Riesz operators with finite rank iterates

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Abstract. Every infinite dimensional Banach space admits Riesz operators that are not finite rank. In this note we discuss conditions under which a Riesz operator, or some power thereof, is a finite rank operator.

AMS Classification: 47B06, 47B10.

Keywords: Riesz operator, finite rank operator, strictly singular operator, finite ascent, finite descent.

Introduction

Each infinite dimensional Banach space admits Riesz operators all of whose powers are infinite rank; see for instance Proposition 2.3, or Examples 2.2 and 4.2 (including the remarks following it) and Theorem 4.5 for examples of such operators defined on particular spaces. We investigate conditions that ensure that a Riesz operator, or some power of it, is finite rank.

This line of research originates in a result of Ghahramani [6, Theorem 1], who proved that a compact homomorphism defined on a C^*-algebra is a finite rank operator. Mathieu [11] generalised this result by proving that a weakly compact homomorphism defined on a C^*-algebra with range in a normed algebra is a finite rank operator. More recently, Koumba and the second named author [7, Example 3.1] have given an example of a homomorphism defined on a C^*-algebra that is a Riesz operator, but not a finite rank operator. However, if a homomorphism T defined on a C^*-algebra is a Riesz operator with finite ascent n, then T^n is a finite rank operator [7, Theorem 3.3]. In the present work, we seek similar results beyond the class of homomorphisms, that is, we consider Riesz operators defined on general Banach spaces.

1 Preliminaries

Let X be a Banach space. Throughout this paper, the Banach algebra of all bounded linear operators on X will be denoted by $\mathcal{L}(X)$ and the closed twosided ideal of all compact operators in $\mathcal{L}(X)$ by $\mathcal{K}(X)$. An operator $T \in \mathcal{L}(X)$ is called a *Riesz operator* if the coset $T + \mathcal{K}(X)$ is quasinilpotent in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$. The collection of these operators will be denoted by $\mathcal{R}(X)$. An operator $T \in \mathcal{L}(X)$ is called an *inessential operator* if the coset $T + \mathcal{K}(X)$ belongs to the radical of the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Hence, every inessential operator is a Riesz operator. The collection of inessential operators on a Banach space is the largest ideal consisting of Riesz operators and this ideal will be denoted by $\mathcal{I}(X)$. An operator $T \in \mathcal{L}(X)$ is called *strictly singular* if there is no infinite dimensional closed subspace Z of X such that $T : Z \to T(Z)$, the restriction of T to Z, is an isomorphism. The closed ideal of strictly singular operators in $\mathcal{L}(X)$ will be denoted by $\mathcal{S}(X)$. An operator $T \in \mathcal{L}(X)$ is called *nuclear* if there are sequences (y_n) in X and (f_n) in X^* such that

$$\sum_{n=1}^{\infty} \|f_n\| \, \|y_n\| < \infty \qquad \text{and} \qquad Tx = \sum_{n=1}^{\infty} f_n(x) y_n \qquad (x \in X).$$

The ideal of nuclear operators in $\mathcal{L}(X)$ will be denoted by $\mathcal{N}(X)$. If $\mathcal{F}(X)$ denotes the ideal of finite rank operators on X, then it is well known that

$$\mathcal{F}(X) \subset \mathcal{N}(X) \subset \mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{I}(X) \subset \mathcal{R}(X). \tag{1.1}$$

Unlike the other sets in (1.1), $\mathcal{R}(X)$ is in general not an ideal. Also, there are examples to illustrate that the inclusions can be proper. However, there are Banach spaces for which some of these ideals coincide. For instance, if H is a Hilbert space then $\mathcal{K}(H) = \mathcal{S}(H) = \mathcal{I}(H)$. We refer the reader to [5] for basic properties of Riesz operators.

