# Universality for random permutations and some other groups 

Mohamed Slim Kammoun* ${ }^{*}$<br>Department of Mathematics and Statistics<br>Lancaster University<br>Lancaster, U.K.<br>m.kammoun@lancaster.ac.uk

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#### Abstract

We present some Markovian approaches to prove universality results for some functions on the symmetric group. Some of those statistics are already studied in [Kammoun, 2018, 2020] but not the general case. We prove, in particular, that the number of occurrences of a vincular patterns satisfies a CLT for conjugation invariant random permutations with few cycles and we improve the results already known for the longest increasing subsequence. The second approach is a suggestion of a generalization to other random permutations and other sets having a similar structure than the symmetric group.


## 1 The ping-pong method

Let $\mathfrak{S}_{n}$ be the group of permutations of $[n]$. For $\sigma \in \mathfrak{S}_{n}$, we denote by $\#(\sigma)$ the number of cycle of $\sigma$. Now let

$$
\mathfrak{S}_{n}^{0}:=\left\{\sigma \in \mathfrak{S}_{n}: \#(\sigma)=1\right\} .
$$

In this section, we are interested in proving universality for conjugation invariant random permutations with few cycles. A sequence of random permutations $\left(\sigma_{n}\right)_{n \geq 1}$ is said to be conjugation invariant if $\sigma_{n}$ is supported on $\mathfrak{S}_{n}$ and

$$
\begin{equation*}
\forall n \geq 1, \forall \sigma \in \mathfrak{S}_{n}, \sigma_{n} \stackrel{d}{=} \sigma^{-1} \sigma_{n} \sigma . \tag{inv}
\end{equation*}
$$

For $\alpha \geq 1$ and $p \in[1, \infty]$, we say that the sequences of random permutations $\left(\sigma_{n}\right)_{n \geq 1}$ satisfies $\mathcal{H}_{i n v, \alpha}^{\mathbb{P}}$ if

$$
\left(\sigma_{n}\right)_{n \geq 1} \text { is conjugation invariant and } \quad \frac{\#\left(\sigma_{n}\right)}{n^{\frac{1}{\alpha}}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \quad\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{P}}\right)
$$

and we say that it satisfies $\mathcal{H}_{i n v, \alpha}^{\mathbb{L}^{p}}$ if

$$
\left(\sigma_{n}\right)_{n \geq 1} \text { is conjugation invariant and } \frac{\#\left(\sigma_{n}\right)}{n^{\frac{1}{\alpha}}} \frac{\mathbb{L}^{p}}{n \rightarrow \infty} 0 . \quad\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{L}^{p}}\right)
$$

[^0]

Figure 1: The directed graph $\mathcal{G}_{\mathfrak{S}_{3}}$

### 1.1 Rebound on the Ewens zero distribution

Given $n \geq 1$ and $E \subset \mathfrak{S}_{n}$, we define

$$
\operatorname{next}(E):=\{\rho \circ(i, j) ; \rho \in E, \#(\rho \circ(i, j))=\#(\rho)-1\} \cup\{\rho \in E ; \#(\rho)=1\}
$$

and

$$
\operatorname{final}(\sigma):=\left\{\begin{array}{ll}
\operatorname{next}^{\#(\sigma)-1}(\{\sigma\}) & \text { if } \#(\sigma)>1 \\
\{\sigma\} & \text { otherwise }
\end{array} .\right.
$$

In other words, $\operatorname{next}(E)$ is the set of permutations obtained by concatenating, if possible, two cycles of some $\sigma \in E$, and final $(\sigma)$ is the set of permutations obtained by concatenating all the cycles of $\sigma$. In particular,

$$
\operatorname{final}(\sigma) \subset \mathfrak{S}_{n}^{0}:=\left\{\sigma \in \mathfrak{S}_{n} ; \#(\sigma)=1\right\}
$$

Let $\mathcal{G}_{\mathfrak{S}_{n}}$ be the directed graph with vertices $\mathfrak{S}_{n}$ and edges $\left\{(\sigma, \rho) ; \sigma \in \mathfrak{S}_{n}, \rho \in \operatorname{next}(\{\sigma\})\right\}$. We represent $\mathcal{G}_{\mathfrak{S}_{3}}$ in Figure 1. $\mathcal{G}_{\mathfrak{S}_{n}}$ can be seen as a directed version of the Cayley graph of $\mathfrak{S}_{n}$ generated by transpositions where the edges are oriented toward the permutations with fewer cycles (the further from the identity according to the graph distance), for which we added loops at the permutations of $\mathfrak{S}_{n}^{0}$. In this first part of this section, we will examine the uniform random walk on $\mathcal{G}_{\mathfrak{S}_{n}}$.

Let $f$ be a function defined on $\mathfrak{S}_{\infty}:=\cup_{i=1}^{\infty} \mathfrak{S}_{n}$ and taking its values in some metric space $\left(F, d_{F}\right)$, for example $\mathbb{Z}, \mathbb{R}, \mathbb{R}^{d}$ or $\mathscr{C}^{0}(\mathbb{R})$. It turns out that the uniform distribution on $\mathfrak{S}_{n}^{0}$, also known as the Ewens distribution ${ }^{1}$ with parameter 0 , is useful to obtain universality results for conjugation invariant permutations if $f$ does not change too much by merging two cycles. More precisely, we define for $1 \leq k \leq n$,

$$
\varepsilon_{n, k}^{\prime}(f):=\max _{\sigma \in \mathfrak{S}_{n}, \#(\sigma)=k} \max _{\rho \in \operatorname{final}(\sigma)} d_{F}(f(\sigma), f(\rho)) .
$$

We present now our main result.

[^1]Theorem 1. Assume that $\left(\sigma_{n}\right)_{n \geq 1}$ and $\left(\sigma_{r e f, n}\right)_{n \geq 1}$ satisfy $\left(\mathcal{H}_{\text {inv }}\right)$. Suppose that there exists $x \in F$ such that

$$
\begin{array}{r}
f\left(\sigma_{r e f, n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x, \\
\varepsilon_{n, \#\left(\sigma_{r e f, n}\right)}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \\
\text { and that } \quad \varepsilon_{n, \#\left(\sigma_{n}\right)}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 . \tag{3}
\end{array}
$$

Then

$$
\begin{equation*}
f\left(\sigma_{n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x \tag{4}
\end{equation*}
$$

Moreover, if the assumptions (1)-(3) hold true for the $\mathbb{L}^{p}$ convergence for some $p \geq 1$ instead of the convergence in probability, then so does (4).

When $F=\mathbb{R}^{d}$, we obtain also the convergence in distribution.
Theorem 2. Assume that $F=\mathbb{R}^{d}$ and that $\left(\sigma_{n}\right)_{n \geq 1}$ and $\left(\sigma_{r e f, n}\right)_{n \geq 1}$ satisfy $\left(\mathcal{H}_{\text {inv }}\right)$. Suppose that (2) and (3) hold true and that there exists a random variable $X$ supported on $F$ such that

$$
f\left(\sigma_{r e f, n}\right) \xrightarrow[n \rightarrow \infty]{d} X
$$

Then

$$
\begin{equation*}
f\left(\sigma_{n}\right) \xrightarrow[n \rightarrow \infty]{d} X \tag{5}
\end{equation*}
$$

Let $\sigma_{u n i f, n}$ and $\sigma_{E w, 0, n}$ be uniform random permutations respectively on $\mathfrak{S}_{n}$ and $\mathfrak{S}_{n}^{0}$. The idea of the proof is to compare both $f\left(\sigma_{n}\right)$ and $f\left(\sigma_{r e f, n}\right)$ with $f\left(\sigma_{E w, 0, n}\right)$. In general, the choice $\sigma_{r e f, n} \stackrel{d}{=} \sigma_{u n i f, n}$ is interesting since, the convergence in (1) is known for many statistics. Moreover, using Proposition 45, we have immediately the following result.

Corollary 3. If $\sigma_{r e f, n} \stackrel{d}{=} \sigma_{u n i f, n}$, in both theorems 1 and 2, the hypothesis (2) can be replaced by the existence of $\kappa>0$ such that

$$
\underset{\left|\frac{k}{\log (n)}-1\right|<\kappa}{\max } \quad \varepsilon_{n, k}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0
$$

We chose to give a very simple version that can be checked easily for many statistics. For almost sure convergence, one can obtain similar results after defining properly the spaces. We will not discuss here this type of convergence. We will give many applications using the following observation.

Remark 4. By the triangle inequality, we have

$$
\varepsilon_{n, k}^{\prime}(f) \leq \sum_{i=2}^{k} \varepsilon_{n, i}(f) \leq(k-1) \varepsilon_{n}(f)
$$

where

$$
\varepsilon_{n, k}(f):=\max _{\sigma \in \mathfrak{S}_{n}, \#(\sigma)=k} \max _{\rho \in \operatorname{next}(\{\sigma\})} d_{F}(f(\sigma), f(\rho)) \quad \text { and } \quad \varepsilon_{n}(f):=\max _{1 \leq k<n} \varepsilon_{k, n}(f) .
$$

Consequently, if there exists some $\alpha \leq 1$ such that

$$
\varepsilon_{n}(f)=O\left(\frac{1}{n^{\frac{1}{\alpha}}}\right)
$$

then $\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{P}}\right)$ implies (3) and $\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{L}^{p}}\right)$ implies the equivalent hypothesis in $\mathbb{L}^{p}$. Moreover, if $\sigma_{r e f, n} \stackrel{d}{=} \sigma_{u n i f, n}$, then Proposition 48 implies (2). We will give some direct applications of this observation in the next subsection.

### 1.2 Some applications

In the next corollary, we will give some applications. The first column of Table 1 contains the function to study. We apologize to the reader because those statistics are not defined yet. One can check the corresponding result in the fifth column for more details.

Corollary 5. For the functions $f$ the distribution $X$ and the real $\alpha$ in Table 1, if $\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{P}}\right)$ is satisfied, then

$$
f\left(\sigma_{n}\right) \xrightarrow[n \rightarrow \infty]{d} X
$$

except for the sixth example where the convergence holds in probability. ${ }^{2}$ For the first and the forth examples the convergence holds also in $\mathbb{L}^{p}$ under $\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{L}^{p}}\right)$. For the fifth example please check the corresponding theorem for more details about the type of convergence.

Note that:

- We give in the third column the inequality we used to obtain our results. Except for the cases where we study the RSK image of the permutation, the longest alternating subsequence and the descent process, the inequality is trivial, but we will prove all the inequalities in the sequel.
- We want to emphasize that these results are just a direct application of theorems 1 and 2. Using more sophisticated controls of the error, one could obtain larger classes of universality as we will detail in the sequel.
- For all our examples, the special case of the Ewens distribution satisfies the hypothesis.


### 1.3 Proof of theorems 1 and 2

Let $\rho_{n}$ be a conjugation invariant random permutation. To prove theorems 1 and 2 , the idea is to modify $\rho_{n}$ to obtain a conjugation invariant random permutation supported on $\mathfrak{S}_{n}^{0}$. We define the following Markov operator $T$ associated to the uniform random walk over $\mathcal{G}_{\mathfrak{S}_{n}}$. Another way to see it is the following: ${ }^{3}$

[^2]| $f(\sigma)$ | X | Error | Hypotheses | Theorem |
| :---: | :---: | :---: | :---: | :---: |
| $\frac{\operatorname{LIS}(\sigma)}{\sqrt{n}}, \frac{\operatorname{LDS}(\sigma)}{\sqrt{n}}$ | 2 | $\varepsilon_{n} \leq \frac{2}{\sqrt{n}}$ | $\begin{aligned} & \left(\mathcal{H}_{i n v, 2}^{p^{p}}\right) \\ & \left(\mathcal{H}_{i n v, 2}^{L^{p}}\right) \\ & \hline \end{aligned}$ | Theorem 8 |
| $\frac{\operatorname{LISC}(\sigma)}{\sqrt{n}}, \frac{\operatorname{LDSC}(\sigma)}{\sqrt{n}}$ | 2 | $\varepsilon_{n} \leq \frac{2}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Corollary 13 |
| $\begin{aligned} & \frac{\operatorname{LIS}(\sigma)-2 \sqrt{n}}{n^{\frac{1}{6}}}, \\ & \frac{\operatorname{LDS}(\sigma)-2 \sqrt{n}}{n^{\frac{1}{6}}} \end{aligned}$ | Tracy-Widom | $\varepsilon_{n} \leq \frac{2}{n^{\frac{1}{6}}}$ | $\left(\mathcal{H}_{i n v, 6}^{\mathbb{P}}\right)$ | Corollary 10 |
| $\frac{\lambda_{i}(\sigma)}{\sqrt{n}}$ | 2 | $\varepsilon_{n} \leq \frac{4}{\sqrt{n}}$ | $\begin{aligned} & \left(\mathcal{H}_{i n v, 2}^{\mathbb{P}_{2}}\right. \\ & \left(\mathcal{H}_{i n v, 2}^{L^{p}}\right) \end{aligned}$ | Proposition 17 |
| $\left(\frac{\lambda_{i}(\sigma)-2 \sqrt{n}}{n^{\frac{1}{6}}}\right)_{1<i<d}$ | Airy ensemble | $\varepsilon_{n} \leq \frac{4}{n^{\frac{1}{6}}}$ | $\left(\mathcal{H}_{\text {inv,6 }}^{\mathbb{P}}\right)$ | Theorem 15 |
| $s \rightarrow \frac{L_{\lambda(\sigma)}(s \sqrt{2 n})}{\sqrt{2 n}}$ | $\Omega$ | $\varepsilon_{n, k}^{\prime} \leq \frac{2 \sqrt{k-1}}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ | Theorem 18 |
| $\frac{\mathcal{K}_{j}(\sigma)}{n^{j}}$ | $\frac{1}{j!^{2}}$ | $\varepsilon_{n} \leq \frac{2 j}{n}$ | $\left(\mathcal{H}_{\text {inv, }}{ }^{\mathbb{P}}\right)$ | Corollary 29 |
| $\frac{\mathcal{K}_{j}(\sigma)-\frac{n^{j}}{(j!)^{2}}}{\sqrt{n}}$ | $\mathcal{N}\left(0, \frac{\binom{4 j-2}{2 j-1}-2\left(\begin{array}{l}\text { 2j-1 }\end{array}\right)^{2}}{2((2 m-1)!)^{2}}\right)$ | $\varepsilon_{n} \leq \frac{2 j}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Corollary 29 |
| $\underline{\mathcal{N}_{\text {exc }}(\sigma)}$ | $\frac{1}{2}$ | $\varepsilon_{n} \leq \frac{4}{n}$ | $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ | Corollary 29 |
| $\frac{\mathcal{N e x c}_{e x c}(\sigma)-\frac{n}{2}}{\sqrt{n}}$ | $\mathcal{N}\left(0, \frac{1}{12}\right)$ | $\varepsilon_{n} \leq \frac{4}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Corollary 29 |
| $\mathbb{1}_{D(\sigma) \subset A}$ | $\operatorname{Ber}\left(\operatorname{det}\left(\left[k_{0}(j-i)\right]_{A}\right)\right)$ | Proposition 32 | $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ | Corollary 34 |
| $\frac{\mathcal{N}_{(\tau, X)}(\sigma)-\frac{n^{p-q}}{p!(p-q)!}}{n^{p-q+\frac{1}{2}}}$ | $\mathcal{N}\left(0, V_{\tau, X}\right)$ | $\varepsilon_{n} \leq \frac{C}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Proposition 31 |

The results below are fully understood in the conjugation invariant case.

| $\frac{\mathcal{N}_{D}(\sigma)}{n}$ | $\frac{1}{2}$ | $\varepsilon_{n} \leq \frac{4}{n}$ | $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ | Corollary 29 |
| :--- | :--- | :--- | :--- | :--- |
| $\frac{\mathcal{N}_{D}(\sigma)-\frac{n}{2}}{\sqrt{n}}$ | $\mathcal{N}\left(0, \frac{1}{12}\right)$ | $\varepsilon_{n} \leq \frac{4}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Corollary 29 |
| $\frac{\mathcal{N}_{\text {peak }(\sigma)}}{n}$ | $\frac{1}{3}$ | $\varepsilon_{n} \leq \frac{6}{n}$ | $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ | Corollary 29 |
| $\frac{\mathcal{N}_{\text {peak } k}(\sigma)-\frac{n}{2}}{\sqrt{n}}$ | $\mathcal{N}\left(0, \frac{2}{45}\right)$ | $\varepsilon_{n} \leq \frac{6}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Corollary 29 |
| $\frac{\operatorname{LAS}(\sigma)}{n}$ | $\frac{2}{3}$ | $\varepsilon_{n} \leq \frac{6}{n}$ | $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ | Corollary 25 |
| $\frac{\operatorname{LAS}(\sigma)-\frac{2 n}{3}}{\sqrt{n}}$ | $\mathcal{N}\left(0, \frac{8}{45}\right)$ | $\varepsilon_{n} \leq \frac{6}{\sqrt{n}}$ | $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ | Corollary 25 |

Table 1: Some examples


Figure 2: The transition probabilities of $T$ for $n=3$

- If the realization $\sigma$ of $\rho_{n}$ has one cycle, $\sigma$ remains unchanged $(T(\sigma)=\sigma)$.
- Otherwise, we choose a couple $(i, j)$ uniformly from the nonempty set

$$
\left\{(i, j): j \notin \mathcal{C}_{i}(\sigma)\right\}
$$

and we take $T(\sigma)=\sigma \circ(i, j)$. Here $\mathcal{C}_{i}(\sigma)$ is the cycle of $\sigma$ containing $i$.
For example, for $n=3$, transition probabilities of $T$ are given in Figure 2.
We denote by $T^{k}\left(\rho_{n}\right)$ the random permutation obtained after applying $k$ times the operator $T$. It is the random permutation obtained after $k$ steps of the uniform random walk on $\mathcal{G}_{\mathfrak{S}_{n}}$ with initial state $\rho_{n}$. Table 2 sums up the evolution of the random walk if we start from the uniform distribution on $\mathfrak{S}_{3}$. Remark that the condition $j \notin \mathcal{C}_{i}(\sigma)$ guarantees that $\#(\sigma \circ(i, j))=\#(\sigma)-1$ since the cycles containing $i$ and $j$ are merged and the remaining of cycles are the same for $\sigma$ and $\sigma \circ(i, j)$.

