Banach Algebras on Groups and Semigroups

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Declaration

I declare that this thesis is my own work, and that the results described here are my own except where stated otherwise. Chapter 5 is based on joint work with my supervisor Niels Laustsen. I declare that I contributed fully to every aspect of this work. Finally, I declare that this thesis has not been submitted for the award of a higher degree elsewhere.

Abstract

This thesis concerns the theory of Banach algebras, particularly those coming from abstract harmonic analysis. The focus for much of the thesis is the theory of the ideals of these algebras. In the final chapter we use semigroup algebras to solve an open probelm in the theory of C*-algebras. Throughout the thesis we are interested in the interplay between abstract algebra and analysis. Chapters 2, 4, and 5 are closely based upon the articles [**88**], [**89**], and [**56**], respectively.

In Chapter 2 we study (algebraic) finite-generation of closed left ideals in Banach algebras. Let G be a locally compact group. We prove that the augmentation ideal in $L^1(G)$ is finitely-generated as a left ideal if and only if G is finite. We then investigate weighted versions of this result, as well as a version for semigroup algebras. Weighted measure algebras are also considered. We are motivated by a recent conjecture of Dales and Żelazko, which states that a unital Banach algebra in which every maximal left ideal is finitely-generated is necessarily finite-dimensional. We prove that this conjecture holds for many of the algebras considered. Finally, we use the theory that we have developed to construct some examples of commutative Banach algebras that relate to a theorem of Gleason.

In Chapter 3 we turn our attention to topological finite-generation of closed left ideals in Banach algebras. We define a Banach algebra to be topologically left Noetherian if every closed left ideal is topologically finitely-generated, and we seek infinite-dimensional examples of such algebras. We show that, given a compact group G, the group algebra $L^1(G)$ is topologically left Noetherian if and only if G is metrisable. For a Banach space E satisying a certain condition we show that the Banach algebra of approximable operators $\mathcal{A}(E)$ is topologically left Noetherian if and only if E' is separable, whereas it is topologically right Noetherian if and only if E is separable.

We also define what it means for a dual Banach algebra to be weak*-topologically left Noetherian, and give examples which satisfy and fail this condition. Along the way, we give classifications of the weak*-closed left ideals in $M(G)$, for G a compact group, and in $\mathcal{B}(E)$, for E a reflexive Banach space with AP.

Chapter 4 looks at the Jacobson radical of the bidual of a Banach algebra. We prove that the bidual of a Beurling algebra on $\mathbb{Z}$, considered as a Banach algebra with the first Arens product, can never be semisimple. We then show that $\operatorname{rad}(\ell^1(\oplus_{i=1}^{\infty}\mathbb{Z})'')$ contains nilpotent elements of every index. Each of these results settles a question of Dales and Lau. Finally we show that there exists a weight ω on $\mathbb{Z}$ such that the bidual of $\ell^1(\mathbb{Z},\omega)$ contains a radical element which is not nilpotent.

In Chapter 5 we move away from the theory of ideals and consider a question about the notion of finiteness in C*-alegebras. We construct a unital pre-C*-algebra A_0 which is stably finite, in the sense that every left invertible square matrix over A_0 is right invertible, while the C*-completion of A_0 contains a non-unitary isometry, and so it is infinite. This answers a question of Choi. The construction is based on semigroup algebras.

Contents

CHAPTER 1

Introduction

In this chapter we shall introduce some notation and basic concepts that we shall use throughout the thesis. The material in this chapter is mostly well-known, and none of it is original. The main purpose is to fix notation and to indicate appropriate sources for background material. In some places, however, we do mention results which are not required in the thesis, but which we feel might interest the reader and offer context.

1.1. Frequently Used Notation and Definitions

We shall denote by $\mathbb{Z}$ the group of integers and by $\mathbb{Z}^+$ the semigroup of non-negative integers $\{0, 1, 2, \ldots\}$. Similarly we write $\mathbb{Z}^- = \{0, -1, -2, \ldots\}$. For us, $\mathbb{N} = \{1, 2, \ldots\}$. Of course, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers.

Let X be any set. We write the identity map $X \to X$ as id_X. If X and Y are two sets, and $f\colon X \to Y$ is any function we write $\operatorname{im} f$ for the image of f. Given a subset $S \subset X$ we write S^c for the complement of S in X. We write χ_S for the indicator function of S. Given $x, y \in X$ we define

$$\mathbb{1}_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases} \tag{1.1}$$

We use this notation in place of the more common Kronecker delta in order to avoid a conflict with our notation for Dirac measures given in (1.3); compare also (1.8).

Let G be a group. We write $H \leqslant G$ to mean that H is a subgroup of G, and we write $[G : H]$ for the index of H in G. Let $X \subset G$. We write $X^{-1} = \{x^{-1} : x \in X\}$, and we say that X is *symmetric* if $X = X^{-1}$. Now suppose that X is a generating set for G. Then we define the *word-length with respect to* X of a group element $u \in G$ by

$$|u|_X := \min\{r \in \mathbb{N} : \text{ there exist } x_1, \ldots, x_r \in X \cup X^{-1} \text{ such that } u = x_1 \cdots x_r\}.$$

When the generating set is clear, we shall usually write $|u|$ in place of $|u|_X$.

In this thesis linear spaces, and in particular algebras, will always be over $\mathbb{C}$ unless otherwise stated. Let E be a linear space. Then, similarly to our notation for subgroups, we write $F \leqslant E$ to indicate that F is a linear subspace of E.

Let K be a locally compact Hausdorff space. We say that a subset of K is *precompact* if it has compact closure. We write $C_0(K)$ for the space of all complex-valued, continuous functions on K which vanish at infinity, and $C_c(K)$ for the linear subspace of $C_0(K)$ of compactly-supported functions. We denote by $C(K)$ the linear space of all continuous functions from K to $\mathbb{C}$. We write $M(K)$ for the set of complex, regular Borel measures on K, which becomes a Banach space under the total variation norm. The dual space of $C_0(K)$ may be identified isometrically with $M(K)$, with the duality given by

$$\langle f, \mu \rangle = \int_K f \,\mathrm{d}\mu \quad (f \in C_0(K),\ \mu \in M(K)). \tag{1.2}$$

We write $\mathscr{B}_K$ for the Borel σ-algebra of K. Given a point $x \in K$ we denote the Dirac measure at x by δ_x. That is

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \qquad (E \in \mathscr{B}_K). \tag{1.3}$$

Now let X be any topological space. Let I be a directed set, $\mathcal{U}$ a filter on I, and $(x_\alpha)_{\alpha \in I}$ a net in X which converges along $\mathcal{U}$. We write $\lim_{\alpha \to \mathcal{U}}$ for the limit of (x_α) along $\mathcal{U}$. In expressions such as $\lim_{\alpha \to \infty} x_\alpha$ the symbol '∞' is understood to represent the Fréchet filter on I. We often write $\lim_{\alpha \to \infty} x_\alpha = \lim_\alpha x_\alpha$. Denoting the topology on X by τ, we sometimes write $\lim_{\tau,\, \alpha \to \mathcal{U}} x_\alpha$ if the topology is ambiguous. For instance, we frequently write things like $\lim_{w*,\alpha} x_\alpha$ when X is a dual Banach space in order to indicate that the limit is taken in the weak*-topology.

1.2. Background From Banach Space Theory

1.2.1. Basic Definitions.

Let E be a Banach space. We use the notation $B_E = \{x \in E : \|x\| \leqslant 1\}$ for the closed unit ball of E. We denote the dual space of E by E', and the second dual, sometimes termed the *bidual*, by E''. We often identify E with its image in E'' under the canonical embedding. We write $\langle x, \lambda \rangle$ for the value of a functional $\lambda \in E'$ applied to $x \in E$. If the exact dual pairing needs clarifying, we sometimes write this as $\langle x, \lambda \rangle_{(E, E')}$. If H is a Hilbert space, we usually write the inner product on H as $\langle \cdot, \cdot \rangle_H$. Given elements $x, y \in H$ we write $x \perp y$ for the statement $\langle x, y \rangle_H = 0$.

Now take subsets $X \subset E$ and $Y \subset E'$. We write

$$X^\perp = \{\lambda \in E' : \langle x, \lambda \rangle = 0 \ (x \in X)\}, \quad Y_\perp = \{x \in E : \langle x, \lambda \rangle = 0 \ (\lambda \in Y)\}.$$

It is well known that, for X and Y as above, we have

$$(Y_\perp)^\perp = \overline{\operatorname{span}}^{w^*} Y, \qquad (X^\perp)_\perp = \overline{\operatorname{span}} X. \tag{1.4}$$

We denote the set of all bounded linear maps $E \to E$ by $\mathcal{B}(E)$. If F is another Banach space, then we denote the set of bounded linear maps $E \to F$ by $\mathcal{B}(E, F)$. If $T \in \mathcal{B}(E, F)$ we write $T' \colon F' \to E'$ for the dual map, and $T'' = (T')'$. We denote by $\mathcal{K}(E, F)$ the set of compact operators $E \to F$, and by $\mathcal{F}(E, F)$ the set of finite-rank operators $E \to F$. We define the *approximable operators* to be the closure of $\mathcal{F}(E, F)$ in $\mathcal{B}(E, F)$, and denote this space by $\mathcal{A}(E, F)$. Each of these spaces is a linear subspace of $\mathcal{B}(E, F)$, and $\mathcal{A}(E, F)$ and $\mathcal{K}(E, F)$ are closed. We write $\mathcal{K}(E) = \mathcal{K}(E, E)$ et cetera. For any Banach space E each of $\mathcal{F}(E)$, $\mathcal{A}(E)$ and $\mathcal{K}(E)$ is an ideal in $\mathcal{B}(E)$.

Given $x \in E$ and $\lambda \in E'$, the notation $x \otimes \lambda$ denotes the member of $\mathcal{B}(E)$ given by

$$x \otimes \lambda \colon y \mapsto \langle y, \lambda \rangle x \quad (y \in E).$$

We denote the projective tensor product of Banach spaces by $\widehat{\otimes}$ (see [**76**] for background on Banach space tensor products). We may identify the set

$$\operatorname{span}\{x \otimes \lambda : x \in E, \lambda \in E'\}$$

with the algebraic tensor product $E \otimes E'$, and therefore with a dense subspace of $E \widehat{\otimes} E'$. However, note that in general it is not possible to identify $E \widehat{\otimes} E'$ with the closure of $\operatorname{span}\{x \otimes \lambda : x \in E, \lambda \in E'\}$ in $\mathcal{B}(E)$, which turns out to be $\mathcal{A}(E)$. In particular, in equation (1.5) below the expression $\sum_{i=1}^{\infty} x_i \otimes \lambda_i$ is understood to represent a member of $E \widehat{\otimes} E'$, not an operator. The subtle issue of representing elements of tensor products of Banach spaces as operators is discussed in more detail in [**65**, Chapter 0, Section b].

Let E be a reflexive Banach space. Then $\mathcal{B}(E)$ may be identified isometrically with $(E \widehat{\otimes} E')'$ via the formula

$$\left\langle \sum_{i=1}^{\infty} x_i \otimes \lambda_i, T \right\rangle_{(E \widehat{\otimes} E', \mathcal{B}(E))} = \sum_{i=1}^{\infty} \langle T x_i, \lambda_i \rangle_{(E, E')}, \tag{1.5}$$

for $T \in \mathcal{B}(E)$ and $\sum_{i=1}^{\infty} x_i \otimes \lambda_i \in E \widehat{\otimes} E'$ [**19**, Proposition A.3.70]. In particular, if we talk about the weak*-topology on $\mathcal{B}(E)$, we always mean the weak*-topology induced by this duality.

Now let E be an arbitrary Banach space. The *strong operator topology*, or the *SOP topology* for short, is the locally convex topology on $\mathcal{B}(E)$ induced by the seminorms $\mathcal{B}(E) \to [0, \infty)$ given by

$$T \mapsto \|Tx\|$$

as x ranges through E. This topology is particularly important when E is a Hilbert space, and as such we shall mention it below when we discuss representation theory in Subsection 1.3.4. However, we also consider this topology on $\mathcal{B}(E)$ for an arbitrary Banach space E in Section 3.6.

1.2.2. The Approximation Property.

A Banach space E is said to have the *approximation property*, or simply *AP*, if, whenever F is another Banach space, we have $\mathcal{A}(F,E) = \mathcal{K}(F,E)$. There is also an equivalent formulation of the approximation property which has some useful generalizations: a Banach space E has AP if and only if, for every compact subset $K \subset E$ and every $\varepsilon > 0$, there exists $T \in \mathcal{F}(E)$ such that $\|Tx - x\| < \varepsilon$ $(x \in K)$ [**59**, Theorem 3.4.32]. We say that E has the *bounded approximation property*, or *BAP*, if there exists a constant $C > 0$ such that the operator T can be chosen to have norm at most C. Clearly BAP implies AP. Moreover, a reflexive Banach space with AP has BAP [**14**, Theorem 3.7]. Many Banach spaces have the bounded approximation property: for instance any Banach space with a Schauder basis [**59**, Theorem 4.1.33] has BAP, and it can be deduced from this that any Hilbert space has BAP. The Banach space $\mathcal{B}(H)$, for H an infinite dimensional Hilbert space, does not even have AP [**82**].

In Subsection 1.4.1 below we define approximate identities. In Chapter 3 we shall be interested in Banach algebras of the form $\mathcal{A}(E)$, for some Banach space E, such that $\mathcal{A}(E)$ contains either a left or a right approximate identity (or both). This is closely related to AP and BAP, as we summarise in the following theorem:

THEOREM 1.2.1. *Let E be a Banach space.*

(i) *The Banach algebra $\mathcal{A}(E)$ has a bounded left approximate identity if and only if E has BAP.*
(ii) *The Banach algebra $\mathcal{A}(E)$ has a bounded (two-sided) approximate identity if and only if E' has BAP.*
(iii) *If E has AP, then the Banach algebra $\mathcal{A}(E)$ has a (possibly unbounded) left approximate identity.*

PROOF. Part (i) follows from [**30**, Theorem 2.6(i)], (ii) follows from [**39**, Theorem 3.3], and (iii) follows from [**30**, Theorem 2.5 (ii)]. $\square$

We do not know whether or not E' having AP is enough to imply that $\mathcal{A}(E)$ has a right approximate identity.

We are also interested in AP in this thesis because of the following result. When E is a Banach space with AP we have $\mathcal{K}(E)' \cong E\widehat{\otimes}E'$ isometrically, with the duality given by

$$\left\langle T, \sum_{i=1}^{\infty} x_i \otimes \lambda_i \right\rangle_{(\mathcal{K}(E),\, E\widehat{\otimes}E')} = \sum_{i=1}^{\infty} \langle Tx_i, \lambda_i \rangle_{(E,\, E')}, \tag{1.6}$$

for $T \in \mathcal{K}(E)$, and $\sum_{i=1}^{\infty} x_i \otimes \lambda_i \in E\widehat{\otimes}E'$ [**19**, A.3.71]. Compare with (1.5). Hence, if E is also reflexive, we have $\mathcal{B}(E) \cong \mathcal{K}(E)''$.

1.3. Background From Abstract Harmonic Analysis

1.3.1. Locally Compact Groups.

A central area of study in this thesis will be the theory of locally compact groups. By a *topological group* we mean a pair (G, τ), where G is a group and τ is a Hausdorff topology on G such that the maps

$$G \times G \to G, \quad (s,t) \mapsto st$$

and

$$G \to G, \quad s \mapsto s^{-1}$$

are continuous. By a *locally compact group* we mean a topological group, whose topology is locally compact. For an introduction to topological and locally compact groups see [**45**] or [**34**].

Every locally compact group G has a positive Borel measure m defined on it which is invariant under left translation, in the sense that

$$m(sE) = m(E) \quad (s \in G,\ E \in \mathscr{B}_G),$$

and also satisfies

(1) $m(U) > 0$ for each non-empty, open set $U \subset G$;

(2) $m(K) < \infty$ for each compact set $K \subset G$;

(3) m is outer regular, i.e.

$$m(E) = \inf\{m(U) : E \subset U,\ U \text{ is open}\} \quad (E \in \mathscr{B}_G);$$

(4) m is inner regular on open sets, i.e. for every open subset U of G we have

$$m(U) = \sup\{m(K) : K \subset U,\ K \text{ is compact}\}.$$

This measure is unique up to a positive scalar multiple and is called the *left Haar measure* on G. In this thesis the left Haar measure on a locally compact group will always be denoted by m. Sometimes we may abbreviate the phrase "left Haar measure" to simply "Haar measure". From now on, given $p \in [1, \infty]$, we write $L^p(G)$ to mean $L^p(G, m)$, and given $f \in L^1(G)$ we usually write $\int_G f(t)\,\mathrm{d}t$ in place of $\int_G f(t)\,\mathrm{d}m(t)$. For a proof of the existence and uniqueness of Haar measure see either [**18**, Chapter 9] or [**34**, Section 2.2]. By (2) above, $m(G) < \infty$ whenever G is compact. When this is the case, we usually scale the Haar measure so that $m(G) = 1$.

Given $t \in G$, the map $E \mapsto m(Et)$, from $\mathscr{B}_G$ to $[0, \infty]$, is easily seen to be another left Haar measure on G so that, by uniqueness, there exists a positive scalar $\Delta(t)$ such that

$$m(Et) = \Delta(t)m(E) \quad (t \in G,\ E \in \mathscr{B}_G).$$

The function Δ is a continuous group homomorphism $G \to (0, \infty)$ which is called the *modular function* of G. We think of the modular function as measuring "how far" the left Haar measure is from being invariant under right translation. A group for which the modular function is identically equal to 1 is called *unimodular*. Compact groups, discrete groups, and locally compact abelian groups are always unimodular. The affine group of the real line is an example of a locally compact group which is not unimodular [**34**, page 48].

Let G be a locally compact group, and let f be an integrable function on G. The translation invariance of the left Haar measure on G implies that we have

$$\int_G f(st)\,\mathrm{d}t = \int_G f(t)\,\mathrm{d}t \quad (s \in G).$$

Moreover, we shall use the following formulae throughout the text without reference; they can be found in [**19**, Lemma 3.3.6]:

$$\int_G f(t)\,\mathrm{d}t = \int_G f(t^{-1})\Delta(t^{-1})\,\mathrm{d}t;$$

$$\int_G f(ts)\,\mathrm{d}t = \Delta(s^{-1})\int_G f(t)\,\mathrm{d}t \quad (s \in G).$$

1.3.2. The Group Algebra and the Measure Algebra.

We now introduce the Banach algebras which will be of most interest to us in this thesis. Let G be a locally compact group, and let $f, g \in L^1(G)$. We define the *convolution of f and g* by

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\,\mathrm{d}t \quad (s \in G), \tag{1.7}$$

and it turns out that this again belongs to $L^1(G)$. In order to be totally rigorous, we should point out that this function is only defined m-almost everywhere, and one can check that the element of $L^1(G)$ that $f * g$ defines does not depend on the functions chosen to represent elements f and g of $L^1(G)$. It can then be shown that convolution defines an algebra multiplication on $L^1(G)$, and that $\|f*g\|_1 \leqslant \|f\|_1\|g\|_1$ $(f, g \in L^1(G))$. Hence $L^1(G)$ is a Banach algebra, which is called *the group algebra of G*. Moreover, the following formulae hold and we shall use them without reference throughout the thesis:

$$(f * g)(s) = \int_G f(st^{-1})g(t)\Delta(t^{-1})\,\mathrm{d}t = \int_G f(st)g(t^{-1})\,\mathrm{d}t \quad (s \in G).$$

We can also define the convolution of two complex, regular Borel measures, using the formula

$$(\mu * \nu)(E) = (\mu \times \nu)(p^{-1}(E)) \quad (\mu, \nu \in M(G),\ E \in \mathscr{B}_G),$$

where $p\colon G \times G \to G$ denotes the multiplication map. Bearing in mind that $M(G)$ may be identified with the dual space of $C_0(G)$, we obtain the following nice formula:

$$\langle f, \mu * \nu \rangle = \int_G \int_G f(st)\, \mathrm{d}\mu(s)\, \mathrm{d}\nu(t) \quad (f \in C_0(G),\ \mu, \nu \in M(G)).$$

In fact this formula gives an alternative way to define the convolution of two measures belonging to $M(G)$ [**81**].

By the Radon–Nikodym Theorem, we may identify $L^1(G)$ with the measures in $M(G)$ which are absolutely continuous with respect to Haar measure, and under this identification the convolution of two elements of $L^1(G)$ regarded as measures coincides with convolution as defined in (1.7). We shall usually not distinguish between $L^1(G)$ and its image inside $M(G)$. In fact $L^1(G)$ is a closed ideal in $M(G)$, and we shall freely use the following formulae without reference:

$$(\mu * f)(s) = \int_G f(t^{-1}s)\, \mathrm{d}\mu(t) \quad (\mu \in M(G),\ f \in L^1(G));$$

$$(f * \mu)(s) = \int_G f(st^{-1})\Delta(t^{-1})\, \mathrm{d}\mu(t) \quad (\mu \in M(G),\ f \in L^1(G)).$$

By a *Banach *-algebra* we mean a Banach algebra A which has an isometric involution defined on it. Given a measure $\mu \in M(G)$ we define $\mu^* \in M(G)$ by

$$\mu^*(E) = \overline{\mu(E^{-1})} \quad (E \in \mathscr{B}_G).$$

The operation $\mu \mapsto \mu^*$ is an involution rendering $M(G)$ a Banach *-algebra, and $L^1(G)$ a *-subalgebra. Given $f \in C_0(G)$ we have

$$\int_G f(t)\, \mathrm{d}\mu^*(t) = \overline{\int_G \overline{f(t^{-1})}\, \mathrm{d}\mu(t)}.$$

Given $f \in L^1(G)$, we find that $f^*(s) = \overline{f(s^{-1})}\Delta(s^{-1})$ $(s \in G)$. We also define the notation $\check{f}(s) = f(s^{-1})$ $(s \in G)$.

1.3.3. Semigroup Algebras and Beurling Algebras.

Before we define Beurling algerbas we define weighted semigroup algebras.

DEFINITION 1.3.1. Let S be a semigroup. Then a *weight* on S is a function $\omega : S \to [1, \infty)$ such that

$$\omega(uv) \leqslant \omega(u)\omega(v) \quad (u, v \in S).$$

In the case where S has an identity e, we insist that $\omega(e) = 1$. Moreover, when G is a locally compact group, weights on G are always assumed to be continuous. A weight ω on a locally compact group G is said to be *symmetric* if $\omega(u) = \omega(u^{-1})$ $(u \in G)$.

Given a semigroup S and a weight ω on S, we define

$$\ell^1(S, \omega) = \left\{ f : S \to \mathbb{C} : \|f\|_\omega := \sum_{u \in S} |f(u)|\omega(u) < \infty \right\}.$$

The set $\ell^1(S, \omega)$ is a Banach space under pointwise operations with the norm given by $\|\cdot\|_\omega$, and a Banach algebra when multiplication is given by convolution, which is defined for $f, g \in \ell^1(S)$ by

$$(f * g)(u) = \sum_{st=u} f(s)g(t) \quad (u \in S).$$

By a *weighted semigroup algebra*, we shall mean a Banach algebra of this form. When ω is identically 1, we write $\ell^1(S)$ in place of $\ell^1(S, \omega)$, and we call such algebras *semigroup algebras*.

We denote by $\mathbb{C}S$ the dense subalgebra of $\ell^1(S)$ consisting of its finitely-supported elements. In fact it is easily seen that $\mathbb{C}S$ is a dense subalgebra of every weighted semigroup algebra on S. Given an element $u \in S$ we write δ_u for the function given by

$$\delta_u(t) = \begin{cases} 1 & \text{if } t = u \\ 0 & \text{otherwise} \end{cases} \qquad (t \in S). \tag{1.8}$$

This does not conflict with the notation defined in (1.3), since S may be thought of as a discrete topological space, and with this point of view the two definitions coincide. We have $\mathbb{C}S = \mathrm{span}\{\delta_u : u \in S\}$, and $\delta_u * \delta_v = \delta_{uv}$ $(u, v \in S)$.

Now suppose that we have a locally compact group G and a weight ω on G. Then we define

$$L^1(G,\omega) = \left\{ f \in L^1(G) : \|f\|_\omega := \int_G |f(t)|\omega(t)\,\mathrm{d}m(t) < \infty \right\},$$

and

$$M(G,\omega) = \left\{ \mu \in M(G) : \|\mu\|_\omega := \int_G \omega(t)\,\mathrm{d}|\mu|(t) < \infty \right\}.$$

The sets $L^1(G,\omega)$ and $M(G,\omega)$ are Banach algebras with respect to convolution multiplication, and point-wise addition and scalar multiplication. Moreover, $L^1(G,\omega)$ is a closed ideal of $M(G,\omega)$. It is a Banach algebra of the form $L^1(G,\omega)$ that we refer to as a *Beurling algebra*, whereas we refer to a Banach algebra of the form $M(G,\omega)$ as a *weighted measure algebra*. Note that Beurling algebras are occasionally referred to as weighted group algebras. When the group G is discrete, both of these definitions coincide with that of $\ell^1(G,\omega)$.

Example 1.3.2. (i) The trivial weight $\omega = 1$ is always a weight on any locally compact group G (or any semigroup), and in this case we recover the group algebra $L^1(G)$.

(ii) Let G be a discrete group, and fix a generating set X. Then

$$u \mapsto (1 + |u|_X)^\alpha, \quad G \to [1,\infty),$$

defines a weight on G for each $\alpha \geqslant 0$. We call this weight a *radial polynomial weight of degree* α.

(iii) With notation as in (ii), the map

$$u \mapsto c^{|u|_X^\beta}, \quad G \to [1, \infty),$$

defines a weight for any $c \geqslant 1$ and $0 < \beta \leqslant 1$. We call this a *radial exponential weight with base* c *and degree* β.

More generally, a weight on a finitely-generated group G is said to be *radial* if there exists a finite generating set X such that $|u|_X = |v|_X$ implies that $\omega(u) = \omega(v)$ for any $u, v \in G$.

When the group G is discrete we usually prefer the notation $\ell^1(G,\omega)$ over $L^1(G,\omega)$. For a general locally compact group G and a weight ω on G, we often write $\ell^1(G,\omega)$ for the Beurling algebra associated with ω and G with its discrete topology. Moreover, the set of discrete measures belonging to $M(G,\omega)$ may be identified with $\ell^1(G,\omega)$. Letting $M_c(G,\omega)$ denote the continuous measures belonging to $M(G,\omega)$, we find that $M_c(G,\omega)$ is a closed ideal in $M(G,\omega)$ and $\ell^1(G,\omega)$ a closed subalgebra, with

$$M(G,\omega) = M_c(G,\omega) \oplus \ell^1(G,\omega),$$

where $\oplus$ denotes the direct sum of Banach spaces. In other words, we have a split exact sequence of Banach algebras

$$0 \to M_c(G,\omega) \to M(G) \to \ell^1(G,\omega) \to 0.$$

In particular

$$M(G,\omega)/M_c(G,\omega) \cong \ell^1(G,\omega). \tag{1.9}$$

(This follows straightforwardly from [**19**, Theorem 3.3.36].)

Let G be a locally compact group, and let ω be a weight on G. We define

$$C_0(G, 1/\omega) = \{f : G \to \mathbb{C} : f/\omega \in C_0(G)\}$$

We define a norm $\|\cdot\|_{\infty,\omega}$ on $C_0(G, 1/\omega)$ by

$$\|f\|_{\infty,\omega} := \sup_{s\in G} \left|\frac{f(s)}{\omega(s)}\right| \quad f \in C_0(G, 1/\omega).$$

The Banach space $M(G, \omega)$ may be identified isometrically with $C_0(G, 1/\omega)'$ via

$$\langle f, \mu\rangle = \int_G f \,\mathrm{d}\mu \quad (f \in C_0(G, 1/\omega),\ \mu \in M(G, \omega)).$$

Now consider a discrete group G, and a weight ω on G. Then we define

$$\ell^\infty(G, 1/\omega) = \left\{f : G \to \mathbb{C} : \|f\|_{\infty,\omega} := \sup_{u\in G} \left|\frac{f(u)}{\omega(u)}\right| < \infty\right\}.$$

This is a Banach space which may be identified isometrically with $\ell^1(G, \omega)'$ via

$$\langle g, f\rangle = \sum_{u\in G} g(u)f(u) \quad (g \in \ell^1(G, \omega),\ f \in \ell^\infty(G, 1/\omega)).$$

For non-discrete G, the space $L^1(G, \omega)'$ may also be identified with a certain weighted L^∞-space, but we shall not use this in this thesis.

1.3.4. Representation Theory.

In what follows, G will be a locally compact group, and given a Hilbert space H we denote the group of unitary operators $H \to H$ by $\mathcal{U}(H)$. We define a *representation of* G to be a pair (π, H_π), where H_π is a Hilbert space, and $\pi : G \to \mathcal{U}(H_\pi)$ is a group homomorphism which is continuous with respect to the given topology on G and the strong operator topology on $\mathcal{U}(H_\pi)$ (many authors refer to this as a "continuous, unitary representation"). A representation (π, H_π) is said to be *irreducible* if it has no non-trivial subrepresentations, *i.e.* if there is no closed linear subspace E of H_π such that $\pi(s)E \subset E$ $(s \in G)$.

Let (π, H_π) and (σ, H_σ) be two representations of G. We say that the two representations are *equivalent* if there exists a surjective isometry $T\colon H_\pi \to H_\sigma$ such that $\sigma(s) \circ T = T \circ \pi(s)$ $(s \in G)$. It is easily checked that this notion of equivalence is an equivalence relation. For each equivalence class of irreducible representations we pick a distinguished member to represent the equivalence class, and we collect these representatives together into a set that we denote by $\widehat{G}$, sometimes called *the unitary dual* of G. Since we do not usually distinguish between representations that are equivalent, we often treat $\widehat{G}$ as if it were the collection of all irreducible representations of G, although, of course, formally it is not. When G is abelian $\widehat{G}$ may be identified with the usual dual group of G consisting of the continuous group homomorphisms $G \to \mathbb{T}$ (see [**72**] for a detailed exposition of locally compact abelian groups and their duals).

Given a representation (π, H_π), there is a bounded algebra homomorphism

$$\pi'\colon M(G) \to \mathcal{B}(H_\pi)$$

such that

$$\langle \pi'(\mu)\xi, \eta\rangle = \int_G \langle \pi(t)\xi, \eta\rangle \,\mathrm{d}\mu(t), \tag{1.10}$$

for every $\mu \in M(G), \xi, \eta \in H_\pi$ (see [**45**, Theorem 22.3 (iii)]). We shall mostly be interested in lifting π to $L^1(G)$ via $\pi'|_{L^1(G)}$. From now on we write $\pi = \pi'|_{L^1(G)}$ in an abuse of notation.

By the Gelfand–Raikov Theorem [**45**, Theorem 22.12], the irreducible representations of G separate the points of G. An important fact about compact groups is that all of their irreducible representations are finite-dimensional [**45**, Theorem 22.13].

1.3.5. The Fourier and Fourier–Stieltjes Algebras.

In this subsection, we shall introduce two more families of Banach algebras associated with locally compact groups, namely the Fourier and Fourier–Stieltjes algebras. These are commutative Banach algebras which capture representation-theoretic information about the underlying group. These algebras were first studied in this level of generality by Eymard in [**32**]. A detailed account of these algebras will soon appear in [**55**].

Let G be a locally compact group. Given $f \in L^1(G)$ define

$$\|f\|_{C^*} = \sup \left\{ \|\pi(f)\| : \pi \in \widehat{G} \right\}.$$

It turns out that this defines a C*-norm on $L^1(G)$. We denote the completion of $L^1(G)$ in the norm $\|\cdot\|_{C^*}$ by $C^*(G)$, *the group C*-algebra of* G (for further details see [**34**, Section 7.1]). This C*-algebra has the property that every representation π of G extends from a *-representation of $L^1(G)$ to a *-representation of $C^*(G)$. We continue to denote this extension by π.

It follows from [**45**, Theorem 22.11] that the representations of G separate the points of $L^1(G)$ (even $M(G)$), and it follows from this fact, and the construction of $C^*(G)$, that they also separate the points of $C^*(G)$.

Let (π, H_π) be a representation of G. Given vectors $\xi, \eta \in H_\pi$ we define a function $\xi *_\pi \eta \colon G \to \mathbb{C}$ by

$$(\xi *_\pi \eta)(s) = \langle \pi(s)\xi, \eta \rangle \quad (s \in G).$$

We define the *Fourier–Stieltjes algebra of* G to be

$$B(G) := \{\xi *_\pi \eta : (\pi, H_\pi) \text{ is a representation of } G,\ \xi, \eta \in H_\pi\}.$$

It can be shown that this is an algebra under point-wise addition and multiplication of functions. Moreover, every element of $B(G)$ acts as a bounded linear functional on $C^*(G)$ via the formula

$$\langle f, \xi *_\pi \eta \rangle_{(C^*(G),\, B(G))} = \langle \pi(f)\xi, \eta \rangle_{H_\pi} \quad (f \in C^*(G)). \tag{1.11}$$

It can be shown that every bounded linear functional of $C^*(G)$ arises in this way, so that we may formally identify $B(G)$ with $C^*(G)'$. We define the norm on $B(G)$ to be the dual space norm inherited from this identification, and it turns out that, with this norm, $B(G)$ becomes not only a Banach algebra, but a dual Banach algebra (as defined in Subsection 1.4.5 below).

We define the *Fourier algebra of* G to be the closed ideal of $B(G)$ given by

$$A(G) := \overline{C_c(G) \cap B(G)}.$$

Many equivalent definitions of $A(G)$ are available (see for example [**32**, Proposition (3.4), Théorème (3.10)]).

Every element s of the group G induces a character on $A(G)$ via evaluation:

$$f \mapsto f(s), \quad A(G) \to \mathbb{C}.$$

In this manner the character space of $A(G)$ may be identified with G as a topological space (see [**32**, Théorème (3.34)], or [**77**]). Since the evaluation maps clearly separate the elements of the Fourier algebra, $A(G)$ may be regarded as a Banach function algebra on G. It is known that $A(G) \subset C_0(G)$ [**32**, Proposition (3.7) 1°].

When G is an abelian group, the Fourier algebra of G may be identified with the group algebra of the dual group $L^1(\widehat{G})$, and likewise $B(G) \cong M(\widehat{G})$. For this reason we often view the theory of the Fourier algebra (now for an arbitrary locally compact group) as being "dual" to the theory of the group algebra. In fact, this can be made precise using the language of Kac algebras [**31**]. To give a very basic example illustrating this point, it is know that, given a locally compact group G, the group algebra $L^1(G)$ is unital if and only if G is discrete [**58**, Theorem 31D]. Therefore we might hope that $A(G)$ should be unital if and only if G is compact, since compactness is the dual notion to discreteness in the theory of locally compact abelian groups. Indeed it is easily checked that this is true.

We shall make use of this heuristic a couple of times in this thesis, and sometimes we are able to "dualise" a proof that works for the Fourier algebra to obtain a proof of a theorem about the group algebra, and vice versa. See, for example, Theorem 2.3.5 and Theorem 2.4.1, or Proposition 3.3.1 and Theorem 3.3.5.

1.4. Background From Banach Algebras

1.4.1. Approximate Identities.

Let A be a semi-topological algebra, that is an algebra with a topology which renders the underlying vector space a topological vector space, and which makes the multiplication separately continuous. We say that a net (e_α) in A is a *left approximate identity* for A if $\lim_\alpha e_\alpha a = a$ for all $a \in A$, and a *right approximate identity* if instead $\lim_\alpha a e_\alpha = a$ for all $a \in A$. We say that (e_α) is an *approximate identity* if it is both a left and a right approximate identity.

Now assume that A is a Banach algebra. Then we say that a net $(e_\alpha) \subset A$ is a *bounded approximate identity* if it is an approximate identity and $\sup_\alpha \|e_\alpha\| < \infty$. Bounded left and right approximate identities are defined analogously. It is known, for example, that $L^1(G)$ always has a bounded approximate identity of bound 1 for any locally compact group G [**19**, Lemma 3.3.22 (i)], as does any C*-algebra [**19**, Lemma 3.2.20]. The former fact shall be important to us in this thesis.

The following result is often known as Cohen's factorisation theorem.

Theorem 1.4.1. *Let A be a Banach algebra with a bounded approximate identity, and let E be a Banach left A-module such that $\overline{\mathrm{span}}\{ax : a \in A,\ x \in E\} = E$. Then for every $x \in E$ there exist $a \in A$ and $y \in E$ such that $x = ay$.*

Proof. See [**19**, Theorem 2.9.24], or [**62**, Theorem 5.2.2]. □

See [**19**, Section 2.9] for more information on approximate identities and factorisation results.

1.4.2. Unitisation.

Let A be a complex algebra. We define the *unitisation of* A (sometimes called the conditional unitisation of A), denoted here by $A^\sharp$, as follows. If A is already unital then we set $A^\sharp = A$, so suppose that A is non-unital. We set

$$A^\sharp = \mathbb{C}1 \oplus A,$$

as a vector space, where 1 is some formal symbol which will act as a unit. We view $A^\sharp$ as an algebra with multiplication given by the formula

$$(\lambda 1 + a)(\mu 1 + b) := \lambda\mu 1 + (\mu a + \lambda b + ab) \quad (\lambda, \mu \in \mathbb{C},\ a, b, \in A).$$

We find that $A^\sharp$ is a unital algebra with identity element 1, and that A embeds into $A^\sharp$ as an ideal via $a \mapsto 01 + a$. From now on we shall always identify A with its image in $A^\sharp$. When $(A, \|\cdot\|)$ is a normed algebra then $A^\sharp$ also becomes a normed algebra, with norm defined by

$$\|\lambda 1 + a\| := |\lambda| + \|a\| \quad (\lambda \in \mathbb{C},\ a \in A).$$

If A is a Banach algebra, then $A^\sharp$ is also.

1.4.3. Multiplier Algebras.

We now describe the so-called multiplier algebra of a Banach algebra A, which is an object that is closely related to the extensions of A. In some ways it is analogous to the concept of the automorphism group of a group in group theory, as we shall explain below.

Let A be a Banach algebra. By a *right multiplier on* A we mean a linear map $R\colon A \to A$ such that $R(ab) = aR(b)$ $(a, b \in A)$. By a *left multiplier on* A we mean a linear map $L\colon A \to A$ such that $L(ab) = L(a)b$ $(a, b \in A)$. By a *multiplier* we mean a pair (L, R), where L is a left multiplier and R is a right multiplier, such that

$$aL(b) = R(a)b \quad (a, b \in A).$$

We say that (L, R) is a *bounded multiplier* if both L and R are bounded. We define the *multiplier algebra* $M(A)$ to be the set of bounded multipliers on A, and note that it inherits the structure of a Banach algebra by regarding it as a closed subalgebra of $\mathcal{B}(A) \oplus_\infty \mathcal{B}(A)^{\mathrm{op}}$ (again, see [**19**, Proposition 2.5.12(i)]). This algebra was originally defined by Johnson in [**49**], who was the first to systematically study multiplier algebras.

The importance of the multiplier algebra lies in the fact that, whenever there is another Banach algebra B such that A may be identified isomorphically with a closed ideal of B, each element $b \in B$ defines a multiplier (L, R) on A, by setting $L\colon a \mapsto ba$ and $R\colon a \mapsto ab$ $(a \in A)$. This induces a bounded homomorphism $B \to M(A)$. The analogy with the automorphism group of a group may now be explained: if G is any group and N a normal subgroup of G, then there is a similar map from G to $\mathrm{Aut}(N)$ defined by conjugation. In general the map $B \to M(A)$ may have a large kernel. Hence, we introduce some further terminology which allows us to avoid trivialities.

An ideal I in A is said to be *left faithful in A* if $xI = 0$ implies that $x = 0$ for every $x \in A$. We define the term *right faithful* similarly, and I is said to be *faithful* if I is both left and right faithful. We say that A is faithful if is a faithful ideal in itself. It is routinely checked that any Banach algebra with an approximate identity is faithful, so that in particular all group algebras are faithful. Also, for any Banach space E, any closed subalgebra of $\mathcal{B}(E)$ containing the finite-rank operators is faithful. By [**19**, Proposition 2.5.12(i)], if A is a faithful Banach algebra, then whenever (L, R) is a multiplier, the maps L and R are automatically continuous, so that the boundedness condition may be dropped in the definition of $M(A)$.

Given $a \in A$ we may define the multiplication maps $L_a, R_a\colon A \to A$ by

$$L_a\colon x \mapsto ax, \quad R_a\colon x \mapsto xa \quad (x \in A).$$

When A is faithful it can be shown that A embeds algebraically into $M(A)$ via $a \mapsto (L_a, R_a)$. When A has a bounded approximate identity this embedding has closed range, and when A has a bounded approximate identity of bound 1 the embedding is isometric.

For faithful Banach algebras, the multiplier algebra is defined by the following universal property, which heuristically speaking says that $M(A)$ is the "largest" Banach algebra containing A as a faithful ideal.