For $T \in \mathcal{L}(X)$, denote the *null space* of T by $N(T)$ and the *range* of T by $R(T)$. The smallest nonnegative integer n such that $N(T^n) = N(T^{n+1})$

is called the *ascent* of T and it is denoted by $\alpha(T)$. If no such n exists, set $\alpha(T) = \infty$. The *descent* of T, $\delta(T)$, is the smallest nonnegative integer n such that $R(T^n) = R(T^{n+1})$. If no such n exists, set $\delta(T) = \infty$. The *nullity* of $T \in \mathcal{L}(X)$ is $n(T) = \dim N(T)$ and the *deficiency* of $T \in \mathcal{L}(X)$ is $d(T) = \dim X/R(T)$. An operator $T \in \mathcal{L}(X)$ is called a *semi Fredholm* operator if it has closed range and it has finite nullity or finite deficiency. It is called an *upper semi Fredholm* operator if it has closed range and finite nullity. It is called a *lower semi Fredholm* operator if it has finite deficiency (in which case $R(T)$ is automatically closed).

2 Riesz operators with finite ascent

This section is motivated by the following result from [7].

Theorem 2.1 ([7], Theorem 2.2) *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a Riesz operator with $\alpha(T) = k < \infty$. If $R(T^k) + N(T^k)$ is closed in X, then T^k is a finite rank operator.*

Our next example illustrates that there exist Riesz operators with finite ascent such that no power of T is finite rank. This shows in particular that the hypothesis that $R(T^k)+N(T^k)$ be closed cannot be omitted in Theorem 2.1. One such example is the Volterra operator.

Example 2.2 *Let $X = C[0,1]$ be the Banach space of all complex valued continuous functions defined on the interval $[0,1]$ and let $T \in \mathcal{L}(X)$ be the Volterra operator given by*

$$(Tf)(x) = \int_0^x f(t)dt$$

for all $f \in X$ and $x \in [0,1]$. Then T is compact, quasinilpotent and injective, and T^k is not finite rank for all $k \in \mathbb{N}$.

In view of [3, Theorem 2.2.5], since T^k is injective and compact for each $k \in \mathbb{N}$, it cannot have closed range.

In fact, any infinite dimensional Banach space admits a nuclear operator T with finite ascent such that no power of T is finite rank, as we shall now show.

Proposition 2.3 *Let X be an infinite dimensional Banach space. Then there exists a nuclear operator T on X such that T has finite ascent and T^n has infinite dimensional range for each $n \in \mathbb{N}$.*

Proof. Since X is infinite dimensional it contains a basic sequence $(b_n)_{n\in\mathbb{N}}$, [10, Theorem 1.a.5]. We may suppose that $\|b_n\| = 1$ for all $n \in \mathbb{N}$. By the Hahn-Banach extension theorem, we may extend the coordinate functionals from the closed span of $\{b_n : n \in \mathbb{N}\}$ to X without increasing their norms; let $f_n \in X^*$ be the extended n^{th} coordinate functional, so that $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$ and $f_n(b_m) = \delta_{m,n}$ for all $m, n \in \mathbb{N}$. We can now define a nuclear operator $T : X \to X$ by

$$Tx = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n} b_n.$$

Then T^m has infinite dimensional range for each $m \in \mathbb{N}$ because $T^m b_n = 2^{-mn} b_n$ for each $n \in \mathbb{N}$.

We claim that $N(T) = N(T^2)$. Only $\supseteq$ requires a proof. Suppose that $x \in N(T^2)$. Then

$$0 = T(Tx) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n} Tb_n = \sum_{n=1}^{\infty} \frac{f_n(x)}{4^n} b_n.$$

Since (b_n) is a basic sequence, this implies that $\frac{f_n(x)}{4^n} = 0$ for each n and hence $f_n(x) = 0$ for each $n \in \mathbb{N}$. Consequently we have $Tx = 0$, so that $x \in N(T)$, which proves the claim. Hence, T has finite ascent; in fact, it is either 0 (if and only if T is injective) or 1. □

3 Riesz operators with finite descent

The purpose of this section is to show that having finite descent is a much stronger condition than having finite ascent for Riesz operators. It follows from the definition of a Riesz operator that every quasinilpotent operator $T \in \mathcal{L}(X)$ is a Riesz operator. However, if T is quasinilpotent and $\delta(T) = p < \infty$, then $T^p = 0$ [1, Exercise V.6.6], and so T^p is a finite rank operator. Our main result is as follows and it illustrates that the previous statement holds in general for Riesz operators.