In particular,

$$
\begin{equation*}
\#\left(T^{i}\left(\rho_{n}\right)\right) \stackrel{\text { a.s }}{=} \max \left(\#\left(\rho_{n}\right)-i, 1\right) \tag{6}
\end{equation*}
$$

The invariant measure of this walk (for conjugation invariant permutations) is trivial.
Lemma 6. If $\rho_{n}$ is a conjugation invariant random permutation of $\mathfrak{S}_{n}$ then the law of $T^{n-1}\left(\rho_{n}\right)^{4}$ is the uniform distribution on $\mathfrak{S}_{n}^{0}$ i.e.

$$
T^{n-1}\left(\rho_{n}\right) \stackrel{d}{=} \sigma_{E w, 0, n}
$$

[^3]|  | $\sigma_{u n i f, 3}$ | $T\left(\sigma_{u n i f, 3}\right)$ | $T^{2}\left(\sigma_{u n i f, 3}\right)$ |
| :--- | :--- | :--- | :--- |
| Id | $1 / 6$ | 0 | 0 |
| $(1,2)$ | $1 / 6$ | $1 / 18$ | 0 |
| $(1,3)$ | $1 / 6$ | $1 / 18$ | 0 |
| $(2,3)$ | $1 / 6$ | $1 / 18$ | 0 |
| $(1,2,3)$ | $1 / 6$ | $5 / 12$ | $1 / 2$ |
| $(1,3,2)$ | $1 / 6$ | $5 / 12$ | $1 / 2$ |

Table 2: Transitions for the $\sigma_{u n i f, 3}$

Proof. First, by construction, if $\rho_{n}$ is conjugation invariant then $T\left(\rho_{n}\right)$ is also conjugation invariant. Indeed, one can see that $T\left(\rho_{n}\right)$ is conjugation invariant since the construction depends only on the cycle structure of $\rho_{n}$ and all the integers between 1 and $n$ play a symmetric role. By iteration, $T^{n-1}\left(\rho_{n}\right)$ is conjugation invariant. Moreover, using (6),

$$
\begin{equation*}
\#\left(T^{n-1}\left(\sigma_{n}\right)\right) \stackrel{a . s}{=} 1 \tag{7}
\end{equation*}
$$

Knowing that all the elements of $\mathfrak{S}_{n}^{0}$ belong to the same conjugacy class, they are equally distributed and Lemme 6 follows from (7).

We now prove theorems 1 and 2 .
Proof of theorems 1 and 2. Equality (6) implies that

$$
T^{n-1}\left(\rho_{n}\right) \stackrel{\text { a.s }}{=} T^{\#\left(\rho_{n}\right)-1}\left(\rho_{n}\right)
$$

Therefore, almost surely,

$$
d_{F}\left(f\left(T^{n-1}\left(\rho_{n}\right)\right), f\left(\rho_{n}\right)\right)=d_{F}\left(f\left(T^{\#\left(\rho_{n}\right)-1}\left(\rho_{n}\right)\right), f\left(\rho_{n}\right)\right) \leq \varepsilon_{n, \#\left(\rho_{n}\right)}^{\prime}
$$

Thus, if $\varepsilon_{n, \#\left(\rho_{n}\right)}^{\prime} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$, then for any $\varepsilon>0$,

$$
\begin{equation*}
\mathbb{P}\left(d_{F}\left(f\left(T^{n-1}\left(\rho_{n}\right)\right), f\left(\rho_{n}\right)\right)>\varepsilon\right) \xrightarrow[n \rightarrow \infty]{ } 0 \tag{8}
\end{equation*}
$$

According to Lemma 6, $T^{n-1}\left(\rho_{n}\right)$ does not depend on the law of $\rho_{n}$. By choosing at first $\rho_{n}=\sigma_{r e f, n},(2)$ then yields

$$
f\left(\sigma_{E w, 0, n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x
$$

By choosing at a second step $\rho_{n}=\sigma_{n}$, we obtain (4) for any $\sigma_{n}$ satisfying the hypothesis of Theorem 1. One can prove Theorem 2 using the same argument.

## 2 Proof of Corollary 5

### 2.1 First application: Longest Increasing Subsequence

Given $\sigma \in \mathfrak{S}_{n}$, a subsequence $\left(\sigma\left(i_{1}\right), \ldots, \sigma\left(i_{k}\right)\right)$ is an increasing (resp. decreasing) subsequence of $\sigma$ of length $k$ if $i_{1}<\cdots<i_{k}$ and $\sigma\left(i_{1}\right)<\cdots<\sigma\left(i_{k}\right)$ (resp. $\left.\sigma\left(i_{1}\right)>\cdots>\sigma\left(i_{k}\right)\right)$. We denote by LIS $(\sigma)$ (resp.
$\operatorname{LDS}(\sigma))$ the length of the longest increasing (resp. decreasing) subsequence of $\sigma^{5}$. For example,

$$
\text { if } \quad \sigma=\left(\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
5 & 3 & 2 & 1 & 4
\end{array}\right), \operatorname{LIS}(\sigma)=2 \text { and } \operatorname{LDS}(\sigma)=4
$$

The study of the limiting behavior of $\operatorname{LIS}\left(\sigma_{u n i f, n}\right)$, where $\sigma_{u n i f, n}$ is a uniform random permutation on $\mathfrak{S}_{n}$, is known as the Ulam's problem (or the Ulam-Hammersley problem): Ulam [1961] conjectured that the limit as $n$ goes to infinity of

$$
\frac{\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{u n i f, n}\right)\right)}{\sqrt{n}}
$$

exists. Using a subadditivity argument, Hammersley [1972] proved this conjecture. He also proved that this convergence holds in probability. Vershik and Kerov [1977] and Logan and Shepp [1977] proved that this limit is equal to 2. An alternative proof is given by Aldous and Diaconis [1995]. The asymptotic fluctuations were studied by Baik, Deift and Johansson. They proved the following result:

Theorem 7. [Baik, Deift, and Johansson, 1999] For all $s \in \mathbb{R}$,

$$
\mathbb{P}\left(\frac{\operatorname{LIS}\left(\sigma_{u n i f, n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \underset{n \rightarrow \infty}{\longrightarrow} F_{2}(s),
$$

where $F_{2}$ is the cumulative distribution function (CDF) of the GUE Tracy-Widom distribution.
For historical details, full proofs and applications, we strongly recommend [Romik, 2015]. Apart from the uniform case, Mueller and Starr [2013] studied the longest increasing subsequence for Mallows' distribution. The case of random involutions is studied by Baik and Rains [2001] who showed that the limiting distribution depends on the number of fixed points and in some regimes, the GOE/GSE TracyWidom distributions appear. They also showed the appearance of a family of probability distributions that interpolate between the GOE and the GSE Tracy-Widom distribution. Mueller and Starr showed that for Mallows' distribution, there is a phase transition between the Gaussian and the Tracy-Widom regimes. In this section, we prove universality results for conjugation invariant random permutations.
Theorem 8. Under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$,

$$
\frac{\operatorname{LIS}\left(\sigma_{n}\right)}{\sqrt{n}} \underset{n \rightarrow \infty}{\mathbb{P}} 2 \text { and } \quad \frac{\operatorname{LDS}\left(\sigma_{n}\right)}{\sqrt{n}} \underset{n \rightarrow \infty}{\mathbb{P}} 2
$$

Moreover, for any $p \in[1, \infty)$, under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{L}^{p}}\right)$,

$$
\frac{\operatorname{LIS}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 2 \quad \text { and } \quad \frac{\operatorname{LDS}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 2
$$

The convergence in probability is stated without proof in [Kammoun, 2018] as it is similar to the proof of [Kammoun, 2018, Theorem 1.2]. For the fluctuations, we have the following result.
Theorem 9. Assume that $\left(\sigma_{n}\right)_{n \geq 1}$ is conjugation invariant and

$$
\begin{equation*}
\frac{1}{n^{\frac{1}{6}}} \min _{1 \leq i \leq n}\left(\left(\sum_{j=1}^{i} \#_{j}\left(\sigma_{n}\right)\right)+\frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} \#_{j}\left(\sigma_{n}\right)\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 . \tag{9}
\end{equation*}
$$

[^4]Then for all $s \in \mathbb{R}$,

$$
\begin{equation*}
\mathbb{P}\left(\frac{\operatorname{LIS}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \underset{n \rightarrow \infty}{\longrightarrow} F_{2}(s) \text { and } \mathbb{P}\left(\frac{\operatorname{LDS}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \underset{n \rightarrow \infty}{\longrightarrow} F_{2}(s) . \tag{TW}
\end{equation*}
$$

Here, $\#_{j}(\sigma)$ is the number of cycles of $\sigma$ of length $j$.
The idea of the proof we give in Subsection 3.3 is to construct a coupling between any distribution satisfying these hypothesises and the uniform distribution in order to use Theorem 7 to obtain first the lower bound then the upper bound. This theorem generalizes the following result.
Corollary 10. [Kammoun, 2018, Theorem 1.2] If $\left(\mathcal{H}_{i n v, 6}^{\mathbb{P}}\right)$ is satisfied then (TW) holds.
The key argument of our proofs is the following lemma:
Lemma 11. For any permutation $\sigma$ and for any transposition $\tau$,

$$
|\operatorname{LIS}(\sigma \circ \tau)-\operatorname{LIS}(\sigma)| \leq 2, \quad|\operatorname{LDS}(\sigma \circ \tau)-\operatorname{LDS}(\sigma)| \leq 2
$$

Proof. Let $\sigma$ be a permutation. By definition of $\operatorname{LIS}(\sigma)$, there exists $i_{1}<i_{2}<\cdots<i_{\operatorname{LIS}(\sigma)}$ such that $\sigma\left(i_{1}\right)<\cdots<\sigma\left(i_{\operatorname{LIS}(\sigma)}\right)$. Let $\tau=(j, k)$ be a transposition and $i_{1}^{\prime}, i_{2}^{\prime}, \ldots, i_{m}^{\prime}$ be the same sequence as $i_{1}, i_{2}, \ldots, i_{\operatorname{LIS}(\sigma)}$ after removing $j$ and $k$ if needed. We have $\sigma\left(i_{1}^{\prime}\right)<\cdots<\sigma\left(i_{m}^{\prime}\right)$. In particular, LIS $(\sigma)-2 \leq$ $m \leq \operatorname{LIS}(\sigma)$. Knowing that $\forall i \notin\{j, k\}, \sigma \circ \tau(i)=\sigma(i)$, then

$$
\sigma \circ \tau\left(i_{1}^{\prime}\right)<\cdots<\sigma \circ \tau\left(i_{m}^{\prime}\right) .
$$

Therefore,

$$
\begin{equation*}
\operatorname{LIS}(\sigma)-\operatorname{LIS}(\sigma \circ \tau) \leq 2 \tag{10}
\end{equation*}
$$

We obtain the second inequality by replacing $\sigma$ by $\sigma \circ \tau$ in (10). For $\operatorname{LDS}(\sigma)$ the proof is similar.
Proof of Theorem 8 and Corollary 10. The main functions we want to study are

$$
f_{\mathrm{LIS} 1}(\sigma):=\frac{\operatorname{LIS}(\sigma)}{\sqrt{n}} \text { and } f_{\mathrm{LIS} 2}(\sigma):=\frac{\operatorname{LIS}(\sigma)-2 \sqrt{n}}{n^{\frac{1}{6}}}
$$

Using Lemma 11 , we have for all $n \geq 3$,

$$
\varepsilon_{n}\left(f_{\mathrm{LIS} 1}\right)=\frac{2}{\sqrt{n}} \text { and } \varepsilon_{n}\left(f_{\mathrm{LIS} 2}\right)=\frac{2}{n^{\frac{1}{6}}},
$$

and one can conclude using theorems 1 and 2 with $\sigma_{r e f, n}=\sigma_{u n i f, n}$ since the uniform case is already studied. Indeed, one can see [Vershik and Kerov, 1977, Logan and Shepp, 1977] for the convergence of $f_{\text {LIS1 }}$ in probability, [Baik, Deift, and Suidan, 2016] for the convergence in $\mathbb{L}^{p}$ of $f_{\text {LIS1 }}$ and [Baik et al., 1999] for the convergence of $f_{\text {LIS2 }}$ in probability. For the $\operatorname{LDS}(\sigma)$, the proof is similar.

A similar application is the length of the longest increasing (resp. decreasing) circular subsequence.
Definition 12. Given $\sigma \in \mathfrak{S}_{\infty}$, a subsequence is said to be increasing (resp. decreasing) circular if it is increasing (resp. decreasing) up to a circular permutation. One can see [Albert et al., 2007] for rigorous definition and more details.

We denote by $\operatorname{LICS}(\sigma)$ (resp. $\operatorname{LDCS}(\sigma))$ the length of the longest increasing (resp. decreasing) circular subsequence.

