ALGEBRAIC SPECTRAL SYNTHESIS AND CRYSTAL RIGIDITY

E. KASTIS AND S.C. POWER

Abstract. A spectral synthesis property is obtained for closed shift-invariant subspaces of vector-valued functions on the lattice $\mathbb{Z}^d$. This result generalises Marcel Lefranc's 1958 theorem for scalar-valued functions. Applications are given to homogeneous systems of multivariable vector-valued discrete difference equations and to the first-order flexibility of crystallographic bar-joint frameworks.

1. Introduction

It is the 60th anniversary of Marcel Lefranc's proof [21] that the discrete group $\mathbb{Z}^d$ admits spectral synthesis. Lefranc showed that every proper closed translation invariant space of complex-valued functions on $\mathbb{Z}^d$ is the closed linear span of a set of exponential monomials. In this context an exponential monomial is a multi-sequence of the form

$$k \to h(k)\omega^k, k \in \mathbb{Z}^d,$$

where h is a multivariable polynomial in d indeterminates and $\omega = (\omega_1, \dots, \omega_d)$ belongs to $(\mathbb{C}\backslash\{0\})^d$. We refer to such a polynomially weighted geometric multi-sequence as a *pg-sequence*, and we refer to ω as its multi-factor. The topology for this setting is the topology of coordinate-wise convergence. Lefranc showed, moreover, that the multi-factors can be chosen from a finite set.

Such spectral synthesis has been examined more recently, in 2007, for general discrete abelian groups, and has been shown to hold if and only if the torsion free rank is finite. See Laczkovich and Székelyhidi [20], [19] for further details.

In the present paper we give a generalisation of Lefranc's theorem for closed invariant subspaces of the space $C(\mathbb{Z}^d; \mathbb{C}^r)$ of vector-valued functions on $\mathbb{Z}^d$ (Theorem 3.10). For the proof we develop module variants of the original arguments for ideals. Moreover we substantially expand and clarify Lefranc's terse arguments which rely in part on some unclear references. We also show that the vectorial Lefranc theorem has immediate implications for the solution of systems of homogeneous discrete multivariable difference equations, and that it provides new methods and results in the analysis of first-order flexibility for crystallographic bar-joint frameworks.

We finish this introduction with some historical remarks. Lefranc's paper appeared in *Comptes Rendus*, having been communicated by Jacques Hadamard, who would have been close to his 93rd birthday. From a retrospective point of view the result can be seen as a highlight. However, at the time and in the ensuing years it seems to have been overshadowed by the development of harmonic analysis on general locally compact groups G, and spectral synthesis in the setting of weak star closed spaces of $L^\infty(G)$, as expounded in Benedetto [5], for example. The only other works of Lefranc that we have found are his 1972 doctoral thesis [22] and a short 1972 article [23], each with the title, *Sur certaines algèbres sur un groupe*. The short article is a *Comptes Rendus*

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note on the nature of idempotents in $B(G)$, the algebra of coefficients of unitary representations of a general group G. The announced result extends a theorem of Paul Cohen in the commutative case to general groups G, and is related to work of Walter Rudin, Henry Helson and Lefranc's PhD advisor, Pierre Eymard. (Eymard's earlier 1964 article [14], *L'Algebre de Fourier d-un groupe localement compact*, centres on analysis and makes no mention of algebraic spectral synthesis.)

Building on particular classical results of Schwartz, Malgrange, Ehrenpreis and Kahane, independent determinations of spectral synthesis were obtained in the setting of general locally compact abelian groups, by Elliott [12], [13], in 1965, and by Gilbert [15], in 1966. However the former articles have some incorrect claims in the case of infinite discrete rank groups, while Gilbert examines restricted contexts. It is interesting that these issues were only resolved in 2007 when Laczkovich and Székelyhidi [20] obtained the characterisation mentioned above, the proof of which makes use of Lefranc's theorem. The only other presentation of Lefranc's proof that we are aware of is in the 2005 article of De Boor and Ron [8], where there are applications to interpolation by multivariate splines.

We have found few biographical details of Marcel Lefranc beyond the fact that he was a professor of Mathematics at the University of Montpellier II, and that before this, in 1957, he was a lecturer there in mathematics and astronomy.

2. Preliminaries

In the main argument it is necessary to move between related modules for several Noetherian rings, namely the polynomial ring $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_d]$, the ring of Laurent polynomials $\mathbb{C}(z) = \mathbb{C}[z_1, \dots, z_d, z_1^{-1}, \dots, z_d^{-1}]$, and rings of formal power series and Taylor series. We first discuss these relationships and other preliminaries.

Definition 2.1. Let R be a Noetherian ring, let L be a submodule of an R-module N, and for $p \in R$, let $\lambda_p : N/L \to N/L$ be multiplication by p. Then L is a *primary submodule* of N if L is proper and for every p the map λ_p is either injective or nilpotent. If $P = \{p \in R : \lambda_p \text{ is nilpotent}\}$ then P is a prime ideal and L is said to be a *P-primary submodule* of N.

The Lasker-Noether theorem states that every submodule of a finitely generated module over a Noetherian ring is a finite intersection of primary submodules.

Definition 2.2. Let $M = Q_1 \cap \cdots \cap Q_s$ be a primary decomposition of the $\mathbb{C}[z]$-module M where Q_i is P_i-primary for distinct primes $P_i, 1 \leq i \leq s$. A *root sequence* for M is a set $\omega(1), \dots, \omega(s)$ of points in $\mathbb{C}^d$ where for each $1 \leq i \leq s$ the point $\omega(i)$ is a *root* of P_i in the sense that $p(\omega(i)) = 0$ for all $p(z)$ in P_i.

