}{dv} = -\frac{d^2\psi_+}{dv^2}\frac{d\psi_-}{dv}\frac{d\Phi_-}{dv}. \tag{4.83}$$

A solution to (4.82) and (4.83) can be found:

$$\begin{aligned}
\Phi_+ &= \Big(\ln\big(e^{c_7(v+c_8)}\big) - \ln\big(-1 + c_2 e^{c_7(v+c_8)}\big)\Big)\frac{c_6}{c_1} + c_{10},\\
\Phi_- &= \Big(-\ln\big(e^{c_1(u+c_3)}\big) + \ln\big(-1 + c_2 e^{c_1(u+c_3)}\big)\Big)\frac{c_6}{c_1} + c_5,\\
\psi_+ &= \frac{\ln\big(-1 + c_2 e^{c_7(v+c_8)}\big)}{c_2} + c_9,\\
\psi_- &= -\frac{\ln\big(-1 + c_2 e^{c_1(u+c_3)}\big)}{c_2} + c_4,
\end{aligned} \tag{4.84}$$

where c_j are constant for $j = 1, 2, .., 10$. Without loss of generality, c_4, c_5, c_9 and c_{10} can be set to zero as only the derivatives of (4.84) are of interest. The effective metrics are quite complicated and no insight is gained from writing them out, so to show their general properties other constants will be set to $c_1 = 1$, $c_2 = 1$, $c_6 = \frac{1}{2}$, $c_7 = 1$, $c_8 = 1$ and $m = 2$. Furthermore, coordinates $\mathfrak{u} = e^u$ and $\mathfrak{v} = e^v$ will be adopted and a 2-dimensional system will be considered. The fields now become

$$d\Phi_0 = \frac{1}{2\mathfrak{u}(\mathfrak{u}-1)}d\mathfrak{u} - \frac{1}{2\mathfrak{v}(\mathfrak{v}-1)}d\mathfrak{v}, \tag{4.85}$$

$$d\psi_0 = -\frac{1}{\mathfrak{u}-1}d\mathfrak{u} + \frac{1}{V-1}dV. \tag{4.86}$$

Note that in order to keep the fields real-valued, $\mathfrak{u} > 1$ and $\mathfrak{v} > 1$ is required, or equivalently $z > |t|$. Firstly consider the effective metric obtained from choosing + in equation (4.71). Figure 4.5 shows a plot of the determinant of this metric and its individual components, and figure 4.6 shows a plot of the Kretschmann scalar. What the plots fail to show is that all of these quantities are divergent at $\mathfrak{u} = 1$ and at $\mathfrak{v} = 1$. There are no other divergences, thus these are the only geometric singularities and there are no horizons at finite $\mathfrak{u}$ and $\mathfrak{v}$. The effective metric obtained from choosing the $-$ sign contains the same naked singularities as the + sign choice. Further naked singularities are found at points that satisfy

$$\begin{aligned} 0 =& \mathfrak{u}^4 - (4\mathfrak{v}-8)\mathfrak{u}^3 + (6\mathfrak{v}^2 - 56\mathfrak{v} + 24)\mathfrak{u}^2 - (4\mathfrak{v}^3 + 56\mathfrak{v}^2 - 80\mathfrak{v} + 32)\mathfrak{u} \\ &+ \mathfrak{v}^4 - 8\mathfrak{v}^3 + 24\mathfrak{v}^2 - 32\mathfrak{v} + 16. \end{aligned} \tag{4.87}$$

Since none of the geometric singularities have horizons at finite $\mathfrak{u}$ and $\mathfrak{v}$, and thus finite u and v, neither of the effective metrics are useful for representing the gravitational field of a black hole.

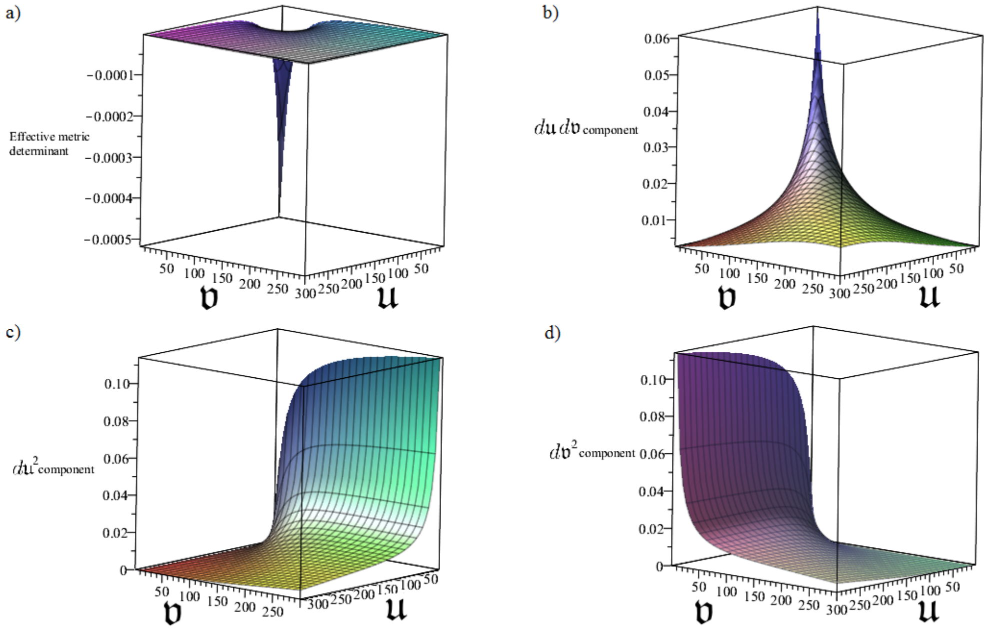

Figure 4.5: Plots of a) determinant, b) $d\mathfrak{u}d\mathfrak{v}$ component, c) $d\mathfrak{u}^2$ component, d) $d\mathfrak{v}^2$ component of the effective metric in equation (4.71) with + sign choice for constants $c_1 = 1$, $c_2 = 1$, $c_6 = \frac{1}{2}$, $c_7 = 1$, $c_8 = 1$ and $m = 2$.

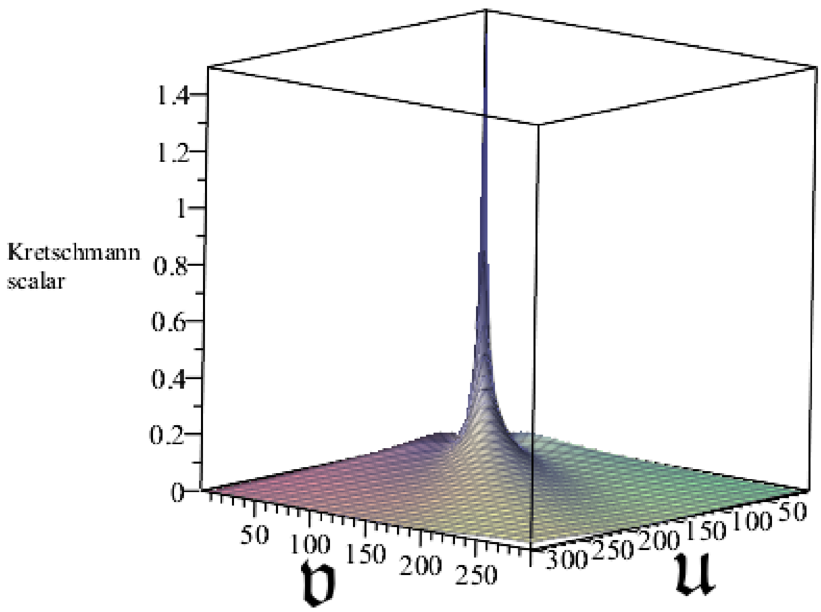

Figure 4.6: Plot of the Kretschmann scalar of the effective metric in equation (4.71) with + sign choice for constants $c_1 = 1$, $c_2 = 1$, $c_6 = \frac{1}{2}$, $c_7 = 1$, $c_8 = 1$ and $m = 2$.

4.2.3 Fields with linear time dependence in spherical coordinates

Suppose that $\Phi_0 = \gamma_\Phi t + h_\Phi(r)$, $\psi_0 = \gamma_\psi t + h_\psi(r)$. The field equations give

$$\partial_r \left(r^2 \partial_r h_\psi (\partial_r h_\Phi)^2\right) = \gamma_\Phi^2 \partial_r (r^2 \partial_r h_\psi), \tag{4.88}$$

$$\partial_r \left(r^2 \partial_r h_\Phi (\partial_r h_\psi)^2\right) = (\gamma_\psi^2 - m^2) \partial_r (r^2 \partial_r h_\Phi). \tag{4.89}$$

These have three solutions:

$$h_\Phi = \frac{c_1}{r} + c_2, \quad h_\psi = c_3, \tag{4.90}$$

$$h_\Phi = c_3, \quad h_\psi = \frac{c_1}{r} + c_2, \tag{4.91}$$

and

$$\begin{aligned} \pm h_\Phi =& \gamma_\Phi r + c_3, \\ \pm h_\psi =& \sqrt{\gamma_\psi^2 r^2 - m^2 r^2 + c_1^2} - c_1 \ln \left(\frac{2c_1^2 + 2c_1 \sqrt{\gamma_\psi^2 r^2 - m^2 r^2 + c_1^2}}{r} \right) + c_2, \end{aligned} \tag{4.92}$$

where c_1, c_2 and c_3 are constants, and the $\pm$ signs in the third solution are not related to the sign in the effective metric and are both independent of each other. For the solution in (4.91) the effective metrics become

$$
\begin{aligned}
g_{\text{eff}} = &- \frac{\zeta r^4}{\zeta^2 r^4 - 2\zeta\gamma_\Phi\gamma_\psi r^4 + c_1^2\gamma_\psi^2} dt \otimes dt + \frac{(\zeta - 2\gamma_\Phi\gamma_\psi) r^4}{\zeta^2 r^4 - 2\zeta\gamma_\Phi\gamma_\psi r^4 + c_1^2\gamma_\psi^2} dr \otimes dr \\
&+ \frac{c_1\gamma_\psi r^2}{\zeta^2 r^4 - 2\zeta\gamma_\Phi\gamma_\psi r^4 + c_1^2\gamma_\psi^2} (dt \otimes dr + dr \otimes dt) \\
&+ \frac{r^2}{\zeta} d\theta \otimes d\theta + \frac{r^2 \sin^2(\theta)}{\zeta} d\varphi \otimes d\varphi,
\end{aligned}
\tag{4.93}
$$

where

$$
\zeta = \pm\sqrt{(d\Phi_0 \cdot d\Phi_0)(d\psi_0 \cdot d\psi_0 + m^2)} = \pm\frac{1}{r^2}\sqrt{(c_1^2 - \gamma_\Phi^2 r^4)(m^2 - \gamma_\psi^2)}. \tag{4.94}
$$

Since $\gamma_\psi^2 > m^2$, (4.94) becomes imaginary for $r^2 \leq \left|\frac{c_1}{\gamma_\Phi}\right|$, thus the metrics are only valid for $r > r_c$, where $r_c = \left|\frac{c_1}{\gamma_\Phi}\right|^{\frac{1}{2}}$. A singularity is obtained when the denominator of the dt^2 component equals zero, which can be achieved if either the plus sign is chosen and γ_Φ and γ_ψ have the same sign, or if the minus sign is chosen and γ_Φ has the opposite sign to γ_ψ. This choice does not change the qualitative properties of the resulting metrics, so only the first case will be discussed. For the positive sign choice, this singularity occurs at

$$
r_h^2 = r_c^2 \frac{m^2}{\sqrt{-2\gamma_\psi^4 + \gamma_\psi^2 m^2 + m^4 + |\gamma_\phi^3|\sqrt{\gamma_\psi^2 - m^2}}}. \tag{4.95}
$$

A plot of the Kretschmann scalar for arbitrarily chosen constants in figure 4.7 shows the singularity at $r = r_h$ is geometrical, that there is another one at $r = r_c$, the curvature between r_c and r_h is high in comparison to the region $r > r_h$, and the metric is asymptotically flat for large r. Those singularities are naked, thus this solution does not relate to a gravitational field of a black hole. When the minus sign is chosen for the metric, it also

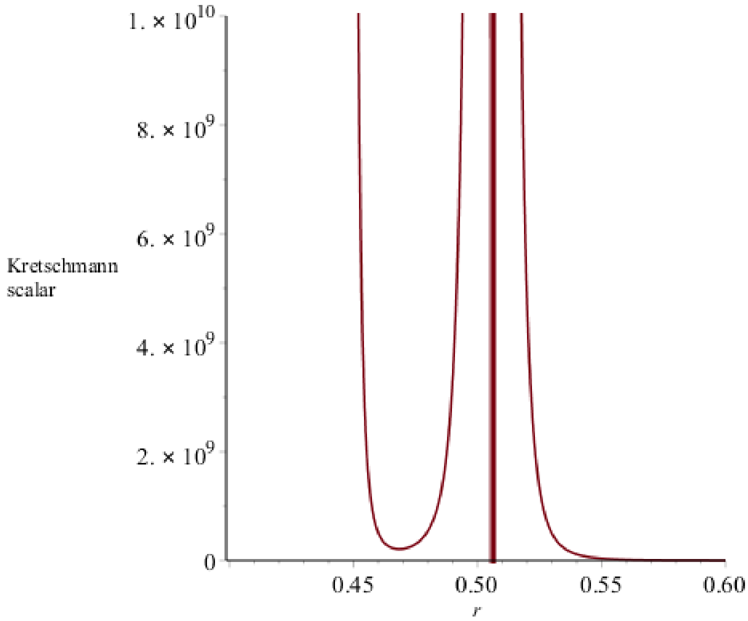

Figure 4.7: Plot of the Kretschmann scalar of the effective metric given in (4.93) with + sign choice, for constants $c_1 = 1$, $\gamma_\Phi = 5$, $\gamma_\psi = 5$ and $m = 4$, giving $r_c \approx 0.447$ and $r_h \approx 0.506$.

exhibits a naked singularity at $r = r_c$ and asymptotic flatness for large r. The second solution given in (4.91) has the same properties as the previous one due to the symmetry of the equations involved, and as such it will be omitted. Finally the last solution given in (4.92) can also be discarded by noting that it leads to $d\psi \cdot d\psi = -m^2$, which forces $\mu = 0$ (i.e. the laser has zero strength) due to equation (3.43).