THEOREM 1.4.2. *Let A be a faithful Banach algebra. Then the multiplier algebra of A is the unique Banach algebra M satisfying:*

(i) *there exists a bounded, injective homomorphism $\alpha_0\colon A \to M$ such that $\alpha_0(A)$ is a faithful ideal in $M(A)$;*

(ii) *whenever there is a bounded, injective homomorphism $\beta\colon A \to B$, for some Banach algebra B, such that $\beta(A)$ is a faithful ideal in B then there exists a unique bounded monomorphism $\theta\colon B \to M$ such that $\theta \circ \beta = \alpha_0$.*

PROOF. The map α_0 in (i) can be taken to be the map $a\colon \mapsto (L_a, R_a)$. The fact that $M(A)$ satisfies (ii) follows from [**13**]. Verifying uniqueness is routine. $\square$

This characterisation of the multiplier algebra is due to Busby [**13**], who defines an analogue of the multiplier algebra, called a *maximal container*, in a much more general, category-theoretic setting. For example, in the category of groups the analogue of being faithful is having trivial centre, and for such groups the maximal container of a group G turns out to be its automorphism group. Busby goes on to show that maximal containers play a central rôle in understanding extensions in the category. Hence, in particular, given Banach algebras A and B, if A is faithful, the theory of the extensions of A by B is intimately connected with $M(A)$.

It is easily seen that, for any Banach algebra A, the pair $(\mathrm{id}_A, \mathrm{id}_A)$ defines a multiplier on A, and that it is a multiplicative unit for $M(A)$. In fact, for faithful Banach algebras, the fact that $M(A)$ is unital can be seen abstractly by using the universal property of Theorem 1.4.2 and the fact that A is a faithful ideal in $A^\sharp$. This leads to $M(A)$ being often described as the "largest" unitisation of A.

Let A be a faithful Banach algebra. We define the *strict topology* on $M(A)$ to be the locally convex topology induced by the seminorms $M(A) \to \mathbb{C}$ given by

$$\mu \mapsto \|a\mu\| \quad \text{and} \quad \mu \mapsto \|\mu a\| \quad (a \in A).$$

Note that in [**19**] the strict topology is referred to as the strong operator topology. If A has an approximate identity, then it is easily seen that A is dense in $M(A)$ with respect to the strict topology.

An important example for us is provided by the beautiful theorem of Wendel which states that, for a locally compact group G, the multiplier algebra of $L^1(G)$ may be identified isometrically with the measure algebra (see [**87**] or [**19**, Theorem 3.3.40]). Another key example is that, for any Banach space E, the multiplier algebras of both $\mathcal{A}(E)$ and $\mathcal{K}(E)$ may be identified isometrically with $\mathcal{B}(E)$ [**62**, 1.7.14]. As a final example, we mention that, if K is any locally compact Hausdorff space, then $M(C_0(K))$ may be identified with $C_b(K)$, the space of bounded, continuous functions on K [**61**, Example 3.1.3].

1.4.4. The Jacobson Radical.

Let A be an algebra, and take $n \in \mathbb{N}$. We say that $a \in A$ is *nilpotent of index* n if $a^n = 0$, but $a^{n-1} \neq 0$. Given a left ideal I of A and $n \in \mathbb{N}$, we write

$$I^n = \operatorname{span}\{a_1 a_2 \cdots a_n : a_1, \dots, a_n \in I\}$$

for the ideal generated by n-fold products of elements of I, and we say that I is *nilpotent of index* n if $I^n = \{0\}$ but $I^{n-1} \neq \{0\}$.

Now let A be a unital Banach algebra. We say that $a \in A$ is *quasi-nilpotent* if its spectrum is zero, or, equivalently, if $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$, and we denote the set of quasi-nilpotent elements of A by $\mathcal{Q}(A)$. Every nilpotent element is also quasi-nilpotent. We define the *Jacobson radical of* A, denoted by $\operatorname{rad}(A)$, to be the largest left ideal of A contained in $\mathcal{Q}(A)$, and it can be shown that

$$\mathrm{rad}\,(A) = \{a \in A : ba \in \mathcal{Q}(A)\ (b \in A)\}.$$

In fact, $\mathrm{rad}\,(A)$ is a closed, two-sided ideal of A, and

$$\mathrm{rad}\,(A) = \{a \in A : ab \in \mathcal{Q}(A)\ (b \in A)\}.$$

For a possibly non-unital Banach algebra A we define $\mathrm{rad}\,(A) := A \cap \mathrm{rad}\,(A^\sharp)$. We often abbreviate the phrase "Jacobson radical" to "radical", and by a *radical element* of A we mean an element of $\mathrm{rad}\,(A)$.

We say that A is *semisimple* if $\mathrm{rad}\,(A) = \{0\}$. For example, any C*-algebra is semisimple [**19**, Corollary 3.2.12]. For us, it is important to note that, for a locally compact group G, the Banach algebras $L^1(G)$ and $M(G)$ are semisimple [**19**, Corollary 3.3.35]. However, it seems to be an open question whether or not Beurling algebras are always semisimple. It is known that they are semisimple in the case that the underlying group is abelian, as well as in the case that the group is arbitrary but the weight is symmetric [**23**, Theorem 7.13].

Many equivalent characterizations of $\mathrm{rad}\,(A)$ are available. We note a few of the important ones for context, although we shall not use them in the thesis (for details see [**19**, Section 1.5]).

THEOREM 1.4.3. *Let A be a Banach algebra. Then the Jacobson radical* $\mathrm{rad}\,(A)$ *is equal to each of the following:*

(1) *the intersection of the maximal modular left ideals of A;*
(2) *the intersection of the maximal modular right ideals of A;*
(3) *the set of elements of A that annihilate every (algebraically) simple left A-module;*
(4) *the set of elements of A that annihilate every (algebraically) simple right A-module.* □

1.4.5. Dual Banach Algebras.

A *dual Banach algebra* is a pair (A, X), where A is a Banach algebra and X is a Banach space, such that X' is isomorphic to A as a Banach space, and such that the multiplication on A is separately continuous in the weak*-topology induced by X. For example, every von Neumann algebra is a dual Banach algebra. Another class of examples is given by $\mathcal{B}(E)$, for E a reflexive Banach space. In this case the predual may be identified with $E\widehat{\otimes}E'$ as in (1.5). An important example for us will be the measure algebra $M(G)$ of a locally compact group G, with predual given by $C_0(G)$ as in (1.2). Similarly the Fourier–Stieltjes algebra $B(G)$ is a dual Banach algebra, with predual $C^*(G)$ as in (1.11). Finally, we remark that, given a Banach algebra A, its bidual A'' is a dual Banach algebra under either Arens product if and only if A is Arens regular (Arens products and Arens regularity are defined in Subsection 1.4.6 below). These examples all appear in [**74**, Example 4.4.2], except for $B(G)$, but it is routine to check that this is a dual Banach algebra. Another natural family of examples is given below:

Proposition 1.4.4. *Let G be a locally compact group, and let ω be a weight on G. Then $(M(G,\omega),\ C_0(G,1/\omega))$ is a dual Banach algebra.*

Proof. By considering compactly supported functions, which form a dense subspace of $C_0(G,1/\omega)$, we see that the formula

$$\langle f, \mu * \nu\rangle = \int_G \int_G f(st)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t) \quad (f \in C_0(G,1/\omega),\ \mu,\nu \in M(G,\omega))$$

continues to hold in the weighted setting. It follows that, given f, μ and ν as in that formula, we have

$$\langle f, \mu * \nu\rangle = \int_G (f \cdot \mu)(t)\,\mathrm{d}\nu(t), \qquad \langle f, \nu * \mu\rangle = \int_G (\mu \cdot f)(t)\,\mathrm{d}\nu(t)$$

where

$$(f \cdot \mu)(s) = \int_G f(ts)\,\mathrm{d}\mu(t) \qquad \text{and} \qquad (\mu \cdot f)(s) = \int_G f(st)\,\mathrm{d}\mu(t),$$

for $s \in G$. If we can show that, for every f and μ, the function $f \cdot \mu \in C_0(G, 1/\omega)$, then it will follow that the map $M(G,\omega) \to \mathbb{C}$ given by $\nu \mapsto \langle f, \mu * \nu\rangle$ is equal to the weak*-continuous map $\nu \mapsto \langle f \cdot \mu, \nu\rangle$. It will then follow that the map $\nu \mapsto \mu * \nu$ is weak*-continuous. Similarly, if we can show that, for every f and μ, the function $\mu \cdot f \in C_0(G, 1/\omega)$, it will follow that multiplication on the right by a fixed element is weak*-continuous, and we will have proven the proposition. We show only that $f \cdot \mu \in C_0(G, 1/\omega)$, the other case being very similar.

To this end, fix $f \in C_0(G, 1/\omega)$, and $\mu \in M(G,\omega)$. We first show that $(f \cdot \mu)/\omega$ is continuous. Let $s \in G$ and let $\varepsilon > 0$. Let F be a compact subset of G such that $\int_{G\setminus F} \omega(t)\,\mathrm{d}|\mu|(t) < \varepsilon$ (which exists because μ is regular and $\|\mu\|_\omega < \infty$). Since ω is continuous, so must f be. Hence $f|_F$ is uniformly continuous, so there exists a compact neighbourhood V of s such that $\sup_F |f(tu) - f(ts)| < \varepsilon$ whenever $u \in V$, and we set $C = \sup_V \omega$. For all $u \in V$ we have

$$\begin{aligned}\left|\int_{G\setminus F} (f(tu) - f(ts))\,\mathrm{d}\mu(t)\right| &\leqslant \int_{G\setminus F} \left|\frac{f(tu)}{\omega(t)}\right| \omega(t)\,\mathrm{d}|\mu|(t) + \int_{G\setminus F} \left|\frac{f(ts)}{\omega(t)}\right| \omega(t)\,\mathrm{d}|\mu|(t)\\ &\leqslant \int_{G\setminus F} \left|\frac{f(tu)}{\omega(tu)}\right| \omega(u)\omega(t)\,\mathrm{d}|\mu|(t)\\ &\qquad\qquad + \int_{G\setminus F} \left|\frac{f(ts)}{\omega(ts)}\right| \omega(s)\omega(t)\,\mathrm{d}|\mu|(t)\\ &\leqslant \|f\|_{\infty,\omega}(\omega(u) + \omega(s))\varepsilon \leqslant 2C\|f\|_{\infty,\omega}\varepsilon.\end{aligned}$$

We also have

$$\begin{aligned}\left|\int_F (f(tu) - f(ts))\,\mathrm{d}\mu(t)\right| &\leqslant \|\mu\|_\omega \sup_{t\in F} \left|\frac{f(tu) - f(ts)}{\omega(t)}\right|\\ &\leqslant \|\mu\|_\omega \sup_{t\in F} |f(tu) - f(ts)| < \|\mu\|_\omega \varepsilon.\end{aligned}$$

It follows that for all $u \in V$ we have

$$|(f \cdot \mu)(u) - (f \cdot \mu)(s)| < \varepsilon(\|\mu\|_\omega + 2C\|f\|_{\infty,\omega}).$$

Since ε was arbitrary, we have shown that $f \cdot \mu$ is continuous, and hence so is $(f \cdot \mu)/\omega$.

Next we show that $(f \cdot \mu)/\omega$ tends to zero at infinity. Let $\varepsilon > 0$, let F be a compact subset of G such that $\int_{G \setminus F} \omega(t)\,\mathrm{d}|\mu|(t) < \varepsilon$, and let E be a compact subset of G such that $\sup_{G \setminus E} |f(s)/\omega(s)| < \varepsilon$. Then for every $s \in G \setminus (F^{-1}E)$ we have

$$\begin{aligned}
\frac{|(f \cdot \mu)(s)|}{\omega(s)} &\leqslant \int_G \frac{|f(ts)|}{\omega(s)}\,\mathrm{d}|\mu|(t) \leqslant \int_G \frac{|f(ts)|}{\omega(ts)}\omega(t)\,\mathrm{d}|\mu|(t) \\
&= \int_{G \setminus F} \frac{|f(ts)|}{\omega(ts)}\omega(t)\,\mathrm{d}|\mu|(t) + \int_F \frac{|f(ts)|}{\omega(ts)}\omega(t)\,\mathrm{d}|\mu|(t) \\
&\leqslant \|f\|_{\infty,\omega} \int_{G \setminus F} \omega(t)\,\mathrm{d}|\mu|(t) + \|\mu\|_\omega \sup_{G \setminus E} \left|\frac{f(s)}{\omega(s)}\right| \\
&\leqslant \varepsilon(\|f\|_{\infty,\omega} + \|\mu\|_\omega).
\end{aligned}$$

As ε was arbitrary, it follows that $f \cdot \mu \in C_0(G, 1/\omega)$. A very similar argument shows that $\mu \cdot f \in C_0(G, 1/\omega)$, and this completes the proof. $\square$

The above proposition does not seem to be stated explicitly anywhere in the literature.

We now give some general background on the topic of dual Banach algebras, although we shall not explicitly use what follows in the thesis. Building on the work of Young [**91**], Daws [**26**] showed that every dual Banach algebra has a weak*-continuous, isometric isomorphism to a weak*-closed subalgebra of $\mathcal{B}(E)$, for some reflexive Banach space E. This result may be thought of as analogous to the famous characterisation due to Sakai of von Neumann algebras as C*-algebras which are isometrically dual Banach spaces. However, in contrast to the situation for von Neumann algebras, the predual of a dual Banach algebra need not be unique. Indeed, consider any Banach space E which has at least two isomorphically distinct preduals (for example ℓ^1) and consider it as a Banach algebra with zero multiplication. Then both preduals render E a dual Banach algebra. In particular, the weak*-topology on a dual Banach algebra may not be unique. A detailed study of such phenomena for more natural Banach algebras, such as $\ell^1(\mathbb{Z})$, has been undertaken: see [**27, 28**].

The term "dual Banach algebra" was first defined in [**73**], although the concept was studied before. Since that time dual Banach algebras have attracted a significant amount of attention, particularly from the harmonic analysis community. One of the most interesting aspects of the theory of dual Banach algebras, although we shall not study it in this thesis, is Connes amenability. This is a certain cohomology condition on a dual Banach algebra which parallels the theory of amenability for ordinary Banach algebras: specifically, a dual Banach algebra (A, X) is Connes amenable if every weak*-continuous derivation from A to a normal dual module is inner. See, for example, [**73, 74**]. One particularly striking result is that the measure algebra of a locally compact group, $M(G)$, is Connes-amenable if and only if G is amenable [**75**]; compare this with the fact that $L^1(G)$ is amenable if and only if G is amenable [**50**], whereas it was shown in [**21**] that $M(G)$ is amenable if and only if G is discrete and amenable, in which case of course $M(G) = L^1(G)$.

1.4.6. Arens Products.

Next we describe Arens products, which give a way to make the bidual of a Banach algebra A again into a Banach algebra by defining an algebra multiplication on A'' with the property that, when A is viewed as a subspace of A'' under the canonical embedding, the new multiplication restricted to A coincides with the original multiplication on A. In fact there are two ways to define such a multiplication.

Arens ([**2**], [**3**]) introduced two products on A'', now denoted by $\Box$ and $\Diamond$, rendering it a Banach algebra. These are called the *first* and *second Arens product* respectively. They are defined in three stages as follows: first we define the action of A on A'; then we define $\Phi \cdot \lambda$ and $\lambda \cdot \Phi$, for $\lambda \in A'$ and $\Phi \in A''$; finally, this allows us to define $\Box$ and

$\Diamond$. The exact formulae are:

$$\langle \lambda \cdot a, b \rangle = \langle \lambda, ab \rangle, \qquad \langle a \cdot \lambda, b \rangle = \langle \lambda, ba \rangle,$$

$$\langle \Phi \cdot \lambda, a \rangle = \langle \Phi, \lambda \cdot a \rangle, \qquad \langle \lambda \cdot \Psi, a \rangle = \langle \Psi, a \cdot \lambda \rangle,$$

$$\langle \Psi \Box \Phi, \lambda \rangle = \langle \Psi, \Phi \cdot \lambda \rangle, \qquad \langle \Psi \Diamond \Phi, \lambda \rangle = \langle \Phi, \lambda \cdot \Psi \rangle,$$

for $\Phi, \Psi \in A'', \lambda \in A', a, b \in A$ (for more details see [**19**, Section 2.6]). Both Arens products have the property that they agree with the original multiplication on A, when A is identified with its image under the canonical embedding into A''. In this thesis, unless we specify otherwise, whenever we talk about the bidual of a Banach algebra we are implicitly considering it as a Banach algebra with the first Arens product. The first Arens product has the property that multiplication by a fixed element on the right is weak*-continuous, whereas the second Arens product has this property on the left. In particular the following formulae hold, for Φ, Ψ elements of A'', and $(a_\alpha), (b_\beta) \subset A$ nets converging in the weak*-topology to Φ and Ψ respectively:

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta, \quad \Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta. \tag{1.12}$$

In these formulae the limits are again taken in the weak*-topology. If $\Box = \Diamond$, we say that A is *Arens regular*, and if the other extreme occurs, namely that

$$\{\Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in A'')\} = A$$

and

$$\{\Phi \in A'' : \Psi \Box \Phi = \Psi \Diamond \Phi \ (\Psi \in A'')\} = A,$$

we say that A is *strongly Arens irregular*. Both of these extremes may occur for Banach algebras of the type considered in Chapter 4, namely those of the form $\ell^1(\mathbb{Z}, \omega)$, as may intermediate cases (see [**23**, Theorem 8.11] and [**23**, Example 9.7]).

Most of our discussion of Arens products will take place in Chapter 4, where we shall consider the second duals of Beurling algebras. However, In Chapter 3 we shall make use of the following result, which also provides a nice example where the Arens products coincide with something familiar.

LEMMA 1.4.5. *Let E be a reflexive Banach space with the approximation property. Then $\mathcal{K}(E)'' = \mathcal{B}(E)$ and both the Arens products coincide with the usual composition of operators in $\mathcal{B}(E)$. In particular, $\mathcal{K}(E)$ is Arens regular.*

PROOF. As we mentioned in Subsection 1.2.2, when E is reflexive with AP $\mathcal{K}(E)''$ may be identified with $\mathcal{B}(E)$, with the dualities given by (1.5) and (1.6). As we noted at the beginning of Subsection 1.4.5, $\mathcal{B}(E)$ is a dual Banach algebra whenever E is reflexive. Hence, in particular composition of operators is separately weak*-continuous. It now follows from (1.12) that both Arens products coincide with composition of operators. $\square$

In fact it follows from [**91**, Theorem 3] that $\mathcal{K}(E)$ is Arens regular if and only if E is reflexive, but without AP we do not know whether $\mathcal{K}(E)''$ is isomorphic to $\mathcal{B}(E)$. See also [**19**, Theorem 2.6.23].

Let A be a Banach algebra. An element $\Phi_0 \in A''$ is called a *mixed identity* if it is a right identity for $(A'', \Box)$ and a left identity for $(A'', \Diamond)$. By [**19**, Proposition 2.9.16], A'' has a mixed identity if and only if A has a bounded approximate identity.

CHAPTER 2

Finitely Generated Left Ideals in Banach Algebras on Groups and Semigroups

2.1. Introduction

This chapter is concerned with finitely-generated ideals in certain Banach algebras, and is based on the paper [**88**]. In this thesis we always understand the phrase "finitely-generated" in the following sense:

Definition 2.1.1. Let A be an algebra and let I be a left ideal in A. We say that I is *finitely-generated* if there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in I$ such that $I = A^\sharp x_1 + \cdots + A^\sharp x_n$.

Note that when this definition is applied to topological algebras we do not take the closure on the right-hand side. In the next chapter we shall study topologically finitely-generated ideals in Banach algebras (defined there). If there is ever danger of confusion we may occasionally write "algebraically finitely-generated" to mean finitely-generated.

The Banach algebras of most interest to us will be those which were defined in Subsections 1.3.2 and 1.3.3 of the introduction. Moreover, we shall mostly focus on the so-called augmentation ideals of these algebras, which we define now.

Let S be a semigroup, and take ω to be a weight on S. We define the *augmentation ideal* of $\ell^1(S,\omega)$ to be

$$\ell^1_0(S,\omega) = \left\{ f \in \ell^1(S,\omega) : \sum_{u \in S} f(u) = 0 \right\}.$$

This is the kernel of the *augmentation character*, which is the map given by

$$f \mapsto \sum_{u \in S} f(u), \quad \ell^1(S, \omega) \to \mathbb{C}.$$

The augmentation ideal is a two-sided ideal of codimension one, and it has analogues in the Beurling algebras and the weighted measure algebras of a locally compact group G, also referred to as the augmentation ideals of those algebras:

$$L_0^1(G, \omega) = \left\{ f \in L^1(G, \omega) : \int_G f \mathrm{d}m = 0 \right\};$$

$$M_0(G, \omega) = \{\mu \in M(G, \omega) : \mu(G) = 0\}.$$

There are also corresponding augmentation characters, given by

$$f \mapsto \int_G f(t) \mathrm{d}m(t), \quad L^1(G, \omega) \to \mathbb{C},$$

and

$$\mu \mapsto \mu(G), \quad M(G, \omega) \to \mathbb{C},$$

respectively. Finally, for a semigroup S, we define

$$\mathbb{C}_0 S = \ell_0^1(S) \cap \mathbb{C}S.$$

One of the central themes of this chapter will be the following question:

QUESTION 2.1.2. Which of the Banach algebras mentioned above have the property that the underlying group or semigroup is finite whenever the augmentation ideal is finitely-generated?

We now give this question some context. In 1974 Sinclair and Tullo [**79**] proved that a left Noetherian Banach algebra, by which we mean a Banach algebra in which all the left ideals are finitely-generated in the sense of Definition 2.1.1, is necessarily finite dimensional. In 2012 Dales and Żelazko [**25**] conjectured the following strengthening of Sinclair and Tullo's result:

CONJECTURE 2.1.3. Let A be a unital Banach algebra in which every maximal left ideal is finitely-generated. Then A is finite dimensional.

It is this conjecture that motivates the inquiries of this chapter. The conjecture is known to be true in the commutative case by a theorem of Ferreira and Tomassini [**33**], and Dales and Żelazko presented a generalization of this result in their paper [**25**]. The conjecture is also known to be true for C*-algebras [**11**], and for $\mathcal{B}(E)$ for many Banach spaces E [**22**]. For instance the conjecture is known to be true when E is a Banach space which is complemented in its bidual and has a Schauder basis, or when $E = c_0(I)$, for I an arbitrary non-empty index set. Moreover, in Corollary 2.2.7 below we show that the conjecture holds for $\mathcal{B}(E)$ whenever E is a reflexive Banach space. However, the conjecture remains open for an arbitrary Banach space E.

We are interested in the conjecture for the Banach algebras arising in harmonic analysis. Our approach is to note that an affirmative answer to Question 2.1.2 for some class of Banach algebras implies that the Dales–Żelazko Conjecture holds for that class. As the Dales–Żelazko conjecture is about unital Banach algebras, all the discrete semigroups that we consider will be monoids, in order to ensure that we are in this setting (note, however, that $\ell^1(S)$ can be unital without S being a monoid; see for instance [**24**, Example 10.15]). However, in Section 2.3 we do prove some results about $L^1(G, \omega)$ for a locally compact group G and a weight ω, an algebra which of course is unital only when G is discrete

We now discuss our main results. Full definitions of the terminology used will be given in the body of the chapter. We begin with the following answer to Question 2.1.2 for group algebras:

COROLLARY 2.1.4. *Let G be a locally compact group. Then $L_0^1(G)$ is finitely-generated if and only if G is finite.*

In particular the Dales–Żelazko conjecture holds for all group algebras. This result follows from Theorems 2.3.2 and 2.3.5, which establish more general results. In particular, Theorem 2.3.5 states that $M_0(G)$ is finitely-generated if and only if G is compact and Theorem 2.3.2 states that, for non-discrete G, $L^1(G)$ has no finitely-generated, closed, maximal left ideals at all. In Section 2.4 we observe that the proofs of these latter results "dualise" to give analogous results about the Fourier and Fourier-Stieltjes algebras (Theorem 2.4.1 and Theorem 2.4.3).

The focus of Section 2.5 is semigroup algebras, and our main result is the following

THEOREM 2.1.5. *Let M be a monoid. Then $\ell_0^1(M)$ is finitely-generated if and only if M is pseudo-finite.*

Here, "pseudo-finite" is a term defined in Section 2.5 which we deem too technical to describe here. For groups (and indeed for weakly right cancellative monoids) pseudo-finiteness coincides with being finite in cardinality, whence the name.

We say that a sequence $(\tau_n) \subset [1, \infty)$ is *tail-preserving* if, for each sequence of complex numbers (x_n), we have $\sum_{n=1}^{\infty} \tau_n \left| \sum_{j=n+1}^{\infty} x_j \right| < \infty$ whenever $\sum_{n=1}^{\infty} \tau_n |x_n| < \infty$. This notion is explored in Section 2.6. In Section 2.7 we prove the following theorem:

THEOREM 2.1.6. *Let G be an infinite, finitely-generated group, with finite, symmetric generating set X. Let ω be a radial weight on G with respect to X, and write τ_n for the value that ω takes on S_n. Then $\ell_0^1(G, \omega)$ is finitely-generated if and only if (τ_n) is tail-preserving.*

Here S_n denotes the set of group elements of word-length exactly n with respect to the fixed generating set X. This implies an affirmative answer to Question 2.1.2 for many weighted group algebras, but also provides examples where the answer is negative:

COROLLARY 2.1.7. *Let G be a finitely-generated, discrete group, and let ω be a weight on G.*

(i) *If ω is either a radial polynomial weight, or a radial exponential weight of degree strictly less than 1, then $\ell_0^1(G,\omega)$ is finitely-generated only if G is finite.*

(ii) *If ω is a radial exponential weight of degree equal to 1, then $\ell_0^1(G,\omega)$ is finitely-generated.*

The proof of this corollary is given in Section 2.7. Finally, in Section 2.8, as an application of the theory developed elsewhere in the chapter, we construct weights ω_1 and ω_2 on $\mathbb{Z}^+$ and $\mathbb{Z}$ respectively for which the Banach algebras $\ell^1(\mathbb{Z}^+,\omega_1)$ and $\ell^1(\mathbb{Z},\omega_2)$ fail to satisfy a converse to Gleason's Theorem on analytic structure (Theorem 2.8.1). We believe that these examples illustrate new phenomena.

2.2. Preliminary Results

In this section we prove some results about finitely-generated left ideals in arbitrary Banach algebras.

LEMMA 2.2.1. *Let A be a Banach algebra, let I be a closed left ideal in A, and let E be a dense subset of I. Suppose that I is finitely-generated. Then I is finitely-generated by elements of E.*

PROOF. Suppose that $I = A^\sharp x_1 + \cdots + A^\sharp x_n$, where $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in I$. Define a map $T : (A^\sharp)^n \to I$ by

$$T : (a_1,\ldots,a_n) \mapsto a_1x_1 + \cdots + a_nx_n.$$

Then T is a bounded linear surjection, and, since the surjections in $\mathcal{B}((A^\sharp)^n, I)$ form an open set [**80**, Lemma 15.3], there exists $\varepsilon > 0$ such that $S \in \mathcal{B}((A^\sharp)^n, I)$ is surjective whenever $\|T - S\| < \varepsilon$. Take $y_1,\ldots,y_n \in E$ with

$$\|y_i - x_i\| < \varepsilon/n \quad (i = 1,\ldots,n).$$

Then we see that the map $(A^\sharp)^n \to I$ defined by

$$(a_1, \dots, a_n) \mapsto a_1 y_1 + \dots + a_n y_n$$

is within ε of T in norm, and hence it is surjective, which implies the result. $\square$

LEMMA 2.2.2. *Let X be a Banach space, with dense linear subspace E, and let Y be a closed linear subspace of X of codimension one. Then $E \cap Y$ is dense in Y.*

PROOF. Since Y is a closed and codimension one subspace, $Y = \ker\varphi$ for some non-zero bounded linear functional φ. Since Y is proper and closed, E is not contained in Y. Hence there exists $x_0 \in E$ such that $\varphi(x_0) = 1$.

Now let $y \in Y$, and take $\varepsilon > 0$. Then there exists $x \in E$ with $\|y - x\| < \varepsilon$. Set $z = x - \varphi(x)x_0$. Then $\varphi(z) = 0$, so that $z \in E \cap Y$. Note that $|\varphi(x)| = |\varphi(y - x)| \leqslant \varepsilon\|\varphi\|$, and hence $\|x - z\| = |\varphi(x)|\|x_0\| \leqslant \varepsilon\|\varphi\|\|x_0\|$, so that $\|y - z\| \leqslant \varepsilon\,(1 + \|\varphi\|\|x_0\|)$. Thus $\overline{E \cap Y} = Y$. $\square$

LEMMA 2.2.3. *Let A be a Banach algebra, and let B be a dense left ideal in A. Let I be a closed, maximal left ideal. Then $B \cap I$ is dense in I.*

PROOF. As I is a closed, maximal left ideal and B is dense in A, B is not contained in I, so that we may choose $b_0 \in B\backslash I$. Consider the left ideal $Ab_0 + I$ of A. As I is maximal, either $Ab_0 + I = I$ or $Ab_0 + I = A$.

In the first case, we see that $ab_0 \in I$ for every $a \in A$, so that $\mathbb{C}b_0 + I$ is a left ideal strictly containing I. This forces $\mathbb{C}b_0 + I = A$, so that I has codimension one. Therefore, in this case, the result follows from Lemma 2.2.2.

Hence we suppose that $Ab_0 + I = A$. Define a map $T : A \to A/I$ by $T : a \mapsto ab_0 + I$. Then T is a bounded linear surjection between Banach spaces, so that, by the open mapping theorem, there exists a constant $C > 0$ such that, for every $y \in A/I$, there exists $x \in A$ with $\|x\| \leqslant C\|y\|$ and $Tx = y$.

Let $a \in I$ and $\varepsilon > 0$ be arbitrary. There exists $b \in B$ with $\|a - b\| < \varepsilon$. It follows that $\|b + I\|_{A/I} \leqslant \varepsilon$, so we can find $a_0 \in A$ with $\|a_0\| \leqslant C\varepsilon$ and $Ta_0 = a_0 b_0 + I = b + I$. Let

$c = b - a_0b_0$. Then $c \in B \cap I$, because B is a left ideal, and $\|b - c\| = \|a_0b_0\| \leqslant C\varepsilon\|b_0\|$. Hence $\|a - c\| \leqslant \varepsilon(1 + C\|b_0\|)$. As a and ε were arbitrary, the result follows. □

COROLLARY 2.2.4. *Let A be a Banach algebra with a dense, proper left ideal. Then:*

(i) *A has no finitely-generated, closed, maximal left ideals;*

(ii) *A has no finitely-generated, closed left ideals of finite codimension.*

PROOF. (i) Assume towards contradiction that I is a finitely-generated, closed, maximal left ideal in A. The algebra A has a proper, dense left ideal B. Then, by Lemma 2.2.3, $B \cap I$ is dense in I, so that, by Lemma 2.2.1, we can find a finite set of generators for I from within B. But then, as B is a left ideal, this forces $I \subset B$, and hence $I = B$ by the maximality of I. But I is closed, whereas B is dense, and both are proper, so we have arrived at a contradiction.

(ii) Let I be a proper, closed left ideal of finite codimension. Then I is contained in some closed maximal left ideal M. We may write $M = I \oplus E$, as linear spaces, for some finite-dimensional space $E \subset A$. If I were finitely-generated, then the generators together with a basis for E would give a finite generating set for M, contradicting (i). Hence I cannot be finitely-generated. □

We note that the above corollary is of limited use since its hypothesis cannot be satisfied in a unital Banach algebra. However, in the non-unital setting it is quite effective, and we shall make use of it in Section 2.3 and Section 2.4. An example of a Banach algebra satisfying the hypothesis of Corollary 2.2.4 coming from outside harmonic analysis is the algebra of approximable operators on an infinite-dimensional Banach space.

We now turn to a result about dual Banach algebras.

PROPOSITION 2.2.5. *Let A be a unital dual Banach algebra. Then every $\|\cdot\|$-closed, finitely-generated left ideal in A is weak*-closed.*

PROOF. Let X be the predual of A, and let I be a closed, finitely-generated left ideal in A. Write

$$I = Ax_1 + \cdots + Ax_n,$$

for some $n \in \mathbb{N}$, and $x_1, \dots, x_n \in I$. Define a linear map

$$S : A^n \to A$$

by

$$S : (a_1, \dots, a_n) \mapsto a_1x_1 + \cdots + a_nx_n.$$

As multiplication in A is separately weak*-continuous, S is a weak*-continuous linear map, and hence $S = T^*$ for some bounded linear map $T : X \to X^n$. We know that $\operatorname{im} S = I$ is closed, implying that $\operatorname{im} T$ is closed, and so we see that $I = \operatorname{im} S$ is weak*-closed, as required. □

LEMMA 2.2.6. *Let A be a unital dual Banach algebra with a proper weak*-dense left ideal. Then A has a maximal left ideal which is not finitely-generated.*

PROOF. Let B be a proper, weak*-dense left ideal in A. Since A is unital, B is contained in some maximal left ideal I of A, which is $\|\cdot\|$-closed. Since I contains B it is also weak*-dense. If I were finitely-generated then, by Proposition 2.2.5, it would be weak*-closed, forcing $I = A$, which contradicts I being a maximal left ideal. □

We note the following corollary of Lemma 2.2.6 here. The result was already known in the case that E has a Schauder basis by [**22**, Theorem 1.4(i)].

COROLLARY 2.2.7. *The family of Banach algebras $\mathcal{B}(E)$, for E a reflexive Banach space, satisfy the Dales-Żelazko conjecture.*

PROOF. Since E is reflexive, $\mathcal{B}(E)$ is a dual Banach algebra. If E is infinite-dimensional, then $\mathcal{F}(E)$ is a proper ideal. Moreover, it is easily checked that $\mathcal{F}(E)_\perp = \{0\}$, implying that $\mathcal{F}(E)$ is weak*-dense in $\mathcal{B}(E)$. Hence, by Lemma 2.2.6, $\mathcal{B}(E)$ has a maximal left ideal which is not finitely-generated whenever it is infinite-dimensional. $\square$

2.3. The Case of a Non-Discrete Locally Compact Group

In this section, we shall consider Question 2.1.2 for $L^1(G, \omega)$ and $M(G, \omega)$, where G is a non-discrete, locally compact group and ω is a weight on G. The first result implies that, if $L^1(G) \subset C(G)$, then G is discrete.

LEMMA 2.3.1. *Let G be a locally compact group. Suppose that, for every precompact, open subset A of G, the function χ_A is equal to a continuous function almost everywhere. Then G is discrete.*

PROOF. Assume to the contrary that G is not discrete. Then by [**20**, Corollary 4.4.4], or [**67**, Theorem 1], G cannot be extremely disconnected, so that there are disjoint open sets A and B and $x_0 \in G$ such that $x_0 \in \overline{A} \cap \overline{B}$. By intersecting with a precompact open neighbourhood of x_0, we may further assume that A is precompact, and thus of finite measure.

Consider the function $h = \chi_A \in L^1(G)$. Then, by hypothesis, there is a continuous function f and a measurable function g such that $\operatorname{supp} g$ is a Haar-null set, with the property that $h = f + g$. In particular, $\operatorname{supp} g$ must have empty interior, so, for any open neighbourhood U of x_0, we can choose $x_U \in U \cap A$ such that $x_U \notin \operatorname{supp} g$. Then (x_U) is a net contained in $A \backslash \operatorname{supp} g$ converging to x_0. Similarly, we may find a net (y_U) contained in $B \backslash \operatorname{supp} g$ converging to x_0. Then $f(x_U) = h(x_U) = 1$ for all U, whereas $f(y_U) = h(y_U) = 0$ for all U. As both nets have the same limit, this contradicts the continuity of f. $\square$

THEOREM 2.3.2. *Let G be a non-discrete, locally compact group, and let ω be a weight on G. Then $L^1(G, \omega)$ has no finitely-generated, closed, maximal left ideals, and no finitely-generated, closed left ideals of finite codimension.*

PROOF. Let $J = L^1(G,\omega) * C_c(G) + C_c(G)$ be the left ideal of $L^1(G,\omega)$ generated by $C_c(G)$. By [**19**, Theorem 3.3.13 (i)], every element of J is continuous, so that, by the previous lemma, J is proper, and of course it is also dense. The result now follows from Corollary 2.2.4. □

When G is a compact group, $L^2(G)$ is a Banach algebra under convolution. A trivial modification of the previous argument shows that, when G is infinite and compact, $L^2(G)$ has no closed, finitely-generated maximal left ideals.

We now turn to the measure algebra. We shall exploit the fact that it is a dual Banach algebra, and make frequent use of (1.4). In particular we shall make use of the following characterisation of the weak*-closed left ideals of a weighted measure algebra. Analogous characterisations exist for the weak*-closed right and two-sided ideals.

LEMMA 2.3.3. *Let G be a locally compact group, and let ω be a weight on G. Then there is a bijective correspondence between the weak*-closed left ideals in $M(G,\omega)$ and the norm-closed subspaces of $C_0(G,1/\omega)$ invariant under left translation. This correspondence is given by*

$$E \mapsto E^{\perp},$$

for E a closed subspace of $C_0(G,1/\omega)$ invariant under left translation.

PROOF. Let E be a closed subspace of $C_0(G,1/\omega)$, invariant under left translation. That $E^{\perp}$ is weak*-closed is clear. We show that it is a left ideal. Let $\mu \in E^{\perp}$. Then for all $f \in E$ and $y \in G$ we have

$$\int_G f(yx)\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}(\delta_y * \mu)(x) = 0. \tag{2.1}$$

Hence $\delta_y * \mu \in E^{\perp}$ for all $y \in G$. That $E^{\perp}$ is a left ideal now follows from weak*-density of the discrete measures in $M(G,\omega)$.

Now suppose that I is a weak*-closed left ideal in $M(G,\omega)$. Set $E = I_{\perp}$. Then, by (1.4), $E^{\perp} = I$. The linear subspace E is clearly closed, and, for $y \in G, \mu \in I$

and $f \in C_0(G, 1/\omega)$, we have $\delta_y * \mu \in I$, so that, by (2.1), $\delta_y * f \in E$. Hence E is left-translation-invariant.

We have shown that the correspondence is well-defined and surjective. To see that it is injective, use (1.4). □

LEMMA 2.3.4. *Let G be a locally compact group. Then $M_0(G)$ is weak*-closed if and only if G is compact.*

PROOF. If G is compact, then $M_0(G) = \{\text{constant functions}\}^{\perp}$, which is weak*-closed.

Assume towards a contradiction that $M_0(G)$ is weak*-closed, but that G is not compact. By Lemma 2.3.3, $E = M_0(G)_{\perp}$ is invariant under left translation, and using the formula $E' \cong M(G)/E^{\perp} = M(G)/M_0(G)$ we see that E has dimension one. So there exists $f \in C_0(G)$ of norm 1 such that $E = \operatorname{span} f$. There exists $x_0 \in G$ such that $|f(x_0)| = 1$. Let K be a compact subset of G such that $|f(x)| < 1/2$ for all $x \in G \backslash K$. Then $Kx_0^{-1} \cup x_0 K^{-1}$ is still compact, so we may choose $y \in G$ not belonging to this set, so that in particular $yx_0, y^{-1}x_0 \notin K$. Then there exists $\lambda \in \mathbb{C} \backslash \{0\}$ such that $\delta_y * f = \lambda f$. Hence

$$|f(yx_0)| = |\lambda||f(x_0)| = |\lambda| < 1/2,$$

whereas

$$1 = |f(x_0)| = |f(yy^{-1}x_0)| = |\lambda||f(y^{-1}x_0)| < 1/2 \cdot 1/2 = 1/4.$$

This contradiction completes the proof. □

The next theorem characterises when $M_0(G)$ is finitely-generated. In particular Question 2.1.2 has a negative answer for the measure algebra.

THEOREM 2.3.5. *Let G be a locally compact group. Then $M_0(G)$ is finitely-generated as a left ideal if and only if G is compact.*

Proof. If G is compact, and m denotes the normalised Haar measure on G, then $m \in M_0(G)$, and it is easily seen, by direct computation, that

$$\mu * m = \varphi_0(\mu)m \quad (\mu \in M(G)).$$

Hence, in particular, m is a right-annihilator of the augmentation ideal, so that, for every $\mu \in M_0(G)$, we have

$$\mu * (\delta_e - m) = \mu * \delta_e - 0 = \mu.$$

Noting that $\delta_e - m \in M_0(G)$, we see that it is an identity element for $M_0(G)$, so that in particular $M_0(G)$ is finitely-generated.

Suppose that $M_0(G)$ is finitely-generated. Then, by Proposition 2.2.5, $M_0(G)$ is weak*-closed, implying that G is compact by Lemma 2.3.4. □

We do not know of a weighted version of this theorem, but when G is discrete $M(G, \omega) = \ell^1(G, \omega)$, and this case will be the focus of Section 2.7, where it seems a very different approach is required as weak*-closure of the augmentation ideal no longer characterises finiteness of the underlying discrete group, and in particular it can happen that $\ell^1_0(G, \omega)$ is weak*-closed, but not finitely-generated.

Note that we have now proven Corollary 2.1.4:

Proof of Corollary 2.1.4. By Theorem 2.3.2, it is enough to consider the discrete case, which follows from Theorem 2.3.5. □

We now prove the Dales–Żelazko conjecture for weighted measure algebras on non-discrete groups. In fact we give two proofs. The first exploits the fact that weighted measure algebras are dual Banach algebras, and does not rely on Lemma 2.3.6 below. The second is a good warm up for the approach taken in Section 2.5 and Section 2.7. We have been unable to fully resolve the discrete version of the conjecture, but again this is addressed in Section 2.7.