Theorem 3.1 *Let T be a Riesz operator on a Banach space X. Then the following three conditions are equivalent:*

(a) *T has finite descent;*

(b) *T has finite ascent and finite descent;*

(c) *Some power of T is finite rank.*

To prove this result we need two lemmas. The first of these is probably well known, but for convenience we outline a proof.

Lemma 3.2 *Let X be a vector space and let $T : X \to X$ be a linear mapping with finite dimensional range. Then T has finite ascent and finite descent.*

Proof. Since $R(T)$ is finite dimensional, it cannot contain an infinite, strictly decreasing sequence of subspaces, so T must have finite descent. Likewise, as $N(T)$ has finite codimension in X, it cannot be contained in an infinite, strictly increasing sequence of subspaces, and therefore T has finite ascent. □

The following result is essentially due to Oudghiri and Zohry [13].

Lemma 3.3 *Let X be a Banach space which admits a semi Fredholm operator that is also a Riesz operator. Then X is finite dimensional.*

Proof. If X admits an upper semi Fredholm operator that is also a Riesz operator, then this is [13, Proposition 2.4].
Otherwise X admits a lower semi Fredholm operator. The dual of this operator is then an upper semi Fredholm operator and Riesz by [12, Theorem

16.4] and [5, Theorem 3.22]. By the first part of the proof X^* is finite dimensional and therefore X is finite dimensional. □

We are now ready to prove Theorem 3.1.

Proof. (a)⇒(c): Suppose that $\delta(T) < \infty$. Set $m = \max\{\delta(T), 1\} \in \mathbb{N}$ and $N = N(T^m)$, and observe that $X = N + R(T^m)$ by [2, Lemma 3.2]. Since T is a Riesz operator, T^m is also a Riesz operator. This together with N being T^m-invariant, implies that the induced operator $S : X/N \to X/N$ defined by $S(x + N) = T^m x + N$ is a Riesz operator, see [5, Theorem 3.23]. Moreover, S is surjective. Indeed, each element of X/N has the form $y + N$ for some $y \in X$. Since $X = N + R(T^m)$, $y = w + T^m x$ for some $w \in N$ and $x \in X$. Then $y - T^m x \in N$, so that $S(x + N) = T^m x + N = y + N$, as desired. This implies that S is a lower semi Fredholm operator. In view of Lemma 3.3, X/N is finite dimensional. Since T^m factors through X/N by the Fundamental isomorphism theorem, we conclude that $R(T^m)$ is finite dimensional.

(b)⇒(a): trivial.

(c)⇒(b): this follows from Lemma 3.2. □

It follows from Theorem 3.1 that the quasinilpotent operator in Example 2.2 and the nuclear operator in Proposition 2.3 do not have finite descent.

4 Riesz operators with closed range

Unlike compact operators with closed range, noncompact Riesz operators with closed range need not be finite rank operators. A simple example to illustrate this is the following.

Example 4.1 *If X is an infinite dimensional Banach space, then the operator $T : X \oplus X \to X \oplus X$ defined by $T(x_1, x_2) = (0, x_1)$ is a Riesz operator with closed range and T is not finite rank. In particular, if X is a Hilbert* *space, then $X \oplus X$ is a Hilbert space and T is a Riesz operator with closed range that is not finite rank.*

The following much more sophisticated example is due to Burlando [4, Example 5.4].

Example 4.2 *Let H be an infinite dimensional Hilbert space. Then there exists a nonnilpotent, quasinilpotent operator $T : H \to H$ such that T^k has closed range for all $k \in \mathbb{N}$.*

We observe that the quasinilpotent, nonnilpotent operator T in Example 4.2 has the property that T^k is not finite rank for all $k \in \mathbb{N}$: indeed, if T^k is finite rank for some $k \in \mathbb{N}$, then Lemma 3.2 implies that T^k has finite descent p (say), and so $T^{kp} = 0$ by [1, Exercise V.6.6]. Our next result provides a condition under which a Riesz operator defined on a Hilbert space and with closed range must be a finite rank operator. Recall that an operator T on a Hilbert space is called normal if $TT^* = T^*T$.