4.3 Laser with varying spot size

The derivation of an effective metric from the field equations (3.49) and (3.50) is very similar to what is discussed in section 4.2, and will be omitted. The difference is that a conformal factor of Λ appears in equation (4.71), thus there are two effective metrics given by

$$g_{\text{eff}}^{-1} = s\Lambda\left(\widetilde{d\Phi_0}\otimes\widetilde{d\psi_0}+\widetilde{d\psi_0}\otimes\widetilde{d\Phi_0}\pm\sqrt{(d\Phi_0\cdot d\Phi_0)(d\psi_0\cdot d\psi_0+1)}g^{-1}\right). \tag{4.96}$$

Note that the dimensionless form has been used, as outlined in Appendix B. The factor of s comes from the freedom of choice of d_{I} in (4.70), and will be assumed to be $s=1$ or $s=-1$, to match the signature of the effective metrics with the background metric. This is equivalent to choosing $d_{\text{I}}^2=1$ rather than simply $d_{\text{I}}=1$. The condition for Lorentzian signature given in (4.73) is still valid, as a conformal factor will not affect it. The advantage of including Λ is that there are two equations for three unspecified fields, thus one of those fields can be freely chosen. It is possible to obtain an effective metric that is conformally related to the Schwarzschild metric, as will be shown in this section. Firstly, suppose that $\Phi_0=\gamma_\Phi t+h_\Phi(z)$, $\psi_0=\gamma_\psi t+h_\psi(z)$, and Λ is some function of z. Then the zeroth order field equations become

$$\begin{aligned} d\left[\Lambda(1-\gamma_\psi^2+\left(\frac{dh_\psi}{dz}\right)^2)(\gamma_\Phi dz+\frac{dh_\Phi}{dz}dt)\right] &= 0\\ \implies \partial_z\left[\Lambda(1-\gamma_\psi^2+\left(\frac{dh_\psi}{dz}\right)^2)\frac{dh_\Phi}{dz}\right] &= 0, \end{aligned} \tag{4.97}$$

$$
\begin{aligned}
d\left[\Lambda(-\gamma_\Phi^2+\left(\frac{dh_\Phi}{dz}\right)^2)(\gamma_\psi dz+\frac{dh_\psi}{dz}dt)\right]=0\\
\implies \partial_z\left[\Lambda(-\gamma_\Phi^2+\left(\frac{dh_\Phi}{dz}\right)^2)\frac{dh_\psi}{dz}\right]=0.
\end{aligned}
\tag{4.98}
$$

Thus these two equations can be written as

$$
\Lambda(-\gamma_-^2+\left(\frac{dh_\psi}{dz}\right)^2)\frac{dh_\Phi}{dz}=c_1.,
\tag{4.99}
$$

$$
\Lambda(-\gamma_\Phi^2+\left(\frac{dh_\Phi}{dz}\right)^2)\frac{dh_\psi}{dz}=c_2.,
\tag{4.100}
$$

where $\gamma_-^2=\gamma_\psi^2-1$, and c_1 and c_2 are constants. This system has one free function, either h_Φ or h_ψ, which will determine (4.96) once specified. This freedom allows for demanding that the ratio of the components of a diagonalised effective metric is proportional to $(1-\frac{z_s}{z})^2$. The effective metric is then conformally related to the Schwarzschild metric with a horizon at $z=z_s$. Introducing $\tau=at+\mathfrak{f}(z)$ for some constant a, and choosing $\mathfrak{f}(z)$ such that the metric becomes diagonal, gives the requirement

$$
-\left(1-\frac{z_s}{z}\right)^2=\frac{a^{-2}\left(2h'_\Phi h'_\psi\pm\sqrt{((h'_\Phi)^2-\gamma_\Phi^2)((h'_\psi)^2-\gamma_-^2)}\right)}{2\gamma_\Phi\gamma_\psi\mp\sqrt{((h'_\Phi)^2-\gamma_\Phi^2)((h'_\psi)^2-\gamma_-^2)}-\frac{(\gamma_\psi h'_\Phi+\gamma_\Phi h'_\psi)^2}{2h'_\Phi h'_\psi\pm\sqrt{((h'_\Phi)^2-\gamma_\Phi^2)((h'_\psi)^2-\gamma_-^2)}}}.
\tag{4.101}
$$

Note that the numerator on the right-hand side is proportional to $g_{\text{eff}}(\partial_\tau,\partial_\tau)$ and the denominator is proportional to $g_{\text{eff}}(\partial_z,\partial_z)$. It is convenient to introduce the scaled variables $h_\Phi=\gamma_\Phi\check{h}_\Phi$ and $h_\psi=\gamma_\psi\check{h}_\psi$. With the introduction of $\epsilon^2 = \frac{\gamma_-^2}{\gamma_\psi^2}$, equation (4.101) becomes

$$-\left(1 - \frac{z_s}{z}\right)^2 = \frac{a^{-2}\left(2\check{h}'_\Phi \check{h}'_\psi \pm \sqrt{(1-(\check{h}'_\Phi)^2)(\epsilon^2 - (\check{h}'_\psi)^2)}\right)}{2 \mp \sqrt{(1-(\check{h}'_\Phi)^2)(\epsilon^2 - (h'_\psi)^2)} - \frac{(\check{h}'_\Phi + \check{h}'_\psi)^2}{2\check{h}'_\Phi \check{h}'_\psi \pm \sqrt{(1-(\check{h}'_\Phi)^2)(\epsilon^2 - (\check{h}'_\psi)^2)}}}. \tag{4.102}$$

Note that

$$(\check{h}'_\psi)^2 < \epsilon^2 < 1, \tag{4.103}$$

where the upper bound comes from the definition of ϵ and the fact that $\gamma_- < \gamma_\psi$, while the lower bound is required so that the metric components do not become imaginary. The field equations (4.99) and (4.100) yield the relationship

$$\beta(\epsilon^2 - (\check{h}'_\psi)^2)\check{h}'_\Phi = (1 - (\check{h}'_\Phi)^2)\check{h}'_\psi, \tag{4.104}$$

between $\check{h}'_\Phi$ and $\check{h}'_\psi$, where $\beta = \frac{c_1}{c_2}$. Introducing $\check{h}'_\psi = \epsilon h$ and choosing $\beta = \frac{1}{\epsilon}$ results in $\check{h}'_\Phi = h$. Also note that $h^2 < 1$ is required due to (4.103). With these simplifications, the components of the inverse of the effective metric in t and z coordinates are

$$g^{-1}_{\text{eff}}(dt, dt) = s\Lambda\gamma_\Phi\gamma_\psi(2 \mp |\epsilon|(1 - h^2)), \tag{4.105}$$

$$g^{-1}_{\text{eff}}(dz, dz) = s\Lambda\gamma_\Phi\gamma_\psi(2\epsilon h^2 \pm |\epsilon|(1 - h^2)), \tag{4.106}$$

$$g^{-1}_{\text{eff}}(dt, dz) = -s\Lambda\gamma_\Phi\gamma_\psi(1 + \epsilon)h, \tag{4.107}$$

with

$$\Lambda^2 = \left(\frac{c_1}{\epsilon h(1 - h^2)}\right)^2. \tag{4.108}$$

Note that by definition $\Lambda > 0$, thus the choice $s = -1$ is required in order to match the signatures of the effective metrics to the background metric. The $\partial_t \otimes \partial_t$ component and the off-diagonal terms, as well as Λ, will be non-zero for all values within the constraints (4.103). Equating $g_{\text{eff}}^{-1}(dz, dz)$ to zero and solving for h leads to a horizon if ϵ is chosen appropriately. The value of h when (4.106) equals zero is

$$h^2 = \frac{|\epsilon|}{|\epsilon| \mp 2\epsilon}. \tag{4.109}$$

When ϵ is positive, for the $+$ sign choice in (4.96) there will be no horizon, while there will be a horizon at $h^2 = \frac{1}{3}$ for the $-$ sign choice. The converse is true for $\epsilon < 0$. Since these are interchangeable, the case of $\epsilon < 0$ will be assumed henceforth. Also note that $\det\left(g_{\text{eff}}^{-1}\right) < 0$ for $-1 < \epsilon < 0$ and $\frac{1}{3} < h^2 < 1$, thus both effective metrics are Lorentzian. This can be seen from plotting the inequality given in (4.73), however it can also be proved algebraically. To show this let $p_\pm = \frac{1}{\Lambda^2 \gamma_\Phi^2 \gamma_\psi^2} \det\left(g_{\text{eff}}^{-1}\right)$ and note:

$$\begin{aligned}
p_+ =& (-3h^4 + 3h^2 - 1)\epsilon^2 + (4h^2 - 2)\epsilon - h^2, \\
p_- =& (2 + |\epsilon|(1 - h^2))(2\epsilon h^2 - |\epsilon|(1 - h^2)) - (1 + \epsilon)^2 h^2 \\
=& -(2 + |\epsilon|(1 - h^2))(-2\epsilon h^2 + |\epsilon|(1 - h^2)) - (1 + \epsilon)^2 h^2.
\end{aligned} \tag{4.110}$$

The result $p_- < 0$ is immediately satisfied and thus $\det\left(g_{\text{eff}}^{-1}\right) < 0$ for the negative sign choice. The derivatives of p_+ are

$$\begin{aligned} \frac{dp_+}{dh} =& -2h((6h^2-3)\epsilon^2 - 4\epsilon + 1) \\ \frac{d^2p_+}{dh^2} =& -24h^2\epsilon^2 \end{aligned} \tag{4.111}$$

At $h^2 = \frac{1}{3}$, $p_+ = -(1+\epsilon)$ and $\frac{dp_+}{dh} = \frac{2}{\sqrt{3}}(\epsilon^2 + 4\epsilon - 1)$. The polynomial $\epsilon^2 + 4\epsilon - 1 = 0$ has solutions at $\epsilon = -2 \pm \sqrt{5}$. Thus $\frac{dp_+}{dh}|_{h^2=\frac{1}{3}} < 0$ for $-1 < \epsilon < 0$, since these roots do not lie between -1 and 0. Furthermore $\frac{d^2p_+}{dh^2} < 0$ for $-1 < \epsilon < 0$ and $\frac{1}{3} < h^2 < 1$, hence p_+ is negative at $h^2 = \frac{1}{3}$ and decreases as h increases. Thus $\det\left(g_{\text{eff}}^{-1}\right) < 0$ for the positive sign choice. Only the + sign effective metric is of interest and will be explored further, as the other one does not contain a horizon. Let $\nu = -\epsilon$ for convenience, with which equation (4.102) can be written as

$$-\left(1 - \frac{z_s}{z}\right)^2 = \frac{a^{-2}\left(1-3h^2\right)^2\nu^2}{\left(2-\nu(1-h^2)\right)\left(-2\nu h^2 + \nu(1-h^2)\right) - (1-\nu)^2 h^2}, \tag{4.112}$$

which will always have a solution in the specified range, because the numerator on the right-hand side is positive and the denominator is always negative because it is the determinant of the effective metric. An expression for a is obtained from (4.112) by matching the limit of $z \to \infty$ to $h \to 1$, yielding

$$a^2 = \left(\frac{2\nu}{1+\nu}\right)^2. \tag{4.113}$$

A solution to equation (4.112) can be found since it is a quadratic equation in h^2, however it is cumbersome and a simpler approach is available. Since the behaviour at large z is of interest, a function that tends to 1 for large z is required. A simple choice for this function is $h = e^{-\frac{1}{z}}$. Inserting $h = e^{-\frac{1}{z}}$ and the solution for a into the right-hand side of equation (4.112), taking the square root and Taylor expanding it in $\frac{1}{z}$ yields $1 - \frac{2}{1+\nu}\frac{1}{z} + \mathcal{O}(z^{-2})$. This allows to match $z_s = 2(1+\nu)^{-1}$ and

$$d\Phi_0 \cdot d\Phi_0 = -\gamma_\Phi^2(1 - e^{-\frac{2}{z}}), \tag{4.114}$$

$$d\psi_0 \cdot d\psi_0 + 1 = -(\gamma_\psi^2 - 1)(1 - e^{-\frac{2}{z}}), \tag{4.115}$$

is obtained. Following the definitions in Appendix B to restore units requires setting

$$l_* = \frac{GM}{c^2}(1+\nu), \tag{4.116}$$

where M is the mass of the effective black hole. This is because $\tilde{z}_s = 2\frac{GM}{c^2}$ and $\tilde{z} = l_\star z$. Now the following relations can be obtained for the spot size $\tilde{\Lambda}$ and dimensionless amplitude a_0 of the laser:

$$\tilde{\Lambda} = l_*^2 \frac{c_1}{\nu e^{-\frac{l_*}{\tilde{z}}}(1 - e^{-2\frac{l_*}{\tilde{z}}})}, \tag{4.117}$$

$$a_0^2 = (\gamma_\psi^2 - 1)(1 - e^{-2\frac{l_*}{\tilde{z}}}) \approx (\gamma_\psi^2 - 1)\frac{2l_*}{\tilde{z}}, \tag{4.118}$$

and the plasma frequency ω_p satisfies $\omega_p^2 \propto \frac{1}{\tilde{z}} + \mathcal{O}(\frac{1}{\tilde{z}^2})$. A system that follows these equations for the laser cross-sectional area $\tilde{\Lambda}$ and dimensionless amplitude a_0 for large $\tilde{z}$ corresponds to an effective black hole with Hawking temperature

$$T_H = \frac{\hbar c^3}{8\pi k_b GM}, \tag{4.119}$$

which can be written as

$$T_H = \frac{\hbar c(1+\nu)}{8\pi k_b l_*}. \tag{4.120}$$

To proceed further, the values of l_* and ν must be fixed using a physical configuration.

