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SLE scaling limits for a Laplacian growth model

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Abstract

We consider a model for planar random growth in which growth on the cluster is concentrated in areas of low harmonic measure. We find that when the concentration is sufficiently strong, the resulting cluster converges to an SLE_4 curve as the size of individual particles tends to 0.

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1 Introduction

1.1 Conformal aggregation

There are many models of random aggregation studied by probabilists: typically we construct a cluster by beginning with some simple initial configuration and adding extra mass at random locations over time. Many of these models involve growth on a lattice, such as diffusion-limited aggregation (DLA) [21] and internal diffusion-limited aggregation [11]; and first passage percolation (FPP) including the Eden model [4]. These models have been used to describe many real-world phenomena.

We are often interested in investigating the large-scale behaviour of these clusters when many particles have been added, such as the result of [11] that an IDLA cluster with many particles, suitably rescaled, converges to a Euclidean disc. However, with models defined on a lattice we often see *anisotropy* in the large-scale behaviour, reflecting the underlying anisotropy of $\mathbb{Z}^d$. For example, it has been shown [2] that a first passage percolation cluster on $\mathbb{Z}^d$ does not converge to a Euclidean ball if d is large enough, and evidence from simulations [5] suggests that large DLA clusters in $\mathbb{Z}^2$ retain the anisotropy of the lattice.

We use models of *conformal growth* to overcome this problem by working in a space without any underlying anistropy, in this case the complex plane $\mathbb{C}$. We typically start with an isotropic seed, such as the disc, and then add particles at positions chosen according to a probability distribution without any natural anisotropy. In this paper we will study the aggregate Loewner evolution (ALE(α, η)) model introduced in [20], which is a generalisation of the Hastings-Levitov process (HL(α)) [7].

In a conformal aggregation model, we add particles to our initial configuration by composing conformal maps from a fixed reference domain to smaller domains. Our initial cluster will be the closed unit disc $K_0 = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. We attach a particle to K_0 by applying a map from $\Delta := \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ to a smaller domain (and then the new cluster will be the complement of the image of Δ). Let $P \subseteq \Delta$ be such that $P \cup \overline{\mathbb{D}}$ is a compact (closed and bounded) and simply-connected subset of $\mathbb{C}$, the point 1 is on the boundary of P, and P is symmetric in the real axis, i.e. $z \in P \iff z^* \in P$. Throughout this paper we will use one-dimensional slits as our particles, i.e. P of the form $(1, 1+d]$, for some $d > 0$.

Definition 1. For a particle shape P as above, by the Riemann mapping theorem there exists a unique bijective conformal map (or *univalent function*)

$$f \colon \Delta \to \Delta \setminus P$$

such that $f(z) = e^{\mathbf{c}} z + O(1)$ near ∞, for some $\mathbf{c} = \mathbf{c}(P) \in \mathbb{R}$.

One benefit of the particle shape $P = (1, 1+d]$ we use in this paper is that we have an explicit expression for $f(z)$ (see [15]). For non-empty P, we call $\mathbf{c}(P) > 0$ the *(logarithmic) capacity* of P. As the name suggests, we can view $\mathbf{c}$ as measuring the "size" of a set in a certain sense.

Definition 2. We will parameterise our function f for $P = (1, 1+d]$ using the capacity $\mathbf{c} = \mathbf{c}(P)$ rather than the length d. Then the length is given by the relationship $4e^{\mathbf{c}} = \frac{(d+2)^2}{d+1}$ (asymptotically $d(\mathbf{c}) \sim 2\mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$), and the preimage of the particle P is $\{e^{i\theta} : -\beta \leq \theta \leq \beta\}$ where $0 < \beta(\mathbf{c}) < \pi$ is uniquely determined by $f(e^{i\beta}) = 1$. We can explicitly calculate β using the expression $e^{i\beta} = 2e^{-\mathbf{c}} - 1 + 2ie^{-\mathbf{c}}\sqrt{e^{\mathbf{c}} - 1}$. Asymptotically $\beta(\mathbf{c}) \sim d(\mathbf{c}) \sim 2\mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$. These explicit expressions are found in [15] and [20].

There is one property of logarithmic capacity which makes it a useful parameterisation for growth models: if P_1 and P_2 are two particles as above with corresponding maps $f_j \colon \Delta \to \Delta \setminus P_j$, then, near infinity, $(f_1 \circ f_2)(z) = e^{\mathbf{c}(P_1)+\mathbf{c}(P_2)}z + O(1)$. Therefore the particle $P_1 \cup f_1(P_2)$ has capacity $\mathbf{c}(P_1) + \mathbf{c}(P_2)$. Although P_2 has been distorted by the later application of f_1, we think of applying the map $f_1 \circ f_2$ as "first attaching P_1, and then attaching P_2". In particular, if the particles we add each have the same capacity $\mathbf{c}$, (in fact we will always attach copies of the same particle at random positions) then the total capacity of the cluster after we have added n particles is $n\mathbf{c}$.

We have maps which can attach one particle, so now we want to be able to build a cluster with multiple particles by composing maps which attach particles in different positions. Fix a particle P as above with $1 \in \overline{P}$, and the corresponding map $f \colon \Delta \to \Delta \setminus P$, with capacity $\mathbf{c} = \mathbf{c}(P)$. We will need to be able to attach the particle P at any point on the circle, not just $e^{i0} = 1$, so for $\theta \in \mathbb{R}$, define the rotated map

$$\begin{aligned} f \colon \Delta &\to \Delta \setminus e^{i\theta}P \\ f^{\theta}(z) &= e^{i\theta} f(e^{-i\theta}z), \end{aligned}$$

and note that it has the same property $f^{\theta}(z) = e^{\mathbf{c}}z + O(1)$ near ∞.

Now we want to attach multiple particles.

Definition 3. Given a sequence of angles $(\theta_n)_{n\in\mathbb{N}}$, write $f_j = f^{\theta_j}$, and define

$$\Phi_n = f_1 \circ f_2 \circ \cdots \circ f_n, \tag{1}$$

and define the nth cluster K_n as the complement of $\Phi_n(\Delta)$, so

$$\Phi_n \colon \Delta \to \mathbb{C}_\infty \setminus K_n.$$

We can now use this setup to construct various models of conformal *random* growth, by choosing the angles $(\theta_n)_{n\in\mathbb{N}}$ according to some stochastic process. For example, if all of the angles θ_n are chosen independently from the uniform distribution on $[0, 2\pi)$, then this corresponds to the HL(0) model of [7]. Note that by conformal invariance of Brownian motion, and the fact that the hitting distribution of a Brownian path on the boundary of Δ is uniform, choosing an angle θ_{n+1} uniformly in $[0, 2\pi)$ corresponds to choosing the point to attach the $(n+1)$th particle according to harmonic measure on the boundary of K_n.

We can also define models of conformal aggregation in which we attach a different particle P_n at the nth step. We may wish to do this because of the distortion of the nth particle by Φ_{n-1}, to make the size of the image of P_n in $K_n = K_{n-1} \cup \Phi_{n-1}(P_n)$ approximately equal to the area of the previously attached particles.

1.2 Aggregate Loewner evolution

The *aggregate Loewner evolution* model introduced in [20] is a conformal aggregation model as in Section 1.1, where we choose the angle sequence $(\theta_n)_{n\in\mathbb{N}}$ such that the attachment angle θ_{n+1} of the $(n+1)$th particle is a random variable whose distribution, conditional on $(\theta_1, \cdots, \theta_n)$, depends on (an approximation of) the density of harmonic measure on the boundary of the nth cluster K_n, and the nth particle we attach has a capacity c_{n+1} which is a function of the density of harmonic measure at the attachment point on ∂K_n. The conditional distribution of θ_{n+1} and the way we obtain c_{n+1} are respectively controlled by the two parameters η and α.

Definition 4. Formally, we choose θ_{n+1} for $n \geq 1$ conditionally on $\theta_1, \cdots, \theta_n$ according to the probability density function

$$h_{n+1}(\theta) = \frac{1}{Z_n} \left| \Phi_n' \left(e^{\sigma + i\theta} \right) \right|^{-\eta}, \theta \in (-\pi, \pi], \tag{2}$$

where $Z_n = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{-\eta} \, \mathrm{d}\theta$ is a normalising factor, and $\sigma = \sigma(\mathbf{c}) > 0$ is a "regularisation parameter" which is a function of $\mathbf{c}$ decaying very quickly as $\mathbf{c} \to 0$.

We introduce a positive σ to avoid evaluating Φ_n' on the boundary of Δ, where it may have poles or zeroes. We can either choose θ_1 uniformly on $\mathbb{T}$, or set $\theta_1 = 0$. Throughout this paper we will take $\theta_1 = 0$, but it is clear that if θ_1 is anything else, then the distribution of $(0, \theta_2 - \theta_1, \theta_3 - \theta_1, \cdots)$ is the same as the conditional distribution of $(\theta_1, \theta_2, \theta_3, \cdots)$ given $\theta_1 = 0$.

After choosing θ_{n+1}, we choose the capacity of the $(n+1)$th particle to be

$$c_{n+1} = \mathbf{c} |\Phi_n'(e^{\sigma + i\theta_{n+1}})|^{-\alpha} \tag{3}$$

where $\mathbf{c}$ is a capacity parameter and $c_1 = \mathbf{c}$, and we will later consider the limit shape of the cluster as $\mathbf{c} \to 0$. Note that $|\Phi_n'(e^{\sigma+i\theta})|$ is approximately the factor by which lengths are distorted by Φ_n near the point $e^{i\theta}$ (this distortion can change more than the size of a particle; we can see in Figure 1 that a simple particle is distorted to a convoluted curve) and so the length of the $(n+1)$th particle, of capacity c_{n+1}, once distorted by the map Φ_n is approximately (as particles of capacity c typically have diameter of order $c^{1/2}$)

$$c_{n+1}^{1/2} |\Phi_n'(e^{\sigma+i\theta_{n+1}})| = \mathbf{c}^{1/2} |\Phi_n'(e^{\sigma+i\theta_{n+1}})|^{1-\alpha/2}.$$

The case of $\alpha = 2$ in the ALE(α, η) model is therefore very interesting as all particles are of approximately the same size, as they are in lattice models. Unfortunately the model has proved difficult to analyse when $\alpha = 2$, and so most results obtained so far have been for the range $0 \leq \alpha < 2$ [14] or only in the case $\alpha = 0$ [17]. In this paper, we will work with the case $\alpha = 0$, which has a number of useful properties which simplify the model. In particular, the total cluster has a deterministic capacity (by additivity, as each particle has capacity $\mathbf{c}$), and we only need to use a single particle shape for a given capacity $\mathbf{c}$.

We have seen how α affects the size of the particles, so will now take a look at how η affects the attachment locations of the particles. The simplest case is when $\eta = 0$, which makes the density (2) constant, and so the angles θ_n are chosen independently and uniformly on $\mathbb{T}$, exactly as in the HL(α) model.

For $\eta \neq 0$, we will also interpret the derivative Φ_n' in terms of harmonic measure. Take a point $z \in \partial(\mathbb{C}_\infty \setminus K_n)$ and write it as $z = \Phi_n(e^{i\theta})$. We want to understand the harmonic measure near z, and so a heuristic calculation, using conformal invariance of Brownian motion and the fact that the harmonic measure on $\mathbb{T}$ is proportional to Lebesgue measure, shows that

$$\mathrm{d}h_{n+1}(\theta) \propto |\Phi_n'(e^{\sigma+i\theta})|^{-\eta}\mathrm{d}\theta \approx |\Phi_n'(e^{\sigma+i\theta})|^{-(1+\eta)}\mathrm{d}z. \tag{4}$$

So, for example, if $\eta > 0$, then h_{n+1} can be thought of as an exaggerated version of harmonic measure on the boundary of K_n, where areas of high harmonic measure are more attractive for the site of the $(n+1)$th particle, and areas of low harmonic measure are less attractive.

One other thing that this heuristic calculation shows is that if $\eta = -1$, then $\mathrm{d}h_{n+1}(\theta)$ approximately corresponds to Lebesgue measure on the boundary of $\partial(\mathbb{C}_\infty \setminus K_n)$, so with $\alpha = 2$, the model ALE$(2, -1)$ is a conformal version of the Eden model.

In this paper we will be taking our particles from a one-parameter family of slits $(1, 1+d]$, $d > 0$. The length d and the capacity $\mathbf{c}$ are related to each other approximately by $d \asymp 2\mathbf{c}^{1/2}$ for small $\mathbf{c}$ (the exact relationship is given by $4e^{\mathbf{c}} = \frac{(d+2)^2}{d+1}$). The harmonic measure for $\overline{\mathbb{D}} \cup (1, 1+d]$ is concentrated around the tip of the slit, so has highest density there, and has the lowest density around the base of the slit. So for large positive values of η, we would expect that the $(n+1)$th particle is likely to attach very close to the tip of the nth particle, which is exactly the result found in [20] for $\eta > 1$. In this paper we will look at the case of large negative η, and we find that as we consider the limit $\mathbf{c} \to 0$, the $(n+1)$th particle is very likely to attach close to the *base* of the nth particle. In Section 1.4 we describe this result in more detail, as well as deducing the consequences for the angle sequence and the cluster, but the essential idea is that for $\eta < -2$, the conditional distribution h_{n+1} of θ_{n+1} given $(\theta_1, \cdots, \theta_n)$ approximates the measure

$$\frac{1}{2}(\delta_{\theta_n - \beta} + \delta_{\theta_n + \beta}),$$

where $\beta = \beta(\mathbf{c})$ is given by $f(e^{i\beta}) = 1$ and $0 < \beta < \pi$.

We also have a parameter σ appearing in (2) and (3), and we will take it to be a function of $\mathbf{c}$, decaying as $\mathbf{c} \to 0$. The ALE(α, η) model can behave differently depending on how quickly σ decays: in [20] the authors find convergence of an ALE($0,\eta$) cluster to a one-dimensional slit when σ decays faster than a certain power of $\mathbf{c}$, but argue that if σ decays particularly slowly then the ALE($0,\eta$) model converges to a disc for any $\eta \in \mathbb{R}$.

In [6] and [9], the authors point out that if $\sigma \asymp \mathbf{c}^{1/2}$, then the distance between the images under f of $e^\sigma\mathbb{T}$ and $\mathbb{T}$ is of the same order as the particle size ($d \sim 2\mathbf{c}^{1/2}$), and so this is a sensible choice for a physical model. In fact, as we will see later, a much smaller σ is needed to obtain our result.

1.3 Loewner's equation and the Schramm-Loewner evolution

In the small-particle limit of the model in this paper, we obtain in distribution a random cluster from the *Schramm-Loewner evolution* (SLE) family. In this section, we will describe what an SLE is, and discuss some of its properties, including some of the facts which we will use to prove convergence in distribution of our cluster to a certain SLE.

The first thing we will introduce is *Loewner's equation*, which describes certain growing families of sets in the complex plane (such as our clusters) as solutions to a differential equation, and encodes all information about the growing shape into a function taking values in the boundary of the domain, $\partial\Delta$.

We will not go into much detail here about Loewner's equation, so for a more detailed treatment of the use of Loewner's equation in aggregation processes, we refer the reader to [1] and [10], or for a more analytic perspective on Loewner's equation, see [3].

As an example, suppose that $\gamma\colon (0,T] \to \Delta$ is a simple continuous curve with $\lim_{t\downarrow 0}\gamma(t) \in \mathbb{T}$, parameterised by capacity so that $\mathbf{c}(\gamma(0,t]) = t$ for all $t \in (0,T]$. Then there exists a continuous *driving function* $\xi\colon [0,T] \to \mathbb{R}$ such that for each $t \in (0,T]$ the unique conformal map $f_t\colon \Delta \to \Delta \setminus \gamma(0,t]$ with $f_t(\infty) = \infty$, $f_t'(\infty) = e^t$ is given by the unique solution to the partial differential equation

$$f_0(z) = z, \quad \frac{\partial}{\partial t}f_t(z) = f_t'(z)p(z,\xi(t)), \quad z \in \Delta \tag{5}$$

where $p\colon \Delta \times \mathbb{C} \to \mathbb{C}$ is a particular continuous function, whose form is not especially important.