Proposition 4.3 *Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be a Riesz operator with closed range. If T is normal, then T is a finite rank operator.*

Proof. Recall that the spectral radius and norm of a normal element in a C^*-algebra are equal. Since T is Riesz, $T + \mathcal{K}(H) \in \mathcal{L}(H)/\mathcal{K}(H)$ is quasinilpotent. It is also normal (because T is), and therefore T is compact. This forces T to be finite rank because it has closed range. □

Recall that an operator $T \in \mathcal{L}(X)$ has a *pseudoinverse* if there exists an operator $S \in \mathcal{L}(X)$ with $T = TST$ and $S = STS$. It is well known that if $T \in \mathcal{L}(X)$ has a pseudoinverse, then $R(T)$ is closed and both $N(T)$ and $R(T)$ are complemented subspaces of X [12, Proposition 31.1].

Proposition 4.4 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be an inessential operator. If T has a pseudoinverse, then T is a finite rank operator.*

Proof. Suppose T is inessential and T has a pseudoinverse $S \in \mathcal{L}(X)$ say, with $T = TST$ and $S = STS$. Since ST is a projection that is also an inessential operator, it is a finite rank operator and so T is a finite rank operator. □

Note that we cannot replace 'inessential' operator in Proposition 4.4 with 'Riesz' operator: Let T be as in Example 4.1 and define an operator S by $S(x_1, x_2) = (x_2, 0)$. Then S is a pseudoinverse for T, but T does not have finite rank.

We have repeatedly used the elementary fact that a compact operator with closed range must be finite rank. It is natural to ask if something similar may be true for any of the larger ideals in the chain (1.1). The following result shows that this is not the case.

Theorem 4.5 *The Banach space $C[0,1]$ admits a strictly singular operator S with closed range and such that no power of S is finite rank. Moreover, S has finite ascent $\alpha(S) = 1$.*

Proof. Since $C[0,1]$ is isomorphic to the Banach space $X = C[0,1] \oplus C[0,1]$ endowed with the norm $\|(x_1, x_2)\| = \max\{\|x_1\|_\infty, \|x_2\|_\infty\}$, it suffices to construct an operator $S \in \mathcal{L}(X)$ with the specified properties. For $j = 1, 2$, let $P_j \in \mathcal{L}(X, C[0,1])$ denote the j^{th} coordinate projection given by

$$P_j(x_1, x_2) = x_j \quad (x_1, x_2) \in C[0,1].$$

It is a standard fact that the Hilbert space ℓ^2 is a quotient of $C[0,1]$ (see [9, Theorem 1.1] or [8, Corollary of Proposition 4]), so we can take a bounded linear surjection $U : C[0,1] \to \ell^2$. Then $UP_1 : X \to \ell^2$ is also surjective. Since the set of bounded linear surjections from one Banach space onto another is open (see for instance [1, Theorem 2.10]), we can find $\varepsilon > 0$ such that every operator $T \in \mathcal{L}(X, \ell^2)$ with $\|T - UP_1\| < \varepsilon$ is surjective.
Every separable Banach space embeds isometrically into $C[0,1]$, so we can

choose a linear isometry $V : \ell^2 \to C[0,1]$. Set $b_n = Ve_n$ for each $n \in \mathbb{N}$, where (e_n) denotes the standard orthonormal basis of ℓ^2, and let $f_n \in C[0,1]^*$ be a norm preserving extension of the n^{th} coordinate functional of the basic sequence (b_m), so that $\|f_n\| = 1$ and $f_n(b_m) = \delta_{m,n}$ for all $m, n \in \mathbb{N}$. We can now define a nuclear operator $R : X \to \ell^2$ of norm at most $\varepsilon/2$ by

$$Rx = \sum_{n=1}^{\infty} \frac{\varepsilon f_n(P_2 x)}{2^{n+1}} e_n.$$

Then $T = UP_1 + R$ is surjective, and hence $S = J_2VT \in \mathcal{L}(X)$ has closed range, where $J_2 : C[0,1] \to X$ is given by $J_2x = (0, x)$ for each $x \in C[0,1]$. Since S factors through the reflexive space ℓ^2, it is weakly compact and therefore strictly singular by a theorem of Pełczyński [14, Theorem 1].
For each $x \in X$, we have $Sx = \sum_{n=1}^{\infty} \alpha_n J_2 b_n$ for