For more details and discussion see Ash [1], as well as Atiyah and MacDonald [2], Krull [18] and Rotman [27]. In particular (Chapter 1 of [1]) a strong form of the Lasker-Noether theorem implies that every finitely generated submodule M of a Noetherian ring over $\mathbb{C}$ has a decomposition as given in Definition 2.2, and this is called a *primary decomposition*. Moreover any such decomposition leads to a *reduced* primary decomposition with distinct primes ideals P_i, and this set of prime ideals is uniquely determined by M.

2.1. Primary ideals in $\mathbb{C}[z]$ and $\mathbb{C}[[z]]$.

Let $\mathbb{C}[[z]]$ be the ring of formal power series in $z_1, \dots, z_n$. We show that a primary $\mathbb{C}[z]$-module in $\mathbb{C}[z] \otimes \mathbb{C}^r$ with root 0 may be recovered from the $\mathbb{C}[[z]]$-module that it generates in $\mathbb{C}[[z]] \otimes \mathbb{C}^r$. We have not found a satisfactory reference for this and so we give a complete proof of Proposition 2.3. This connection plays a key role in Section 3.2.

Write $\mathbb{C}[z]_{(z)}$ for the ring of rational functions in $z_1, \dots, z_d$ that are continuous on some neighbourhood of 0. The notation reflects the fact that if (z) is the ideal in $\mathbb{C}[z]$ generated by $z_1, ..., z_d$

then the set $S = \mathbb{C}[z]\backslash(z)$ is multiplicative and $\mathbb{C}[z]_{(z)}$ is the localization $S^{-1}\mathbb{C}[z]$. Since (z) is maximal, and therefore prime, $\mathbb{C}[z]_{(z)}$ is a Noetherian local ring with unique maximal ideal $m_{(z)} = (z)\mathbb{C}[z]_{(z)}$. The ring $\mathbb{C}[[z]]$ is also a Noetherian local ring, with unique maximal ideal $m_{[z]} = (z)\mathbb{C}[[z]]$. Thus we have the natural ring inclusions

$$\mathbb{C}[z] \subset \mathbb{C}[z]_{(z)} \subset \mathbb{C}[[z]].$$

That these rings are Noetherian is discussed in Atiyah and MacDonald [2], for example.

Let Q be a finitely generated submodule of $\mathbb{C}[z]\otimes\mathbb{C}^r$, let $R[Q] := \mathbb{C}[z]_{(z)}\cdot Q$ be the corresponding $\mathbb{C}[z]_{(z)}$-module in $\mathbb{C}[z]_{(z)} \otimes \mathbb{C}^r$, and let $S[Q] = \mathbb{C}[[z]] \cdot Q$ be the corresponding $\mathbb{C}[[z]]$-module in $\mathbb{C}[[z]] \otimes \mathbb{C}^r$.

Proposition 2.3. *Let Q be a primary submodule in $\mathbb{C}[z] \otimes \mathbb{C}^r$ with associated root* 0. *Then $Q = S[Q] \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)$.*

For the proof we use a preliminary lemma which depends on the following Krull intersection theorem [1], [2].

Theorem 2.4. *Let R be a Noetherian local ring with maximal ideal m and let N be a finitely generated R-module. Then $\bigcap_{n=1}^{\infty} m^n N = \{0\}$.*

Lemma 2.5. *Let Q be a $\mathbb{C}[z]$-module in $\mathbb{C}[z]_{(z)} \otimes \mathbb{C}^r$. Then $R[Q] = S[Q] \cap (\mathbb{C}[z]_{(z)} \otimes \mathbb{C}^r)$.*

Proof. The inclusion of $R[Q]$ in the intersection is elementary. On the other hand the intersection is equal to the set

$$\left\{ P = \sum_{i=1}^{N} g_i f_i \in \mathbb{C}[z]_{(z)} \otimes \mathbb{C}^r : g_i \in \mathbb{C}[[z]], f_i \in Q \right\}.$$

Write $g_i = g_{i,0} + r_i$ where $g_{i,0}$ is the partial sum of the series for g_i for terms of total degree less than M. Then the element $P_0 = \sum_i g_{i,0} f_i$ belongs to $R[Q]$. Also the element $P_r = \sum_i r_i f_i = P - \sum_i g_{i,0} f_i$ belongs to $\mathbb{C}[z]_{(z)} \otimes \mathbb{C}^r$. Observe that P_r also belongs to $m^M_{[[z]]} \otimes \mathbb{C}^r$ and so it belongs to $m^M_{(z)} \otimes \mathbb{C}^r$. Thus P lies in the intersection

$$\bigcap_{M=0}^{\infty} (R[Q] + m^M_{(z)} \otimes \mathbb{C}^r). \tag{1}$$

By the Krull intersection theorem

$$\bigcap_{M=0}^{\infty} m^M_{(z)}((\mathbb{C}[z]_{(z)} \otimes \mathbb{C}^r)/R[Q]) = \{0\}$$

and so the intersection of (1) is equal to $R[Q]$, and the lemma follows. □

Proof of Lemma 2.3. By the previous lemma it suffices to show Q is equal to $R[Q] \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)$, which is the set

$$\left\{ h = \sum g_i f_i \in \mathbb{C}[z] \otimes \mathbb{C}^r : g_i \in \mathbb{C}[z]_{(z)}, f_i \in Q \right\}.$$

Let h belong to this set. Then h is equal to the finite sum $\sum_i \frac{p_i}{q_i} f_i = \sum a_i f_i / \prod q_i$, where $p_i, q_i \in \mathbb{C}[z]$ for all i. Thus $\sum_i a_i f_i = (\prod q_i) h \in Q$.