There is a converse to this, which states that for any continuous or càdlàg function $\xi\colon [0,T] \to \mathbb{R}$ there is a unique solution $(f_t)_{t\in[0,T]}$ to (5) where each f_t is the conformal map $\Delta \to \Delta \setminus K_t$ for some bounded set K_t. The family $(K_t)_{t\in[0,T]}$ satisfies $K_0 = \emptyset$, $K_s \subsetneq K_t$ for $0 \le s < t \le T$, $\overline{K_t} \cap \Delta = K_t$ for all t, and $\Delta \setminus K_t$ is simply connected. However, in general it is not true that K_t is the image of a simple curve γ, even when ξ is continuous.

One very useful property of Loewner's equation for this paper is the continuity of the map $\xi \mapsto \overline{\mathbb{D}} \cup K_T$, which we discuss in a remark in the next section.

We will now introduce the *Schramm-Loewner evolution* family of random clusters to which the limit of our process belongs. Above we said that every continuous $\xi\colon [0,T] \to \mathbb{R}$

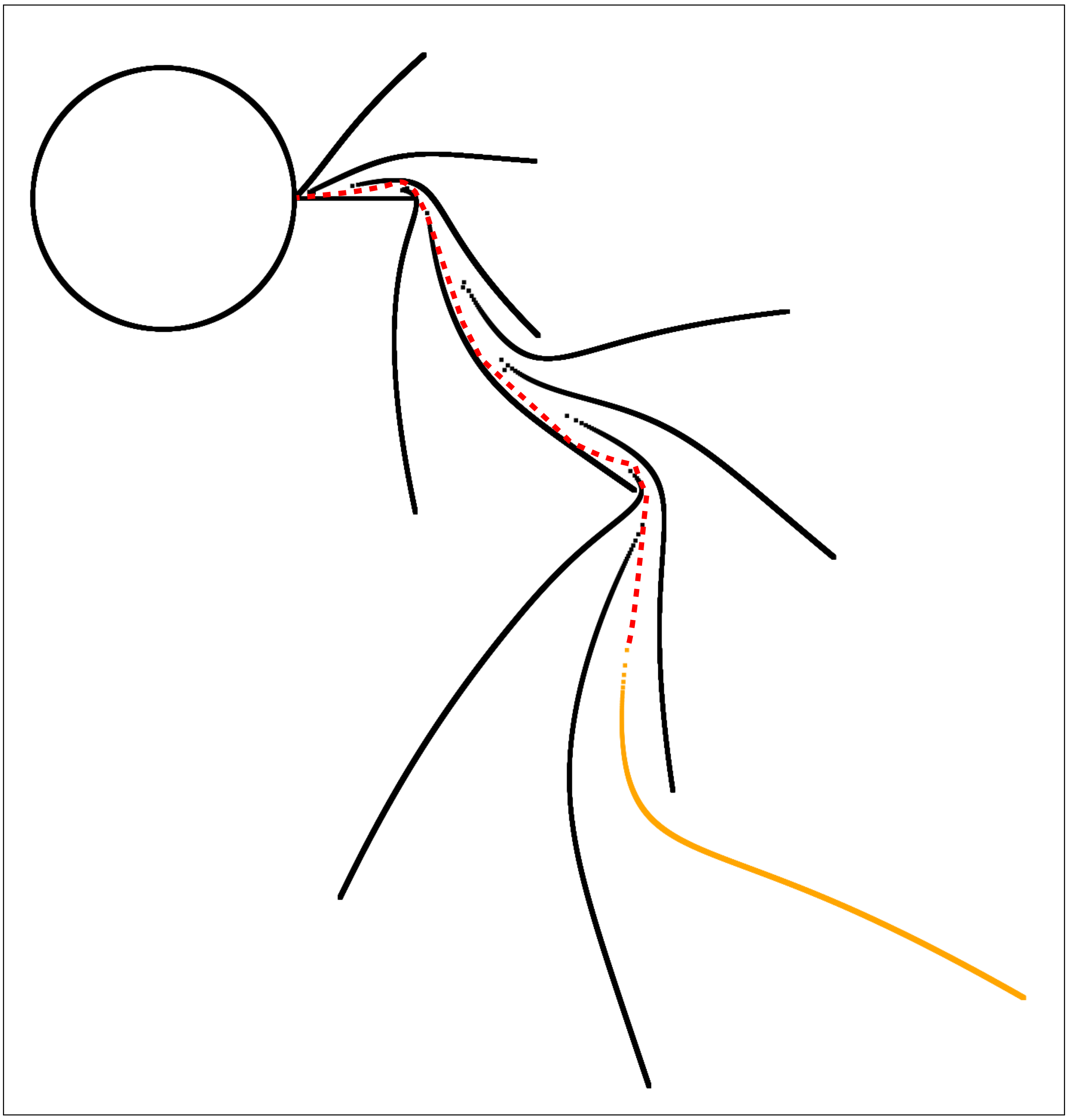

Figure 1: We can see in this figure that the final particle (the rightmost, in orange) of the cluster K_n is highly distorted by the application of the first $n-1$ maps $f_{n-1}, f_{n-2}, \cdots, f_1$. The distortion is much greater near the base of the particle: we have had to fill in a guess (the red dashed line) for the behaviour of the particle deep into the cluster, as the distortion is so large there that we are unable to find the exact location of enough points to draw a sensible diagram. In fact, the red dashed section corresponds to only 1/500 000th the length of the original, undistorted slit.

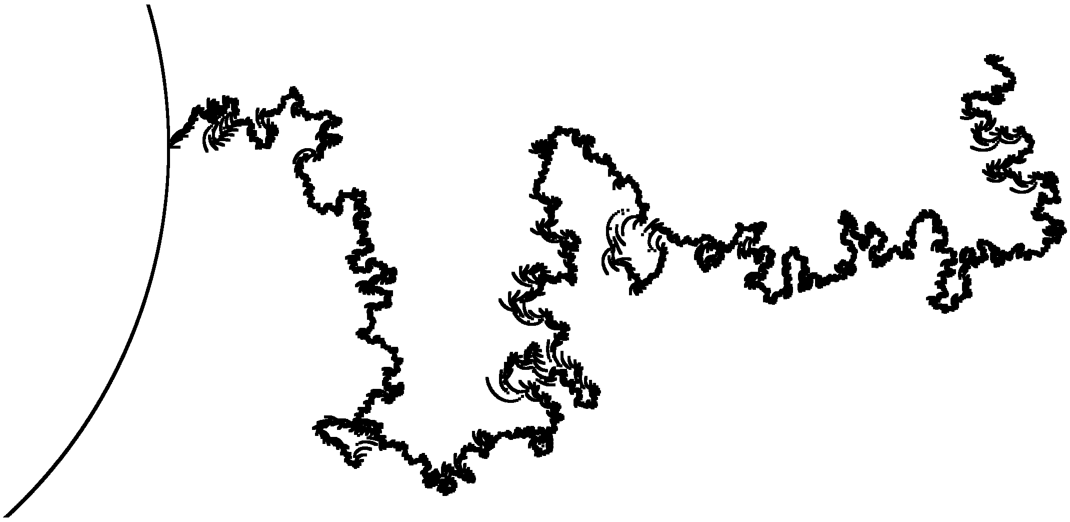

Figure 2: One cluster of the ALE$(0, -\infty)$ process described below, with 3000 particles each of capacity $\mathbf{c} = 0.0001$. An SLE_4 cluster is a simple curve with Hausdorff dimension 3/2, and this diagram appear to show that our process converges to a fractal curve, which we claim is SLE_4.

corresponds, by solving (5), to an increasing family of hulls $(K_t)_{t\in[0,T]}$. If we allow ξ to be a stochastic process, then $\left(\overline{\mathbb{D}} \cup K_t\right)_{t\in[0,T]}$ will also be a stochastic process taking values in $\mathcal{K}$, the space of compact subsets of $\mathbb{C}$.

Consider the driving function $(U_t)_{t\in[0,T]} = (\sqrt{\kappa} B_t)_{t\in[0,T]}$, where B is a standard Brownian motion and $\kappa > 0$ is a constant. Then the solution to (5) with $\xi = U$ is the *Schramm-Loewner evolution* with parameter κ (SLE_κ).

Schramm-Loewner evolutions arise as both the scaling limit of discrete models such as the loop-erased random walk, which converges to an SLE_2 curve [12], or the percolation interface and SLE_6 [19]; and there are many links between the SLE and other continuous random objects, such as the links between the paths of Brownian motion and SLE_6 [13], and between the Gaussian free field and SLE_4 [18].

The Schramm-Loewner evolution has a number of interesting properties which we will not discuss in detail in this paper, so see [10] for more detail on the SLE and many of its properties. One fact we do use is that for $\kappa \leq 4$, the family of hulls $(K_t)_{t\in[0,T]}$ is given by $K_t = \gamma(0, t]$ for some simple curve γ with probability 1 (and for $\kappa > 4$ this is almost surely not the case). We will show that the clusters of our ALE$(0, \eta)$ process converge to an SLE_4 cluster, so this above fact means that the scaling limit of our clusters is a random *simple curve.*

1.4 Our results

In this paper, we study the ALE model defined in Section 1.2 with $\alpha = 0$ and large negative values of the parameter η, which controls the influence of harmonic measure on our attachment locations. For convenience we will often write $\nu = -\eta > 0$. Our main result describes the limit of the process as the particle size tends to 0:

Theorem 5. *Fix some $T > 0$. For $\eta < -2$ and if $\sigma(\mathbf{c}) \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$ for all $\mathbf{c} < 1$, then the corresponding ALE$(0, \eta)$ cluster with $N = \lfloor T/\mathbf{c} \rfloor$ particles, each of capacity $\mathbf{c}$, $K_N^{\mathbf{c}}$, converges in distribution as $\mathbf{c} \to 0$ to a radial SLE$_4$ curve of capacity T. This convergence is as a random variable in the space $\mathcal{K}$ of compact subsets of $\mathbb{C}$ containing 0, equipped with the Carathéodory topology.*

Remark. For a discussion of the Carathéodory topology on $\mathcal{K}$, see Section 3.1 of [3]. Note in particular that convergence of a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{K}$ is equivalent to uniform convergence on compact subsets of Δ of the corresponding maps $\Psi_n : \Delta \to \mathbb{C}_\infty \setminus C_n$.

Remark. We can use *Loewner's equation* to encode the cluster $K_N^{\mathbf{c}}$ by a càdlàg function $\xi^{\mathbf{c}} : [0, T] \to \mathbb{R}$, and then as the map $D[0, T] \to \mathcal{K}$ sending a function to the corresponding hull is continuous (see [8]), to show convergence of $K_N^{\mathbf{c}}$ to an SLE$_4$ in distribution, we only need to establish that $\xi^{\mathbf{c}}$ converges in distribution to $2B$, where B is a standard Brownian motion. Theorem 5 is therefore a corollary of the following result:

Proposition 6. *For η, σ as in Theorem 5, let $(\theta_n^{\mathbf{c}})_{n\geq 1}$ be the sequence of angles we obtain from the* ALE$(0, \eta)$ *process with capacity parameter $\mathbf{c}$. Let $\xi_t^{\mathbf{c}} = \theta_{\lfloor t/\mathbf{c} \rfloor}^{\mathbf{c}}$ for $t \geq 0$. Then for any fixed $T > 0$,*

$$(\xi_t^{\mathbf{c}})_{t\in[0,T]} \to (2B_t)_{t\in[0,T]} \text{ in distribution as } \mathbf{c} \to 0,$$

as a random variable in the Skorokhod space $D[0, T]$.

1.5 Structure of paper

Our proof of Proposition 6 will involve showing that the distribution of θ_{n+1}, conditional on $(\theta_1, \cdots, \theta_n)$, converges to $\frac{1}{2}(\delta_{\theta_n+\beta} + \delta_{\theta_n-\beta})$, and so the whole path $\xi^{\mathbf{c}}$ converges to the same limit as a simple random walk with step length $\beta \sim 2\mathbf{c}^{1/2}$.

Firstly, we will give a heuristic argument which illustrates the idea behind our proof. Consider the "$\eta = -\infty$, $\sigma = 0$" model where the distribution of θ_{n+1} conditional on $\theta_1, \cdots, \theta_n$ is

$$h_{n+1} = \frac{1}{k} \sum_{i=1}^{k} \delta_{p_i}. \tag{6}$$

where $p_1, \cdots, p_k \in \mathbb{T}$ are the "strongest" poles of $|\Phi_n'|$, i.e. the set of points such that

$$\liminf_{\sigma\to 0} \inf_{z\in\mathbb{T}} \frac{|\Phi_n'(e^\sigma p_j)|}{|\Phi_n'(e^\sigma z)|} > 0.$$

Then we have $\theta_1 = 0$, and $|f_1'|$ only has two poles: at $e^{\pm i\beta}$, where $\beta = \beta(\mathbf{c}) \sim 2\mathbf{c}^{1/2}$ is the angle distance between θ_1 and each of its two preimages under f_1. Hence $h_2 =$

Figure 3: Left: the one-slit cluster of our process with 1000 points in red sampled according to harmonic measure on the boundary. Right: the three-slit cluster of the process with 10 000 points sampled according to harmonic measure. Note in the second image that there are almost no points landing near the base of the most recent (longest) particle.

$\frac{1}{2}(\delta_{-\beta} + \delta_{+\beta})$. Then we will look at h_3 by examining $|\Phi_2'|$. Without loss of generality, $\theta_2 = \beta$. By the chain rule,

$$|\Phi_2'(z)| = |(f_1 \circ f_2)'(z)| = |f_1'(f_2(z))| \times |f_2'(z)|,$$

so there are three poles which may contribute to h_3: $e^{i(\theta_2 \pm \beta)}$, which are poles for both $f_1' \circ f_2$ and f_2', and $\widehat{z} = f_2^{-1}(e^{-i\beta})$, which is a pole of $f_1' \circ f_2$, but is not a pole of f_2'. A simple calculation shows that

$$\liminf_{\sigma \to 0} \frac{|\Phi_2'(e^{\sigma}\widehat{z})|}{|\Phi_2'(e^{\sigma + i(\theta_2 + \beta)})|} = 0,$$

and so $h_3 = \frac{1}{2}(\delta_{\theta_2 - \beta} + \delta_{\theta_2 + \beta})$.

Similarly, we can show $h_{n+1} = \frac{1}{2}(\delta_{\theta_n - \beta} + \delta_{\theta_n + \beta})$ for every n, and so the sequence of angles $(\theta_n)_{n \geq 1}$ is just a simple symmetric random walk, with steps of length $\beta \sim 2\mathbf{c}^{1/2}$.

Our approach for finite $\eta < -2$ will therefore be to find a small upper bound on h_{n+1} away from the poles of Φ_n' to deduce that h_{n+1} is an approximation to (6). Then we show separately that the contribution to $\int_{\mathbb{T}} h_{n+1}(\theta)\, \mathrm{d}\theta$ from poles other than $e^{i(\theta_n \pm \beta)}$ is small.

In the actual model with $-\infty < \eta < -2$, we can only show that h_{n+1} *approximates* $\frac{1}{2}(\delta_{\theta_n - \beta} + \delta_{\theta_n + \beta})$ as $\mathbf{c} \to 0$. However, weak convergence of these measures is not enough to prove Proposition 6, so we will need to introduce some extra notation to describe the possible behaviour of the process $(\theta_n)_{n \geq 1}$, and make precise the way in which its steps converge to the SSRW steps as above.