Corollary 13. If $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$ is satisfied then

$$
\frac{\operatorname{LICS}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 \text { and } \quad \frac{\operatorname{LDCS}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2
$$

Proof. The uniform case is proved in [Albert et al., 2007, Theorem 1] for the LICS and the case of the LDCS can be obtained by composition by the permutation $i \mapsto n-i+1$. Moreover, using the same argument as for the LIS in Lemma 11, we have

$$
|\operatorname{LICS}(\sigma \circ \tau)-\operatorname{LICS}(\sigma)| \leq 2 \quad \text { and } \quad|\operatorname{LDCS}(\sigma \circ \tau)-\operatorname{LDCS}(\sigma)| \leq 2
$$

which concludes the proof using Theorem 1.
We will give now a generalization for the universality for the LCS. Given $\sigma \in \mathfrak{S}_{n}$, let $\left(\lambda_{i}(\sigma)\right)_{i \geq 1}$ and $\left(\lambda_{i}^{\prime}(\sigma)\right)_{i \geq 1}$ be respectively the shape of image of $\sigma$ by the RSK correspondence and its transpose. One way to define it is the following. Let

$$
\begin{aligned}
\mathfrak{I}_{1}(\sigma) & :=\{s \subset\{1,2, \ldots, n\} ; \forall i, j \in s,(i-j)(\sigma(i)-\sigma(j)) \geq 0\}, \\
\mathfrak{D}_{1}(\sigma) & :=\{s \subset\{1,2, \ldots, n\} ; \forall i, j \in s,(i-j)(\sigma(i)-\sigma(j)) \leq 0\}, \\
\mathfrak{I}_{k+1}(\sigma) & :=\left\{s \cup s^{\prime}, s \in \mathfrak{I}_{k}, s^{\prime} \in \mathfrak{I}_{1}\right\}, \\
\mathfrak{D}_{k+1}(\sigma) & :=\left\{s \cup s^{\prime}, s \in \mathfrak{D}_{k}, s^{\prime} \in \mathfrak{D}_{1}\right\} .
\end{aligned}
$$

For example, for

$$
\sigma_{e x, 3}:=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right), \quad \mathfrak{I}_{1}\left(\sigma_{e x, 3}\right)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\}\}
$$

and

$$
\mathfrak{I}_{2}\left(\sigma_{e x, 3}\right)=\mathfrak{D}_{2}\left(\sigma_{e x, 3}\right)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\} .
$$

The RSK image is defined as follows. For any permutation $\sigma \in \mathfrak{S}_{n}$,

$$
\begin{equation*}
\max _{s \in \mathfrak{I}_{i}(\sigma)} \operatorname{card}(s)=\sum_{k=1}^{i} \lambda_{k}(\sigma), \quad \max _{s \in \mathfrak{D}_{i}(\sigma)} \operatorname{card}(s)=\sum_{k=1}^{i} \lambda_{k}^{\prime}(\sigma) . \tag{11}
\end{equation*}
$$

In particular,

$$
\max _{s \in \mathfrak{I}_{1}(\sigma)} \operatorname{card}(s)=\lambda_{1}(\sigma)=\operatorname{LIS}(\sigma), \quad \max _{s \in \mathfrak{D}_{1}(\sigma)} \operatorname{card}(s)=\lambda_{1}^{\prime}(\sigma)=\operatorname{LDS}(\sigma)
$$

We strongly recommend [Sagan, 2001] equivalent constructions.
A more general version of the result of Theorem 7 is the following.
Theorem 14. [Borodin et al., 2000, Theorem 5][Johansson, 2001, Theorem 1.4] For all real numbers $s_{1}, s_{2}, \ldots, s_{k}$,

$$
\lim _{n \rightarrow \infty} \mathbb{P}\left(\forall i \leq k, \frac{\lambda_{i}\left(\sigma_{u n i f, n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s_{i}\right)=F_{2, k}\left(s_{1}, s_{2}, \ldots, s_{k}\right)
$$

For the permutations satisfying the same assumptions as in Theorem 9, we have the same asymptotic as in the uniform setting at the edge.

Theorem 15. Assume that $\left(\sigma_{n}\right)_{n \geq 1}$ is conjugation invariant and

$$
\begin{equation*}
\frac{1}{n^{\frac{1}{6}}} \min _{1 \leq i \leq n}\left(\left(\sum_{j=1}^{i} \#_{j}\left(\sigma_{n}\right)\right)+\frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} \#_{j}\left(\sigma_{n}\right)\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \tag{12}
\end{equation*}
$$

Then for all positive integer $k$, for all real numbers $s_{1}, s_{2}, \ldots, s_{k}$,

$$
\begin{align*}
\lim _{n \rightarrow \infty} \mathbb{P}\left(\forall i \leq k, \frac{\lambda_{i}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s_{i}\right) & =\lim _{n \rightarrow \infty} \mathbb{P}\left(\forall i \leq k, \frac{\lambda_{i}^{\prime}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s_{i}\right) \\
& =F_{2, k}\left(s_{1}, s_{2}, \ldots, s_{k}\right) . \tag{Ai}
\end{align*}
$$

Before proving this result, we recall first an already known weaker version.
Proposition 16. [Kammoun, 2018] If $\left(\mathcal{H}_{i n v, 6}^{\mathbb{P}}\right)$ is satisfied then (Ai) holds true.
Under weaker assumptions, one can still prove the first order convergence.
Proposition 17. If ( $\mathcal{H}_{i n v, 2}^{\mathbb{P}}$ ) is satisfied then for any $i \geq 1$

$$
\frac{\lambda_{i}\left(\sigma_{n}\right)}{\sqrt{n}} \underset{n \rightarrow \infty}{\mathbb{P}} 2 \quad \text { and } \quad \frac{\lambda_{i}^{\prime}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 .
$$

Moreover, for any $p \in[1, \infty)$, under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{L}^{p}}\right)$,

$$
\frac{\lambda_{i}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 2 \quad \text { and } \quad \frac{\lambda_{i}^{\prime}\left(\sigma_{n}\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 2 .
$$

Corollary 10 (resp. Theorem 8) is a direct application of Proposition 16 (resp. Proposition 17) for $k=1$ (resp. $i=1$ ). We will prove first in the next subsection propositions 16 and 17 as they are direct applications of Theorem 1.
The typical shape of $\left(\lambda_{i}\left(\sigma_{u n i f, n}\right)\right)_{i \geq 1}$ seen as young diagram was studied separately by Logan and Shepp [1977] and Vershik and Kerov [1977]. Stronger results are proved by Vershik and Kerov [1985]. In 1993, Kerov studied the limiting fluctuations but did not publish his results. See [Ivanov and Olshanski, 2002] for further details. Let $L_{\lambda(\sigma)}$ be the height function of $\lambda(\sigma)=\left(\lambda_{i}(\sigma)\right)_{i \geq 1}$ rotated by $\frac{5 \pi}{4}$ and extended by the function $x \mapsto|x|$ to obtain a function defined on $\mathbb{R}$. For example, if $\lambda(\sigma)=(7,5,2,1,1, \underline{0})$ the associated function $L_{\lambda(\sigma)}$ is represented by Figure 3. A direct application of Theorem 1 is the following.

Theorem 18. [Kammoun, 2018] Under $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$,

$$
\begin{equation*}
\sup _{s \in \mathbb{R}}\left|\frac{1}{\sqrt{2 n}} L_{\lambda\left(\sigma_{n}\right)}(s \sqrt{2 n})-\Omega(s)\right| \underset{n \rightarrow \infty}{\mathbb{P}} 0 \tag{VKLS}
\end{equation*}
$$

where,

$$
\Omega(s):= \begin{cases}\frac{2}{\pi}\left(s \arcsin (s)+\sqrt{1-s^{2}}\right) & \text { if }|s|<1 \\ |s| & \text { if }|s| \geq 1\end{cases}
$$



Figure 3: $\quad L_{(7,5,2,1,1, \mathbf{0})}$

Proof. We wan to apply Theorem 1. Let now $F$ be the set of continual diagrams i.e. the set of 1-Lipschitz real functions $g$ from $\mathbb{R}$ to $\mathbb{R}_{+}$such that $\exists a, b>0$ s.t. $\forall x \notin[-b, b], g(x)=|x-a|$. For $g, h \in F$, we denote by $d_{F}(g, h)=\sup _{\mathbb{R}}|h-g|$. For $\sigma \in \mathfrak{S}_{n}, f(\sigma)$ is the function $s \rightarrow \frac{L_{\lambda(\sigma)}(s \sqrt{2 n})}{\sqrt{2 n}}$. So that $f$ is a function from $\mathfrak{S}_{\infty}$ taking values in the metric space $\left(F, d_{F}\right)$. If we choose $\sigma_{r e f, n}=\sigma_{u n i f, n}$ and $x$ to be the function $\Omega$, the convergence

$$
f\left(\sigma_{r e f, n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x
$$

is proven by Logan and Shepp [1977] and Vershik and Kerov [1977]. Using [Kammoun, 2018, Lemma 3.7.], for any $1 \leq k \leq n$,

$$
\begin{equation*}
\varepsilon_{n, k}^{\prime}(f) \leq 2 \sqrt{\frac{k-1}{n}} \tag{13}
\end{equation*}
$$

So that Theorem 1 gives the conclusion.

### 2.2 Proof of propositions 16 and 17

Lemma 19. For any permutation $\sigma$ and any transposition $\tau$,

$$
\begin{equation*}
\left|\sum_{k=1}^{i} \lambda_{k}(\sigma)-\lambda_{k}(\sigma \circ \tau)\right| \leq 2 \quad \text { and } \quad\left|\sum_{k=1}^{i} \lambda_{k}^{\prime}(\sigma)-\lambda_{k}^{\prime}(\sigma \circ \tau)\right| \leq 2 . \tag{14}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\left|\lambda_{i}(\sigma)-\lambda_{i}(\sigma \circ \tau)\right| \leq 4 \quad \text { and } \quad\left|\lambda_{i}^{\prime}(\sigma)-\lambda_{i}^{\prime}(\sigma \circ \tau)\right| \leq 4 \tag{15}
\end{equation*}
$$

Proof. Let $\sigma$ be a permutation and $\tau=(l, m)$ be a transposition. We have then for all integer $i$,

$$
\left\{s \backslash\{l, m\}, s \in \mathfrak{I}_{i}(\sigma)\right\} \subset \mathfrak{I}_{i}(\sigma \circ \tau)
$$

and similarly

$$
\left\{s \backslash\{l, m\}, s \in \mathfrak{D}_{i}(\sigma)\right\} \subset \mathfrak{D}_{i}(\sigma \circ \tau) .
$$

Consequently, using (11),

$$
\sum_{k=1}^{i} \lambda_{k}(\sigma)-\lambda_{k}(\sigma \circ \tau) \geq-2, \quad \sum_{k=1}^{i} \lambda_{k}^{\prime}(\sigma)-\lambda_{k}^{\prime}(\sigma \circ \tau) \geq-2
$$

Using the same argument with $\sigma \circ \tau$ instead of $\sigma$, (14) follows. Moreover, since

$$
\lambda_{i+1}=\sum_{k=1}^{i+1} \lambda_{k}-\sum_{k=1}^{i} \lambda_{k}, \quad \lambda_{i+1}^{\prime}=\sum_{k=1}^{i+1} \lambda_{k}^{\prime}-\sum_{k=1}^{i} \lambda_{k}^{\prime},
$$

the triangle inequality yields (15).
Using (15), Propositions 16 and 17 are direct applications of Theorem 1.

### 2.3 Second application: Longest Alternating Subsequence

A more tricky application is the length of the Longest Alternating Subsequence. This is a special case of a large class of statistics we will present in the next subsection.
Definition 20. Given $\sigma \in \mathfrak{S}_{n},\left(\sigma\left(i_{1}\right), \sigma\left(i_{2}\right), \ldots, \sigma_{n}\left(i_{k}\right)\right)$ is said to be an alternating subsequence of $\sigma$ of length $k$ if $i_{1}<i_{2}<\cdots<i_{k}$ and $\sigma\left(i_{1}\right)>\sigma\left(i_{2}\right)<\sigma\left(i_{3}\right)>\ldots \sigma\left(i_{k}\right)$. We denote by LAS $(\sigma)$ the length of the longest alternating subsequence of $\sigma$.

The uniform case is already studied in [Stanley, 2010, Romik, 2011]. We have the two following results.
Proposition 21. [Stanley, 2010, Page 17] For $n \geq 2$,

$$
\mathbb{E}\left(\operatorname{LAS}\left(\sigma_{u n i f, n}\right)\right)=\frac{2 n}{3}+\frac{1}{6}
$$

and for $n \geq 4$,

$$
\operatorname{Var}\left(\operatorname{LAS}\left(\sigma_{u n i f, n}\right)\right)=\frac{8 n}{45}-\frac{13}{180}
$$

Proposition 22. [Romik, 2011, Proposition 4]

$$
\frac{\operatorname{LAS}\left(\sigma_{u n i f, n}\right)-\frac{2}{3} n}{\sqrt{n}} \underset{n \rightarrow \infty}{d} \mathcal{N}\left(0, \frac{8}{45}\right) .
$$

Here, $\mathcal{N}\left(m, \sigma^{2}\right)$ is the normal distribution. We also make use of the following result.
Proposition 23. [Romik, 2011, Corollary 2]

$$
\operatorname{LAS}(\sigma)=1+\sum_{i=1}^{n-1} M_{k}(\sigma)
$$

where

$$
M_{1}(\sigma)=\mathbb{1}_{\sigma(1)>\sigma(2)}
$$

and for $1<k<n$,

$$
M_{k}(\sigma)=\mathbb{1}_{\sigma(k-1)>\sigma(k)<\sigma(k+1)}+\mathbb{1}_{\sigma(k-1)<\sigma(k)>\sigma(k+1)} .
$$

This yields the following.
Lemma 24. For any $\sigma \in \mathfrak{S}_{n}$ and $1 \leq i, j \leq n$,

$$
|\operatorname{LAS}(\sigma)-\operatorname{LAS}(\sigma \circ(i, j))| \leq 6
$$

Proof. Let $1 \leq k<n$. If $\min (|k-i|,|k-j|) \geq 2$, then $M_{k}(\sigma)=M_{k}(\sigma \circ(i, j))$ and consequently,

$$
\begin{aligned}
|\operatorname{LAS}(\sigma)-\operatorname{LAS}(\sigma \circ(i, j))| & =\left|\sum_{k \in(\{i-1, i, i+1\} \cup\{j-1, j, j+1\}) \cap\{1, \ldots, n-1\}} M_{k}(\sigma)-M_{k}(\sigma \circ(i, j))\right| \\
& \leq \sum_{k \in(\{i-1, i, i+1\} \cup\{j-1, j, j+1\}) \cap\{1, \ldots, n-1\}}\left|M_{k}(\sigma)-M_{k}(\sigma \circ(i, j))\right| \\
& \leq \sum_{k \in(\{i-1, i, i+1\} \cup\{j-1, j, j+1\}) \cap\{1, \ldots, n-1\}} 1 \\
& =\operatorname{card}((\{i-1, i, i+1\} \cup\{j-1, j, j+1\}) \cap\{1, \ldots, n-1\}) \\
& \leq 6 .
\end{aligned}
$$

Consequently, we have the next corollary.
Corollary 25. - Under $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$, we have

$$
\begin{equation*}
\frac{\operatorname{LAS}\left(\sigma_{n}\right)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{2}{3} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left(\operatorname{LAS}\left(\sigma_{n}\right)\right)=\frac{2}{3} n+o(n) \tag{17}
\end{equation*}
$$

- Under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$, we have

$$
\begin{equation*}
\frac{\operatorname{LAS}\left(\sigma_{n}\right)-\frac{2}{3} n}{\sqrt{n}} \underset{n \rightarrow \infty}{d} \mathcal{N}\left(0, \frac{8}{45}\right) . \tag{18}
\end{equation*}
$$

Proof of Corollary 25. Let $f_{L A S 1}$ and $f_{L A S 2}$ be the two functions defined on $\mathfrak{S}_{\infty}$ by: For $\sigma \in \mathfrak{S}_{n}$,

$$
f_{L A S 1}(\sigma):=\frac{\operatorname{LAS}(\sigma)}{n} \quad \text { and } \quad f_{L A S 2}(\sigma):=\frac{\operatorname{LAS}(\sigma)-\frac{2}{3} n}{\sqrt{n}}
$$

By Lemma 24, we obtain $\varepsilon_{n}\left(f_{L A S 1}\right) \leq \frac{6}{n}$ and $\varepsilon_{n}\left(f_{L A S 2}\right) \leq \frac{6}{\sqrt{n}}$. Thus (16) and (18) follow from theorems 1 and 2. Moreover, since $\frac{\operatorname{LAS}\left(\sigma_{n}\right)}{n} \in(0,1],(17)$ is a direct consequence of (16).

### 2.4 Local statistics

Definition 26. Given $k \geq 1$, we call a function $f$ defined on $\mathfrak{S}_{\infty}$ a local function of type $k$, and we write $f \in \mathcal{L} o c_{k}$, if there exist a positive integer $m \geq 1$, a Boolean function $g$ defined on $\mathbb{N}^{(m+1) k}$ such that, for any $n \geq k+m-1$ and any $\sigma \in \mathfrak{S}_{n}$,

$$
f(\sigma)=\sum_{1 \leq i_{1}<\cdots<i_{k} \leq n} g\left(i_{1}, \ldots, i_{k}, \sigma\left(i_{1}\right), \sigma\left(i_{1}-1\right), \ldots, \sigma\left(i_{1}-m+1\right), \sigma\left(i_{2}\right), \ldots, \sigma\left(i_{k}-m+1\right)\right) .
$$

We used the convention $\sigma(i)=0$ when $i \leq 0$.
Here are some examples of local statistics.

- The number of fixed points:

By choosing $k=m=1$ and $g(x, y)=\mathbb{1}_{x=y}$, we obtain that $\operatorname{tr} \in \mathcal{L} o c_{1}$.