On the other hand, since Q is a primary $\mathbb{C}[z]$-module, the map

$$\lambda_{\prod q_i} : (\mathbb{C}[z] \otimes \mathbb{C}^r)/Q \to (\mathbb{C}[z] \otimes \mathbb{C}^r)/Q$$

is either nilpotent or injective. Since $\prod q_i$ does not vanish at the origin the map is not nilpotent and so it follows that $h \in Q$. □

3. Shift-invariant subspaces of $C(\mathbb{Z}^d; \mathbb{C}^r)$

Let $r \geq 1$ and let $C(\mathbb{Z}^d; \mathbb{C}^r)$ be the topological vector space of vector-valued functions $u : \mathbb{Z}^d \to \mathbb{C}^r$ with the topology of coordinatewise convergence. We also write this space as $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$. Let $e_1, \dots, e_d$ be the generators of $\mathbb{Z}^d$ and let $W_i, 1 \leq i \leq d$, be the forward shift operators, so that $(W_i u)(k) = u(k - e_i)$, for all k and each i. A subspace A of $C(\mathbb{Z}^d; \mathbb{C}^r)$ is said to be an *invariant subspace* if it is invariant for the shift operators and their inverses, or equivalently if $W_i A = A$ for each i.

3.1. $\mathbb{C}(z)$**-modules and their reflexivity.** There is a bilinear pairing $\langle p, u \rangle : \mathbb{C}(z) \times C(\mathbb{Z}^d) \to \mathbb{C}$ such that, for $p(z) = \sum_k a_k z^k$ in $\mathbb{C}(z)$ and $u = (u_k)_{k \in \mathbb{Z}^d}$ in $C(\mathbb{Z}^d)$, $\langle p, u \rangle = \sum_k a_k u_k$. Similarly, considering $C(\mathbb{Z}^d; \mathbb{C}^r)$ as the space $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$, for $p = (p_i) \in \mathbb{C}(z) \otimes \mathbb{C}^r$ and $u = (u_i) \in C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ we have the corresponding pairing $\langle p, u \rangle : \mathbb{C}(z) \otimes \mathbb{C}^r \times C(\mathbb{Z}^d) \otimes \mathbb{C}^r \to \mathbb{C}$, where

$$\langle p, u \rangle = \langle (p_i), (u_i) \rangle = \sum_{i=1}^{r} \langle p_i, u_i \rangle.$$

With this pairing the vector space dual of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ can be identified with $\mathbb{C}(z) \otimes \mathbb{C}^r$. Also, with the same pairing the dual space of the vector space $\mathbb{C}(z) \otimes \mathbb{C}^r$ is identified with $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$. Thus both spaces are reflexive, that is, equal to their double dual, in the category of vector spaces. These dual space identifications also hold in the category of linear topological spaces when each is endowed with the topology of coordinatewise convergence, since all linear functionals are automatically continuous with these topologies.

For a subspace A of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ we write $B = A^\perp$ for the annihilator in $\mathbb{C}(z) \otimes \mathbb{C}^r$ with respect to the pairing. Thus

$$B = \{p \in \mathbb{C}(z) \otimes \mathbb{C}^r : \langle p, u \rangle = 0, \text{ for all } u \in A\}.$$

Similarly for a subspace B of $\mathbb{C}(z) \otimes \mathbb{C}^r$ we write $B^\perp$ for the annihilator in $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ with respect to the same pairing.

Lemma 3.1. *Let A be a closed subspace of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ and let M be a closed subspace of $\mathbb{C}(z) \otimes \mathbb{C}^r$. Then $A = (A^\perp)^\perp$ and $M = (M^\perp)^\perp$.*

Proof. This follows from the dual space identifications and from the Hahn-Banach theorem for topological vector spaces ([10], IV. 3.15). □

The following lemma provides a route for the analysis of shift-invariant subspaces A in terms of the structure of their uniquely associated $\mathbb{C}(z)$-modules $B = A^\perp$. Note that it follows from the Noetherian property that $\mathbb{C}(z)$-modules in $\mathbb{C}(\mathbb{Z}^d) \otimes \mathbb{C}^r$ are necessarily closed.

Lemma 3.2. *A closed subspace A in $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ is an invariant subspace if and only if $A^\perp$ is a $\mathbb{C}(z)$-submodule of the module $\mathbb{C}(z) \otimes \mathbb{C}^r$.*

Proof. For all $a \in A$, $b \in B = A^\perp$ and $1 \leq i \leq d$ we have $\langle W_i a, b \rangle = \langle a, z_i^{-1} b \rangle$ and the lemma follows. □

3.2. **Primary decompositions of Noetherian modules.** The Lasker-Noether theorem for a nonzero finitely generated module M over a Noetherian ring R ensures that M is an intersection of a finite sequence of primary modules, $Q_1, \dots, Q_s$, where Q_i is P_i-primary for distinct prime ideals $P_1, \dots, P_s$.