Definition 7. For a small $D = D(\mathbf{c})$ (which we will specify later), define the stopping time

$$\tau_D := \inf\{n \geq 2 : \min(|\theta_n - (\theta_{n-1} + \beta)|, |\theta_n - (\theta_{n-1} - \beta)|) > D\}.$$

Remark. Given that $n < \tau_D$, we have a lot of information about the angle sequence $(\theta_1, \cdots, \theta_n)$, and so can say quite a lot about the conditional distribution of θ_{n+1}. In particular, we can say that the probability that $n+1 = \tau_D$ is very low, and that the distribution of $\theta_{n+1} - \theta_n$ is (approximately) symmetric. The results of all the following sections will be used to establish these two facts.

Theorem 8. *Suppose that $\nu > 2$. There exists a constant $A > 0$ depending only on ν and T such that when $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, then for $D = \mathbf{c}^{9/2}\sigma^{1/2}$, whenever $n < N \wedge \tau_D$ and $\mathbf{c}$ is sufficiently small,*

$$\int_{F_n} h_{n+1}(\theta)\,\mathrm{d}\theta \leq A\mathbf{c}^4 \tag{7}$$

with probability 1*, where $F_n = \{\theta \in \mathbb{T} : |\theta - (\theta_n + \beta)| \geq D$ and $|\theta - (\theta_n - \beta)| \geq D\}$, and with probability* 1

$$\left| \int_{\theta_n+\beta-D}^{\theta_n+\beta+D} h_{n+1}(\theta)\,\mathrm{d}\theta - \int_{\theta_n-\beta-D}^{\theta_n-\beta+D} h_{n+1}(\theta)\,\mathrm{d}\theta \right| \leq A\mathbf{c}^{11/4}. \tag{8}$$

In Section 2 we prove a number of technical results about the positions of the images and preimages of points $w \in \Delta$ under the maps f_j, $\Phi_n = f_1 \circ \cdots \circ f_n$, and $\Phi_{j,n} = \Phi_j^{-1} \circ \Phi_n$ when w is close to the poles of Φ_n'. When dealing with points away from these poles, we make extensive use of results from [20]. Our estimates for the positions of these images will be useful when we find upper bounds on the derivative $|\Phi_n'(w)| = |f_n'(w)| \times |f_{n-1}'(\Phi_{n-1,n}(w))| \times \cdots \times |f_1'(\Phi_{1,n}(w))|$, using lower bounds on the distance between $\Phi_{j,n}(w)$ and the poles of f_j'.

In Section 3.1 we integrate the pre-normalised density $|\Phi_n'(e^{\sigma+i\theta})|^\nu$ over the regions around $\theta_n \pm \beta$, and so obtain a lower bound on

$$Z_n = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^\nu\,\mathrm{d}\theta.$$

In Section 3.2 and Section 4 we find upper bounds on $|\Phi_n'(e^{\sigma+i\theta})|$ for $\theta \in F_n$, and so using the lower bound on Z_n we can establish the bound (7).

In Section 3.3 we establish the technical results needed to prove (8).

Remark. In our proof of Theorem 8, the convergence of h_{n+1} to $\frac{1}{2}(\delta_{\theta_n+\beta} + \delta_{\theta_n-\beta})$ does not rely on the convergence of $h_1, \cdots, h_n$ to these *symmetric* discrete measures, only that $n < \tau_D$. If we were to use the fact that the angle sequence up until time n is very close to a simple symmetric random walk, then some properties (such as the fact that the longest interval on which a SSRW is monotone has length of order $O(\log n)$) would allow us to optimise our choice of σ further than we have. However, for the convergence of our cluster to an SLE_4 curve, we do require a σ which decays at least as quickly as $\mathbf{c}^{1/\mathbf{c}}$, which is already much faster than the fixed power of $\mathbf{c}$ used in [20] and elsewhere, so we have not attempted to optimise our choice of $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$.

1.6 Table of notation

As we introduce a lot of notation in this paper, we will give a list here so that it is possible to look up any notation appearing in any section without searching for where it was introduced.

Subsets of the complex plane

$\mathbb{C}_\infty$ The Riemann sphere, $\mathbb{C} \cup \{\infty\}$

$\mathbb{D}$ The open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

$\overline{\mathbb{D}}$ The closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$.

Δ The exterior disc $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$.

$\mathbb{T}$ The unit circle $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$. We will often abuse notation and identify $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$.

∂U The boundary of a set $U \subseteq \mathbb{C}_\infty$, defined as $\partial U = \overline{U} \setminus U^\circ$.

Conformal maps

f The conformal map $f_{\mathbf{c}} \colon \Delta \to \Delta \setminus (1, 1 + d(\mathbf{c})]$ which we say attaches a particle to the unit circle at the point 1.

f_j Given a sequence of angles $(\theta_j)_{j\geq 1}$, f_j attaches a particle to the unit circle at the point $e^{i\theta_j}$: $f_j(z) := e^{i\theta_j} f(e^{-i\theta_j} z)$.

β The distance from 1 of the points which are sent to the base of the particle by f. Defined uniquely as $\beta = \beta(\mathbf{c}) \in (0, \pi)$ such that $f_{\mathbf{c}}(e^{\pm i\beta}) = 1$, and obeys $\beta \sim 2\mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$.

d The length of the particle attached by f, defined by $f_{\mathbf{c}}(1) = 1 + d(\mathbf{c})$. Obeys $d \sim \beta \sim 2\mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$.

Φ_n The conformal map which attaches the entire cluster of n particles to the unit circle at the point 1. Constructed as $f_1 \circ f_2 \circ \cdots \circ f_n$.

$\Phi_{j,n}$ The conformal map which attaches only the most recent $n-j$ particles to the unit circle. $\Phi_{j,n} = \Phi_j^{-1} \circ \Phi_n$.

Model parameters

η The parameter controlling the relationship between our attachment distributions and the harmonic measure on the boundary of the cluster. Throughout this paper we take $\eta < -2$.

ν We write $\nu = -\eta$. Note that $\nu > 2$ throughout.

T The total capacity of our cluster, fixed throughout.

$\mathbf{c}$ The capacity of each individual particle attached to the cluster. We consider in this paper the limit $\mathbf{c} \to 0$, so all the following parameters are functions of $\mathbf{c}$.

σ A regularisation parameter, used so that we do not evaluate our conformal maps Φ_n' at their poles on $\mathbb{T}$, instead evaluating everything on $e^{\sigma}\mathbb{T}$. We take σ to be a function of $\mathbf{c}$, decaying very rapidly as $\mathbf{c} \to 0$: $\sigma \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$

L The maximum distance of z from $e^{i(\theta_n \pm \beta)}$ at which we rely on the estimates for $|\Phi_{j,n}(z) - e^{i\theta_{j+1}}|$ we obtain in the proof of Theorem 15. We take L to be a function of $\mathbf{c}$ which does not decay as rapidly as σ: $L = \mathbf{c}^{2^{N+1}}$.

D A bound on $\min_{\pm} |\theta_{n+1} - (\theta_n \pm \beta)|$ which holds with high probability. If this distance exceeds D, we stop the process. We can take $D = \mathbf{c}^{9/2}\sigma^{1/2}$.

Points in $\mathbb{T}$

$\theta_j^{\top}$ The point in $\mathbb{T}$ which θ_j was "supposed to" attach nearby to, i.e. the unique choice of $\theta_{j-1} \pm \beta$ which θ_j is within D of (if θ_j is not within D of either, we will have stopped the process at time $\tau_D \leq j$).

$\theta_j^{\perp}$ The choice of $\theta_{j-1} \pm \beta$ which isn't $\theta_j^{\top}$.

$\widehat{z}_j^n$ The point on $\mathbb{T}$ corresponding to the base of the jth particle in the cluster K_n, for $1 \leq j \leq n-1$. Given by $\widehat{z}_j^n := \Phi_{j,n}^{-1}(e^{i\theta_{j+1}^{\perp}})$. See Figure 5 for an illustration. We refer to the points on $\mathbb{T}$ close to $\widehat{z}_j^n$ for some j as *singular points* for h_{n+1}, and points away from all $\widehat{z}_j^n$ as *regular points*.

Probabilistic objects

h_{n+1} The density of the distribution on $\mathbb{T}$ of θ_{n+1}, conditional on $\theta_1, \cdots, \theta_n$. Given by $h_{n+1}(\theta) \propto |\Phi_n'(e^{\sigma+i\theta})|^{\nu}$.

Z_n The normalising factor for h_{n+1}. Given by $Z_n := \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{\nu} \, \mathrm{d}\theta$.

$\mathbb{P}$ The law of $(\theta_n)_{n\in\mathbb{N}}$. Implicitly depends on $\mathbf{c}$ and σ.

τ_D The first time at which some θ_{n+1} is further than D from both of $\theta_n \pm \beta$. We stop the process when this happens, but show in Section 3 and Section 4 that with high probability $\tau_D > N := \lfloor T/\mathbf{c} \rfloor$.

Approximations and bounds

We will use the following notation when we have two functions depending on a parameter x which is converging to some $x_0 \in \mathbb{R} \cup \{\pm\infty\}$, and we want to say the two functions are similar in some way, or that one bounds the other.

$f(x) \sim g(x)$ The ratio $\frac{f(x)}{g(x)} \to 1$ as $x \to x_0$.

$f(x) = O(g(x))$	The ratio $\left\|\frac{f(x)}{g(x)}\right\|$ is bounded above as $x \to x_0$, so there exists a constant $C > 0$ such that $\|f(x)\| \leq C\|g(x)\|$ in a neighbourhood of x_0. The constant C should not depend on any other parameter or variable, other than η which is fixed throughout the paper. If the value of C does depend on a parameter ρ, we will write $f(x) = O_\rho(g(x))$. Throughout this paper we hold T and $\nu = -\eta$ fixed, so we may occasionally omit these as subscripts when the constant depends on them.
$f(x) = o(g(x))$	The ratio $\left\|\frac{f(x)}{g(x)}\right\| \to 0$ as $x \to x_0$.

When f and g are non-negative (particularly when they are probabilities or densities), we may use the following alternative notations.

$f(x) \lesssim g(x)$	The same as $f(x) = O(g(x))$, i.e. there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ in a neighbourhood of x_0.
$f(x) \ll g(x)$	The same as $f(x) = o(g(x))$, i.e. $f(x)/g(x) \to 0$ as $x \to x_0$.
$f(x) \asymp g(x)$	Both $f(x) = O(g(x))$ and $g(x) = O(f(x))$, i.e. there exists constants $C_1, C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ in a neighbourhood of x_0.

Finally, we may write $f(x) \approx g(x)$, but this will only be used informally to mean that f and g behave similarly in some sense.

2 Spatial distortion of points

There are several steps we need to establish our upper bound on $\int h_{n+1}(\theta)\,\mathrm{d}\theta$ in (7), including precise estimates for $|\Phi_n'|$ near its poles. We can decompose the derivative

$$\Phi_n'(w) = \prod_{j=0}^{n-1} f_{n-j}'(\Phi_{n-j,n}(w)) \tag{9}$$

where

$$\Phi_{k,n} := \Phi_k^{-1} \circ \Phi_n = f_{k+1} \circ f_{k+2} \circ \cdots \circ f_n. \tag{10}$$

Then we have precise estimates on $|f'|$ near to its poles $e^{\pm i\beta}$, and upper bounds away from these poles, and so we write

$$|\Phi_n'(w)| = \prod_{j=0}^{n-1} \left| f'\left(e^{-i\theta_{n-j}} \Phi_{n-j,n}(w)\right)\right|. \tag{11}$$

We will show that if w is close to one of $e^{i(\theta_n \pm \beta)}$, then for each j, the point $e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)$ is close to a pole of $|f'|$, and we will derive specific estimates on the distance in terms of the distance $|w - e^{i(\theta_n \pm\beta)}|$. Conversely, we will show that the only way for *every* image $e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)$ to be close to a pole is for w to be close to $e^{i(\theta_n\pm\beta)}$, and so the measure $\mathrm{d}h_{n+1}$ is concentrated around $\theta_n + \beta$ and $\theta_n - \beta$.

Firstly, we will establish an estimate for $|f'|$ close to its poles $e^{\pm i\beta}$, and a universal upper bound away from these two points.

Lemma 9. *There are universal constants $A_1, A_2 > 0$ such that for all $\mathbf{c} < 1$, for $w \in \Delta$, if $|w - e^{i\beta}| \le \frac{3}{4}\beta$, then*

$$A_1 \frac{\beta^{1/2}}{|w - e^{i\beta}|^{1/2}} \le |f_{\mathbf{c}}'(w)| \le A_2 \frac{\beta^{1/2}}{|w - e^{i\beta}|^{1/2}}, \tag{12}$$

and similarly if $|w - e^{-i\beta}| \le \frac{3}{4}\beta$.

Moreover, there is a third constant A_3 such that if $\min\{|w - e^{i\beta}|, |w - e^{-i\beta}|\} > \frac{3}{4}\beta$, then

$$|f_{\mathbf{c}}'(w)| \le A_3.$$

Proof. See Lemma 5 of [20]. □

This lemma tells us that $|\Phi_n'(w)|$ will be large only when many of the points $e^{-i\theta_{n-j}}\Phi_{n-j,n}(w)$ in (11) are close to one of the poles $e^{\pm i\beta}$. We will next introduce some technical estimates which will allow us to determine for which points w this is true.

Remark. If we imagine an idealised path in which $|\theta_{i+1} - \theta_i| = \beta$ for all i, then $f_n(e^{i(\theta_n\pm\beta)}) = e^{i\theta_{n-1}}$, and $f_{n-1}(e^{i\theta_{n-1}}) = e^{i\theta_{n-2}}$, and so on. Hence $\Phi_{n-j,n}(e^{i(\theta_n\pm\beta)}) = e^{i\theta_{n-j+1}} = e^{i(\theta_{n-j}+s_{n-j}\beta)}$, where $s_{n-j} \in \{\pm 1\}$. So if a point w is close to one of $e^{i(\theta_n\pm\beta)}$ then, as f is continuous when extended to $\overline{\Delta}$, each of the points in (11) is close to $e^{is_{n-j}\beta}$, but continuity alone does not allow us to make precise what we mean by "w is close to $e^{i(\theta_n \pm \beta)}$", so to estimate the size of $|\Phi'_n(w)|$, we need a precise estimate for $|f(w) - f(e^{i\beta})|$ in terms of $|w - e^{i\beta}|$.

Lemma 10. *For $w \in \Delta$, for all $\mathbf{c} < 1$, if $|w - e^{i\beta}| \leq \beta/2$, then*

$$|f_{\mathbf{c}}(w) - 1| = 2(e^{\mathbf{c}} - 1)^{1/4}|w - e^{i\beta}|^{1/2} \times \left(1 + O\left[\frac{|w - e^{i\beta}|}{\mathbf{c}^{1/2}} \vee \mathbf{c}^{1/4}|w - e^{i\beta}|^{1/2}\right]\right). \tag{13}$$

Proof. We will work with the half-plane slit map $\widetilde{f}_{\mathbf{c}} \colon \mathbb{H} \to \mathbb{H} \setminus (0, i\sqrt{1 - e^{-\mathbf{c}}}\,]$ by conjugating f with the Möbius map $m_{\mathbb{H}} \colon \Delta \to \mathbb{H}$ given by

$$m_{\mathbb{H}}(w) = i\frac{w - 1}{w + 1}, \tag{14}$$

and its inverse

$$m_{\Delta}(z) := m_{\mathbb{H}}^{-1}(z) = \frac{1 - iz}{1 + iz}. \tag{15}$$

The benefit of this is that $\widetilde{f}_{\mathbf{c}}$ has a simple explicit form:

$$\widetilde{f}_{\mathbf{c}}(\zeta) = e^{-\mathbf{c}/2}\sqrt{\zeta^2 - (e^{\mathbf{c}} - 1)} \tag{16}$$

where the branch of the square root is given by $\arg \colon \mathbb{C} \setminus [0, \infty) \to (0, 2\pi)$, so we write

$$f_{\mathbf{c}} = m_{\Delta} \circ \widetilde{f}_{\mathbf{c}} \circ m_{\mathbb{H}}$$

and will derive a separate estimate for each of the three maps.