4.3.1 Application of the result

The Hawking temperature can be calculated if the initial dimensionless amplitude $a_0|_S$ and the initial laser cross-sectional area $\tilde{\Lambda}|_S$ are known. Here $|_S$ indicates evaluation at $z = z_S$. By matching $a_0|_S$ to equation (4.118) a value for γ_ψ is obtained, and hence ν. Note that it is possible to solve for γ_ψ algebraically, however the solution is cumbersome and no insight is gained, and as such it will be omitted. Choosing

$$c_1 = \nu[e^{-\frac{l_*}{z}}(1 - e^{-2\frac{l_*}{z}})]|_S, \tag{4.121}$$

gives $l_\star = \sqrt{\tilde{\Lambda}}|_S$ using (4.117), and the initial laser width w_0 can be expressed as $w_0 = \sqrt{\tilde{\Lambda}}|_S$. Now ν and l_* are known, thus M is known from equation (4.116), and Hawking temperature follows. As an example the parameters $a_0|_S \approx 0.7$ and $w_0 \approx 30$ μm are achievable [53] for maintaining an intense near-IR laser pulse. These parameters result in $\gamma_\psi \approx 1.15$, $\nu \approx 0.49$, $c_1 \approx 0.18$, $M \approx 2.71 \times 10^{22}$ kg, and the associated Hawking temperature is $T_H \approx 4.52$ K. Note that these values were obtained with asymptotically expanded a_0. Without the expansion the quantities are $\gamma_\psi \approx 1.27$, $\nu \approx 0.61$, $c_1 \approx 0.22$, $M \approx 2.50 \times 10^{22}$ kg and $T_H \approx 4.89$ K. This temperature is very small regardless. However, inspection of (4.116) reveals that the mass of the effective black hole only depends on γ_ψ through ν, and $\nu < 1$, thus no matter how big $a_0|_S$ is, it will have small impact on the Hawking temperature. But the mass of the effective black hole depends linearly on l_*. This suggests that w_0 contributes much more significantly to the Hawking temperature than a_0 does. Indeed as $a_0|_S$ gets smaller, γ_ψ gets closer to 1 and thus ν gets closer to 0. This lowers the Hawking temperature by a factor of 2 from the maximum value which coincides with $\nu = 1$. On the other hand, T_H is inversely proportional to w_0. Thus decreasing w_0 by any factor will increase the Hawking temperature by the same factor. For example, pulses with waist of ≈ 100 nm are experimentally achievable [56]. Even for a small $a_0|_S$, and thus $\nu \approx 0$, the Hawking temperature evaluates to ≈ 760 K. Note however that there is no plasma in Ref. [56] and the pulse length is 210 μm, while the model used in this section is only applicable when w_0 is greater than pulse length. Nevertheless even a pulse width of a few μm and high intensity would yield $T_H \approx 100$ K.

Chapter 5

Effective field theory of laser-driven plasma

In this chapter the quantum effects of the previously derived action given in (3.48) will be explored through the 1-loop effective action. The field equations will be derived in section 5.1. In section 5.2 the fields will be assumed to be linear in Minkowski coordinates and they will be perturbed in order to investigate the effects of the quantum fluctuations. In section 5.3 the perturbations will be assumed to be of the form of a plane wave leading to a dispersion relation. Finally the effects of this on a Gaussian wave packet will be investigated in section 5.3.1.

The spot size of the laser pulse will be assumed to be constant, hence (3.48) can be written as

$$S[\Phi, \psi] = \frac{1}{2} \int_{\mathcal{M}} (d\Phi \cdot d\Phi)(d\psi \cdot d\psi + 1) \# 1, \tag{5.1}$$

using the dimensionless forms of Φ and ψ as outlined in Appendix B. The 1-loop effective action Γ is given by [12]

$$\Gamma[\Phi,\psi] = S[\Phi,\psi] - i\ln\left\{\int \mathcal{D}\vec{f}\exp\left(i\Lambda_{total}[\vec{\mathfrak{f}}]\right)\right\}, \tag{5.2}$$

where

$$\Lambda_{total}[\vec{\mathfrak{f}}] = \frac{1}{2}\iiiint dzdtdz'dt'\frac{\delta^2 S}{\delta^2\varphi_A(z,t)\delta^2\varphi_B(z',t')}\mathfrak{f}_A(z,t)\mathfrak{f}_B(z',t'), \tag{5.3}$$

with the indices A, B ranging over 1, 2, $\varphi_1 = \Phi$, $\varphi_2 = \psi$ and $\vec{\mathfrak{f}} = (\mathfrak{f}_1\ \mathfrak{f}_2)^T$, T denoting matrix transposition. Note that the dimensionless factor Λ introduced in equation (3.41) and the functional Λ_{total} are unrelated. The relevant functional derivatives are

$$\begin{aligned}
\frac{\delta S}{\delta\Phi(z,t)\delta\Phi(z',t')} =& \partial_t(\partial_t\delta^{(2)}(d\psi\cdot d\psi+1)) - \partial_z(\partial_z\delta^{(2)}(d\psi\cdot d\psi+1)),\\
\frac{\delta S}{\delta\psi(z,t)\delta\psi(z',t')} =& \partial_t(\partial_t\delta^{(2)}(d\Phi\cdot d\Phi)) - \partial_z(\partial_z\delta^{(2)}(d\Phi\cdot d\Phi)),\\
\frac{\delta S}{\delta\Phi(z,t)\delta\psi(z',t')} =& 2\partial_t(\partial_t\Phi(\partial_z\psi\partial_z\delta^{(2)} - \partial_t\psi\partial_t\delta^{(2)})) - 2\partial_z(\partial_z\Phi(\partial_z\psi\partial_z\delta^{(2)} - \partial_t\psi\partial_t\delta^{(2)})),
\end{aligned} \tag{5.4}$$

where $\delta^{(2)}$ is a product of two Dirac delta functions $\delta(x-x')$ and $\delta(t-t')$, giving

$$\begin{aligned}
\Lambda_{total}[\vec{\mathfrak{f}}] = \int_{\mathcal{M}}\frac{1}{2}[&(d\Phi\cdot d\Phi)(d\mathfrak{f}_2\cdot d\mathfrak{f}_2) + (d\psi\cdot d\psi+1)(d\mathfrak{f}_1\cdot d\mathfrak{f}_1)\\
&+4(d\Phi\cdot d\mathfrak{f}_1)(d\psi\cdot d\mathfrak{f}_2)]\#1,
\end{aligned} \tag{5.5}$$

or equivalently

$$\Lambda_{total}[\vec{\mathfrak{f}}] = -\int dtdx\frac{1}{2}\vec{\mathfrak{f}}^\dagger\mathcal{O}\vec{\mathfrak{f}}, \tag{5.6}$$

where the operator $\mathcal{O}$ is given by

$$\mathcal{O}\vec{\mathfrak{f}} = \partial_\mu \begin{pmatrix} (\eta^{\sigma\tau}\partial_\sigma\psi\partial_\tau\psi + 1)\eta^{\mu\nu} & 2\eta^{\mu\sigma}\partial_\sigma\Phi\eta^{\nu\tau}\partial_\tau\psi \\ 2\eta^{\mu\sigma}\partial_\sigma\psi\eta^{\nu\tau}\partial_\tau\Phi & \eta^{\sigma\tau}\partial_\sigma\Phi\partial_\tau\Phi\eta^{\mu\nu} \end{pmatrix} \partial_\nu\vec{\mathfrak{f}}. \tag{5.7}$$

Here $\eta^{\mu\nu} = \mathrm{Diag}(-1, 1)$. By definition, the path integral $\int \mathcal{D}\vec{f} \exp\left(i\Lambda_{total}[\vec{\mathfrak{f}}]\right)$ is equal to $\mathrm{Det}(-i\mathcal{O})^{-\frac{1}{2}}$, where $\mathrm{Det}(-i\mathcal{O})$ is the functional determinant [7,8] of the operator $-i\mathcal{O}$, which is equal to the product of its eigenvalues. It is difficult to directly calculate this analytically, however it is greatly simplified when the fields Φ and ψ are linear functions of z and t. In this case, the matrix in (5.7) is constant, thus the eigenfunctions of the operator are of the form $(a\, b)^T \exp(il_\mu x^\mu)$, where a, b and l_μ are constant. The eigenvalues of the matrix in (5.7) arise in pairs, where each pair λ_l^+ and λ_l^- corresponds to each l_μ and must satisfy

$$\lambda_l^+\lambda_l^- = (\eta^{\sigma\tau}\partial_\sigma\psi\partial_\tau\psi+1)\eta^{\gamma\delta}\partial_\gamma\Phi\partial_\delta\Phi(\eta^{\mu\nu}l_\mu l_\nu)^2 - 4(\eta^{\mu\sigma}\partial_\sigma\Phi\eta^{\nu\tau}\partial_\tau\Phi l_\mu l_\nu)^2. \tag{5.8}$$

Equation (5.8) can be readily factorised to $\lambda_l^+\lambda_l^- = (\mathcal{A}_+^{\mu\nu}l_\mu l_\nu)(\mathcal{A}_-^{\sigma\tau}l_\sigma l_\tau)$, where

$$\begin{aligned} \mathcal{A}_+^{\mu\nu} =& s\left(\sqrt{(\eta^{\sigma\tau}\partial_\sigma\psi\partial_\tau\psi + 1)\eta^{\gamma\delta}\partial_\gamma\Phi\partial_\delta\Phi}\eta^{\mu\nu} + \eta^{\mu\sigma}\eta^{\nu\tau}(\partial_\sigma\Phi\partial_\tau\psi + \partial_\tau\Phi\partial_\sigma\psi)\right), \\ \mathcal{A}_-^{\mu\nu} =& s\left(\sqrt{(\eta^{\sigma\tau}\partial_\sigma\psi\partial_\tau\psi + 1)\eta^{\gamma\delta}\partial_\gamma\Phi\partial_\delta\Phi}\eta^{\mu\nu} - \eta^{\mu\sigma}\eta^{\nu\tau}(\partial_\sigma\Phi\partial_\tau\psi + \partial_\tau\Phi\partial_\sigma\psi)\right), \end{aligned} \tag{5.9}$$

where $s = \pm 1$ accounts for the overall sign of both tensors. Note that these two tensors are exactly the same as the effective metrics found in section 4.2, up to a sign. Now, due to the product of eigenvalues being commutative, the functional determinant can be expressed as $\mathrm{Det}(-i\mathcal{O}) = \mathrm{Det}(-i\mathcal{O}_+)\,\mathrm{Det}(-i\mathcal{O}_-)$, where

$$\mathcal{O}_+\mathfrak{f} = -\partial_\mu(\mathcal{A}_+^{\mu\nu}\partial_\nu\mathfrak{f}), \qquad \mathcal{O}_-\mathfrak{f} = -\partial_\mu(\mathcal{A}_-^{\mu\nu}\partial_\nu\mathfrak{f}). \tag{5.10}$$

The path integral can also be factorised to

$$\int \mathcal{D}\vec{f}\exp\left(i\Lambda_{total}[\vec{\mathfrak{f}}]\right) = \left\{\int \mathcal{D}\vec{f}\exp\left(i\Lambda_+[\vec{\mathfrak{f}}]\right)\right\}\left\{\int \mathcal{D}\vec{f}\exp\left(i\Lambda_-[\vec{\mathfrak{f}}]\right)\right\}, \tag{5.11}$$

where

$$\Lambda_+[\mathfrak{f}] = \int dtdz\frac{1}{2}\mathcal{A}_+^{\mu\nu}\partial_\mu\mathfrak{f}\partial_\nu\mathfrak{f}, \qquad \Lambda_-[\mathfrak{f}] = \int dtdz\frac{1}{2}\mathcal{A}_-^{\mu\nu}\partial_\mu\mathfrak{f}\partial_\nu\mathfrak{f}. \tag{5.12}$$