Lemma 3.3. *The Laurent polynomial ring $\mathbb{C}(z)$ is a Noetherian ring.*

Proof. The argument is elementary. (Alternatively, if S is the multiplicative subset $\{z^k : k \in \mathbb{Z}^d_+\}$ then the ring $\mathbb{C}(z)$ is isomorphic to the localization $S^{-1}\mathbb{C}[z]$, and so is Noetherian by [27], Corollary 10.20.) □

For the rest of this section we let B be a proper $\mathbb{C}(z)$-module in $\mathbb{C}(z) \otimes \mathbb{C}^r$ with primary decomposition

$$B = Q_1 \cap \cdots \cap Q_s$$

as above, where the $\mathbb{C}(z)$-modules Q_i are P_i-primary. Also, we write $\mathbb{C}_*$ for $\mathbb{C}\backslash\{0\}$.

Lemma 3.4. *Fix i, with $1 \leq i \leq s$. Then there exists a point $\omega(i) \in \mathbb{C}^d_*$ such that if $p(z)$ is a polynomial in $P^*_i = P_i \cap \mathbb{C}[z]$ then $p(\omega(i)) = 0$.*

Proof. To see this note that the complex variety $V(P^*_i)$ is nonempty by Hilbert's Nullstellensatz [2], since P_i and hence P^*_i is a proper ideal. Moreover, there is a point $\omega(i)$ in this variety which is in $\mathbb{C}^d_*$. Indeed, if this were not the case then the monomial $z_1 \cdots z_d$ would be zero on the variety of P_i. It then follows from the strong Nullstellensatz ([27], Theorem 5.99) that for some index ρ the power $(z_1 z_2 \dots z_d)^\rho$ is in P^*_i. This implies $P_i = \mathbb{C}(z)$ which is a contradiction. □

Write B^* for the $\mathbb{C}[z]$-module $B \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)$ and note that B is recoverable from B^* as the set of elements $z^k p(z)$ with $p(z)$ in B^* and $k \in \mathbb{Z}^d$. It follows from this that we have the decomposition

$$B^* = Q^*_1 \cap \cdots \cap Q^*_s$$

where the implied modules Q^*_i (the intersections $Q_i \cap \mathbb{C}[z] \otimes \mathbb{C}^r$) are P^*_i-primary $\mathbb{C}[z]$-modules with distinct prime ideals P^*_i. Moreover each prime ideal P^*_i has a root $\omega(i)$ in $\mathbb{C}^d_*$ (rather than $\mathbb{C}^d$).

Finally, we obtain a reinterpretation of this primary decomposition for B^* in terms of modules for formal Taylor series.

For each $i = 1, \dots, s$ and associated root $\omega(i) \in \mathbb{C}^d_*$, as above, let ${Q_i}^{*b}$ be the "big" $\mathbb{C}_{\omega(i)}[[z]]$-module generated by the module ${Q_i}^*$, where $\mathbb{C}_{\omega(i)}[[z]]$ is the ring of formal Taylor series in the variables $z_1 - \omega(i)_1, z_2 - \omega(i)_2, \dots, z_d - \omega(i)_d$. Since Q^*_i is a primary module for the polynomial ring $\mathbb{C}[z]$ with root $\omega(i)$ it follows from Proposition 2.3 that $Q^*_i = {Q_i}^{*b} \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)$.

Thus B^* is the set of polynomials $p(z)$ in $\mathbb{C}[z] \otimes \mathbb{C}^r$ which lie in the big module ${Q_i}^{*b}$ for each i, and so

$$B^* = (Q^{*b}_1 \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)) \cap \cdots \cap (Q^{*b}_s \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)). \tag{2}$$

The rationale for considering this form of primary decomposition is that the rings $\mathbb{C}_{\omega(i)}[[z]]$ and their finitely generated modules in $\mathbb{C}_{\omega(i)}[[z]] \otimes \mathbb{C}^r$ have dual spaces consisting of *finitely* supported functionals. This follows in the same way as the duality between $\mathbb{C}(z)$ and $C(\mathbb{Z}^d)$. At the same time these finitely supported functionals may be represented in different ways, as we see in Proposition 3.6.

3.3. **Modules and duality for formal Taylor series.** We first recall Lefranc's differential operator formalism for scalar-valued trigonometric polynomials, as expressed in the next lemma.

Let $s_i \in \mathbb{N}$ and let $z_i^{[s_i]} = (z_i + 1)(z_i + 2) \dots (z_i + s_i)$. A polynomial $q(z) \in \mathbb{C}[z]$ may be written uniquely as

$$q(z) = \sum \beta_j z^{[j]}$$

where $[j] = ([j_1], \dots, [j_d])$ and (β_j) is a finitely nonzero multi-sequence with support in $\mathbb{Z}^d_+$.

Lemma 3.5. *Let* $p(z) = \sum a_k z^k \in \mathbb{C}[z]$ *and let* $e_{\omega,q}$ *be a pg-sequence in* $C(\mathbb{Z}^d)$. *Then*

$$\langle p(z), e_{\omega,q}\rangle = \sum_k a_k q(k)\omega_1^{k_1}\cdots\omega_d^{k_d} = \left[\sum_j \beta_j \partial_j (p(z)z^j)\right]_{z=\omega} \tag{3}$$

where ∂_j *is the partial derivative for the multi-index* $j \in \mathbb{Z}_+^d$.