As w is close to $e^{i\beta} = 2e^{-\mathbf{c}} - 1 + 2ie^{-\mathbf{c}}\sqrt{e^{\mathbf{c}} - 1}$, we will expand each map about the images (given by a simple calculation) $m_{\mathbb{H}}(e^{i\beta}) = -\sqrt{e^{\mathbf{c}} - 1}$, $\widetilde{f}_{\mathbf{c}}(-\sqrt{e^{\mathbf{c}} - 1}) = 0$, and $m_{\Delta}(0) = 1$. Our calculations will show that m_{Δ} and $m_{\mathbb{H}}$ behave like scaling by a constant close to the relevant points, and that the behaviour of $f_{\mathbf{c}}$ seen in (13) is due to the behaviour of $\widetilde{f}_{\mathbf{c}}$ close to $\pm\sqrt{e^{\mathbf{c}} - 1}$.

First, when $w = e^{i\beta} + \delta$,

$$\begin{aligned} |m_{\mathbb{H}}(w) - m_{\mathbb{H}}(e^{i\beta})| &= \left|\frac{e^{i\beta} - 1 + \delta}{e^{i\beta} + 1 + \delta} - \frac{e^{i\beta} - 1}{e^{i\beta} + 1}\right| \\ &= \left|\frac{2\delta}{(e^{i\beta} + 1 + \delta)(e^{i\beta} + 1)}\right| \\ &= \frac{1}{2}e^{\mathbf{c}}|\delta|(1 + O(|\delta|)) \end{aligned} \tag{17}$$

since a simple calculation shows that $|e^{i\beta} + 1|^2 = 4e^{-\mathbf{c}}$.

Next, we will evaluate $\widetilde{f}_{\mathbf{c}}$ at a point close to one of the two preimages of 0, $\pm\sqrt{e^{\mathbf{c}} - 1}$.

$$\begin{aligned} \left|\widetilde{f}_{\mathbf{c}}(\pm\sqrt{e^{\mathbf{c}} - 1} + \lambda)\right| &= e^{-\mathbf{c}/2}\left|\sqrt{\pm 2\sqrt{e^{\mathbf{c}} - 1}\lambda + \lambda^2}\right| \\ &= \sqrt{2}e^{-\mathbf{c}/2}(e^{\mathbf{c}} - 1)^{1/4}|\lambda|^{1/2}\left(1 + O\left(\frac{|\lambda|}{\mathbf{c}^{1/2}}\right)\right). \end{aligned} \tag{18}$$

Finally, for a small $z \in \mathbb{H}$,

$$|m_\Delta(z) - 1| = \left|\frac{1-iz}{1+iz} - 1\right| = \left|\frac{-2iz}{1+iz}\right| = 2|z|(1 + O(|z|)). \tag{19}$$

Then for w close to $e^{i\beta}$, applying (17), (18) and (19) in turn, we obtain

$$\begin{aligned}|f(w) - 1| &= 2(e^{\mathbf{c}} - 1)^{1/4}|w - e^{i\beta}|^{1/2} \\ &\quad \times \left(1 + O\left(\frac{|w - e^{i\beta}|}{\mathbf{c}^{1/2}}\right)\right)\left(1 + O\left(\mathbf{c}^{1/4}|w - e^{i\beta}|^{1/2}\right)\right).\end{aligned}$$

Then for $\mathbf{c}^{3/2} \le |w - e^{i\beta}| \le \beta/2$, we have the estimate (13) with error term of order $\mathbf{c}^{-1/2}|w - e^{i\beta}|$, and for $|w - e^{i\beta}| \le \mathbf{c}^{3/2}$ the error term has order $\mathbf{c}^{1/4}|w - e^{i\beta}|^{1/2}$. □

Remark. Unlike most results in this section, we will not use the following lemma at all in the next section, but it will be very useful in Section 4.2. We include it here and omit the proof as it is very similar to Lemma 10.

Lemma 11. *For all* $\mathbf{c} < 1$, *if* $z \in \Delta \setminus (1, 1 + d(\mathbf{c})]$ *has* $|z - 1| \le \mathbf{c}$, *then*

$$\min_{\pm} |f^{-1}(z) - e^{\pm i\beta}| = \frac{|z-1|^2}{4(e^{\mathbf{c}} - 1)^{1/2}}\left(1 + O\left(|z - 1|\right)\right).$$

Now we have all the technical results we need in order to prove our lower bound on $|\Phi_n'(w)|$ when w is close to one of the two "most recent basepoints" $e^{i(\theta_n \pm \beta)}$. We will derive the bound itself in Section 3.1, and here we will show that each of the points $\Phi_{n-j,n}(w)$ in (11) is close to $e^{i\theta_{n-j+1}}$.

Proposition 12. *Let* $L = L(\mathbf{c}, N) = \mathbf{c}^{2^{N+1}}$, *and let* $n < N \wedge \tau_D$. *If* $\delta := \min |w - e^{i(\theta_n \pm \beta)}| \le L$, *and* $|w| \ge e^{\sigma}$, *then for all* $1 \le j \le n$,

$$\left|\Phi_{n-j,n}(w) - e^{i\theta^{\top}_{n-j+1}}\right| = \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-j})}\delta^{2^{-j}}(1 + O(\mathbf{c}^4)). \tag{20}$$

Before we begin the proof we will introduce some notation in order to make the argument easier to follow.

Definition 13. By definition of τ_D, for each $n < \tau_D$ one of the two angles $\theta_{n-1} \pm \beta$ is within distance D of θ_n. We will call the closer of the two angles $\theta_n^{\top}$, and the other angle $\theta_n^{\perp}$.

Proof of Proposition 12. We will proceed by induction on j. For $j = 1$, the estimate (20) follows directly from Lemma 10. For a given $1 \le j \le n-1$, assume that

$$|\Phi_{n-j,n}(w) - e^{i\theta^{\top}_{n-j+1}}| = \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-j})}\delta^{2^{-j}}(1 + O(\mathbf{c}^4)),$$

(as $|\theta_n - \theta_n^\top| < D \ll \mathbf{c}^4$, this certainly holds for $j = 1$) and then by the triangle inequality, since $|e^{i\theta_{n-j}} - e^{i\theta_{n-j}^\top}| \leq |\theta_{n-j} - \theta_{n-j}^\top| < D$,

$$|\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| - D \leq |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}^\top}| \leq |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| + D.$$

Now by Lemma 10,

$$\begin{aligned}
&|\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| = |f_{n-j}(\Phi_{n-j,n}(w)) - f_{n-j}(e^{i\theta_{n-j+1}^\top})| \\
&= |f(e^{-i\theta_{n-j+1}^\top}\Phi_{n-j,n}(w)) - 1| \\
&= 2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}|e^{-i\theta_{n-j+1}^\top}\Phi_{n-j,n}(w) - 1|^{1/2}(1 + O(\mathbf{c}^{1/4}|e^{-i\theta_{n-j+1}^\top}\Phi_{n-j,n}(w) - 1|^{1/2})) \\
&= \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{1+(1-2^{-j})} \delta^{2^{-(j+1)}}(1 + O(\mathbf{c}^4))(1 + O(\mathbf{c}^{3/8}\delta^{2^{-(j+1)}})) \\
&= \left[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}\right]^{2(1-2^{-(j+1)})} \delta^{2^{-(j+1)}}(1 + O(\mathbf{c}^4))
\end{aligned}$$

and the second error term is absorbed since $\delta^{2^{-(j+1)}} \leq L^{2^{-(j+1)}} \leq \mathbf{c}^4$.

Now as $\delta = |w - e^{i(\theta_n \pm \beta)}| \geq |w| - 1 \geq \sigma$, and $D \sim \mathbf{c}^{9/2}\sigma^{1/2}$ (see Section 1.6), we have

$$\begin{aligned}
|\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}^\top}| &= |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| \left(1 + O\left(\frac{D}{\mathbf{c}^{\frac{1}{2}(1-2^{-(j+1)})}\delta^{2^{-(j+1)}}}\right)\right) \\
&= |\Phi_{n-j-1,n}(w) - e^{i\theta_{n-j}}| \left(1 + O\left(\mathbf{c}^4\sigma^{1/4}\right)\right),
\end{aligned}$$

and hence our result holds for all $1 \leq j \leq n$ by induction. □

3 The newest basepoints

3.1 A lower bound on the normalising factor

We defined in (2) the density function $h_{n+1}(\theta)$ and the nth normalising factor

$$Z_n = \int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^{-\eta}\, \mathrm{d}\theta. \tag{21}$$

If we are going to find upper bounds on h_{n+1} by bounding $|\Phi_n'|$, then we will need to have some lower bound on the normalising factor Z_n. In this section, we will obtain a lower bound on Z_n, and it will give us our upper bound on h_{n+1} in Section 4.2. First, we will need a good estimate for $|\Phi_n'|$ around the poles $e^{i(\theta_n \pm \beta)}$.

Lemma 14. *Let $n < \lfloor T/\mathbf{c} \rfloor \wedge \tau_D$. There are constants $A_1, A_2 > 0$ such that for any $\mathbf{c} < 1$, whenever $|\varphi| < L$,*

$$A_1^n \frac{\mathbf{c}^{\frac{1}{2}(1-2^{-n})}}{(\sigma^2 + \varphi^2)^{\frac{1}{2}(1-2^{-n})}} \leq \left|\Phi_n'\left(e^{\sigma+i(\theta_n \pm \beta + \varphi)}\right)\right| \leq A_2^n \frac{\mathbf{c}^{\frac{1}{2}(1-2^{-n})}}{(\sigma^2 + \varphi^2)^{\frac{1}{2}(1-2^{-n})}}$$

provided that $\sigma = \sigma(\mathbf{c}) \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$.

Proof. For $|\varphi| < L$, without loss of generality take $\theta = \theta_n + \beta + \varphi$. Since $\Phi_n = f_1 \circ \cdots \circ f_n$, by the chain rule,

$$|\Phi_n'(e^{\sigma+i\theta})| = \prod_{j=0}^{n-1} \left| f'\left(e^{-i\theta_{n-j}} \Phi_{n-j,n}(e^{\sigma+i\theta})\right)\right|,$$

where $\Phi_{k,n} = \Phi_k^{-1} \circ \Phi_n = f_{k+1} \circ f_{k+2} \circ \cdots \circ f_n$.

By Proposition 12, if $\delta := |e^{\sigma+i\theta} - e^{i(\theta_n+\beta)}| < L$, then for all $1 \le j \le n-1$, $|\Phi_{n-j,n}(e^{\sigma+i\theta}) - e^{i\theta^\top_{n-j+1}}| = [2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}]^{2(1-2^{-j})} \delta^{2^{-j}}(1 + O(\mathbf{c}^4))$, and so by Lemma 9 (the above estimate shows that $e^{-i\theta_{n-j}}\Phi_{n-j,n}(e^{\sigma+i\theta})$ is close enough to one of $e^{\pm i\beta}$ to apply this lemma),

$$\begin{aligned}\left| f'\left(e^{-\theta_{n-j}} \Phi_{n-j,n}(e^{\sigma+i\theta})\right)\right| &\asymp \beta^{1/2} |\Phi_{n-j,n}(e^{\sigma+i\theta}) - e^{i\theta^\top_{n-j+1}}|^{-1/2} \\ &= \beta^{1/2}[2(e^{\mathbf{c}} - 1)^{\frac{1}{4}}]^{-(1-2^{-j})} \delta^{-2^{-j-1}}(1 + O(\mathbf{c}^4)) \\ &\asymp \mathbf{c}^{2^{-(j+2)}} \delta^{-2^{-(j+1)}}.\end{aligned}$$

For $j = 0$, as $\Phi_{n,n}$ is the identity map,

$$\begin{aligned}|f'(e^{-i\theta_n}\Phi_{n,n}(e^{\sigma+i\theta}))| = |f'(e^{\sigma+i(\theta-\theta_n)})| &\asymp A_1 \beta^{1/2} \delta^{-1/2} \\ &\asymp A\mathbf{c}^{1/4}\delta^{-1/2}.\end{aligned}$$

Now if we combine the bounds for each term in the above product for $|\Phi_n'(e^{\sigma+i\theta})|$, we have

$$\begin{aligned}|\Phi_n'(e^{\sigma+i\theta})| &\ge \prod_{j=0}^{n-1} \left(A_1 \mathbf{c}^{2^{-(j+2)}} \delta^{-2^{-(j+1)}}\right) \\ &= A_1^n \mathbf{c}^{\frac{1}{2}(1-2^{-n})} \delta^{-(1-2^{-n})}.\end{aligned}$$

and a similar upper bound. Finally, δ is given by

$$\begin{aligned}\delta &= |e^{\sigma+i\theta} - e^{i(\theta_n+\beta)}| \\ &= |e^{\sigma+i\varphi} - 1| \\ &= (\sigma^2 + \varphi^2)^{1/2}(1 + O(\sigma + |\varphi|)) \\ &\asymp (\sigma^2 + \varphi^2)^{1/2},\end{aligned}$$

and so, modifying the constants as necessary, we have our result. □

We can now obtain our lower bound on the normalising factor.

Proposition 15. *If $\nu > 2$, then there exists a constant A depending only on ν such that for any fixed $T > 0$, for sufficiently small $\mathbf{c}$ and for $n < \lfloor T/\mathbf{c} \rfloor \wedge \tau_D$,*

$$Z_n \ge A^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})} \sigma^{-[\nu(1-2^{-n})-1]} \tag{22}$$

provided that $\sigma = \sigma(\mathbf{c}) \le \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$.

Proof. The normalising factor Z_n is given by the integral $\int_{\mathbb{T}} |\Phi_n'(e^{\sigma+i\theta})|^\nu \, \mathrm{d}\theta$, and Lemma 14 gives us a lower bound on the integrand for θ close to $\theta_n + \beta$:

$$|\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu \geq A^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}(\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})}$$

when $|\varphi| < L$.

We will now integrate our lower bound over the interval $(\theta_n + \beta - L, \theta_n + \beta + L)$. First, note that

$$\begin{aligned}\int_{-L}^{L} (\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})} \, \mathrm{d}\varphi &= \int_{-L/\sigma}^{L/\sigma} (\sigma^2+\sigma^2 x^2)^{-\frac{\nu}{2}(1-2^{-n})} \, \sigma \mathrm{d}x \\ &= \sigma^{1-\nu(1-2^{-n})} \int_{-L/\sigma}^{L/\sigma} \frac{\mathrm{d}x}{(1+x^2)^{\frac{\nu}{2}(1-2^{-n})}} \\ &\geq A'\sigma^{1-\nu(1-2^{-n})}\end{aligned}$$

for a constant A', since the integral term on the right hand side is increasing as $\mathbf{c} \to 0$ because $\sigma \ll L$. Note that this all remains true for any $\eta < 0$, and the fact that $\eta < -2$ will only be necessary in Section 3.2.

Finally, we can put together our bounds (and modify our constant A) to get

$$\begin{aligned}\int_{\theta_n+\beta-L}^{\theta_n+\beta+L} |\Phi_n'(e^{\sigma+i\theta})|^\nu \, \mathrm{d}\theta &\geq A^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})} \int_{-L}^{L} (\sigma^2+\varphi^2)^{\frac{\nu}{2}(1-2^{-n})} \, \mathrm{d}\varphi \\ &\geq A^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})} \sigma^{1-\nu(1-2^{-n})}\end{aligned}$$

as required. □

3.2 Concentration about each basepoint

Most of our upper bounds on $|\Phi_n'|$ will be established in Section 4, but we will find one here as it uses the estimates from the previous section. Using the terminology we introduce in Section 4 and illustrate in Figure 4, in this section we look at *singular points* which are within L of one of the "main" poles $e^{i(\theta_n \pm \beta)}$ so the estimate of Lemma 14 is valid, but are not within D of these poles.