LEMMA 2.3.6. *Let G be a discrete group, and ω a weight on G. Suppose that $\ell^1_0(G,\omega)$ is finitely-generated as a left ideal. Then G is finitely-generated.*

PROOF. Suppose $\ell^1_0(G,\omega)$ is generated by $h_1,\ldots,h_n \in \ell^1_0(G,\omega)$. By Lemma 2.2.1 we may assume each h_i is finitely-supported. Let H be the subgroup of G generated by $\bigcup_{i=1}^n \operatorname{supp} h_i$. We show that $H = G$. Let $g \in \ell^1(G,\omega)$. Then, for each $i \in \{1,\ldots,n\}$,

$$\sum_{u\in H}(g * h_i)(u) = \sum_{u\in H}\sum_{st=u} g(s)h_i(t) = \sum_{s\in G} g(s) \sum_{t\in s^{-1}H} h_i(t) = 0, \tag{2.2}$$

where the final equality holds because either $s \notin H$, in which case $s^{-1}H$ is disjoint from $\operatorname{supp} h_i$, or else $s^{-1}H = H \supset \operatorname{supp} h_i$, in which case $h_i \in \ell^1_0(G,\omega)$ implies that $\sum_{t\in H} h_i(t) = 0$. Since the functions h_i generate $\ell^1_0(G,\omega)$ it follows that $\sum_{u\in H} f(u) = 0$ for every $f \in \ell^1_0(G,\omega)$. This clearly forces $H = G$, as claimed. □

THEOREM 2.3.7. *The Dales–Żelazko conjecture holds for the algebra $M(G,\omega)$, whenever G is a non-discrete locally compact group, and ω is a weight on G.*

PROOF 1. Proposition 1.4.4 implies that $M(G,\omega)$ is a unital dual Banach algebra, and, since G is non-discrete, $L^1(G,\omega)$ is a proper, weak*-dense ideal in $M(G,\omega)$. The conjecture now follows from Lemma 2.2.6. □

PROOF 2. By (1.9) $\ell^1(G,\omega)$ is the quotient of $M(G,\omega)$ by the closed ideal consisting of the continuous measures belonging to $M(G,\omega)$. As G is non-discrete, it is uncountable, and hence, by Lemma 2.3.6, $\ell^1_0(G,\omega)$ is not finitely-generated as a left ideal. Taking the preimage of this ideal under the quotient map gives a codimension 1 ideal of $M(G,\omega)$, and this ideal is not finitely-generated as a left ideal. □

2.4. Interlude on the Fourier Algebra

In this section we prove analogues of Corollary 2.1.4 and Theorem 2.3.5 for the Fourier and Fourier-Stieltjes algebras. We define ideals

$$B_0(G) := \{f \in B(G) : f(e) = 0\}$$

and

$$A_0(G) := \{f \in A(G) : f(e) = 0\}.$$

These are the analogues of the augmentation ideal for $A(G)$ and $B(G)$. (Beware that this conflicts with another common notation, where authors define $B_0(G) = C_0(G) \cap B(G)$). Our main results of this section are Theorem 2.4.3(ii), which says that $A_0(G)$ is finitely-generated if and only if G is finite, and Theorem 2.4.1, which says that $B_0(G)$ is finitely-generated if and only if G is discrete. The proof of Theorem 2.4.1 may be thought of, heuristically, as "dual" to the proof of Theorem 2.3.5; likewise the proof of Theorem 2.4.3(ii) is "dual" to that of Corollary 2.1.4.

THEOREM 2.4.1. *Let G be a locally compact group. Then $B_0(G)$ is finitely-generated if and only if G is discrete.*

PROOF. If G is discrete then, by (the easy direction of) Host's idempotent theorem [**47**], we have $\chi_{G\backslash\{e\}} \in B(G)$, and clearly $B_0(G) = B(G)\chi_{G\backslash\{e\}}$.

Suppose that $B_0(G)$ is finitely-generated. Our plan is to show that this forces $C^*(G)$ to be unital, which implies that G is discrete by [**60**]. Since $B(G)$ is a dual Banach algebra, Proposition 2.2.5 implies that $B_0(G)$ is weak*-closed. Let $E = B_0(G)_\perp \subset C^*(G)$. Since $E^\perp = B_0(G) \neq B(G)$, we must have $E \neq \{0\}$ (in fact E must be 1-dimensional). Let $f \in E\backslash\{0\}$. We shall show that F may be scaled to be a unit for $C^*(G)$.

Observe that

$$B_0(G) = \{\xi *_\pi \eta : \langle \xi, \eta \rangle_{H_\pi} = 0\}$$

so that

$$\begin{aligned}E = \{f \in C^*(G) :\ &\langle \xi *_\pi \eta, f\rangle = 0 \text{ for every representation } (\pi, H_\pi),\\ &\qquad\qquad \text{and every } \xi, \eta \in H_\pi \text{ such that } \xi \perp \eta\}\\ = \{f \in C^*(G) :\ &\langle \pi(f)\xi, \eta\rangle_{H_\pi} = 0 \text{ for every representation } (\pi, H_\pi),\\ &\qquad\qquad \text{and every } \xi, \eta \in H_\pi \text{ such that } \xi \perp \eta\}.\end{aligned}$$

Now fix a representation of G, say (π, H_π). Then we see that $\pi(f)\xi \perp \eta$ whenever $\xi, \eta \in H_\pi$ satisfy $\xi \perp \eta$. It follows that

$$\pi(f)\xi \in \{\xi\}^{\perp\perp} = \operatorname{span}\xi \quad (\xi \in H_\pi). \tag{2.3}$$

Let (e_i) be a (possibly uncountably) orthonormal basis for H_π. Then, by (2.3), for each i there exists a scalar λ_i such that $\pi(f)e_i = \lambda_i e_i$. Given indices i, j there must also exist a scalar λ such that $\pi(f)(e_i + e_j) = \lambda(e_i + e_j)$, so that

$$\lambda e_i + \lambda e_j = \pi(f)(e_i + e_j) = \pi(f)e_i + \pi(f)e_j = \lambda_i e_i + \lambda_j e_j,$$

which implies that $\lambda_i = \lambda = \lambda_j$. Hence all of the scalars λ_i are the same, so that $\pi(f)$ acts as a scalar multiple of the identity on H_π.

Now take two representations (π, H_π) and (σ, H_σ), and let $\xi_\pi \in H_\pi$ and $\xi_\sigma \in H_\sigma$. We consider the direct sum representation $\pi \oplus \sigma$. Again using (2.3) we have scalars $\lambda_\pi, \lambda_\sigma$ and $\lambda_{\pi\oplus\sigma}$ such that $\pi(f) = \lambda_\pi \operatorname{id}_{H_\pi}, \sigma(f) = \lambda_\sigma \operatorname{id}_{H_\sigma}$ and $(\pi \oplus \sigma)(f) = \lambda_{\pi\oplus\sigma} \operatorname{id}_{H_\pi \oplus H_\sigma}$. Observe that

$$\lambda_{\pi\oplus\sigma}(\xi_\pi, \xi_\sigma) = (\pi \oplus \sigma)(f)(\xi_\pi, \xi_\sigma) = (\lambda_\pi\xi_\pi, \lambda_\sigma\xi_\sigma),$$

so that $\lambda_\pi = \lambda_{\pi\oplus\sigma} = \lambda_\sigma$. Hence f acts as the same scalar under every representation. Moreover, as $f \neq 0$, and the representations of G separate the points of $C^*(G)$, this scalar is non-zero so that, by scaling, we may assume that $\pi(f)$ is the identity for every representation π.

Let $g \in C^*(G)$. Then for every representation π of G we have $\pi(fg) = \pi(g) = \pi(gf)$, so that, agian using the fact that the representations of G separate the points of $C^*(G)$, we have $fg = g = gf$. As g was arbitrary it follows that f is an identity element for $C^*(G)$, so that G is discrete by [**60**]. □

Now we turn to the Fourier algebra.

LEMMA 2.4.2. *Let G be a locally compact group. We have $C_c(G) \cap A(G) = A(G)$ if and only if G is compact.*

PROOF. Assume that G is not compact, and let K be a compact neighbourhood of the identity in G. Let $H = \bigcup_{i=1}^{\infty} K^i$. Then H is a clopen subgroup of G.

First suppose that H is compact. Then, as G is not compact, we must have $[G : H] = \infty$, so that we can find a sequence of group elements $t_1, t_2, \ldots \in G$ such that $t_1H, t_2H, \ldots$ are all distinct cosets. By [**32**, Lemme (3.2)], we may find non-negative functions $f_i \in A(G)$ $(i \in \mathbb{N})$ such that

$$f_i(s) = \begin{cases} 1 & s \in t_iH \\ 0 & s \notin t_iH \end{cases} \qquad (i \in \mathbb{N}).$$

Let

$$f = \sum_{i=1}^{\infty} 2^{-i} \frac{f_i}{\|f_i\|} \in A(G).$$

Then the support of f is $\bigcup_{i=1}^{\infty} t_iH$, which is not compact.

Now suppose instead that H is not compact. Then, again using [**32**, Lemme (3.2)], we can find functions non-negative $f_i \in A(G)$ $(i \in \mathbb{N})$ such that

$$f_i(s) = \begin{cases} 1 & s \in K^i \\ 0 & s \notin H \end{cases} \qquad (i \in \mathbb{N}).$$

Let $f = \sum_{i=1}^{\infty} 2^{-i} \frac{f_i}{\|f_i\|} \in A(G)$. Then the support of f is H, which is not compact by supposition. □

THEOREM 2.4.3. *Let G be a locally compact group.*

(i) *If G is non-compact then $A(G)$ has no finitely-generated ideals of finite codimension.*

(ii) *The ideal $A_0(G)$ is finitely-generated if and only if G is finite.*

PROOF. (i) By Lemma 2.4.2, $C_c(G) \cap A(G)$ is a proper dense ideal in $A(G)$. Hence (i) follows from Corollary 2.2.4(ii).

(ii) Suppose that $A_0(G)$ is finitely-generated. Then by (i) G is compact, so that $A(G) = B(G)$. Hence, by Theorem 2.4.1, G is discrete, forcing G to be finite. The converse is trivial. □

Remark. Observe that, by Theorem 2.4.3(ii), the conclusion of the Dales-Żelazko conjecture holds for $A(G)$ even though the hypothesis that the algebra be unital is not satisfied unless G is compact. Indeed, since $A(G)$ is not necessarily unital, this result is not covered by [**25**]. However $B(G)$ is covered by [**25**], so that the Dales-Żelazko conjecture holds for Fourier-Stieltjes algebras.

2.5. The Case of a Discrete Monoid

We begin this section with some definitions, which generalise ideas such as word-length in group theory to the context of an arbitrary monoid. By a *monoid* we mean a semigroup possessing an identity element e. Let M be a monoid, and let E be a subset of M. Then for $x \in M$ we define

$$E \cdot x = \{ux : u \in E\}, \quad x \cdot E = \{xu : u \in E\},$$

and

$$E \cdot x^{-1} = \{u \in M : ux \in E\}, \quad x^{-1} \cdot E = \{u \in M : xu \in E\}.$$

We abbreviate $\{u\} \cdot x^{-1}$ to $u \cdot x^{-1}$, and similarly $u \cdot x$ represents the set $\{u\} \cdot x$. The important thing to note in these definitions is that there may not be an element x^{-1}, and that $u \cdot x^{-1}$ represents not an element but a set, which in general may be infinite or empty. Also, be aware that '$\cdot$' is not necessarily associative: $(x \cdot y^{-1}) \cdot z^{-1}$ is meaningful whereas $x \cdot (y^{-1} \cdot z^{-1})$ is not.

Now let $X \subset M$, and fix $u \in M$. We say that a finite sequence $(z_i)_{i=1}^n$ in M is an *ancestry for u with respect to X* if $z_1 = u, z_n = e$, and, for each $i \in \mathbb{N}$ with $1 < i \leqslant n$, there exists $x \in X$ such that either $z_i x = z_{i-1}$ or $z_i = z_{i-1} x$.

Denote by H_X the set of elements of M which have an ancestry with respect to X. Then

$$H_X = \{e\} \cup \left(\bigcup_{n \in \mathbb{N}, x_1, \ldots, x_n \in X} \ \bigcup_{(\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n} (\ldots((e \cdot x_1^{\varepsilon_1}) \cdot x_2^{\varepsilon_2}) \cdot \ldots \cdot x_n^{\varepsilon_n}) \right).$$

We say that the monoid M is *pseudo-generated* by X if $M = H_X$; this is the same notion as what is termed being *right unitarily generated by X* in [**52**]. Observe that when M is not just a monoid but a group, M is pseudo-generated by X if and only if it is generated by X. We say that M is *finitely pseudo-generated* if M is pseudo-generated by some finite set X.

Given a subset X of M we set $B_0 = \{e\}$ and for each $n \in \mathbb{N}$ we set

$$B_n = \{e\} \cup \left(\bigcup_{x_1, \ldots, x_k \in X,\, k \leqslant n} \ \bigcup_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{\pm 1\}^k} (\ldots((e \cdot x_1^{\varepsilon_1}) \cdot x_2^{\varepsilon_2}) \cdot \ldots \cdot x_k^{\varepsilon_k}) \right)$$

and

$$S_n = B_n \backslash B_{n-1}. \tag{2.4}$$

The set B_n consists of those u in M which have an ancestry of length at most n with respect to X. Of course the sets B_n and S_n depend on X, but we suppress this in the notation as X is usually clear from the context. Finally, we say that M is *pseudo-finite* if there is some $n \in \mathbb{N}$ and a finite subset X of M such that every element of M has an ancestry with respect to X of length at most n, or equivalently if $M = B_n$. Again, for a group M, M is pseudo-finite if and only if it is finite.

To see an example of a monoid which is pseudo-finite, but not finite, take any infinite monoid M and add a zero θ to obtain $M^0 = M \cup \{\theta\}$. (This is a new monoid in which the multiplication restricted to M coincides with the original multiplication, and otherwise is determined by $a\theta = \theta = \theta a$ for all $a \in M^0$). Then

$$M^0 = \theta \cdot \theta^{-1},$$

so that M^0 is pseudo-finite. Incidentally, this also furnishes us with an example where associativity of '$\cdot$' fails, even though all expressions involved are meaningful: we have $(\theta \cdot \theta^{-1}) \cdot e = M^0$, whereas $\theta \cdot (\theta^{-1} \cdot e) = \varnothing$.

In the next two lemmas we establish a version of Lemma 2.3.6 for monoids.

Lemma 2.5.1. *Let M be a monoid, and let $X \subset M$. Then we have*

$$H_X \cdot u, H_X \cdot u^{-1} \subset H_X \quad (u \in H_X).$$

Proof. To see this, we define $H_0 = \{e\} \cup X$, and subsequently

$$H_k = \left(\bigcup_{x \in X} H_{k-1} \cdot x \right) \cup \left(\bigcup_{x \in X} H_{k-1} \cdot x^{-1} \right)$$

for $k \in \mathbb{N}$. It is easily seen that

$$H_X = \bigcup_{k=0}^{\infty} H_k.$$

We establish the lemma by induction on k such that $u \in H_k$. The case $k = 0$ follows just from the definition of H_X. So suppose that $k > 0$. Then either $u = zx$ or $ux = z$ for some $z \in H_{k-1}$ and $x \in X$. Consider the first case, and let $h \in H_X$. Then $hu = hzx$. By the induction hypothesis $hz \in H_X$, and hence $hu = hzx \in H_X$ by the case $k = 0$. Similarly, if $y \in M$ is such that $yu = yzx \in H_X$, then (again using the case $k = 0$) we have $yz \in H_X$, and so $y \in H_X$ by the induction hypothesis applied to z.

Similar considerations apply in the case where u has the property that $ux = z$ for some $z \in H_{k-1}$ and some $x \in X$, and we see that in either case $H_X \cdot u$, $H_X \cdot u^{-1} \subset H_X$, completing the induction. $\square$

LEMMA 2.5.2. *Let M be a monoid, let ω be a weight on M, and suppose that $\ell^1_0(M, \omega)$ is finitely-generated as a left ideal in $\ell^1(M, \omega)$. Then M is finitely pseudo-generated.*

PROOF. Write $A = \ell^1(M, \omega)$. Since $\mathbb{C}_0 M$ is dense in $\ell^1_0(M, \omega)$, by Lemma 2.2.1 we may suppose that

$$\ell^1_0(M, \omega) = A * h_1 + \cdots + A * h_n \tag{2.5}$$

for some $h_1, \ldots, h_n \in \mathbb{C}_0 M$. Set

$$X = \bigcup_{i=1}^{n} \operatorname{supp} h_i,$$

so that X is a finite set. We shall complete the proof by showing that X pseudo-generates M.

Write $H = H_X$. We observe that, for $s \in M$, if $s^{-1} \cdot H \cap H \neq \varnothing$, then $s \in H$. Indeed, suppose that $u \in s^{-1} \cdot H \cap H$. Then $su \in H$, and hence $s \in H \cdot u^{-1}$, which is a subset of H by Lemma 2.5.1.

Now let $g \in A$ be arbitrary. Then, for every $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned}\sum_{u \in H} (g * h_i)(u) &= \sum_{u \in H} \sum_{st = u} g(s) h_i(t) = \sum_{u \in H} \sum_{s \in M} \sum_{t \in s^{-1} \cdot u} g(s) h_i(t) \\ &= \sum_{s \in M} \left(g(s) \sum_{t \in s^{-1} \cdot H} h_i(t) \right) = \sum_{s \in H} \left(g(s) \sum_{t \in s^{-1} \cdot H} h_i(t) \right),\end{aligned}$$

where the last equality holds because $s^{-1} \cdot H \cap \operatorname{supp} h_i \subset s^{-1} \cdot H \cap H = \varnothing$ whenever $s \notin H$. However, when $s \in H$, then, for every $x \in \operatorname{supp} h_i$, we have $sx \in H$ by Lemma

2.5.1, which implies that $\operatorname{supp} h_i \subset s^{-1} \cdot H$. It follows that

$$\sum_{t \in s^{-1} \cdot H} h_i(t) = 0$$

because $h_i \in \mathbb{C}_0 M$. Hence

$$\sum_{u \in H} (g * h_i)(u) = 0.$$

By (2.5), this implies that

$$\sum_{u \in H} f(u) = 0$$

for every $f \in \ell^1_0(M)$. But this clearly forces $M = H$, as required. $\square$

Suppose that a monoid M is pseudo-generated by a set X. Given $f \in \ell^1(M)$, we define a sequence of scalars $(\sigma_n(f))$ by

$$\sigma_n(f) = \sum_{u \in B_n} f(u).$$

LEMMA 2.5.3. *Let M be a monoid and $X \subset M$. Let the sets B_n in the definition of σ_n refer to X. Then, for every $g \in \ell^1(M)$ and every $x \in X$ we have*

$$\sum_{n=1}^{\infty} |\sigma_n(g * (\delta_e - \delta_x))| < \infty.$$

PROOF. Write $\sigma_n = \sigma_n(g * (\delta_e - \delta_x))$. Since

$$g * (\delta_e - \delta_x) = \sum_{u \in M} g(u)\delta_u - g(u)\delta_{ux},$$

it follows that

$$\sigma_n = \sum_{u \in B_n} g(u) - \sum_{v \in B_n \cdot x^{-1}} g(v).$$

If $u \in B_{n-1}$, then $ux \in B_n$, implying that $B_{n-1} \subset B_n \cap B_n \cdot x^{-1}$. Hence

$$\begin{aligned}\sigma_n &= \sum_{u \in B_n \setminus B_{n-1}} g(u) + \sum_{u \in B_{n-1}} g(u) - \left(\sum_{u \in B_n \cdot x^{-1} \setminus B_{n-1}} g(u) + \sum_{u \in B_{n-1}} g(u) \right) \\ &= \sum_{u \in S_n} g(u) - \sum_{v \in B_n \cdot x^{-1} \setminus B_{n-1}} g(v).\end{aligned}$$

Notice that $B_n \cdot x^{-1} \subseteq B_{n+1}$, so that

$$B_n \cdot x^{-1} \backslash B_{n-1} \subseteq B_{n+1} \backslash B_{n-1} = S_n \cup S_{n+1}.$$

Hence

$$|\sigma_n| \leqslant \sum_{u \in S_n} |g(u)| + \sum_{u \in S_n \cup S_{n+1}} |g(u)| = 2 \sum_{u \in S_n} |g(u)| + \sum_{u \in S_{n+1}} |g(u)|,$$

so that $\sum\limits_{n=1}^{\infty} |\sigma_n| \leqslant 3 \sum\limits_{u \in M} |g(u)| < \infty$, using the fact that the sets S_n are pairwise disjoint. □

We shall now prove Theorem 2.1.5 in the next two propositions.

PROPOSITION 2.5.4. *Let M be a monoid such that $\ell_0^1(M)$ is finitely-generated as a left ideal. Then M is pseudo-finite.*

PROOF. By Lemmas 2.2.1 and 2.2.2, $\ell_0^1(M)$ is generated by finitely many elements of $\mathbb{C}_0 M$. Suppose that $h = \sum\limits_{i=1}^{N} \alpha_i \delta_{u_i}$ is one of these generators, where $N \in \mathbb{N}$ and $u_1, \ldots, u_N \in M$. Then a simple calculation exploiting the fact that $\sum_{i=1}^{N} \alpha_i = 0$ shows that

$$h = \sum_{i=1}^{N-1} \left(\sum_{j=1}^{i} \alpha_j \right) (\delta_{u_i} - \delta_{u_{i+1}}).$$

Writing $\delta_{u_i} - \delta_{u_{i+1}} = (\delta_e - \delta_{u_{i+1}}) - (\delta_e - \delta_{u_i})$ shows that

$$h = \sum_{i=1}^{N} \beta_i (\delta_e - \delta_{u_i})$$

for some $\beta_1, \ldots, \beta_N \in \mathbb{C}$. It follows that there is some finite subset Y of M such that $\ell^1(M)$ is generated by elements of the form $\delta_e - \delta_u$ $(u \in Y)$.

By Lemma 2.5.2, M is pseudo-generated by some finite set X. Enlarging X if necessary, we may suppose that $Y \subset X$. It then follows from Lemma 2.5.3 that $(\sigma_n(f)) \in \ell^1(\mathbb{N})$ for every $f \in \ell_0^1(M)$, since now every element of $\ell_0^1(M)$ is a linear combination of elements of the form considered in that lemma. We now show that this gives a contradiction in the case where M is not pseudo-finite by constructing an element f of $\ell^1_0(M)$ for which $(\sigma_n(f)) \notin \ell^1(\mathbb{N})$.

Assume that M is not pseudo-finite. Then no B_n is the whole of M, but, by the definition of X, $\bigcup_{n=1}^{\infty} B_n = M$, so there exists an increasing sequence (n_k) of natural numbers such that $B_{n_{k-1}} \subsetneq B_{n_k}$ for every $k \in \mathbb{N}$. Select $u_k \in B_{n_k} \backslash B_{n_{k-1}}$ $(k \in \mathbb{N})$. Let $\zeta = \sum_{j=1}^{\infty} 1/j^2$, and define $f \in \ell^1_0(M)$ by $f(e) = \zeta, f(u_k) = -1/k^2$ and $f(u) = 0$ otherwise. Then

$$\sigma_{n_k}(f) = \zeta - \sum_{j=1}^{k} \frac{1}{j^2} = \sum_{j=k+1}^{\infty} \frac{1}{j^2} \geqslant \frac{1}{k} \quad (k \in \mathbb{N}).$$

Hence $\sum_{k=1}^{\infty} |\sigma_{n_k}(f)| = \infty$, so that $(\sigma_n(f)) \notin \ell^1(\mathbb{N})$, as required. $\square$

The converse of Proposition 2.5.4 is also true, as we shall now prove, completing the proof of Theorem 2.1.5.

PROPOSITION 2.5.5. *Let M be a pseudo-finite monoid. Then $\ell^1_0(M)$ is finitely-generated.*

PROOF. Let $X = \{x_1, \dots, x_r\}$ be a finite pseudo-generating set for M such that $B_n = M$ for some $n \in \mathbb{N}$. For $k \in \mathbb{N}$, we define

$$\Lambda_k = \{f \in \ell^1_0(M) : \operatorname{supp} f \subset B_k\},$$

and use induction on k to show that Λ_k is contained in a finitely-generated ideal which is contained in $\ell^1_0(M)$.

Write $A = \ell^1(M)$, and denote the augmentation character on $\ell^1(M)$ by φ_0. For $f \in \Lambda_1$, we may write

$$\begin{aligned} f &= f(e)\delta_e + \sum_{i=1}^{r} f(x_i)\delta_{x_i} \\ &= f(e)(\delta_e - \delta_{x_1}) + (f(e) + f(x_1))(\delta_{x_1} - \delta_{x_2}) + \\ &\quad \cdots + (f(e) + \cdots + f(x_{r-1}))(\delta_{x_{r-1}} - \delta_{x_r}). \end{aligned}$$

It follows that $\Lambda_1 \subset A * (\delta_e - \delta_{x_1}) + \cdots + A * (\delta_{x_{r-1}} - \delta_{x_r})$. This establishes the base case.

Consider $k > 1$. By the induction hypothesis, there exist $m \in \mathbb{N}$ and $p_1, \ldots, p_m \in \ell_0^1(M)$ such that

$$\Lambda_{k-1} \subset A * p_1 + \cdots + A * p_m.$$

Write B_k as

$$B_k = \{e\} \cup \left(\bigcup_{i=1}^{r} B_{k-1} \cdot x_i \right) \cup \left(\bigcup_{i=1}^{r} B_{k-1} \cdot x_i^{-1} \right).$$

Write $f \in \Lambda_k$ as

$$f = f(e)\delta_e + g_1 + \cdots + g_r + h_1 + \cdots + h_r,$$

where $\operatorname{supp} g_i \subset B_{k-1} \cdot x_i$ and $\operatorname{supp} h_i \subset B_{k-1} \cdot x_i^{-1}$. Then

$$\begin{aligned} f = \sum_{i=1}^{r} (g_i - \varphi_0(g_i)\delta_{x_i}) + \sum_{i=1}^{r} (h_i - \varphi_0(h_i)\delta_e) \\ + \sum_{i=1}^{r} \varphi_0(g_i)\delta_{x_i} + \left(f(e) + \sum_{i=1}^{r} \varphi_0(h_i) \right) \delta_e. \end{aligned}$$

We note that

$$\sum_{i=1}^{r} \varphi_0(g_i)\delta_{x_i} + \left(f(e) + \sum_{i=1}^{r} \varphi_0(h_i) \right) \delta_e \in A * (\delta_e - \delta_{x_1}) + \cdots + A * (\delta_{x_{r-1}} - \delta_{x_r})$$

by the base case. Fix $i \in \{1, \ldots, r\}$. Each $u \in B_{k-1} \cdot x_i$ can be written $u = u'x_i$ for some $u' \in B_{k-1}$ (which depends on u, and may not be unique), and we calculate that

$$g_i = \sum_{u \in B_{k-1} \cdot x_i} g_i(u)\delta_{u'x_i} = g_i' * \delta_{x_i},$$

where $g_i' = \sum_{u \in B_{k-1} \cdot x_i} g_i(u)\delta_{u'}$. Moreover,

$$g_i - \varphi_0(g_i)\delta_{x_i} = (g_i' - \varphi_0(g_i)\delta_e) * \delta_{x_i}.$$

The support of $g_i' - \varphi_0(g_i)\delta_e$ is contained in B_{k-1}, and so, by the induction hypothesis, we have

$$g_i' - \varphi_0(g_i)\delta_e \in A * p_1 + \cdots + A * p_m,$$

whence

$$g_i - \varphi_0(g_i)\delta_{x_i} \in A * p_1 * \delta_{x_i} + \cdots + A * p_m * \delta_{x_i}.$$

Now consider $h_i - \varphi_0(h_1)\delta_e$. We have

$$h_i * \delta_{x_i} = \sum_{u \in B_{k-1} \cdot x_i^{-1}} h_i(u)\delta_{ux_i},$$

so that $\operatorname{supp}(h_i * \delta_{x_i}) \subset B_{k-1}$ and, in particular, $\operatorname{supp}(h_i * \delta_{x_i} - \varphi_0(h_i)\delta_{x_i}) \subset B_{k-1}$ (as $k \geqslant 2$). It then follows from the induction hypothesis that

$$(h_i - \varphi_0(h_i)\delta_e) * \delta_{x_i} = a_1 * p_1 + \cdots + a_m * p_m$$

for some $a_1, \ldots, a_m \in A$. So

$$\begin{aligned} h_i - \varphi_0(h_i)\delta_e &= (h_i - \varphi_0(h_i)\delta_e) * (\delta_e - \delta_{x_i}) + a_1 * p_1 + \cdots + a_m * p_m \\ &\in A * (\delta_e - \delta_{x_i}) + A * p_1 + \cdots + A * p_m. \end{aligned}$$

We now conclude that

$$\Lambda_k \subset \sum_{i=1}^{m} A * p_i + \sum_{i,j} A * p_i * \delta_{x_j} + \sum_{i=1}^{r} A * (\delta_e - \delta_{x_i}) + \sum_{i=1}^{r-1} A * (\delta_{x_i} - \delta_{x_{i+1}}).$$

This completes the induction. When $k = n$, we obtain the theorem. □

We recall the following standard definitions:

Definition 2.5.6. Let M be a monoid. Then:

(i) M is *right cancellative* if $a = b$ whenever $ax = bx$ $(a, b, x \in M)$;

(ii) M is *weakly right cancellative* if, for every $a, x \in M$, the set $a \cdot x^{-1}$ is finite.

It is easily seen from the definitions that a weakly right cancellative monoid is pseudo-finite if and only if it is finite. Hence, Question 2.1.2 and the Dales–Żelazko conjecture both have answers in the affirmative for the class of Banach algebras of the form $\ell^1(M)$, where M is a weakly right cancellative monoid. However, it remains open whether the Dales–Żelazko conjecture holds for $\ell^1(M)$ for an arbitrary monoid M.

2.6. τ-Summable Sequences

In this section $\tau = (\tau_n)$ will always be a sequence of real numbers, all at least 1. We say that a sequence of complex numbers (x_n) τ-*summable* if

$$\sum_{n=1}^{\infty} \tau_n |x_n| < \infty.$$

Note that if (x_n) is τ-summable for some τ, then in particular $(x_n) \in \ell^1$.

We say that τ is *tail-preserving* if the sequence $\left(\sum_{j=n+1}^{\infty} x_j\right)$ is τ-summable whenever (x_n) is τ-summable. For example, the constant 1 sequence is not tail-preserving (as can be seen by considering, for instance, the sequence $x_n = 1/n^2$ $(n \in \mathbb{N})$), but it will be a consequence of Proposition 2.6.1, below, that $\tau_n = c^n$ is tail-preserving for each $c > 1$. The main result of this section is an intrinsic characterization of tail-preserving sequences, given in Proposition 2.6.1. The results of this section will underlie our main line of attack when we consider questions involving weights on discrete groups in Sections 2.7 and 2.8.

Our approach is to consider the Banach spaces $\ell^1(\tau)$, defined by

$$\ell^1(\tau) = \left\{ (x_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} \tau_n |x_n| < \infty \right\},$$

with the norm given by

$$\|(x_n)\|_\tau = \sum_{n=1}^{\infty} \tau_n |x_n|,$$

so that $\ell^1(\tau)$ is exactly the set of τ-summable sequences. Each space $\ell^1(\tau)$ is in fact isometrically isomorphic to ℓ^1.

PROPOSITION 2.6.1. *Let $\tau = (\tau_n)$ be a sequence in $[1, \infty)$. Then the following are equivalent:*

(a) *τ is tail-preserving;*

(b) *there exists a constant $D > 0$ such that*

$$\tau_{n+1} \geqslant D \sum_{j=1}^{n} \tau_j \quad (n \in \mathbb{N}); \tag{2.6}$$

(c) $\liminf_n \left(\tau_{n+1} \Big/ \sum_{i=1}^{n} \tau_i \right) > 0.$

PROOF. The equivalence of (b) and (c) is clear. We show the equivalence of (a) and (b).

Given $x = (x_n) \in \ell^1(\tau)$ we write $T(x)$ for the sequence

$$T(x) = \left(\sum_{j=n+1}^{\infty} x_j \right).$$

Clearly the condition that τ is tail-preserving is equivalent to the statement that T defines a map $\ell^1(\tau) \to \ell^1(\tau)$. We begin by showing that, in fact, this is equivalent to the statement that T defines a *bounded* linear map $\ell^1(\tau) \to \ell^1(\tau)$. One implication is trivial, and the other is an application of the Closed Graph Theorem.

Indeed, suppose that τ is tail preserving, and let $(x^{(i)})$ be a sequence of elements of $\ell^1(\tau)$ converging to zero, with the property that $\left(T(x^{(i)})\right)$ converges to some point $y \in \ell^1(\tau)$. Let $\varepsilon > 0$, and let i be large enough that both $\|x^{(i)}\|_\tau < \varepsilon$ and $\|T(x^{(i)}) - y\|_\tau < \varepsilon$. Observe that, for all $n \in \mathbb{N}$, we have

$$\left| \sum_{j=n+1}^{\infty} x_j^{(i)} \right| \leqslant \sum_{j=1}^{\infty} |x_j^{(i)}| \leqslant \|x^{(i)}\|_\tau < \varepsilon.$$

Also,

$$\sum_{n=1}^{\infty} \tau_n \left| \left(\sum_{j=n+1}^{\infty} x_j^{(i)} \right) - y_n \right| = \left\| T(x^{(i)}) - y \right\|_\tau < \varepsilon$$

implies that, for each $n \in \mathbb{N}$, we have

$$\left| \sum_{j=n+1}^{\infty} x_j^{(i)} - y_n \right| < \varepsilon.$$

Hence, for each $n \in \mathbb{N}$,

$$|y_n| \leqslant \left| \left(\sum_{j=n+1}^{\infty} x_j^{(i)} \right) - y_n \right| + \left| \sum_{j=n+1}^{\infty} x_j^{(i)} \right| < 2\varepsilon.$$

As n and ε were arbitrary, this forces $y = 0$. Hence, by the Closed Graph Theorem, T is bounded.

We now complete the proof. Clearly T is bounded if and only if it is bounded on the non-negative real sequences belonging to $\ell^1(\tau)$. Note that, by changing the order of summation, we have

$$\sum_{n=1}^{\infty} \tau_n \sum_{j=n+1}^{\infty} x_j = \sum_{j=2}^{\infty} x_j \sum_{n=1}^{j-1} \tau_j,$$

for any non-negative $x \in \ell^1(\tau)$. Hence T is bounded if and only if there exists $D > 0$ such that

$$\sum_{j=2}^{\infty} x_j \sum_{n=1}^{j-1} \tau_n \leqslant D \sum_{n=1}^{\infty} \tau_n x_n$$

for every non-negative $x \in \ell^1(\tau)$, which is evidently equivalent to

$$\sum_{n=1}^{j-1} \tau_n \leqslant D\tau_j \quad (j \in \mathbb{N}).$$

This establishes the equivalence of (a) and (b). □

As we remarked above, it is an immediate consequence of this proposition that the sequence (c^n) is tail-preserving for each $c > 1$.

The following lemma concerns the growth of tail-preserving sequences. Part (ii) implies that, if (τ_n) is tail-preserving and $\tau'_n \geqslant \tau_n$ for all n, then (τ'_n) is not necessarily tail-preserving.

LEMMA 2.6.2. (i) *Let* $\tau = (\tau_n)$ *be a tail-preserving sequence, and let* $D > 0$ *satisfy* (2.6). *Then*

$$\tau_{j+1} \geqslant D(D+1)^{j-1}\tau_1 \quad (j \in \mathbb{N}).$$

(ii) *Let* $\rho > 1$. *There exists a sequence* $(\tau_n) \subset [1, \infty)$ *such that* $\rho^n \leqslant \tau_n$ *for all* $n \in \mathbb{N}$, *but* (τ_n) *is not tail-preserving.*

PROOF. (i) We proceed by induction on $j \in \mathbb{N}$. The case $j = 1$ is immediate from (2.6). Now suppose that $j > 1$, and assume that the result holds for all $i < j$. Then we have

$$\begin{aligned}\tau_{j+1} \geqslant D\sum_{i=1}^{j}\tau_i &\geqslant D[D(D+1)^{j-2} + \cdots + D(D+1) + D + 1]\tau_1 \\ &= D(D+1)^{j-1}\tau_1.\end{aligned}$$

Hence the result also holds for j.

(ii) Define integers n_k recursively by $n_1 = 1$ and $n_k = n_{k-1} + k + 1$ for $k \geqslant 2$. Then define

$$\tau_j = \rho^{n_k+1} \quad (n_{k-1} + 1 < j \leqslant n_k + 1).$$

Then clearly $\tau_j \geqslant \rho^j$ for all $j \in \mathbb{N}$, and

$$\frac{\tau_{n_k+1}}{\sum_{j=1}^{n_k}\tau_j} \leqslant \frac{\tau_{n_k+1}}{\sum_{j=n_{k-1}+2}^{n_k}\tau_j} = \frac{1}{k} \to 0.$$

Hence (τ_n) violates condition (c) of Proposition 2.6.1, so cannot be tail-preserving. $\square$

2.7. Weighted Discrete Groups

In this section G will denote a discrete group, with finite generating set X, and ω will be a weight on G. Without loss of generality we may suppose that X is symmetric (we recall that a subset X of a group G is *symmetric* if $X = X^{-1}$). We shall consider whether $\ell_0^1(G, \omega)$ is finitely-generated. We note that when considering Question 2.1.2 and Conjecture 2.1.3 for $L^1(G, \omega)$, Theorem 2.3.2 and Lemma 2.3.6 allow us to reduce to this setting. As noted at the end of Section 2.3, similar remarks pertain to $M(G,\omega)$.

We define a sequence of real numbers, all at least 1, by

$$\tau_n = \min_{u \in S_n} \omega(u), \tag{2.7}$$

where S_n is defined by (2.4). As we are now in the group setting, S_n is exactly the the set of group elements of word-length n with respect to X. We write

$$C = \max_{x \in X} \omega(x). \tag{2.8}$$

LEMMA 2.7.1. *With τ_n $(n \in \mathbb{N})$ and C defined by (2.7) and (2.8), respectively, we have $\tau_n \leqslant C\tau_{n+1}$ for all $n \in \mathbb{N}$.*

PROOF. For each $n \in \mathbb{N}$, take $y_n \in S_n$ satisfying $\omega(y_n) = \tau_n$. Then $y_{n+1} = zx$ for some $z \in S_n$ and some $x \in X$, so

$$\tau_n = \omega(y_n) \leqslant \omega(z) = \omega(y_{n+1}x^{-1}) \leqslant C\omega(y_{n+1}) = C\tau_{n+1},$$

giving the result. $\square$

In the next lemma, notice that parts (i) and (ii) depend on the weight having the specified properties, whereas part (iii) is a purely algebraic result that can be applied more broadly. In fact, Lemma 2.7.2(iii) is well known; see e.g. [**63**, Chapter 3, Lemma 1.1]. We include a short proof for the convenience of the reader.

LEMMA 2.7.2. *Let G be a group with finite generating set X, and denote word-length with respect to X by $|\cdot|$. Let ω be a radial weight on G, and denote by τ_n the value that ω takes on S_n. Assume that (τ_n) is tail-preserving, and let $D > 0$ be a constant as in (2.6). Consider $\mathbb{C}G \subset \ell^1(G,\omega)$.*

(i) *Let $u \in G$ be expressed as $u = y_1 \cdots y_n$ for $y_1, \ldots, y_n \in X$, where $n = |u|$. Then*

$$\delta_e - \delta_u = \sum_{x \in X} f_x * (\delta_e - \delta_x)$$

for some $f_x \in \mathbb{C}G$ $(x \in X)$ *each of which may be taken to have the form*

$$f_x = \sum_{j=0}^{n-1} a_x^{(j)},$$

where each $a_x^{(j)}$ *is either* 0 *or* $\delta_{y_1 \cdots y_j}$ *in the case that* $j \neq 0$, *and either* 0 *or* δ_e *in the case that* $j = 0$.

(ii) *Each* f_x *in* (i) *satisfies*

$$\|f_x\| \leqslant \frac{1}{D}\omega(u) \quad (x \in X). \tag{2.9}$$

(iii) *As a left ideal in* $\mathbb{C}G$, $\mathbb{C}_0 G$ *is generated by the elements*

$$\delta_e - \delta_x \quad (x \in X).$$

Proof. (i) We proceed by induction on $n = |u|$. The case $n = 1$ is trivial, so suppose that $n > 1$. Set $v = y_2 \cdots y_n \in S_{n-1}$. By the induction hypothesis applied to v,

$$\delta_e - \delta_v = \sum_{x \in X} g_x * (\delta_e - \delta_x),$$

where

$$g_x = \sum_{j=0}^{n-2} b_x^{(j)} \quad (x \in X)$$

and each $b_x^{(j)}$ is either 0, $\delta_{y_2 \cdots y_{j+1}}$ or δ_e. We have

$$\begin{aligned} \delta_e - \delta_u = \delta_{y_1} * (\delta_e - \delta_v) + (\delta_e - \delta_{y_1}) &= \delta_{y_1} * \sum_{x \in X} g_x * (\delta_e - \delta_x) + (\delta_e - \delta_{y_1}) \\ &= \sum_{x \neq y_1} \delta_{y_1} * g_x * (\delta_e - \delta_x) + (\delta_{y_1} * g_{y_1} + \delta_e) * (\delta_e - \delta_{y_1}). \end{aligned}$$

We define

$$f_x = \begin{cases} \delta_{y_1} * g_x & (x \neq y_1), \\ \delta_{y_1} * g_{y_1} + \delta_e & (x = y_1), \end{cases}$$

and check that each f_x can be written in the required form. To see this, set

$$a_x^{(j)} = \begin{cases} \delta_{y_1} * b_x^{(j-1)} & (j = 1, \ldots, n-1), \\ 0 & (x \neq y_1,\ j = 0), \\ \delta_e & (x = y_1,\ j = 0). \end{cases}$$

It is easily checked that each $a_x^{(j)}$ has the required form, and that $f_x = \sum_{j=0}^{n-1} a_x^{(j)}$ $(x \in X)$. This completes the induction.