Proof. Note first that for $p(z) = z^l$, a monomial in $\mathbb{C}[z]$, we have

$$\partial_j(p(z)z^j) = \partial_j(z^l z^j) = \left[\prod_{i=1}^d (l_i + j_i)(l_i + j_i - 1)\cdots(l_i + 1)\right] z^l = l^{[j]} z^l.$$

Thus, for $q(z) = z^{[j]}$ we have

$$[\partial_j(p(z)z^j)]_{z=\omega} = q(l)\omega^l = \langle z^l, (q(k)\omega^k)\rangle = \langle p(z), e_{\omega,q}\rangle.$$

(The pairing here is for $\mathbb{C}(z)$ and its dual space although we are restricting consideration to polynomials $p(z)$.) Since the partial differential operators are linear on $\mathbb{C}[z]$ it follows that the right hand side of the desired equality is linear in $p(z)$. It then follows, by linearity, that the equality holds also for general polynomials $q(z)$. □

For $\omega \in \mathbb{C}_*^r$ and (β_j) a finitely nonzero sequence write $L_{\omega,\beta}$ for the *differential operator functional* on the vector space $\mathbb{C}_\omega[[z]]$ of formal power series in $z_1 - \omega_1, \dots, z_d - \omega_d$ which is given by

$$L_{\omega,\beta} : s(z) \to \left[\sum \beta_j \partial_j (s(z)z^j)\right]_{z=\omega}.$$

Proposition 3.6. *The vector space dual of the power series ring* $\mathbb{C}_\omega[[z]]$ *is the space of differential operator functionals* $L_{\omega,\beta}$.

Proof. The dual space of $\mathbb{C}_\omega[[z]]$ is the space of finite linear combinations of the natural coefficient functionals. Thus it will be enough to show that for each j the j^{th}-coefficient evaluation functional F_j, for $j \in \mathbb{Z}_+^d$, is given by a differential operator functional $L_{\omega,\beta}$. Here F_j is defined by linearity and the requirement, in multinomial notation, is that $F_j((z-\omega)^k) = \delta_{j,k}$ for $k \in \mathbb{Z}_+^d$. Order $\mathbb{Z}_+^d$ and the corresponding monomials lexicographically. Evidently for $j = (0,\dots,0)$ the first functional F_j is a differential operator functional. We argue by induction on the lexicographic order. Fix $l \in \mathbb{Z}_+^d$ and let β be the sequence $(\delta_{l,k})_k$. Then

$$\left[\sum \beta_j \partial_j (s(z)z^j)\right]_{z=\omega} = \partial_l(s(z)z^l)_{z=\omega} = (\partial_l s)(\omega)\omega^l + F(s(z))$$

where F is a linear functional which is in the linear span of the functionals F_j where $j < l$. Thus

$$L_{\omega,\beta}(s(z)) = cF_l(s(z)) + F(s(z))$$

where $c = \omega^l$ is nonzero and it follows from the induction hypothesis that F_l has the desired form. □

Returning to vector-valued polynomials note that the vector space dual $(\mathbb{C}_\omega[[z]] \otimes \mathbb{C}^r)'$ is naturally identifiable with $(\mathbb{C}_\omega[[z]]') \otimes \mathbb{C}^r$ where $\mathbb{C}_\omega[[z]]'$ is the dual space of $\mathbb{C}_\omega[[z]]$. Thus we can identify $(\mathbb{C}_\omega[[z]] \otimes \mathbb{C}^r)'$ with the space of r-tuples

$$L_{\omega,\underline{\beta}} = (L_{\omega,\beta^1}, \dots, L_{\omega,\beta^r})$$

associated with the set of finite multi-sequences $\underline{\beta} = (\beta^1, \dots, \beta^r)$ where each $\beta^i = (\beta^i_k)$ is a finitely nonzero multi-sequence. The vector version of equation (3) takes the form

$$\langle p(z), u_{\omega,\underline{q}}\rangle = L_{\omega,\underline{\beta}}(p), \quad p(z) \in \mathbb{C}[z] \otimes \mathbb{C}^r, \tag{4}$$

where $u_{\omega,\underline{q}}$ is the *vectorial pg-sequence*

$$u_{\omega,\underline{q}} : k \to (e_{\omega,q_1}(k), \dots, e_{\omega,q_r}(k)).$$

and $\underline{q} = (q_1(z), \dots, q_r(z))$ is the vector of polynomials associated with $\underline{\beta}$. In view of Proposition 3.6 we can extend this pairing to a pairing

$$\langle \cdot, \cdot \rangle_\omega : (\mathbb{C}_\omega[[z]] \otimes \mathbb{C}^r) \times \{u_{\omega,\underline{q}} : q(z) \in \mathbb{C}[z] \otimes \mathbb{C}^r\} \to \mathbb{C}$$

by defining

$$\langle s(z), u_{\omega,\underline{q}} \rangle_\omega := L_{\omega,\underline{\beta}}(s(z)) \tag{5}$$

where $s(z) \in \mathbb{C}_\omega[[z]] \otimes \mathbb{C}^r$. In this way we describe the dual of the power series space $\mathbb{C}_\omega[[z]] \otimes \mathbb{C}^r$ in terms which extend the pairing of the submodule $\mathbb{C}[z] \otimes \mathbb{C}^r$ with vectorial *pg*-sequences (rather than in terms of sequences with finite support).