Proposition 16. *Let $n < \lfloor T/\mathbf{c} \rfloor \wedge \tau_D$. For $\sigma(\mathbf{c}) \leq \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, then with $L = \mathbf{c}^{2^{N+1}}$ and $D = \mathbf{c}^{9/2}\sigma^{1/2} \ll L$,*

$$\frac{1}{Z_n} \int_{[-L,L]\setminus[-D,D]} |\Phi_n'(e^{\sigma+i(\theta_n\pm\beta+\varphi)})|^\nu \, \mathrm{d}\varphi = o(\mathbf{c}^4)$$

as $\mathbf{c} \to 0$.

Proof. Using the symmetry of our upper bound in Lemma 14, it is enough to show that $\int_D^L |\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu \, \mathrm{d}\varphi \ll \mathbf{c}^2 Z_n$. We have, modifying the constant A_2 where necessary,

$$\begin{aligned}
\int_D^L |\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu \,\mathrm{d}\varphi &\le A_2^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})} \int_D^L (\sigma^2+\varphi^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\mathrm{d}\varphi \\
&= A_2^n \frac{\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}}{\sigma^{\nu(1-2^{-n})-1}} \int_{D/\sigma}^{L/\sigma} (1+x^2)^{-\frac{\nu}{2}(1-2^{-n})}\,\mathrm{d}x \\
&\le A_2^n \frac{\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}}{\sigma^{\nu(1-2^{-n})-1}} \int_{D/\sigma}^{L/\sigma} x^{-\nu(1-2^{-n})}\,\mathrm{d}x \\
&= A_2^n \frac{\mathbf{c}^{\frac{\nu}{2}(1-2^{-n})}}{D^{\nu(1-2^{-n})-1}},
\end{aligned}$$

and so, using our lower bound on Z_n,

$$\begin{aligned}
\frac{\int_D^L |\Phi_n'(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu \,\mathrm{d}\varphi}{Z_n} &\le (A_2/A)^n \left(\frac{\sigma}{D}\right)^{\nu(1-2^{-n})-1} \\
&= (A_2/A)^n \left(\mathbf{c}^{-9/2}\sigma^{1/2}\right)^{\nu(1-2^{-n})-1}
\end{aligned}$$

which, since $\nu(1-2^{-n})-1 \ge \frac{1}{2}\nu - 1 > 0$, decays faster than any power of $\mathbf{c}$ as $\mathbf{c} \to 0$. □

Note that the above proof is the only place in which we use that $\eta < -2$. If $-2 \le \eta < 0$, then early on in the process we will attach the nth particle where $n = \tau_D$, and the inductive arguments we used in the proofs of Proposition 12 and Lemma 14 will fail. It then becomes extremely difficult to say how the process behaves, but the scaling limit as $\mathbf{c} \to 0$ is unlikely to be described by the Schramm-Loewner evolution.

3.3 Symmetry of the two most recent basepoints

There are two parts to the statement in Theorem 8 about convergence of h_{n+1} to the discrete measure $\frac{1}{2}(\delta_{\theta_n-\beta} + \delta_{\theta_n+\beta})$: the previous two sections and Section 4 establish that h_{n+1} is concentrated very tightly around $\theta_n \pm \beta$, and we will show here that the weight given to each of these two points is approximately equal.

Remark. Unlike the results from the previous two sections, the following proposition is not inductive, i.e. as long as $n < \lfloor T/\mathbf{c} \rfloor \wedge \tau_D$, the previous choices of angle do not have to have been symmetric, so it would still apply in the extreme case where $(\theta_n)_{n\in\mathbb{N}}$ is close to an arithmetic progression: $\theta_2 \approx \theta_1 + \beta$, $\theta_3 \approx \theta_2 + \beta, \cdots, \theta_n \approx \theta_{n-1} + \beta$.

Proposition 17. *Let $n < \lfloor T/\mathbf{c} \rfloor \wedge \tau_D$. Then*

$$\sup_{|\varphi|<D} \left| \log\left(\frac{\left|\Phi_n'\left(e^{\sigma+i(\theta_n+\beta+\varphi)}\right)\right|}{\left|\Phi_n'\left(e^{\sigma+i(\theta_n-\beta-\varphi)}\right)\right|} \right) \right| \le A\mathbf{c}^{11/4}$$

for some constant A depending only on T.

Proof. Let $z_\pm = \exp\left(\sigma + i\left[\theta_n \pm (\beta+\varphi)\right]\right)$ for $|\varphi| < D$, and write $\lambda_\pm = z_\pm - e^{i(\theta_n \pm \beta)}$. We can then write

$$\log\left(\frac{|\Phi_n'(z_+)|}{|\Phi_n'(z_-)|}\right) = \sum_{j=0}^{n-1} \log\left(\frac{|f_{n-j}'(\Phi_{n-j,n}(z_+))|}{|f_{n-j}'(\Phi_{n-j,n}(z_-))|}\right) \tag{23}$$

and so we can estimate each term in (23) separately.

The $j=0$ term is exactly 0, by the symmetry of $|f_n'|$ about θ_n.

For $1 \le j \le n-1$, we will use Lemma 4 of [20], which states that $f'(z) = \frac{f(z)}{z}\frac{z-1}{(z-e^{i\beta})^{1/2}(z-e^{-i\beta})^{1/2}}$, to compare the two derivatives in the jth term of (23). Write $z_\pm^j = \Phi_{n-j,n}(z_\pm)$, then the jth term in (23) is

$$|f_{n-j}'(z_\pm^j)| = \frac{|z_\pm^{j+1}|}{|z_\pm^j|}\frac{|z_\pm^j - e^{i\theta_{n-j}}|}{|z_\pm^j - e^{i\theta_{n-j+1}^\perp}|^{1/2}|z_\pm^j - e^{i\theta_{n-j+1}^\top}|^{1/2}} \tag{24}$$

There will be some telescoping in the product which allows us to find

$$\prod_{j=1}^{n-1}\frac{|z_\pm^{j+1}|}{|z_\pm^j|} = \frac{|z_\pm^n|}{|z_\pm^1|}.$$

Then recall that in Section 2 we derived estimates for the distance of $z_\pm^n$ from $e^{i\theta_{n-j+1}^\top}$ in terms of $|\lambda_\pm|$. So by Proposition 12, as $e^{i\theta_1^\top} = 1$,

$$|z_\pm^n - 1| = \left[2(e^{\mathbf{c}}-1)^{\frac{1}{4}}\right]^{2(1-2^{-n})}|\lambda_\pm|^{2^{-n}}(1+O(\mathbf{c}^4)) = O(\mathbf{c}^{17/4})$$

since $|\lambda_\pm|^{2^{-n}} \lesssim D^{2^{-n}} \ll L^{2^{-n}} \le \mathbf{c}^4$. Therefore $|z_\pm^n| = 1 + O(\mathbf{c}^{17/4})$, and similarly $|z_\pm^1| = 1 + O(\mathbf{c}^{17/4})$.

Having dealt with the first fraction in all derivatives (24) at once, we will tackle the remaining terms individually for each $1 \le j \le n-1$.

First note that by definition of $\theta_-^\top$, $|e^{i\theta_{n-j+1}^\top} - e^{i\theta_{n-j}}| = |e^{i\beta}-1|$. Hence, using Proposition 12 again,

$$\begin{aligned}|z_\pm^j - e^{i\theta_{n-j}}| &= |e^{i\theta_{n-j+1}^\top} - e^{i\theta_{n-j}}|\left[1 + O\left(\frac{|z_\pm^j - e^{i\theta_{n-j+1}^\top}|}{|e^{i\theta_{n-j+1}^\top} - e^{i\theta_{n-j}}|}\right)\right]\\ &= |e^{i\beta}-1|\left[1+O\left(\mathbf{c}^{-2^{-(j+1)}}|\lambda_\pm|^{2^{-j}}\right)\right]\\ &= |e^{i\beta}-1|\left[1+O\left(\mathbf{c}^{15/4}\right)\right]\end{aligned}$$

since $|\lambda_\pm|^{2^{-j}} \ll L^{2^{-(n-1)}} \le \mathbf{c}^4$.

Similarly,

$$|z_\pm^j - e^{i\theta_{n-j+1}^\perp}| = |e^{2i\beta}-1|(1+O(\mathbf{c}^{15/4})),$$

and finally, directly from Proposition 12,

$$|z^j_\pm - e^{i\theta^\top_{n-j+1}}| = \left[2(e^{\mathbf{c}}-1)^{\frac{1}{4}}\right]^{2(1-2^{-j})} |\lambda_\pm|^{2^{-j}}(1+O(\mathbf{c}^4)).$$

Note that for the three estimates we just found, the only part which depends on the choice of $\pm$ is the error term (as $|\lambda_+| = |\lambda_-|$). Hence the part of the ratio of $|f'_{n-j}(z^j_+)|$ to $|f'_{n-j}(z^j_-)|$ which comes from the second fraction in (24) is just $1+O(\mathbf{c}^{15/4})$.

We can therefore find a constant A (which does not depend on n or φ) such that for each $1 \le j \le n-1$, $\left|\log\left(\frac{|f'_{n-j}(z^j_+)|}{|f'_{n-j}(z^j_-)|}\right)\right| \le A\mathbf{c}^{15/4}$. As there are $O_T(\mathbf{c}^{-1})$ such terms in the product (23), we can obtain the result we wanted:

$$\left|\log\left(\frac{|\Phi'_n(z_+)|}{|\Phi'_n(z_-)|}\right)\right| = O_T(\mathbf{c}^{11/4}). \qquad \square$$

Now we can deduce that h_{n+1} gives (asymptotically) the same measure to the sets $(\theta_n+\beta-D, \theta_n+\beta+D)$ and $(\theta_n-\beta-D, \theta_n-\beta+D)$.

Remark. Recall that earlier we used the heuristic argument that if $\eta = -\infty$ (so we choose from points with the highest-order pole), then we attach the $(n+1)$th particle to one of $\theta_n \pm \beta$, with equal probability. With finite $\eta < -2$, the derivative $|\Phi'_n|$ in fact differs slightly at each of $e^{\sigma+i(\theta_n+\beta)}$ and $e^{\sigma+i(\theta_n-\beta)}$, and so choosing to attach a particle at $e^{i\theta}$ for θ maximising $|\Phi'_n(e^{\sigma+i\theta})|$ leads to a deterministic process rather than our SLE_4 limit.

However, when we have a finite $\eta < -2$, integrating over the range $(-D, D)$ around each $\theta_n \pm \beta$ means that only the asymptotic behaviour of $|\Phi'_n|$ needs to be the same in order to have symmetry between the two points $\theta_n \pm \beta$.

Corollary 18. *For $n < \lfloor T/\mathbf{c}\rfloor \wedge \tau_D$,*

$$\left|\int_{-D}^{D} h_{n+1}(\theta_n+\beta+\varphi)\,\mathrm{d}\varphi - \int_{-D}^{D} h_{n+1}(\theta_n-\beta-\varphi)\,\mathrm{d}\varphi\right| = O_T(\mathbf{c}^{11/4}). \tag{25}$$

Proof. From Proposition 17, we have

$$\begin{aligned}
&\int_{-D}^{D} h_{n+1}(\theta_n+\beta+\varphi)\,\mathrm{d}\varphi - \int_{-D}^{D} h_{n+1}(\theta_n-\beta-\varphi)\,\mathrm{d}\varphi \\
&\quad= \frac{1}{Z_n}\int_{-D}^{D}\left(|\Phi'_n(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu - |\Phi'_n(e^{\sigma+i(\theta_n-\beta-\varphi)})|^\nu\right)\mathrm{d}\varphi \\
&\quad= \frac{1}{Z_n}\int_{-D}^{D}\left(|\Phi'_n(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu - e^{O_T(\mathbf{c}^{11/4})}|\Phi'_n(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu\right)\mathrm{d}\varphi \\
&\quad= O_T\left(\mathbf{c}^{11/4}\frac{\int_{-D}^{D}|\Phi'_n(e^{\sigma+i(\theta_n+\beta+\varphi)})|^\nu\,\mathrm{d}\varphi}{Z_n}\right)
\end{aligned}$$

which is just $O_T(\mathbf{c}^{11/4})$ by definition of Z_n. $\square$

4 Analysis of the density away from the main basepoints

In this section, we will classify the points $\theta \in \mathbb{T}$ with $|\theta - (\theta_n \pm \beta)| \geq D$ (i.e. the set F_n from Theorem 8) into *regular points* R_n where $h_{n+1}(\theta) \ll 1$, and *singular points* S_n where $h_{n+1}(\theta) \gtrsim 1$. We make this classification based on how close the image $\Phi_n(e^{\sigma+i\theta})$ is to the common basepoint of the cluster, which is the image of all the poles of Φ_n', as we can see in Figure 4.

In Section 4.1 we make this classification explicit and establish a bound on h_{n+1} for the regular points. In Section 4.2 we analyse the singular points more carefully and establish an upper bound on $\int_{S_n} h_{n+1}(\theta)\,\mathrm{d}\theta$ using similar techniques to those which gave us a lower bound for $\int_{\mathbb{T}\setminus F_n} h_{n+1}(\theta)\,\mathrm{d}\theta$ in Section 3.1.

4.1 Regular points

In this section, we will establish a criterion for $\theta \in \mathbb{T}$ to be in our set of *regular* points for which $h_{n+1}(\theta) \ll 1$, based on the position of $\Phi_n(e^{\sigma+i\theta})$, as shown in Figure 4.

We will first derive an upper bound on $|\Phi_n'(w)|$ in terms of $|\Phi_n(w) - 1|$, so we can classify $w \in \Delta$ as a regular point using the distance of its image $\Phi_n(w)$ from 1.

Proposition 19. *Let $n < N(\mathbf{c}) \wedge \tau_D$. For $\theta \in \mathbb{R}$, let $w = \exp(\sigma + i\theta)$.*

For any function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $D^{2^{-N}}/\beta \leq a(\mathbf{c}) \leq \mathbf{c}^{3/2}$ for all $0 < \mathbf{c} < 1$, if

$$|\Phi_n(w) - 1| \geq \beta a(\mathbf{c}) \tag{26}$$

then, for sufficiently small $\mathbf{c}$,

$$|\Phi_n'(w)| \leq A^n \beta^{n/2} \left(\frac{a(\mathbf{c})}{8}\right)^{-\frac{1}{2}(2^n-1)} \tag{27}$$

where A is a universal constant independent of a.

Proof. We will use the estimate (13) from Lemma 10. For convenience, let $z = \Phi_n(w)$, and we will estimate $|\Phi_n'(w)| = |(\Phi_n^{-1})'(z)|^{-1}$ by using (11) and estimating each term separately, using Lemma 10 to obtain estimates on $\Phi_{n-j,n}(w) = \Phi_{n-j}^{-1}(z)$ by induction on j.

First we claim that for $A(\mathbf{c}) \leq \mathbf{c}^{1/2}$, and $\zeta \in \Delta\setminus(1, 1+d(\mathbf{c})]$, if we have $|\zeta - 1| \geq \beta A(\mathbf{c})$, then

$$\min_{\pm}(|f^{-1}(\zeta) - e^{\pm i\beta}|) \geq \frac{1}{4}\beta A(\mathbf{c})^2 \tag{28}$$

for all $\mathbf{c} < c_0$, where $c_0 > 0$ is a universal constant which doesn't depend on A.