While this derivation requires the fields Φ and ψ to be linear functions of the Minkowski coordinates, it is also valid when Φ and ψ are sufficiently slowly varying. This approach is analogous to the common usage of the Euler-Heisenberg Lagrangian even when the background electric and magnetic fields are not constant. The form of Γ can be obtained when (5.11) is expressed in terms of a massless field theory on a dilatonic curved background, as discussed in section 2.2.3. This identification requires $\mathcal{A}_+^{\mu\nu}$ and $\mathcal{A}_-^{\mu\nu}$ to be Lorentzian, hence the constraint given in (4.73) applies. The pairs of metrics $g_{\mu\nu}^+$, $g_{\mu\nu}^-$ and dilatons φ_+, φ_- are

$$g_+^{\mu\nu} = \frac{\mathcal{A}_+^{\mu\nu}}{\sqrt{A_+}}, \quad \varphi_+ = -\frac{1}{4}\ln(A_+), \quad g_-^{\mu\nu} = \frac{\mathcal{A}_-^{\mu\nu}}{\sqrt{A_-}}, \quad \varphi_- = -\frac{1}{4}\ln(A_-), \tag{5.13}$$

where A_+ and A_- are the determinants of $\eta_{\mu\sigma}\mathcal{A}_+^{\sigma\nu}$ and $\eta_{\mu\sigma}\mathcal{A}_-^{\sigma\nu}$ respectively. Thus (5.12) can be written as

$$\begin{aligned}\Lambda_+[\mathfrak{f}] &= \int d^2x\sqrt{-g^+}\frac{1}{2}\exp\{-2\varphi_+\}g_+^{\mu\nu}\partial_\mu\mathfrak{f}\partial_\nu\mathfrak{f}, \\ \Lambda_-[\mathfrak{f}] &= \int d^2x\sqrt{-g^-}\frac{1}{2}\exp\{-2\varphi_-\}g_-^{\mu\nu}\partial_\mu\mathfrak{f}\partial_\nu\mathfrak{f},\end{aligned} \tag{5.14}$$

where g^+ and g^- are determinants of $g_{\mu\nu}^+$ and $g_{\mu\nu}^-$ respectively, and $g^+ = g^- = \eta$ by construction. Following the discussion in section 2.2.3, the effective action is given by

$$\Gamma[\Phi, \psi] = S[\Phi, \psi] + w_+ + w_-, \tag{5.15}$$

where $w_+ = w[g_{\mu\nu}^+, \varphi_+, \mu_+]$ and $w_- = w[g_{\mu\nu}^-, \varphi_-, \mu_-]$.

5.1 Field equations

It is not straightforward to extract the field equations from varying $\Gamma[\Phi, \psi]$ immediately due to the amount of substitutions made. To obtain them, the variations of different terms will be presented, building up to expressing the field equations in a condensed form. Starting with functional derivatives of $w[g, \phi, \mu]$ (see equation (2.57)), it is relatively easy to show that

$$\frac{12\pi}{\sqrt{-g}}\frac{\delta w}{\delta\phi} = 3\nabla_\mu\Box^{-1}R\nabla^\mu\phi + (3\Box^{-1}R + 1 + 6\ln\mu)\Box\phi + R. \tag{5.16}$$

The functional derivative of w with respect to $g_{\mu\nu}$ is more involved, as $\Box^{-1}$, R and $\sqrt{-g}$ contain the metric. The calculation will be performed for the first term of w, displaying all the tools necessary to obtain the whole expression, however the remaining terms will be omitted for brevity and the final result will be stated. Firstly let I be given by

$$\begin{aligned} I =& \delta \int d^2x \sqrt{-g} R \Box^{-1} R \\ =& \int d^2x \Big\{ \delta(\sqrt{-g}) R \Box^{-1} R + \sqrt{-g} R \delta(\Box^{-1}) R + \sqrt{-g} \delta R \Box^{-1} R + \sqrt{-g} R \Box^{-1} \delta R \Big\} \\ =& \int d^2x \Big\{ \delta(\sqrt{-g}) R \Box^{-1} R + \sqrt{-g} R \delta(\Box^{-1}) R + 2\sqrt{-g} \delta R \Box^{-1} R \Big\}, \end{aligned} \tag{5.17}$$

where the last step utilised the identity given by

$$\begin{aligned} \int_{\mathcal{M}} d^2x \sqrt{-g} h_1 \Box^{-1} h_2 = \int_{\mathcal{M}} d^2x \sqrt{-g} \Box H_1 H_2 = \int_{\mathcal{M}} d^2x \sqrt{-g} H_1 \Box H_2 \\ = \int_{\mathcal{M}} d^2x \sqrt{-g} h_2 \Box^{-1} h_1, \end{aligned} \tag{5.18}$$

where $H_1 = \Box h_1$ and $H_2 = \Box h_2$ for some scalar fields h_1 and h_2, and integration by parts has been used. It can be shown that $\delta R = R_{\mu\nu}\delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu}$ and $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}$, also

$$\begin{aligned} \int d^2x \sqrt{-g} h_1 \delta(\Box) h_2 =& \delta \int d^2x \sqrt{-g} h_1 \Box h_2 - \int d^2x \delta(\sqrt{-g}) h_1 \Box h_2 \\ =& -\delta \int d^2x \sqrt{-g} g^{\mu\nu} \nabla_\mu h_1 \nabla_\nu h_2 - \int d^2x \delta(\sqrt{-g}) h_1 \Box h_2 \\ =& -\int d^2x \sqrt{-g} (\nabla_\mu h_1 \nabla_\nu h_2 - \frac{1}{2} g_{\mu\nu} \nabla h_1 \cdot \nabla h_2) \delta g^{\mu\nu} \\ & - \int d^2x \delta(\sqrt{-g}) h_1 \Box h_2 \end{aligned} \tag{5.19}$$

where h_1 and h_2 are again arbitrary scalar fields and integration by parts has been used. It is worth noting that this result holds even if h_1 and h_2 depend on the metric; the extra terms that appear in the first two lines of working (due to δh_1 and δh_2 being non-zero) cancel in the final result. Now I can be evaluated to

$$I = \int d^2x\sqrt{-g}\Big\{2G_{\mu\nu}\Box^{-1}R - 2\nabla_\mu\nabla_\nu\Box^{-1}R + 2g_{\mu\nu}R + \nabla_\mu\Box^{-1}R\nabla_\nu\Box^{-1}R - \frac{1}{2}g_{\mu\nu}(\nabla\Box^{-1}R)^2\Big\}\delta g^{\mu\nu}, \tag{5.20}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, however in two dimensions $G_{\mu\nu} = 0$ [57]. Using the methods above yields

$$\begin{aligned}\frac{48\pi}{\sqrt{-g}}\frac{\delta w}{\delta g^{\mu\nu}} = &- \nabla_\mu\nabla_\nu\Box^{-1}R + g_{\mu\nu}R + \frac{1}{2}\nabla_\mu\Box^{-1}R\nabla_\nu\Box^{-1}R - \frac{1}{4}g_{\mu\nu}(\nabla\Box^{-1}R)^2 \\ &- 4\nabla_\mu\nabla_\nu\phi + 4g_{\mu\nu}\Box\phi - 6\nabla_\mu\phi\nabla_\nu\phi\Box^{-1}R + 3g_{\mu\nu}(\nabla\phi)^2\Box^{-1}R \\ &- 6\nabla_{(\mu}\Box^{-1}(\nabla\phi)^2\nabla_{\nu)}\Box^{-1}R + 3g_{\mu\nu}g^{\sigma\omega}\nabla_\sigma\Box^{-1}(\nabla\phi)^2\nabla_\omega\Box^{-1}R \\ &+ 6\nabla_\mu\nabla_\nu\Box^{-1}(\nabla\phi)^2 - 2(1+6\ln\mu)\nabla_\mu\phi\nabla_\nu\phi \\ &+ (-5+6\ln\mu)g_{\mu\nu}(\nabla\phi)^2,\end{aligned} \tag{5.21}$$

where the parentheses enclosing indices denote tensor symmetrization given by $2\nabla_{(\mu}X\nabla_{\nu)}Y = \nabla_\mu X\nabla_\nu Y + \nabla_\nu X\nabla_\mu Y$. The functional derivatives of w_+ or w_- are simply the variations of w evaluated at $\phi = \varphi_+$ and $g^{\mu\nu} = g_+^{\mu\nu}$, or $\phi = \varphi_-$ and $g^{\mu\nu} = g_-^{\mu\nu}$ respectively. It is now also possible to determine the functional derivatives of w_+ and w_- with respect to $\mathcal{A}_+^{\mu\nu}$ and $\mathcal{A}_-^{\mu\nu}$ by introducing $\mathcal{A}^{\mu\nu} = e^{-2\phi}g^{\mu\nu}$, where similarly to (5.13), $\phi = -\frac{1}{4}\ln\mathcal{A}$ and $\mathcal{A}$ is the determinant of $\eta_{\mu\sigma}\mathcal{A}^{\sigma\nu}$. The variations now become $\delta g^{\mu\nu} = e^{2\phi}\delta\mathcal{A}^{\mu\nu} +$

$2g^{\mu\nu}\delta\phi$ and $4\delta\phi = -e^{2\phi}g_{\mu\nu}\delta\mathcal{A}^{\mu\nu}$, thus

$$
\begin{aligned}
\delta w &= \int d^2x \left(\frac{\delta w}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\delta w}{\delta \phi} \delta\phi \right) \\
&= \int d^2x \left\{ \frac{\delta w}{\delta g^{\mu\nu}} - \frac{1}{4} \left(2\frac{\delta w}{\delta g^{\sigma\tau}} g^{\sigma\tau} + \frac{\delta w}{\delta\phi} \right) g_{\mu\nu} \right\} e^{2\phi} \delta\mathcal{A}^{\mu\nu}.
\end{aligned}
\tag{5.22}
$$

Exploiting the fact that the determinant g of $g_{\mu\nu}$ and the determinant η of $\eta_{\mu\nu}$ satisfies $g = \eta$ gives

$$
\frac{1}{\sqrt{-\eta}} \frac{\delta w}{\delta\mathcal{A}^{\mu\nu}} = \frac{1}{\sqrt{-g}} \left\{ \frac{\delta w}{\delta g^{\mu\nu}} - \frac{1}{4} \left(2\frac{\delta w}{\delta g^{\sigma\tau}} g^{\sigma\tau} + \frac{\delta w}{\delta\phi} \right) g_{\mu\nu} \right\} e^{2\phi}, \tag{5.23}
$$

where again the properties of δw_+ and δw_- follow through suitable substitutions. Recalling that $s^2 = 1$, equation (5.9) yields

$$
\begin{aligned}
s\delta\mathcal{A}_+^{\mu\nu} &= \left(-\sqrt{\frac{\varepsilon}{\alpha}}\eta^{\mu\nu}\beta^\sigma + 2\zeta^{(\mu}\eta^{\nu)\sigma} \right) \partial_\sigma\delta\Phi + \left(-\sqrt{\frac{\varepsilon}{\alpha}}\eta^{\mu\nu}\zeta^\sigma + 2\beta^{(\mu}\eta^{\nu)\sigma} \right) \partial_\sigma\delta\psi, \\
s\delta\mathcal{A}_-^{\mu\nu} &= \left(-\sqrt{\frac{\varepsilon}{\alpha}}\eta^{\mu\nu}\beta^\sigma - 2\zeta^{(\mu}\eta^{\nu)\sigma} \right) \partial_\sigma\delta\Phi + \left(-\sqrt{\frac{\varepsilon}{\alpha}}\eta^{\mu\nu}\zeta^\sigma - 2\beta^{(\mu}\eta^{\nu)\sigma} \right) \partial_\sigma\delta\psi,
\end{aligned}
\tag{5.24}
$$

where

$$
\alpha = \eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi, \quad \beta^\mu = \eta^{\mu\nu}\partial_\nu\Phi, \quad \varepsilon = \eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + 1, \quad \zeta^\mu = \eta^{\mu\nu}\partial_\nu\psi. \tag{5.25}
$$

Thus the field equations arising from varying $\Gamma[\Phi,\psi]$ with respect to Φ and ψ can be written in concise form as

$$
\nabla_\sigma^{(\eta)}(\varepsilon\beta^\sigma + \mathcal{B}_+^\sigma + \mathcal{B}_-^\sigma) = 0, \quad \nabla_\sigma^{(\eta)}(\alpha\zeta^\sigma + \mathcal{C}_+^\sigma + \mathcal{C}_-^\sigma) = 0, \tag{5.26}
$$

respectively, where

$$
\begin{aligned}
\mathcal{B}^{\sigma}_{+} =& \frac{s}{\sqrt{-\eta}} \frac{\delta w_{+}}{\delta \mathcal{A}^{\mu\nu}_{+}} \left(-\sqrt{\frac{\varepsilon}{\alpha}} \eta^{\mu\nu} \beta^{\sigma} + 2\zeta^{\mu} \eta^{\nu\sigma} \right), \\
\mathcal{B}^{\sigma}_{-} =& \frac{s}{\sqrt{-\eta}} \frac{\delta w_{-}}{\delta \mathcal{A}^{\mu\nu}_{-}} \left(-\sqrt{\frac{\varepsilon}{\alpha}} \eta^{\mu\nu} \beta^{\sigma} - 2\zeta^{\mu} \eta^{\nu\sigma} \right),
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
\mathcal{C}^{\sigma}_{+} =& \frac{s}{\sqrt{-\eta}} \frac{\delta w_{+}}{\delta \mathcal{A}^{\mu\nu}_{+}} \left(-\sqrt{\frac{\alpha}{\varepsilon}} \eta^{\mu\nu} \zeta^{\sigma} + 2\beta^{\mu} \eta^{\nu\sigma} \right), \\
\mathcal{C}^{\sigma}_{-} =& \frac{s}{\sqrt{-\eta}} \frac{\delta w_{-}}{\delta \mathcal{A}^{\mu\nu}_{-}} \left(-\sqrt{\frac{\alpha}{\varepsilon}} \eta^{\mu\nu} \zeta^{\sigma} - 2\zeta^{\mu} \eta^{\nu\sigma} \right).
\end{aligned}
\tag{5.28}
$$

The terms $\nabla^{(\eta)}_{\sigma}(\varepsilon\beta^{\sigma})$ and $\nabla^{(\eta)}_{\sigma}(\alpha\zeta^{\sigma})$ come from variation of $S[\Phi, \psi]$.