(ii) Using part (i), we see that

$$\|f_x\| = \sum_{j=0}^{n-1} \|a_x^{(j)}\| \quad (x \in X)$$

and, since, for each $x \in X$, every non-zero $a_x^{(j)}$ is δ_w for some $w \in S_j$, we have

$$\|f_x\| \leqslant \sum_{j=0}^{n-1} \tau_j \leqslant \frac{1}{D}\tau_n = \frac{1}{D}\omega(u),$$

as required.

(iii) Let $\sum\limits_{i=0}^{N} \alpha_i \delta_{u_i} \in \mathbb{C}_0 G$. A simple calculation shows that

$$\sum_{i=0}^{N} \alpha_i \delta_{u_i} = \sum_{i=0}^{N} \left(\sum_{j=0}^{i} \alpha_j \right) (\delta_{u_i} - \delta_{u_{i+1}}).$$

Moreover, for each $i \in \mathbb{N}$, we have $\delta_{u_i} - \delta_{u_{i+1}} = (\delta_e - \delta_{u_{i+1}}) - (\delta_e - \delta_{u_i})$, so that the result follows from (i). $\square$

By analogy to our approach in Section 2.5, we associate to each function $f \in \ell^1(G, \omega)$, a complex-valued sequence $(\sigma_n(f))$, defined by

$$\sigma_n(f) = \sum_{u \in B_n} f(u). \tag{2.10}$$

LEMMA 2.7.3. *Let G be a group generated by a finite, symmetric set X, and let ω be a weight on G. Let $\tau = (\tau_n)$ be defined by (2.7). Let $g \in \ell^1(G, \omega)$ and $x \in X$, and* *write*

$$\sigma_n = \sigma_n[g * (\delta_e - \delta_x)] \quad (n \in \mathbb{N}).$$

Then $(\sigma_n) \in \ell^1(\tau)$.

PROOF. Begin by repeating exactly the argument from the beginning of the proof of Lemma 2.5.3, to obtain

$$\sigma_n = \sum_{u \in S_n} g(u) - \sum_{u \in B_n x^{-1} \backslash B_{n-1}} g(u).$$

Again we have $B_n x^{-1} \backslash B_{n-1} \subseteq S_n \cup S_{n+1}$. Taking C as in (2.8), we compute

$$\begin{aligned}
\tau_n |\sigma_n| &\leqslant 2 \sum_{u \in S_n} |g(u)| \tau_n + \sum_{u \in S_{n+1}} |g(u)| \tau_n \\
&\leqslant 2 \sum_{u \in S_n} |g(u)| \tau_n + C \sum_{u \in S_{n+1}} |g(u)| \tau_{n+1} \\
&\leqslant 2 \sum_{u \in S_n} |g(u)| \omega(u) + C \sum_{u \in S_{n+1}} |g(u)| \omega(u),
\end{aligned}$$

where we have used Lemma 2.7.1 in the second line, and (2.7) in the third line. Since the sets S_n are pairwise disjoint, we conclude that

$$\sum_{n=1}^{\infty} \tau_n |\sigma_n| \leqslant (2 + C) \|g\|_\omega < \infty.$$

Hence $(\sigma_n) \in \ell^1(\tau)$, as claimed. □

The following gives a strategy for showing that $\ell^1_0(G, \omega)$ fails to be finitely-generated, for finitely-generated groups G and certain weights ω on G.

THEOREM 2.7.4. *Let G be an infinite group generated by the finite, symmetric set X, and let ω be a weight on G. Let $\tau = (\tau_n)$ be defined by* (2.7)*. Suppose that $\ell^1_0(G, \omega)$ is finitely-generated. Then τ is tail-preserving.*

PROOF. By Lemmas 2.2.1 and 2.2.2, we may suppose that $\ell^1_0(G, \omega)$ is generated by a finite subset of $\mathbb{C}_0 G$, and hence, by Lemma 2.7.2(iii), we may suppose that each generator has the form $\delta_e - \delta_x$, for some $x \in X$. Therefore every element of $\ell^1_0(G,\omega)$ is a finite linear combination of elements of the form $g * (\delta_e - \delta_x)$, where $g \in \ell^1(G,\omega)$ and $x \in X$, and so, by Lemma 2.7.3, $(\sigma_n(f)) \in \ell^1(\tau)$ for every $f \in \ell^1_0(G,\omega)$.

Assume for contradiction that τ fails to be tail-preserving. Then there exists a sequence (α_n) of non-negative reals such that $(\alpha_n) \in \ell^1(\tau)$, but such that

$$\left(\sum_{j=n+1}^{\infty} \alpha_j\right) \notin \ell^1(\tau).$$

For $n \in \mathbb{N}$, let $y_n \in S_n$ satisfy $\omega(y_n) = \tau_n$, and define

$$f = \zeta\delta_e - \sum_{j=1}^{\infty} \alpha_n \delta_{y_n},$$

where $\zeta = \sum\limits_{j=1}^{\infty} \alpha_j$. Then f is well defined, because $(\alpha_n) \in \ell^1(\tau)$, and clearly $\varphi_0(f) = 0$. However,

$$\sigma_n(f) = \zeta - \sum_{j=1}^{n} \alpha_j = \sum_{j=n+1}^{\infty} \alpha_j,$$

so that $(\sigma_n(f)) \notin \ell^1(\tau)$ by the choice of α, contradicting Lemma 2.7.3. □

We are now ready to prove Theorem 2.1.6, which completely characterises finite generation of the augmentation ideal in the case where the weight is radial. In particular, this characterisation establishes the Dales–Żelazko conjecture for $\ell^1(G,\omega)$ for many groups G and weights ω.

Proof of Theorem 2.1.6. If $\ell^1_0(G,\omega)$ is finitely-generated then, by Theorem 2.7.4, (τ_n) is tail-preserving .

Suppose that (τ_n) is tail-preserving. Write $X = \{x_1, \ldots, x_r\}$ and enumerate G as $G = \{u_0 = e, u_1, u_2, \ldots\}$. Let $f = \sum_{n=0}^{\infty} \alpha_n \delta_{u_n} \in \ell^1_0(G,\omega)$, and let $D > 0$ be as in (2.6). By Lemma 2.7.2, for each $n \in \mathbb{N}$, there exist $g_n^{(1)}, \ldots, g_n^{(r)} \in \mathbb{C}G$ such that $\delta_e - \delta_{u_n} = \sum_{i=1}^{r} g_n^{(i)} * (\delta_e - \delta_{x_i})$ and

$$\|g_n^{(i)}\| \leqslant \frac{1}{D}\|\delta_{u_n}\| \quad (i = 1, \ldots, r).$$

This implies that, for each $i = 1, \ldots, r$, we may define an element of $\ell^1(G, \omega)$ by

$$s^{(i)} = -\sum_{n=1}^{\infty} \alpha_n g_n^{(i)}.$$

Then

$$\begin{aligned} f &= \sum_{n=0}^{\infty} \alpha_n \delta_{u_n} - \left(\sum_{n=0}^{\infty} \alpha_n\right) \delta_e = -\sum_{n=1}^{\infty} \alpha_n (\delta_e - \delta_{u_n}) \\ &= -\sum_{n=1}^{\infty} \alpha_n \left(\sum_{i=1}^{r} g_n^{(i)} * (\delta_e - \delta_{x_i})\right) = \sum_{i=1}^{r} s^{(i)} * (\delta_e - \delta_{x_i}). \end{aligned}$$

As f was arbitrary, it follows that the elements $\delta_e - \delta_{x_1}, \ldots, \delta_e - \delta_{x_r}$ together generate $\ell_0^1(G, \omega)$. □

We now prove Corollary 2.1.7, part (ii) of which shows that it can happen that $\ell_0^1(G, \omega)$ is finitely-generated, for certain infinite groups G and certain weights ω.

Proof of Corollary 2.1.7. (i) Lemma 2.6.2(i) implies that, for such a weight, the sequence (τ_n) defined in Theorem 2.1.6 is not tail-preserving, and the result follows from that theorem.

(ii) By Theorem 2.6.1, the sequence (τ_n) of Theorem 2.1.6 is tail-preserving. □

Let G be a discrete group, and G' its commutator subgroup. We conclude this section by remarking that, if $[G : G'] = \infty$, then $\ell^1(G, \omega)$ safisfies the Dales–Żelazko conjecture for every weight ω. The reasoning is as follows. As the conjecture holds for commutative Banach algebras [**33**, Corollary 1.7], the conjecture holds for $\ell^1(H, \omega)$ whenever H is an abelian group and ω is a weight on H. Then, by [**68**, Theorem 3.1.13], given G and ω, there exists a weight $\widetilde{\omega}$ on G/G' such that $\ell^1(G/G', \widetilde{\omega})$ is a quotient of $\ell^1(G, \omega)$. Finally, by the commutative result, there is some maximal ideal in $\ell^1(G/G', \widetilde{\omega})$ which is not finitely-generated, and taking its preimage under the quotient map gives a maximal left ideal in $\ell^1(G, \omega)$ which is not finitely-generated. However, we have not been able to establish the Dales–Żelazko conjecture for an arbitrary weighted group algebra.

2.8. Examples on $\mathbb{Z}$ and $\mathbb{Z}^+$

In this section we look at some specific examples of weighted algebras on $\mathbb{Z}$ and $\mathbb{Z}^+$, and consider how they fit into the more general theory of maximal ideals in commutative Banach algebras. When convenient, we shall sometimes write ω_n in place of $\omega(n)$.

For a commutative Banach algebra A we shall denote the character space of A by Φ_A, and for an element $a \in A$, we shall denote by $\widehat{a}$ its Gelfand transform. We first recall Gleason's Theorem [**80**, Theorem 15.2]:

THEOREM 2.8.1. *Let A be a commutative Banach algebra, with unit 1 and take $\varphi_0 \in \Phi_A$. Suppose that $\ker \varphi_0$ is finitely-generated by $g_1, \ldots, g_n$, and take $\gamma : \Phi_A \to \mathbb{C}^n$ to be the map given by*

$$\gamma(\varphi) = (\varphi(g_1), \ldots, \varphi(g_n)) \quad (\varphi \in \Phi_A).$$

Then there is a neighbourhood Ω of 0 in $\mathbb{C}^n$ such that:

(i) *γ is a homeomorphism of $\gamma^{-1}(\Omega)$ onto an analytic variety E of Ω;*

(ii) *for every $a \in A$, there is a holomorphic function F on Ω such that $\widehat{a} = F \circ \gamma$ on $\gamma^{-1}(\Omega)$;*

(iii) *if $\varphi \in \gamma^{-1}(\Omega)$, then $\ker \varphi$ is finitely-generated by*

$$g_1 - \varphi(g_1)1, \ldots, g_n - \varphi(g_n)1.$$

It is natural to wonder whether there are circumstances under which a converse holds. For instance, suppose we have a commutative Banach algebra A such that there is an open subset U of the character space, which is homeomorphic to an open subset of $\mathbb{C}^n$, and such that $\widehat{a}$ is holomorphic on U under this identification for every $a \in A$. Does it then follow that the maximal ideals corresponding to points of U are finitely-generated? T. T. Reed gave an example [**80**, Example 15.9] which shows that this need not be true in general, even for uniform algebras. We note that the character space in Reed's example is very complicated. In this section we give two examples of commutative Banach algebras for which the converse to Gleason's Theorem fails to hold, and whose character spaces are the disc and the annulus respectively. The first (Theorem 2.8.4) shows that there is no general converse to Gleason's Theorem for the class of natural Banach function algebras on simply connected compact plane sets. The second (Theorem 2.8.6) shows that there is no general converse to Gleason's Theorem for the class of weighted abelian group algebras. Interestingly, these examples rely on constructing counterparts to the sequence (τ_n) of Lemma 2.6.2(ii) satisfying the additional constraints that the sequence must now be a weight on $\mathbb{Z}^+$ in Theorem 2.8.4, and a weight on $\mathbb{Z}^+$ admitting an extension to $\mathbb{Z}$ in Theorem 2.8.6.

We note that many authors have considered a related question, known as Gleason's Problem, which may be stated as follows: let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $R(\Omega)$ be a ring of holomorphic functions on Ω containing the polynomials. Given $p = (p_1, \ldots, p_n) \in \Omega$, is the ideal consisting of those functions in $R(\Omega)$ which vanish at p generated by

$$(z - p_1), \ldots, (z - p_n)?$$

The cases of $A(\Omega)$, the algebra of functions which are holomorphic on Ω and continuous on its closure, and $H^\infty(\Omega)$, the algebra of bounded, holomorphic functions on Ω, are considered to be of particular interest. See e.g. [**53, 57**].

Before we construct our examples, we first recall some facts about weights on $\mathbb{Z}$ and $\mathbb{Z}^+$; see [**19**, Section 4.6] for more details.

Let ω be a weight on $\mathbb{Z}$. The character space of $\ell^1(\mathbb{Z}, \omega)$ may be identified with the annulus $\{z \in \mathbb{C} : \rho_1 \leqslant |z| \leqslant \rho_2\}$, where

$$\rho_1 = \lim_{n\to\infty} \omega_{-n}^{-1/n} \qquad \text{and} \qquad \rho_2 = \lim_{n\to\infty} \omega_n^{1/n}.$$

The identification is given by $\varphi \mapsto \varphi(\delta_1)$, for φ a character. Note that $\rho_1 \leqslant 1 \leqslant \rho_2$. As it is easily seen to be semi-simple, $\ell^1(\mathbb{Z}, \omega)$ may be thought of as a Banach function algebra on the annulus, and in fact these functions are all holomorphic on the interior of the annulus. We denote by M_z the maximal ideal corresponding to the point z of the annulus, and observe that the augmentation ideal is M_1.

Now instead let ω be a weight on $\mathbb{Z}^+$. The situation for $\ell^1(\mathbb{Z}^+, \omega)$ is analogous to the situation above. Now the character space is identified with the disc $\{z \in \mathbb{C} : |z| \leqslant \rho\}$, where $\rho = \lim_{n\to\infty} \omega_n^{1/n}$, and $\ell^1(\mathbb{Z}^+, \omega)$ may be considered as a Banach function algebra on this set, with the property that each of its elements is holomorphic on the interior. In this context M_z denotes the maximal ideal corresponding to the point z of the disc.

Before giving our examples we characterise those weights for which $\ell^1_0(\mathbb{Z}, \omega)$ is finitely-generated. Note that this is a slight improvement, for the group $\mathbb{Z}$, on Theorem 2.1.6 since we no longer need to assume that the weight is radial.

THEOREM 2.8.2. *Let ω be a weight on $\mathbb{Z}$. Then $\ell^1_0(\mathbb{Z}, \omega)$ is finitely-generated if and only if both sequences $(\omega_n)_{n\in\mathbb{N}}$ and $(\omega_{-n})_{n\in\mathbb{N}}$ are tail-preserving.*

PROOF. Set $A = \ell^1(\mathbb{Z}, \omega)$. Suppose that $(\omega_n)_{n\in\mathbb{N}}$ is not tail-preserving. Then we can repeat the proof of Theorem 2.7.4 with $G = \mathbb{Z}$ essentially unchanged, except that now we insist that all functions appearing in it have support contained in $\mathbb{Z}^+$, to show that $\ell^1_0(\mathbb{Z}, \omega)$ is not finitely-generated. By symmetry, the same conclusion holds if instead $(\omega_{-n})_{n\in\mathbb{N}}$ fails to be tail-preserving.

Now suppose that $(\omega_n)_{n\in\mathbb{N}}$ and $(\omega_{-n})_{n\in\mathbb{N}}$ are both tail-preserving. Let $f \in \ell^1_0(\mathbb{Z}, \omega)$, and suppose for the moment that $\operatorname{supp} f \subset \mathbb{Z}^+$. Then we have

$$\sum_{n=0}^{\infty} \omega_n \left| \sum_{i=1}^{n} f(i) \right| = \sum_{n=0}^{\infty} \omega_n \left| \sum_{i=n+1}^{\infty} f(i) \right| < \infty,$$

and so we may define $g \in A$ by

$$g = -\sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} f(i) \right) \delta_n.$$

Then

$$g * (\delta_1 - \delta_0) = -\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} f(i)\right)(\delta_{n+1} - \delta_n)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} f(i) - \sum_{i=0}^{n-1} f(i)\right)\delta_n = \sum_{n=0}^{\infty} f(n)\delta_n = f.$$

Hence

$$f = g * (\delta_1 - \delta_0) \in A * (\delta_1 - \delta_0).$$

A similar argument shows that, if $\operatorname{supp} f \subset \mathbb{Z}^-$, then

$$f \in A * (\delta_{-1} - \delta_0).$$

But any $f \in \ell_0^1(\mathbb{Z}, \omega)$ can be written as $f = f_1 + f_2$ for $f_1, f_2 \in \ell_0^1(\mathbb{Z}, \omega)$, with $\operatorname{supp} f_1 \subset \mathbb{Z}^+$ and $\operatorname{supp} f_2 \subset \mathbb{Z}^-$, and so we see that

$$\ell_0^1(\mathbb{Z}, \omega) = A * (\delta_1 - \delta_0) + A * (\delta_{-1} - \delta_0)$$

is finitely-generated, as required $\square$

We now construct the first of our special weights described at the beginning of the section. This is a weight on $\mathbb{Z}^+$ such that neither the augmentation ideal nor M_0 are finitely-generated.

LEMMA 2.8.3. *Let $\rho > 1$. Then there exists a weight ω on $\mathbb{Z}^+$, satisfying*

$$\lim_{n\to\infty} \omega_n^{1/n} = \rho$$

such that there exists a strictly increasing sequence of natural numbers (n_k) with

$$\frac{\omega_{n_k+1}}{\omega_{n_k}} \leqslant \frac{\rho+1}{k} \quad (k \in \mathbb{N}). \tag{2.11}$$

PROOF. First, we define inductively a non-increasing null sequence $(\varepsilon(n))$ of positive reals, as follows. Set $\varepsilon(0) = 1$. Since $\lim_{n\to\infty}(1/n)^{1/n} = 1$, we can find an integer n_1

such that

$$0 < \left(\frac{1}{n_1}\right)^{1/n_1} (\rho + 1) - \rho.$$

Define $\varepsilon(n) = 1$ for $n \leqslant n_1$, and then choose $\varepsilon(n_1 + 1)$ such that

$$0 < \varepsilon(n_1 + 1) < \left(\frac{1}{n_1}\right)^{1/n_1} (\rho + 1) - \rho.$$

Note also that $\varepsilon(n_1 + 1) < 1$.

Now take $k \geqslant 2$, and suppose that we have already defined a strictly increasing sequence of integers $n_1, \dots, n_{k-1}$, and defined $\varepsilon(n)$ for $n \leqslant n_{k-1} + 1$. Then choose $n_k \in \mathbb{N}$, with $n_k > n_{k-1}$ and such that

$$0 < \left(\frac{1}{n_k}\right)^{1/n_k} (\rho + \varepsilon(n_{k-1} + 1)) - \rho.$$

Define $\varepsilon(n) = \varepsilon(n_{k-1} + 1)$ for $n_{k-1} + 1 < n \leqslant n_k$, and then choose $\varepsilon(n_k + 1)$ such that

$$0 < \varepsilon(n_k + 1) < \left(\frac{1}{n_k}\right)^{1/n_k} (\rho + \varepsilon(n_{k-1} + 1)) - \rho,$$

whilst ensuring that $\varepsilon(n_k + 1) < \min\{1/k, \varepsilon(n_k)\}$. This completes the inductive construction of ε.

Now define

$$\omega_n = (\rho + \varepsilon(n))^n \quad (n \in \mathbb{Z}^+).$$

Then $\omega := (\omega_n)$ is a weight on $\mathbb{Z}^+$, because

$$\begin{aligned} \omega_{m+n} &= (\rho + \varepsilon(m + n))^m (\rho + \varepsilon(m + n))^n \\ &\leqslant (\rho + \varepsilon(m))^m (\rho + \varepsilon(n))^n = \omega_m \omega_n \quad (m, n \in \mathbb{Z}^+), \end{aligned}$$

where we have used the fact that ε is non-increasing. As $\lim\limits_{n\to\infty} \varepsilon(n) = 0$, we have $\lim\limits_{n\to\infty} \omega_n^{1/n} = \rho$. It remains to show that (2.11) holds.

For $k \in \mathbb{N}$, we have

$$\frac{\omega_{n_k+1}}{\omega_{n_k}} \leqslant (\rho + 1) \frac{(\rho + \varepsilon(n_k + 1))^{n_k}}{(\rho + \varepsilon(n_k))^{n_k}}.$$

However

$$\rho + \varepsilon(n_k + 1) < \left(\frac{1}{n_k}\right)^{1/n_k} (\rho + \varepsilon(n_{k-1} + 1)) = \left(\frac{1}{n_k}\right)^{1/n_k} (\rho + \varepsilon(n_k)),$$

which implies that

$$\frac{(\rho + \varepsilon(n_k + 1))^{n_k}}{(\rho + \varepsilon(n_k))^{n_k}} < \frac{1}{n_k} \leqslant \frac{1}{k},$$

and (2.11) now follows. □

As $\lim\limits_{n\to\infty} \omega_n^{1/n} = \inf\limits_{n\in\mathbb{N}} \omega_n^{1/n}$ by [**19**, Proposition A.1.26(iii)], the weight constructed in Lemma 2.8.3 satisfies $\omega_n \geqslant \rho^n$ $(n \in \mathbb{N})$. However, Proposition 2.6.1 implies that ω is not tail-preserving, as

$$\liminf_n \left(\omega_{n+1} \left(\sum_{j=1}^n \omega_j \right)^{-1} \right) \leqslant \liminf_n \frac{\omega_{n+1}}{\omega_n} = 0.$$

Hence we have a version of Lemma 2.6.2(ii) in which the sequence is also a weight.

Theorem 2.8.4. *Let ω denote the weight constructed in Lemma* 2.8.3*. Then neither M_1 nor M_0 is finitely-generated, even though both 0 and 1 correspond to interior points of the character space.*

Proof. Set $A = \ell^1(\mathbb{Z}^+, \omega)$ and assume towards a contradiction that M_0 is finitely-generated. Note that $M_0 = \{f \in A : f(0) = 0\}$, so that every finitely supported element of M_0 is of the form $g * \delta_1$, for some $g \in A$. By Lemmas 2.2.1 and 2.2.2, we may suppose that the generators of M_0 have finite support, and as they also lie in M_0, we may factor out a δ_1 from each one. It follows that $M_0 = A * \delta_1$. Define a sequence of non-negative reals by

$$\alpha_j = \begin{cases} (k\omega_{n_k})^{-1} & \text{if } j = n_k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f = \sum\limits_{j=1}^{\infty} \alpha_j \delta_j$. Then by (2.11) we have

$$\sum_{j=0}^{\infty} |\alpha_j|\omega_j = \sum_{k=1}^{\infty} \frac{\omega_{n_k+1}}{k\omega_{n_k}} \leqslant (\rho+1) \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

This shows that $f \in A$, and so clearly $f \in M_0$. Assume that $f = g * \delta_1$ for some $g \in A$. Then g must satisfy $g(j-1) = f(j)$ $(j \in \mathbb{N})$. However,

$$\sum_{j=1}^{\infty} |f(j)|\omega_{j-1} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so that $g \notin A$.

The case of M_1 is very similar. This time we know that, if M_1 is finitely-generated, it must equal $A * (\delta_0 - \delta_1)$. By the remark preceding the theorem ω is not tail-preserving, and so there exists some sequence $(\alpha_n) \in \ell^1(\omega)$, such that

$$\sum_{n=1}^{\infty} \omega_n \left| \sum_{j=n+1}^{\infty} \alpha_j \right| = \infty.$$

Let $\zeta = \sum_{n=1}^{\infty} \alpha_n$, and let $f = \zeta\delta_0 - \sum_{n=1}^{\infty} \alpha_n \delta_n$. Then $f \in M_1$. Assume that $f = g * (\delta_0 - \delta_1)$ for some $g \in A$. A short calculation implies that

$$g(n) = \sum_{j=0}^{n} f(j) = -\sum_{j=n+1}^{\infty} \alpha_j$$

for all $n \geqslant 1$, contradicting the fact that $\sum_{n=0}^{\infty} \omega_n |g(n)| < \infty$. $\square$

We remark that a weight ω on $\mathbb{Z}^+$ extends to a weight on $\mathbb{Z}$ if and only if $\sup_{n\in\mathbb{N}} \omega_n/\omega_{n+1} < \infty$. The "only if" direction of this implication just follows from submultiplicativity of the weight at -1. For the "if" direction, set $C = \sup_{n\in\mathbb{N}} \omega_n/\omega_{n+1}$. Then it is routine to verify that $\omega_{-n} = C^n\omega_n$ $(n \in \mathbb{N})$ defines an extension. It follows from this observation that the weight constructed in Lemma 2.8.3 admits no extension to $\mathbb{Z}$. However, a different construction does allow us to do something similar on $\mathbb{Z}$.

LEMMA 2.8.5. *Let $\rho > 1$. Then there exists a weight ω on $\mathbb{Z}$ satisfying $\lim_{n\to\infty} \omega_n^{1/n} = \rho$ and $\lim_{n\to\infty} \omega_{-n}^{-1/n} < 1$, but such that $(\omega_n)_{n\in\mathbb{N}}$ is not tail-preserving.*

PROOF. With the preceding remark in mind, we construct first a weight γ on $\mathbb{Z}^+$ satisfying $\sup_{n\in\mathbb{N}}\{\gamma_n/\gamma_{n+1}\} \leqslant \rho+1$, which ensures that γ extends to a weight on $\mathbb{Z}$. In the end we shall define ω by $\omega_n = \rho^{|n|}\gamma_n$.

We set $n_k = 2^k - 1$ ($k \in \mathbb{N}$). We define γ on $\{0, 1, 2, 3\}$ by

$$\gamma_0 = 1,\ \gamma_1 = \rho+1,\ \gamma_2 = (\rho+1)^2,\ \gamma_3 = \rho+1.$$

We then recursively define

$$\gamma_j = (\rho+1)\gamma_{j-n_k} \quad (n_k \leqslant j < n_{k+1},\ k \geqslant 2).$$

We observe that

$$\gamma_{n_k - i} = (\rho+1)^{i+1} \quad (0 \leqslant i \leqslant k-1,\ k \geqslant 2). \tag{2.12}$$

This follows by an easy induction on k. Indeed, the base case can be seen to hold by inspection, and for $k \geqslant 3$ we see that

$$\begin{aligned}\gamma_{n_k - i} &= (\rho+1)\gamma_{n_k - i - n_{k-1}} = (\rho+1)\gamma_{n_{k-1}+1-i}\\ &= (\rho+1)(\rho+1)^i = (\rho+1)^{i+1} \quad (1 \leqslant i \leqslant k-1)\end{aligned}$$

and

$$\gamma_{n_k} = (\rho+1)(\gamma_{n_k - n_k}) = (\rho+1)\gamma_0 = (\rho+1).$$

We now claim that

$$\gamma_j \leqslant (\rho+1)\gamma_{j+1} \quad (j \in \mathbb{N}). \tag{2.13}$$

Again, this can be seen by inspection for $j \leqslant n_2$, and we then proceed by induction on k. Indeed, if $j \in [n_k, n_{k+1} - 2]$ then

$$\frac{\gamma_j}{\gamma_{j+1}} = \frac{(\rho+1)\gamma_{j-n_k}}{(\rho+1)\gamma_{j+1-n_k}} = \frac{\gamma_{j-n_k}}{\gamma_{j+1-n_k}} \leqslant (\rho+1).$$

When $j = n_{k+1} - 1$ and $j + 1 = n_{k+1}$, then, by (2.12), we have

$$\frac{\gamma_j}{\gamma_{j+1}} = \frac{(\rho+1)^2}{(\rho+1)} = \rho + 1,$$

establishing the claim.

Now we are ready to prove that γ really is a weight. That γ is submultiplicative on $\{0, 1, 2, 3\}$ can be seen by inspection. Let $i, j \in \mathbb{N}$, with $i \leqslant j$, and let $k \in \mathbb{N}$ satisfy $i + j \in [n_{k+1}, n_{k+2})$. We proceed by induction on $i + j$. If $j \leqslant n_k$ then $i + j \leqslant 2n_k < n_{k+1}$, so we must have $j \geqslant n_k + 1$. There are three cases. Firstly, if $j \geqslant n_{k+1}$, then

$$\gamma_{i+j} = (\rho+1)\gamma_{i+j-n_{k+1}} \leqslant (\rho+1)\gamma_i\gamma_{j-n_{k+1}} = \gamma_i\gamma_j.$$

If instead $j < n_{k+1}$, but $i \geqslant n_k$, then

$$\begin{aligned}\gamma_{i+j} &= (\rho+1)\gamma_{i+j-n_{k+1}} = (\rho+1)\gamma_{(i-n_k)+(j-1-n_k)} \\ &\leqslant (\rho+1)\gamma_{i-n_k}\gamma_{j-1-n_k} = \frac{1}{\rho+1}\gamma_i\gamma_{j-1} \leqslant \gamma_i\gamma_j,\end{aligned}$$

by (2.13). Finally, suppose that $i < n_k$ and $n_k \leqslant j < n_{k+1}$. In this case we have $i + j - 2^k < n_{k+1}$, and, since $i + j \geqslant n_{k+1}$, we also have

$$i + j - 2^k \geqslant n_{k+1} - 2^k = n_k,$$

so that $i + j - 2^k \in [n_k, n_{k+1})$. Then the formula $i + j - n_{k+1} = i + j - (n_k + 2^k)$ implies that

$$\gamma_{i+j} = (\rho+1)\gamma_{i+j-n_{k+1}} = (\rho+1)\gamma_{i+j-2^k-n_k} = \gamma_{i+j-2^k}.$$

Therefore

$$\begin{aligned}\gamma_{i+j} &= \gamma_{i+j-2^k} \leqslant \gamma_{i-1}\gamma_{j+1-2^k} = \gamma_{i-1}\gamma_{j-n_k} \leqslant (\rho+1)\gamma_i\gamma_{j-n_k} \quad \text{by (2.13)} \\ &= \gamma_i\gamma_j.\end{aligned}$$

This concludes the proof that γ is a weight. By (2.13), it extends to a weight on $\mathbb{Z}$, which we also denote by γ.

Define $\omega = (\omega_n)$ by $\omega_n = \rho^{|n|}\gamma_n$ $(n \in \mathbb{Z})$. As $\gamma_{n_k}^{1/n_k} = (\rho+1)^{1/n_k}$ for all $k \geqslant 2$, we must have $\lim_{n\to\infty} \gamma_n^{1/n} = 1$, and hence $\lim_{n\to\infty} \omega_n^{1/n} = \rho$. Furthermore,

$$\lim_{n\to\infty} \omega_{-n}^{-1/n} = \frac{1}{\rho} \lim_{n\to\infty} \gamma_{-n}^{-1/n} \leqslant \frac{1}{\rho} < 1,$$

as required.

It remains to show that $(\omega_n)_{n\in\mathbb{N}}$ is not tail-preserving. We compute

$$\frac{\omega_{n_k}}{\sum_{j=1}^{n_k-1} \omega_j} \leqslant \frac{\omega_{n_k}}{\omega_{n_k-(k-1)}} = \frac{\rho^{n_k}(\rho+1)}{\rho^{n_k+1-k}(\rho+1)^k} = \left(\frac{\rho}{\rho+1}\right)^{k-1},$$

which tends to 0 as k goes to infinity. In particular, $(\omega_n)_{n\in\mathbb{N}}$ violates (2.6), so it is not tail-preserving. ☐

Theorem 2.8.6. *Let ω be the weight constructed in Lemma* 2.8.5. *Then the augmentation ideal $\ell_0^1(\mathbb{Z}, \omega)$ fails to be finitely-generated, despite corresponding to an interior point of the character space.*

Proof. By construction M_1 corresponds to an interior point of the annulus. Now apply Theorem 2.8.2. ☐

CHAPTER 3

Topologically Finitely-Generated Left Ideals

3.1. Introduction

In Chapter 2 we discussed Sinclair and Tullo's theorem [**79**] that a left Noetherian Banach algebra is finite-dimensional. We then went on to prove new results which illustrate the fact that algebraic finite-generation of left ideals in a Banach algebra is a very strong condition. Perhaps a notion better suited to the study of Banach algebras is topological finite-generation. In this chapter we aim to complement the discussion of the first chapter by considering Banach algebras in which every closed left ideal is topologically finitely-generated. We call such Banach algebras *topologically left Noetherian*. In contrast to Sinclair and Tullo's result we find that there are many natural examples of infinite-dimensional, topologically left Noetherian Banach algebras. Moreover, this property often captures interesting properties of some underlying object: for example, given a compact group G the group algebra $L^1(G)$ is topologically left Noetherian if and only if G is metrisable (Theorem 3.3.5); meanwhile, given a Banach space E with the approximation property, the algebra of compact operators $\mathcal{K}(E)$ is topologically left Noetherian if and only if E' is separable (Theorem 3.5.9).

In some situations, however, it seems that in order to get interesting examples and capture interesting properties of the underlying group or Banach space, it is better to consider topological finite-generation in some topology other than the norm topology. For instance, given a compact group G, the measure algebra $M(G)$ is $\|\cdot\|$-topologically left Noetherian if and only if G is finite (Proposition 3.3.6), whereas it is weak*-topologically left Noetherian whenever G is metrisable (Corollary 3.4.13). Hence part of this chapter is devoted to weak*-topological Noetherianity of dual Banach algebras. In Section 3.6 we shall also consider SOP-topological left Noetherianity of $\mathcal{B}(E)$ for a Banach space E, and show that this is equivalent to a natural condition on the Banach space (Corollary 3.6.5(ii)).

In this chapter we shall also take an interest in classifying the closed left ideals in certain families of Banach and semi-topological algebras. In some cases this is a necessary step towards our results about topological left Noetherianity, but these classification results are also of interest in their own right. Our main result of this nature is a classification of the closed left and right ideals in the Banach algebra of approximable operators $\mathcal{A}(E)$, for any Banach space E satisfying a certain condition (Theorem 3.5.4 and Theorem 3.5.10 respectively)[1] . Our classification holds, for instance, whenever the dual of the Banach space has the bounded approximation property. We also give classifications of the weak*-closed left ideals of the measure algebra of a compact group (Theorem 3.4.17), and of the weak*-closed left and right ideals of $\mathcal{B}(E)$, for E a reflexive Banach space with the approximation property (Theorem 3.6.7). In addition, we give a classification of the SOP-closed left ideals of $\mathcal{B}(E)$ for an arbitrary Banach space E (Theorem 3.6.2).

3.2. General Theory

Recall that a *semi-topological algebra* is a pair (A, τ), where A is an algebra, and τ is a topology on A such that $(A, +, \tau)$ is a topological vector space, and such that multiplication on A is separately continuous. For example, a dual Banach algebra with its weak*-topology is a semi-topological algebra.

Let (A, τ) be a semi-topological algebra. Let I be a closed left ideal in A, and let $n \in \mathbb{N}$. We say that I is *τ-topologically generated* by elements $x_1, \ldots, x_n \in I$ if

$$I = \overline{A^\sharp x_1 + \cdots + A^\sharp x_n}.$$

We say that I is *τ-topologically finitely-generated* if there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in I$ which τ-topologically generate I. We say that A is *τ-topologically left Noetherian* if

[1]Since proving Theorem 3.5.4 we have found that there is a very similar result to Theorem 3.5.4 already in the literature due to Grønbæk [**38**, Proposition 7.3]. We acknowledge this in Section 3.5.

every closed left ideal of A is τ-topologically finitely-generated. For example, we shall often discuss weak*-topologically left Noetherian dual Banach algebras. When the topology is clear we may simply speak of "topologically finitely-generated left ideals" et cetera, without naming the topology referred to. In the context of Banach algebras, if there is more than one topology under discussion, phrases such as "topologically finitely-generated left ideal" and "topologically left Noetherian Banach algebra" are understood to refer to the norm topology.

Analogously we may define *τ-topologically finitely-generated right ideals*, as well as *τ-topologically right Noetherian algebras*. If the algebra in question is commutative we usually drop the words "left" and "right".

We note that when a semi-topological algebra A has a left approximate identity we have

$$\overline{A^\sharp x_1 + \cdots + A^\sharp x_n} = \overline{Ax_1 + \cdots + Ax_n},$$

for each $n \in \mathbb{N}$ and each $x_1, \ldots, x_n \in A$. When this is the case we usually drop the unitisations in order to ease notation. For example, in the proof of Theorem 3.3.5 below, we shall write $\overline{L^1(G) * g}$ in place of $\overline{L^1(G)^\sharp * g}$, for G a locally compact group and $g \in L^1(G)$.

The following lemma will be invaluable throughout this chapter.

LEMMA 3.2.1. *Let A be a semi-topological algebra with a left approximate identity. Let J be a dense right ideal of A. Then J intersects every closed left ideal of A densely.*

PROOF. Let (e_α) be a left approximate identity for A. Given an open neighbourhood of the origin U and an index α choose $f_{\alpha,U} \in J$ such that $e_\alpha - f_{\alpha,U} \in U$. Note that $(f_{\alpha,U})$ is a net, where the underlying directed set is ordered by $(\alpha, U) \leqslant (\beta, V)$ if and only if $\alpha \leqslant \beta$ and $V \subset U$. We see that $(f_{\alpha,U})$ is a left approximate identity for A contained in J: let $a \in A$, let U be an arbitrary open neighbourhood of the origin, let V_0 be an open neighbourhood of the origin such that $V_0 + V_0 a \subset U$, and let α_0 be such that $e_\alpha a - a \in V_0$ for all $\alpha \geqslant \alpha_0$. Then, for all open neighbourhoods of the origin

$V \subset V_0$ and all $\alpha \geqslant \alpha_0$, we have

$$a - f_{\alpha,V}a = (a - e_\alpha a) + (e_\alpha a - f_{\alpha,V}a) \in V_0 + V_0 a \subset U.$$

Let I be a closed left ideal in A and let $a \in I$. Then for every index (α, U) we have $f_{\alpha,U}a \in J \cap I$. Since $a = \lim_{(\alpha,U)} f_{\alpha,U}a \in \overline{J \cap I}$, and a was arbitrary, it follows that $\overline{J \cap I} = I$, as required. □

Next we show that topological left Noetherianity is stable under taking quotients and extensions.

LEMMA 3.2.2. *Let A be a semi-topological algebra, and let I be a closed (two-sided) ideal in A.*

(i) *If A is topologically left Noetherian then so is A/I.*

(ii) *Suppose that both I and A/I are topologically left Noetherian. Then so is A.*

(iii) *A is topologically left Noetherian if and only if $A^\sharp$ is topologically left Noetherian.*

PROOF. (i) Let J be a closed left ideal in A/I and let $q\colon A \to A/I$ denote the quotient map. Let $\widetilde{J} = q^{-1}(J)$. Since A is topologically left Noetherian, there exists $n \in \mathbb{N}$ and there exist $x_1, \ldots, x_n \in \widetilde{J}$ such that $\widetilde{J} = \overline{A^\sharp x_1 + \cdots + A^\sharp x_n}$. It follows from the continuity and surjectivity of q that J is topologically generated by $q(x_1), \ldots, q(x_n)$.

(ii) Let J be a closed left ideal in A. Since I is topologically left Noetherian there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in J$ such that $J \cap I = \overline{I^\sharp x_1 + \cdots + I^\sharp x_n}$. Moreover, since $J/(I \cap J)$ is topologically isomorphic as a left A-module to $(J + I)/I$, which is a topologically finitely-generated closed left ideal of A/I, there must exist $m \in \mathbb{N}$ and $y_1, \ldots, y_m \in J$ such that $y_1 + I \cap J, \ldots, y_m + I \cap J$ topologically generate $J/(I \cap J)$.