To see this, suppose that $|f^{-1}(\zeta) - e^{i\beta}| < \frac{1}{4}\beta A(\mathbf{c})^2$. Then by Lemma 10, setting

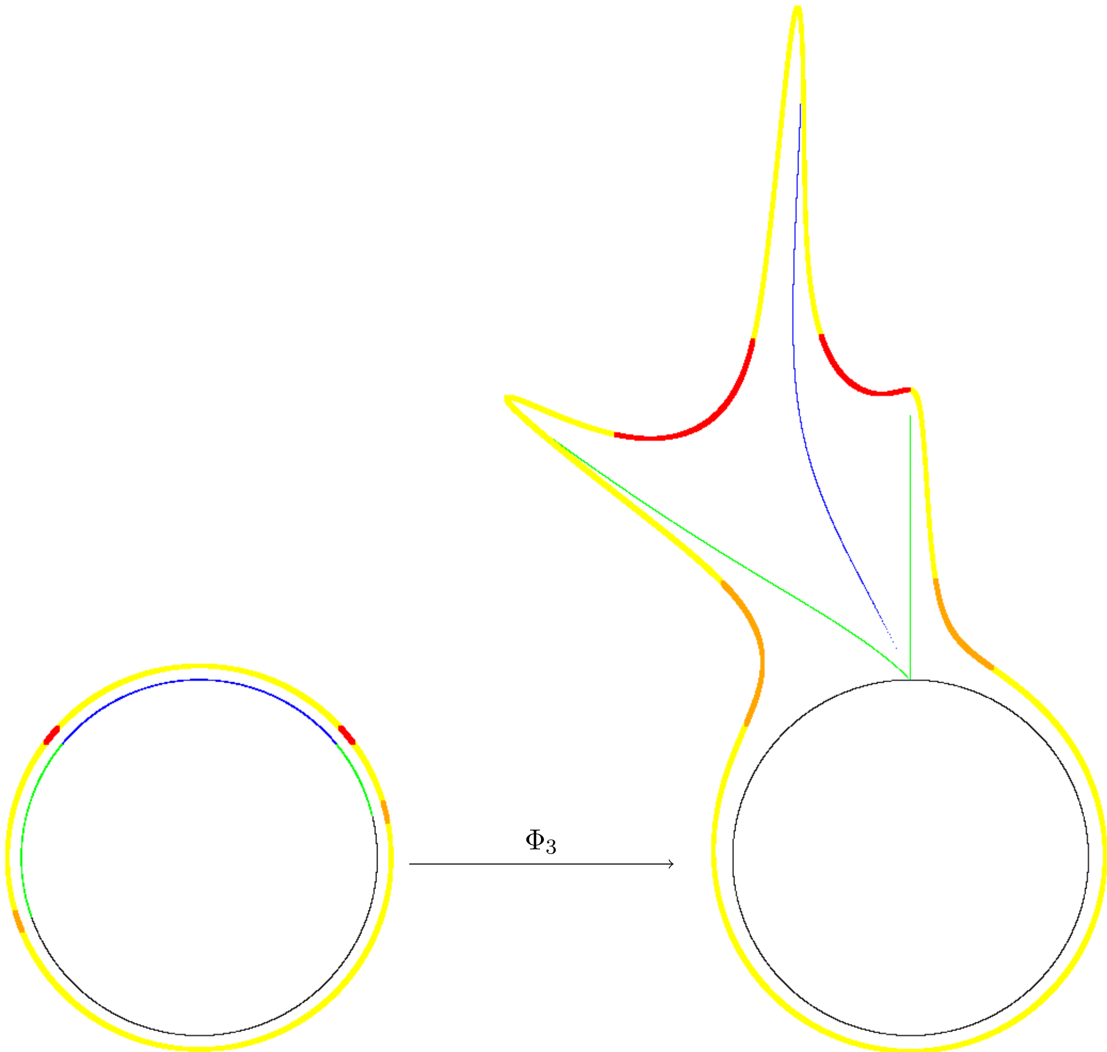

Figure 4: We can see on the left the three types of points in $e^\sigma\mathbb{T}$ for the three-slit cluster: we have the *singular points* in red and orange and the *regular points* in yellow. The right hand side of the diagram shows that a point on $e^\sigma\mathbb{T}$ is classified as regular if its image under Φ_n is far from the common basepoint (Proposition 19 in Section 4.1 shows that this implies $h_{n+1} \ll 1$), and the singular points are further classified into the two main (red) arcs containing $e^{i(\theta_n \pm \beta)}$, and the other (orange) singular points. We have $h_{n+1} \gtrsim 1$ for all singular points, but we obtained a lower bound on the integral of $|\Phi_n'|$ over the red regions in Section 3.1, and we will find an upper bound on the integral of this derivative over the orange regions in Section 4.2. Note that the choice of σ we have used for this diagram is around $\mathbf{c}^2$ rather than the much smaller $\mathbf{c}^{2^{1/\mathbf{c}}}$, which is necessary to make the envelope $\Phi_3(e^\sigma\mathbb{T})$ clear, but does mean that some "regular" points are closer to the common basepoints than the red "singular" points. With a sufficiently small σ this isn't the case.

$\varepsilon = 2^{1/4} - 1 > 0$, for sufficiently small $\mathbf{c}$,

$$\begin{aligned}|\zeta - 1| &= |f(f^{-1}(\zeta)) - f(e^{i\beta})| \\ &= 2(e^{\mathbf{c}} - 1)^{1/4}|f^{-1}(\zeta) - e^{i\beta}|^{1/2}(1 + O\left(A(\mathbf{c})^2 \vee \mathbf{c}^{1/2}A(\mathbf{c})\right)) \\ &< 2(\beta/2)^{1/2}(1+\varepsilon)\frac{1}{2}\beta^{1/2}A(\mathbf{c})(1+\varepsilon) \\ &= \beta A(\mathbf{c}),\end{aligned}$$

so we have shown the contrapositive for our claim.

The derivative $|\Phi_n'(w)|$ is decomposed in (11) into the product of terms of the form $\left|f'(e^{-i\theta_k}\Phi_{k,n}(w))\right|$, and so we can find an upper bound on $|\Phi_n'(w)|$ by obtaining lower bounds on each $|\Phi_{k,n}(w) - e^{i(\theta_k \pm \beta)}| = |\Phi_k^{-1}(z) - e^{i(\theta_k \pm \beta)}|$ for $0 \leq k \leq n-1$ and applying Lemma 9.

We claim that, for each $0 \leq k \leq n-1$,

$$|\Phi_k^{-1}(z) - e^{i\theta_{k+1}}| \geq \beta \times 8\left(\frac{a(\mathbf{c})}{8}\right)^{2^k} \tag{29}$$

and we will show this using induction. For $k = 0$, (29) is exactly the assumption (26) of this proposition. For $k \geq 1$, we assume as the induction step that

$$|\Phi_{k-1}^{-1}(z) - e^{i\theta_k}| \geq \beta \times 8\left(\frac{a(\mathbf{c})}{8}\right)^{2^{k-1}}$$

and aim to obtain (29) by applying (28).

Taking $A(\mathbf{c}) = 8\left(\frac{a(\mathbf{c})}{8}\right)^{2^{k-1}}$ in (28) gives us

$$|\Phi_k^{-1}(z) - e^{i\theta_{k+1}^\top}| \geq \beta \times 16\left(\frac{a(\mathbf{c})}{8}\right)^{2^k},$$

and so since $8\beta\left(\frac{a(\mathbf{c})}{8}\right)^{2^k} \geq 2D$ when $k \leq N \wedge \tau_D$ (for $\mathbf{c}$ sufficiently small),

$$\begin{aligned}|\Phi_k^{-1}(z) - e^{i\theta_{k+1}}| &\geq |\Phi_k^{-1}(z) - e^{i\theta_{k+1}^\top}| - |e^{i\theta_{k+1}} - e^{i\theta_{k+1}^\top}| \\ &\geq 16\beta\left(\frac{a(\mathbf{c})}{8}\right)^{2^k} - 2D \\ &\geq 8\beta\left(\frac{a(\mathbf{c})}{8}\right)^{2^k},\end{aligned}$$

verifying (29).

Then (29) tells us, using (28), that for each $0 \leq k \leq n-1$,

$$|\Phi_k^{-1}(z) - e^{i(\theta_k \pm \beta)}| \geq \beta \times 16\left(\frac{a(\mathbf{c})}{8}\right)^{2^k}, \tag{30}$$

and so, by Lemma 9, for $\mathbf{c}$ sufficiently small,

$$\begin{aligned} |\Phi_n'(w)| &= \prod_{k=0}^{n-1} |f_{k+1}'(\Phi_k^{-1}(z))| \\ &\le A^n \beta^{n/2} \prod_{k=1}^{n-1} \left(\beta^{1/2} \left[\beta \times 16 \left(\frac{a(\mathbf{c})}{8} \right)^{2^k} \right]^{-1/2} \right) \\ &= (A/4)^n \beta^{n/2} \left(\frac{a(\mathbf{c})}{8} \right)^{-\frac{1}{2}(2^n-1)} \end{aligned}$$

for a universal constant A. □

In the next section we will use these results with $a(\mathbf{c})$ equal to $\frac{L}{4\beta}$. We can easily check now that if we use this choice of a in Proposition 19 then, comparing (27) with (22), if σ decays as fast as $\mathbf{c}^{2^{2^N}}$ then $|\Phi_n'(z)|^\nu$ is far smaller than $\mathbf{c}Z_n$, for z away from the preimages of $e^{i\theta_1}$, and so if we classify our regular points as those θ for which $|\Phi_n(e^{\sigma+i\theta}) - 1| \ge \frac{L}{4}$ then we do have $\sup_{\theta \in R_n} h_{n+1}(\theta) \ll 1$.

4.2 Old singular points

In Section 3, we established a lower bound on the nth normalising factor Z_n. So to show that the probability is low that the $(n+1)$th particle is attached at a point in $E \subseteq \mathbb{T}$, we need to find an upper bound on $\int_E |\Phi_n'(e^{\sigma+i\theta})|^\nu \, \mathrm{d}\theta$.

We did this over certain regions in Section 4.1 by finding a bound $|\Phi_n'(e^{\sigma+i\theta})|^\nu \ll \mathbf{c}Z_n$. In this section we will consider *singular* points where we can have $|\Phi_n'(e^{\sigma+i\theta})|^\nu \gg Z_n$. However, if we look at Figure 4 we can see that not all singular points are close to the preimages $\theta_n \pm \beta$ of the base of the most recent particle; there are singular points at the preimages of the base of each particle. We will therefore need to estimate the integrand $|\Phi_n'|^\nu$ more carefully, and show that when integrated over the singular points around these old bases and normalised by Z_n, the resulting probability is small.

The first thing we need to do is to describe precisely which points we are integrating over. We have previously classified our points into regular points R_n and singular points S_n by looking at the distance $|\Phi_n(w) - 1|$. Points are singular when $|\Phi_n(w) - 1| < \beta a(\mathbf{c})$ (for an $a(\mathbf{c})$ we will specify later), and we will find a way of differentiating between the "new" singular points around the preimages of the nth particle's base and the "older" singular points around the preimages of the other particles' bases. To make this clear, we will first give names to all of these preimages.

Firstly, we have the two "most attractive" points: the preimages of the base of the most recent (nth) slit. We will call these two points $\widehat{z}_\pm^n = e^{i(\theta_n \pm \beta)}$. Now the other points correspond to the bases of the $n-1$ other slits in the cluster, and we will denote them by $\widehat{z}_j^n$ for $1 \le j \le n-1$. The base of the first slit is the image under f_1 of the choice of $e^{i(\theta_2 \pm \beta)}$ which is *not* close to $e^{i\theta_2}$. We defined this in Definition 13 to be $e^{i\theta_2^\perp}$, and so the

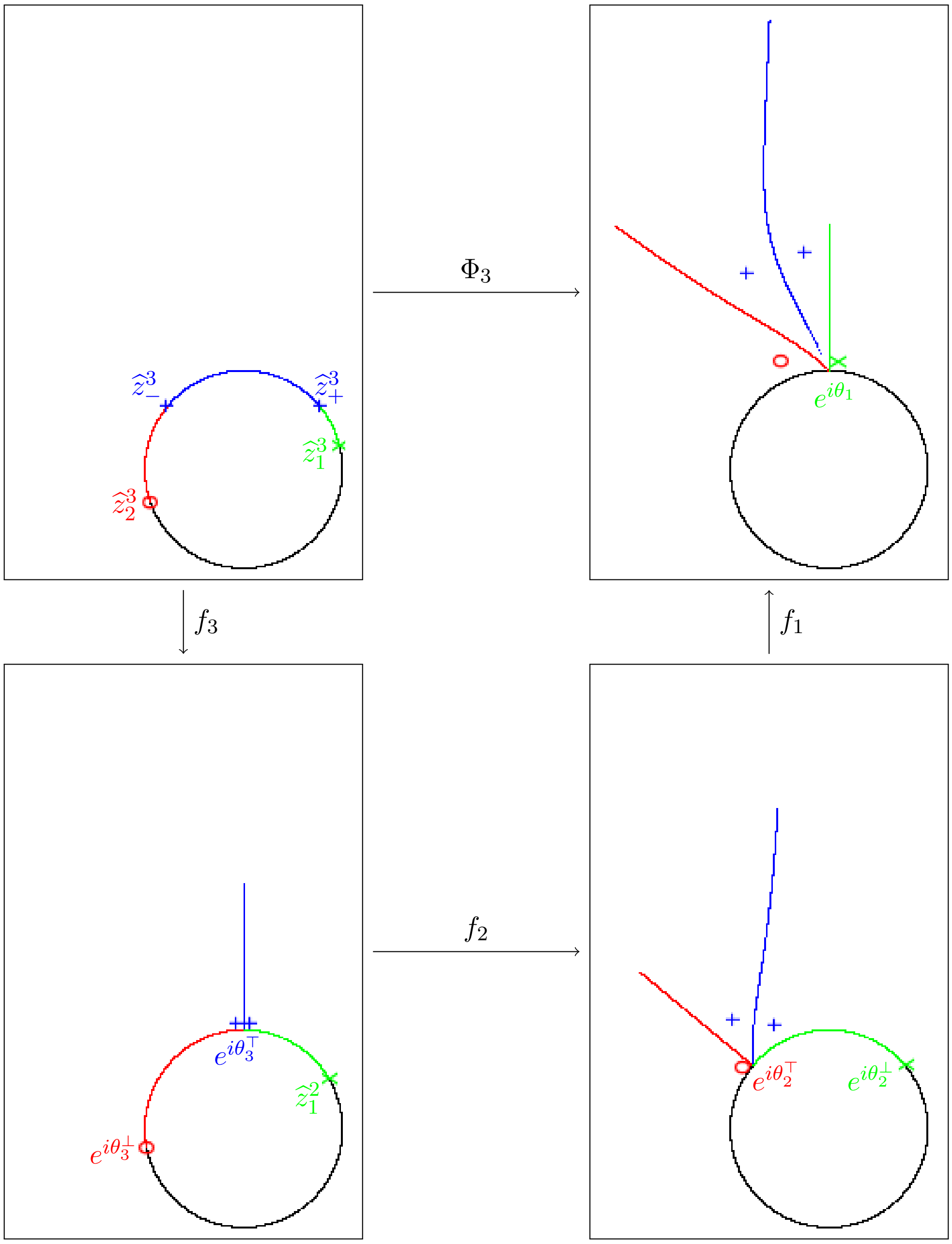

Figure 5: The construction of a cluster with three particles by composing the three maps f_3, f_2 and f_1. The top left diagram has labelled the four poles $\widehat{z}^3_\pm$, $\widehat{z}^3_2$ and $\widehat{z}^3_1$ of Φ_3' with text, and the markers $+$, $\times$ and $\circ$ have been used to track the images of $e^\sigma \widehat{z}$ for each pole $\widehat{z}$. By following the preimages of each point in the upper-right diagram through each map f_1, f_2 and f_3, we can see how we defined the "lesser" poles $\widehat{z}^3_2$ and $\widehat{z}^3_1$: for example, in the lower-right diagram $e^{i\theta_2^\perp}$ is a pole of f_1', its preimage under f_2 is $\widehat{z}^2_1$, and the preimage of $\widehat{z}^2_1$ under f_3 is $\widehat{z}^3_1$.

point sent to the base of the first slit by Φ_n is the preimage under $f_2 \circ \cdots \circ f_n = \Phi_{1,n}$ of $e^{i\theta_2^\perp}$, so set $\widehat{z}_1^n = \Phi_{1,n}^{-1}(e^{i\theta_2^\perp})$.