- $\#_{k} \in \mathcal{L} O c_{k}$ and $\sigma \mapsto \operatorname{tr}\left(\sigma^{k}\right) \in \mathcal{L} O c_{k}$.
- The number of $j$-exceedances ${ }^{6}$ :

For $j \in \mathbb{N}$ fixed, we define for $\sigma \in \mathfrak{S}_{n}$ and, we define

$$
\mathcal{N}_{\text {exc }_{j}}(\sigma):=\operatorname{card}\left(\left\{i, \sigma_{i} \geq i+j\right\}\right) .
$$

We choose again $k=m=1$ and $g(x, y)=\mathbb{1}_{x+j \leq y}$ and we obtain again $\mathcal{N}_{e x c_{j}} \in \mathcal{L} O c_{1}$.

- Longest alternating subsequence (LAS):

LAS $\in \mathcal{L} o c_{1}$. This is a direct application of Proposition 23. Here, $k=1, m=3$ and

$$
g\left(i, y_{1}, y_{2}, y_{3}\right)=\left\{\begin{array}{ll}
0 & \text { if } i=0 \\
1 & \text { if } i=1 \\
\mathbb{1}_{y_{2}>y_{1}} & \text { if } i=2 \\
\mathbb{1}_{l<k>j}+\mathbb{1}_{y_{3}>y_{2}<y_{1}} & \text { if } i>2
\end{array} .\right.
$$

- Number of peaks:

For $\sigma \in \mathfrak{S}_{n}$, we define

$$
\mathcal{N}_{\text {peak }}(\sigma):=\operatorname{card}(\{1<i<n, \sigma(i-1)<\sigma(i)>\sigma(i+1)\}) .
$$

We choose again $k=1, m=3$ and $g\left(x, y_{1}, y_{2}, y_{3}\right)=\mathbb{1}_{x \geq 3} \mathbb{1}_{y_{1}<y_{2}>y_{3}}$ and we obtain again $\mathcal{N}_{\text {peak }} \in$ $\mathcal{L} c_{1}$.

- Number of $j$-descents:

For $j \geq 1, \sigma \in \mathfrak{S}_{n}$, we define

$$
\mathcal{N}_{D_{j}}(\sigma):=\operatorname{card}\{1 \leq i \leq n-1, \sigma(i+1)+j \leq \sigma(i)\} .
$$

We choose $k=1, m=2$ and $g\left(x, y_{1}, y_{2}\right)=\mathbb{1}_{x \geq 2} \mathbb{1}_{y_{2} \geq y_{1}+j}$ and we obtain again $\mathcal{N}_{D_{j}} \in \mathcal{L} o c_{1}$.
When $j=1$, the 1 -descents are known as the descents. We also set

$$
\mathcal{N}_{D}(\sigma):=\operatorname{card}\{1 \leq i \leq n-1, \sigma(i+1)<\sigma(i)\}=\mathcal{N}_{D_{1}}(\sigma) .
$$

[^5]- Number of inversions and $m$-clicks of the permutation graph:

Definition 27. Let $\sigma \in \mathfrak{S}_{n}$. Let $\mathfrak{G}(\sigma)=\left(V_{\mathfrak{G}(\sigma)}, E_{\mathfrak{G}(\sigma)}\right)^{7}$ be the permutation graph of $\sigma$ defined by

$$
V_{\mathfrak{G}(\sigma)}=\{1, \ldots, n\} \text { and } E_{\mathfrak{G}(\sigma)}=\{(i, j) \in\{1,2, \ldots, n\} ;(\sigma(i)-\sigma(j))(i-j)<0\} .
$$

For example, $E_{\mathfrak{G}(\sigma)}=\emptyset$ if and only if $\sigma=I d_{n}$ and for the permutation $\sigma: i \mapsto n-i+1, \mathfrak{G}(\sigma)$ is the complete graph with $n$ vertices.
Given $j \geq 2$, we denote by

$$
\mathcal{K}_{j}(\sigma):=\operatorname{card}\left(\left\{\left(i_{1}, i_{2}, \ldots, i_{j}\right) ; 1 \leq i_{1}<\cdots<i_{j} \leq n, \sigma\left(i_{1}\right)>\cdots>\sigma\left(i_{j}\right)\right\}\right)
$$

the number of $j$-clicks of $\mathfrak{G}(\sigma)^{8}$. In particular, $\mathcal{K}_{2}(\sigma)$ is the number of inversions of $\sigma$. One can easily check that with $\mathcal{K}_{j} \in \mathcal{L} o c_{j}$. Here,

$$
g\left(x_{1}, \ldots, x_{j}, y_{1}, \ldots, y_{j}\right)=\mathbb{1}_{y_{1}>y_{2}>\cdots>y_{j}} .
$$

- Let $d_{k}(\sigma):=\operatorname{card}\left(\left\{i ;(i, k) \in E_{\mathfrak{G}(\sigma)}\right\}\right)$ be the degree of the vertex $k$ in $\mathfrak{G}(\sigma)$. We have $d_{k}(\sigma) \in \mathcal{L} o c_{2}$.

Proposition 28. Given $k \geq 1, f \in \mathcal{L o c}_{k}$, a random real variable $X, k-1<\gamma \leq k$ and $\left(a_{n}\right)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}}$ such that

$$
\frac{f\left(\sigma_{u n i f, n}\right)-a_{n}}{n^{\gamma}} \xrightarrow[n \rightarrow \infty]{d} X,
$$

if $\left(\mathcal{H}_{i n v, \frac{1}{\gamma-k+1}}^{\mathbb{P}}\right)$ holds then

$$
\frac{f\left(\sigma_{n}\right)-a_{n}}{n^{\gamma}} \xrightarrow[n \rightarrow \infty]{d} X .
$$

Proof. By counting the number of possible choices of $1 \leq i_{1}<i_{2}, \cdots<i_{k} \leq n$ such that $\{i, j\} \cap\left\{i_{1}, \ldots, i_{1}-\right.$ $\left.m+1, i_{2}, \ldots, i_{k}-m+1\right\} \neq \emptyset$, it is easy to see that for any permutation $\sigma \in \mathfrak{S}_{n}$ and any transposition $(i, j)$ we have

$$
|f(\sigma(i, j))-f(\sigma)| \leq \frac{2 k m(n-1)!}{(k-1)!(n-k)!} \leq 2 k m n^{k-1}
$$

Consequently for $h=\frac{f-a_{n}}{n \gamma}, \varepsilon_{n}(h) \leq 2 k n^{k-\gamma-1} m$ and one can conclude using Remark 4.
One can then easily apply this result combined with the discussion in the previous subsection to our local statistics.

[^6]Corollary 29. Under $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$, we have for any $j \geq 2$,

$$
\begin{aligned}
& \xrightarrow[\mathcal{N}_{D_{j}}\left(\sigma_{n}\right)]{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathbb{L}^{1}}{2}} \frac{1}{2}, \\
& \frac{\mathcal{N}_{D}\left(\sigma_{n}\right)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{1}} \frac{1}{2}, \\
& \frac{\mathcal{K}_{j}\left(\sigma_{n}\right)}{n^{m}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{1}} \frac{1}{(m!)^{2}}, \\
& \frac{\mathcal{N}_{\text {exc }}\left(\sigma_{n}\right)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{1}} \frac{1}{2}, \\
& \frac{\mathcal{N}_{\text {peak }\left(\sigma_{n}\right)}}{n \rightarrow \infty} \frac{\mathbb{L}^{1}}{n}
\end{aligned}
$$

Moreover, under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$, we have for any $j \geq 2$,

$$
\begin{aligned}
& \underset{\sqrt{n}}{\mathcal{N}_{D_{j}}\left(\sigma_{n}\right)-\frac{n}{2}} \underset{n \rightarrow \infty}{d} \mathcal{N}\left(0, \frac{1}{12}\right), \\
& \frac{\mathcal{N}_{D}\left(\sigma_{n}\right)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{1}{12}\right), \\
& \frac{\mathcal{K}_{j}\left(\sigma_{n}\right)-\frac{n^{j}}{(j!)^{2}}}{n^{j-\frac{1}{2}}} \underset{n \rightarrow \infty}{d} \mathcal{N}\left(0, v_{j}\right), \\
& \frac{\mathcal{N}_{\text {exc }}\left(\sigma_{n}\right)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, \frac{1}{12}\right), \\
& \frac{\mathcal{N}_{\text {peak }}\left(\sigma_{n}\right)-\frac{n}{2}}{\sqrt{n}} \underset{n \rightarrow \infty}{d} \mathcal{N}\left(0, \frac{2}{45}\right),
\end{aligned}
$$

where

$$
v_{j}=\frac{\binom{4 j-2}{2 j-1}-2\binom{2 j-1}{j}^{2}}{2((2 m-1)!)^{2}}
$$

The uniform case for $\mathcal{N}_{D}, \mathcal{N}_{\text {peak }}, \mathcal{K}_{j}$ and $\mathcal{N}_{\text {exc }}$ has already been studied. One can find a proof respectively in [Kim and Lee, 2020], [Fulman et al., 2019], [Gürerk et al., 2019] and [Féray, 2013]. For the conjugation invariant case, as we explained before, $\mathcal{N}_{D}$ and $\mathcal{N}_{\text {peak }}$ are fully understood but, to the best knowledge of the author, it is not the case for $\mathcal{K}_{j}$ and $\mathcal{N}_{\text {exc }}$. For $\mathcal{N}_{\text {exc }}$, the special case of the Ewens distribution is studied in [Féray, 2013]. Moreover, the results for $\mathcal{N}_{D_{j}}$ and $\mathcal{N}_{\text {exc }}^{j}$ are direct consequences of respectively $\mathcal{N}_{D}$ and $\mathcal{N}_{\text {exc1 }}$ since for any conjugation invariant random permutation $\sigma_{n}$,

$$
0 \leq \mathbb{E}\left(\mathcal{N}_{D}\left(\sigma_{n}\right)-\mathcal{N}_{D_{j}}\left(\sigma_{n}\right)\right)=\frac{(j-1)(n-j-1)\left(1-\mathbb{P}\left(\sigma_{n}(1)=1\right)\right)}{n-1} \leq j-1
$$

and

$$
0 \leq \mathbb{E}\left(\mathcal{N}_{e x c_{1}}\left(\sigma_{n}\right)-\mathcal{N}_{e x c_{j}}\left(\sigma_{n}\right)\right) \leq j-1 .
$$

### 2.5 Number of occurrences of a vincular permutation pattern

Vincular Patterns also known as dashed patterns are introduces by Babson and Steingrímsson [2000]. We use the same definition as in [Féray, 2013].

Definition 30. A vincular pattern of size $p$ is a couple $(\tau, X)$ such that $\tau \in \mathfrak{S}_{p}$ and $X \subset[p-1]$. Given $\sigma \in \mathfrak{S}_{\infty}$, an occurrence of $(\tau, X)$ is a list $i_{1}<\cdots<i_{p}$ such that

- $i_{x+1}=i_{x}+1$ for any $x \in X$.
- $\left(\sigma\left(i_{1}\right), \ldots, \sigma\left(i_{p}\right)\right)$ is in the same relative order as $\left(\tau\left(i_{1}\right), \ldots, \tau\left(i_{p}\right)\right)$.

We denote by $\mathcal{N}_{(\tau, X)}(\sigma)$ the number of occurrences of $(\tau, X)$ in $\sigma$.
When $X=\emptyset,(\tau, X)$ is said to be a classic pattern. Here is some examples of vincular patterns:

- $\mathcal{N}_{(21, \emptyset)}=\mathcal{N}_{i n v}$
- $\mathcal{N}_{(21,\{1\})}=\mathcal{N}_{D}$
- $\mathcal{N}_{(j \ldots 21, \emptyset)}=\mathcal{K}_{j}$
- $\mathcal{N}_{(132,\{1,2\})}+\mathcal{N}_{(231,\{1,2\})}=\mathcal{N}_{\text {peak }}$.

Remark that for any $(\tau, X), \mathcal{N}_{(\tau, X)} \in \mathcal{L} c_{p} \cap \mathcal{L} o c_{p-\operatorname{card}(X)}$.
For the uniform case, Bóna [2010], Janson, Luczak, and Rucinski [2011] and Hofer [2017] proved respectively a CLT for monotone, classic and vincular patterns. Féray [2013] gives a generalization for the Ewens distribution. In particular, Hofer [2017] proved that for any $\tau \in \mathfrak{S}_{p}$ and any $X \subset[p-1]$,

$$
\frac{\mathcal{N}_{(\tau, X)}\left(\sigma_{u n i f, n}\right)-\frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \underset{n \rightarrow \infty}{d} \mathcal{N}\left(0, V_{\tau, X}\right) .
$$

Here, $q=\operatorname{card}(X)$ and $V_{\tau, X}>0$. Using Proposition 28, we have immediately the following.
Proposition 31. Under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$, for any $\tau \in \mathfrak{S}_{p}$ and any $X \subset[p-1]$

$$
\frac{\mathcal{N}_{(\tau, X)}\left(\sigma_{n}\right)-\frac{n^{p-q}}{p^{p-p-q)!}}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}\left(0, V_{\tau, X}\right) .
$$

Here, $q=\operatorname{card}(X)$ and $V_{\tau, X}>0$.

## 3 Further discussion and improved bounds

### 3.1 Universality for $\widetilde{\mathcal{L} O C}$

We denote by $\widetilde{\mathcal{L o c}}$ the set of local functions $f$ of any type associated with a Boolean function $g$ such that

$$
\begin{equation*}
\operatorname{card}\left(\left\{i \in \mathbb{N}^{*} ; \max _{I \in \mathbb{N}^{k-1}} \max _{J \in \mathbb{N}^{m k}} g(I, i, J)=1\right\}\right)<\infty . \tag{19}
\end{equation*}
$$

For this class, it is simple to obtain the convergence of the expectation. It can be seen as a macroscopic universality result.

Let $A \subset \mathbb{N}^{*}$ be finite, $n>\max (A)$ and $\left(\sigma_{n}\right)_{n \geq 1}$ satisfying $\left(\mathcal{H}_{i n v}\right)$. Using again the random walk associated to $T$ and seeing that

$$
\mathbb{P}\left(\exists i \in\left\{i_{1}-i_{2} ; i_{1} \in A, 0 \leq i_{2}<m-1\right\},\left(T^{n-1}\left(\sigma_{n}\right)\right)(i) \neq \sigma_{n}(i)\right) \leq \frac{2 \#\left(\sigma_{n}\right) \operatorname{card}(A) m}{n}
$$

we obtain the following.
Proposition 32. Given $f \in \widetilde{\mathcal{L} \text { Oc }}$ and assuming that $\left(\sigma_{n}\right)_{n \geq 1}$ and $\left(\sigma_{r e f, n}\right)_{n \geq 1}$ satisfy $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$ we have

$$
\mathbb{E}\left(f\left(\sigma_{n}\right)\right)-\mathbb{E}\left(f\left(\sigma_{r e f, n}\right)\right) \xrightarrow[n \rightarrow \infty]{\longrightarrow} 0
$$

Moreover, if $f\left(\sigma_{r e f, n}\right)$ converges in distribution then $f\left(\sigma_{n}\right)$ does also converge to the same limit.
We give now an application: Let $n$ be a positive integer and $\sigma \in \mathfrak{S}_{n}$, we define

$$
\begin{equation*}
D(\sigma):=\{i \in\{1, \ldots, n-1\} ; \sigma(i+1)<\sigma(i)\} . \tag{20}
\end{equation*}
$$

When $\sigma$ is random, $D(\sigma)$ is known as a descent process.

Given $A \subset \mathbb{N}^{*}$ finite, if we introduce

$$
\begin{equation*}
D^{A}(\sigma):=\mathbb{1}_{A \subset D(\sigma)} \tag{21}
\end{equation*}
$$

then $D^{A} \in \mathcal{L} \mathcal{L o c}_{|A|} \cap \widetilde{\mathcal{L} o c}$. Here,

$$
g\left(x_{1}, x_{2}, \ldots, x_{|A|}, y_{1}, y_{1}^{\prime}, y_{2}, \ldots, y_{|A|}, y_{|A|}^{\prime}\right)=\mathbb{1}_{A=\left\{x_{i}-1,1 \leq i \leq|A|\right\}} \prod_{i=1}^{|A|} \mathbb{1}_{y_{i}<y_{i}^{\prime}} .
$$

We further investigate the descent process. First, the descent process is well understood in the uniform case.