The next lemma follows readily as a corollary of Proposition 3.6 and the previous observations and is a module variant of a key lemma in Lefranc's argument [21] for ideals. The term "orthogonal" is in reference to the extended bilinear pairing above in the case $\omega = \omega(i)$.

Lemma 3.7. *Let $\omega(i)$ be a root in $\mathbb{C}_*^d$ for Q_i^*, as above. Then a vector-valued polynomial $p(z)$ in $\mathbb{C}[z] \otimes \mathbb{C}^r$ belongs to Q_i^* if and only if it is orthogonal to each vectorial pg-sequence $u_{\omega(i),h}$ which is orthogonal to Q_i^{*b}.*

Proof. Let $p(z)$ be a polynomial in $\mathbb{C}_{\omega(i)}[[z]] \otimes \mathbb{C}^r$ that is orthogonal to all vectorial *pg*-sequences that are orthogonal to Q_i^{*b}. Suppose that $p(z)$ is not in $Q_i^* = Q_i^{*b} \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)$. Then by the Hahn-Banach theorem there is a continuous linear functional that separates them, which is contradiction since all such functionals are given by the differential operator functionals. $\square$

3.4. **Shift-invariant subspaces.** The next two lemmas enable the transference of orthogonality and dual space density results between modules in $\mathbb{C}[z] \otimes \mathbb{C}^r$ and modules in $\mathbb{C}(z) \otimes \mathbb{C}^r$.

Lemma 3.8. *The vectorial pg-sequence $u_{\omega,\underline{h}}$ is orthogonal to the $\mathbb{C}(z)$-module B if and only if it is orthogonal to the $\mathbb{C}[z]$-module B^*.*

Proof. Note that for fixed $p(z) = (p_1(z), \dots, p_r(z))$ in B^* and fixed $h(z) = (h_1(z), \dots, h_r(z))$ in $\mathbb{C}[z] \otimes \mathbb{C}^r$ we have

$$\langle z^i p(z), u_{\omega,h} \rangle = \sum_{t=1}^{r} \langle z^i p_t(z), (h_t(k)\omega^k)_k \rangle = \pi(i)\omega^i.$$

for some polynomial $\pi(z)$. This is clear if the polynomials p_t, h_t are monomials and so it follows in general by linearity. If these terms are zero for all $i \in \mathbb{Z}_+^d$ then $\pi(i)$ is zero for all such i and so $\pi(z)$ is the zero polynomial, and hence the terms are equal to zero for all $i \in \mathbb{Z}^d$. Since B is the union of the spaces $z^i B^*$, for all multi-indices i, the lemma follows. $\square$

Lemma 3.9. *Let A be a closed invariant subspace of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ and let $A_+ \subseteq C(\mathbb{Z}_+^d) \otimes \mathbb{C}^r$ be the set of restrictions of sequences u in A. Also, let $\mathcal{P}$ be an invariant linear space of vectorial pg-sequences in A whose restrictions to $\mathbb{Z}_+^d$ form a dense set in A_+. Then $\mathcal{P}$ is dense in A.*

Proof. Identify A_+ with the corresponding set of $\mathbb{Z}^d$-sequences (w_k) which are zero if $k \notin \mathbb{Z}_+^d$. Similarly define $\mathcal{P}_+$. Since A is shift-invariant, each $u \in A$ is the limit of a sequence of elements of the form $(W_1 \cdots W_d)^{-n}(u^n)_+$, with $u^n \in A$. By the hypotheses, each $(u^n)_+$ is approximable by elements w_+ of $\mathcal{P}_+$ where w is a linear combination of *pg*-sequences in A. It follows that u is also approximable by the corresponding sequence of elements $(W_1 \cdots W_d)^{-n} w$ in A. Since these elements are linear combinations of *pg*-sequences in A the lemma follows. $\square$

Theorem 3.10. *Let A be a closed invariant subspace of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$. Then there is a finite set of geometric indices such that A is the closed linear span of the vectorial pg-sequences in A with geometric indices in this set.*

Proof. Let B be the annihilator of A with associated $\mathbb{C}[z]$-module B^*. By (2) we have the decomposition

$$B^* = (Q_1^{*b} \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)) \cap \cdots \cap (Q_s^{*b} \cap (\mathbb{C}[z] \otimes \mathbb{C}^r))$$

associated with any choice of roots $\omega(1), \dots, \omega(s)$ for the associated primary ideals Q_i. By Lemma 3.7 a vector polynomial $p(z)$ lies in B^* if and only if for each $1 \leq i \leq s$ it is orthogonal to every vectorial pg-sequence $u_{\omega(i),h}$ which is orthogonal to Q_i^*. It follows that the set of all the functionals L in $(\mathbb{C}[z] \otimes \mathbb{C}^r)'$ of the form

$$L_{\omega(i),h} : p(z) \to \langle p, u_{\omega(i),h} \rangle, \quad h \in \mathbb{C}[z] \otimes \mathbb{C}^r, 1 \leq i \leq s,$$

determine membership in B^*. That is, if $L(p(z)) = 0$ for all such L with $L(Q_i^{*b}) = 0$, for all $i \in 1, ..., s$ then $p(z) \in B^*$. By the reflexivity of $\mathbb{C}[z] \otimes \mathbb{C}^r$ it also follows that the subset of functionals which annihilate B^* has dense linear span in $(B^*)^\perp$. Let us write $\mathcal{S}_+$ for this subset and $\mathcal{S}$ for the set of corresponding functionals on $\mathbb{C}(z) \otimes \mathbb{C}^r$.