In general, when the jth slit is attached to the cluster by f_j, there are two points which are mapped to the base of the slit: $e^{i\theta_{j+1}^\top}$ (where the later slits are also attached), and $e^{i\theta_{j+1}^\perp}$, which has nothing else attached to it. Therefore, the point sent to the base of the jth slit by Φ_n is the preimage of $e^{i\theta_{j+1}^\perp}$ under $f_{j+1} \circ \cdots \circ f_n$. We can see this illustrated in Figure 5.

Definition 20. The base of the jth slit for $1 \le j \le n-1$ is the image of

$$\widehat{z}_j^n := \Phi_{j,n}^{-1}\left(e^{i\theta_{j+1}^\perp}\right) \tag{31}$$

under Φ_n.

Note that for all $n < N \wedge \tau_D$ and $1 \le j \le n-1$,

$$f_n(\widehat{z}_j^n) = \widehat{z}_j^{n-1}, \tag{32}$$

where we adopt the convention that $\widehat{z}_{n-1}^{n-1} = e^{i\theta_n^\perp}$.

Remark. We will bound $|\Phi_n'(w)|$ above when w is close to $\widehat{z}_j^n$, so first we will have to show that these points $\widehat{z}_j^n$ for $1 \le j \le n-1$ are not close to the points $e^{i(\theta_n \pm \beta)}$ where we have already shown $|\Phi_n'|$ is large.

Lemma 21. *For $n < N \wedge \tau_D$ and $1 \le j \le n-1$,*

$$|e^{i(\theta_n \pm \beta)} - \widehat{z}_j^n| \ge \mathbf{c}^{2^{n-j}},$$

when $\mathbf{c}$ *is sufficiently small.*

Proof. Assume for contradiction that $|e^{i(\theta_n+\beta)} - \widehat{z}_j^n| < \mathbf{c}^{2^{n-j}}$. By Lemma 10,

$$\begin{aligned}|e^{i\theta_n} - \widehat{z}_j^{n-1}| &= |f_n(e^{i(\theta_n+\beta)}) - f_n(\widehat{z}_j^n)| \\ &= 2(e^{\mathbf{c}}-1)^{1/4}\mathbf{c}^{2^{n-j-1}}\left(1 + O\left(\mathbf{c}^{1/4}\mathbf{c}^{2^{n-j-1}}\right)\right) \\ &< \frac{1}{2}\mathbf{c}^{2^{n-j-1}}\end{aligned}$$

for $\mathbf{c}$ smaller than some universal c_0 (with $(c_0-1)^{1/4} < 1/4$, and small enough to make the error term irrelevant), and so

$$|e^{i\theta_n^\top} - \widehat{z}_j^{n-1}| \le |e^{i\theta_n^\top} - e^{i\theta_n}| + |e^{i\theta_n} - \widehat{z}_j^{n-1}| < \mathbf{c}^{2^{n-j-1}}, \tag{33}$$

since $|e^{i\theta_n^\top} - e^{i\theta_n}| \lesssim D \ll \mathbf{c}^{2^{n-j-1}}$. Then, as $\theta_n^\top = \theta_{n-1} \pm \beta$ for some choice of $\pm$, we can apply this argument repeatedly until we arrive at $|e^{i\theta_{j+1}^\top} - \widehat{z}_j^j| < \mathbf{c}^{2^{j-j}} = \mathbf{c}$. But as we noted after (32), $\widehat{z}_j^j = e^{i\theta_{j+1}^\perp}$, and $|e^{i\theta_{j+1}^\top} - e^{i\theta_{j+1}^\perp}| \sim 4\mathbf{c}^{1/2} \gg \mathbf{c}$, and so we have our contradiction. □

Remark. In fact the lower bound in Lemma 21 is fairly generous; it would take only a small amount of extra work in the proof above to get a tighter bound of $\mathbf{c}^{2^{n-j-1}}$, and we could improve this even further as we used the weak bound $(e^{\mathbf{c}}-1)^{1/4} < \frac{1}{4}$ in the initial calculation. However, all we need from Lemma 21 is a bound which decays more slowly than $L = \mathbf{c}^{2^{N+1}}$, and so we have chosen the bound which leads to the simplest possible proof.

Remark. The following corollary (which we will not prove) is not used in the proof of our main results, but does answer a question we may worry about: if we know that w is within L of some $\widehat{z}_j^n$, then is that j uniquely determined?

Corollary 22. *For $n < N \wedge \tau_D$, if $1 \leq j < k \leq n-1$, then*

$$|\widehat{z}_j^n - \widehat{z}_k^n| \geq \mathbf{c}^{2^{n-j}}$$

for sufficiently small $\mathbf{c}$.

Remark. The next result will be useful in telling us for which points $\theta \in \mathbb{T}$ we can bound $|\Phi_n'(e^{\sigma+i\theta})|$ above using Proposition 19, and will later help us locate those points for which Proposition 19 does not provide an upper bound.

Lemma 23. *Suppose that $n < N \wedge \tau_D$, and let $w \in \Delta$. For all $\mathbf{c}$ sufficiently small, if $|\Phi_n(w) - 1| \leq \frac{L}{4}$, then either $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \leq L$, or there exists some $1 \leq j \leq n-1$ such that*

$$|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}| \leq \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}.$$

Proof. Suppose that there is no such j. We will show that $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \leq L$. We claim that $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\top}}| \leq \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$ for all $0 \leq j \leq n-1$ (where $\Phi_{0,n} = \Phi_n$ and $\theta_1^{\top} = \theta_1 = 0$). For $j = 0$ the claim is the true by assumption, and if the claim is true for $0 \leq j < n-1$, then by Lemma 11, as $|\Phi_{j,n} - e^{i\theta_{j+1}}| \leq \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j} + |e^{i\theta_{j+1}^{\top}} - e^{i\theta_{j+1}}| \leq \frac{\beta}{2}\left(\frac{L}{\beta}\right)^{2^j}$, for sufficiently small $\mathbf{c}$,

$$\begin{aligned}
\min(|\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\top}}|, |\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\perp}}|) &\leq \frac{\frac{1}{4}\beta^2\left(\frac{L}{\beta}\right)^{2^{j+1}}}{4(e^{\mathbf{c}}-1)^{1/2}}\left(1 + \frac{1}{2}\right) \\
&= \frac{3\beta/2}{4(e^{\mathbf{c}}-1)^{1/2}} \times \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j} \\
&\leq \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}
\end{aligned}$$

since $\beta \sim 2(e^{\mathbf{c}}-1)^{1/2}$ for small $\mathbf{c}$. But we supposed at the start of this proof that $|\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\perp}}| > \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^{j+1}}$, and so the above shows that $|\Phi_{j+1,n}(w) - e^{i\theta_{j+2}^{\top}}| \leq$

$\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^{j+1}}$, and by induction our claim holds. Finally, one more application of Lemma 11 after the $j = n-1$ case of our claim, $|\Phi_{n-1,n}(w) - e^{i\theta_n}| \le \frac{\beta}{2}\left(\frac{L}{\beta}\right)^{2^{n-1}}$, tells us that $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \le \frac{3\beta/2}{16(e^{\mathbf{c}}-1)^{1/2}} \beta \left(\frac{L}{\beta}\right)^{2^n} \ll L$, as required. □

Remark. We intend to use this lemma to find a precise expression for our set S_n of singular points and then we can make a precise estimate on the size of $|\Phi_n'(e^{\sigma+i\theta})|$ for $\theta \in S_n$ as we did in Lemma 14. For a singular point w, Lemma 23 tells us that for some j, $\Phi_{j,n}(w)$ is close to $e^{i\theta_{j+1}^{\perp}}$, and we now need to turn that into an estimate for the distance between w and $\Phi_{j,n}^{-1}(e^{i\theta_{j+1}^{\perp}}) = \widehat{z}_j^n$.

Corollary 24. *Suppose that $n < N \wedge \tau_D$, and let $w \in \Delta$. For all $\mathbf{c}$ sufficiently small, if $|\Phi_n(w) - 1| \le \frac{L}{4}$ then either $\min_{\pm} |w - e^{i(\theta_n \pm \beta)}| \le L$ or there exists some $1 \le j \le n-1$ such that*

$$|w - \widehat{z}_j^n| \le A^{n-j} \frac{\beta}{4} \left(\frac{L}{\beta}\right)^{2^j},$$

where A is some universal constant.

Proof. To deduce this from Lemma 23, we need only show that there is some constant A such that $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^{\perp}}| \le \frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j} \implies |w - \widehat{z}_j^n| \le A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$. Fix some $1 \le j \le n-1$. We will show that for $j \le k \le n-1$, $|\Phi_{k+1,n}(w) - \widehat{z}_j^{k+1}| \le A|\Phi_{k,n}(w) - \widehat{z}_j^k|$.

Fix a path $\gamma : (0,1] \to \Delta$ with $\lim_{\varepsilon \downarrow 0} \gamma(\varepsilon) = \widehat{z}_j^k$, $\gamma(1) = \Phi_{k,n}(w)$, and $|\gamma(t) - \widehat{z}_j^k| \le |\Phi_{k,n}(w) - \widehat{z}_j^k|$ for all $t \in (0,1]$. We can also choose γ in such a way that it has arc length $\ell := \int_\gamma |\mathrm{d}z| \le 2|\Phi_{k,n}(w) - \widehat{z}_j^k|$. By the fundamental theorem of calculus,

$$\begin{aligned} |\Phi_{k+1,n}(w) - \widehat{z}_j^{k+1}| &= |f_{k+1}^{-1}(\Phi_{k,n}(w)) - f_{k+1}^{-1}(\widehat{z}_j^k)| \\ &= \left| \int_\gamma (f_{k+1}^{-1})'(\zeta)\,\mathrm{d}\zeta \right| \\ &\le \ell \times \sup_{\zeta \in \gamma(0,1]} |(f_{k+1}^{-1})'(\zeta)| \\ &= \frac{\ell}{\inf_{\omega \in f_{k+1}^{-1}(\gamma(0,1])} |f_{k+1}'(\omega)|}. \end{aligned}$$

Now there must be some constant $M \ge 1$ such that $|\omega - e^{i\theta_{k+1}}| \ge \beta/M$ for all $\omega \in f_{k+1}^{-1}(\gamma(0,1])$ (otherwise, if $|\omega - e^{i\theta_{k+1}}| < \beta/M$, then it is easy to check using the explicit form of $f_{\mathbf{c}}$ from [15] that $|f_{k+1}(\omega) - e^{i\theta_{k+1}}(1+d)| = O(\beta/M^2)$, and so $|\widehat{z}_j^k - e^{i\theta_{k+1}}(1+d)| \le |f_{k+1}(\omega) - e^{i\theta_{k+1}}(1+d)| + |f_{k+1}(\omega) - \widehat{z}_j^k| \le \frac{1}{2}d$ for M sufficiently large, contradicting $\widehat{z}_j^k \in \mathbb{T}$). There is therefore, by Lemma 9, some constant A such that $\inf_{\omega \in f_{k+1}^{-1}(\gamma(0,1])} |f_{k+1}'(\omega)| \ge 2A^{-1}$.

We therefore obtain

$$|\Phi_{k+1,n}(w) - \widehat{z}_j^{k+1}| \leq A|\Phi_{k,n}(w) - \widehat{z}_j^k| \tag{34}$$

for all $j \leq k \leq n-1$, and so

$$|w - \widehat{z}_j^n| = |\Phi_{n,n}(w) - \widehat{z}_j^n| \leq A^{n-j}|\Phi_{j,n}(w) - \widehat{z}_j^j| \leq A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j},$$

as required. □

If we let L_j^n be the upper bound in Corollary 24, then we can now classify $\theta \in \mathbb{T}$ as regular or singular based only on its location: if $e^{\sigma+i\theta} \in \Delta$ is within L_j^n of $\widehat{z}_j^n$ for some j, then θ is *singular*, and otherwise it is *regular*. Now that we know where the singular points are, we can find a precise estimate for $|\Phi_n'|$ on S_n as we did in Lemma 14. The proof for this estimate will also be similar to the proof of Lemma 14.

Lemma 25. *Let $n < N \wedge \tau_D$, and $1 \leq j \leq n-1$. If $\mathbf{c}$ is sufficiently small, then for all $w \in \Delta$ with $|w| = e^\sigma$ and $|w - \widehat{z}_j^n| \leq A^{n-j}\frac{\beta}{4}\left(\frac{L}{\beta}\right)^{2^j}$, for A as in Corollary 24, we have*

$$|\Phi_n'(w)| \leq B^n \mathbf{c}^{\frac{n-j}{4}+1}\mathbf{c}^{\frac{1}{2}(1-2^{-j})}\frac{1}{\mathbf{c}^{2^{n-j}}}|w - \widehat{z}_j^n|^{-(1-2^{-j})}$$

where B is a universal constant.

Proof. We will complete the proof by finding bounds on $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^\perp}|$; an upper bound to show $|\Phi_{j,n}'(w)|$ is small, and a lower bound to show $|\Phi_j'(\Phi_{j,n}(w))|$ is small. The rest of the proof will be similar to the way we deduced Lemma 14 from Proposition 12.

First, we will estimate the positions of $\Phi_{n-1,n}(w), \Phi_{n-2,n}(w), \cdots, \Phi_{j,n}(w)$. As in the proof of Corollary 24, for $j+1 \leq k \leq n$,

$$\begin{aligned}|\Phi_{k-1,n}(w) - \widehat{z}_j^{k-1}| &= |f_k(\Phi_{k,n}(w)) - f_k(\widehat{z}_j^k)| \\ &\leq 2|\Phi_{k,n}(w) - \widehat{z}_j^k| \times \sup_{|\zeta - \widehat{z}_j^k| \leq |\Phi_{k,n}(w) - \widehat{z}_j^k|}|f_k'(\zeta)|,\end{aligned} \tag{35}$$

so we need only bound $|f_k'(\zeta)|$ for ζ close to $\widehat{z}_j^k$. We will also need inductively that $|\Phi_{k,n}(w) - \widehat{z}_j^k|$ is small in order to say that ζ *is* close to $\widehat{z}_j^k$.

Claim. For $j+1 \leq k \leq n$, $|\Phi_{k,n}(w) - \widehat{z}_j^k| \leq A^{n-j}\mathbf{c}^{3\times 2^n}$ for sufficiently small $\mathbf{c}$.