Theorem 33. [Borodin et al., 2010, Theorem 5.1] For any positive integer $n$ and any $A \subset\{1,2, \ldots, n-1\}$,

$$
\mathbb{P}\left(A \subset D\left(\sigma_{u n i f, n}\right)\right)=\operatorname{det}\left(\left[k_{0}(j-i)\right]_{i, j \in A}\right),
$$

where,

$$
\sum_{i \in \mathbb{Z}} k_{0}(i) z^{i}=\frac{1}{1-e^{z}} .
$$

We say that the descent process is determinantal with kernel $K_{0}(i, j):=k_{0}(j-i)$.

In the non-uniform setting, the descent process is already studied for the Mallows law with Kendall tau metric: it is also determinantal with different kernels, see [Borodin et al., 2010, Proposition 5.2]. We showed in [Kammoun, 2018] that for a large class of random permutations, the limiting descent process is determinantal with the same kernel as the uniform setting. We will detail a weaker result than [Kammoun, 2018].

Corollary 34. Under $\left(\mathcal{H}_{i n v, 1}^{\mathbb{P}}\right)$, for any finite set $A \subset \mathbb{N}^{*}$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}\left(A \subset D\left(\sigma_{n}\right)\right)=\operatorname{det}\left(\left[k_{0}(j-i)\right]_{i, j \in A}\right) . \tag{DPP}
\end{equation*}
$$

Proof. Just apply Proposition 32 for the statistic $D^{A}$ defined in (21).
The same argument can be applied for other local statistics but not necessarily in $\widetilde{\mathcal{L o c}}$. For example, we have similar results for the degree of vertices of the permutation graph.
Proposition 35. Under ( $\mathcal{H}_{i n v, 1}^{\mathbb{P}}$ ),

$$
\frac{d_{k}\left(\sigma_{n}\right)}{n} \underset{n \rightarrow \infty}{\mathbb{P}} \frac{1}{2}, \quad \xrightarrow[\frac{d \frac{n}{2}}{}\left(\sigma_{n}\right)]{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2}, \quad \frac{d_{n}\left(\sigma_{n}\right)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2}
$$

Moreover, under $\left(\mathcal{H}_{i n v, 2}^{\mathbb{P}}\right)$,

$$
\frac{d \frac{n}{2}\left(\sigma_{n}\right)-\frac{n}{2}}{2 \sqrt{n}} \underset{n \rightarrow \infty}{d} \mathcal{N}(U, 1-U), \quad \frac{d_{n}\left(\sigma_{n}\right)-\frac{n}{2}}{\sqrt{n}} \underset{n \rightarrow \infty}{d} \mathcal{N}(0,6), \quad \frac{d_{k}\left(\sigma_{n}\right)-\frac{n}{2}}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0,6),
$$

where $U$ is a uniform random variable on $[0,1]$.
Note that $d_{k}$ is a local statistic for fixed $k$ but it is not the case for $d_{n}$. The uniform case is already studied by Gürerk et al. [2019]. The problem for $d_{n}$ is that for any $2<k<n, \varepsilon_{n}\left(d_{n}\right)=n-1$ since $d_{n}\left(I d_{n}\right)=0$ and $d_{n}((n, 1))=n-1$ and thus we cannot apply directly our previous approach. The idea of the proof is the following. If we condition on the event

$$
E_{n}=\left\{T^{1}, T^{2}, \ldots, T^{n} \text { do not change } \sigma_{n}(n)\right\}
$$

then $d_{n}$ changes at most by 2 every time we apply $T$ and one concludes easily since

$$
\mathbb{P}\left(E_{n}\right) \geq 1-2 \frac{\mathbb{E}\left(\#\left(\sigma_{n}\right)\right)}{n}
$$

### 3.2 A lower bound for fluctuations

For some statistics, one can obtain a better lower bound by using a different way to go from $\sigma_{E w, 0, n}$ to $\sigma_{n}$. Unlike the previous examples, the control of the error may depend on the statistic. Our first example is the longest increasing subsequence. We give a lower bound for the fluctuations for a conjugation invariant random permutation. Using this inverse walk one can obtain the following results.

Proposition 36. If $\left(\mathcal{H}_{i n v, \frac{3}{2}}^{\mathbb{P}}\right)$ is satisfied, then for any $k \geq 1$, for any $s_{1}, \ldots, s_{k} \in \mathbb{R}$,

$$
\limsup _{n \rightarrow \infty} \mathbb{P}\left(\forall i \leq k^{\prime}, \frac{\lambda_{i}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s_{i}\right) \leq F_{2}\left(s_{1}, s_{2}, \ldots, s_{k}\right) .
$$

In particular,

$$
\limsup _{n \rightarrow \infty} \mathbb{P}\left(\frac{\operatorname{LIS}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \leq F_{2}(s)
$$



Figure 4: The transition probabilities of $T_{\sigma_{u n i f, 3}}$

To do so, we define a new Markov operator. Let $\sigma \in \mathfrak{S}_{n}^{0}, \lambda \in \mathbb{Y}_{n}$ and $i \in\{1, \ldots, n\}^{9}$. We define $\mathfrak{T}_{i, \lambda}(\sigma):=\left(\sigma^{\lambda_{1}+1}(i), \ldots, \sigma^{\lambda_{1}+\lambda_{2}}(i)\right) \ldots\left(\sigma^{\sum_{j=1}^{\ell(\lambda)-1} \lambda_{j}}(i), \ldots, \sigma^{n}(i)\right)$. Now let $\sigma_{n}$ be a conjugation invariant random permutation and let $T_{\sigma_{n}}$ be the Markov operator defined on $\mathfrak{S}_{n}^{0}$ as follows. Starting from $\sigma \in$ $\mathfrak{S}_{n}^{0}$, choose $i$ uniformly in $\{1, \ldots, n\}$ and $\lambda$ randomly according to the distribution of $\hat{\lambda}\left(\sigma_{n}\right)^{10}$ and then $T_{\sigma_{n}}(\sigma)$ returns $\mathfrak{T}_{i, \lambda}(\sigma) .{ }^{11}$ For example, the transition probabilities of $T_{\sigma_{u n i f, 3}}$. are shown in Figure 4. By construction, $\hat{\lambda}\left(T_{\sigma_{n}}(\sigma)\right)=\lambda$ and thus, for any cyclic permutation $\sigma \in \mathfrak{S}_{n}^{0}$,

$$
\hat{\lambda}\left(T_{\sigma_{n}}(\sigma)\right) \stackrel{d}{=} \hat{\lambda}\left(\sigma_{n}\right) .
$$

This yields,

$$
\hat{\lambda}\left(T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right)\right) \stackrel{d}{=} \hat{\lambda}\left(\sigma_{n}\right) .
$$

Finally, since the construction depends only on the cycle structure, $T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right)$ is conjugation invariant and

$$
\begin{equation*}
T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right) \stackrel{d}{=} \sigma_{n} . \tag{22}
\end{equation*}
$$

Our main argument is the following lemma.

[^7]Lemma 37. For any permutation $\rho \in \mathfrak{S}_{n}^{0}$, for any conjugation invariant random permutation $\sigma_{n}$, for any positive integer $k$, almost surely

$$
\begin{aligned}
\mathbb{E}\left(\left(\sum_{i=1}^{k} \lambda_{j}\left(T_{\sigma_{n}}(\rho)\right)-\lambda_{j}(\rho)\right)_{-} \mid \#\left(T_{\sigma_{n}}(\rho)\right)\right) & \leq \frac{\#\left(T_{\sigma_{n}}(\rho)\right)}{n} \sum_{j=1}^{k} \lambda_{i}(\rho) \\
& =\frac{d}{=} \frac{\#\left(\sigma_{n}\right)}{n} \sum_{i=1}^{k} \lambda_{i}(\rho) .
\end{aligned}
$$

Proof. Let $i_{1}<i_{2}<\cdots<i_{\sum_{i=1}^{k} \lambda_{i}(\rho)}$ such that $\left\{i_{1}, i_{2}, \cdots<i_{\sum_{i=1}^{k} \lambda_{i}(\rho)}\right\} \subset \mathfrak{I}_{k}(\rho)$. We have then for any permutation $\rho^{\prime}$,

$$
\left\{i_{1}, i_{2}, \cdots<i_{\sum_{i=1}^{k} \lambda_{i}(\rho)}\right\} \cap\left\{i, \rho^{\prime}(i)=\rho(i)\right\} \subset \mathfrak{I}_{k}\left(\rho^{\prime}\right)
$$

and then

$$
\begin{equation*}
\left(\sum_{j=1}^{k} \lambda_{j}\left(\rho^{\prime}\right)-\lambda_{j}(\rho)\right)_{-} \leq \operatorname{card}\left\{j \leq \sum_{i=1}^{k} \lambda_{i}(\rho) ; \rho\left(i_{j}\right) \neq \rho^{\prime}\left(i_{j}\right)\right\} . \tag{23}
\end{equation*}
$$

Consequently, almost surely

$$
\begin{aligned}
\mathbb{E}\left(\left(\sum_{j=1}^{k} \lambda_{j}\left(T_{\sigma_{n}}(\rho)\right)-\lambda_{j}(\rho)\right)_{-} \mid \#\left(T_{\sigma_{n}}(\rho)\right)\right) & \leq \sum_{j=1}^{\sum_{i=1}^{k} \lambda_{i}(\rho)} \mathbb{E}\left(\mathbb{1}_{\rho(j) \neq \rho^{\prime}(j)} \mid \#\left(T_{\sigma_{n}}(\rho)\right)\right) \\
& =\sum_{i=1}^{k} \lambda_{i}(\rho) \frac{\#\left(T_{\sigma_{n}}(\rho)\right)}{n} .
\end{aligned}
$$

Proof of Proposition 36. For any $\varepsilon>0$ there exists $n_{0}$ such that

$$
\mathbb{P}\left(\sum_{i=1}^{k} \lambda_{i}\left(\sigma_{E w, 0, n}\right)<9 k \sqrt{n}\right) \geq \sqrt{1-\varepsilon}
$$

and by hypothesis for any $\varepsilon^{\prime}>0$ there exist $n_{1}>$ such that for any $n>n_{1}$

$$
\mathbb{P}\left(\#\left(\sigma_{n}\right)<\varepsilon^{\prime} \frac{n^{\frac{2}{3}}}{9 k}\right)>\sqrt{1-\varepsilon}
$$

Consequently,

$$
\mathbb{P}\left(\frac{\mathbb{E}\left(\left(\sum_{j=1}^{k} \lambda_{j}\left(T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right)\right)-\lambda_{j}\left(\sigma_{E w, 0, n}\right)\right)_{-} \mid \#\left(T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right)\right)\right)}{n^{\frac{1}{6}}}<\varepsilon^{\prime}\right)>1-\varepsilon
$$

This yields

$$
\frac{\left(\sum_{j=1}^{k} \lambda_{j}\left(T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right)\right)-\lambda_{j}\left(\sigma_{E w, 0, n}\right)\right)_{-} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, ~ n^{\frac{1}{6}}}{}
$$

which concludes the proof since $T_{\sigma_{n}}\left(\sigma_{E w, 0, n}\right) \stackrel{d}{=} \sigma_{n}$.

### 3.3 Proof of Theorems 9 and of Proposition 15

Since Theorems 9 is the particular case $k=1$ of Proposition 15, we will prove only Proposition 15. Moreover $\left(\mathcal{H}_{i n v, \frac{3}{2}}^{\mathbb{P}}\right)$ implies clearly (12) and consequently, the first bound of Proposition 15 is a direct application of Proposition 36. So it is sufficient to prove that under $\left(\mathcal{H}_{\text {inv }}\right)$ and (12), we have

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \mathbb{P}\left(\forall i \leq k^{\prime}, \frac{\lambda_{i}\left(\sigma_{n}\right)-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq s_{i}\right) \geq F_{2}\left(s_{1}, s_{2}, \ldots, s_{k}\right) . \tag{24}
\end{equation*}
$$

Sketch of proof. We will not go trough all the details since we have already presented similar techniques many times. The idea is to modify the random walk associated to $T$ as following. Given $1 \leq j \leq n-1$, we define $\hat{T}_{j}$ the Markov operator as following. $\hat{T}_{j}(\sigma)$ is a permutation chosen uniformly at random among the permutations obtained by merging all cycles of length less than $j$ to (one of) the biggest cycles of $\sigma$ to obtain a permutation with cycles of length more than $j$. Since this construction depends only on the cycle structure, under $\left(\mathcal{H}_{\text {inv }}\right), \hat{T}_{j}\left(\sigma_{n}\right)$ is conjugation invariant. Therefore $T^{n}\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)$ is distributed according to $E w(0)$. Similarly to the previous proofs, we have

$$
\left.\mathbb{E}\left(\left(\sum_{i=1}^{k} \lambda_{j}\left(T^{n}\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)\right)-\lambda_{j}\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)\right)\right)_{-} \mid \#\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)\right) \leq \frac{\#\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)}{j} \sum_{i=1}^{k} \lambda_{i}\left(\hat{T}_{j}\left(\sigma_{n}\right)\right) .
$$

Let $\left(j_{n}\right)_{n>1}$ be such that

$$
\frac{1}{n^{\frac{1}{6}}}\left(\left(\sum_{k=1}^{j_{n}} \not \#_{k}\left(\sigma_{n}\right)\right)+\frac{\sqrt{n}}{j_{n}} \sum_{k=j_{n}+1}^{n} \#_{k}\left(\sigma_{n}\right)\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 .
$$

We have then $\left(T_{j_{n}}\left(\sigma_{n}\right)\right)_{n \geq 1}$ satisfies $\left(\mathcal{H}_{i n v, 6}^{\mathbb{P}}\right)$,

$$
\frac{\sum_{i=1}^{k} \lambda_{i}\left(\hat{T}_{j_{n}}\left(\sigma_{n}\right)\right)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 k
$$

and

$$
\frac{\left.\mathbb{E}\left(\left(\sum_{i=1}^{k} \lambda_{j}\left(T^{n}\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)\right)-\lambda_{j}\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)\right)\right)_{-} \mid \#\left(\hat{T}_{j}\left(\sigma_{n}\right)\right)\right)}{n^{\frac{1}{6}}} \underset{n \rightarrow \infty}{\mathbb{P}} 0 .
$$

This yields (24).

### 3.4 Lower bound for the longest increasing subsequence

Proposition 38. If $\left(\sigma_{n}\right)_{n \geq 1}$ is conjugation invariant then for any $\varepsilon>0$,

$$
\mathbb{P}\left(\operatorname{LIS}\left(\sigma_{n}\right)>(2 \sqrt{13}-6-\varepsilon) \sqrt{n}\right) \underset{n \rightarrow \infty}{\longrightarrow} 1
$$

This yields the following lower bound

$$
\liminf _{n \rightarrow \infty} \frac{\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right)\right)}{\sqrt{n}} \geq 2 \sqrt{13}-6 \simeq 1.21 \ldots
$$

Motivated by a conjecture of Bukh and Zhou [2016], the author tried in a previous work to prove an asymptotic lower bound on the expectation of the longest increasing subsequence of a conjugation invariant random permutation without cycle conditions. In particular, under the same hypothesis, it was proved in [Kammoun, 2020] that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right)\right)}{\sqrt{n}} \geq 2 \sqrt{\theta} \simeq 0.564 \ldots \tag{25}
\end{equation*}
$$

where $\theta$ is the unique solution of $G(2 \sqrt{x})=\frac{2+x}{12}$,

$$
\begin{align*}
G:=[0,2] & \rightarrow\left[0, \frac{1}{2}\right] \\
x & \mapsto \int_{-1}^{1}\left(\Omega(s)-\left|s+\frac{x}{2}\right|-\frac{x}{2}\right)_{+} \mathrm{d} s \tag{26}
\end{align*}
$$

and

$$
\Omega(s):= \begin{cases}\frac{2}{\pi}\left(s \arcsin (s)+\sqrt{1-s^{2}}\right) & \text { if }|s|<1 \\ |s| & \text { if }|s| \geq 1\end{cases}
$$

Sketch of the proof of Proposition 38. The proof is an adaptation of the proof of [Kammoun, 2020, Thm 1] Before we start, let

$$
\theta^{\prime}:=4-\sqrt{13} \text { and } \theta^{\prime \prime}:=2\left(1-\theta^{\prime}\right)=2 \sqrt{6 \theta^{\prime}-2}=2 \sqrt{13}-6=1.21 \ldots
$$

In this proof, we use the following convention. Let $A, B \subset \mathfrak{S}_{n}$ and $f: \mathfrak{S}_{n} \rightarrow \mathbb{R}$. If $\mathbb{P}\left(\sigma_{n} \in A\right)=0$, we assign $\mathbb{P}\left(\sigma_{n} \in B \mid \sigma_{n} \in A\right)=0$ and $\mathbb{E}\left(f\left(\sigma_{n}\right) \mid \sigma_{n} \in A\right)=0$.