By Lemma 3.8 the set $\mathcal{S}$ consists of the differential operator functionals that annihilate B. In particular $\mathcal{S}$ is an invariant set for the shift operators and their inverses. By Lemma 3.9 it follows that the linear span of this set is dense in A, as desired. □

Remark 3.11. Proposition 3.6, in the scalar case, identifies the dual space of the power series ring as a space of differential operator functionals. In combination with the dual space identifications in Section 3.1 and the Hahn-Banach theorem this identification shows that the differential operator functionals determine ideal membership. The following theorem is a version of this which depends on a variant differential operator formalism giving constant coefficient differential operator functionals. (See also the somewhat cryptic indications in Eisenbud [11].) We also remark that Laczkovich [19] has obtained a generalisation of this for rings with countably many variables and differential operators which are infinite sums.

Theorem 3.12. *Let $\mathcal{I}$ be an ideal in the ring $\mathbb{C}[z]$ and let $p(z)$ be a polynomial in $\mathbb{C}[z]$ which is not in $\mathcal{I}$. Then there is a constant coefficient linear differential operator $D = \sum_{k \in \mathbb{Z}^d} c_k \partial^k$ and $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{C}^d$ such that $Df(\omega) = 0$ for $f \in I$ and $Dp(\omega) \neq 0$.*

4. Difference equations and crystal flexibility

Polynomially weighted geometric multi-sequences appear naturally in the classical solution of finite systems of inhomogeneous difference equations in the discrete setting of multivariable sequences. See de Boor et. al. [7], for example. Nevertheless the density of their linear span in the space of all solutions of a homogeneous system is much more subtle and in fact depends on Lefranc's theorem. We can obtain now the following vector-valued form of this.

Theorem 4.1. *Let A be the space of solutions of a finite homogeneous system of vector-valued multivariable difference equations. Then the set of vectorial pg-sequences in A has dense linear span in A.*

The proof follows from Theorem 3.10. Indeed let Δ_i be the difference operator $W_i^{-1} - I$ where, as before, W_i is the forward shift operator associated with the i^{th} coordinate, with $1 \leq i \leq d$. Then, for a multivariable vector-valued polynomial $p(z)$, the space of vector-valued multi-sequence solutions u to the equation $p(\Delta)u = 0$ is shift-invariant.

We next recall crystallographic bar-joint frameworks and their flex spaces [3], [9], [24].

A *crystal framework* $\mathcal{C}$ in $\mathbb{R}^d$ is a bar-joint framework (G,p), where $G=(V,E)$ is a countable simple graph and $p:V\to\mathbb{R}^d$ is an injective translationally periodic placement of the vertices as joints $p(v)$. It is assumed, moreover, that the periodicity is determined by a basis of d linearly independent vectors and that the corresponding translation classes for the joints and bars are finite in number. The assumption that $p:V\to\mathbb{R}^d$ is injective is not essential although with this relaxation one should assume that each bar $p(v)p(w)$ has positive length $\|p(v)-p(w)\|$.

The complex infinitesimal flex space $\mathcal{F}(\mathcal{C};\mathbb{C})$ is the vector space of $\mathbb{C}^d$-valued functions u on the set of joints which satisfy the first-order flex conditions

$$(u(p(v))-u(p(w)))\cdot(p(v)-p(w))=0,\quad vw\in E. \tag{6}$$

Also, $\mathcal{C}$ is said to be *infinitesimally rigid* if every infinitesimal flex is a velocity field of rigid motion type. The rigid motion velocity fields form a subspace $\mathcal{F}_{\rm rig}(\mathcal{C};\mathbb{C})$ of $\mathcal{F}(\mathcal{C};\mathbb{C})$ spanned by infinitesimal translations and infinitesimal rotations. These definitions follow the usual definitions for finite bar-joint frameworks [16].

Coordinates for the space $\mathcal{V}(\mathcal{C};\mathbb{C})$ of all velocity fields may be introduced, first, by making a (possibly different) choice of d linearly independent periodicity vectors for $\mathcal{C}$, which we shall denote as

$$\underline{a}=\{a_1,\dots,a_d\},$$

and, second, by choosing finite sets, F_v and F_e respectively, for the corresponding translation classes of the joints and the bars. With $n=|F_v|$ we may label the joints of $\mathcal{C}$ as $p_{\kappa,k}$, where $1\le\kappa\le n$ and $p_{\kappa,k}$ is the translate $p_{\kappa,0}+k_1a_1+\dots+k_da_d$. The velocity fields are therefore functions $u\in C(\mathbb{Z}^d;\mathbb{C}^{dn})$, where $u(0)$ is the composite velocity vector for the set of joints $p_{\kappa,0}$ in F_v. We say that u is a *pg-flex* with geometric factor ω if it is a first-order flex of the form $u_{\omega,h}$.

Theorem 4.2. *Let $\mathcal{C}$ be a crystallographic bar-joint framework. Then the vector space of complex infinitesimal flexes is the closed linear span of the pg-flexes.*

Proof. It is sufficient to note that $\mathcal{F}(\mathcal{C};\mathbb{C})$ is a translation invariant closed subspace of $C(\mathbb{Z}^d;\mathbb{C}^{dn})$ and to apply Theorem 3.10. □

A velocity field u is *geometric* if for some $\omega\in\mathbb{C}^d$ we have $u(k)=\omega^ku(0)$, for all $k\in\mathbb{Z}^d$, and is a *geometric flex* if it is also an infinitesimal flex. These are the *pg*-flexes for which h is a constant vector-valued polynomial. Also we say that u is a *linear* (resp. *quadratic*) *pg*-flex if the total degree of $h\in\mathbb{C}[z]\otimes\mathbb{C}^{dn}$ is 1 (resp. 2). Also, when $\omega=\underline{1}$ we refer to the flex $u=u_{\omega,h}$ as a *polynomially weighted periodic flex*, and as a *strictly periodic flex* if, moreover, h is constant.