The claim is true for $k = n$, as $|w - \widehat{z}_j^n| \leq A^{n-j}\mathbf{c}^{1/2}\left(\frac{1}{2}\mathbf{c}^{2^{n+1}-1/2}\right)^{2^j} \leq A^{n-j}\mathbf{c}^{2^{n+j+1}-2^{j-1}} \leq A^{n-j}\mathbf{c}^{2^{n+2}-2^n}$. Then, if the claim holds for all $l \geq k$, we have

$$|\Phi_{l,n}(w) - \widehat{z}_j^l| \leq A^{n-j}\mathbf{c}^{3\times 2^n} \leq \frac{1}{2}\mathbf{c}^{2^{l-j}}$$

for all sufficiently small $\mathbf{c}$, and so, by Lemma 21 and the triangle inequality, for all ζ such that $|\zeta - \widehat{z}_j^l| \leq |\Phi_{l,n}(w) - \widehat{z}_j^l|$, we have $\min_\pm |\zeta - e^{i(\theta_l \pm \beta)}| \geq \frac{1}{2}\mathbf{c}^{2^{l-j}}$. Hence by Lemma 9,

$$|f_k'(\zeta)| \leq A_2 \frac{\mathbf{c}^{1/2}}{\mathbf{c}^{2^{l-j-1}}}$$

Therefore, by (35),

$$\begin{aligned}
|\Phi_{k-1,n}(w) - \widehat{z}_j^{k-1}| &\leq 2^{n-k+1}|\Phi_{n,n}(w) - \widehat{z}_j^n| \times \prod_{l=k}^{n} \left(A_2 \mathbf{c}^{\frac{1}{2}-2^{l-j-1}}\right) \\
&\leq (2A_2)^{n-k+1} A^{n-j} \frac{\beta}{4} \left(\frac{L}{\beta}\right)^{2^j} \mathbf{c}^{\frac{n-k+1}{4}} \mathbf{c}^{-\sum_{l=k-j-1}^{n-j-1} 2^l} \\
&\leq \left[(2A_2)^{n-k+1} \mathbf{c}^{\frac{n-k+1}{4}}\right] A^{n-j} \left(\mathbf{c}^{2^{n+1}-\frac{1}{2}}\right)^{2^j} \mathbf{c}^{-(2^{n-j}-2^{k-j-1})} \\
&\leq A^{n-j} \mathbf{c}^{2^{n+j+1}-2^{j-1}-2^{n-j}+2^{k-j-1}} \\
&\leq A^{n-j} \mathbf{c}^{2^{n+2}-2^{n-1}-2^{n-1}} \\
&= A^{n-j} \mathbf{c}^{3\times 2^n},
\end{aligned}$$

and so our claim holds by induction.

We can also see, from the same computation, that

$$|\Phi_{j,n}(w) - e^{i\theta_{j+1}^\perp}| = |\Phi_{j,n}(w) - \widehat{z}_j^j| \leq \mathbf{c}^{3\times 2^n}. \tag{36}$$

Then for each $j+1 \leq k \leq n$, as $\mathbf{c}^{3\times 2^n} \leq \frac{1}{2}\mathbf{c}^{2^{k-j}}$, we have by the triangle inequality and Lemma 21 that $|\Phi_{k,n}(w) - e^{i(\theta_k \pm \beta)}| \geq \frac{1}{2}\mathbf{c}^{2^{k-j}}$, and so by Lemma 9,

$$\begin{aligned}
|\Phi_{j,n}'(w)| &= \prod_{k=j+1}^{n} |f_k'(\Phi_{k,n}(w))| \\
&\leq \prod_{k=j+1}^{n} A_2 \frac{\beta^{1/2}}{(\frac{1}{2}\mathbf{c}^{2^{k-j}})^{1/2}} \\
&\leq (2A_2)^{n-j} \mathbf{c}^{\frac{n-j}{4} - \sum_{k=0}^{n-j-1} 2^k} \\
&= (2A_2)^{n-j} \mathbf{c}^{\frac{n-j}{4} - 2^{n-j}+1}
\end{aligned} \tag{37}$$

for sufficiently small $\mathbf{c}$.

We will next establish an upper bound on $|\Phi_j'(\Phi_{j,n}(w))|$. By the arguments used to prove Corollary 24, we have a lower bound on $|\Phi_{j,n}(w) - e^{i\theta_{j+1}^\perp}|$ as well as the upper bound we just established:

$$|\Phi_{j,n}(w) - e^{i\theta_{j+1}^\perp}| \geq A^{-(n-j)}|w - \widehat{z}_j^n|, \tag{38}$$

where A is a constant. The upper bound in (36) is less than $\mathbf{c}^{2^{n+1}}$, and so we can apply (the proof of) Lemma 14 to say

$$|\Phi_j'(\Phi_{j,n}(w))| \leq \frac{(A')^j}{A^{\frac{n-j}{2}}} \frac{\mathbf{c}^{\frac{1}{2}(1-2^{-j})}}{|w - \widehat{z}_j^n|^{1-2^{-j}}}, \tag{39}$$

and so we can combine (37) and (39) to obtain

$$\begin{aligned} |\Phi_n'(w)| &= |\Phi_{j,n}'(w)| \times |\Phi_j'(\Phi_{j,n}(w))| \\ &\leq \left(\frac{2A_2}{\sqrt{A}}\right)^{n-j} (A')^j \mathbf{c}^{\frac{n-j}{4} - 2^{n-j} + 1} \mathbf{c}^{\frac{1}{2}(1-2^{-j})} |w - \widehat{z}_j^n|^{-(1-2^{-j})} \\ &\leq (A'')^n \mathbf{c}^{\frac{n-j}{4}+1} \mathbf{c}^{\frac{1}{2}(1-2^{-j})} \frac{1}{\mathbf{c}^{2^{n-j}}} |w - \widehat{z}_j^n|^{-(1-2^{-j})} \end{aligned}$$

where $A'' = \max(\frac{2A_2}{\sqrt{A}}, A')$ is a constant. □

Corollary 26. *Let $n < N \wedge \tau_D$, and $1 \leq j \leq n-1$. Then for $L_j^n = A^{n-j} \frac{\beta}{4} \left(\frac{L}{\beta}\right)^{2^j}$, we have*

$$\int_{-L_j^n}^{L_j^n} |\Phi_n'(\widehat{z}_j^n e^{\sigma + i\varphi})|^\nu \, \mathrm{d}\varphi \leq B_\nu^n \frac{\mathbf{c}^{\nu(\frac{n-j}{4}+1)}}{\mathbf{c}^{\nu 2^{n-j}}} \mathbf{c}^{\frac{\nu}{2}(1-2^{-j})} \sigma^{-\left[\nu(1-2^{-j})-1\right]} \tag{40}$$

where B_ν is a constant depending only on ν.

Proof. As $|\widehat{z}_j^n e^{\sigma+i\varphi} - \widehat{z}_j^n| \asymp (\sigma^2 + \varphi^2)^{1/2}$, the bound follows immediately from Lemma 25 (in the same way as we obtained Proposition 15 from Lemma 14). □

5 Proof of main results

With the results of the previous sections, we are finally ready to prove our main scaling limit result, that the cluster $K_N^{\mathbf{c}}$ converges in distribution, as $\mathbf{c} \to 0$, to an SLE_4 cluster. To help picture the sets $S_{n,j}$ and R_n, it may be useful to refer to Figure 4.

Proof of Theorem 8. We want to show that $h_{n+1}(F_n) = \int_{F_n} h_{n+1}(\theta)\,\mathrm{d}\theta$ is small, and so we will decompose F_n into several sets.

Let $R_n = \{\theta \in \mathbb{T} : |\Phi_n(e^{\sigma+i\theta}) - 1| > \frac{L}{4}\}$, $S_n = F_n \setminus R_n$. We will further decompose S_n: let $T_n = \{\theta \in S_n : D < \min_\pm |e^{\sigma+i\theta} - e^{i(\theta_n \pm \beta)}| \leq L\}$, and for $1 \leq j \leq n-1$, let $S_{n,j} = \{\theta \in S_n : |e^{\sigma+i\theta} - \widehat{z}_j^n| \leq L_j^n\}$, where L_j^n is the bound appearing in Corollary 24, then Corollary 24 tells us that $S_n = T_n \cup \left(\bigcup_{j=1}^{n-1} S_{n,j}\right)$. We can then split the integral as

$$h_{n+1}(F_n) \leq h_{n+1}(R_n) + h_{n+1}(T_n) + \sum_{j=1}^{n-1} h_{n+1}(S_{n,j}). \tag{41}$$

We showed in Section 3.2 that $h_{n+1}(T_n) = o(\mathbf{c}^4)$, and so we only need to bound $h_{n+1}(R_n)$ and each $h_{n+1}(S_{n,j})$. Bounding $h_{n+1}(R_n)$ is simple using Proposition 19, as for any $\theta \in R_n$, $|\Phi_n'(e^{\sigma+i\theta})| \le A^n \beta^{n/2} \left(\frac{L}{32\beta}\right)^{-\frac{1}{2}(2^n-1)} \ll \mathbf{c}^4 Z_n$, and so $h_{n+1}(R_n) = o(\mathbf{c}^4)$. Finally, we will bound $h_{n+1}(S_{n,j})$. Using the bounds from Proposition 15 and Corollary 26, we have

$$\begin{aligned}
h_{n+1}(S_{n,j}) &\asymp \frac{1}{Z_n}\int_{-L_j^n}^{L_j^n} |\Phi_n'(\widehat{z}_j^n e^{\sigma+i\varphi})|^\nu \,\mathrm{d}\varphi \\
&\le \frac{B_\nu^n \frac{\mathbf{c}^{\nu(\frac{n-j}{4}+1)}}{\mathbf{c}^{\nu 2^{n-j}}} \mathbf{c}^{\frac{\nu}{2}(1-2^{-j})} \sigma^{-\left[\nu(1-2^{-j})-1\right]}}{A^n \mathbf{c}^{\frac{\nu}{2}(1-2^{-n})} \sigma^{-[\nu(1-2^{-n})-1]}} \\
&= \left(\frac{B_\nu}{A}\right)^n \underbrace{\mathbf{c}^{\nu(\frac{n-j}{4}+1)} \mathbf{c}^{-\frac{\nu}{2}(2^{-j}-2^{-n})}}_{o(\mathbf{c}^5)} \mathbf{c}^{-\nu 2^{n-j}} \sigma^{\nu(2^{-j}-2^{-n})} \\
&\ll \mathbf{c}^5 \left(\frac{B_\nu}{A}\right)^n \mathbf{c}^{-\nu 2^{n-j}} \sigma^{\nu 2^{-n}},
\end{aligned}$$

then as $\sigma \le \mathbf{c}^{2^{2^{1/\mathbf{c}}}}$, we have $\mathbf{c}^{-\nu 2^{n-j}} \sigma^{\nu 2^{-n}} \le \mathbf{c}^{\nu\left(2^{2^{1/\mathbf{c}}-n} - 2^{n-j}\right)} \le \mathbf{c}^{\nu\left(2^{2^{1/\mathbf{c}}-N} - 2^{N}\right)}$ which decays faster than exponentially in N. Therefore $h_{n+1}(S_{n,j}) = o_T(\mathbf{c}^5)$, and so $\sum_{j=1}^{n-1} h_{n+1}(S_{n,j}) = o_T(\mathbf{c}^4)$, establishing (7). The second bound, (8), comes immediately from Corollary 18. □

Remark. We have now seen that $(\theta_n^{\mathbf{c}})_{n \le \lfloor T/\mathbf{c} \rfloor}$ is very close to a simple symmetric random walk with step length $\beta \sim 2\mathbf{c}^{1/2}$, and so we expect $(\xi_t^{\mathbf{c}})_{t\in[0,T]} = (\theta_{\lfloor t/\mathbf{c}\rfloor})_{t\in[0,T]}$ will converge in distribution to $(2B_t)_{t\in[0,T]}$, where B is a standard Brownian motion. We can use a result on convergence of near-martingales to establish this rigorously.

Proof of Corollary 6. Corollary 3.8 of [16] gives us three conditions to check in order to obtain that as $\mathbf{c} \to 0$, $\xi^{\mathbf{c}} \to 2B$ in distribution with respect to the topology of $D[0,T]$; for any $t \in [0,T]$ we need all of the following to converge in probability:

$$\sum_{j=1}^{\lfloor t/\mathbf{c}\rfloor} \int_{-\pi}^{\pi} \varphi^2 h_{j+1}(\theta_j+\varphi) 1[|\varphi| > \varepsilon] \,\mathrm{d}\varphi \to 0 \text{ for any } \varepsilon > 0; \tag{42}$$

$$\sum_{j=1}^{\lfloor t/\mathbf{c}\rfloor} \int_{-\pi}^{\pi} \varphi^2 h_{j+1}(\theta_j+\varphi) \,\mathrm{d}\varphi \to 4t; \tag{43}$$

$$\sum_{j=1}^{\lfloor t/\mathbf{c}\rfloor} \left| \int_{-\pi}^{\pi} \varphi h_{j+1}(\theta_j+\varphi) \,\mathrm{d}\varphi \right| \to 0. \tag{44}$$

The jth term in the sum (42) is bounded by $\pi^2 \mathbb{P}(|\theta_{j+1}-\theta_j| > \varepsilon | (\theta_1, \cdots, \theta_j))$, which, once $\mathbf{c}$ is small enough that $\beta + D < \varepsilon$, is bounded by $\pi^2 \mathbb{P}[\tau_D \le j+1]$. Then (7) tells us that for all $1 \leq j \leq \lfloor T/\mathbf{c} \rfloor$, $\mathbb{P}[\tau_D \leq j+1] \leq A\mathbf{c}^4 + \mathbb{P}[\tau_D \leq j]$ almost surely, and so $\mathbb{P}[\tau_D \leq j+1] \leq A(j+1)\mathbf{c}^4 \leq AT\mathbf{c}^3$, which clearly converges to 0 in probability as $\mathbf{c} \to 0$.

Next, in (43), we know that h_{j+1} approximates $\frac{1}{2}(\delta_{\theta_j - \beta} + \delta_{\theta_j + \beta})$, and so we can write

$$\begin{aligned}
\int_{-\pi}^{\pi} \varphi^2 h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi &= \int_{\beta - D}^{\beta + D} \varphi^2 h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi + \int_{-\beta - D}^{-\beta + D} \varphi^2 h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi + E_j \\
&= (\beta + O(D))^2 \int_{\mathbb{T} \setminus F_j} h_{j+1}(\theta) \,\mathrm{d}\theta + E_j \\
&= \beta^2 + O(\beta D) + E_j',
\end{aligned}$$

where E_j' is the sum of $\int_{F_j} \theta^2 h_{j+1}(\theta) \,\mathrm{d}\theta \leq \pi^2 \mathbb{P}[\tau_D \leq j+1] + \pi^2 A\mathbf{c}^4 = O_T(\mathbf{c}^3)$ and $\int_{F_j} h_{j+1}(\theta) \,\mathrm{d}\theta = O_T(\mathbf{c}^3)$, and so

$$\begin{aligned}
\sum_{j=1}^{\lfloor t/\mathbf{c} \rfloor} \int_{-\pi}^{\pi} \varphi^2 h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi &= \lfloor t/\mathbf{c} \rfloor \beta^2 + O\left(\frac{\beta D}{\mathbf{c}}\right) + O(\mathbf{c}^2) \\
&= \lfloor t/\mathbf{c} \rfloor 4\mathbf{c}(1 + O(\mathbf{c}^{1/2})) + O\left(\frac{\beta D}{\mathbf{c}}\right) + O(\mathbf{c}^2) \\
&\to 4t
\end{aligned}$$

in probability as $\mathbf{c} \to 0$.

Finally, for the symmetry condition (44), combining (7) and (8) gives us

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} \varphi h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi \right| &\leq \left| \int_{\beta - D}^{\beta + D} \varphi h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi + \int_{-\beta - D}^{-\beta + D} \varphi h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi \right| + E_j \\
&= (\beta + O(D)) \left| \int_{\theta_n + \beta - D}^{\theta_n + \beta + D} h_{j+1}(\theta) \,\mathrm{d}\theta - \int_{\theta_n - \beta - D}^{\theta_n - \beta + D} h_{j+1}(\theta) \,\mathrm{d}\theta \right| + E_j \\
&= O(\mathbf{c}^{11/4}),
\end{aligned}$$

and so we have a bound

$$\sum_{j=1}^{\lfloor t/\mathbf{c} \rfloor} \left| \int_{-\pi}^{\pi} \varphi h_{j+1}(\theta_j + \varphi) \,\mathrm{d}\varphi \right| = O_T(\mathbf{c}^{7/4})$$

and so (44) tends to zero in probability as $\mathbf{c} \to 0$. □

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