5.2 Perturbations of the linear solution

The simplest solution of the field equations is given by the case when Φ and ψ are linear functions of the Minkowski coordinates t and z. In this case α and ε are constant, meaning $\mathcal{A}^{\mu\nu}_{+}$, $\mathcal{A}^{\mu\nu}_{-}$, β^{μ} and ζ^{μ} are constant. Therefore $g^{\mu\nu}_{+}$, $g^{\mu\nu}_{-}$, φ_{+} and φ_{-} are also constant, and in addition $R^{+} = R^{-} = 0$. Hence (5.16) and (5.21) are zero, and thus (5.26) is satisfied. To investigate the effects of the quantum fluctuations on this solution, introduce

$$
g^{\mu\nu} = \bar{g}^{\mu\nu} + g^{\mu\nu}_{(1)}, \quad \phi = \bar{\phi} + \phi_{(1)}, \tag{5.29}
$$

where barred characters indicate quantities associated with the unperturbed solution. The functional derivatives of w obtained in (5.16) and (5.21) become

$$
\frac{12\pi}{\sqrt{-g}} \frac{\delta w}{\delta \phi} = R_{(1)} + (1 + 6 \ln \mu) \bar{\Box} \phi_{(1)}, \tag{5.30}
$$

$$\frac{48\pi}{\sqrt{-g}}\frac{\delta w}{\delta\phi} = -\partial_\mu\partial_\nu\bar{\Box}^{-1}R_{(1)} + \bar{g}_{\mu\nu}R_{(1)} - 4\partial_\mu\partial_\nu\phi_{(1)} + 4\bar{g}_{\mu\nu}\bar{\Box}\phi_{(1)}, \tag{5.31}$$

respectively, to first order in the perturbations $g_{(1)}$ and $\phi_{(1)}$, and the perturbation of the curvature scalar is given by

$$R_{(1)} = -\partial_\mu\partial_\nu g^{\mu\nu}_{(1)} + \bar{g}_{\mu\nu}\bar{\Box}g^{\mu\nu}_{(1)}. \tag{5.32}$$

With suitable substitutions, the field equations (5.26) become

$$\begin{aligned}
0 =& \bar{\varepsilon}\Box_{(\eta)}\Phi_{(1)} + 2\bar{\beta}^\mu\bar{\zeta}^\nu\partial_\mu\partial_\nu\psi_{(1)} \\
&- \frac{s}{96\pi}\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}^\sigma\partial_\sigma\big\{e^{2\bar{\varphi}_+}[-2(\Box_{(\eta)}/\bar{\Box}^+)R^+_{(1)} - c^+R^+_{(1)} - 8\Box_{(\eta)}\varphi^+_{(1)} \\
&\qquad\qquad + 2c^+(1-6\ln\mu_+)\bar{\Box}^+\varphi^+_{(1)}] \\
&\qquad + e^{2\bar{\varphi}_-}[-2(\Box_{(\eta)}/\bar{\Box}^-)R^-_{(1)} - c^-R^-_{(1)} - 8\Box_{(\eta)}\varphi^-_{(1)} \\
&\qquad\qquad + 2c^-(1-6\ln\mu_-)\bar{\Box}^-\varphi^-_{(1)}]\big\} \\
&+ \frac{s}{48\pi}\bar{\zeta}^\mu\big\{e^{2\bar{\varphi}_+}[-2(\Box_{(\eta)}/\bar{\Box}^+)\partial_\mu R^+_{(1)} - \bar{g}^+_{\mu\nu}\partial^\nu R^+_{(1)} - 8\Box_{(\eta)}\varphi^+_{(1)} \\
&\qquad\qquad + 2(1-6\ln\mu_+)\bar{g}^+_{\mu\nu}\partial^\nu\varphi^+_{(1)}] \\
&\qquad - e^{2\bar{\varphi}_-}[-2(\Box_{(\eta)}/\bar{\Box}^-)\partial_\mu R^-_{(1)} - \bar{g}^-_{\mu\nu}\partial^\nu R^-_{(1)} - 8\Box_{(\eta)}\varphi^-_{(1)} \\
&\qquad\qquad + 2(1-6\ln\mu_-)\bar{g}^-_{\mu\nu}\partial^\nu\varphi^-_{(1)}]\big\},
\end{aligned} \tag{5.33}$$

$$
\begin{aligned}
0 =&\bar{\alpha}\Box_{(\eta)}\psi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_\mu\partial_\nu\Phi_{(1)} \\
&- \frac{s}{96\pi}\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}^{\sigma}\partial_\sigma\big\{e^{2\bar{\varphi}_+}[-2(\Box_{(\eta)}/\bar{\Box}^+)R^+_{(1)} - c^+R^+_{(1)} - 8\Box_{(\eta)}\varphi^+_{(1)} \\
&\qquad\qquad + 2c^+(1-6\ln\mu_+)\bar{\Box}^+\varphi^+_{(1)}] \\
&\qquad + e^{2\bar{\varphi}_-}[-2(\Box_{(\eta)}/\bar{\Box}^-)R^-_{(1)} - c^-R^-_{(1)} - 8\Box_{(\eta)}\varphi^-_{(1)} \\
&\qquad\qquad + 2c^-(1-6\ln\mu_-)\bar{\Box}^-\varphi^-_{(1)}]\big\} \qquad (5.34)\\
&+ \frac{s}{48\pi}\bar{\beta}^{\mu}\big\{e^{2\bar{\varphi}_+}[-2(\Box_{(\eta)}/\bar{\Box}^+)\partial_\mu R^+_{(1)} - \bar{g}^+_{\mu\nu}\partial^\nu R^+_{(1)} - 8\Box_{(\eta)}\varphi^+_{(1)} \\
&\qquad\qquad + 2(1-6\ln\mu_+)\bar{g}^+_{\mu\nu}\partial^\nu\varphi^+_{(1)}] \\
&\qquad - e^{2\bar{\varphi}_-}[-2(\Box_{(\eta)}/\bar{\Box}^-)\partial_\mu R^-_{(1)} - \bar{g}^-_{\mu\nu}\partial^\nu R^-_{(1)} - 8\Box_{(\eta)}\varphi^-_{(1)} \\
&\qquad\qquad + (1-6\ln\mu_-)\bar{g}^-_{\mu\nu}\partial^\nu\varphi^-_{(1)}]\big\},
\end{aligned}
$$

where $c^+ = \bar{g}^{\mu\nu}_+\eta_{\mu\nu}$, $c^- = \bar{g}^{\mu\nu}_-\eta_{\mu\nu}$ and $\Box_{(\eta)} = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Expressing the metric and dilaton perturbations in terms of perturbed fields gives

$$
g^{\mu\nu}_{+(1)} = 2\varphi^+_{(1)}\bar{g}^+_{\mu\nu} + e^{2\bar{\varphi}_+}\bar{\mathcal{A}}^{\mu\nu}_{+(1)}, \quad g^{\mu\nu}_{-(1)} = 2\varphi^-_{(1)}\bar{g}^-_{\mu\nu} + e^{2\bar{\varphi}_-}\bar{\mathcal{A}}^{\mu\nu}_{-(1)}, \qquad (5.35)
$$

$$
4\varphi^+_{(1)} = -e^{2\bar{\varphi}_+}\bar{g}^+_{\mu\nu}\bar{\mathcal{A}}^{\mu\nu}_{+(1)}, \quad 4\varphi^-_{(1)} = -e^{2\bar{\varphi}_-}\bar{g}^-_{\mu\nu}\bar{\mathcal{A}}^{\mu\nu}_{-(1)}, \qquad (5.36)
$$

where

$$
\begin{aligned}
s\bar{\mathcal{A}}^{\mu\nu}_{+(1)} &= \left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\eta^{\mu\nu}\bar{\beta}^{\sigma} + 2\bar{\zeta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_\sigma\Phi_{(1)} + \left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\eta^{\mu\nu}\bar{\zeta}^{\sigma} + 2\bar{\beta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_\sigma\psi_{(1)}, \\
s\bar{\mathcal{A}}^{\mu\nu}_{-(1)} &= \left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\eta^{\mu\nu}\bar{\beta}^{\sigma} - 2\bar{\zeta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_\sigma\Phi_{(1)} + \left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\eta^{\mu\nu}\bar{\zeta}^{\sigma} - 2\bar{\beta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_\sigma\psi_{(1)}.
\end{aligned}
\qquad (5.37)
$$

5.3 Plane-wave perturbations and their dispersion relations

By inspection of (5.33) and (5.34), the classical behaviour of the field perturbations satisfies

$$\bar{\varepsilon}\Box_{(\eta)}\Phi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\psi_{(1)} = 0, \quad \bar{\varepsilon}\Box_{(\eta)}\psi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\Phi_{(1)} = 0. \tag{5.38}$$

Requiring the perturbations to be of the form of a plane-wave, $\Phi_{(1)} \propto e^{ikx}$ and $\psi_{(1)} \propto e^{ikx}$, where $kx \equiv k_{\mu}x^{\mu}$, leads to the dispersion relation

$$\bar{\mathcal{A}}_{+}^{\mu\nu}k_{\mu}k_{\nu}\bar{\mathcal{A}}_{-}^{\sigma\tau}k_{\sigma}k_{\tau} = 0, \tag{5.39}$$

where $s\bar{\mathcal{A}}_{+}^{\mu\nu} = \sqrt{\bar{\alpha}\bar{\varepsilon}}\eta^{\mu\nu} + 2\bar{\beta}^{(\mu}\zeta^{\nu)}$ and $s\bar{\mathcal{A}}_{-}^{\mu\nu} = \sqrt{\bar{\alpha}\bar{\varepsilon}}\eta^{\mu\nu} - 2\bar{\beta}^{(\mu}\zeta^{\nu)}$. Furthermore (5.38) is linear in $\Phi_{(1)}$ and $\psi_{(1)}$, thus the following substitutions can be performed in (5.33) and (5.34): $\partial_{\mu} \mapsto ik_{\mu}$, $\Box_{(\eta)} \mapsto -\eta^{\mu\nu}k_{\mu}k_{\nu} \equiv -k\cdot k$, $\bar{\Box}^{+} \mapsto -\bar{g}_{+}^{\mu\nu}k_{\mu}k_{\nu}$, $\bar{\Box}^{-} \mapsto -\bar{g}_{-}^{\mu\nu}k_{\mu}k_{\nu}$, $1/\bar{\Box}^{+} \mapsto -1/(\bar{g}_{+}^{\mu\nu}k_{\mu}k_{\nu})$ and $1/\bar{\Box}^{-} \mapsto -1/(\bar{g}_{-}^{\mu\nu}k_{\mu}k_{\nu})$. Terms containing $1/\bar{\Box}^{+}$ and $1/\bar{\Box}^{-}$ dominate the quantum corrections if $\bar{\mathcal{A}}_{+}^{\mu\nu}k_{\mu}k_{\nu} = 0$ and $\bar{\mathcal{A}}_{-}^{\mu\nu}k_{\mu}k_{\nu} = 0$ respectively. Close to the classical solution $\bar{\mathcal{A}}_{+}^{\mu\nu}k_{\mu}k_{\nu} = 0$ the field equations (5.33) and (5.34) approximate to

$$0 = \bar{g}_{+}^{\mu\nu}k_{\mu}k_{\nu}\left(\bar{\varepsilon}k\cdot k\Phi_{(1)} + 2\bar{\beta}k\bar{\zeta}k\psi_{(1)}\right) + is\frac{e^{2\bar{\varphi}_{+}}}{48\pi}\left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}k + 2\bar{\zeta}k\right)k\cdot kR_{(1)}^{+}, \tag{5.40}$$

$$0 = \bar{g}_{+}^{\mu\nu} k_\mu k_\nu \left(\bar{\alpha} k \cdot k \psi_{(1)} + 2\bar{\beta}k\bar{\zeta}k\Phi_{(1)} \right) + is\frac{e^{2\bar{\varphi}_+}}{48\pi} \left(-\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}k + 2\bar{\beta}k \right) k \cdot k R^{+}_{(1)}, \tag{5.41}$$

respectively, where $\bar{\beta}k \equiv \beta^\mu k_\mu$ and $\bar{\zeta}k \equiv \zeta^\mu k_\mu$, and $R^{+}_{(1)} = k_\mu k_\nu \bar{g}^{\mu\nu}_{+(1)}$ follows from equation (5.32). Furthermore $\bar{g}^{\mu\nu}_{+(1)} = e^{2\bar{\varphi}_+}\mathcal{A}^{\mu\nu}_{+(1)}$, hence

$$sR^{+}_{(1)} = \left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}k + 2\bar{\zeta}k \right) ik \cdot ke^{2\bar{\varphi}_+}\Phi_{(1)} + \left(-\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}k + 2\bar{\beta}k \right) ik \cdot ke^{2\bar{\varphi}_+}\psi_{(1)}. \tag{5.42}$$