We claim that $x_1, \ldots, x_n, y_1, \ldots, y_m$ topologically generate J. Since J was arbitrary this will complete the proof of (ii). Denote the quotient map $A \to A/I \cap J$ by q. Let $z \in J$, let U be any open neighbourhood of 0, and let V be an open neighbourhood of 0 such that $V + V \subset U$. Since q is an open map, $q(V)$ is an open neighbourhood of 0 in $A/I \cap J$, so that, by the above, there exist $a_1, \dots, a_m \in A^\sharp$ such that

$$q(z) - (a_1 q(y_1) + \cdots + a_m q(y_m)) \in q(V).$$

It follows that there exists $w \in I \cap J$ such that $z - (a_1 y_1 + \cdots + a_m y_m) - w \in V$. Furthermore there exist $b_1, \dots, b_n \in A^\sharp$ such that $w - b_1 x_1 + \cdots + b_n x_n \in V$. Hence

$$\begin{aligned} z &- (a_1 y_1 + \cdots + a_m y_m + b_1 x_1 + \cdots + b_n x_n) \\ &= z - (a_1 y_1 + \cdots + a_m y_m) - w + w - (b_1 x_1 + \cdots + b_n x_n) \in V + V \subset U, \end{aligned}$$

and as U was arbitrary this proves the claim.

(iii) We may suppose that A is non-unital for otherwise the result is trivial. Suppose that A is topologically left Noetherian. Then, since $A^\sharp/A \cong \mathbb{C}$ is topologically left Noetherian, it follows from (ii) that $A^\sharp$ is also. The converse follows from the fact that every closed left ideal in A is also a closed left ideal in $A^\sharp$. $\square$

3.3. Examples From Abstract Harmonic Analysis

It is surely easiest to determine whether or not a Banach algebra is topologically left Noetherian when we know what its closed left ideals are. Fortunately, this is the case for the group algebra of a compact group, as well as for the Fourier algebra of a discrete group. We shall show below that, for a compact group G, the group algebra $L^1(G)$ is topologically left Noetherian if and only if G is metrisable, whereas, for a discrete group G, the Fourier algebra $A(G)$ is topologically Noetherian if and only if G is countable. The proofs of both statements are similar, but the latter is easier, so we start there.

PROPOSITION 3.3.1. *Let G be a discrete group. Then $A(G)$ is topologically Noetherian if and only if G is countable.*

PROOF. Given $E \subset G$, write $I(E) = \{f \in A(G) : f(x) = 0, x \in E\}$. It is well known that, since G is discrete, the closed ideals of $A(G)$ are all of the from $I(E)$ for some subset E of G: see, for instance, [**46**, Theorem 39.18].

Suppose first that G is countable and let $I \lhd A(G)$ be closed. Let $E \subset G$ be such that $I = I(E)$, and enumerate $G \backslash E = \{x_1, x_2, \ldots, \}$. Since G is discrete the point mass δ_x belongs to $A(G)$ for each $x \in G$, since this may be realised as $\delta_e *_\lambda \delta_x$, where λ here denotes the left regular representation. Hence $g = \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_{x_n} \in A(G)$. It is clear that $\operatorname{supp} g = G \backslash E$, and hence that

$$\left\{ x \in G : f(x) = 0 \text{ for every } f \in \overline{A(G)^\sharp g} \right\} = E.$$

It follows from the classification of the closed ideals of $A(G)$ given above that $I = \overline{A(G)^\sharp g}$. As I was arbitrary we conclude that $A(G)$ is topologically Noetherian.

Now suppose that $A(G)$ is topologically Noetherian. Let

$$I = \{f \in A(G) : f(e) = 0\}.$$

Then there exist $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in I$ such that $I = \overline{A(G)^\sharp h_1 + \cdots + A(G)^\sharp h_n}$. Since $A(G) \subset c_0(G)$, every function in $A(G)$ must have countable support. Hence $\mathcal{S} := \bigcup_{i=1}^{n} \operatorname{supp} h_i$ is a countable set. Every $f \in A(G)^\sharp h_1 + \cdots + A(G)^\sharp h_n$ has $\operatorname{supp} f \subset \mathcal{S}$, and of course, after taking closures, we see that this must hold for every $f \in I$. This clearly forces $\mathcal{S} = G \backslash \{e\}$, so that G must be countable. $\square$

We now recall some facts about compact groups. Firstly, when G is compact each representation in $\widehat{G}$ is finite-dimensional. Secondly, for G a compact group the closed left ideals of $L^1(G)$ have the following characterisation [**46**, Theorem 38.13]:

THEOREM 3.3.2. *Let G be a compact group, and let I be a closed left ideal in $L^1(G)$. Then there exist linear subspaces $E_\pi \subset H_\pi$ $(\pi \in \widehat{G})$ such that*

$$I = \left\{ f \in L^1(G) : \pi(f)(E_\pi) = 0, \ \pi \in \widehat{G} \right\}.$$

Let G be a compact group. Given $\pi \in \widehat{G}$ we write $T_\pi(G) = \operatorname{span}\{\xi *_\pi \eta : \xi, \eta \in H_\pi\}$, and we write $T(G) = \operatorname{span}\{\xi *_\pi \eta : \xi, \eta \in H_\pi, \pi \in \widehat{G}\}$. We recall the following facts about these spaces from [**45, 46**]:

THEOREM 3.3.3. *Let G be a compact group.*

(i) *Let $\sigma, \pi \in \widehat{G}$ with $\sigma \neq \pi$. Then $\sigma(\xi *_\pi \eta) = 0$.*

(ii) *The linear space $T(G)$ is a dense ideal in $L^1(G)$.*

(iii) *For each $\pi \in \widehat{G}$ the space $T_\pi(G)$ is an ideal in $L^1(G)$, and as an algebra $T_\pi(G) \cong M_{d_\pi}(G)$, where d_π denotes the dimension of H_π.*

PROOF. It follows from equation (1.10) that

$$\langle \sigma(f)\zeta_1, \zeta_2 \rangle = \int_G f(t) \langle \sigma(t)\zeta_1, \zeta_2 \rangle \, \mathrm{d}t,$$

for $f \in L^1(G), \sigma \in \widehat{G}$ and $\zeta_1, \zeta_2 \in H_\sigma$. Part (i) follows from this and the orthogonality relations [**46**, Theorem 27.20 (iii)]. Part (ii) follows from [**46**, Theorem 27.20, Lemma 31.4], and part (iii) follows from [**46**, Theorem 27.21]. □

We also record the following result here, although we shall not require it in a proof until Section 3.4.

LEMMA 3.3.4. *Distinct choices of linear subspace in Theorem 3.3.2 give rise to distinct ideals.*

PROOF. Suppose that $(E_\pi)_{\pi \in \widehat{G}}$ and $(F_\pi)_{\pi \in \widehat{G}}$ are two distinct choices of linear subspaces of the Hilbert spaces $(H_\pi)_{\pi \in \widehat{G}}$. Then there exists $\sigma \in \widehat{G}$ such that $E_\sigma \neq F_\sigma$. Without loss of generality $F_\sigma \nsubseteq E_\sigma$. Let n be the dimension of E_σ and let m be the dimension of H_σ. Choose an orthonormal basis $\eta_1, \dots, \eta_m$ for H_σ such that $\eta_1, \dots, \eta_n$ is a basis for E_σ and $\eta_{n+1} \dots \eta_m$ is a basis for $E_\sigma^\perp$. Let $f = \sum_{i=n+1}^m \overline{\eta_i *_\sigma \eta_i} \in L^1(G)$. Then it follows from Theorem 3.3.3(i) that $\pi(f) = 0$ for $\pi \neq \sigma$. Moreover, using (1.10) and the orthogonality relations [**46**, Theorem 27.20 (iii)], we see that $\sigma(f)$ is the orthogonal projection onto $E_\sigma^\perp$. Hence $\pi(f)(E_\pi) = 0$ $(\pi \in \widehat{G})$, but taking $\xi \in F_\sigma \backslash E_\sigma$ we have $\sigma(f)\xi \neq 0$. □

We can now prove our theorem. The equivalence of conditions (b) and (c) has surely been noticed before, but we include a short proof for the convenience of the reader.

THEOREM 3.3.5. *Let G be a compact group. Then the following are equivalent:*

(a) *$L^1(G)$ is topologically left Noetherian;*

(b) *$\widehat{G}$ is countable;*

(c) *G is metrisable.*

PROOF. We first demonstrate that (b) implies (c). Our method is to show that G is first countable, which will implie that G is metrisable by [**45**, Theorem 8.3]. Indeed, it follows from Tannaka-Krein duality [**51**] that the topology on G is the initial topology induced by its irreducible representations, and as such has a base given by sets of the form

$$U(\pi_1, \ldots, \pi_n; \varepsilon; t) := \{s \in G : \|\pi_i(t) - \pi_i(s)\| < \varepsilon, i = 1, \ldots, n\},$$

where $\varepsilon > 0, t \in G$, and $(\pi_1, H_1), \ldots, (\pi_n, H_n) \in \widehat{G}$. Hence, if $\widehat{G}$ is countable, for every $t \in G$ the sets $U(\pi_1, \ldots, \pi_n; 1/m; t)$ $(m \in \mathbb{N}, \pi_1, \ldots, \pi_n \in \widehat{G})$ form a countable neighbourhood base at t, and so G is first-countable.

Now suppose instead that G is metrisable. Then $C(G)$ is separable. Since the infinity norm dominates the L^2-norm for a compact space, and since $C(G)$ is dense in $L^2(G)$, it follows that $L^2(G)$ is separable. By [**46**, Theorem 27.40]

$$L^2(G) \cong \bigoplus_{\pi \in \widehat{G}} H_\pi,$$

(where $\oplus$ denotes the direct sum of Hilbert spaces) which is clearly separable only if $\widehat{G}$ is countable. Hence (c) implies (b).

Next we show that (b) implies (a). Suppose that $\widehat{G}$ is countable. By Theorem 3.3.3(ii) $T(G)$ is a dense ideal in $L^1(G)$ so that, by Lemma 3.2.1, $\overline{I \cap T(G)} = I$ for every closed left ideal I in $L^1(G)$.

Fix a closed left ideal I in $L^1(G)$. By Theorem 3.3.2 there exist linear subspaces $E_\pi \subset H_\pi$ $(\pi \in \widehat{G})$ such that

$$I = \left\{ f \in L^1(G) : \pi(f)(E_\pi) = 0,\ \pi \in \widehat{G} \right\}.$$

By Theorem 3.3.3(iii), for each $\pi \in \widehat{G}$ we have $T_\pi(G) \cong M_{d_\pi}(\mathbb{C})$, where d_π is the dimension of H_π, and since $I \cap T_\pi(G)$ is a left ideal in $T_\pi(G)$ there must be an idempotent $P_\pi \in T_\pi(G)$ such that $I \cap T_\pi(G) = T_\pi(G) * P_\pi$. Set $\alpha_\pi = \|P_\pi\|^{-1}$. Enumerate $\widehat{G} = \{\pi_1, \pi_2, \ldots\}$, and define

$$g = \sum_{i=1}^{\infty} \frac{1}{i^2} \alpha_{\pi_i} P_{\pi_i} \in L^1(G),$$

which belongs to I because each P_{π_i} does, and I is closed.

We claim that $I = \overline{L^1(G) * g}$. Indeed, $I \supset \overline{L^1(G) * g}$ because $g \in I$. For the reverse inclusion we show that, for $j \in \mathbb{N}$ and $\xi \in H_{\pi_j}$, we have $\pi_j(f)(\xi) = 0$ for all $f \in \overline{L^1(G) * g}$ if and only if $\xi \in E_{\pi_j}$. The claim then follows from Theorem 3.3.2. Indeed, if $f \in \overline{L^1(G) * g}$ then $\pi_j(f)(\xi) = 0$ because $f \in I$. On the other hand if $\xi \in H_{\pi_j} \backslash E_{\pi_j}$ then $\pi_j(P_{\pi_j})(\xi) \neq 0$, whereas $\pi_i(P_{\pi_j}) = 0$ for $i \neq j$ by Theorem 3.3.3(i), which implies that $\pi_j(g)\xi = \frac{1}{j^2} \alpha_{\pi_j} \pi_j(P_{\pi_j})(\xi) \neq 0$. This establishes the claim.

Finally we show that (a) implies (b). Assume that $L^1(G)$ is topologically left Noetherian. Then there exist $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in L^1_0(G)$ such that

$$L^1_0(G) = \overline{L^1(G) * g_1 + \cdots + L^1(G) * g_r}.$$

For each $n \in \mathbb{N}$ there exist $t_n^{(i)} \in T(G)$ $(i = 1, \ldots, r)$ such that

$$\|t_n^{(i)} - g_i\| < \frac{1}{n} \quad (i = 1, \ldots, r).$$

Let $\mathcal{S}$ be the set

$$\mathcal{S} = \left\{\pi \in \widehat{G} : \text{ there exist } i, n \in \mathbb{N} \text{ such that } \pi\left(t_n^{(i)}\right) \neq 0\right\} \cup \{1\},$$

where 1 denotes the trivial representation. We see that $\mathcal{S}$ is countable because, by Theorem 3.3.3(i), each function $t_n^{(i)}$ satisfies $\pi(t_n^{(i)}) \neq 0$ for at most finitely many $\pi \in \widehat{G}$. We shall show that $\mathcal{S} = \widehat{G}$.

Assume instead that there exists some $\pi \in \widehat{G}\backslash\mathcal{S}$, and let u be the identity element of $T_\pi(G)$. For $\sigma \in \widehat{G}\backslash\{\pi\}$ we have $\sigma(u) = 0$, whereas $\pi\left(t_n^{(i)}\right) = 0$ for every $n \in \mathbb{N}$ and every $i = 1, \ldots, r$. Hence $\sigma\left(t_n^{(i)} * u\right) = 0$ $(\sigma \in \widehat{G}, n \in \mathbb{N}, i \in \{1, \ldots, r\})$, which implies that $t_n^{(i)} * u = 0$ for every $n \in \mathbb{N}$ and $i = 1, \ldots, r$.

By taking the limit as n goes to infinity, this shows that $g_i * u = 0$ $(i = 1, \ldots, r)$, and hence that $f * u = 0$ for every $f \in L_0^1(G)$. However, because $1 \in \mathcal{S}$, Theorem 3.3.3(i) implies that $u \in L_0^1(G)$. Since u was chosen to be an identity $u * u = u \neq 0$. This contradiction implies that $\widehat{G} = \mathcal{S}$, as claimed. □

We conjecture that in fact G is compact whenever $L^1(G)$ is topologically left Noetherian, and hence metrisable by the previous Theorem. Indeed, this is the case when G is abelian by a theorem of Atzmon [**5**] (note that Atzmon says "finitely-generated" where we say "topologically finitely-generated"). In [**5**] Atzmon points out the relationship between questions about topologically finitely-generated ideals of $L^1(G)$ and difficult questions about spectral synthesis. In the light of this our conjecture seems daunting, and we do not attempt to prove it here.

The next proposition suggests to us that weak*-topological Noetherianity is a more interesting notion for the measure algebra of a locally compact group G than $\|\cdot\|$-topological Noetherianity, and we explore this in the next section.

PROPOSITION 3.3.6. *Let G be a locally compact group such that $M(G)$ is topologically left Noetherian. Then G is countable. If, in addition, G is either compact or abelian, then G is finite.*

PROOF. Suppose that $M(G)$ is topologically left Noetherian. Then, by Lemma 3.2.2 (i), so are its quotients, whence $\ell^1(G_d)$ is topologically left Noetherian, where G_d denotes the group G with the discrete topology. It follows that

$$\ell^1_0(G_d) = \overline{\ell^1(G_d) * g_1 + \cdots + \ell^1(G_d) * g_n},$$

for some $n \in \mathbb{N}$ and some $g_1, \ldots, g_n \in \ell^1_0(G_d)$. Let H be the subgroup of G generated by the supports of the functions $g_1, \ldots, g_n$. This is a countable set. Define $\sigma \colon \ell^1(G_d) \to \mathbb{C}$ by

$$\sigma \colon f \mapsto \sum_{x \in H} f(x) \quad (f \in \ell^1(G_d)).$$

Then, by the calculation (2.2) performed in Lemma 2.3.6,

$$\sigma(f) = 0 \quad (f \in \ell^1(G_d) * g_1 + \cdots + \ell^1(G_d) * g_n),$$

and hence, since σ is clearly bounded, $\sigma(f) = 0$ for every $f \in \ell^1_0(G_d)$. This forces $G = H$. Hence G is countable.

A countable locally compact group is always discrete, so that if it is also compact it must be finite. If G is abelian, then the fact that $\ell^1(G_d)$ is topologically Noetherian implies that G is finite by [**5**, Theorem 1.1]. $\square$

Note that if our conjecture that G is compact whenever $L^1(G)$ is topologically left Noetherian is correct, then the above proof actually shows that G is finite whenever $M(G)$ is topologically left Noetherian.

3.4. Multiplier Algebras and Dual Banach Algebras

In this section we consider Banach algebras whose multiplier algebras are dual Banach algebras. We first develop some general theory and then go on to prove some results concerning the weak*-closed left ideals of such multiplier algebras. We shall observe that, as a consequence of our general theory and Theorem 3.3.5, the measure algebra of a locally compact group G is weak*-topologically left Noetherian whenever G is compact and metrisable. The results of this section will be useful again when we consider weak*-topological left Noetherianity of $\mathcal{B}(E)$, with E a reflexive Banach space, in Section 3.6.

Let A be a faithful Banach algebra and denote the multiplier algebra of A by $M(A)$. We always regard A as an ideal in $M(A)$. The bidual A'' admits left and right actions of $M(A)$ rendering it a Banach $M(A)$-bimodule, denoted by

$$\Phi \mapsto \Phi \Box \mu \qquad \text{and} \qquad \Phi \mapsto \mu \Diamond \Phi,$$

and defined by

$$\langle \Phi \Box \mu, \lambda \rangle = \langle \Phi, \mu \cdot \lambda \rangle, \qquad \langle \mu \Diamond \Phi, \lambda \rangle = \langle \Phi, \lambda \cdot \mu \rangle,$$
$$\langle a, \mu \cdot \lambda \rangle = \langle a\mu, \lambda \rangle, \qquad \langle a, \lambda \cdot \mu \rangle = \langle \mu a, \lambda \rangle,$$

for $\Phi \in A'', \lambda \in A', a \in A, \mu \in M(A)$. When we view A as sitting inside $M(A)$ we recover the first and second Arens products, which we also denote by $\Box$ and $\Diamond$.

Now suppose that A has a bounded approximate identity, and let Φ_0 be a mixed identity for A'', which exists by [**19**, Proposition 2.9.16 (iii)]. Then it is well known that we have embeddings L and R of $M(A)$ into $(A'', \Box)$ and $(A'', \Diamond)$, respectively, given by

$$L\colon \mu \mapsto \Phi_0 \Box \mu \qquad \text{and} \qquad R\colon \mu \mapsto \mu \Diamond \Phi_0,$$

for $\mu \in M(A)$ (see [**19**, Theorem 2.9.49(iii)] for the details in the special case that $\|\Phi_0\| = 1$; the general case is very similar). Note that L and R are just the identity map when restricted to A. Moreover, whenever $a \in A$ and $\mu \in M(A)$ we have

$$a \Box L(\mu) = a \Box \Phi_0 \Box \mu = a \Box \mu = a\mu, \tag{3.1}$$

and similarly

$$R(\mu) \Box a = \mu a. \tag{3.2}$$

In this section we shall be interested in Banach algebras with a bounded approximate identity whose multiplier algebras are also dual Banach algebras. Moreover, we shall focus on cases where the dual structure and multiplier structure are compatible in some sense. It can be checked that $\Phi_0 \square A''$ is a dual Banach space with predual $A' \cdot A$, and similarly $A'' \diamond \Phi_0$ may be identified with the dual of $A \cdot A'$. The $(A' \cdot A,\, \Phi_0 \square A'')$-duality coincides with the $(A',\, A'')$-duality in the sense that

$$\langle \lambda, \Psi \rangle_{(A' \cdot A,\, \Phi_0 \square A'')} = \langle \lambda, \Psi \rangle_{(A',\, A'')} \qquad (\lambda \in A' \cdot A,\ \ \Psi \in \Phi_0 \square A''),$$

and similarly for the $(A \cdot A',\, A \diamond \Phi_0)$-duality. Of course, $L(M(A)) \subset \Phi_0 \square A''$ and $R(M(A)) \subset A'' \diamond \Phi_0$. Also, by Cohen's factorisation theorem, we have $A' \cdot A = \overline{\mathrm{span}}(A' \cdot A)$ and $A \cdot A' = \overline{\mathrm{span}}(A \cdot A')$.

We next define what what we shall call in this thesis an Ülger algebra. This is a non-commutative version of a condition considered by Ülger in [**84**], in which the condition was applied to commutative, semisimple Banach algebras.

Definition 3.4.1. We say that a Banach algebra A is an *Ülger algebra* if

(1) A has a bounded approximate identity;
(2) $M(A)$ is a dual Banach algebra, with predual X say;
(3) there are bounded module maps $\iota_L \colon X \to A \cdot A'$ and $\iota_R \colon X \to A' \cdot A$ which are bounded below, such that under the map L the $(X, M(A))$-duality is identified with the $(\iota_L(X), L(M(A)))$-duality, and under the map R the $(X, M(A))$ duality is identified with the $(\iota_R(X), R(M(A)))$ duality, i.e.

$$\langle x, \mu \rangle = \langle \iota_L(x), L(\mu) \rangle = \langle \iota_R(x), R(\mu) \rangle \quad (\mu \in M(A), x \in X). \tag{3.3}$$

Examples of Ülger algebras include $A_p(G)$ for G a locally compact amenable group, and $p \in (1, \infty)$, with $X = PF_p(G)$, the space of pseudo-functions on G (see [**84**, page 99]). Below we shall show that, for any locally compact group G, the group algebra $L^1(G)$, whose multiplier algebra may be identified with the measure algebra $M(G)$, is an Ülger algebra (Proposition 3.4.4), as well as $\mathcal{K}(E)$, for any reflexive Banach space

E with the approximation property (Corollary 3.4.9). Compare with our discussion of these algebras in Subsections 1.4.3 and 1.4.5 of the introduction.

The following lemma is often useful.

LEMMA 3.4.2. *Let A be a Banach algebra with a bounded approximate identity. Then the following formulae hold for $a \in A, \mu \in M(A)$ and $\lambda \in A'$:*

$$\langle \lambda \cdot a, L(\mu) \rangle_{(A', A'')} = \langle a, \mu \cdot \lambda \rangle_{(A, A')} \tag{3.4}$$

$$\langle a \cdot \lambda, R(\mu) \rangle_{(A', A'')} = \langle a, \lambda \cdot \mu \rangle_{(A, A')}. \tag{3.5}$$

PROOF. Let $a \in A, \mu \in M(A), \lambda \in A'$. Then

$$\begin{aligned} \langle \lambda \cdot a, L(\mu) \rangle_{(A', A'')} &= \langle \lambda \cdot a, \Phi_0 \Box \mu \rangle_{(A', A'')} = \langle \mu \cdot \lambda \cdot a, \Phi_0 \rangle_{(A', A'')} \\ &= \langle \mu \cdot \lambda, a \Box \Phi_0 \rangle_{(A', A'')} = \langle a, \mu \cdot \lambda \rangle_{(A, A')}. \end{aligned}$$

The other identity is proved similarly. $\Box$

Remark. It follows from the above lemma that, although the maps L and R may depend on the choice of the mixed identity Φ_0, the definition of an Ülger algebra does not. Indeed, suppose that we have two mixed identities Φ_1 and Φ_2, and corresponding maps L_1 and R_1, and L_2 and R_2. Then Lemma 3.4.2 implies that

$$\langle \lambda \cdot a, L_1(\mu) \rangle = \langle \lambda \cdot a, L_2(\mu) \rangle \quad \text{and} \quad \langle a \cdot \lambda, R_1(\mu) \rangle = \langle a \cdot \lambda, R_2(\mu) \rangle,$$

for $a \in A, \lambda \in A'$, and $\mu \in M(A)$. Hence if A is an Ülger algebra with respect to Φ_1, we have $\langle x, \mu \rangle = \langle \iota_L(x), L_1(\mu) \rangle = \langle \iota_L(x), L_2(\mu) \rangle$ $(x \in X, \mu \in M(A))$, and similarly for R_1 and R_2, so that A is also an Ülger algebra with respect to Φ_2 for the same choices of maps ι_L and ι_R.

We wish to verify that the group algebra of a locally compact group is an Ülger algebra. For this we shall need the following lemma which is surely well known. We include a short proof for the reader's convenience.

LEMMA 3.4.3. *The action of $L^1(G)$ on $L^\infty(G)$ as its dual is given by*

$$f \cdot \phi = \phi * \check{f} \qquad \text{and} \qquad \phi \cdot f = (\widecheck{\Delta f}) * \phi,$$

where Δ denotes the modular function, and $f \in L^1(G)$ and $\phi \in L^\infty(G)$. Here, Δf denotes the pointwise product of the modular function with f.

PROOF. Let $g \in L^1(G)$. Then, for $f \in L^1(G)$ and $\phi \in L^\infty(G)$, we have

$$\begin{aligned} \langle g, \phi \cdot f \rangle = \langle f * g, \phi \rangle &= \int_G \int_G f(st^{-1}) g(t) \Delta(t^{-1}) \phi(s) \, \mathrm{d}t \, \mathrm{d}s \\ &= \int_G g(t) \int_G f(st^{-1}) \Delta(t^{-1}) \phi(s) \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_G g(t) \int_G (\widecheck{\Delta f})(ts^{-1}) \Delta(s^{-1}) \phi(s) \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_G g(t) [(\widecheck{\Delta f}) * \phi](t) \, \mathrm{d}t. \end{aligned}$$

It follows that $\phi \cdot f = (\widecheck{\Delta f}) * \phi$. The other formula is proved in a similar fashion, but the calculation is slightly simplified by the absence of the modular function. $\square$

PROPOSITION 3.4.4. *Let G be a locally compact group. Then $L^1(G)$ is an Ülger algebra.*

PROOF. That $L^1(G)$ has a bounded approximate identity is well known. Recall that we may identify the multiplier algebra of $L^1(G)$ with the measure algebra $M(G)$, and that $M(G)$ is a dual Banach algebra, with predual $C_0(G)$.

We claim that $C_0(G) \subset (L^\infty(G) \cdot L^1(G)) \cap (L^1(G) \cdot L^\infty(G))$, so that part (2) of Definition 3.4.1 is satisfied with $\iota_L = \iota_R$ taken to be the inclusion map. By [**32**, Proposition (3.4)] we have that $A(G) = \overline{\operatorname{span}}\{g_1 * \check{g}_2 : g_1, g_2 \in C_c(G)\}$, where the closure is taken in the $A(G)$-norm. Because this norm dominates the infinity norm, we have

$$\begin{aligned} C_0(G) = \overline{A(G)}^{\|\cdot\|_\infty} &\subset \overline{\operatorname{span}}^{\|\cdot\|_\infty}\{\phi * \check{f} : \phi, f \in C_c(G)\} \\ &= \overline{\operatorname{span}}\{f \cdot \phi : f, \phi \in C_c(G)\} \\ &\subset \overline{L^1(G) \cdot L^\infty(G)} = L^1(G) \cdot L^\infty(G). \end{aligned}$$

The other inclusion is demonstrated using the same idea, but now the formula for $\phi \cdot f$ involves the modular function. Indeed, we have

$$\begin{aligned} C_0(G) &= \overline{A(G)}^{\|\cdot\|_\infty} \subset \overline{\operatorname{span}}^{\|\cdot\|_\infty}\{g_1 * g_2 : g_1, g_2 \in C_c(G)\} \\ &= \overline{\operatorname{span}}\{(\widecheck{\Delta f}) * \phi : f, \phi \in C_c(G)\} = \overline{\operatorname{span}}\{\phi \cdot f : f, \phi \in C_c(G)\} \\ &\subset L^\infty(G) \cdot L^1(G), \end{aligned}$$

as required.

It remains to show that part (3) of the definition holds, with $\iota_L = \iota_R$ taken to be the inclusion map. Fix $\mu \in M(G)$ and $g \in C_0(G)$. We have to show that

$$\langle g, \mu \rangle_{(C_0(G),\, M(G))} = \langle g, L(\mu) \rangle_{(L^\infty(G),\, L^1(G)'')}.$$

By the above, we can write $g = \phi \cdot f$, for some $\phi \in L^\infty(G)$ and $f \in L^1(G)$. By Lemma 3.4.3, $\langle \phi \cdot f, L(\mu) \rangle_{(L^\infty(G),\, L^1(G)'')} = \langle f * \mu, \phi \rangle_{(L^1(G),\, L^\infty(G))}$, so it remains to show that

$$\langle \phi \cdot f, \mu \rangle_{(C_0(G),\, M(G))} = \langle f * \mu, \phi \rangle_{(L^1(G),\, L^\infty(G))}.$$

This is equivalent to showing that

$$\int_G (\phi \cdot f)(t) \, \mathrm{d}\mu(t) = \int_G (f * \mu)(t) \phi(t) \, \mathrm{d}m(t),$$

where m denotes the left Haar measure on the group. By Lemma 3.4.3 the left-hand side is equal to

$$\begin{aligned}\int_G [(\widecheck{\Delta f}) * \phi](t)\,\mathrm{d}\mu(t) &= \int_G \int_G (\widecheck{\Delta f})(ts^{-1})\phi(s)\Delta(s^{-1})\,\mathrm{d}m(s)\,\mathrm{d}\mu(t)\\ &= \int_G \int_G \Delta(st^{-1})f(st^{-1})\phi(s)\Delta(s^{-1})\,\mathrm{d}m(s)\,\mathrm{d}\mu(t)\\ &= \int_G \int_G f(st^{-1})\phi(s)\Delta(t^{-1})\,\mathrm{d}m(s)\,\mathrm{d}\mu(t)\\ &= \int_G \phi(s)\int_G f(st^{-1})\Delta(t^{-1})\,\mathrm{d}\mu(t)\,\mathrm{d}m(s)\\ &= \langle f * \mu, \phi\rangle_{(L^1(G),\,L^\infty(G))}.\end{aligned}$$

It remains to show that $\langle g, \mu\rangle_{(C_0(G),\,M(G))} = \langle g, R(\mu)\rangle_{(L^\infty(G),\,L^1(G)'')}$, and for this we write $g = f \cdot \phi$ for $\phi \in L^\infty(G)$ and $f \in L^1(G)$. We shall show that

$$\langle f \cdot \phi, \mu\rangle_{(C_0(G),\,M(G))} = \langle \mu * f, \phi\rangle_{(L^1(G),\,L^\infty(G))},$$

or, in other words, that

$$\int_G (f \cdot \phi)(t)\,\mathrm{d}\mu(t) = \int_G (\mu * f)(t)\phi(t)\,\mathrm{d}m(t).$$

This time the left-hand side is equal to

$$\begin{aligned}\int_G (\phi * \check{f})(t)\,\mathrm{d}\mu(t) &= \int_G \int_G \phi(s)\check{f}(s^{-1}t)\,\mathrm{d}m(s)\,\mathrm{d}\mu(t)\\ &= \int_G \int_G \phi(s)f(t^{-1}s)\,\mathrm{d}\mu(t)\ \mathrm{d}m(s)\\ &= \int_G \phi(s)(\mu * f)(s)\,\mathrm{d}m(s),\end{aligned}$$

as required. This completes the proof. $\square$

The next lemma lists some basic properties of Ülger algebras.

Lemma 3.4.5. *Let A be an Ülger algebra with bounded approximate identity (e_α). Then:*

(i) *A is weak*-dense in $M(A)$;*

(ii) *The weak*-topology on $M(A)$ is coarser than the strict topology.*

PROOF. (i) Let $x \in A_\perp \subset X$. Then for every $a \in A$ we have $\langle \iota(x), a\rangle_{(A',A)} = \langle x, a\rangle_{(X, M(A))} = 0$, so that $\iota(x) = 0$, forcing $x = 0$. As x was arbitrary, we have shown that $A_\perp = \{0\}$, and hence that $\overline{A}^{w^*} = (A_\perp)^\perp = \{0\}^\perp = M(A)$.

(ii) Let $(\mu_\alpha) \subset M(A)$ be a net converging to some $\mu \in M(A)$ in the strict topology. Given $x \in X$, there exist $a \in A$ and $\lambda \in A'$ such that $\iota(x) = \lambda \cdot a$. Therefore

$$\langle x, \mu_\alpha\rangle = \langle \lambda \cdot a, L(\mu_\alpha)\rangle = \langle \lambda, aL(\mu_\alpha)\rangle = \langle \lambda, a\mu_\alpha\rangle,$$

which converges to $\langle \lambda, a\mu\rangle = \langle x, \mu\rangle$. As x was arbitrary $\lim_{w*,\alpha} \mu_\alpha = \mu$. ☐

DEFINITION 3.4.6. Let A be an Ülger algebra. We say that A is *strongly Ülger* if the map L is $\sigma(M(A), X)$-$\sigma(\Phi_0\Box A'',\, A'\cdot A)$ continuous and the map R is $\sigma(M(A),\, X)$-$\sigma(A''\Diamond\Phi_0,\, A\cdot A')$ continuous.

In this thesis we shall consider the ideal structure of strongly Ülger algebras, but we note that they appear to have interesting properties more broadly and are worthy of further study. In the papers [**43**] and [**44**] Hayati and Amini consider Connes amenability of certain multiplier algebras which are also dual Banach algebras. Although their framework is different to ours, their proof of [**43**, Theorem 3.1] can be lifted with only trivial modifications to show that, if A is a strongly Ülger algebra, then A is amenable if and only if $M(A)$ is Connes amenable.

For finding examples of strongly Ülger algebras, the following lemma is quite useful.

LEMMA 3.4.7. *Let A be an Ülger algebra, and suppose that the maps ι_L and ι_R in Definition 3.4.1 are surjective. Then A is strongly Ülger.*

PROOF. We show that the map L is continuous in the appropriate sense, the argument for R being very similar. Suppose (μ_α) is a net in $M(A)$, converging in the weak*-topology to some element $\mu \in M(A)$. Let $a \in A$, and let $\lambda \in A'$. There exists

$x \in X$ such that $\iota_L(x) = \lambda \cdot a$, so we have

$$\langle \lambda \cdot a, L(\mu_\alpha) \rangle = \langle \iota_L(x), L(\mu_\alpha) \rangle = \langle x, \mu_\alpha \rangle,$$

which converges to $\langle x, \mu \rangle = \langle \lambda \cdot a, L(\mu) \rangle$. As λ and a were arbitrary, it follows that $L(\mu_\alpha)$ converges to $L(\mu)$ in the $\sigma(\Phi_0 \Box A'', A' \cdot A)$-topology. We have shown that L is $\sigma(M(A), X)$-$\sigma(\Phi_0 \Box A'', A' \cdot A)$ continuous. The argument for R is analogous. $\Box$

This gives us the following family of examples of strongly Ülger algebras.

LEMMA 3.4.8. *Let A be a Banach algebra with a bounded approximate identity which is Arens regular and an ideal in its bidual. Then A is a strongly Ülger algebra.*

PROOF. By [**54**, Theorem 3.9] A'' may be identified with $M(A)$. Arens regularity implies that A'' is a dual Banach algebra with predual A'. The criteria set out in Definition 3.4.1 now follow trivially, setting $X = A'$ and $\iota_L = \iota_R = \mathrm{id}_{A'}$. As the maps ι_L and ι_R are surjective, A'' is strongly Ülger by Lemma 3.4.7. $\Box$

It follows from Lemma 3.4.8 that $c_0(\mathbb{N})$ is an example of a strongly Ülger algebra. A family of examples that will be important to us in the Section 5 is the following:

COROLLARY 3.4.9. *Let E be a reflexive Banach space with the approximation property. Then $\mathcal{K}(E)$ is a strongly Ülger algebra.*

PROOF. By [**91**, Theorem 3] $\mathcal{A}(E) = \mathcal{K}(E)$ is Arens regular. Moreover $\mathcal{K}(E)'' = \mathcal{B}(E)$ and by Lemma 1.4.5 the Arens product coincides with the usual composition of operators, so that we see that $\mathcal{K}(E)$ is an ideal in its bidual. Hence the result follows from the previous lemma. $\Box$

One of the most useful properties of strongly Ülger algebras is summarised in the following lemma.

LEMMA 3.4.10. *Let A be a strongly Ülger algebra. Then for each $a \in A$ the maps $M(A) \to A$ given by $\mu \mapsto a\mu$ and $\mu \mapsto \mu a$ are weak*-weakly continuous, and hence weakly compact.*

PROOF. To verify weak*-weak continuity of the map $\mu \mapsto a\mu$ we compose with an arbitrary $\lambda \in A'$, and observe that the result is a weak*-continuous linear funtional on $M(A)$. Indeed, by Lemma 3.4.2,

$$\langle a\mu, \lambda \rangle = \langle \lambda \cdot a, L(\mu) \rangle \quad (\mu \in M(A)),$$

and the map $\mu \mapsto \langle \lambda \cdot a, L(\mu) \rangle$ is weak*-continuous by hypothesis. The case of the other map is similar. $\square$

Unfortunately, the group algebra is usually not strongly Ülger.

PROPOSITION 3.4.11. *Let G be a locally compact group. The Banach algebra $L^1(G)$ is strongly Ülger if and only if G is compact.*

PROOF. First assume that G is compact. Then by [**34**, Proposition 2.39(d)] we have $\phi * f, f * \phi \in C(G)$ for every $\phi \in L^\infty(G), f \in L^1(G)$, so that in fact $C(G) = L^1(G) \cdot L^\infty(G) = L^\infty(G) \cdot L^1(G)$. Hence, by Lemma 3.4.7, $L^1(G)$ is strongly Ülger.

Now assume that $L^1(G)$ is strongly Ülger. Then whenever $f \in L^1(G)\backslash\{0\}$ Lemma 3.4.10 implies that the maps $L^1(G) \to L^1(G)$ given by

$$L_f \colon g \mapsto g * f \qquad \text{and} \qquad R_f \colon g \mapsto f * g$$

are weakly compact. Hence $L_f''(A''), R_f''(A'') \subset A$ by [**59**, Theorem 3.5.8]. Observing that $L_f'' \colon \Psi \mapsto f \Box \Psi$ $(\Psi \in A'')$, we see that $L^1(G)$ is a right ideal its bidual, and similarly it is a left ideal. Hence, by [**40**] G is compact. $\square$

We now come to some results which describe how our different versions of Noetherianity play out in the setting of Ülger algebras. The first hypothesis of the following proposition is satisfied whenever A is an Ülger algebra by Lemma 3.4.5(i).

PROPOSITION 3.4.12. *Let A be a Banach algebra with a bounded approximate identity such that $M(A)$ admits the structure of a dual Banach algebra in such a way that A is weak*-dense in $M(A)$. Suppose that for every closed left ideal I in A there* *exists* $n \in \mathbb{N}$ *and there exist* $\mu_1, \ldots, \mu_n \in M(A)$ *such that* $I = \overline{A^\sharp\mu_1 + \cdots + A^\sharp\mu_n}$. *Then* $M(A)$ *is weak*-topologically left Noetherian. In particular,* $M(A)$ *is weak*-topologically left Noetherian whenever* A *is* $\|\cdot\|$*-topologically left Noetherian.*

Proof. Let I be a weak*-closed left ideal of $M(A)$. Since A is weak*-dense in $M(A)$, which is unital, Lemma 3.2.1 implies that $A \cap I$ is weak*-dense in I. On the other hand, $A \cap I$ is a closed left ideal in A, so there exists $n \in \mathbb{N}$, and there exist $\mu_1, \ldots, \mu_n \in M(A)$ such that $A \cap I = \overline{A^\sharp\mu_1 + \cdots + A^\sharp\mu_n}$. It follows that

$$I = \overline{A^\sharp\mu_1 + \cdots + A^\sharp\mu_n}^{w^*} = \overline{M(A)\mu_1 + \cdots + M(A)\mu_n}^{w^*}.$$

As I was arbitrary the result follows. □

We are now able to give an interesting family of examples of weak*-topologically left Noetherian dual Banach algebras which (by Proposition 3.3.6) are not usually $\|\cdot\|$-topologically left Noetherian.

Corollary 3.4.13. *Let* G *be a compact, metrisable group. Then* $M(G)$ *is weak*-topologically left Noetherian.*

Proof. By Proposition 3.4.4, Lemma 3.4.5, and Theorem 3.3.5, $L^1(G)$ satisfies the hypothesis of Proposition 3.4.12. The result now follows from that Proposition. □

For strongly Ülger algebras there is a bijective correspondence between the closed left ideals of A and the weak*-closed left ideals of $M(A)$ as we describe below in Corollary 3.4.15. In Section 3.6 this will allow us to classify the weak*-closed left and right ideals of $\mathcal{B}(E)$, for E a reflexive Banach space with the approximation property, in Theorem 3.6.7. In Theorem 3.4.17 below, this will allow us to classify the weak*-closed left ideals of the measure algebra of a compact group.

Lemma 3.4.14. *Let* I *be a closed left ideal of a strongly Ülger algebra* A*, and let* $\mu \in \overline{I}^{w^*} \subset M(A)$*. Then* $A\mu \subset I$*.*

PROOF. Let (μ_α) be a net in I converging to μ in the weak*-topology and let $a \in A$. For each index α we have $a\mu_\alpha \in I$. Since A is strongly Ülger, Lemma 3.4.10 implies that the map $\nu \mapsto a\nu,\ M(A) \to A$, is weak*-weakly continuous, so that net $a\mu_\alpha$ converges weakly to $a\mu$ in A. Hence $a\mu \in \overline{I}^w = I$. As a was arbitrary, the result follows. □

PROPOSITION 3.4.15. *Let A be a strongly Ülger algebra. The map*

$$I \mapsto \overline{I}^{w*},$$

defines a bijective correspondence between closed left ideals in A and weak-closed left ideals in $M(A)$. The inverse is given by*

$$J \mapsto A \cap J,$$

for J a weak-closed left ideal in $M(A)$.*

PROOF. First we take an arbitrary closed left ideal I in A and show that $A \cap \overline{I}^{w*} = I$. Certainly $I \subset A \cap \overline{I}^{w*}$. Let $a \in A \cap \overline{I}^{w*}$. Then by Lemma 3.4.14 we have $Aa \subset I$. Since A has a bounded approximate identity, this implies that $a \in I$. As a was arbitrary, we must have $I = A \cap \overline{I}^{w*}$.