The *geometric flex spectrum* of $\mathcal{C}$ associated with the periodic structure $\underline{a}$ is the set $\Gamma(\mathcal{C})$ of multi-factors $\omega\in\mathbb{C}^d_*$ for which there exists a nonzero geometric flex with multi-factor ω. This generalisation of the rigid unit mode (RUM) spectrum of $\mathcal{C}$ is introduced in [4] in connection with the analysis of localised infinitesimal flexes.

We may now use the previous theorem and the following degree reduction principle to obtain a characterisation of first-order rigidity.

Consider the ordering on $\mathbb{Z}^d_+$ for which $j<j'$ if $|j|<|j'|$, or $|j|=|j'|$ and j precedes j' in the lexicographic order. The monomials z^j inherit this total ordering and we define the multi-degree $\deg(p)$ of $p\in\mathbb{C}[z]$ as the multi-index j of the highest monomial of $p(z)$, and we define the total degree of $p(z)$ as $j_1+\dots+j_d$. Similarly the multi-degree and the total degree of $h(z)\in\mathbb{C}[z]\otimes\mathbb{C}^r$ are defined as the maximum of the corresponding degrees of the coordinate functions of h.

Lemma 4.3. *Let $u=u_{\omega,h}$ be a pg-sequence and let A_0 be the (unclosed) linear span of the $\mathbb{Z}^d$-translates of u. If $\delta(h)\ge 1$ then there is a pg-sequence $u_{\omega,h'}$ in A_0 with $\delta(h')=\delta(h)-1$.*

Proof. Let $j = (j_1, \dots, j_d)$ be the multi-degree $\deg(h)$ and let j_i be the first nonzero index of j. Then we may take

$$h'(z) = h(z_1, \dots, z_i + 1, \dots, z_d) - h(z)$$

□

Theorem 4.4. *A crystallographic bar-joint framework $\mathcal{C}$ is first-order rigid if and only if the following 3 conditions hold.*

(a) $\Gamma(\mathcal{C}) = \{\underline{1}\}$.
(b) Each strictly periodic flex and each linearly weighted periodic flex is a rigid motion flex.
(c) There are no quadratically weighted periodic flexes.

Proof. The necessity of the conditions is clear and so we assume that (a), (b) and (c) hold. By Theorem 4.2 $\mathcal{F}(\mathcal{C}, \mathbb{C})$ is the span of *pg*-flexes. Note first that (a) and the degree reduction lemma imply that there can be no nonzero *pg*-flex with $\omega \neq \underline{1}$. It suffices then to show that every nonzero *pg*-flex of the form $u = u_{\underline{1},h}$ is of rigid motion type. By (b) and (c) it follows that $\delta(u) \geq 3$. Now repeated application of the lemma leads to a *pg*-flex of total degree 2, contradicting (c). □

We remark that the requirements in (b) are equivalent to flexible lattice periodic rigidity ([9], [25]), and that (b) and (c) are each equivalent to maximal rank conditions for a finite matrix. In fact we show elsewhere that further arguments show that condition (c) is redundant.

The proof above also applies, with minor changes, to crystallographic bar-joint frameworks in the non-Euclidean spaces $(\mathbb{R}^d, \|\cdot\|_q)$, where $\|\cdot\|_q$ is the classical q-norm, for $1 \leq q < \infty, q \neq 2$. In this setting the space of rigid motion infinitesimal flexes reduces to the d-dimensional space of infinitesimal translations and we have the following characterisation of first-order rigidity. The condition on the bars in Theorem 4.5 guarantees the well-definedness of the flex equations which now take the form

$$(u(p(v)) - u(p(w))) \cdot (p(v) - p(w))^{(q-1)} = 0, \quad vw \in E, \tag{7}$$

where, for a vector $x = (x_1, \dots, x_d)$ in $\mathbb{R}^d$ we write

$$x^{(q-1)} = (\operatorname{sgn} x_1 |x_1|^{q-1}, \dots, \operatorname{sgn} x_d |x_d|^{q-1}).$$

Further details for such flex equations are given in [17].

Theorem 4.5. *Let $\mathcal{C}$ be a crystallographic bar-joint framework in the non-Euclidean space $(\mathbb{R}^d, \|\cdot\|_q)$, where $q \neq 2$ and $1 \leq q < \infty$. Suppose moreover that no bar vector $p(v) - p(w)$ of the framework is orthogonal to a principal axis. Then $\mathcal{C}$ is first-order rigid if and only if the following 3 conditions hold.*

(a) $\Gamma(\mathcal{C}) = \{\underline{1}\}$.
(b) Each strictly periodic flex is an infinitesimal translation.
(c) There are no linearly weighted periodic flexes.

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Dept. Math. Stats., Lancaster University, Lancaster LA1 4YF, U.K.
E-mail address: l.kastis@lancaster.ac.uk
E-mail address: s.power@lancaster.ac.uk