Introducing

$$a = s\left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}k + 2\bar{\zeta}k \right) k \cdot ke^{2\bar{\varphi}_+}, \tag{5.43}$$

$$b = s\left(-\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}k + 2\bar{\beta}k \right) k \cdot ke^{2\bar{\varphi}_+}, \tag{5.44}$$

the field equations become

$$0 = \bar{g}_{+}^{\mu\nu} k_\mu k_\nu \left(\bar{\varepsilon} k \cdot k \Phi_{(1)} + 2\bar{\beta}k\bar{\zeta}k\psi_{(1)} \right) - \frac{1}{48\pi}a\left(a\Phi_{(1)} + b\psi_{(1)} \right), \tag{5.45}$$

$$0 = \bar{g}_{+}^{\mu\nu} k_\mu k_\nu \left(\bar{\alpha} k \cdot k \psi_{(1)} + 2\bar{\beta}k\bar{\zeta}k\Phi_{(1)} \right) - \frac{1}{48\pi}b\left(a\Phi_{(1)} + b\psi_{(1)} \right). \tag{5.46}$$

These can be written in matrix form as

$$\begin{pmatrix} \bar{g}_{+}^{\mu\nu} k_\mu k_\nu \bar{\varepsilon} k \cdot k - \frac{a^2}{48\pi} & 2\bar{g}_{+}^{\mu\nu} k_\mu k_\nu \bar{\beta}k\bar{\zeta}k - \frac{ab}{48\pi} \\ 2\bar{g}_{+}^{\mu\nu} k_\mu k_\nu \bar{\beta}k\bar{\zeta}k - \frac{ab}{48\pi} & \bar{g}_{+}^{\mu\nu} k_\mu k_\nu \bar{\alpha} k \cdot k - \frac{b^2}{48\pi} \end{pmatrix} \begin{pmatrix} \Phi_{(1)} \\ \psi_{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.47}$$

Requiring the determinant of this matrix to equal zero yields

$$\begin{aligned} 0 =& (\bar{g}_{+}^{\mu\nu} k_\mu k_\nu)^2 [\bar{\varepsilon}\bar{\alpha}(k \cdot k)^2 - (2\bar{\beta}k\bar{\zeta}k)^2] \\ &- \frac{1}{48\pi}[k \cdot k(a^2\bar{\alpha} + b^2\bar{\varepsilon}) - 4ab\bar{\beta}k\bar{\zeta}k]\bar{g}_{+}^{\mu\nu} k_\mu k_\nu. \end{aligned} \tag{5.48}$$

From (5.13) it follows that $\bar{g}_+^{\mu\nu}k_\mu k_\nu = e^{2\bar{\varphi}_+}\bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu$, furthermore $\bar{\varepsilon}\bar{\alpha}(k\cdot k)^2 - (2\bar{\beta}k\bar{\zeta}k)^2 = \bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu \bar{\mathcal{A}}_-^{\sigma\tau}k_\sigma k_\tau$, thus

$$0 = e^{2\bar{\varphi}_+}(\bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu)^2\bar{\mathcal{A}}_-^{\sigma\tau}k_\sigma k_\tau - \frac{1}{48\pi}[k\cdot k(a^2\bar{\alpha}+b^2\bar{\varepsilon}) - 4ab\bar{\beta}k\bar{\zeta}k]. \tag{5.49}$$

Utilising $\bar{\mathcal{A}}_-^{\mu\nu}k_\mu k_\nu = 2s\sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k - \bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu$ and $2\bar{\beta}k\bar{\zeta}k = s\bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu - \sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k$, and keeping first two lowest order terms in k_μ gives

$$0 = 2se^{2\bar{\varphi}_+}(\bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu)^2\sqrt{\bar{\alpha}\bar{\varepsilon}} - \frac{1}{48\pi}[(a^2\bar{\alpha}+b^2\bar{\varepsilon}) + 2ab\sqrt{\bar{\alpha}\bar{\varepsilon}}]. \tag{5.50}$$

Dividing by $2\sqrt{\bar{\alpha}\bar{\varepsilon}}$ and factorising the last term gives

$$e^{2\bar{\varphi}_+}(\bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu)^2 = -\frac{s}{96\pi}\left(a\left(\frac{\bar{\alpha}}{\bar{\varepsilon}}\right)^{\frac{1}{4}} - b\left(\frac{\bar{\varepsilon}}{\bar{\alpha}}\right)^{\frac{1}{4}}\right)^2. \tag{5.51}$$

Demanding that both sides are real-valued when taking the square root requires choosing $s=-1$. By definition $e^{-4\bar{\varphi}_+} = (\bar{\alpha}\bar{\varepsilon}+\bar{\beta}\cdot\bar{\zeta})^2 - \bar{\beta}\cdot\bar{\beta}\bar{\zeta}\cdot\bar{\zeta}$. Finally substituting a, b and $\bar{\mathcal{A}}_+^{\mu\nu}k_\mu k_\nu$ back and taking the square root yields

$$|\sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k + 2\bar{\beta}k\bar{\zeta}k| = \check{\epsilon}\frac{3}{4\sqrt{6\pi}}\frac{|k\cdot k|}{[(\sqrt{\bar{\alpha}\bar{\varepsilon}}+\bar{\beta}\cdot\bar{\zeta})^2 - \bar{\beta}\cdot\bar{\beta}\bar{\zeta}\cdot\bar{\zeta}]^{\frac{1}{4}}}\left|\left(\frac{\bar{\alpha}}{\bar{\varepsilon}}\right)^{\frac{1}{4}}\bar{\zeta}k - \left(\frac{\bar{\varepsilon}}{\bar{\alpha}}\right)^{\frac{1}{4}}\bar{\beta}k\right|, \tag{5.52}$$

where $\check{\epsilon}$ is a parameter tracking the order of k and will be set to unity at the end. A similar approach can be taken with the field equations close to the $\bar{\mathcal{A}}_-^{\mu\nu}k_\mu k_\nu = 0$ solution with similar results, the difference being $a = s\left(-\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}k - 2\bar{\zeta}k\right)k\cdot ke^{2\bar{\varphi}_+}$, $b = s\left(-\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}k - 2\bar{\beta}k\right)k\cdot ke^{2\bar{\varphi}_+}$, $2\bar{\beta}k\bar{\zeta}k = -s\bar{\mathcal{A}}_-^{\mu\nu}k_\mu k_\nu + \sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k$, $e^{-4\bar{\varphi}_-} = (\bar{\alpha}\bar{\varepsilon} - \bar{\beta}\cdot\bar{\zeta})^2 - \bar{\beta}\cdot\bar{\beta}\bar{\zeta}\cdot\bar{\zeta}$, and again requiring $s=-1$, yielding

$$|\sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k-2\bar{\beta}k\bar{\zeta}k|=\check{\epsilon}\frac{3}{4\sqrt{6\pi}}\frac{|k\cdot k|}{[(\sqrt{\bar{\alpha}\bar{\varepsilon}}-\bar{\beta}\cdot\bar{\zeta})^2-\bar{\beta}\cdot\bar{\beta}\bar{\zeta}\cdot\bar{\zeta}]^{\frac{1}{4}}}\left|\left(\frac{\bar{\alpha}}{\bar{\varepsilon}}\right)^{\frac{1}{4}}\bar{\zeta}k+\left(\frac{\bar{\varepsilon}}{\bar{\alpha}}\right)^{\frac{1}{4}}\bar{\beta}k\right|. \tag{5.53}$$

The justification of (5.1) requires the laser frequency to be the highest frequency in the unperturbed case, hence equations (5.52) and (5.53) will be investigated in the ultrarelativistic limit. To do so, let the timelike vector $\bar{\beta}^\mu$ be decomposed as

$$\bar{\beta}^\mu=\frac{1}{\epsilon}\bar{\beta}^\mu_{[-1]}+\epsilon\bar{\beta}^\mu_{[1]}, \tag{5.54}$$

where $\bar{\beta}^\mu_{[-1]}$ and $\bar{\beta}^\mu_{[-1]}$ are lightlike, and ϵ is a positive parameter used for tracking the order of perturbations and will be set to unity at the end. It is useful to correlate ϵ and $\check{\epsilon}$: inspection of (5.52) and (5.53) with (5.54) suggests $\check{\epsilon}=\sqrt{\epsilon^{2p-1}}$ for some positive integer p. Note that $p\gg 1$ since the quantum corrections should be much smaller than the deviation of β^μ from a null vector. Utilising $\bar{\alpha}=\bar{\beta}\cdot\bar{\beta}=2\bar{\beta}_{[-1]}\cdot\bar{\beta}_{[1]}$ and $\bar{\varepsilon}=\bar{\zeta}\cdot\bar{\zeta}+1$, (5.52) and (5.53) become

$$|\epsilon\sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k+\left(\bar{\beta}_{[-1]}k+\epsilon^2\bar{\beta}_{[1]}k\right)\bar{\zeta}k|=\epsilon^p\frac{3|k\cdot k|\left(\frac{\bar{\varepsilon}}{\bar{\alpha}}\right)^{\frac{1}{4}}\left|\bar{\beta}_{[-1]}k\right|}{4\sqrt{6\pi}\sqrt{|\bar{\beta}_{[-1]}\cdot\bar{\zeta}|}}, \tag{5.55}$$

$$|\epsilon\sqrt{\bar{\alpha}\bar{\varepsilon}}k\cdot k-\left(\bar{\beta}_{[-1]}k+\epsilon^2\bar{\beta}_{[1]}k\right)\bar{\zeta}k|=\epsilon^p\frac{3|k\cdot k|\left(\frac{\bar{\varepsilon}}{\bar{\alpha}}\right)^{\frac{1}{4}}\left|\bar{\beta}_{[-1]}k\right|}{4\sqrt{6\pi}\sqrt{|\bar{\beta}_{[-1]}\cdot\bar{\zeta}|}}, \tag{5.56}$$

where $\mathcal{O}(\epsilon^{p+1})$ has been set to zero. To the lowest order in ϵ both of these are solved by

$$\bar{\beta}_{[-1]}k_{[0]}\bar{\zeta}k_{[0]}=0, \tag{5.57}$$

where $k_\mu = k_{[0]\mu} + \epsilon k_{[1]\mu} + \mathcal{O}(\epsilon^2)$ has been introduced. This can be solved by $\bar{\beta}_{[-1]}k_{[0]} = 0$ or $\bar{\zeta}k_{[0]} = 0$. Firstly consider $\bar{\beta}_{[-1]}k_{[0]} = 0$: since $\bar{\beta}^\mu_{[-1]}$ is null, $k^\mu_{[0]}$ must also be null and proportional to $\bar{\beta}^\mu_{[-1]}$, where $k^\mu_{[0]} = \eta^{\mu\nu}k_{[0]\mu}$. To first order in ϵ equations (5.55) and (5.56) both give

$$\bar{\beta}_{[-1]}k_{[1]}\bar{\zeta}k_{[0]} = 0, \tag{5.58}$$

thus $k_{[1]\mu}$ is proportional to $k_{[0]\mu}$. Up to first order in ϵ, the phase speed of the perturbations $\phi_{(1)}$ and $\psi_{(1)}$ is the speed of light in the vacuum. Now consider the $\bar{\zeta}k_{[0]} = 0$ case. To analyse this solution in the first order of ϵ, it is useful to introduce a timelike unit vector $\mathfrak{n}^\mu = \bar{\zeta}^\mu/\sqrt{-\zeta\cdot\zeta}$ and a spacelike unit vector $\mathfrak{n}_\perp$ orthogonal to $\mathfrak{n}$. By using $\eta^{\mu\nu} = -\mathfrak{n}^\mu\mathfrak{n}^\nu + \mathfrak{n}^\mu_\perp\mathfrak{n}^\nu_\perp$, equations (5.55) and (5.56) to first order in ϵ give

$$\mathfrak{n}k_{[1]} = -\sqrt{\frac{\bar{\alpha}\bar{\varepsilon}}{-\bar{\zeta}\cdot\bar{\zeta}}}\frac{\mathfrak{n}_\perp k_{[0]}}{2\bar{\beta}_{[-1]}\cdot\mathfrak{n}_\perp}, \tag{5.59}$$

$$\mathfrak{n}k_{[1]} = \sqrt{\frac{\bar{\alpha}\bar{\varepsilon}}{-\bar{\zeta}\cdot\bar{\zeta}}}\frac{\mathfrak{n}_\perp k_{[0]}}{2\bar{\beta}_{[-1]}\cdot\mathfrak{n}_\perp}, \tag{5.60}$$

respectively. Note that $\mathfrak{n}_\perp k$ is proportional to the wavenumber of the perturbations $\Phi_{(1)}$ and $\psi_{(1)}$ in the rest frame of the plasma electrons, thus up to $(p-1)$th order there is no dispersion, as (5.55) and (5.56) are second-order homogeneous polynomials in k_μ. Corrections due to quantum fluctuations only contribute at pth order and above in ϵ. For the $\bar{\beta}_{[1]}k_{[0]} = 0$ case, the right-hand side of (5.55) and (5.56) can be shown to be of order ϵ^{p+1}. Thus the quantum corrections are too small to be captured by the analysis. However when $\bar{\zeta}k_{[0]} = 0$,

$$\mathfrak{n}k = \pm\nu\mathfrak{n}_\perp k - \epsilon^p \frac{3}{8\sqrt{6\pi|\mathfrak{n}\cdot\bar{\beta}_{[-1]}|}} \left(\frac{\bar{\varepsilon}}{\bar{\alpha}|\bar{\zeta}\cdot\bar{\zeta}|^3}\right)^{\frac{1}{4}} (\mathfrak{n}_\perp k)^2 + \mathcal{O}(\epsilon^{p+1}), \tag{5.61}$$

where the $-$ sign corresponds to (5.55) and the $+$ sign corresponds to (5.56), and the constant ν is given by

$$\nu = \epsilon\sqrt{\frac{\bar{\alpha}\bar{\varepsilon}}{-\bar{\zeta}\cdot\bar{\zeta}}}\frac{1}{2\bar{\beta}_{[-1]}\cdot\mathfrak{n}_\perp} + \mathcal{O}(\epsilon^2). \tag{5.62}$$