It remains to show that, given a weak*-closed left ideal J of $M(A)$, we have $\overline{A \cap J}^{w*} = J$, and this follows from Lemma 3.2.1. □

Using Proposition 3.4.15 we are able to classify the weak*-closed left ideals of $M(G)$, for G a compact group. Let G be a compact group and suppose the for each $\pi \in \widehat{G}$ we have chosen a linear subspace $E_\pi \leqslant H_\pi$. Then we define

$$J[(E_\pi)_{\pi\in\widehat{G}}] := \left\{\mu \in M(G) : \pi(\mu)(E_\pi) = 0,\ \pi \in \widehat{G}\right\}.$$

We shall show that these are exactly the weak*-closed left ideals of $M(G)$.

LEMMA 3.4.16. *Let G be a compact group and let $E_\pi \leqslant H_\pi$ $(\pi \in \widehat{G})$. Then*

$$\overline{\operatorname{span}}\left\{\xi *_\pi \eta : \pi \in \widehat{G},\, \xi \in E_\pi,\, \eta \in H_\pi\right\}^\perp = J[(E_\pi)_{\pi\in\widehat{G}}]. \tag{3.6}$$

PROOF. Let $\pi \in \widehat{G}$, $\eta \in H_\pi$ and $\xi \in E_\pi$. We calculate that

$$\langle \pi(\mu)\xi, \eta\rangle_{H_\pi} = \int_G \langle \pi(t)\xi, \eta\rangle \,\mathrm{d}\mu(t) = \langle \xi *_\pi \eta, \mu\rangle.$$

It follows that $\pi(\mu)(\xi) = 0$ for every $\pi \in \widehat{G}$ and $\xi \in E_\pi$ if and only if $\langle \xi *_\pi \eta, \mu\rangle = 0$ for every $\pi \in \widehat{G}, \eta \in H_\pi$ and $\xi \in E_\pi$. The result follows. $\square$

THEOREM 3.4.17. *Let G be a compact group. Then the weak*-closed left ideals of $M(G)$ are given by $J[(E_\pi)_{\pi\in\widehat{G}}]$, as $(E_\pi)_{\pi\in\widehat{G}}$ runs over the possible choices of linear subspaces $E_\pi \leqslant H_\pi$ $(\pi \in \widehat{G})$. Moreover, distinct choices of the subspaces $(E_\pi)_{\pi\in\widehat{G}}$ yield distinct ideals $J[(E_\pi)_{\pi\in\widehat{G}}]$.*

PROOF. By Proposition 3.4.11, $L^1(G)$ is a strongly Ülger Banach algebra, so, by Proposition 3.4.15, there is a bijection Λ from the set of weak*-closed left ideals of $M(G)$ to the set of $\|\cdot\|$-closed left ideals of $L^1(G)$ given by

$$\Lambda \colon I \mapsto I \cap L^1(G),$$

for I a weak*-closed left ideal in $M(G)$. By Lemma 3.4.16 each space $J[(E_\pi)_{\pi\in\widehat{G}}]$ is weak*-closed, and it is easily checked that it is a left ideal. Moreover, by Theorem 3.3.2, each closed left ideal of $L^1(G)$ has the form $L^1(G) \cap J[(E_\pi)_{\pi\in\widehat{G}}]$, for some choice of subspaces $E_\pi \leqslant H_\pi$ $(\pi \in \widehat{G})$. Hence Λ is surjective when restricted to the set

$$\left\{J[(E_\pi)_{\pi\in\widehat{G}}] : E_\pi \leqslant H_\pi,\, \pi \in \widehat{G}\right\}.$$

Since Λ is a bijection, it follows that this set must be the full set of weak*-closed left ideals of $M(G)$. Finally, it follows from Lemma 3.3.4 and the injectivity of Λ that different choices of subspaces give different left ideals. $\square$

COROLLARY 3.4.18. *Let G be a compact group, and let $X \subset C(G)$ be a closed linear subspace, which is invariant under left translation. Then there exists a unique choice of linear subspaces $E_\pi \leqslant H_\pi$ $(\pi \in \widehat{G})$ such that*

$$X = \overline{\operatorname{span}}\left\{\xi *_\pi \eta : \pi \in \widehat{G},\, \xi \in E_\pi,\, \eta \in H_\pi\right\}.$$

PROOF. By Lemma 2.3.3 X has the form $I_\perp$, for some weak*-closed ideal I of $M(G)$. It now follows from Theorem 3.4.17 and Lemma 3.4.16, that X has the given form. □

Finally we show that for strongly Ülger algebras weak*-topological left Noetherianity of $M(A)$ can be characterised in terms of a $\|\cdot\|$-topological condition on A.

PROPOSITION 3.4.19. *Let A be a strongly Ülger algebra. Then $M(A)$ is weak*-topologically left Noetherian if and only if for every closed left ideal I in A has the form*

$$I = \overline{A\mu_1 + \cdots + A\mu_n},$$

for some $n \in \mathbb{N}$, and some $\mu_1, \ldots, \mu_n \in M(A)$.

PROOF. The "if" direction follows from Proposition 3.4.12. Conversely, suppose that $M(A)$ is weak*-topologically left Noetherian, and let I be a closed left ideal in A. Then there exist $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in M(A)$ such that

$$\overline{I}^{w*} = \overline{M(A)\mu_1 + \cdots + M(A)\mu_n}^{w*} = \overline{A\mu_1 + \cdots + A\mu_n}^{w*},$$

where we have used Lemma 3.4.5(i) to get the second equality. Hence, by applying Proposition 3.4.15 twice, we obtain

$$I = \overline{I}^{w*} \cap A = \overline{A\mu_1 + \cdots + A\mu_n}^{w*} \cap A = \overline{A\mu_1 + \cdots + A\mu_n}.$$

The result follows. □

3.5. Left and Right Ideals of Approximable Operators on a Banach Space

In this section we classify the closed left ideals and closed right ideals of $\mathcal{A}(E)$, for E belonging to a large class of Banach spaces that includes those Banach spaces E such that E' has BAP. Specifically, we require that $\mathcal{A}(E)$ has a left approximate identity for the former classification, and a right approximate identity for the latter (compare this with Theorem 1.2.1, which details some of the relationships between approximate identities for $\mathcal{A}(E)$ and approximation properties of E). We then use this characterisation to determine when $\mathcal{A}(E)$ is topologically left and right Noetherian for such Banach spaces. Of course when E has the approximation property, we have $\mathcal{A}(E) = \mathcal{K}(E)$.

Since proving this result we have become aware of a very similar classification of the closed left ideals of $\mathcal{K}(E)$, for E a Banach space with AP, due to Grønbæk [**38**, Proposition 7.3]. Indeed, Grønbæk's proof is very similar to ours. Hence, our classification of the closed left ideals is not really new. However, we feel that our exposition gives a slightly more detailed picture than Grønbæk's, so we have included it anyway. Moreover, Grønbæk says nothing about closed right ideals.

Let E be a Banach space, and let A be a closed subalgebra of $\mathcal{B}(E)$. Given closed linear subspaces $F \subset E'$ and $G \subset E$ we define

$$\mathscr{L}_A(F) = \{T \in A : \operatorname{im} T' \subset F\} \tag{3.7}$$

and

$$\mathscr{R}_A(G) = \{T \in A : \operatorname{im} T \subset G\}. \tag{3.8}$$

These define families of closed left and right ideals respectively. We also define a family of closed left ideals by

$$\mathscr{I}_A(G) = \{T \in A : \ker T \supset G\}, \tag{3.9}$$

where G is a closed linear subspace of E. When the ambient algebra A is unambiguous we shall often drop the subscript and simply write $\mathscr{L}(F)$, $\mathscr{R}(G)$, and $\mathscr{I}(G)$. Usually A will be either $\mathcal{A}(E)$ or $\mathcal{B}(E)$. We shall show that when $\mathcal{A}(E)$ has a left approximate identity then every closed left ideal has the form $\mathscr{L}_{\mathcal{A}(E)}(F)$, for some closed linear subspace $F \subset E'$ (Theorem 3.5.4). Similarly, when $\mathcal{A}(E)$ has a right approximate identity every closed right ideal of $\mathcal{A}(E)$ has the form $\mathscr{R}_{\mathcal{A}(E)}(G)$, for some closed linear subspace G of E (Theorem 3.5.10).

We begin by verifying that the sets defined in (3.7) really are closed left ideals of A. The proof that the sets defined in (3.8) and (3.9) are closed right and left ideals, respectively, is totally routine, and we leave it to the reader.

LEMMA 3.5.1. *Let E be a Banach space, and let A be a closed subalgebra of $\mathcal{B}(E)$. For each closed linear subspace F in E' the set $\mathscr{L}_A(F)$ is a closed left ideal in A.*

PROOF. It is clear that $\mathscr{L}_A(F)$ is a linear subspace. Let $T \in \mathscr{L}_A(F)$, and let $S \in A$. Then $\operatorname{im}(S \circ T)' = \operatorname{im}(T' \circ S') \subset \operatorname{im} T' \subset F$, so that $S \circ T \in \mathscr{L}_A(F)$. Suppose (T_n) is a sequence in $\mathscr{L}_A(F)$ converging to some $T \in A$. Then for each $\lambda \in E'$ we know that $T_n'\lambda \in F$, which implies that $T'\lambda = \lim_{n\to\infty} T_n'\lambda \in F$. As λ was arbitrary it follows that $\operatorname{im} T' \subset F$, and hence $T \in \mathscr{L}_A(F)$. □

In what follows, given a Banach space E and $X \subset \mathcal{B}(E)$ we write $E' \circ X$ for the set $\{\lambda \circ T : \lambda \in E', T \in X\} = \bigcup_{T\in X} \operatorname{im} T'$. Sets of this form will be important because they give a way to recover the closed linear subspace appearing in (3.7): more precisely we shall show that, given a closed left ideal I of $\mathcal{A}(E)$, the set $F = E' \circ I$ is a closed linear subspace of E', and moreover $I = \mathscr{L}(F)$.

LEMMA 3.5.2. *Let E be a Banach space, and let I be a left ideal in $\mathcal{A}(E)$. Then $x \otimes \lambda \in I$ whenever $x \in E$ and $\lambda \in E' \circ I$.*

PROOF. We can write $\lambda = \varphi \circ T$, for some $\varphi \in E'$, and some $T \in I$. Then $x \otimes \lambda = (x \otimes \varphi) \circ T \in I$. □

LEMMA 3.5.3. *Let E be a Banach space, and let I be a closed left ideal in $\mathcal{A}(E)$. Then $E' \circ I$ is a closed linear subspace of E'.*

PROOF. It is clear that $E' \circ I$ is closed under scalar multiplication. We may assume that E is non-zero and fix $x \in E\backslash\{0\}$ and $\eta \in E'$ satisfying $\eta(x) = 1$. Let $\lambda_1, \lambda_2 \in E' \circ I$. By Lemma 3.5.2, $x \otimes \lambda_1, x \otimes \lambda_2 \in I$, so that

$$\eta \circ (x \otimes \lambda_1 + x \otimes \lambda_2) = \lambda_1 + \lambda_2 \in E' \circ I.$$

Hence $E' \circ I$ is closed under addition.

Let $(\lambda_n) \subset E' \circ I$ be a sequence converging in norm to some $\lambda \in E'$, and let x and η be as above. We have $\lim_{n\to\infty} x \otimes \lambda_n = x \otimes \lambda$ in $\mathcal{A}(E)$. Moreover, by Lemma 3.5.2 we have $x \otimes \lambda_n \in I$ $(n \in \mathbb{N})$, so that $x \otimes \lambda \in I$, since I is closed. Hence $\eta \circ (x \otimes \lambda) = \lambda \in E' \circ I$. We have shown that $E' \circ I$ is closed. $\square$

We write $\mathbf{SUB}(E)$ for the set of all closed linear subspaces of a Banach space E. Similarly, given a Banach algebra A we write $\mathbf{CLI}(A)$ for the set of closed left ideals of A. Both of these sets are lattices when ordered by inclusion. We can now state and prove our first classification result precisely. Recall also that we write $\mathbb{1}_{ij}$ for the Kronecker delta, as in (1.1).

THEOREM 3.5.4. *Let E be a Banach space such that $\mathcal{A}(E)$ has a left approximate identity. Then the map*

$$\Theta : (\mathbf{SUB}(E'), \subset) \to (\mathbf{CLI}(\mathcal{A}(E)), \subset), \quad F \mapsto \mathscr{L}(F)$$

is a lattice isomorphism, with inverse given by

$$\widehat{\Theta} \colon I \mapsto E' \circ I, \quad (I \in \mathbf{CLI}(\mathcal{A}(E))).$$

PROOF. The maps Θ and $\widehat{\Theta}$ have the specified codomains by Lemma 3.5.1 and Lemma 3.5.3, respectively, and it is clear that they respect inclusion. Since isomorphisms of posets preserve the lattice structure, once we have shown that these maps are mutually inverse it will follow that they are lattice isomorphisms.

Let I be a closed left ideal in $\mathcal{A}(E)$, and set $F = \widehat{\Theta}(I)$. We show that $I = (\Theta \circ \widehat{\Theta})(I)$, i.e. that $I = \mathscr{L}(F)$. Noting that $F = \{T'\lambda : \lambda \in E', T \in I\}$, it is clear that $I \subset \mathscr{L}(F)$.

To show the reverse inclusion, we note that, by Lemma 3.2.1, the finite-rank operators intersect $\mathscr{L}(F)$ densely, so that it is sufficient to show that $\mathcal{F}(E) \cap \mathscr{L}(F) \subset I$. Let $T \in \mathscr{L}(F)$ be finite-rank, and write $T = \sum_{i=1}^N x_i \otimes \lambda_i$, for some $N \in \mathbb{N}$, $x_1, \ldots, x_N \in E$, and $\lambda_1, \ldots, \lambda_N \in E'$. We may assume that the vectors $x_1, \ldots, x_N$ are linearly independent, so that there exist $\eta_1, \ldots \eta_N \in E'$ such that $\eta_i(x_j) = \mathbb{1}_{ij}$. Then $T'\eta_i = \lambda_i$ $(i = 1, \ldots, N)$, so that each λ_i belongs to F. Therefore, by Lemma 3.5.2, each $x_i \otimes \lambda_i$ belongs to I, and hence so does T. We have shown that $\widehat{\Theta}$ is a right inverse for Θ.

Now let F be a closed subspace of E', and consider $G = (\widehat{\Theta} \circ \Theta)(F) = E' \circ \mathscr{L}(F)$. Let $\lambda \in F$. Then, picking $x \in E\backslash\{0\}$ and $\eta \in E'$ such that $\eta(x) = 1$, we see that $x \otimes \lambda \in \mathscr{L}(F)$, and hence $\lambda = \eta \circ (x \otimes \lambda) \in G$. We conclude that $F \subset G$. Conversely whenever we have $\lambda = \varphi \circ T \in G$, for some $\varphi \in E'$ and $T \in \mathscr{L}(F)$, we have $\lambda = T'\varphi \in F$. Hence $G = F$, and we have shown that $\widehat{\Theta}$ is a left inverse for Θ. $\square$

The proof of the next corollary makes use of the following formula, valid for any (possibly infinite) collection of bounded linear operators (T_i) on a Banach space E:

$$\overline{\operatorname{span}}_i^{w^*}(\operatorname{im} T_i') = \left(\bigcap_i \ker T_i\right)^{\perp}. \tag{3.10}$$

COROLLARY 3.5.5. *Let E be a reflexive Banach space such that $\mathcal{A}(E)$ has a left approximate identity. Then there is a lattice anti-isomorphism*

$$\Phi : (\mathbf{SUB}(E), \subset) \to (\mathbf{CLI}(\mathcal{A}(E)), \subset)$$

given by

$$\Phi : F \mapsto \mathscr{I}_{\mathcal{A}(E)}(F),$$

with inverse given by

$$\widehat{\Phi} : I \mapsto \bigcap_{T \in I} \ker T,$$

for $I \in \mathbf{CLI}(\mathcal{A}(E))$.

Proof. As in the proof of the previous theorem, it is sufficient to show that Φ and $\widehat{\Phi}$ are mutually inverse poset anti-isomorphisms. Let Θ and $\widehat{\Theta}$ be as in Theorem 3.5.4 and define $\Psi : \mathbf{SUB}(E) \to \mathbf{SUB}(E')$ by $\Psi : F \mapsto F^{\perp}$, for F a closed linear subspace of E. Then, by reflexivity, the map Ψ is an anti-isomorphism of posets, with inverse given by $\Psi^{-1} \colon G \mapsto G_{\perp}$ $(G \in \mathbf{SUB}(E'))$.

By (3.10), for $T \in \mathcal{A}(E)$ we have $(\ker T)^{\perp} = \overline{\operatorname{im} T'}^{w^*} = \overline{\operatorname{im} T'}^{w} = \overline{\operatorname{im} T'}$, where we have used the fact that the weak and weak*-topologies coincide for a reflexive Banach space, and then Mazur's Theorem. Hence, by applying Ψ^{-1} to this equality, we have $\ker T = \left(\overline{\operatorname{im} T'}\right)_{\perp}$. It follows that, for any closed $F \leqslant E$,

$$\begin{aligned} \mathscr{L}(F^{\perp}) &= \{T \in \mathcal{A}(E) : \operatorname{im} T' \subset F^{\perp}\} = \{T \in \mathcal{A}(E) : \overline{\operatorname{im} T'} \subset F^{\perp}\} \\ &= \{T \in \mathcal{A}(E) : \left(\overline{\operatorname{im} T'}\right)_{\perp} \supset F\} = \{T \in \mathcal{A}(E) : \ker T \supset F\} = \Phi(F), \end{aligned}$$

which is equivalent to saying that $\Phi = \Theta \circ \Psi$. Since Θ is a poset isomorphism, and Ψ is a poset anti-isomorphism, we see that Φ is an anti-isomorphism of posets, and it remains to show that its inverse is given by $\widehat{\Phi}$. Indeed, we have $\Psi^{-1} \circ \widehat{\Theta} = \widehat{\Phi}$:

$$\begin{aligned} \left(\Psi \circ \widehat{\Phi}\right)(I) &= \left(\bigcap_{T \in I} \ker T\right)^{\perp} = \overline{\operatorname{span}}_{T \in I}\left(\operatorname{im} T'\right) \\ &= \overline{\operatorname{span}}\{\lambda \circ T : \lambda \in E', T \in I\} = E' \circ I = \widehat{\Theta}(I), \end{aligned}$$

where we have used (3.10) and Mazur's Theorem in the first line, and Lemma 3.5.3 in the second. Hence Φ is invertible, with inverse given by $\widehat{\Phi}$. $\square$

Remark. Corollary 3.5.5 is a generalisation of the well-known classification of the left ideals of $M_n(\mathbb{C})$, for $n \in \mathbb{N}$ (see, e.g., [**48**, Exercise 3, pg. 173]).

Now that we have a classification of the closed left ideals of $\mathcal{A}(E)$ we approach the question of when $\mathcal{A}(E)$ is topologically left Noetherian. The next lemma gives a more explicit description of the correspondence of Theorem 3.5.4 for a topologically finitely-generated left ideal in terms of its generators.

LEMMA 3.5.6. *Let E be a Banach space. Let $n \in \mathbb{N}$, let $T_1, \dots, T_n \in \mathcal{A}(E)$, and let $I = \overline{\mathcal{A}(E)^\sharp T_1 + \cdots + \mathcal{A}(E)^\sharp T_n}$. Then*

$$E' \circ I = \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}.$$

PROOF. As $E' \circ I = \bigcup_{T \in I} \operatorname{im} T'$ we have $E' \circ I \supset \operatorname{im} T_i'$ $(i = 1, \dots n)$. Since $E' \circ I$ is a closed linear subspace, it follows that $E' \circ I \supset \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}$.

For the reverse inclusion, let $S \in I$ and let $\lambda \in E'$. There are sequences

$$(R_1^{(j)})_j, \dots, (R_n^{(j)})_j \subset \mathcal{A}(E) + \mathbb{C}\,\mathrm{id}_E$$

such that

$$S = \lim_{j \to \infty} \left(R_1^{(j)} \circ T_1 + \cdots + R_n^{(j)} \circ T_n \right).$$

Then

$$\begin{aligned} \lambda \circ S &= \lim_{j \to \infty} \left(\lambda \circ (R_1^j \circ T_1) + \cdots + \lambda \circ (R_n^{(j)} \circ T_n) \right) \\ &= \lim_{j \to \infty} \left(T_1'(\lambda_1 \circ R_1^{(j)}) + \cdots + T_n'(\lambda \circ R_n^{(j)}) \right) \in \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}. \end{aligned}$$

As λ and S were arbitrary, this concludes the proof. $\square$

The next corollary gives a partial characterisation of when $\mathcal{A}(E)$ is topologically left Noetherian. The full characterisation will be given in Theorem 3.5.9.

COROLLARY 3.5.7. *Let E be a Banach space such that $\mathcal{A}(E)$ has a left approximate identity.*

(i) *Let $F \subset E$ be a closed linear subspace. Then $\mathscr{L}(F)$ is topologically generated by $T_1, \dots, T_n \in \mathcal{A}(E)$ if and only if*

$$F = \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}. \tag{3.11}$$

(ii) *The algebra is $\mathcal{A}(E)$ is topologically left Noetherian if and only if every closed linear subspace of E' has the form* (3.11), *for some $n \in \mathbb{N}$ and $T_1, \dots, T_n \in \mathcal{A}(E)$.*

Proof. (i) Suppose that $\mathscr{L}(F) = \overline{\mathcal{A}(E)T_1 + \cdots + \mathcal{A}(E)T_n}$, for some $T_1, \dots, T_n \in \mathcal{A}(E)$. Then by Lemma 3.5.6

$$E' \circ \mathscr{L}(F) = \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'},$$

so that, by Theorem 3.5.4, $F = \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}$.

Conversely, suppose that there are maps $T_1, \dots, T_n \in \mathcal{A}(E)$ such that F has the form (3.11). Consider the left ideal

$$I = \overline{\mathcal{A}(E)T_1 + \cdots + \mathcal{A}(E)T_n}.$$

By Lemma 3.5.6 we have $E' \circ I = F$, and so by Theorem 3.5.4 we have $I = \mathscr{L}(E' \circ I) = \mathscr{L}(F)$. Hence

$$\mathscr{L}(F) = \overline{\mathcal{A}(E)T_1 + \cdots + \mathcal{A}(E)T_n},$$

as required.

(ii) This is clear from (i) and Theorem 3.5.4. $\square$

In the proof of the next lemma we use the fact that every infinite-dimensional Banach space contains a basic sequence [**59**, Theorem 4.1.30].

Lemma 3.5.8. *Let E be a Banach space, and let $F \subset E'$ be a closed, separable linear subspace. Then there exists $T \in \mathcal{A}(E)$ such that $\overline{\operatorname{im} T'} = F$.*

PROOF. We may suppose that E is infinite-dimensional, since otherwise the lemma follows from routine linear algebra. Let $\{\lambda_n : n \in \mathbb{N}\}$ be a dense subset of B_F, and let (b_n) be a normalised basic sequence in E. Let $(\beta_n) \subset E'$ satisfy $\langle b_i, \beta_j \rangle = \mathbb{1}_{ij}$ $(i, j \in \mathbb{N})$. Define $T = \sum_{n=1}^{\infty} 2^{-n} b_n \otimes \lambda_n$. The operator T is a limit of finite-rank operators and

$$T'\varphi = \sum_{n=1}^{\infty} 2^{-n} \varphi(b_n)\lambda_n \quad (\varphi \in E').$$

Certainly $\overline{\operatorname{im} T'} \subset F$. Observing that $T'(2^i \beta_i) = \lambda_i$ $(i \in \mathbb{N})$, we see that $\overline{\operatorname{im} T'} = F$, as required. $\square$

We can now give our characterisation of topological left Noetherianity for $\mathcal{A}(E)$. We notice that our proof actually implies that for these Banach algebras topological left Noetherianity is equivalent to every closed left ideal being topologically singly generated.

THEOREM 3.5.9. *Let E be a Banach space such that $\mathcal{A}(E)$ has a left approximate identity. Then the following are equivalent:*

(a) *the Banach algebra $\mathcal{A}(E)$ is topologically left Noetherian;*

(b) *every closed left ideal of $\mathcal{A}(E)$ is topologically singly-generated;*

(c) *the space E' is separable.*

PROOF. It is trivial that (b) implies (a). To see that (c) implies (b), note that, by Theorem 3.5.4, every closed left ideal of $\mathcal{A}(E)$ has the form $\mathscr{L}(F)$, for some closed linear subspace F in E'. Fixing $F \in \mathbf{SUB}(E')$, by Lemma 3.5.8 there exists $T \in \mathcal{A}(E)$ such that $F = \overline{\operatorname{im} T'}$, which implies that $\mathscr{L}(F) = \overline{\mathcal{A}(E)T}$, by Corollary 3.5.7(i).

We show that (a) implies (c) to complete the proof. Suppose that $\mathcal{A}(E)$ is topologically left Noetherian. Then in particular $\mathcal{A}(E) = \overline{\mathcal{A}(E)T_1 + \cdots + \mathcal{A}(E)T_n}$ for some $T_1, \ldots, T_n \in \mathcal{A}(E)$. Observing that $\mathscr{L}(E') = \mathcal{A}(E)$, Lemma 3.5.6 implies that $E' = \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}$. Since each operator T_i is compact, so is each T_i', implying that each space $\operatorname{im} T_i'$ is separable. It follows that $E' = \overline{\operatorname{im} T_1' + \cdots + \operatorname{im} T_n'}$ is separable. $\square$

Remark. Let E_{AH} be the Banach space constructed by Argyros and Haydon in [**4**] with the property that $\mathcal{B}(E_{AH}) = \mathbb{C}\,\mathrm{id}_{E_{AH}} + \mathcal{K}(E_{AH})$. Since E_{AH} is a predual of ℓ^1, which has BAP, it satisfies the assumptions of Theorem 3.5.9 by Theorem 1.2.1(ii). Furthermore $\mathcal{A}(E_{AH}) = \mathcal{K}(E_{AH})$. Since $\mathcal{B}(E_{AH}) = \mathcal{K}(E_{AH})^\sharp$, Theorem 3.5.9 and Lemma 3.2.2(iii) implie that $\mathcal{B}(E_{AH})$ is $\|\cdot\|$-topologically left Noetherian.

We can give a very similar treatment of the closed right ideals of $\mathcal{A}(E)$; in fact this case is a little simpler. Observe that our hypothesis on $\mathcal{A}(E)$ changes from possessing a left approximate identity to possessing a right approximate identity. We denote the set of closed right ideals of a Banach algebra A by $\mathbf{CRI}(A)$.

THEOREM 3.5.10. *Let E be a Banach space such that $\mathcal{A}(E)$ has a right approximate identity. There is a lattice isomorphism $\Xi : (\mathbf{SUB}(E), \subset) \to (\mathbf{CRI}(E), \subset)$ given by*

$$\Xi : F \mapsto \mathscr{R}(F),$$

with inverse given by

$$\widehat{\Xi} : I \mapsto \overline{\mathrm{span}}_{T \in I}(\mathrm{im}\, T) \quad (I \in \mathbf{CRI}(\mathcal{A}(E))).$$

PROOF. It is clear that Ξ and $\widehat{\Xi}$ are inclusion preserving. Since a poset isomorphism between lattices preserves the lattice strucure, once we have shown that Ξ and $\widehat{\Xi}$ are mutually inverse it will follow that they are lattice isomorphisms.

Let F be a closed linear subspace of E and set $G = \widehat{\Xi}(\mathscr{R}(F))$. It is immediate from the definitions that $G \subset F$. Moreover, given $x \in F$, by considering $x \otimes \lambda$ for some $\lambda \in E \backslash \{0\}$ we see that $x \in G$. Hence $F = G$, and, since F was arbitrary, this shows that $\widehat{\Xi} \circ \Xi$ is the identity map.

Let I be a closed right ideal in $\mathcal{A}(E)$, and set $F = \widehat{\Xi}(I)$. It is clear that $I \subset \mathscr{R}(F)$. By Lemma 3.2.1 the finite-rank operators intersect $\mathscr{R}(F)$ densely, so in order to check the reverse inclusion it is sufficient to show that $\mathcal{F}(E) \cap \mathscr{R}(F) \subset I$. Let $T \in \mathcal{F}(E) \cap \mathscr{R}(F)$. Then we can write $T = \sum_{i=1}^n x_i \otimes \lambda_i$, for some $n \in \mathbb{N}$, some

$x_1, \ldots, x_n \in \operatorname{im} T$ and some $\lambda_1, \ldots, \lambda_n \in E'$. Fix $i \in \{1, \ldots, n\}$. Then $x_i \in F$ so there exists a sequence $(y_j) \subset \operatorname{span}_{U \in I}(\operatorname{im} U)$ such that $\lim_{j\to\infty} y_j = x_i$. Moreover, for each j we can write $y_j = S_1^{(j)} z_1 + \cdots + S_{k_j}^{(j)} z_{k_j}$, for some $k_j \in \mathbb{N}$, some $S_1^{(j)}, \ldots, S_{k_j}^{(j)} \in I$, and some $z_1, \ldots, z_{k_j} \in E$. For each j, and each $p = 1, \ldots, k_j$ we have $\left(S_p^{(j)} z_j\right) \otimes \lambda_i = S_p^{(j)} \circ (z_p \otimes \lambda_i) \in I$. Hence $y_j \otimes \lambda_i \in I$ for each j, so that, taking the limit as j goes to infinity, $x_i \otimes \lambda_i \in I$. As i was arbitrary it follows that $T \in I$. Hence we have shown that $I = \mathscr{R}(F)$. As I was arbitrary, we have shown that $\Xi \circ \widehat{\Xi}$ is the identity map. $\square$

Now we set out to characterise when $\mathcal{A}(E)$ is topologically right Noetherian, for E a Banach space as in Theorem 3.5.10.

Lemma 3.5.11. *Let E and $\widehat{\Xi}$ be as in Theorem 3.5.10. Let $T_1, \ldots, T_n \in \mathcal{A}(E)$ and let $I = \overline{T_1\mathcal{A}(E) + \cdots + T_n\mathcal{A}(E)}$. Then*

$$\widehat{\Xi}(I) = \overline{\operatorname{im} T_1 + \cdots + \operatorname{im} T_n}.$$

Proof. Since each T_i $(i = 1, \ldots, n)$ belongs to I we have $\overline{\operatorname{im} T_1 + \cdots + \operatorname{im} T_n} \subset \widehat{\Xi}(I)$. Let $x \in \widehat{\Xi}(I)$, and let $\varepsilon > 0$. Then, by the definition of $\widehat{\Xi}$, there exist $m \in \mathbb{N}$, $S_1, \ldots, S_m \in I$, and $y_1, \ldots, y_m \in E$ such that

$$\|x - (S_1 y_1 + \cdots + S_m y_m)\| < \varepsilon.$$

Since $T_1\mathcal{A}(E) + \cdots + T_n\mathcal{A}(E)$ is dense in I, we may in fact suppose that

$$S_1, \ldots, S_m \in T_1\mathcal{A}(E) + \cdots + T_n\mathcal{A}(E),$$

so that $S_1 y_1 + \cdots + S_m y_m \in \operatorname{im} T_1 + \cdots + \operatorname{im} T_n$. As ε was arbitrary we see that $x \in \overline{\operatorname{im} T_1 + \cdots + \operatorname{im} T_n}$. The result now follows. $\square$

Lemma 3.5.12. *Let E be a Banach space, and let F be any separable, closed linear subspace of E. Then there exists an approximable linear map from E to F with dense range.*

Proof. We may suppose that E is infinite-dimensional. Let $(x_n) \subset F$ be dense. Since E is infinite-dimensional, there exists a normalised basic sequence $(b_n) \subset E$. Let $(\beta_n) \subset E'$ be a bounded sequence satisfying $\langle b_i, \beta_j \rangle = \mathbb{1}_{ij}$ $(i, j \in \mathbb{N})$, which we can obtain by taking the coordinate functionals for (b_n) and extending them using the Hahn-Banach Theorem. Define $T: E \to F$ by $T = \sum_{n=1}^{\infty} 2^{-n} x_n \otimes \beta_n$. Then T is a limit of finite-rank operators, and $T(2^i b_i) = x_i$ $(i \in \mathbb{N})$ implies that T has dense range. □

Theorem 3.5.13. *Let E be a Banach space such that $\mathcal{A}(E)$ has a right approximate identity. Then the following are equivalent:*

(a) *the Banach algebra $\mathcal{A}(E)$ is topologically right Noetherian;*

(b) *every closed right ideal of $\mathcal{A}(E)$ is topologically singly-generated;*

(c) *the space E is separable.*

Proof. It is trivial that (b) implies (a). We show that (a) implies (c). Suppose that $\mathcal{A}(E)$ is topologically right Noetherian. Then $\mathcal{A}(E) = \mathscr{R}(E)$ is topologically finitely-generated so that, by Lemma 3.5.11, there exist $n \in \mathbb{N}$ and $T_1, \ldots, T_n \in \mathcal{A}(E)$ such that $E = \overline{\operatorname{im} T_1 + \cdots + \operatorname{im} T_n}$. Since each operator T_i $(i = 1, \ldots, n)$ is compact, its image is separable, and hence so is E.

Now suppose instead that E is separable, and let I be a closed right ideal in $\mathcal{A}(E)$. Then, by Theorem 3.5.10, $I = \mathscr{R}(F)$ for some $F \in \mathbf{SUB}(E)$. By Lemma 3.5.12 there exists $T \in \mathcal{A}(E)$ with $\overline{\operatorname{im} T} = F$. By Lemma 3.5.11 we have $\widehat{\Xi}\left(\overline{T\mathcal{A}(E)}\right) = \overline{\operatorname{im} T} = F$, so that, by Theorem 3.5.10, $I = \mathscr{R}(F) = \overline{T\mathcal{A}(E)}$. Since I was arbitrary, this shows that (c) implies (b). □

Remark. Consider $\mathcal{K}(\ell^1)$. Of course, $(\ell^1)' \cong \ell^\infty$, which has BAP by [**90**, Example 5(a), Chapter II E], so that $\mathcal{K}(\ell^1)$ has an approximate identity by Theorem 1.2.1(ii). By Theorem 3.5.9 and Theorem 3.5.13 $\mathcal{K}(\ell^1)$ is an example of a Banach algebra which is topologically right Noetherian, but not topologically left Noetherian.

3.6. Left Ideals of $\mathcal{B}(E)$, Closed in Various Topologies

In this section we consider left ideals of the Banach algebra $\mathcal{B}(E)$, for E a Banach space, which are closed in either the strong operator (SOP) topology or, in the case that $\mathcal{B}(E)$ is a dual Banach algebra, the weak*-topology. We first classify the SOP-closed left ideals of $\mathcal{B}(E)$, for E an arbitrary Banach space (Theorem 3.6.2), and show that SOP-topological left Noetherianity of $\mathcal{B}(E)$ is equivalent to asking that every closed linear subspace of E can be realised as the intersection of the kernels of finitely many bounded linear operators on E (Corollary 3.6.5). We then recall that $\mathcal{B}(E)$ is a dual Banach algebra whenever E is reflexive, with predual $E\widehat{\otimes}E'$, and we observe that, when E also has the approximation property, results from Section 3.4 and Section 3.5 give a classification of the weak*-closed left ideals of $\mathcal{B}(E)$ (Theorem 3.6.7). Finally, we give an example of a dual Banach algebra of the form $\mathcal{B}(E)$ which fails to be weak*-topologically left Noetherian (Theorem 3.6.12).

We begin with our classification of the SOP-closed left ideals in $\mathcal{B}(E)$ which states that, given a Banach space E, these left ideals are exactly the left ideals

$$\mathscr{I}(F) = \{T \in \mathcal{B}(E) : \ker T \supset F\}$$

defined in (3.9), as F runs through the closed linear subspaces of F. It is routinely checked that each set $\mathscr{I}(F)$ is a left ideal, and it is SOP-closed since it is the intersection of the kernels of the SOP-continuous maps $\mathcal{B}(E) \to E$ given by $T \mapsto Tx$, as x runs through the elements of F.

LEMMA 3.6.1. *Let E be a Banach space. Let I be a left ideal in $\mathcal{B}(E)$, and let $F = \bigcap_{T\in I} \ker T$. Then I acts algebraically irreducibly on E/F via*

$$T \cdot (x + F) = Tx + F \quad (T \in I).$$

PROOF. The action is well defined since, by definition, $\ker T \supset F$ for every $T \in I$. Let $x \in E\backslash F$. Then, again by the definition of F, there exists $T \in I$ such that $Tx \neq 0$.

Let $y \in E$, and let $\eta \in E'$ satisfy $\langle Tx, \eta\rangle = 1$. Define $S = (y \otimes \eta) \circ T \in I$ and observe that $Sx = y$, and hence $S \cdot (x + F) = y + F$. It follows that the action of I on E/F is algebraically irreducible. $\square$

THEOREM 3.6.2. *Let E be a Banach space. The map Ω from $\mathbf{SUB}(E)$ to the set of* SOP*-closed left ideals of $\mathcal{B}(E)$ given by*

$$\Omega\colon F \mapsto \mathscr{I}(F) \quad (F \in \mathbf{SUB}(E))$$

is a lattice anti-isomorphism, with inverse given by

$$\widehat{\Omega}\colon I \mapsto \bigcap_{T \in I} \ker T,$$

for I a SOP*-closed left ideal of $\mathcal{B}(E)$.*

PROOF. We write $q\colon E \to E/F$ for the quotient map. It is clear that Ω and $\widehat{\Omega}$ are anti-homomorphisms of posets. Hence once we have show that they are bijections it will follows that they are lattice anti-isomorphisms. We first show that $\widehat{\Omega} \circ \Omega$ is the identity map. Indeed, let $F \in \mathbf{SUB}(E)$. Then by definition $F \subset \bigcap_{T \in \mathscr{I}(F)} \ker T$. Suppose that $x \in E \backslash F$, and let $\eta \in (E/F)'$ satisfy $\langle q(x), \eta\rangle = 1$. Then $x \otimes q'(\eta) \in \mathscr{I}(F)$ but

$$[x \otimes q'(\eta)](x) = x \neq 0,$$

so that $x \notin \bigcap_{T \in \mathscr{I}(F)} \ker T$. Hence we must have $F = \bigcap_{T \in \mathscr{I}(F)} \ker T$, as required.

It remains to prove that $\Omega \circ \widehat{\Omega}$ is the identity map. Let I be a SOP-closed left ideal in $\mathcal{B}(E)$, and let $F = \bigcap_{T \in I} \ker T$. We must show that $I = \mathscr{I}(F)$. Clearly $I \subset \mathscr{I}(F)$, so that it remains to show the reverse inclusion. To this end let $S \in \mathscr{I}(F)$ be arbitrary. We shall show that there exist nets $(R_\alpha) \subset \mathcal{F}(E)$ and $(T_\alpha) \subset I$ such that $\lim_{\mathrm{SOP}, \alpha} R_\alpha \circ T_\alpha = S$. The indexing set of the nets will be the collection of non-zero, finite-dimensional subspaces of E/F.

Let $\alpha \neq \{0\}$ be a finite-dimensional subspace of E/F, of dimension n say, and let $z_1, \ldots, z_n$ be such that $\{z_1 + F, \ldots, z_n + F\}$ is a basis for α. Since, by Lemma 3.6.1,

I acts irreducibly on E/F, by [**62**, Theorem 4.2.13] there exists $T_\alpha \in I$ such that $\{T_\alpha(z_1) + F, \dots, T_\alpha(z_n) + F\}$ is linearly independent. By the Hahn–Banach Theorem there exist $\eta_{\alpha,1}, \dots, \eta_{\alpha,n} \in E'$ such that

$$\langle T_\alpha z_i, \eta_{\alpha,j} \rangle = \mathbb{1}_{i,j} \quad (i, j = 1, \dots, n).$$

Define

$$R_\alpha = \sum_{i=1}^n S(z_i) \otimes \eta_{\alpha,i}.$$

Then $(R_\alpha \circ T_\alpha)(z_i) = S(z_i)$ $(i = 1, \dots, n)$. Since $\ker R_\alpha \circ T_\alpha, \ker S \supset F$, it follows that

$$(R_\alpha \circ T_\alpha)|_{q^{-1}(\alpha)} = S|_{q^{-1}(\alpha)}\,. \tag{3.12}$$

Let $z \in E$ and let $\alpha_0 = \operatorname{span}\{z + F\}$. Then whenever $\alpha \supset \alpha_0$ is a finite-dimensional linear subspace of E/F we have $(R_\alpha \circ T_\alpha)(z) = S(z)$ by (3.12). Hence $\lim_\alpha (R_\alpha \circ T_\alpha)(z) = S(z)$, so that $S = \lim_{\mathrm{SOP},\,\alpha}(R_\alpha \circ T_\alpha) \in I$, as required. $\square$

Given a Banach space E, it seems a natural question to ask which of the closed linear subspaces of E can be realised as the kernel of some bounded linear operator $E \to E$. We show that this question can be rephrased in terms of the SOP-closed ideals of $\mathcal{B}(E)$.