We have

$$
\begin{aligned}
\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right)\right) & =\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right) \mid \#_{1}\left(\sigma_{n}\right)<\theta^{\prime \prime} \sqrt{n}\right) \mathbb{P}\left(\#_{1}\left(\sigma_{n}\right)<\theta^{\prime \prime} \sqrt{n}\right) \\
& +\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right) \mid \#_{1}\left(\sigma_{n}\right) \geq \theta^{\prime \prime} \sqrt{n}\right) \mathbb{P}\left(\#_{1}\left(\sigma_{n}\right) \geq \theta^{\prime \prime} \sqrt{n}\right) \\
& \geq E\left(\operatorname{LIS}\left(\sigma_{n}\right) \mid \#_{1}\left(\sigma_{n}\right)<\theta^{\prime \prime} \sqrt{n}\right) \mathbb{P}\left(\#_{1}\left(\sigma_{n}\right)<\theta^{\prime \prime} \sqrt{n}\right) \\
& +\theta^{\prime \prime} \sqrt{n} \mathbb{P}\left(\#_{1}\left(\sigma_{n}\right) \geq \theta^{\prime \prime} \sqrt{n}\right)
\end{aligned}
$$

Since the condition on the fixed points is conjugation invariant, it is sufficient to prove this result in the case where almost surely $\#_{1}\left(\sigma_{n}\right)<\theta^{\prime \prime} \sqrt{n}$. Using the same argument and since the condition on the number of cycles is conjugation invariant, it is sufficient to prove this result in the two particular cases.

- If almost surely $\#\left(\sigma_{n}\right)>n \theta^{\prime}$.

We recall that

$$
\#_{1}\left(\sigma^{2}\right) \geq 6 \#(\sigma)-3 \#_{1}(\sigma)-2 n
$$

Consequently, under the condition $\#_{1}\left(\sigma_{n}\right)<\theta^{\prime \prime} \sqrt{n}$, almost surely,

$$
\#_{1}\left(\sigma_{n}^{2}\right)>n\left(6 \theta^{\prime}-2\right)-3 \theta^{\prime} \sqrt{n}
$$

We can then conclude by [Kammoun, 2020, Proposition 15] that

$$
\liminf _{n \rightarrow \infty} \frac{\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right)\right)}{\sqrt{n\left(6 \theta^{\prime}-2\right)-3 \theta^{\prime} \sqrt{n}}} \geq 2
$$

Thus,

$$
\liminf _{n \rightarrow \infty} \frac{\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right)\right)}{\sqrt{n}} \geq 2 \sqrt{6 \theta^{\prime}-2}=\theta^{\prime \prime}
$$

- If almost surely $\#\left(\sigma_{n}\right) \leq n \theta^{\prime}$. Using Lemma 37 for $k=1$, we obtain that for any $\varepsilon, \varepsilon^{\prime}>0$ there exists $n_{0}$ such that for any $n>n_{0}$, for any conjugation invariant random permutation $\sigma_{n}$ such that almost surely $\#\left(\sigma_{n}\right) \leq n \theta^{\prime}$,

$$
\mathbb{P}\left(\operatorname{LIS}\left(\sigma_{n}\right)>2 \sqrt{n}\left(1-\theta^{\prime}-\varepsilon\right)\right)>1-\varepsilon^{\prime} .
$$

Consequently,

$$
\liminf _{n \rightarrow \infty} \frac{\mathbb{E}\left(\operatorname{LIS}\left(\sigma_{n}\right)\right)}{\sqrt{n}} \geq 2\left(1-\theta^{\prime}\right)=\theta^{\prime \prime}
$$

This concludes the proof.

## 4 Other groups

### 4.1 General idea and main results

The same technique of proof we presented in Section 1 can be applied to other sets having a similar structure to the symmetric group. We will give applications in the next subsection. In general, one can apply the same techniques when there exists a "nice" sequence of undirected graphs $G:=\left(G_{n}=\right.$ $\left.\left(V_{n}, E_{n}\right)\right)_{n \geq 1}{ }^{12}$ such that ${ }^{13}$.

$$
\begin{equation*}
\forall n \geq 1, G_{n} \text { is locally finite. } \tag{27}
\end{equation*}
$$

$$
\begin{equation*}
\forall n \geq 1 \text {, there exists a countable set } I_{n} \text { and finite sets }\left(V_{n}^{i}\right)_{i \in I_{n}} \text { such that } V_{n}=\sqcup_{i \in I_{n}} V_{n}^{i} \text {. } \tag{28}
\end{equation*}
$$

For any $n \geq 1$, for any $i, j \in I_{n}$, for any $\sigma_{1}, \sigma_{2} \in V_{n}^{i}$,

$$
\begin{equation*}
\operatorname{card}\left(\left\{\sigma^{\prime} \in V_{n}^{j} ;\left(\sigma^{\prime}, \sigma_{1}\right) \in E_{n}\right\}\right)=\operatorname{card}\left(\left\{\sigma^{\prime} \in V_{n}^{j} ;\left(\sigma^{\prime}, \sigma_{2}\right) \in E_{n}\right\}\right)=: \mathrm{e}_{j, i} . \tag{29}
\end{equation*}
$$

i.e. the number of neighbors in $V_{n}^{j}$ of any element of $V_{n}^{i}$ only depends on $(i, j)$; we denote it by $\mathrm{e}_{i, j}$. We denote by $\widetilde{E_{n}}:=\left\{(i, j) \in I_{n}^{2} ; \mathrm{e}_{i, j}>0\right\}$ and by $\widetilde{G_{n}}:=\left(I_{n}, \widetilde{E_{n}}\right)$ the classes graph. We need moreover in the sequel of this Section 1 that

$$
\begin{equation*}
\forall n \geq 1, \text { the classes graph } \widetilde{G_{n}} \text { is connected. } \tag{30}
\end{equation*}
$$

In the sequel of this section, we assume (27)- (30).
For example, if $G_{n}$ is the Cayley graph of the symmetric group generated by transpositions we have

[^8]

Figure 5: The classes graph for the Cayley graph of $\mathfrak{S}_{n}$ generated by transpositions for $n=4$

- $V_{n}=\mathfrak{S}_{n}$
- $E_{n}=\left\{(\sigma, \sigma \circ(i, j)) ; \sigma \in \mathfrak{S}_{n}, i \neq j\right\}$
- $I_{n}=\mathbb{Y}_{n}$ (the set of Young diagrams of size $n$ ).
- $V_{n}^{i}=\left\{\sigma \in \mathfrak{S}_{n} ; \hat{\lambda}(\sigma)=i\right\}$,
- $\widetilde{E_{n}}$ the set of couples of Young diagrams such that one can obtain one from the other by concatenating two arrows. For example, for $n=4$, we obtain the classes graph in Figure 5.

With analogy with Section 1, we will now construct a new directed graph for which we will consider the uniform random walk. Let $d_{G_{n}}$ be the usual graph distance and for $\sigma \in V_{n}$, we denote by $\operatorname{Class}(\sigma)$ the unique $i \in I_{n}$ such that $j \in V_{n}^{i}$.

Let $\left(i_{n}^{*}\right)_{n \geq 1} \in \prod_{n \geq 1} I_{n}$ be a "nice" sequence of classes. We denote by $\underline{d}(\sigma):=\min _{\rho \in V_{n}^{i_{n}^{*}}} d_{G_{n}}(\sigma, \rho)$. The random walk we use to prove universality will be the uniform random walk on the directed graph $G_{n}^{\prime}:=\left(V_{n}, E_{n}^{\prime}\right)$ where

$$
E_{n}^{\prime}=\left\{\left(\sigma_{1}, \sigma_{2}\right) \in E_{n} ; \underline{d}\left(\sigma_{2}\right)=\underline{d}\left(\sigma_{1}\right)-1\right\} \cup\left\{(\sigma, \sigma), \sigma \in V_{n}^{i_{n}^{*}}\right\} .
$$

Back to the example of the Cayley graph of the symmetric group generated by transpositions we have

- Class $(\sigma)=\hat{\lambda}(\sigma)$,
- $i_{n}^{*}=(n, \underline{0})$ is the Young diagram with a unique row of length $n$,
- $\left(V_{n}, E_{n}^{\prime}\right)=\mathcal{G}_{\mathfrak{S}_{n}}{ }^{14}$,
- $\underline{d}(\sigma)=\#(\sigma)-1$.

With analogy with the symmetric group, let $T_{G_{n}^{\prime}}$ be the Markov operator associated to the uniform random walk on $G_{n}^{\prime}, V_{\infty}:=\cup_{n \geq 1} V_{n}$ and $f$ be a function defined on $V_{\infty}$ and having values on some metric space $\left(F, d_{F}\right)$. With analogy with Section 1 , for $S \subset V_{n}$ and $\sigma \in V_{n}$, let

$$
\operatorname{next}(S):=\left\{\sigma_{2} ; \sigma_{1} \in S \text { and }\left(\sigma_{1}, \sigma_{2}\right) \in E_{n}^{\prime}\right\}
$$

[^9]\[

\operatorname{final}(\sigma):= $$
\begin{cases}\operatorname{next} \underline{d}(\sigma)(\{\sigma\}) & \text { if } \underline{d}(\sigma)>1 \\ \{\sigma\} & \text { otherwise }\end{cases}
$$
\]

and for $i \in I_{n}$ and $p \geq 1$, we define

$$
\begin{aligned}
\underline{\varepsilon}_{n, i, p}(f) & :=\left(\sum_{\sigma \in V_{n}^{i}} \sum_{\rho \in \operatorname{next}(\{\sigma\})} \frac{\left(d_{F}(f(\sigma), f(\rho))\right)^{p}}{\operatorname{card}\left(V_{n}^{i}\right) \operatorname{card}(\operatorname{next}(\{\sigma\}))}\right)^{\frac{1}{p}} \\
\underline{\varepsilon}_{n, p}(f) & :=\sup _{i \in I_{n}} \underline{\varepsilon}_{n, i, p}(f) \\
\underline{\varepsilon}_{n, i, \infty}(f) & :=\max _{\sigma \in V_{n}^{i}} \max _{\rho \in \operatorname{next}(\{\sigma\})} d_{F}(f(\sigma), f(\rho)) \\
\underline{\varepsilon}_{n, \infty}(f) & :=\sup _{i \in I_{n}} \varepsilon_{n, i, \infty}(f) \\
\underline{\varepsilon}_{n, i, p}^{\prime}(f) & :=\left(\sum_{\sigma \in V_{n}^{i}} \sum_{\rho \in \operatorname{final}(\sigma)} \frac{\left(d_{F}(f(\sigma), f(\rho))\right)^{p}}{\operatorname{card}\left(V_{n}^{i}\right) \operatorname{card}(\operatorname{final}(\sigma))}\right)^{\frac{1}{p}} \\
\underline{\varepsilon}_{n, i, \infty}^{\prime}(f) & :=\max _{\sigma \in V_{n}^{i}} \max _{\rho \in \operatorname{final}(\sigma)} d_{F}(f(\sigma), f(\rho)) .
\end{aligned}
$$

Finally, let $\left(\sigma_{n}\right)_{n \geq 1}$ be a sequence of random variables such that $\sigma_{n}$ is supported on $V_{n}$. We say that $\sigma_{n}$ is $G_{n}$ invariant (with respect to the partition $\left.\left\{V_{n}^{i}\right\}_{i \in I_{n}}\right)^{15}$ if for any $i \in I_{n}$ and any $\sigma, \rho \in V_{n}^{i}$

$$
\mathbb{P}\left(\sigma_{n}=\sigma\right)=\mathbb{P}\left(\sigma_{n}=\rho\right),
$$

and we say that $\left(\sigma_{n}\right)_{n \geq 1}$ is $G$-invariant if $\sigma_{n}$ is $G_{n}$-invariant $\forall n \geq 1$.
Definition 39. For $\alpha>0$ and $p \in[1, \infty]$, we say that $\left(\sigma_{n}\right)_{n \geq 1}$ satisfies $\mathcal{H}_{G-i n v, \alpha}^{\mathbb{P}}$ if

$$
\left(\sigma_{n}\right)_{n \geq 1} \text { is G-invariant and } \frac{\underline{d}\left(\sigma_{n}\right)}{n^{\frac{1}{\alpha}}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \quad\left(\mathcal{H}_{G-i n v, \alpha}^{\mathbb{P}}\right)
$$

we say that it satisfies $\mathcal{H}_{G-\text { inv, } \alpha}^{\mathbb{L}^{p}}$ if

$$
\left(\sigma_{n}\right)_{n \geq 1} \text { is G-invariant and } \frac{\underline{d}\left(\sigma_{n}\right)}{n^{\frac{1}{\alpha}}} \frac{\mathbb{L}^{p}}{n \rightarrow \infty} 0 . \quad\left(\mathcal{H}_{G-i n v, \alpha}^{\mathbb{L}^{p}}\right)
$$

Interesting results can be obtained if the graph satisfies an additional symmetry property:
For any $\sigma_{1} \in V_{n}$, for any $\sigma_{2}, \sigma_{3} \in \operatorname{final}\left(\sigma_{1}\right)$, the number of paths in $G_{n}^{\prime}$ of length $\underline{d}(\sigma)$ from $\sigma_{1}$ to $\sigma_{2}$ is equal to that from $\sigma_{1}$ to $\sigma_{3}$ i.e. $A_{G_{n}^{\prime}}$ the adjacency matrix of $G_{n}^{\prime}$ satisfies the following:

$$
\begin{equation*}
\forall \sigma_{1} \in \mathfrak{S}_{n}, \exists c_{\sigma_{1}} \in \mathbb{N} \text { such that } \forall \rho \in \mathfrak{S}_{n}, A_{G_{n}^{\prime}}^{\frac{d}{(\sigma)}}\left(\sigma_{1}, \rho\right)=c_{\sigma_{1}} \mathbb{1}_{\rho \in \operatorname{final}\left(\sigma_{1}\right)} \tag{31}
\end{equation*}
$$

In particular, we have the following:
Lemma 40. Under (27)-(31), for any $\left(\sigma_{n}\right)_{n \geq 1} G$-invariant, for any $\mathbb{P} \in[1, \infty[$,

$$
\mathbb{E}\left(\left(d_{F}\left(f\left(\sigma_{n}\right), f\left(T_{\bar{G}_{n}^{\prime}}^{\underline{d}\left(\sigma_{n}\right)}\left(\sigma_{n}\right)\right)\right)\right)^{p}\right)=\mathbb{E}\left(\left(\underline{\varepsilon}_{n, \operatorname{Class}\left(\sigma_{n}\right), p}^{\prime}\right)^{p}\right) .
$$

[^10]Proof. For any random variable $\sigma_{n}$, we have

$$
\begin{aligned}
\mathbb{E}\left(\left(d_{F}\left(f\left(\sigma_{n}\right), f\left(T_{\bar{G}_{n}^{\prime}}^{d\left(\sigma_{n}\right)}\left(\sigma_{n}\right)\right)\right)\right)^{p}\right) & =\mathbb{E}\left(\mathbb{E}\left(\left(d_{F}\left(f\left(\sigma_{n}\right), f\left(T_{\bar{G}_{n}^{\prime}}^{d\left(\sigma_{n}\right)}\left(\sigma_{n}\right)\right)\right)\right)^{p} \mid \sigma_{n}\right)\right) \\
& =\sum_{i \in I_{n}} \sum_{\sigma \in V_{n}^{i}} \mathbb{P}\left(\sigma_{n}=\sigma\right) \mathbb{E}\left(\left(d_{F}\left(f\left(\sigma_{n}\right), f\left(T_{\bar{G}_{n}^{\prime}}^{d\left(\sigma_{n}\right)}\left(\sigma_{n}\right)\right)\right)\right)^{p} \mid \sigma_{n}=\sigma\right) .
\end{aligned}
$$