It can be shown that $|\bar{\beta}_{[-1]}\cdot\mathfrak{n}| = |\bar{\beta}_{[-1]}\cdot\mathfrak{n}_\perp| = |\partial_0\Phi_{(0)}|$. The sign of the quantum corrections will be chosen such that its contribution to the frequency of the perturbation is positive. Introducing the angular frequency $\omega = |\mathfrak{n}k|$ and wavenumber $\kappa = |\mathfrak{n}_\perp k|$ of the perturbations in the rest frame of plasma ions, and setting $\mathcal{O}(\epsilon^{p+1})$ to zero, ϵ to unity, $\Phi_{(0)} = \omega_0 t + k_0 z$, $d\Phi_{(0)}\cdot d\Phi_{(0)} = -\omega_p^2$ with ω_p being the plasma frequency, and restoring units as outlined in Appendix B finally yields

$$\omega = v\kappa + \frac{3}{8}\sqrt{\frac{\hbar e^2}{6\pi\varepsilon_0 m^2 c^3\tilde{\Lambda}}}\left(\frac{a_0^2}{(a_0^2+1)^3}\right)^{\frac{1}{4}}\frac{c^2\kappa^2}{\sqrt{\omega_0\omega_p}}, \tag{5.63}$$

with

$$v = \frac{\omega_p c}{2\omega_0}\frac{a_0}{\sqrt{a_0^2+1}}. \tag{5.64}$$

Equation (5.63) can be written as

$$\omega = v\kappa + \sqrt{\frac{3\alpha}{128\pi^2}}\frac{\lambda_e}{w_0}\left(\frac{a_0^2}{(a_0^2+1)^3}\right)^{\frac{1}{4}}\frac{c^2\kappa^2}{\sqrt{\omega_0\omega_p}}, \tag{5.65}$$

where λ_e is the Compton wavelength of an electron, α is the fine-structure constant, and the width of the laser pulse $w_0 = \sqrt{\tilde{\Lambda}}$ has been introduced. Note that the standard dispersion relation for a cold plasma in the absence of a laser pulse $\omega^2 = \omega_p^2 + c^2k^2$ is not recovered in the limit $a_0 \to 0$. This is because both the laser and the plasma are indispensable parts of the underlying theory, as equations (3.49) and (3.50) both vanish if either is field is zero.

5.3.1 Gaussian wave packet

This section will show the implications of the dispersion relation found in (5.65) for a Gaussian pulse, suggesting that quantum fluctuations could play a significant role in the evolution of an underdense plasma driven by an x-ray laser pulse. Consider a Gaussian wave packet given by

$$\Xi(t,z) = \int_{-\infty}^{\infty} dk e^{i(kz-\omega t)} \exp\left(-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}\right), \tag{5.66}$$

where ω is a function of k, and σ controls the width of the packet at $t = 0$. Taylor expanding $\omega = \omega(k_0)+\omega'(k_0)(k-k_0)+\frac{1}{2}\omega''(k_0)(k-k_0)^2+\mathcal{O}((k-k_0)^3)$, where prime denotes derivative with respect to k, yields

$$|\Xi| = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{(\omega''(k_0))^2t^2+\frac{1}{\sigma^4}}}\exp\left\{-\frac{1}{2}\frac{(z-\omega'(k_0)t)^2}{\sigma^2(\omega''(k_0))^2t^2+\frac{1}{\sigma^2}}\right\}. \tag{5.67}$$

Thus the quantity

$$\Delta = \sqrt{\sigma^2(\omega''(k_0))^2t^2+\frac{1}{\sigma^2}} \tag{5.68}$$

is an estimate of half of the width of a pulse subject to the dispersion relation given by $\omega(k)$. Let $\Delta_0 = \frac{1}{\sigma}$, and $\frac{\Delta}{\Delta_0} = 1+\delta$ for some small parameter δ, since the contributions of the quantum fluctuations are small. The characteristic time scale τ over which the length of a Gaussian wave packet increases by a small amount δ due to quantum fluctuations is obtained from (5.68):

$$\tau \sim \sqrt{\frac{128\pi^2}{3\alpha}\frac{w_0}{\lambda_e}}\left(\frac{a_0^2}{(a_0^2+1)^3}\right)^{-\frac{1}{4}}\frac{\sqrt{\omega_0\omega_p}}{c^2}\sigma^2\frac{\sqrt{(1+\delta)^2-1}}{2}. \tag{5.69}$$

Let $\breve{\tau} = \frac{\omega_0}{2\pi}\tau$ and $\breve{\sigma} = \frac{\sigma}{\lambda_0}$ be normalised quantities with respect to the laser period and laser wavelength $\lambda_0 = \frac{2\pi c}{\omega_0}$ respectively. The wavelength of the laser pulse is required to be the shortest classical wavelength in this model, thus $\breve{\sigma} > 1$. Considering $\delta \ll 1$ gives

$$\breve{\tau} \gtrsim 1509\left(\frac{a_0^2}{(a_0^2+1)^3}\right)^{-\frac{1}{4}}\sqrt{\frac{w_0^2\lambda_0\delta}{2\lambda_e^2\lambda_p}}, \tag{5.70}$$

where $\lambda_p = \frac{2\pi c}{\omega_p}$. It is possible for an intense laser pulse to propagate through an underdense plasma over distances that are many multiples of the classical Rayleigh length. Equation (5.70) along with the number of oscillations $\frac{N\pi w_0^2}{\lambda_0^2}$ corresponding to N multiples of the Rayleigh length of a laser beam yields the upper bound

$$\delta \lesssim 8.7\times 10^{-6}\sqrt{\frac{a_0^2}{(a_0^2+1)^3}}\frac{N^2 w_0^2\lambda_e^2\lambda_p}{\lambda_0^5}. \tag{5.71}$$

This suggests that the effects due to quantum fluctuations will not be detectable in experiments based on near-IR lasers. As an example, it is possible to maintain an intense near-IR laser pulse with $\lambda_0 = 800$ nm and $w_0 = \lambda_p = 30$ μm over tens of Rayleigh lengths [53]. Even though the dimensionless laser amplitude $a_0 \approx 0.7$ is achievable using high-power near-IR lasers, these parameters yield $\delta \lesssim 2.6 \times 10^{-9}$ for $N = 40$, which is experimentally unresolvable. However, similarly to the results of section 4.3.1, the role of a_0 is not that significant. Strong dependence of (5.71) on λ_0 suggests that x-ray lasers may lead to an experimentally accessible measurement. For example taking $\lambda_0 = 10$ nm, $\lambda_p = 100$ μm, $w_0 = 100$ μm, $a_0 \approx 6 \times 10^{-5}$ yields $\delta \lesssim 0.05$. Thus it may be possible to investigate this result with the use of an x-ray laser, such as the European XFEL [5].

Chapter 6

Conclusion

The main aim of this thesis was to establish a link between laser-driven plasma and analogue Hawking radiation. While this has been achieved, other results were obtained along the way, as will be summarised below.

Two field equations for a laser-driven plasma were derived in chapter 3 in terms of the laser phase Φ, 4-momentum potential ψ of the plasma electrons and the energy density λ. Furthermore, utilising the dispersion relations of Φ and ψ in the minimal energy density case, these field equations were shown to be readily expressible in terms of Φ and ψ only. This system was also reduced to two dimensions with the introduction of Λ, related to the cross-sectional area of the system. It was shown that the field equations for the minimal energy density system can be motivated from scalar quantum electrodynamics.

Following that, effective metrics were derived from the perturbations of the field equations in chapter 4 in various regimes. Firstly the field equations were considered separately in section 4.1. It was shown that regardless of the choice of energy density, the equation involving ψ does not lead to Lorentzian effective metrics. However, the Φ equation was more robust. It was shown that utilising minimal energy density leads yet again to a non-Lorentzian effective metric, but assuming $\Phi = \gamma t + h(z)$ for some general energy density does indeed give a Lorentzian effective metric. It was shown that with suitable substitutions it is the homogeneous plane wave metric. Such a metric is conformally flat; reducing the system to two dimensions allowed for investigation of the Unruh effect, leading to non-trivial motion of the accelerated observer in the laboratory frame, with the freedom of choice of $h(z)$. Lastly it was briefly explained that considering a spherical system does not readily lead to an analogue of a physically interesting spacetime.

In section 4.2, the field equations were considered together with a general energy density. It was shown that in order to derive an effective metric, the perturbations of the fields are required to have high frequency. With that, the only physically sensible energy density giving rise to two effective metrics was found to be that of the minimal energy density. Firstly the case of the fields being linear in Minkowski coordinates was considered. This led to flat metric, which again allowed for investigating the Unruh effect, and led to non-uniform acceleration in the laboratory frame. Following that, the fields were considered as solutions of the 2-dimensional wave equation. While the field equations were solvable, the effective metrics had undesirable properties and as such were discarded. A spherical system was also investigated with both fields having the form of $\gamma_{\Phi/\psi} t + h_{\Phi/\psi}(r)$. Two distinct solutions were found for the h functions, however one displayed naked singularities while the other was not physically viable.

The final regime considered effective metrics on 2-dimensional spacetime where the cross-sectional area of the laser pulse (spot size) is not constant. The spot size appeared in the field equations as well as being the conformal factor of the effective metrics. Both fields were assumed to be of the form $\gamma_{\Phi/\psi}t + h_{\Phi/\psi}(z)$, and a relation between h_Φ and h_ψ was found in terms of a function h. It was shown that there exists a singularity in one of the effective metrics, and parameters were matched such that the ratio of the components of the effective metrics was that of the Schwarzschild metric. The resulting Hawking temperature depends on the initial dimensionless laser amplitude a_0 and the waist of the laser $w_0 = \sqrt{\tilde{\Lambda}}$. The initial amplitude has a small effect on the Hawking temperature compared to the waist. It was shown that the Hawking temperature resulting from using an intense near-IR laser is about 4.5 K. However, the temperature is inversely proportional to the waist, which significantly increases the feasibility of detecting it as the waist gets smaller.

Finally chapter 5 explored the quantum effects of the action for underdense plasma in two dimensions for a constant spot size. The analysis included quantum backreaction of the system; backreaction is an important area of study of analogue evaporating black hole systems. The 1-loop effective action was shown to be expressible in terms of a massless field theory on a dilatonic curved background for fields that are linear in the Minkowski coordinates. However it was argued that the fields may have a more general form, thus the field equations were subsequently derived for a general case. With these, the linear solution was perturbed and two distinct dispersion relations were derived, describing dynamical perturbations of a uniform underdense laser-driven plasma. One of the dispersion relations describes propagation in the same direction, at essentially the same phase speed, as the laser beam. The remaining dispersion relation is associated with perturbations that co-propagate and counter-propagate with the laser beam, but at a much slower speed than the laser beam. None of the modes are dispersive without quantum corrections, and the modes that propagate at essentially the same speed as the laser beam are non-dispersive even when quantum effects are included. The effect of the non-trivial dispersion relation on a Gaussian wave packet was then analysed. It was shown that for an near-IR laser the effect is negligible, while for an x-ray laser the width of the packet increases by about 5% over a distance corresponding to 40 Rayleigh lengths of the laser.

6.1 Future work

There are several avenues for undertaking future work. Understanding how to set up and measure the vacuum of an effective metric, and that of an accelerated observer, would be an important step in finding a way to probe the Unruh effect found in section 4.2.1. Investigating phenomena other than Hawking radiation may also be feasible, such as naked singularities as displayed by several of the presented effective metrics, or analogue string theory through homogeneous plane wave effective metrics found in section 4.1.3. The simplest underdense plasma model was heavily focused on in this thesis, but it may be possible to find a manageable system in a more general case.

Lastly, a comparison of the analogue Hawking temperature found in section 4.3 and typical plasma temperatures suggests that a detailed model of the laser-driven plasma is needed to confidently identify signatures of the analogue Hawking effect. The temperature of the plasma electrons in a laser-driven plasma accelerator is $\sim 5 \times 10^5$ K [53], which is $\sim \times 10^5$ times larger than the expected analogue Hawking temperature. Even so, for comparison, it is claimed [40, 41] that an analogue Hawking temperature of 1.2 nK has been measured in an atomic Bose-Einstein condensate. Whilst it is clear that identifying the analogue Hawking effect in a laser-driven plasma accelerator is a significant challenge, the fact that the results show that its analogue Hawking temperature is ten orders of magnitude larger than that of a Bose-Einstein condensate suggests that further investigation is deserved.