LEMMA 3.6.3. *Let E be a Banach space, let $n \in \mathbb{N}$, and let $T_1, \dots, T_n \in \mathcal{B}(E)$. Set*

$$I = \overline{\mathcal{B}(E)T_1 + \cdots + \mathcal{B}(E)T_n}^{\mathrm{SOP}}.$$

Then $\bigcap_{T \in I} \ker T = \bigcap_{i=1}^n \ker T_i$.

PROOF. Let $F = \bigcap_{i=1}^n \ker T_i$ and let $G = \bigcap_{T \in I} \ker T$. Since each T_i $(i = 1, \dots, n)$ belongs to I, $F \supset G$. Since evaluation at any point in E is SOP-continuous, $\ker T \supset F$ for each $T \in I$, so that $F = G$. $\square$

Proposition 3.6.4. *Let E be a Banach space and let F be a closed linear subspace of E. The left ideal $\mathscr{I}(F)$ is* SOP*-topologically generated by operators $T_1, \dots, T_n \in \mathcal{B}(E)$ if and only if $\bigcap_{i=1}^{n} \ker T_i = F$.*

Proof. First suppose that $\mathscr{I}(F) = \overline{\mathcal{B}(E)T_1 + \cdots + \mathcal{B}(E)T_n}^{\mathrm{SOP}}$. By Theorem 3.6.2, $F = \bigcap_{T \in \mathscr{I}(F)} \ker T$, so that, by Lemma 3.6.3, $F = \bigcap_{i=1}^{n} \ker T_i$.

Now suppose instead that we have $T_1, \dots, T_n \in \mathcal{B}(E)$ with $\bigcap_{i=1}^{n} \ker T_i = F$. Then $J := \overline{\mathcal{B}(E)T_1 + \cdots + \mathcal{B}(E)T_n}^{\mathrm{SOP}}$ is a SOP-closed left ideal with $\bigcap_{T \in J} \ker T = F$, by Lemma 3.6.3. Hence, by Theorem 3.6.2, $J = \mathscr{I}(F)$, as required. □

We now give our characterisation of SOP-topological left Noetherianty of $\mathcal{B}(E)$.

Corollary 3.6.5. *Let E be a Banach space.*

(i) *The Banach algebra $\mathcal{B}(E)$ has the property that every SOP-closed left ideal is SOP-topologically generated by a single element if and only if every closed linear subspace of E can be realised as the kernel of some operator in $\mathcal{B}(E)$.*

(ii) *The Banach algebra $\mathcal{B}(E)$ is SOP-topologically left Noetherian if and only if, given a closed linear subspace F of E, there exist $n \in \mathbb{N}$ and $T_1, \dots, T_n \in \mathcal{B}(E)$ such that $\bigcap_{i=1}^{n} \ker T_i = F$.*

Proof. This follows from Proposition 3.6.4 and Theorem 3.6.2. □

We now turn our attention to reflexive Banach spaces E and consider the Banach algebra $\mathcal{B}(E)$ with its weak*-topology. First of all we observe that our earlier work gives us many examples of these algebras which are weak*-topologically left and right Noetherian.

Proposition 3.6.6. *Let E be a separable, reflexive Banach space with the approximation property. Then the dual Banach algebra $\mathcal{B}(E)$ is weak*-topologically left and right Noetherian.*

Proof. By Corollary 3.4.9, the Banach algebra $\mathcal{A}(E) = \mathcal{K}(E)$ is an Ülger algebra for such Banach spaces. Hence, by Proposition 3.4.12, its multiplier algebra, which may be identified with $\mathcal{B}(E)$, is weak*-topologically left Noetherian whenever $\mathcal{K}(E)$ is $\|\cdot\|$-topologically left Noetherian. Since $\mathcal{K}(E)$ is an Ülger algebra it has an approximate identity [2], so that it is $\|\cdot\|$-topologically left Noetherian whenever E', or equivalently E, is separable by Theorem 3.5.9. Similarly, $\mathcal{B}(E)$ is weak*-topologically right Noetherian by Theorem 3.5.13. □

Remark. We observe that this corollary cannot be strengthened to an "if and only if" statement because $\mathcal{B}(H)$ is always weak*-topologically left Noetherian for any Hilbert space H, as is any von Neumann algebra [**83**, Proposition 3.12].

Our earlier work also allows us to classify the weak*-closed left and right ideals for these algebras. We note that for any reflexive Banach space E, possibly without the approximation property, and for any closed linear subspace $F \subset E$ the left ideal $\mathscr{I}_{\mathcal{B}(E)}(F)$ is weak*-closed since we have

$$\mathscr{I}_{\mathcal{B}(E)}(F) = \{x \otimes \lambda : x \in F,\ \lambda \in E'\}^{\perp},$$

where $x \otimes \lambda$ denotes an element of the predual $E \widehat{\otimes} E'$. Similarly we have

$$\mathscr{R}_{\mathcal{B}(E)}(F) = \{x \otimes \lambda : x \in E,\ \lambda \in F^{\perp}\}^{\perp},$$

so that these right ideals are weak*-closed.

Theorem 3.6.7. *Let E be a reflexive Banach space with the approximation property. Then the weak*-closed left ideals are exactly given by $\mathscr{I}_{\mathcal{B}(E)}(F)$, as F runs through* $\mathbf{SUB}(E)$*. The weak*-closed right ideals are given by $\mathscr{R}_{\mathcal{B}(E)}(F)$, as F runs through* $\mathbf{SUB}(E)$.

[2]Alternatively this can be seen directly as follows: since E has is reflexive with AP, it has BAP, implying that E' has BAP. Now we may apply Theorem 1.2.1

PROOF. By Corollary 3.4.9, $\mathcal{K}(E)$ is a strongly Ülger algebra for such Banach spaces. Hence, by Proposition 3.4.15, there is a bijection Λ from the set of weak*-closed left ideals of $\mathcal{B}(E)$ to $\mathbf{CLI}(\mathcal{K}(E))$ given by

$$\Lambda\colon I \mapsto I \cap \mathcal{K}(E).$$

For each $F \in \mathbf{SUB}(E)$, the ideal $\mathscr{I}_{\mathcal{B}(E)}(F)$ is weak*-closed by the remarks preceding the theorem. Clearly

$$\mathscr{I}_{\mathcal{B}(E)}(F) \cap \mathcal{K}(E) = \mathscr{I}_{\mathcal{K}(E)}(F) \quad (F \in \mathbf{SUB}(E)),$$

so that, by Corollary 3.5.5, the map Λ is surjective when restricted to the set

$$\{\mathscr{I}_{\mathcal{B}(E)}(F) : F \in \mathbf{SUB}(E)\}.$$

Since Λ is a bijection, this forces this set to be the full set of weak*-closed left ideals of $\mathcal{B}(E)$. This concludes the proof of the result about weak*-closed left ideals.

A similar argument, using Theorem 3.5.10, gives the result about weak*-closed right ideals. Alternatively, one could use the fact that $\mathcal{B}(E)$ and $\mathcal{B}(E')$ are anti-isomorphic as dual Banach algebras via $T \mapsto T'$, so that the weak*-closed right ideals of $\mathcal{B}(E)$ correspond the weak*-closed left ideals on $\mathcal{B}(E')$. $\square$

Remark. By Theorem 3.6.2 and the previous theorem, if E is a reflexive Banach space with AP, then the weak*-closed and SOP-closed left ideals of $\mathcal{B}(E)$ coincide. We know of no abstract proof of this fact that avoids simply classifying both types of left ideals and observing that they are the same.

We now give an example of a dual Banach algebra of the form $\mathcal{B}(E)$ which is not weak*-topologically left Noetherian. I must thank my doctoral supervisor Niels Laustsen for pointing out this example of a Banach space, and for the subsequent discussion that lead to the proof of Theorem 3.6.12. The Banach space in question will be the dual of a certain space that we denote by E_W, which is an example due to Wark [**86**], who adapted a construction of Shelah and Steprāns [**78**]. The Banach space E_W is a reflexive Banach space with the property that it is non-separable but

$$\mathcal{B}(E_W) = \mathbb{C}\,\mathrm{id}_{E_W} + \mathscr{X}(E_W), \tag{3.13}$$

where $\mathscr{X}(E_W)$ denotes the set of operators on E_W with separable range. We would like to thank Hugh Wark for pointing out to us that the space E_W has AP.

LEMMA 3.6.8. *The Banach space E_W has the approximation property. Hence so does E_W', and moreover $\mathcal{A}(E_W')$ is a strongly Ülger algebra.*

PROOF. The space E_W has a transfinite basis, and as such has AP by [**71**]. To see the second statement apply Corollary 3.4.9. □

We recall some notions from Banach space theory that we shall require in what follows. Let E be a Banach space. A *biorthogonal system* in E is a set

$$\{(x_\gamma, \lambda_\gamma) : \gamma \in \Gamma\} \subset E \times E',$$

for some indexing set Γ, with the property that

$$\langle x_\alpha, \lambda_\beta \rangle = \mathbb{1}_{\alpha,\beta} \quad (\alpha, \beta \in \Gamma).$$

A biorthogonal system $\{(x_\gamma, \lambda_\gamma) : \gamma \in \Gamma\}$ is said to be *bounded* if

$$\sup\{\|x_\gamma\|, \|\lambda_\gamma\| : \gamma \in \Gamma\} < \infty.$$

A *Markushevich basis* for a Banach space E is a biorthogonal system $\{(x_\gamma, \lambda_\gamma) : \gamma \in \Gamma\}$ in E such that $\{\lambda_\gamma : \gamma \in \Gamma\}$ separates the points of E and such that $\overline{\mathrm{span}}\{x_\gamma : \gamma \in \Gamma\} = E$. For an in-depth discussion of Markushevich bases see [**42**], in which a Markushevich basis is referred to as an "M-basis".

LEMMA 3.6.9. *Let E be a Banach space containing an uncountable, bounded biorthogonal system. Then E contains a closed linear subspace F such that both F and E/F are non-separable*

PROOF. Let $\{(x_\gamma, \lambda_\gamma) : \gamma \in \Gamma\}$ be an uncountable, bounded biorthogonal system in E. Since Γ is uncountable, it has an uncountable subset Γ_0 such that $\Gamma\backslash\Gamma_0$ is also uncountable. Set $F = \overline{\operatorname{span}}\{x_\gamma : \gamma \in \Gamma_0\}$, and set $C = \sup\{\|x_\gamma\|, \|\lambda_\gamma\| : \gamma \in \Gamma\}$. The subspace F is non-separable since $\{x_\gamma : \gamma \in \Gamma_0\}$ is an uncountable set satisfying

$$\|x_\alpha - x_\beta\| \geqslant \frac{1}{C}|\langle x_\alpha - x_\beta, \lambda_\alpha\rangle| = \frac{1}{C} \quad (\alpha, \beta \in \Gamma_0, \alpha \neq \beta).$$

Let $q\colon E \to E/F$ denote the quotient map. It is well known that the dual map $q'\colon (E/F)' \to E'$ is an isometry with image equal to $F^\perp$. Each functional λ_γ, for $\gamma \notin \Gamma_0$, clearly belongs to $F^\perp$ so that, for each $\gamma \in \Gamma\backslash\Gamma_0$ there exists $g_\gamma \in (E/F)'$ such that $q'(g_\gamma) = \lambda_\gamma$ and such that $\|g_\gamma\| = \|\lambda_\gamma\|$. We now see that $\{q(x_\gamma) : \gamma \in \Gamma\backslash\Gamma_0\}$ is an uncountable $1/C$-separated subset of E/F because

$$\begin{aligned}\|q(x_\alpha) - q(x_\beta)\| &\geqslant \frac{1}{C}|\langle q(x_\alpha) - q(x_\beta), g_\alpha\rangle| = \frac{1}{C}|\langle x_\alpha - x_\beta, q'(g_\alpha)\rangle| \\ &= \frac{1}{C}|\langle x_\alpha - x_\beta, \lambda_\alpha\rangle| = \frac{1}{C}.\end{aligned}$$

It follows that E/F is non-separable. □

LEMMA 3.6.10. *Let E be a non-separable, reflexive Banach space. Then E contains a closed linear subspace F such that both F and E/F are non-separable.*

PROOF. By [**42**, Theorem 5.1] every reflexive Banach space has a Markushevich basis. By [**41**, Theorem 5] it follows that every reflexive Banach space has a bounded Markushevich basis, so that, in particular, E has a bounded Markushevich basis, say $\{(x_\gamma, f_\gamma) : \gamma \in \Gamma\}$. (Please note that a flawed proof of this theorem was given in [**66**] and [**42**].) Since, by the definition of a Markushevich basis, $\overline{\operatorname{span}}\{x_\gamma : \gamma \in \Gamma\} = E$, and E is non-separable, the set $\{(x_\gamma, f_\gamma) : \gamma \in \Gamma\}$ must be uncountable. In particular this set is a bounded, uncountable biorthogonal system in E, so the result now follows from Lemma 3.6.9. □

PROPOSITION 3.6.11. *Let F be a subspace of E_W with the property that both F and E_W/F are non-separable. Then F cannot be written as $\overline{\operatorname{im} T_1 + \cdots + \operatorname{im} T_n}$, for any $n \in \mathbb{N}$ and $T_1, \ldots, T_n \in \mathcal{B}(E_W)$.*

PROOF. Assume towards a contradiction that there exists $n \in \mathbb{N}$ and there exist $T_1, \ldots, T_n \in \mathcal{B}(E_W)$ such that $F = \overline{\operatorname{im} T_1 + \cdots + \operatorname{im} T_n}$. By (3.13) there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $S_1, \ldots, S_n \in \mathscr{X}(E_W)$ such that

$$T_i = \alpha_i \mathrm{id}_{E_W} + S_i \quad (i = 1, \ldots, n).$$

If every α_i equals zero, then $F = \overline{\operatorname{im} S_1 + \cdots + \operatorname{im} S_n}$, which is separable, contradicting our assumption on F. Hence, without loss of generality, we may assume that $\alpha_1 \neq 0$. Let $x \in E_W$. Then $T_1 x = \alpha_1 x + S_1 x$, implying that

$$x = \frac{1}{\alpha_1}(T_1 x - S_1 x) \in F + \overline{\operatorname{im} S_1}.$$

As x was arbitrary, it follows that $E_W = F + \overline{\operatorname{im} S_1}$, so that

$$E_W/F = \frac{\left(F + \overline{\operatorname{im} S_1}\right)}{F} \cong \frac{\overline{\operatorname{im} S_1}}{\left(\overline{\operatorname{im} S_1} \cap F\right)}.$$

This implies that E_W/F is separable, and this contradiction completes the proof. □

We can now prove our theorem.

THEOREM 3.6.12. *The dual Banach algebra $\mathcal{B}(E_W')$ is not weak*-topologically left Noetherian.*

PROOF. Let F be a closed linear subspace of E_W such that both F and E_W/F are non-separable, which exists by Lemma 3.6.10. Observe that, $\mathscr{I}(F^\perp)$ is a weak*-closed left ideal of $\mathcal{B}(E_W')$ by Lemma 3.6.8 and Theorem 3.6.7. We shall show that this ideal fails to be weak*-topologically finitely-generated. Assume towards a contradiction that there exist $n \in \mathbb{N}$ and $T_1, \ldots, T_n \in \mathcal{B}(E'_W)$ such that

$$\mathscr{I}(F^\perp) = \overline{\mathcal{B}(E'_W)T_1 + \cdots + \mathcal{B}(E'_W)T_n}^{w^*}.$$

Using an almost identical argument to that given in the proof of Lemma 3.6.3, it is then easily checked that

$$\bigcap_{T \in \mathscr{I}(F^\perp)} \ker T = \bigcap_{i=1}^{n} \ker T_i.$$

This implies that $F^\perp = \bigcap_{i=1}^n \ker T_i$ (by, for example, Theorem 3.6.2). When we identify E''_W with E_W, this is equivalent to the statement that

$$F = \overline{\operatorname{im} T'_1 + \cdots + \operatorname{im} T'_n},$$

by (3.10) and Mazur's Theorem. However, this cannot occur by Proposition 3.6.11. $\square$

Remark. This is the only example that we know of a dual Banach algebra which is not weak*-topologically left Noetherian. It would be interesting to know if there are examples of the form $M(G)$ or $B(G)$, for a locally compact group G.

CHAPTER 4

The Radical of the Bidual of a Beurling Algebra

4.1. Introduction

In this chapter we study the Jacobson radical $\operatorname{rad}(\ell^1(G,\omega)'', \Box)$, for G a discrete group, and ω a weight on G. The chapter is based on [**89**]. The focus will be on the cases where either $\omega = 1$, in which case we are in fact studying the bidual of the group algebra $\ell^1(G)$, or where the weight is non-trivial but $G = \mathbb{Z}$. Our main results will be solutions to two questions posed by Dales and Lau in [**23**].

The study of the radicals of the biduals of Banach algebras goes back at least to Civin and Yood's paper [**16**], where it was shown that if G is either a locally compact, non-discrete, abelian group, or a discrete, soluble, infinite group, then $\operatorname{rad}(L^1(G)'') \neq \{0\}$. Civin and Yood's results have since been extended to show that $\operatorname{rad}(L^1(G)'')$ is not only non-zero, but non-separable, whenever G is discrete and amenable ([**36**], [**64**, 7.31(iii)]) or non-discrete [**37**]. The study has not been restricted to those Banach algebras coming from abstract harmonic analysis. One particularly striking result is a theorem of Daws and Read [**29**] which states that, for $1 < p < \infty$, the algebra $\mathcal{B}(\ell^p)''$ is semisimple if and only if $p = 2$.

A study of $\operatorname{rad}(\ell^1(G,\omega)'')$ for G a discrete group and ω a weight on G was undertaken by Dales and Lau in [**23**]. In the list of open problems at the end of their memoir the authors ask whether $\ell^1(\mathbb{Z},\omega)''$ can ever be semisimple [**23**, Chapter 14, Question 6]. In Section 4.3 we shall prove that the answer to this question is negative:

Theorem 4.1.1. *Let ω be a weight on $\mathbb{Z}$. Then* $\operatorname{rad}(\ell^1(\mathbb{Z},\omega)'') \neq \{0\}$.

A key observation of Civin and Yood (see [**16**, Theorem 3.1]) is that, for an amenable group G, the difference of any two invariant means on $\ell^\infty(G)$ always belongs to the radical of $\ell^1(G)''$, and this idea is what lies behind many of the subsequent results mentioned above. Note that the set of invariant means on an infinite, amenable group G is known to have cardinality $2^{2^{|G|}}$ [**64**, Corollary (7.8)]. Dales and Lau developed a weighted version of this argument in [**23**, Theorem 8.27], and invariant means are also at the centre of our proof of Theorem 4.1.1.

In each of the works [**16**], [**36**] and [**37**], whenever an element of the radical of the bidual of some group algebra is constructed it is nilpotent of index 2. This is an artifact of the method of invariant means. Moreover, it follows from [**23**, Proposition 2.16] and [**23**, Theorem 8.11] that, for a discrete group G, if ω is a weight on G such that $\ell^1(G,\omega)$ is semisimple and Arens regular, then $\operatorname{rad}(\ell^1(G,\omega)'')^{\Box 2} = \{0\}$. To see that this is a large class of examples consider [**23**, Theorem 7.13] and [**23**, Theorem 8.11]. In [**23**, Chapter 14, Question 3], Dales and Lau ask, amongst other things, whether or not we always have $\operatorname{rad}(L^1(G)'')^{\Box 2} = \{0\}$, for G a locally compact group. It also seems that until now it was not known whether or not $\operatorname{rad}(L^1(G,\omega)'')$ is always nilpotent, for G a locally compact group and ω a weight on G, although there is an example of a weight on $\mathbb{Z}$ in [**23**, Example 9.15] for which this radical cubes to zero, but has non-zero square. In Section 4.4 we shall answer both of these questions in the negative by proving the following:

THEOREM 4.1.2. *Let* $G = \oplus_{i=1}^{\infty}\mathbb{Z}$. *Then* $\operatorname{rad}(\ell^1(G)'')$ *contains nilpotent elements of every index.*

Here we understand $\oplus_{i=1}^{\infty}\mathbb{Z}$ to consist of integer sequences which are eventually zero, so that our example is a countable abelian group.

We note that by a theorem of Grabiner [**35**], Theorem 4.1.2 implies that the radical of $\ell^1(\oplus_{i=1}^{\infty}\mathbb{Z})''$ contains non-nilpotent elements. In Section 4.5, we obtain a similar result on $\mathbb{Z}$, but this time involving a weight.

THEOREM 4.1.3. *There exists a weight* ω *on* $\mathbb{Z}$ *such that* $\operatorname{rad}(\ell^1(\mathbb{Z},\omega)'')$ *contains non-nilpotent elements.*

In the light of Theorem 4.1.2, it would be interesting to ask whether or not there exists any locally compact group G for which $\operatorname{rad}(L^1(G)'')^{\Box 2} = \{0\}$. We do not know of such a group, and we are unable to say whether or not this happens for $G = \mathbb{Z}$.

4.2. Repeated Limit Notation

In this short section we fix some notation relating to repeated limits. This will be useful to us in this chapter, for instance, when we are considering powers of some element of a bidual of a Banach algebra, which has been defined as the weak*-limits of some net in the Banach algebra itself. Let X and Y be topological spaces, let I be a directed set, and let $\mathcal{U}$ be a filter on I. Let $(x_\alpha)_{\alpha\in I}$ be a net in X, let $r \in \mathbb{N}$, and let $f : X^r \to Y$ be a function. Then we define

$$\lim_{\underline{\alpha}\to\mathcal{U}}{}^{(r)} f(x_{\alpha_1}, \ldots, x_{\alpha_r}) = \lim_{\alpha_1\to\mathcal{U}} \cdots \lim_{\alpha_r\to\mathcal{U}} f(x_{\alpha_1}, \ldots, x_{\alpha_r}),$$

whenever the repeated limit exists. We define

$$\limsup_{\underline{\alpha}\to\mathcal{U}}{}^{(r)} f(x_{\alpha_1}, \ldots, x_{\alpha_r})$$

analogously. Suppose now that we have two directed sets I and J and two filters: $\mathcal{U}$ on I and $\mathcal{V}$ on J. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\beta)_{\beta\in J}$ be two nets in X, let $r \in \mathbb{N}$, and let $f : X^{2r} \to Y$. Then we define

$$\lim_{\underline{\alpha}\to\mathcal{U},\,\underline{\beta}\to\mathcal{V}}{}^{(r)} f(x_{\alpha_1}, y_{\beta_1}, \ldots, x_{\alpha_r}, y_{\beta_r}) =$$

$$\lim_{\alpha_1\to\mathcal{U}} \lim_{\beta_1\to\mathcal{V}} \cdots \lim_{\alpha_r\to\mathcal{U}} \lim_{\beta_r\to\mathcal{V}} f(x_{\alpha_1}, y_{\beta_1}, \ldots, x_{\alpha_r}, y_{\beta_r}),$$

whenever the limit exists. It is important to note that the choice of directed set in the above repeated limit alternates. In expressions of the form $\lim_{\underline{\alpha}\to\infty}{}^{(r)} f(x_{\alpha_1}, \ldots, x_{\alpha_r})$ the symbol '∞' is understood to represent the Fréchet filter on the directed set.

4.3. $\ell^1(\mathbb{Z},\omega)''$ is Not Semisimple

In this section we shall prove Theorem 4.1.1. Throughout ω will be a weight on $\mathbb{Z}$, and we shall write $A_\omega = \ell^1(\mathbb{Z},\omega)$. We shall write $\rho_\omega = \lim_{n\to\infty} \omega_n^{1/n}$, and recall that $\rho_\omega = \inf_{n\in\mathbb{N}} \omega_n^{1/n}$ [**19**, Proposition A.1.26(iii)]. In an abuse of notation, we shall write $1 \in \ell^\infty(\mathbb{Z})$ for the sequence which is constantly 1. Note that this is the augmentation character (see Section 2.1) when regarded as an element of A_ω'. We define

$$I_\omega = \{\Lambda \in A_\omega'' : \delta_n \Box \Lambda = \Lambda \ (n \in \mathbb{Z}),\ \langle \Lambda, 1\rangle = 0\}.$$

By [**23**, Proposition 8.23] I_ω is an ideal of A_ω'', satisfying $I_\omega^{\Box 2} = \{0\}$, so that $I_\omega \subset \operatorname{rad}(A_\omega'')$. Our strategy will be to reduce to a setting in which we can show that $I_\omega \neq \{0\}$. Our argument is an adaptation of [**23**, Theorem 8.27].

Let $\Lambda \in \ell^\infty(\mathbb{Z},1/\omega)'$. We say that Λ is *positive*, written $\Lambda \geqslant 0$, if $\langle \Lambda, f\rangle \geqslant 0$ whenever $f \geqslant 0$ ($f \in \ell^\infty(\mathbb{Z},1/\omega)$), and we say that Λ is a *mean* if $\Lambda \geqslant 0$ and $\|\Lambda\| = 1$. We say that a mean $\Lambda \in \ell^\infty(\mathbb{Z},1/\omega)'$ is an *invariant mean* if $\delta_n \Box \Lambda = \Lambda$ ($n \in \mathbb{Z}$).

LEMMA 4.3.1. *Let ω be a weight on $\mathbb{Z}$ and let $\Lambda \in \ell^\infty(\mathbb{Z},1/\omega)'$ be positive. Then $\|\Lambda\| = \langle \Lambda, \omega\rangle$.*

PROOF. This follows by considering the positive isometric Banach space isomorphism $T\colon \ell^\infty(\mathbb{Z},1/\omega) \to \ell^\infty(\mathbb{Z})$ given by $T(f) = f/\omega$ ($f \in \ell^\infty(\mathbb{Z},1/\omega)$), and then using the facts that the formula holds in the C*-algebra $\ell^\infty(\mathbb{Z})$ and that $T(\omega) = 1$. $\Box$

In what follows, given $E \subset \mathbb{N}$ we denote the complement of E by E^c.

LEMMA 4.3.2. *Let ω be a weight on $\mathbb{Z}$, and suppose that $\rho_\omega = 1$. Then there exist at least two distinct invariant means Λ and M on $\ell^\infty(\mathbb{Z},1/\omega)$ such that $\langle \Lambda, 1\rangle = \langle M, 1\rangle$.*

PROOF. Since $\inf_{n\in\mathbb{N}} \omega_n^{1/n} = 1$, Lemma 2.6.2(i) implies that the sequence $(\omega_n)_{n\in\mathbb{N}}$ is not tail-preserving. Hence, by Proposition 2.6.1, there exists a strictly increasing sequence (n_k) of integers such that $n_0 = 0, n_1 = 1$ and such that

$$\lim_{k\to\infty} \omega_{n_k}/(\omega_0 + \cdots + \omega_{n_k}) = 0. \tag{4.1}$$

(In fact, an inspection of the proof of Lemma 2.6.2(i) shows that this fact follows directly from the calculation performed there, and so we do not really need Proposition 2.6.1 here). By passing to a subsequence if necessary we may suppose that

$$\lim_{k\to\infty}(n_k+1)/(\omega_0+\cdots+\omega_{n_k})$$

exists.

Set $C_k=\omega_0+\cdots+\omega_{n_k}$, and define $\Lambda_k=\frac{1}{C_k}(\delta_0+\cdots+\delta_{n_k})$; we regard each Λ_k as an element of A''_ω. Notice that, for each fixed $i\in\mathbb{N}$, we have

$$\lim_{k\to\infty} C_i/C_k=0. \tag{4.2}$$

We shall first show that the sequence (Λ_k) does not converge when considered as a sequence in A''_ω with the weak*-topology. This will then allow us to use two different ultrafilters in such a way as to obtain distinct limits of (Λ_k), and these limits will turn out to be our invariant means. To achieve this, we shall inductively construct a function $\psi\colon\mathbb{Z}\to\mathbb{C}$ and choose non-negative integers

$$s_1<t_1<s_2<t_2<\cdots<s_k<t_k<\cdots$$

such that

$$|\langle\Lambda_{s_j},\psi\rangle|<\frac{1}{4},\qquad |\langle\Lambda_{t_j},\psi\rangle|>\frac{3}{4}\quad (j\in\mathbb{N}) \tag{4.3}$$

and

$$0\leqslant\psi(i)\leqslant\omega_i+1\quad (i\in\mathbb{Z}). \tag{4.4}$$

Since (4.4) ensures that $\psi\in\ell^\infty(\mathbb{Z},1/\omega)$, this will indeed show that (Λ_k) is weak*-divergent. We set $s_1=0$ and $t_1=1$, and define $\psi(i)=0$ $(i\leqslant 0)$ and $\psi(1)=C_1$, and observe that this ensures that (4.3) holds for $j=1$, and that (4.4) holds for all $i\leqslant 1$.

Now assume inductively that we have found $s_1 < t_1 < \cdots < s_k < t_k$, and defined ψ up to n_{t_k} in such a way that (4.3) holds for $j = 1, \ldots, k$, and such that (4.4) holds for $i \leqslant n_{t_k}$. By (4.2), we may choose $s_{k+1} > t_k$ such that

$$\frac{C_{t_k}}{C_{s_{k+1}}} < \frac{1}{4}|\langle \Lambda_{t_k}, \psi\rangle|^{-1};$$

we then define $\psi(i) = 0$ ($n_{t_k} < i \leqslant n_{s_{k+1}}$), and note that (4.4) holds trivially for these values of i. Then

$$|\langle \Lambda_{s_{k+1}}, \psi\rangle| = \frac{C_{t_k}}{C_{s_{k+1}}}|\langle \Lambda_{t_k}, \psi\rangle| < \frac{1}{4},$$

as required.

Again using (4.2), we may choose $t_{k+1} > s_{k+1}$ such that $C_{s_{k+1}}/C_{t_{k+1}} < 1/8$, so that

$$\frac{\omega(n_{s_{k+1}} + 1) + \cdots + \omega(n_{t_{k+1}})}{C_{t_{k+1}}} > \frac{7}{8}.$$

Set $\psi(i) = \omega_i$ ($n_{s_{k+1}} < i \leqslant n_{t_{k+1}}$), and note that (4.4) continues to hold. Then

$$\begin{aligned} |\langle \Lambda_{t_{k+1}}, \psi\rangle| &= \left|\frac{\omega(n_{s_{k+1}} + 1) + \cdots + \omega(n_{t_{k+1}})}{C_{t_{k+1}}} + \frac{C_{s_{k+1}}}{C_{t_{k+1}}}\langle \Lambda_{s_{k+1}}, \psi\rangle\right| \\ &> \frac{7}{8} - \frac{1}{8}\cdot\frac{1}{4} > \frac{3}{4}. \end{aligned}$$

The induction continues.

Let $\mathcal{F}$ denote the Fréchet filter on $\mathbb{N}$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and set $\Lambda = \lim_{k\to\mathcal{U}} \Lambda_k$ (the limit being taken in the weak*-topology on A_ω''). We have shown that the sequence (Λ_k) is not convergent in the weak*-topology on A_ω'', and so it follows that there exists a weak*-open neighbourhood $\mathcal{O}$ of Λ such that $E := \{k \in \mathbb{N} : \Lambda_k \in \mathcal{O}\} \notin \mathcal{F}$. As $E \notin \mathcal{F}$, the set E^c is infinite, so that $E^c \cap A \neq \varnothing$ ($A \in \mathcal{F}$). Hence there exists a free ultrafilter $\mathcal{V}$ on $\mathbb{N}$ containing E^c and $\mathcal{F}$. Let $M = \lim_{k\to\mathcal{V}} \Lambda_k$. Since $E^c \in \mathcal{V}$, we have $E \notin \mathcal{V}$, so that $\Lambda \neq M$.

Next we show that Λ and M are invariant means on $\ell^\infty(\mathbb{Z},1/\omega)$. Let $f \in \ell^\infty(\mathbb{Z},1/\omega)$. Then

$$\begin{aligned}|\langle \delta_1 \Box \Lambda_k - \Lambda_k, f\rangle| &= \frac{1}{C_k}|(f(1)+\cdots+f(n_k+1)) - (f(0)+\cdots+f(n_k))| \\ &= \frac{1}{C_k}|f(n_k+1) - f(0)| \\ &\leqslant \frac{1}{C_k}\|f\|(\omega(1)\omega(n_k)+\omega(0)),\end{aligned}$$

which, by (4.1), tends to zero as $k \to \infty$. Hence

$$\delta_1 \Box \Lambda - \Lambda = \lim_{k\to\mathcal{U}}(\delta_1 \Box \Lambda_k - \Lambda_k) = 0,$$

and a similar calculation shows that $\delta_{-1}\Box\Lambda = \Lambda$ as well. It follows that Λ is invariant. That $\Lambda \geqslant 0$ is clear, and hence, by Lemma 4.3.1,

$$\|\Lambda\| = \langle \Lambda, \omega\rangle = \lim_{k\to\mathcal{U}}\langle \Lambda_k, \omega\rangle = 1.$$

Hence Λ is an invariant mean, as claimed. The same argument shows that M is also an invariant mean.

Finally, we calculate that

$$\langle \Lambda, 1\rangle = \lim_{k\to\mathcal{U}}\langle \Lambda_k, 1\rangle = \lim_{k\to\infty}(n_k+1)/C_k = \lim_{k\to\mathcal{V}}\langle \Lambda_k, 1\rangle = \langle M, 1\rangle,$$

as required. □

We now prove the main result of this section.

Proof of Theorem 4.1.1. Let $\rho = \rho_\omega$, and let $\gamma_n = \omega_n/\rho^n$ $(n \in \mathbb{Z})$. Then γ is a weight on $\mathbb{Z}$, and $T\colon (f(n)) \mapsto (\rho^n f(n))$ defines an (isometric) isomorphism of Banach algebras $A_\gamma \to A_\omega$. The weight γ satisfies the hypothesis of Lemma 4.3.2, so that there exist distinct invariant means Λ and M on A_γ'' as in that lemma. Then $\langle \Lambda - M, 1\rangle = 0$, so that $\Lambda - M \in I_\gamma\backslash\{0\}$. Hence, by [**23**, Proposition 8.23], $\operatorname{rad}(A_\gamma'') \neq \{0\}$, so that $\operatorname{rad}(A_\omega'') \neq \{0\}$. □

Remark. A trivial modification of the proof of Theorem 4.1.1 shows that in fact we also have $\operatorname{rad}(\ell^1(\mathbb{Z}^+,\omega)'') \neq \{0\}$ for every weight ω on $\mathbb{Z}^+$.

Remark. Since A_ω is commutative, $(A''_\omega,\Diamond) = (A''_\omega,\Box)^{\mathrm{op}}$, and it follows that $\operatorname{rad}(A''_\omega,\Diamond) = \operatorname{rad}(A''_\omega,\Box)$, so that $(A''_\omega,\Diamond)$ is never semisimple either.

4.4. The Radical of $\ell^1(\oplus_{i=1}^{\infty}\mathbb{Z})''$

In this section we prove Theorem 4.1.2. In addition we observe in Corollary 4.4.5 that there are many non-amenable groups G for which $\operatorname{rad}(\ell^1(G)'') \neq \{0\}$. Ideals of the following form will be central to both of these arguments.

Definition 4.4.1. Let G be a group, let $\theta\colon \ell^1(G) \to \ell^1(G)$ be a bounded algebra homomorphism, and let $J \subset \ell^1(G)''$ be an ideal. We define

$$I(\theta,J) = \{\Phi \in \ell^1(G)'' : \delta_s\Box\Phi = \theta(\delta_s)\Box\Phi \ (s \in G),\ \theta''(\Phi) \in J\}.$$

Proposition 4.4.2. *Let G,θ and J be as in Definition 4.4.1. Then $I(\theta,J)$ is an ideal in $\ell^1(G)''$.*

Proof. Let $\Phi \in I(\theta,J)$, and let $s,t \in G$. Then

$$\delta_s\Box(\delta_t\Box\Phi) = \delta_{st}\Box\Phi = \theta(\delta_{st})\Box\Phi = \theta(\delta_s)\Box\theta(\delta_t)\Box\Phi = \theta(\delta_s)\Box(\delta_t\Box\Phi).$$

By taking linear combinations and weak*-limits, we may conclude that

$$\delta_s\Box\Psi\Box\Phi = \theta(\delta_s)\Box\Psi\Box\Phi$$

for every $\Psi \in \ell^1(G)''$ and every $s \in G$. It is clear that $\delta_s\Box\Phi\Box\Psi = \theta(\delta_s)\Box\Phi\Box\Psi$ for every $\Psi \in \ell^1(G)''$. Since J is an ideal and $\theta''(\Phi) \in J$, we have $\theta''(\Psi\Box\Phi) = \theta''(\Psi)\Box\theta''(\Phi) \in J$ and $\theta''(\Phi\Box\Psi) = \theta''(\Phi)\Box\theta''(\Psi) \in J$ for every $\Psi \in \ell^1(G)''$. Finally, we note that $I(\theta,J)$ is clearly a linear space. We have shown that $I(\theta,J)$ is an ideal in $\ell^1(G)''$. $\Box$

Lemma 4.4.3. *Let G,θ and J be as in Definition 4.4.1. Then:*

(i) *if* $\Phi \in I(\theta, J)$ *and* $\Psi \in \ell^1(G)''$, *then* $\Psi \Box \Phi = \theta''(\Psi) \Box \Phi$*;*

(ii) *if* J *is nilpotent of index* n*, then* $I(\theta, J)$ *is nilpotent of index at most* $n + 1$*.*

Proof. (i) This follows from the identity $\delta_s \Box \Phi = \theta''(\delta_s) \Box \Phi$, and the fact that θ'' is linear and weak*-continuous.

(ii) Given $\Phi_1, \ldots, \Phi_{n+1} \in I(\theta, J)$, we have

$$\begin{aligned} \Phi_1 \Box \cdots \Box \Phi_{n+1} &= \theta''(\Phi_1 \Box \cdots \Box \Phi_n) \Box \Phi_{n+1} \\ &= \theta''(\Phi_1) \Box \cdots \Box \theta''(\Phi_n) \Box \Phi_{n+1} = 0 \end{aligned}$$

because $\theta''(\Phi_1), \ldots, \theta''(\Phi_n) \in J$. As $\Phi_1, \ldots, \Phi_{n+1}$ were arbitrary, this shows that $I(\theta, J)^{\Box(n+1)} = \{0\}$. $\Box$

The key idea in the proof of Theorem 4.1.2 is to use invariant means coming from each of the copies of $\mathbb{Z}$ in the direct sum to build more complicated radical elements in $\ell^1(\oplus_{i=1}^\infty \mathbb{Z})''$. We shall use the following lemma. Recall that, for a group G with subgroups N and H, where N is normal in G, we say that N *is complemented by* H if $H \cap N = \{e\}$ and $G = HN$. In this case every element of G may be written uniquely as hn, for some $h \in H$ and some $n \in N$, and the map $G \to G$ defined by $hn \mapsto h$ is a group homomorphism.

Lemma 4.4.4. *Let* G *be a group with a normal, amenable subgroup* N *which is complemented by a subgroup* H*. Let* $\pi \colon \ell^1(G) \to \ell^1(G)$ *be the bounded algebra homomorphism defined by* $\pi(\delta_{hn}) = \delta_h$ $(h \in H, n \in N)$ *and let* $\iota \colon \ell^1(N) \to \ell^1(G)$ *denote the inclusion map. Let* M *be an invariant mean on* $\ell^\infty(N)$*, and write* $\widetilde{M} = \iota''(M)$*. Then* $\widetilde{M}$ *satisfies:*

$$\delta_s \Box \widetilde{M} = \pi(\delta_s) \Box \widetilde{M} \quad (s \in G); \tag{4.5}$$

$$\pi''(\widetilde{M}) = \delta_e. \tag{4.6}$$

Proof. For every $n \in N$ and every $f \in \ell^1(N)$, we have $\delta_n * \iota(f) = \iota(\delta_n * f)$, and so, by taking weak*-limits, we see that $\delta_n \Box \iota''(\Phi) = \iota''(\delta_n \Box \Phi)$ for all $\Phi \in \ell^1(N)''$. An arbitrary element $s \in G$ may be written as $s = hn$ for some $h \in H$ and $n \in N$, and so

$$\begin{aligned}\delta_s \Box \widetilde{M} &= \delta_h \Box \delta_n \Box \iota''(M) = \delta_h \Box \iota''(\delta_n \Box M)\\ &= \delta_h \Box \iota''(M) = \pi(\delta_s) \Box \widetilde{M}.\end{aligned}$$

Hence (4.5) holds.

Define $\varphi_0 \colon \ell^1(N) \to \ell^1(N)$ by $\varphi_0 \colon f \mapsto \langle f, 1\rangle \delta_e$ $(f \in \ell^1(N))$. It is easily verified that $\pi \circ \iota = \iota \circ \varphi_0$, and so $\pi'' \circ \iota'' = \iota'' \circ \varphi_0''$. We also have $\varphi_0''(\Phi) = \langle \Phi, 1\rangle \delta_e$ $(\Phi \in \ell^1(N)'')$. Hence

$$\pi''(\widetilde{M}) = (\pi'' \circ \iota'')(M) = (\iota'' \circ \varphi_0'')(M) = \iota''(\langle M, 1\rangle \delta_e) = \delta_e,$$

establishing (4.6). $\square$

We have not seen in the literature any instance of a discrete, non-amenable group G for which it is known that $\operatorname{rad}(\ell^1(G)'') \neq \{0\}$. However the next corollary gives a large class of easy examples of such groups.