If $\left(\sigma_{n}\right)_{n \geq 1}$ is $G$-invariant, then $\mathbb{P}\left(\sigma_{n}=\sigma\right)=\frac{1}{\operatorname{card}(\operatorname{Class}(\sigma))} \mathbb{P}\left(\operatorname{Class}\left(\sigma_{n}\right) \operatorname{Class}(\sigma)\right)$. Moreover, under (31),

$$
\mathbb{E}\left(\left(d_{F}\left(f\left(\sigma_{n}\right), f\left(T_{\bar{G}_{n}^{\prime}}^{d\left(\sigma_{n}\right)}\left(\sigma_{n}\right)\right)\right)^{p} \mid \sigma_{n}=\sigma\right)=\frac{\sum_{\rho \in \operatorname{final}(\sigma)}\left(d_{F}(f(\sigma), f(\rho))\right)^{p}}{\operatorname{card}(\operatorname{final}(\sigma))}\right.
$$

Consequently, one can conclude since

$$
\begin{aligned}
\mathbb{E}\left(\left(\underline{\varepsilon}_{n, \operatorname{Class}\left(\sigma_{n}\right), p}^{\prime}\right)^{p}\right)= & \mathbb{E}\left(\mathbb{E}\left(\left(\underline{\varepsilon}_{n, \operatorname{Class}\left(\sigma_{n}\right), p}^{\prime}\right)^{p}\right) \mid \operatorname{Class}\left(\sigma_{n}\right)\right) \\
& =\sum_{i \in I_{n}} \mathbb{P}\left(\operatorname{Class}\left(\sigma_{n}\right)=i\right)\left(\underline{\varepsilon}_{n, i, p}^{\prime}\right)^{p} \\
& =\sum_{i \in I_{n}} \mathbb{P}\left(\operatorname{Class}\left(\sigma_{n}\right)=i\right) \sum_{\sigma \in V_{n}^{i}} \sum_{p \in \operatorname{final}(\sigma)} \frac{\left(d_{F}(f(\sigma), f(\rho))\right)^{p}}{\operatorname{card}\left(V_{n}^{i}\right) \operatorname{card}(\operatorname{final}(\sigma))} .
\end{aligned}
$$

Similarly, one can prove the following.
Lemma 41. Under (27)-(30), $\left(\sigma_{n}\right)_{n \geq 1}$ is $G$-invariant, for $n \geq 1$, for any $\mathbb{P} \in[1, \infty[$,

$$
\mathbb{E}\left(\left(d_{F}\left(f\left(\sigma_{n}\right), f\left(T_{G_{n}^{\prime}}\right)\left(\sigma_{n}\right)\right)\right)^{p}\right)=\mathbb{E}\left(\left(\underline{\varepsilon}_{n, \text { Class }\left(\sigma_{n}\right), p}\right)^{p}\right)
$$

This gives as a universality result.
Theorem 42. Assume that (27)-(30) and that $\left(\sigma_{n}\right)_{n \geq 1}$ and $\left(\sigma_{r e f, n}\right)_{n \geq 1}$ are $G$-invariant. Suppose that there exists some deterministic $x \in F$ and $p \in[1, \infty[$ such that

$$
\begin{array}{cc}
f\left(\sigma_{r e f, n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x & \left(\text { resp. } f\left(\sigma_{\text {ref }, n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} x\right), \\
\underline{\varepsilon}_{n, \text { Class }\left(\sigma_{\text {ref }, n}\right), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 & \left(\text { resp. } \underline{\varepsilon}_{n, \text { Class }\left(\sigma_{r e f, n}\right), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 0\right) \tag{32}
\end{array}
$$

and

$$
\begin{equation*}
\underline{\varepsilon}_{n, \text { Class }\left(\sigma_{n}\right), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad\left(\text { resp. } \underline{\varepsilon}_{n, \text { Class }\left(\sigma_{n}\right), \infty}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 0\right) . \tag{33}
\end{equation*}
$$

Then

$$
f\left(\sigma_{n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} x \quad\left(\text { resp. } f\left(\sigma_{n}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} x\right) .
$$

Moreover, under (31), (32) and (33) can be replaced by

$$
\underline{\varepsilon}_{n, \text { Class }\left(\sigma_{r e f, n}\right), 1}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad\left(\operatorname{resp} . \underline{\varepsilon}_{n, \operatorname{Class}\left(\sigma_{r e f, n}\right), p}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 0\right)
$$

and

$$
\underline{\varepsilon}_{n, \operatorname{Class}\left(\sigma_{n}\right), 1}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad\left(\text { resp. } \underline{\varepsilon}_{n, \operatorname{Class}\left(\sigma_{n}\right), p}^{\prime}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^{p}} 0\right) .
$$

Idea of the proof. The proof is identical to that of theorems 1 and 2. Indeed, (29) guarantees that under the $G$-invariance, for any $n \geq 1, T_{G_{n}^{\prime}}\left(\sigma_{n}\right)$ is $G_{n}$ invariant and by construction almost surely

$$
\underline{d}\left(T_{G_{n}^{\prime}}\left(\sigma_{n}\right)\right)=\max \left(0, \underline{d}\left(\sigma_{n}\right)-1\right) .
$$

Consequently, by induction, $T_{\bar{G}_{n}^{\prime}}^{d\left(\sigma_{n}\right)}\left(\sigma_{n}\right)$ is distributed according to the uniform distribution on $V_{n}^{i_{n}^{*}}$ and almost surely

$$
d_{F}\left(f\left(T_{G_{n}^{\prime}}^{\underline{d}\left(\sigma_{n}\right)}\left(\sigma_{n}\right)\right), f\left(\sigma_{n}\right)\right) \leq \underline{\varepsilon}_{n, \operatorname{class}\left(\sigma_{n}\right), \infty}(f) .
$$

Similarly to Remark 4, by the triangle inequality and using that the arithmetic mean is smaller than the $p$-mean, we have ${ }^{16}$

$$
\left(\underline{\varepsilon}_{n, k, p}(f)\right)^{p} \leq \sum_{i=1}^{\underline{d}(k)} \max _{j ; \underline{d}(j)=i}\left(\underline{\varepsilon}_{n, j, p}^{p}(f)\right) \leq \underline{d}(k) \underline{\varepsilon}_{n, p}^{p}(f) .
$$

Consequently, if there exists $\alpha>0$ such that

$$
\underline{\varepsilon}_{n, p}^{p}(f)=O\left(\frac{1}{n^{\frac{1}{\alpha}}}\right),
$$

then one can obtain (32) and (33) for the equivalent classes of $\left(\mathcal{H}_{G-i n v, \alpha}^{\mathbb{P}}\right)\left(\operatorname{resp} .\left(\mathcal{H}_{G-i n v, \alpha}^{\mathbb{L}^{p}}\right)\right)$.

### 4.2 Some examples of finite graphs

In general, Cayley graphs are good candidates. An interesting case is when there exists $\left(i_{n}^{*}\right)_{n \geq 1} \in \prod_{n \geq 1} I_{n}$ such that

$$
\frac{1}{\operatorname{card}\left(V_{n}\right)} \sum_{\sigma \in V_{n}} \min _{\sigma^{\prime} \in V_{n}^{i_{n}^{*}}} d_{G_{n}}\left(\sigma, \sigma^{\prime}\right)=o\left(\max _{\sigma_{1}, \sigma_{2} \in V_{n}} d_{G_{n}}\left(\sigma_{1}, \sigma_{2}\right)\right),
$$

in this case, the comparison with the uniform distribution can be done for reasonable statistics. The first four examples we give are different ways to apply our results to the symmetric group. The other four examples are different graphs. Our eight examples satisfy (27)- (31). In the first two examples we will give in details the different objects, for the other we will give only $G_{n}, I_{n} V_{n}^{i}$ and $i_{n}^{*}$. The others can be obtained easily by applying the definitions.

- The Cayley graph of symmetric group generated by transpositions: We recall that
$-\left(V_{n}, E_{n}^{\prime}\right)=\mathcal{G}_{\mathfrak{S}_{n}}$,
- $I_{n}=\mathbb{Y}_{n}$,
$-V_{n}^{i}=\left\{\sigma \in \mathfrak{S}_{n} ; \hat{\lambda}(\sigma)=i\right\}$,
$-\operatorname{Class}(\sigma)=\hat{\lambda}(\sigma)$,
$-i_{n}^{*}=(n, \underline{0})$ the Young diagram with a unique row of length $n$,
$-\underline{d}(\sigma)=\#(\sigma)-1$,

[^11]We have then the following.

$$
\begin{aligned}
\frac{1}{\operatorname{card}\left(V_{n}\right)} \sum_{\sigma \in V_{n}} \min _{\sigma^{\prime} \in V_{n}^{*}} d_{G_{n}}(\sigma, \sigma *) & =\mathbb{E}\left(\#\left(\sigma_{u n i f, n}\right)-1\right) \\
& =\sum_{k=2}^{n} \frac{1}{k}=o(n-1)=o\left(\max _{\sigma_{1}, \sigma_{2} \in V_{n}} d_{G_{n}}\left(\sigma_{1}, \sigma_{2}\right)\right) .
\end{aligned}
$$

- Even permutations: A permutation $\sigma \in \mathfrak{S}_{n}$ is said to be even if $n-\#(\sigma)$ is even. Cycles of length 3 are a generator of $\mathfrak{S}_{n}$. When $n$ is odd, $\mathfrak{S}_{n}^{0}$ is a subset of the set of even permutations. One can choose for example.
- $G_{n}$ the Cayley graph of $\mathfrak{S}_{2 n+1}$ generated by cycles of length 3
$-I_{n}=\left\{\lambda \in \mathbb{Y}_{2 n+1} ; \ell(\lambda) \equiv 1(\bmod 2)\right\}$,
$-V_{n}^{i}=\left\{\sigma \in \mathfrak{S}_{2 n+1}, \hat{\lambda}(\sigma)=i\right\}$,
$-\operatorname{Class}(\sigma)=\hat{\lambda}(\sigma)$,
$-i_{n}^{*}=(2 n+1, \underline{0})$,
$-\underline{d}(\sigma)=\frac{\#(\sigma)+1}{2}$.
- $\mathfrak{S}_{n}$ seen as a Coxeter group: Here we take the right (or the left) Cayley graph generated by transpositions of type $(i, i+1) .{ }^{17}$ In this case we have:
- $G_{n}$ the right (or the left) Cayley graph of $\mathfrak{S}_{n}$ generated by $\{(i, i+1) ; 1 \leq i \leq n-1\}$.
$-I_{n}=\left\{0,1, \ldots, \frac{n(n-1)}{2}\right\}$,
- $V_{n}^{i}=\left\{\sigma ; \mathcal{K}_{2}(\sigma)=i\right\}$, where we recall that $\mathcal{K}_{2}(\sigma)$ is the number of inversions of $\sigma$.
$-\operatorname{Class}(\sigma)=\mathcal{K}_{2}(\sigma)$,
$-i_{n}^{*}=\left\lceil\frac{n^{2}}{4}\right\rceil$,
$-\underline{d}(\sigma)=\left|\left\lceil\frac{n^{2}}{4}\right\rceil-\mathcal{K}_{2}(\sigma)\right|$.
For example, $G_{3}^{\prime}$ is represented in Figure 6. Corollary 29 guarantees that $i_{n}^{*}=\left\lceil\frac{n^{2}}{4}\right\rceil$ is a good candidate if we want to compare with the uniform distribution. But also it is possible to choose $i_{n}^{*}=0$ when looking for universality results for random permutations close to the identity. For this graph, the Mallows law with Kendall tau distance is $G_{n}$-invariant and one can obtain a first order universality for all local statistics we already studied in the previous sections and for the limiting shape ${ }^{18}$. The second order fails.
- Using the same previous graph (same $G_{n}$ ) but with only two classes even and odd permutations ${ }^{19}$ i.e. $I_{n}=\{$ even, odd $\}$ we obtain that, if $f\left(\sigma_{n}\right)$ converges in probability (or $\mathbb{L}^{1}$ ) when $\sigma_{n}$ follows one of these three distributions
- Uniform law of $\mathfrak{S}_{n}$

[^12]

Figure 6: $G_{3}^{\prime}$ obtained by the transpositions $(1,2)$ and $(2,3)$

- Uniform law of even permutations
- Uniform law of odd permutations
it converges also for the two others as soon as

$$
\frac{\min \left(\sum_{\sigma \in \mathfrak{S}_{n}, 1 \leq i<n} d_{F}(f(\sigma \circ(i, i+1)), f(\sigma)) ; \sum_{\sigma \in \mathfrak{S}_{n}, 1 \leq i<n} d_{F}(f((i, i+1) \circ \sigma), f(\sigma))\right)}{n!(n-1)}=o(1) .
$$

- Another possible application is the hypercube $(\mathbb{Z} / 2 \mathbb{Z})^{2 n}$. In this case, we set
- $G_{n}=(\mathbb{Z} / 2 \mathbb{Z})^{2 n}$
- $I_{n}=\{0,1, \ldots, 2 n\}$
- $V_{n}^{i}$ is the set of edges of the graph such that the graph distance from $(0, \ldots, 0)$ is $i$.
$-i_{n}^{*}=n$.
In this case,

$$
\begin{aligned}
\frac{1}{\operatorname{card}\left(V_{n}\right)} \sum_{\sigma \in V_{n}} \min _{\sigma^{\prime} \in V_{n}^{i^{*}}} d_{G_{n}}(\sigma, \sigma *) & =\frac{\left.\sum_{k=0}^{2 n} \left\lvert\, \begin{array}{c}
2 n \\
k
\end{array}\right.\right)(k-n) \mid}{4^{n}} \\
& \leq \sqrt{\frac{\sum_{k=0}^{2 n}\binom{2 n}{k}(k-n)^{2}}{4^{n}}} \\
& =\sqrt{\frac{n}{2}} \\
& =o(n)=o\left(\max _{\sigma_{1}, \sigma_{2} \in V_{n}} d_{G_{n}}\left(\sigma_{1}, \sigma_{2}\right)\right) .
\end{aligned}
$$

- $(\mathbb{Z} / d \mathbb{Z})^{n d}$ : Let $\mathcal{R}_{n}$ be the equivalent relation defined as follows: For any

$$
x=\left(x_{i}\right)_{1 \leq i \leq n d}, y=\left(y_{i}\right)_{1 \leq i \leq n d} \in(\mathbb{Z} / d \mathbb{Z})^{n d}, x \mathcal{R}_{n} y \Leftrightarrow \exists \sigma \in \mathfrak{S}_{n d}, y=\left(x_{\sigma(i)}\right)_{1 \leq i \leq n d} .
$$

$\mathcal{R}_{n}$ define naturally the classes of the vertices. The central limit theorem guarantees that the class $i_{n}^{*}$ where we have exactly $n$ coordinates equal to $k$ for any $k$ in $\mathbb{Z} / d \mathbb{Z}$. is a good candidate ${ }^{20}$.