Bibliography

[1] J. Kormendy and L. C. Ho, "Coevolution (Or Not) of Supermassive Black Holes and Host Galaxies", Annu. Rev. Astron. Astrophys. 51 (2013) 511.

[2] A. M. Ghez et al., "Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits", Astrophys. J. 689 (2008) 1044.

[3] S. W. Hawking, "Particle creation by black holes", Commun. Math. Phys. 43 (1975) 199.

[4] https://www.eli-beams.eu/

[5] https://www.xfel.eu/

[6] W. G. Unruh, "Notes on black-hole evaporation", Phys. Rev. D 14 (1976) 870.

[7] N. D. Birrell, P. C. W. Davies, "Quantum Fields in Curved Space", Cambridge University Press, 1982.

[8] L. Parker and D. Toms, "Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity", Cambridge University Press, 2009.

[9] R. Brout et al., “A Primer for black hole quantum physics”, Phys. Rep. 260 (1995) 329.

[10] T. Jacobson and G. Kang, “Conformal Invariance of Black Hole Temperature”, Class. Quant. Grav. 10 (1993) L201.

[11] W. Kummer and D. V. Vassilevich, “Effective action and Hawking radiation for dilaton coupled scalars in two dimensions”, Phys. Rev. D 60 (1999) 084021.

[12] H. Kleinert, “Path integrals in quantum mechanics, statistics, polymer physics, and financial markets”, World Scientific, 2009.

[13] M. Forger and H. Römer, “Currents and the energy-momentum tensor in classical field theory: a fresh look at an old problem”, Ann. Phys. 309 (2004) 306.

[14] P. B. Gikey, “Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem”, CRC Press, Boca Raton, 1994.

[15] W. Kummer, H. Liebel and D. V. Vassilevich, “Hawking Radiation for Non-Minimally Coupled Matter from Generalized 2-D Black Hole Models”, Mod. Phys. Lett. A 12 (1997) 2683.

[16] W. G. Unruh, “Experimental Black-Hole Evaporation?”, Phys. Rev. Lett. 46 (1981) 1351.

[17] T. A. Jacobson, “Black-hole evaporation and ultrashort distances”, Phys. Rev. D 44 (1991) 1731.

[18] C. Barceló, S. Liberati and M. Visser, "Analogue Gravity", Living Rev. Relativ. 14 (2011) 3.

[19] G.Rousseaux et al., "Horizon effects with surface waves on moving water", New J. Phys. 12 (2010) 095018.

[20] M. Visser and S. Weinfurtner, "Analogue spacetimes: Toy models for 'quantum gravity' ", in "From Quantum to Emergent Gravity: Theory and Phenomenology", Proceedings of Science, SISSA, Trieste (2007).

[21] F. W. Hehl, Y. N. Obukhov and G. F. Rubilar, "Spacetime metric from linear electrodynamics II", arXiv:gr-qc/9911096.

[22] M. Novello et al., "Geometrical aspects of light propagation in nonlinear electrodynamics", Phys. Rev. D 61 (2000) 045001.

[23] F. de Felice, "On the gravitational field acting as an optical medium", Gen. Rel. Grav. 2 (1971) 347.

[24] D. Faccio, "Laser pulse analogues for gravity and analogue Hawking radiation", Contemp. Phys. 53 (2012) 97.

[25] C. Barceló, S. Liberati and M. Visser, "Analogue gravity from Bose-Einstein condensates", Class. Quant. Grav. 18 (2001) 1137.

[26] P. Chen and G. Mourou, "Accelerating Plasma Mirrors to Investigate the Black Hole Information Loss Paradox", Phys. Rev. Lett 118 (2017) 045001.

[27] G. Rousseaux and Y. Stepanyants, “Modeling of Bose–Einstein Condensation in a Water Tank”, in “Nonlinear Waves and Pattern Dynamics”, Springer International Publishing, 2018.

[28] C. Barceló, S. Liberati, M. Visser, “Analog gravity from field theory normal modes?”, Class. Quant. Grav. 18 (2001) 3595.

[29] C. Barceló, S. Liberati, M. Visser, “Refringence, field theory and normal modes”, Class. Quant. Grav. 19 (2002) 2961.

[30] L. C. Garcia de Andrade, “Non-Riemannian vortex geometry of rotational viscous fluids and breaking of the acoustic Lorentz invariance”, Phys. Lett. A 339 (2005) 188.

[31] T. G. Philbin et al., “Fiber-Optical Analog of the Event Horizon”, Science 319 (2008) 1367.

[32] G. Rousseaux et al., “Observation of negative-frequency waves in a water tank: a classical analogue to the Hawking effect?”, New J. Phys. 10 (2008) 053015.

[33] O. Lahav et al., “Realization of a Sonic Black Hole Analog in a Bose-Einstein Condensate”, Phys. Rev. Lett. 105 (2010) 240401.

[34] S. Weinfurtner et al., “Measurement of Stimulated Hawking Emission in an Analogue System”, Phys. Rev. Lett. 106 (2011) 021302.

[35] R. Balbinot et al., “Nonlocal density correlations as a signature of Hawking radiation from acoustic black holes”, Phys. Rev. A 78 (2008) 021603(R).

[36] I. Carusotto et al., “Numerical observation of Hawking radiation from acoustic black holes in atomic Bose–Einstein condensates”, New J. Phys. 10 (2008) 103001.

[37] J. R. M. de Nova, F. Sols, and I. Zapata, “Violation of Cauchy-Schwarz inequalities by spontaneous Hawking radiation in resonant boson structures”, Phys. Rev. A 89 (2014) 043808.

[38] X. Busch and R. Parentani, “Quantum entanglement in analogue Hawking radiation: When is the final state nonseparable?”, Phys. Rev. D 89 (2014) 105024.

[39] S. Finazzi and I. Carusotto, “Entangled phonons in atomic Bose-Einstein condensates”, Phys. Rev. A 90 (2014) 033607.

[40] J. Steinhauer, “Observation of quantum Hawking radiation and its entanglement in an analogue black hole”, Nat. Phys. 12 (2016) 959.

[41] J. R. M. de Nova et al., “Observation of thermal Hawking radiation and its temperature in an analogue black hole”, Nature 569 (2019) 688.

[42] U. Leonhardt, “Questioning the Recent Observation of Quantum Hawking Radiation”, Ann. Phys. (Berlin) 530 (2018) 1700114.

[43] J. Drori et al., “Observation of Stimulated Hawking Radiation in an Optical Analogue”, Phys. Rev. Lett. 122 (2019) 010404.

[44] A. Choudhary and F. König, “Efficient frequency shifting of dispersive waves at solitons”, Opt. Express 20 (2012) 5538.

[45] T. Torres et al., “Rotational superradiant scattering in a vortex flow”, Nat. Phys. 13 (2017) 833.

[46] T. Torres et al., “Waves on a vortex: rays, rings and resonances”, J. Fluid Mech. 857 (2018) 291.

[47] H. Goodhew et al., “Backreaction in an analogue black hole experiment”, arXiv:1905.03045 [gr-qc].

[48] E. A. Startsev and C. J. McKinstrie, “Multiple scale derivation of the relativistic ponderomotive force”, Phys. Rev. E 55 (1997) 7527.

[49] I. Y. Dodin, N. J. Fisch and G. M. Fraiman, “Drift Lagrangian for a Relativistic Particle in an Intense Laser Field”, Jetp Lett. 78 (2003) 202.

[50] D. Bauer, P. Mulser and W. -H. Steeb, “Relativistic Ponderomotive Force, Uphill Acceleration, and Transition to Chaos”, Phys. Rev. Lett. 75 (1995) 4622.

[51] T. W. B. Kibble, “Frequency Shift in High-Intensity Compton Scattering”, Phys. Rev. 138 (1965) B740.

[52] T. W. B. Kibble, “Mutual Refraction of Electrons and Photons”, Phys. Rev. 150 (1966) 1060.

[53] E. Esarey, C. B. Schroeder and W.P. Leemans, “Physics of laser-driven plasma-based electron accelerators”, Rev. Mod. Phys. 81 (2009) 1229.

[54] G. W. Gibbons, “Quantized fields propagating in plane-wave spacetimes”, Commun. Math. Phys. 45 (1975) 191.

[55] G. Papadopoulos, J. G. Russo and A. A. Tseytlin, "Solvable model of strings in a time-dependent plane-wave background", Class. Quant. Grav. 20 (2003) 969.

[56] H. Yoneda et al., "Atomic inner-shell laser at 1.5-ångström wavelength pumped by an X-ray free-electron laser", Nature 524(7566) (2015) 446.

[57] A. D. Boozer, "General relativity in (1 + 1) dimensions", Eur. J. Phys. 29 (2008) 319.

Appendix A

Identities

A smooth map $f : \mathcal{M} \to \mathcal{N}$ between two manifolds $\mathcal{M}$ and $\mathcal{N}$ induces a pull-back map $f^* : \mathcal{N} \to \mathcal{M}$ with the following properties:

$$\begin{aligned} f^* h \equiv& h \circ f, \\ f^*(\alpha + \beta) =& f^*\alpha + f^*\beta, \\ f^*(\alpha \wedge \beta) =& f^*\alpha \wedge f^*\beta, \\ f^* d =& d f^*, \end{aligned} \tag{A.1}$$

where h is a 0-form, α and β are p-forms.

Stokes' theorem is given by

$$\int_{\mathcal{S}} d\alpha = \int_{\partial\mathcal{S}} \alpha, \tag{A.2}$$

for any p-form alpha. Whenever this is used, all quantities will be assumed to have compact support, meaning they vanish at the boundary of $\mathcal{S}$,thus

$$\int_{\mathcal{S}} d\alpha = 0, \tag{A.3}$$

for all α.

Appendix B

Dimensionless variables and restoring units

The process of restoration of dimensionful variables for results obtained from the action (3.48) will be presented. This action is used in section 4.3 and chapter 5. Firstly, the action written out with all the physical constants is

$$S = \frac{\varepsilon_0}{q^2 c^2} \int d\tilde{t} d\tilde{z} \frac{1}{2} \tilde{\Lambda} \left\{ ((\partial_{\tilde{t}} \tilde{\Phi})^2 - c^2 (\partial_{\tilde{z}} \tilde{\Phi})^2)((\partial_{\tilde{t}} \tilde{\psi})^2 - c^2 (\partial_{\tilde{z}} \tilde{\psi})^2 - m^2 c^4) \right\}, \quad \text{(B.1)}$$

where $\sim$ indicates quantities without any substitutions, ϵ_0 is the permittivity of free space, c is the speed of light and q is the elementary charge. The constant coefficient in (B.1) ensures that S has the correct physical dimension. Introducing

$$\tilde{t} = \frac{l_*}{c} t, \quad \tilde{z} = l_* z, \quad \tilde{\psi} = mcl_* \psi, \quad \text{(B.2)}$$

where l_* has units of length, yields

$$S = \frac{\varepsilon_0 m^2 c^3}{q^2} \int dt dz \frac{1}{2} \tilde{\Lambda} \Big\{ (d\Phi \cdot d\Phi)(d\psi \cdot d\psi + 1) \Big\}. \tag{B.3}$$

The dot product is taken with respect to the metric $g = -dt \otimes dt + dz \otimes dz$. If the spot size is not constant then two further substitutions are required:

$$\tilde{\Phi} = \sqrt{\frac{\hbar e^2}{\varepsilon_0 m^2 c^3 l_*^2}} \Phi, \quad \tilde{\Lambda} = \Lambda l_*^2, \tag{B.4}$$

resulting in

$$S = \hbar \int dt dz \frac{1}{2} \Lambda (d\Phi \cdot d\Phi)(d\psi \cdot d\psi + 1). \tag{B.5}$$

However if the spot size is constant,

$$\tilde{\Phi} = \sqrt{\frac{\hbar e^2}{\varepsilon_0 m^2 c^3 \tilde{\Lambda}}} \Phi \tag{B.6}$$

suffices, resulting in

$$S = \hbar \int dt dz \frac{1}{2} (d\Phi \cdot d\Phi)(d\psi \cdot d\psi + 1). \tag{B.7}$$

Identifying

$$(\partial_{\tilde{t}} \tilde{\Phi})^2 - c^2 (\partial_{\tilde{z}} \tilde{\Phi})^2 = \omega_p^2, \tag{B.8}$$

$$(\partial_{\tilde{t}} \tilde{\psi})^2 - c^2 (\partial_{\tilde{z}} \tilde{\psi})^2 - m^2 c^4 = m^2 c^4 a_0^2, \tag{B.9}$$

where ω_p is the plasma frequency and a_0 is the dimensionless amplitude, allows to express relevant observables with restored units as

$$\omega_p^2 = -\frac{K^2 c^2}{l_*^2} d\Phi \cdot d\Phi, \tag{B.10}$$

$$a_0^2 = -d\psi \cdot d\psi - 1, \tag{B.11}$$

where $\tilde{\Phi} = K\Phi$, with

$$K = \sqrt{\frac{\hbar e^2}{\varepsilon_0 m^2 c^3 \tilde{\Lambda}}}, \tag{B.12}$$

when $\tilde{\Lambda}$ is constant, and

$$K = \sqrt{\frac{\hbar e^2}{\varepsilon_0 m^2 c^3 l_*}}, \tag{B.13}$$

when $\tilde{\Lambda}$ is not constant.