$$\alpha_n = (UP_1x \mid e_n) + \frac{\varepsilon f_n(P_2x)}{2^{n+1}} \qquad (n \in \mathbb{N}),$$

where $(\cdot \mid \cdot)$ denotes the inner product on ℓ^2. By iteration, we obtain

$$S^k x = \sum_{n=1}^{\infty} \alpha_n \Big(\frac{\varepsilon}{2^{n+1}}\Big)^{k-1} J_2 b_n \qquad (k \in \mathbb{N}).$$

First, this shows that $S^k J_2 b_n = (\varepsilon/2^{n+1})^k J_2 b_n$ for each $k, n \in \mathbb{N}$, so that S^k has infinite dimensional range for each $k \in \mathbb{N}$. Second, we see that

$$\begin{aligned} S^2x = 0 \quad &\iff \quad \alpha_n \frac{\varepsilon}{2^{n+1}} = 0 \quad (n \in \mathbb{N}) \\ &\iff \quad \alpha_n = 0 \quad (n \in \mathbb{N}) \quad \iff \quad Sx = 0, \end{aligned}$$

so that S has finite ascent and $\alpha(S) \leq 1$. We cannot have $\alpha(S) = 0$ because S is not injective; indeed, the surjection T cannot be injective (because X is not isomorphic to ℓ^2), and $N(T) = N(S)$. $\square$

Question 4.6 *Does there exist a strictly singular (or inessential) operator S on a Banach space such that S has finite ascent and S^k has closed, infinite dimensional range for each $k \in \mathbb{N}$?*

We remark that the quasinilpotent operator in Burlando's example 4.2 cannot be inessential or strictly singular because $\mathcal{K}(H) = \mathcal{S}(H) = \mathcal{I}(H)$ for a Hilbert space H.

Acknowledgements. The research presented in this paper was initiated during a visit of the first named author to the University of Johannesburg in January 2017. We are pleased to acknowledge the award of a Scheme 5 grant from the London Mathematical Society that made this visit possible. The first named author also wishes to thank his hosts for a very enjoyable and productive time in Johannesburg.

References

[1] Y.A. Abramovich and C.D. Aliprantis, *An Invitation to Operator Theory*, Graduate Studies in Mathematics Volume 50, American Mathematical Society, Providence Rhode Island 2002.

[2] P. Aiena, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic Publishers, Dordrecht 2010.

[3] B. Aupetit, *A Primer on Spectral Theory*, Springer-Verlag, New York 1991.

[4] L. Burlando, *On nilpotent operators*, Studia Mathematica **166** (2005), 101-129.

[5] H.R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, London, New York, San Francisco 1978.

[6] F. Ghahramani, *Compact homomorphisms of C^*-algebras*, Proceedings of the American Mathematical Society **103** (1988), 458-461.

[7] U. Koumba and H. Raubenheimer, *Finite rank Riesz operators*, Glasgow Mathematical Journal **56** (2014), 183-185.

[8] H.E. Lacey, *Separable quotients of Banach Spaces*, Annals of the Brazilian Academy of Sciences **44** (1972), 185-189.

[9] H.E. Lacey and P.D. Morris, *On Spaces of Type $A(K)$ and their Duals*, Proceedings of the American Mathematical Society **23** (1969), 151-157.

[10] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92, Springer-Verlag, Berlin-New York 1977.

[11] M. Mathieu, *Weakly compact homomorphisms*, Proceedings of the American Mathematical Society **107** (1989), 761-762.

[12] V. Müller, *Spectral Theory of Linear Operarors and Spectral Systems in Banach Algebras*, Birkhäuser, Basel, Boston, Berlin 2007.

[13] M. Oudghiri and M. Zohry, *Riesz operators and the complete norm topology*, Journal of Mathematical Analysis and Applications **306** (2005), 663-668.

[14] A. Pełczyński, *On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in $C(S)$-spaces*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques **13** (1965), 31-36.

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