- Let $\left(H_{n}\right)_{n \geq 1}$ be a sequence of non-commutative and finite groups and $\left(A_{n}\right)_{n \geq 1}$ such that $A_{n}$ is a conjugation invariant subset of $H_{n}{ }^{21}$.
- $G_{n}$ be the Cayley graph generated by $H_{n}$.
- $I_{n}$ is the set of conjugacy classes
$-V_{n}^{i}=i$
In this case, $G$-invariant random variables are conjugation invariant variables. The choice of $i_{n}^{*}$ is specific to the choice of $G_{n}$.
- Dihedral group $\mathbb{D}_{2 n}{ }^{22}$ with $n \geq 3$ : The Dihedral group $\mathbb{D}_{2 n}$ is defined via its representation $<$ $\sigma, \mu \mid \sigma^{2}, \mu^{2},(\mu \sigma)^{n}>^{23}$. This representation shows that $\mathbb{D}_{2 n}$ is a Coxeter group. For our study, one can admit that

$$
\mathbb{D}_{2 n}=\left\{\mathrm{s}_{0}, \ldots, \mathrm{~s}_{n-1}, \mathrm{r}_{0}, \ldots, \mathrm{r}_{n-1}\right\}
$$

and

$$
\mathrm{r}_{i} \mathrm{r}_{j}=\mathrm{r}_{i+j}, \quad \mathrm{r}_{i} \mathrm{~s}_{j}=\mathrm{s}_{i+j}, \quad \mathrm{~s}_{i} \mathrm{r}_{j}=\mathrm{s}_{i-j}, \quad \mathrm{~s}_{i} \mathrm{~s}_{j}=\mathrm{r}_{i-j} .
$$

Here, $(i, j)$ are in $\mathbb{Z} / n \mathbb{Z}$. One can choose either

- $G_{n}$ : the Cayley graph generated by $\left\{s_{i}, 0 \leq i \leq n\right\}$,
- $I_{n}=\{\mathrm{r}, \mathrm{s}\}$,
- $V_{n}^{s}=\left\{s_{i}, 0 \leq i \leq n\right\}$ is the set of transpositions and $V_{n}^{r}=\left\{r_{i}, 0 \leq i \leq n\right\}$ is the set of rotations,
$-i_{n}^{*}=r$
or keep the same graph and choose conjugacy classes as classes (as in the previous examples) ${ }^{24}$. In the second case, we require that $f\left(\sigma_{n}\right)$ convergences for any sequence of rotations and a transposition does not change a lot the statistic.
- Colored permutations: A less trivial example is the set of signed permutations and more generally the set of colored permutations. Given two positive integers $n$ and $m$, a colored permutation is a map $\pi=(\sigma, \phi)$ such that $\sigma \in \mathfrak{S}_{n}$ and $\phi \in\{1, \ldots, n\}^{\{1, \ldots, m\}}$. A subsequence $\pi\left(x_{1}\right), \ldots, \pi\left(x_{k}\right)$ of $\pi$ is called increasing of length $m(k-1)+p$ if $\sigma\left(x_{1}\right)<\sigma\left(x_{2}\right)<\cdots<\sigma\left(x_{k}\right)$ and $\phi\left(x_{1}\right)=\phi\left(x_{2}\right)=\cdots=\phi\left(x_{k}\right)=p$. We denote by $\operatorname{LIS}(\pi)$ the length of a longest increasing subsequence.

[^13]Theorem 43. Let $\left(\pi_{n}=\left(\sigma_{n}, \phi_{n}\right)\right)_{n \geq 1}$ be a sequence of random colored permutations and assume that:
$-\sigma_{n}$ is independent of $\phi_{n}$,

- $\phi_{n}$ is distributed according to the uniform distribution,
$-\sigma_{n}$ is conjugation invariant,
$-\frac{\# \sigma_{n}}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0$.
then,

$$
\begin{equation*}
\mathbb{P}\left(\frac{\operatorname{LIS}\left(\pi_{n}\right)-2 \sqrt{n m}}{m^{\frac{2}{3}} \sqrt[6]{n m}}<s\right) \rightarrow F_{2}^{m}(s) \tag{34}
\end{equation*}
$$

Proof. The uniform case is proved by Borodin [1999]. To apply our theorem, choose the graph where two colored permutations are related by an edge if only the first components differ by a transposition i.e.
$-E_{n}:=\{((\sigma, \phi),(\sigma \circ(i, j), \phi)) ; i \neq j\}$,
$-I_{n}=\mathbb{Y}_{n}$,
$-V_{n}^{i}=\{(\sigma, \phi) ; \bar{\sigma}=i\}$,
$-i_{n}^{*}=(n, \underline{0})$.

For our examples, a trivial example of $G$-invariant elements is the uniform measure, or the uniform measure on a given class. Since $\underline{d}$ is constant in classes, a natural way to generalize Ewens measures is the following. Given $q \in \mathbb{R}_{+}$, the probability measure satisfying

$$
\mathbb{P}\left(\rho_{G, q, n}=\sigma\right)=\frac{q^{\underline{d}(\sigma)}}{\sum_{\sigma^{\prime} \in V_{n}} q^{\underline{d}\left(\sigma^{\prime}\right)}}
$$

is $G$-invariant and for any statistic $f$ such that $f\left(\rho_{G, 0, n}\right)$ converges, one can obtain a non-empty universality result around $\rho_{G, 0, n}$ since

$$
\operatorname{err}_{n}:=q \mapsto \mathbb{E}\left(d_{F}\left(f\left(T^{\underline{d}\left(\rho_{G, q, n}\right)}\left(\rho_{G, q, n}\right), f\left(\rho_{G, q, n}\right)\right)\right)\right)
$$

is continuous and $\operatorname{err}_{n}(0)=0$. In fact, in the case of permutations, Ewens and Mallows measures with Kendall tau distance are particular case of $\rho_{G, q, n}$.

### 4.3 Infinite case

We take now $G_{n}=G$ an infinite graph. Example of "nice graphs":

- The infinite $d$-regular tree $\mathfrak{T}_{d}$.
- The set of words of a finite alphabet of length $d$.
- The free group $\mathcal{F}_{d}$ with its natural Cayley graph.
- The Cayley graph of $\mathcal{B}_{d}$, the Artin Braid group.
- The Cayley graph of an infinite and finitely generated group $\left.H=<x_{1}, x_{2}, \ldots, x_{n}\right\rangle$.

The classes here are indexed by $\mathbb{N}$ according to the distance to the root (or the identity). Let $G$ be such that

$$
0<\liminf _{n \rightarrow \infty} \frac{\log (\operatorname{card}(\{x ; \underline{d}(x)=n\}))}{n}=\limsup _{n \rightarrow \infty} \frac{\log (\operatorname{card}(\{x ; \underline{d}(x)=n\}))}{n}=\log (\lambda)<\infty .
$$

It is the case for the first three examples. Let $f$ be a statistic such that $f\left(\sigma_{n}\right)$ convergences for the uniform law on $V^{n}=V_{n}^{n}$ and $\sum_{i=1}^{\infty} \underline{\varepsilon}_{n, i, \infty}^{\prime}(f)<\infty$. We obtain then that $f\left(\sigma_{n}\right)$ converges for the Mallows law when its parameter goes to $\lambda$. More generally, it converges for any distribution such that Class $(i)$ converges in probability to infinity.

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## A Ewens measures

Definition 44. Let $\theta$ be a non-negative real number. We say that a random permutation $\sigma_{E w, \theta, n}$ follows the Ewens distribution with parameter $\theta$ if for all $\sigma \in \mathfrak{S}_{n}$,

$$
\begin{equation*}
\mathbb{P}\left(\sigma_{E w, \theta, n}=\sigma\right)=\frac{\theta^{\#(\sigma)-1}}{\prod_{i=1}^{n-1}(\theta+i)} . \tag{35}
\end{equation*}
$$

Note that when $\theta=1$, the Ewens distribution is just the uniform distribution on $\mathfrak{S}_{n}$, whereas when $\theta=0$ we have the uniform distribution on permutations having a unique cycle. For general $\theta$, the Ewens distribution is clearly conjugation invariant since it only involves the cycle structure of $\theta$.

We want to recall the interpretation of the Ewens distribution via a nice stochastic process known as "the Chinese restaurant process". Suppose that there are an infinite number of circular tables with infinite capacity.

- At $t=0$, all tables are empty.
- At $t=1$, the person " 1 " comes and sits in the first table.

- At $t=2$, the person " 2 " comes and sits in the table near person 1 with probability $\frac{1}{1+\theta}$

and sits alone in a new table with probability $\frac{\theta}{1+\theta}$.

- At $t=n$, the person " $n$ " comes, she/he chooses to sit alone in a new table with probability $\frac{\theta}{\theta+n-1}$ and in an occupied table $i$ with probability $\frac{\left|B_{i}\right|}{\theta+n-1}$, where $B_{i}$ is the number of persons at the table $i$. In this case, she/he chooses her/his position uniformly in gaps between two persons.

For example, if we have the following configuration ${ }^{25}$,


[^14]at $t=5$, the probability to switch to each of the following configurations



is $\frac{1}{\theta+4}$ and the probability to switch to

is equal to $\frac{\theta}{4+\theta}$.
To obtain the associated permutation to a configuration one reads the elements on each non-empty circle counterclockwise to get a cycle. For example, to the configuration

we associate the permutation $(1,4,2)(3,5)$.
Using the Chinese restaurant process description of the Ewens distribution, it is obvious to see that the number of cycles $\#\left(\sigma_{E w, \theta, n}\right)$ is the sum of $n$ independent Bernoulli random variables with parameters $\left\{\frac{\theta}{\theta+i}\right\}_{0 \leq i \leq n-1}$. For further reading, we recommend [Aldous, 1985, McCullagh, 2011, Chafaï et al., 2013]. In particular, we have the following classic result.

## Proposition 45.

$$
\mathbb{E}\left(\#_{1}\left(\sigma_{E w, \theta, n}\right)\right)=\frac{n \theta}{n-1+\theta} \quad \text { and } \quad \mathbb{E}\left(\#\left(\sigma_{E w, \theta, n}\right)\right)=1+\sum_{i=2}^{n} \frac{\theta}{i-1+\theta} \leq 2+\theta \log (n) .
$$

In particular, for uniform distribution, we have

$$
\begin{equation*}
\frac{\#\left(\sigma_{u n i f, n}\right)}{\log (n)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 1 \tag{36}
\end{equation*}
$$

Proof. Since the number of cycles of the uniform law is the sum of $n$ independent random Bernoulli variables of parameters $1, \frac{1}{2}, \ldots, \frac{1}{n}$ and using Chebyshev's inequality, we obtain

$$
\mathbb{P}\left(\left|\frac{\#\left(\sigma_{u n i f, n}\right)}{\log (n)}-1\right|>\alpha\right) \leq \frac{\frac{\sum_{i=1}^{n} \frac{i-1}{i^{2}}}{\log (n)^{2}}}{\left(\alpha+1-\frac{\sum_{i=1}^{n} \frac{1}{i}}{\log n}\right)^{2}}=O\left(\frac{1}{\log (n)}\right) .
$$

Remark 46. This convergence holds almost surely. The proof uses martingale techniques. One can find a proof of this result in [Chafaï et al., 2013].

One can now apply our results using the following two results.
Corollary 47. Let $\left(\theta_{n}\right)_{n \geq 1}$ be a sequence of non-negative real numbers such that:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\theta_{n} \log (n)}{n^{\frac{1}{\alpha}}}=0 \tag{37}
\end{equation*}
$$

Then $\left(\sigma_{E w, \theta_{n}, n}\right)_{n \geq 1}$ satisfies $\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{P}}\right)$.
Proposition 48. For any $\theta \geq 0, \alpha>0$ and $p \geq\left[1, \infty\left[,\left(\sigma_{E w, \theta, n}\right)_{n \geq 1}\right.\right.$ satisfies $\left(\mathcal{H}_{i n v, \alpha}^{\mathbb{T}^{p}}\right)$.
Proof. Using Bernstein inequality, if $\theta \geq 1$,

$$
\begin{aligned}
\mathbb{P}\left(\#\left(\sigma_{E w, \theta, n}\right)>(3 p+1) \theta \log (n)+2\right) & \leq \mathbb{P}\left(\#\left(\sigma_{E w, \theta, n}\right)>\mathbb{E}\left(\#\left(\sigma_{E w, \theta, n}\right)\right)+3 p \theta \log (n)\right) \\
& \leq \exp \left(\frac{-\frac{9}{2} \theta^{2} p^{2} \log (n)^{2}}{\operatorname{var}\left(\#\left(\sigma_{E w, \theta, n}\right)\right)+\frac{3 p}{3} \theta \log (n)}\right) \\
& \leq \exp \left(\frac{-\frac{9}{2} \theta^{2} p^{2} \log (n)^{2}}{(p+1) \theta \log (n)+2}\right)=O\left(n^{-\frac{9}{4} \theta \frac{p^{2}}{p+1}}\right)=O\left(n^{-\frac{9 p}{8}}\right) .
\end{aligned}
$$

Consequently,

$$
\mathbb{E}\left(\#\left(\sigma_{E w, \theta, n}\right)^{p}\right) \leq((3 p+1) \theta \log (n)+2)^{p}+n^{p} O\left(n^{-\frac{9 p}{8}}\right)=O\left(\log ^{p}(n)\right) .
$$

When $\theta<1$, one can conclude since $\mathbb{E}\left(\#\left(\sigma_{E w, \theta, n}\right)^{p}\right)<\mathbb{E}\left(\#\left(\sigma_{E w, 1, n}\right)^{p}\right)$.


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    ${ }^{\dagger}$ Partially supported by a Leverhulme Trust Research Project Grant RPG-2020-103.

[^1]:    ${ }^{1}$ See Appendix A for more details.

[^2]:    ${ }^{2}$ In the space of continual diagrams i.e. the set of 1-Lipschitz real functions $f$ such that outside one compact, $f(x)=$ $|x-a|$. One can see [Kerov, 1993, Sodin, 2017] for more details for continual diagrams. We will use as distance, $d_{F}(f, g)=$ $\sup _{x \in \mathbb{R}}|f(x)-g(x)|$ which is finite since both functions are continuous and outside one compact of $\mathbb{R}, f-g$ is constant.
    ${ }^{3}$ Slightly different Markov operators have already been studied in [Kammoun, 2018, 2020], we modify a little the two operators presented in the cited papers to obtain a uniform random walk easy to generalize to other sets. The three operators coincide when $n \leq 3$.

[^3]:    ${ }^{4}$ After all, a drunk and lost man who is driving on a two-way road (the Cayley graph of $\mathfrak{S}_{n}$ ) needs $n \log (n)$ steps to be close to his destination and will never attend it but if he drives in a one-way road, he needs at most $n$ step to be sure to arrive to destination. In both cases, it is dangerous for a drunk man to drive.

[^4]:    ${ }^{5}$ There is a language abuse here: a longest increasing subsequence may not be unique but its length is always defined.

[^5]:    ${ }^{6}$ In the literature, $j$-exceedances is sometimes defined by the condition $\sigma_{i} \geq i+j$ and othertimes by $\sigma_{i}=i+j$. In both cases, the number $j$-exceedances is a local statistic but only the first case is in interest for our purpose.

[^6]:    ${ }^{7}$ Fun fact 1: the application $\sigma \mapsto \mathfrak{G}(\sigma)$ is injective.
    ${ }^{8}$ This a special case of the number of occurrences of a pattern in a permutation. In general, the number of occurrences of any pattern is a local statistic.

[^7]:    ${ }^{9}$ We recall that $\mathfrak{S}_{n}^{0}$ is the set of cyclic permutations.
    ${ }^{10} \hat{\lambda}(\sigma)$ is the cycle structure of $\sigma$.
    ${ }^{11}$ Here we define a different Markov operator for every distribution.

[^8]:    ${ }^{12}$ We use the usual notations i.e. $V_{n}$ is the set of vertices and $E_{n}$ is the set of edges.
    ${ }^{13}$ We use $\sqcup$ to denote disjoint union.

[^9]:    ${ }^{14} \mathcal{G}_{\mathfrak{G}_{n}}$ is defined in Section 1

[^10]:    ${ }^{15}$ we omit this precision when it is clear from the context.

[^11]:    ${ }^{16}$ There is here a notation abuse. Since $\underline{d}$ is constant in any class, we denote by $\underline{d}(k), \underline{d}(\sigma)$ for some $\sigma \in k$.

[^12]:    ${ }^{17}$ Fun fact 2: depending on the choose of the right or the left composition, one can obtain a different universality theorem. The classes are the same but the graph (and consequently error controls) are different.
    ${ }^{18}$ We apologize again to the reader because it is not defined yet.
    ${ }^{19}$ Here, the choice of $i_{n}^{*}$ is not important but the reader can take $i_{n}^{*}=$ even.

[^13]:    ${ }^{20}$ Fun fact 3: by choosing fixed and different proportions of every element of $\mathbb{Z} / d \mathbb{Z}$ for $i_{n}^{*}$, one can obtain different universality result.
    ${ }^{21}$ i.e. if $\sigma \in A_{n}$ then $\bar{\sigma} \subset H_{n}$.
    ${ }^{22}$ This is a typical "bad" Cayley graph since its diameter is bounded (equal to 2 ) and consequently the universality result is trivial.
    ${ }^{23}$ This notation is classic to define groups. It means in our case that $\mathbb{D}_{2 n}$ is isomorphic to the group generated by $\sigma$ and $\mu$ such that $\sigma^{2}=\mu^{2}=(\mu \sigma)^{n}=1$
    ${ }^{24}$ There are $n+1$ or $n+2$ depending on the parity of $n$.

[^14]:    ${ }^{25}$ we omitted empty tables