COROLLARY 4.4.5. *Let G be a group with an infinite, amenable, complemented, normal subgroup N. Then* $|\operatorname{rad}(\ell^1(G)'')| \geqslant 2^{2^{|N|}}$.

PROOF. Let M_1, M_2 be two invariant means in $\ell^1(N)''$, and let ι and π be as in Lemma 4.4.4 . Then by that lemma $\iota''(M_1 - M_2) \in I(\pi, 0)$, which is a nilpotent ideal by Lemma 4.4.3(ii). The result now follows from the injectivity of ι'' and [**64**, Theorem 7.26]. $\square$

We now prove our main theorem.

PROOF OF THEOREM 4.1.2. Let $G = \oplus_{i=1}^{\infty}\mathbb{Z}$, and, given $i \in \mathbb{N}$, write G_i for the i^{th} copy of $\mathbb{Z}$ appearing in this direct sum. Let $\pi_i \colon G \to G$ be the homomorphism which "deletes" the i^{th} coordinate, that is

$$\pi_i \colon (n_1, n_2, \ldots) \mapsto (n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots).$$

Each map π_i gives rise to a bounded homomorphism $\ell^1(G) \to \ell^1(G)$, which we also denote by π_i, given by

$$\pi_i\colon f \mapsto \sum_{s\in G} f(s)\delta_{\pi_i(s)} \quad (f \in \ell^1(G)).$$

Similarly, we write $\iota_i\colon G_i \to G$ for the inclusion map of groups, and $\iota_i\colon \ell^1(G_i) \to \ell^1(G)$ for the inclusion of algebras which it induces.

Define a sequence of ideals I_j in $\ell^1(G)''$ by $I_1 = I(\pi_1, 0)$ and

$$I_j = I(\pi_j, I_{j-1}) \quad (j \geqslant 2).$$

By Lemma 4.4.3(ii), each I_j is nilpotent of index at most $j+1$ and the strategy of the proof is to show that the index is exactly $j+1$.

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Given $i \in \mathbb{N}$ and $n \in \mathbb{Z}$ we write

$$\delta_n^{(i)} = \delta_{(0,\dots,0,n,0,\dots)},$$

where n appears in the i^{th} place. Given $j \in \mathbb{N}$ we define elements $\sigma_j, M_j \in \ell^1(G)''$ to be the weak*-limits $\sigma_j = \lim_{k\to\mathcal{U}} \sigma_{j,k}$ and $M_j = \lim_{k\to\mathcal{U}} M_{j,k}$, where

$$M_{j,k} = \frac{1}{k}\sum_{i=1}^{k} \delta_i^{(j)} \quad (j,k \in \mathbb{N}),$$

and

$$\sigma_{j,k} = \frac{1}{k}\sum_{i=1}^{k} \left(\delta_i^{(j)} - \delta_{-i}^{(j)}\right) \quad (j,k \in \mathbb{N}).$$

We claim that, for each $j \in \mathbb{N}$, M_j and σ_j satisfy:

(4.7) $$\delta_s \Box M_j = \pi_j(\delta_s)\Box M_j \ (s \in G);$$

(4.8) $$\pi_j''(M_j) = \delta_{(0,0,\dots)};$$

(4.9) $$\pi_i''(\sigma_j) = \sigma_j \text{ and } \pi_i''(M_j) = M_j \ (i \neq j);$$

(4.10) $$\sigma_j \in I(\pi_j, 0).$$

Since $\pi_i''(M_{j,k}) = M_{j,k}$ and $\pi_i''(\sigma_{j,k}) = \sigma_{j,k}$ $(k \in \mathbb{N}, i \neq j)$, (4.9) follows from the weak*-continuity of π_j''. We observe that M_j is the image of an invariant mean on $\mathbb{Z}$ under ι_j'', so that we may apply Lemma 4.4.4 to obtain (4.7) and (4.8). Similarly, σ_j is the image of the difference of two invariant means on $\mathbb{Z}$ under ι_j, so that Lemma 4.4.4 implies that $\delta_s \Box \sigma_j = \pi_j(\delta_s)\Box\sigma_j$ $(s \in G)$ and $\pi_j''(\sigma_j) = 0$, so that (4.10) holds.

We demonstrate that

$$\sigma_1 \Box \sigma_2 \Box \cdots \Box \sigma_j \neq 0 \quad (j \in \mathbb{N}). \tag{4.11}$$

To see this, define $h \in \ell^\infty(G)$ by

$$h(n_1, n_2, \ldots) = \begin{cases} 1 & \text{if } n_i \geqslant 0 \text{ for all } i \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have

$$\langle \sigma_{i,k}, h \rangle = 1 \quad (i, k \in \mathbb{N}). \tag{4.12}$$

It is easily checked that

$$\langle \pi_i(\delta_s) * \iota_i(\delta_t), h \rangle = \langle \pi_i(\delta_s), h \rangle \langle \iota_i(\delta_t), h \rangle \tag{4.13}$$

for every $s \in G$ and $t \in G_i$: indeed, given s and t, observe that $\pi_i(\delta_s)$ is equal to δ_u, for some $u \in G$ with i^{th} coordinate equal to zero, whereas $\iota_i(\delta_t)$ is of the form δ_v, for some $v \in G$ which is zero in every other coordinate. It follows that $\pi_i(\delta_s) * \iota_i(\delta_t)$ has the property that all of its coordinates are non-negative if and only if both $\pi_i(\delta_s)$ and $\iota_i(\delta_t)$ separately have this property. Hence $\langle \pi_i(\delta_s) * \iota_i(\delta_t), h \rangle = 1$ if and only if $\langle \pi_i(\delta_s), h \rangle = \langle \iota_i(\delta_t), h \rangle = 1$, and equals 0 otherwise. Equation (4.13) follows. This equation implies that

$$\langle \pi_i(f) * \iota_i(g), h \rangle = \langle \pi_i(f), h \rangle \langle \iota_i(g), h \rangle \quad (f \in \ell^1(G),\ g \in \ell^1(G_i)). \tag{4.14}$$

Given $i, k \in \mathbb{N}$, the element $\sigma_{i,k}$ belongs to the image of ι_i, and, together with (4.9), (4.14) and (4.12), this allows us to conclude that, for all $k_1, \ldots, k_j \in \mathbb{N}$, we have

$$\begin{aligned}
&\langle \sigma_{1,k_1} * \sigma_{2,k_2} * \cdots * \sigma_{j,k_j}, h\rangle \\
&\qquad = \langle \pi_j(\sigma_{1,k_1} * \cdots * \sigma_{j-1,k_{j-1}}) * \sigma_{j,k_j}, h\rangle \\
&\qquad = \langle \pi_j(\sigma_{1,k_1} * \cdots * \sigma_{j-1,k_{j-1}}), h\rangle\langle \sigma_{j,k_j}, h\rangle \\
&\qquad = \langle \pi_{j-1}(\sigma_{1,k_1} * \cdots * \sigma_{j-2,k_{j-2}}) * \sigma_{j-1,k_{j-1}}, h\rangle\langle \sigma_{j,k_j}, h\rangle = \cdots \\
&\qquad = \langle \sigma_{1,k_1}, h\rangle\langle \sigma_{2,k_2}, h\rangle \cdots \langle \sigma_{j,k_j}, h\rangle = 1.
\end{aligned}$$

Therefore

$$\langle \sigma_1 \Box \sigma_2 \Box \cdots \Box \sigma_j, h\rangle = \lim_{\underline{k} \to \mathcal{U}}{}^{(j)} \left\langle \sigma_{1,k_1} * \sigma_{2,k_2} * \cdots * \sigma_{j,k_j}, h\right\rangle = 1.$$

Equation (4.11) follows.

We now come to the main argument of the proof. We recursively define $\Lambda_j \in \ell^1(G)''$ by $\Lambda_1 = \sigma_1$ and

$$\Lambda_j = M_j \Box \Lambda_{j-1} + \sigma_j \quad (j \geqslant 2).$$

We shall show inductively that each Λ_j satisfies:

$$\Lambda_j \in I_j; \tag{4.15}$$

$$\Lambda_j^{\Box j} = \sigma_1 \Box \sigma_2 \Box \cdots \Box \sigma_j; \tag{4.16}$$

$$\pi_i''(\Lambda_j) = \Lambda_j \ (i > j). \tag{4.17}$$

Since by Lemma 4.4.3(ii) $I_j^{\Box(j+1)} = \{0\}$, and by (4.11) $\sigma_1 \Box \sigma_2 \Box \cdots \Box \sigma_j \neq 0$, this will give the result. The base case of the induction holds by (4.10) and (4.9).

Now assume that the hypothesis holds up to $j-1$. It follows from (4.10) and (4.7) that $\delta_s \Box \Lambda_j = \pi_j(\delta_s) \Box \Lambda_j$ $(s \in G)$. Moreover, by (4.8), (4.10), and (4.17) applied to

Λ_{j-1}, we have

$$\pi_j''(\Lambda_j) = \pi_j''(M_j)\Box\pi_j''(\Lambda_{j-1}) + \pi_j''(\sigma_j) = \Lambda_{j-1}, \tag{4.18}$$

so that, by the induction hypothesis, $\pi_j''(\Lambda_j) \in I_{j-1}$. Hence (4.15) holds. We see that (4.17) holds for a given $i > j$ because it holds for each of M_j, σ_j and Λ_{j-1} by (4.9) and the induction hypothesis. Finally, we verify (4.16):

$$\begin{aligned}\Lambda_j^{\Box j} &= \pi_j''(\Lambda_j)^{\Box(j-1)}\Box\Lambda_j = \Lambda_{j-1}^{\Box(j-1)}\Box\Lambda_j \\ &= \Lambda_{j-1}^{\Box(j-1)}\Box M_j\Box\Lambda_{j-1} + \Lambda_{j-1}^{\Box(j-1)}\Box\sigma_j = \sigma_1\Box\sigma_2\Box\cdots\Box\sigma_{j-1}\Box\sigma_j,\end{aligned}$$

where we have used Lemma 4.4.3(i) and (4.18) in the first line, and the fact that $I_{j-1}^{\Box j} = \{0\}$ in the second line to get $\Lambda_{j-1}^{\Box(j-1)}\Box M_j\Box\Lambda_{j-1} = 0$.

This completes the proof. □

Remark. A simpler version of the above argument shows that $\ell^1(\mathbb{Z}^2)''$ contains a radical element which is nilpotent of index 3, which is enough to resolve Dales and Lau's question of whether the radical of $L^1(G)''$, for G a locally compact group, always has zero square [**23**, Chapter 14, Question 3]. Specifically, this may be achieved by terminating the induction at $j = 2$, and otherwise making trivial alterations.

COROLLARY 4.4.6. *The radical of $\ell^1(\oplus_{i=1}^{\infty}\mathbb{Z})''$ contains non-nilpotent elements.*

PROOF. By a theorem of Grabiner [**35**], if every element of $\mathrm{rad}\,(\ell^1(\oplus_{i=1}^{\infty}\mathbb{Z})'')$ were nilpotent, then there would be a uniform bound on the index of nilpotency. Hence, by Theorem 4.1.2, $\mathrm{rad}\,(\ell^1(\oplus_{i=1}^{\infty}\mathbb{Z})'')$ must contain non-nilpotent elements. □

4.5. A Weight ω for Which $\mathrm{rad}\,(\ell^1(\mathbb{Z},\omega)'')$ is Not Nilpotent

In this section we shall prove Theorem 4.1.3. We shall also prove a related result, as a sort of warm up: namely Proposition 4.5.5, which states that, for every $q \in \mathbb{N}$ at least 2, there is a weight ω_q on $\mathbb{Z}$ such that $\mathrm{rad}\,(\ell^1(\mathbb{Z},\omega_q)'')$ contains a nilpotent element of index exactly q. The proof of Theorem 4.1.3 does not rely on that of Proposition 4.5.5, although there are some common ideas involved.

Given a weight ω on $\mathbb{Z}$ and $r \in \mathbb{N}$, we define $\Omega_\omega^{(r)} : \mathbb{Z}^r \to (0,1]$ by

$$\Omega_\omega^{(r)}(n_1,\dots,n_r) = \frac{\omega(n_1+n_2+\cdots+n_r)}{\omega(n_1)\omega(n_2)\cdots\omega(n_r)} \quad (n_1,\dots,n_r \in \mathbb{Z})$$

(compare with [**23**, Equation 8.7]). Often we simply write $\Omega^{(r)}$ when the weight ω is clear. As in Section 4.3 we write $A_\omega = \ell^1(\mathbb{Z},\omega)$.

Our main tool will be Proposition 4.5.1. Recall that we denote the unit ball of a Banach space E by B_E.

PROPOSITION 4.5.1. *Let ω be a weight on $\mathbb{Z}$ and suppose that there is some sequence $(n_k) \subset \mathbb{Z}$ such that*

$$\liminf_{r\to\infty} \limsup_{\underline{k}\to\infty}{}^{(r)} \left[\Omega^{(r)}(n_{k_1},\dots,n_{k_r})\right]^{1/r} = 0. \tag{4.19}$$

Let Φ be a weak-accumulation point of $\{\delta_{n_k}/\omega(n_k) : k \in \mathbb{N}\}$. Then $\Phi \in \mathrm{rad}\,(A_\omega'')\backslash\{0\}$. Furthermore, if there is some $q \in \mathbb{N}$ such that*

$$\limsup_{\underline{k}\to\infty}{}^{(q)} \Omega^{(q)}(n_{k_1},\dots,n_{k_q}) = 0 \tag{4.20}$$

then the left ideal generated by Φ is nilpotent of index at most q.

PROOF. There exists some free filter $\mathcal{U}$ on $\mathbb{N}$ such that

$$\Phi = \lim_{k\to\mathcal{U}} \frac{1}{\omega(n_k)}\delta_{n_k},$$

where the limit is taken in the weak*-topology. Let $\Psi \in B_{A_\omega''}$. Then there exists a net (a_α) in B_{A_ω} such that $\lim_{w*,\alpha} a_\alpha = \Psi$. Let $\lambda \in B_{A_\omega'}$. Then, for each $r \in \mathbb{N}$, we have

$$
\begin{aligned}
(4.21)\qquad |\langle (\Psi\Box\Phi)^{\Box r}, \lambda\rangle| &= \lim_{\underline{\alpha}\to\infty,\,\underline{k}\to\mathcal{U}}{}^{(r)} \left|\frac{\langle a_{\alpha_1} * \delta_{n_{k_1}} * \cdots * a_{\alpha_r} * \delta_{n_{k_r}}, \lambda\rangle}{\omega(n_{k_1})\cdots\omega(n_{k_r})}\right| \\
&= \lim_{\underline{\alpha}\to\infty,\,\underline{k}\to\mathcal{U}}{}^{(r)} \left|\frac{\langle a_{\alpha_1} * \cdots * a_{\alpha_r} * \delta_{n_{k_1}} * \cdots * \delta_{n_{k_r}}, \lambda\rangle}{\omega(n_{k_1})\cdots\omega(n_{k_r})}\right| \\
&\leqslant \limsup_{\underline{k}\to\infty}{}^{(r)} \left\|\frac{\delta_{n_{k_1}+\cdots+n_{k_r}}}{\omega(n_{k_1})\cdots\omega(n_{k_r})}\right\| \\
&= \limsup_{\underline{k}\to\infty}{}^{(r)} \Omega^{(r)}(n_{k_1},\dots,n_{k_r}).
\end{aligned}
$$

Hence

$$
\|(\Psi\Box\Phi)^{\Box r}\|^{1/r} = \sup_{\lambda\in B_{A'_\omega}} |\langle(\Psi\Box\Phi)^{\Box r},\lambda\rangle|^{1/r} \leqslant \limsup_{\underline{k}\to\infty}{}^{(r)} \left[\Omega^{(r)}(n_{k_1},\dots,n_{k_r})\right]^{1/r},
$$

and so $\lim_{r\to\infty}\|(\Psi\Box\Phi)^{\Box r}\|^{1/r} = 0$ by (4.19). Therefore $\Psi\Box\Phi \in \mathcal{Q}(A''_\omega)$. As Ψ was arbitrary, it follows that $\Phi \in \operatorname{rad}(A''_\omega)$. Moreover, $\Phi \neq 0$ because

$$
\langle \Phi, \omega\rangle = \lim_{k\to\mathcal{U}} \left\langle \frac{\delta_{n_k}}{\omega(n_k)}, \omega \right\rangle = 1.
$$

If, further, (4.20) holds, then (4.21) with $r = q$ implies that $(\Psi\Box\Phi)^{\Box q} = 0$. This completes the proof. $\square$

Remark. Observe that, since ω is submultiplicative, and Ω takes values in $[0,1]$, the formula $\Omega^{(i)} \leqslant \Omega^{(j)}$ holds pointwise whenever $i \leqslant j$. Hence (4.20) implies (4.19).

In their memoir [**23**], Dales and Lau put forward a candidate for a weight ω such that A''_ω is semisimple. They attribute this weight to Feinstein. In light of Theorem 4.1.1 this cannot be the case, but Proposition 4.5.1 gives us a second way to see this.

PROPOSITION 4.5.2. *Let ω denote the so-called Feinstein weight, studied in* [**23**, Example 9.17]. *Then* $\operatorname{rad}(A''_\omega) \neq \{0\}$.

PROOF. Let $X = \{\pm 2^k : k \in \mathbb{N}\}$, and recall that ω is defined by $\omega(n) = e^{|n|_X}$. Let $n_k = 2^{2k} + 2^{2k-2} + \cdots + 1$, as in [**23**]. It was shown there that $\omega(n_k) = e^{k+1}$. Let

$k_1 \geqslant k_2$ be natural numbers. Then

$$\begin{aligned} n_{k_1} + n_{k_2} &= 2^{2k_1} + \cdots + 2^{2k_2+2} + 2 \cdot 2^{2k_2} + \cdots 2 \cdot 1 \\ &= 2^{2k_1} + \cdots + 2^{2k_2+2} + 2^{2k_2+1} + \cdots + 2 \end{aligned}$$

so that $|n_{k_1} + n_{k_2}|_X \leqslant k_1 + 1$. Hence $\Omega^{(2)}(n_{k_2}, n_{k_1}) \leqslant \frac{e^{k_1+1}}{e^{k_1+1}e^{k_2+1}} = e^{-(k_2+1)}$, and it follows that $\lim_{k_2\to\infty} \lim_{k_1\to\infty} \Omega^{(2)}(n_{k_2}, n_{k_1}) = 0$. Now apply Proposition 4.5.1. □

In Section 1.1 word-length of a group element with respect to a generating was defined, and in Example 1.3.2 we described how a given generating set gives rise to certain weights on the group via the associated word-length function. In what follows we shall use infinite generating sets for $\mathbb{Z}$ to construct weights. In this context the word-length of an integer n with respect to a (possibly infinite) generating set $X \subset \mathbb{Z}$ is given by the formula

$$|n|_X = \min\left\{ r : n = \sum_{i=1}^{r} \varepsilon_i s_i, \text{ for some } s_1, \ldots, s_r \in X, \varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\} \right\}.$$

Recall that, for any generating set $X \subset \mathbb{Z}$, the function $n \mapsto \mathrm{e}^{|n|_X}$ defines a weight on $\mathbb{Z}$.

Let $q \in \mathbb{N}$. We now set about showing that there is a weight ω_q on $\mathbb{Z}$ such that $\operatorname{rad}(A''_{\omega_q})$ contains a nilpotent element of index exactly q. Throughout we set $m = 2(q-1)$, $X_q = \{\pm m^k : k \in \mathbb{N}\}$, and $\eta_q(n) = |n|_{X_q}$ and $\omega_q(n) = e^{\eta_q(n)}$ $(n \in \mathbb{Z})$. We also define a sequence of integers (s_k) by

$$s_k = m^{2k} + m^{2k-2} + \cdots + 1 \quad (k \in \mathbb{Z}^+).$$

LEMMA 4.5.3. *Let d be an integer with $1 \leqslant d \leqslant m-1$. The equation $cm + c' = d - m^2$, for $c, c' \in \{-(m-1), \ldots, m-2, m-1\}$, has only the solution $c = -(m-1), c' = d - m$.*

PROOF. (i) Suppose that c and c' are integers satisfying $cm + c' = d - m^2$ which lie between $-(m-1)$ and $(m-1)$. We have $m^2 - d = |cm + c'| \leqslant |c|m + (m-1)$, which forces $|c| = (m-1)$ (since otherwise $|c|m + (m-1) \leqslant (m-2)m + m - 1 = m^2 - (m+1) < m^2 - d$), and, since c must be negative, we have $c = -(m-1)$. The value of c' is then determined by the equation. $\square$

LEMMA 4.5.4. *Given an integer $1 \leqslant d \leqslant q-1$ and $k_1, \dots, k_d \in \mathbb{N}$ we have*

$$\eta_q(s_{k_1} + \cdots + s_{k_d}) = k_1 + \cdots + k_d + d.$$

PROOF. We shall assume that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d$. Certainly

$$\eta_q(s_{k_1} + \cdots + s_{k_d}) \leqslant \eta_q(s_{k_1}) + \cdots + \eta_q(s_{k_d}) \leqslant k_1 + \cdots + k_d + d.$$

To get the lower bound we proceed by induction on k_1. In what follows we shall interpret $s_{-1} = 0$, and for the base of our induction we shall take all of the cases in which $k_1, \dots, k_d \in \{-1, 0\}$, each of which holds trivially.

Assume that $k_1 > 0, k_2, \dots, k_d \geqslant 0$. Write

$$s_{k_1} + \cdots + s_{k_d} = \sum_{j=1}^{p} c_j m^{a_j},$$

for some natural numbers $a_1 > a_2 > \cdots > a_p$, and some non-zero integers $c_1, \dots, c_p$ satisfying $\sum_{j=1}^{p} |c_j| = \eta_q(s_{k_1} + \cdots + s_{k_d})$. It follows from the minimality of $\sum_{j=1}^{p} |c_j|$ that $c_1, \dots, c_p \in \{\pm 1, \pm 2, \dots, \pm(m-1)\}$. As $s_{k_1} + \cdots + s_{k_d} \equiv d \pmod{m}$ we have $a_p = 0$. Suppose that $c_p = d$. Then, as $s_{k_1} + \cdots + s_{k_d} \equiv d \pmod{m^2}$, we must have $a_{p-1} \geqslant 2$. Recall that we understand $s_{-1} = 0$, and compute

$$s_{k_1 - 1} + \cdots + s_{k_d - 1} = \frac{1}{m^2}(s_{k_1} + \cdots + s_{k_d} - d) = \sum_{j=1}^{p-1} c_j m^{a_j - 2},$$

so that, by the induction hypothesis,

$$\sum_{j=1}^{p-1} |c_j| \geqslant (k_1 - 1) + \cdots + (k_d - 1) + d,$$

which implies that

$$\sum_{j=1}^{p} |c_j| \geqslant (k_1-1)+\cdots+(k_d-1)+d+d = k_1+\cdots+k_d+d.$$

Now suppose that $c_p \neq d$. Then, as $s_{k_1}+\cdots+s_{k_d} \equiv d \pmod{m^2}$, we must have $a_{p-1}=1$. Since $c_{p-1}m+c_p \equiv d \pmod{m^2}$ and $|c_{p-1}m+c_p| \leqslant m^2-1$, we must have either $c_{p-1}m+c_p=d$ or $c_{p-1}m+c_p=d-m^2$.

In the case where $c_{p-1}m+c_p=d$ we repeat the argument that we used when $c_p=d$ to get

$$\sum_{j=1}^{p-2} |c_j| \geqslant (k_1-1)+\cdots+(k_d-1)+d.$$

Then $c_{p-1}m+c_p=d, c_{p-1}\neq 0$ forces $c_{p-1}=1$ and $c_p=d-m$, so that $|c_{p-1}|+|c_p| = 1+m-d \geqslant d$, since $2d \leqslant m$ by hypothesis. Thus

$$\sum_{j=1}^{p} |c_j| \geqslant (k_1-1)+\cdots+(k_d-1)+d+d = k_1+\cdots+k_d+d.$$

We now turn to the case where $c_{p-1}m+c_p=d-m^2$, and compute

$$\begin{aligned} s_{k_1-1}+\cdots+s_{k_d-1} &= \frac{1}{m^2}(s_{k_1}+\cdots+s_{k_d}-d) \\ &= \sum_{j=1}^{p-2} c_j m^{a_j-2} + \frac{1}{m^2}(c_{p-1}m+c_p-d) = \sum_{j=1}^{p-2} c_j m^{a_j-2} - 1. \end{aligned}$$

It follows that

$$\sum_{j=1}^{p-2} |c_j| + 1 \geqslant (k_1-1)+\cdots+(k_d-1)+d. \tag{4.22}$$

By Lemma 4.5.3 we have $c_{p-1}=-(m-1)$ and $c_p=d-m$, so that $|c_{p-1}|+|c_p| = 2m-1-d$. Since $q \geqslant 2$, we have $d \leqslant q-1 \leqslant 2q-3 = m-1$, so that $|c_{p-1}|+|c_p| = 2m-1-d \geqslant d+1$. Hence by (4.22),

$$\sum_{j=1}^{p} |c_j| \geqslant (k_1-1)+\cdots+(k_d-1)+d-1+(d+1) = k_1+\cdots+k_d+d.$$

This completes the proof. $\square$

PROPOSITION 4.5.5. *The Banach algebra A''_{ω_q} contains a radical element which is nilpotent of index exactly q.*

PROOF. Let Φ be a weak*-accumulation point of $\{\delta_{s_k}/\omega_q(s_k) : k \in \mathbb{N}\}$. For each $j \in \mathbb{N}$ we have

$$\begin{aligned} qs_j &= (m-(q-2))s_j \\ &= m^{2j+1} - (q-2)m^{2j} + m^{2j-1} - (q-2)m^{2j-2} + \cdots + (m-(q-2)), \end{aligned}$$

so that

$$\eta_q(qs_j) \leqslant (j+1) + (j+1)(q-2) = (j+1)(q-1),$$

and so that, for natural numbers $k_1 \geqslant \ldots \geqslant k_q > j$, we have

$$\begin{aligned} \eta_q(s_{k_1} + \cdots + s_{k_q}) &= \eta_q((s_{k_1} - s_j) + \cdots + (s_{k_q} - s_j) + qs_j) \\ &\leqslant \eta_q(s_{k_1} - s_j) + \cdots + \eta_q(s_{k_q} - s_j) + \eta_q(qs_j) \\ &\leqslant (k_1 - j) + \cdots + (k_q - j) + (j+1)(q-1) \\ &= k_1 + \cdots + k_q - j + q - 1. \end{aligned}$$

Using Lemma 4.5.4 to get $\omega_q(s_{k_i}) = e^{k_i+1}$ $(i = 1, \ldots, q)$, we then have

$$\Omega^{(q)}(s_{k_1}, \ldots, s_{k_q}) \leqslant \frac{e^{k_1+\cdots+k_m-j+q-1}}{e^{k_1+1}\cdots e^{k_q+1}} \leqslant e^{-j},$$

which implies that

$$\lim_{\underline{k}\to\infty}{}^{(q)}\,\Omega^{(q)}(s_{k_1}, \ldots, s_{k_q}) = 0.$$

By Proposition 4.5.1 $\Phi \in \operatorname{rad}(A''_{\omega_q})$ and $\Phi^{\Box q} = 0$. However, by Lemma 4.5.4, we have

$$\begin{aligned} \left\langle \Phi^{\Box(q-1)}, \omega_q \right\rangle &= \lim_{\underline{k}\to\mathcal{U}}{}^{(q-1)} \frac{1}{\omega_q(s_{k_1})\cdots\omega_q(s_{k_{q-1}})} \langle \delta_{s_{k_1}+\cdots+s_{k_{q-1}}}, \omega_q \rangle \\ &= \lim_{\underline{k}\to\mathcal{U}}{}^{(q-1)} \frac{e^{k_1+k_2+\cdots+k_{q-1}+q-1}}{e^{k_1+1}\cdots e^{k_{q-1}+1}} = 1, \end{aligned}$$

where $\mathcal{U}$ is some filter satisfying $\Phi = \lim_{k\to\mathcal{U}} \delta_{s_k}/\omega_q(s_k)$ in the weak*-topology, so that $\Phi^{\Box(q-1)} \neq 0$, as required. $\Box$

We now turn to our main example. The weight in Theorem 4.1.3 will be defined as follows. We let

$$X_\infty = \left\{2^{k^2} : k \in \mathbb{Z}^+\right\},$$

set $\eta(n) = |n|_{X_\infty}$, and define our weight by $\omega(n) = \mathrm{e}^{\eta(n)}$ $(n \in \mathbb{Z})$. We also define a sequence of integers (n_k) by

$$n_k = 2^{k^2} + 2^{(k-1)^2} + \cdots + 1 \quad (k \in \mathbb{N}).$$

LEMMA 4.5.6. *We have* $\eta(n_k) = k+1$ $(k \in \mathbb{N})$.

PROOF. We proceed by induction on $k \in \mathbb{N}$, the base case being trivial. Take $k > 1$, and assume that the lemma holds for $k-1$. That $\eta(n_k) \leqslant k+1$ is clear from the definitions, and so it remains to show that $\eta(n_k) \geqslant k+1$. Observe that, for all $k \in \mathbb{N}$, we have

$$k2^{(k-1)^2} = \frac{k}{2^{2k-1}} 2^{k^2} < 2^{k^2}. \tag{4.23}$$

Assume towards a contradiction that $\eta(n_k) < k+1$. Then we can write $n_k = \sum_{i=1}^p c_i 2^{a_i^2}$, for some $p \in \mathbb{N}, c_1, \ldots, c_p \in \mathbb{Z}\backslash\{0\}$, and $a_1, \ldots, a_p \in \mathbb{Z}^+$ such that $\sum_{i=1}^p |c_i| < k+1$. We may suppose that $a_1 > a_2 > \cdots > a_p$. We first show that $a_1 = k$. If $a_1 \leqslant k-1$, we find that

$$n_k = \left|\sum_{i=1}^p c_i 2^{a_i^2}\right| \leqslant \left(\sum_{i=1}^p |c_i|\right) 2^{(k-1)^2} \leqslant k2^{(k-1)^2} < n_k$$

by (4.23), a contradiction. Similarly, if $a_1 \geqslant k+1$, we have

$$\begin{aligned} n_k = \left|\sum_{i=1}^{p} c_i 2^{a_i^2}\right| &\geqslant |c_1| 2^{{a_1}^2} - \left(\sum_{i=2}^{p} |c_i|\right) 2^{(a_1-1)^2} \geqslant 2^{{a_1}^2} - (k-1)2^{(a_1-1)^2} \\ &> a_1 2^{(a_1-1)^2} - (k-1)2^{(a_1-1)^2} \\ &= (a_1+1-k)2^{(a_1-1)^2} \geqslant 2\cdot 2^{k^2} > n_k, \end{aligned}$$

where we have used (4.23) to obtain the second line. Hence in either case we get a contradiction, so we must have $a_1 = k$, as claimed.

Observe that $c_1 > 0$, since otherwise

$$\sum_{i=1}^{p} c_i 2^{a_i^2} \leqslant -2^{k^2} + \sum_{i=2}^{p} |c_i| 2^{a_i^2} \leqslant -2^{k^2} + (k-1)2^{(k-1)^2} < 0.$$

Hence we have deduced that

$$2^{k^2} + n_{k-1} = n_k = 2^{k^2} + (c_1-1)2^{k^2} + \sum_{i=2}^{p} c_i 2^{a_i^2},$$

which implies that

$$n_{k-1} = (c_1-1)2^{k^2} + \sum_{i=2}^{p} c_i 2^{a_i^2},$$

and this contradicts the induction hypothesis, since $c_1 - 1 + \sum_{i=2}^{p} |c_i| < k$. $\square$

LEMMA 4.5.7. *Let $j \in \mathbb{N}$, and set $r = r(j) = 2^{2j+1}$. Then, for all $k_1, \ldots, k_r \geqslant j$, we have*

$$\left[\Omega^{(r)}(n_{k_1}, \ldots, n_{k_r})\right]^{1/r} \leqslant \mathrm{e}^{-j}.$$

PROOF. First of all, we compute

$$\begin{aligned} r(j) n_j = 2^{2j+1} \sum_{i=0}^{j} 2^{(j-i)^2} &= \sum_{i=0}^{j} 2^{2j+1+j^2-2ij+i^2} \\ &= \sum_{i=0}^{j} 2^{2i} 2^{-2i+2j-2ij+j^2+1+i^2} = \sum_{i=0}^{j} 2^{2i} \cdot 2^{(j+1-i)^2}. \end{aligned}$$

This implies that

$$\eta(rn_j) \leqslant \sum_{i=0}^{j} 2^{2i} = \frac{1}{3}(2^{2j}-1) \leqslant r,$$

so that, for all $k_1, \ldots, k_r \geqslant j$, we have

$$\begin{aligned}\eta(n_{k_1}+\cdots+n_{k_r}) &= \eta[(n_{k_1}-n_j)+\cdots+(n_{k_r}-n_j)+rn_j]\\ &\leqslant \eta(n_{k_1}-n_j)+\cdots+\eta(n_{k_r}-n_j)+\eta(rn_j)\\ &\leqslant (k_1-j)+\cdots+(k_r-j)+r\\ &= k_1+\cdots+k_r-(j-1)r.\end{aligned}$$

Hence, by Lemma 4.5.6, we have

$$\begin{aligned}\left[\Omega^{(r)}(n_{k_1},\ldots,n_{k_r})\right]^{1/r} &\leqslant \left[\frac{\mathrm{e}^{k_1+\cdots+k_r-(j-1)r}}{\mathrm{e}^{k_1+1}\cdots\mathrm{e}^{k_r+1}}\right]^{1/r}\\ &= \left[\mathrm{e}^{-(j-1)r-r}\right]^{1/r} = \mathrm{e}^{-j},\end{aligned}$$

as required. □

LEMMA 4.5.8. *Fix $j \in \mathbb{N}$, and set $r = 2^{2j+1}$. Let $J \in \mathbb{N}$ satisfy $J \geqslant j$ and*

$$2^{2k-1} > rk + 2r \quad (k \geqslant J).$$

Then, for all $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r \geqslant J$, we have

$$\eta(n_{k_1}+\cdots+n_{k_r}) \geqslant k_1+k_2+\cdots+k_r - rJ.$$

PROOF. Note that, by our hypothesis on J, whenever $k \geqslant J$ we have $2^{k^2-(k-1)^2} > rk + 2r$, which implies that

$$2^{k^2} > rk2^{(k-1)^2} + r2^{(k-1)^2+1} \quad (k \geqslant J). \tag{4.24}$$

We proceed by induction on $k_1 \geqslant J$, with the base case corresponding to the case where $k_1 = J$, and hence also $k_2 = \cdots = k_r = J$. Therefore the base hypothesis merely states that $\eta(rn_J) \geqslant 0$, which is true.

Suppose that $k_1 > J$, and assume that the lemma holds for all smaller values of $k_1 \geqslant J$. Assume towards a contradiction that there exist $k_2, \dots, k_r \in \mathbb{Z}$ such that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r \geqslant J$, and such that

$$\eta(n_{k_1} + \cdots + n_{k_r}) < k_1 + \cdots + k_r - rJ.$$

Then we may write

$$n_{k_1} + \cdots + n_{k_r} = \sum_{i=1}^{p} c_i 2^{a_i^2}$$

for some $p \in \mathbb{N}$, some $a_1, \dots, a_p \in \mathbb{Z}^+$, and some $c_1, \dots, c_p \in \mathbb{Z}\backslash\{0\}$, satisfying $a_1 > a_2 > \cdots > a_p$ and $\sum_{i=1}^{p} |c_i| < k_1 + \cdots + k_r - rJ$. Note that we have

$$\left|\sum_{i=2}^{p} c_i 2^{a_i^2}\right| < (k_1 + \cdots + k_r - rJ)2^{a_2^2} \leqslant k_1 r 2^{(a_1-1)^2}. \tag{4.25}$$

We claim that $a_1 = k_1$. Assume instead that $a_1 \geqslant k_1 + 1$. Then, using (4.24) and (4.25), we have

$$\begin{aligned}\left|\sum_{i=1}^{p} c_i 2^{a_i^2}\right| &> |c_1| 2^{a_1^2} - \left|\sum_{i=2}^{p} c_i 2^{a_i^2}\right| \\ &\geqslant 2^{a_1^2} - rk_1 2^{(a_1-1)^2} > r2^{(a_1-1)^2+1} \\ &\geqslant r2^{k_1^2+1} \geqslant n_{k_1} + \cdots + n_{k_r},\end{aligned}$$

a contradiction. If, on the other hand, we assume that $a_1 \leqslant k_1 - 1$, then (4.24) implies that

$$\left|\sum_{i=1}^{p} c_i 2^{a_i^2}\right| \leqslant (k_1 + \cdots + k_r)2^{(k_1-1)^2} \leqslant rk_1 2^{(k_1-1)^2} < n_{k_1},$$

a contradiction. Hence $a_1 = k_1$, as claimed.

Let $d \in \mathbb{N}$ be maximal such that $k_d = k_1$. We claim that $c_1 \geqslant d$. Firstly, if c_1 were negative, we would have

$$\sum_{i=1}^{p} c_i 2^{a_i^2} \leqslant -2^{k_1^2} + \left|\sum_{i=2}^{p} c_i 2^{a_i^2}\right| \leqslant -2^{k_1^2} + rk_1 2^{(k_1-1)^2} < 0,$$

by (4.24), a contradiction. Hence c_1 must be positive. Suppose that $c_1 \leqslant d-1$. Then, using (4.24) to obtain the second line, we would have

$$\sum_{i=1}^{p} c_i 2^{a_i^2} \leqslant (d-1)2^{k_1^2} + \left|\sum_{i=2}^{p} c_i 2^{a_i^2}\right| \leqslant (d-1)2^{k_1^2} + rk_1 2^{(k_1-1)^2}$$
$$< n_{k_1} + \cdots + n_{k_d} \leqslant n_{k_1} + \cdots + n_{k_r},$$

again a contradiction. Hence we must have $c_1 \geqslant d$, as claimed.

We now complete the proof. We have

$$\begin{aligned} c_1 2^{k_1^2} + \sum_{i=2}^{p} c_i 2^{a_i^2} &= n_{k_1} + \cdots + n_{k_r} \\ &= dn_{k_1} + n_{k_{d+1}} + \cdots + n_{k_r} \\ &= d2^{k_1^2} + n_{k_1-1} + \cdots + n_{k_d-1} + n_{k_{d+1}} + \cdots + n_{k_r}, \end{aligned}$$

which implies that

$$(c_1 - d)2^{k_1^2} + \sum_{i=2}^{p} c_i 2^{a_i^2} = n_{k_1-1} + \cdots + n_{k_d-1} + n_{k_{d+1}} + \cdots + n_{k_r}.$$

But

$$(c_1 - d) + \sum_{i=2}^{p} |c_i| < (k_1 - 1) + \cdots + (k_d - 1) + k_{d+1} + \cdots + k_r - Jr,$$

which contradicts the induction hypothesis applied to $k_1 - 1$. □

COROLLARY 4.5.9. *Fix $j \in \mathbb{N}$, set $r = 2^{2j+1}$, and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then* $\lim_{\underline{k}\to\mathcal{U}}{}^{(r)}\Omega^{(r)}(n_{k_1}, \ldots, n_{k_r}) > 0$.

PROOF. Let J be as in Lemma 4.5.8. Then, for all $k_1, \ldots, k_r \geqslant J$, we have

$$\eta(n_{k_1} + \cdots + n_{k_r}) \geqslant k_1 + \cdots + k_r - rJ,$$

which, when combined with Lemma 4.5.6, implies that

$$\Omega^{(r)}(n_{k_1}, \ldots, n_{k_r}) \geqslant \frac{\mathrm{e}^{k_1+\cdots+k_r-rJ}}{\mathrm{e}^{k_1+1}\cdots\mathrm{e}^{k_r+1}} = \mathrm{e}^{-r(J+1)} > 0.$$

This implies the result. $\square$

We can now prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Let Φ be a weak*-accumulation point of

$$\{\delta_{n_k}/\omega(n_k) : k \in \mathbb{N}\}.$$

Then, by Proposition 4.5.1 and Lemma 4.5.7, $\Phi \in \mathrm{rad}\,(A''_\omega)$.

Take $j \in \mathbb{N}$ and set $r = 2^{2j+1}$. Let $\mathcal{U}$ be a filter on $\mathbb{N}$ such that Φ is equal to the weak*-limit $\lim_{k\to\mathcal{U}} \delta_{n_k}/\omega(n_k)$. Then, by Corollary 4.5.9,

$$\begin{aligned}\left\langle \Phi^{\Box r}, \omega \right\rangle &= \lim_{\underline{k}\to\mathcal{U}}{}^{(r)} \left\langle \frac{1}{\omega(n_{k_1})\cdots\omega(n_{k_r})}\delta_{n_{k_1}+\cdots+n_{k_r}}, \omega \right\rangle \\ &= \lim_{\underline{k}\to\mathcal{U}}{}^{(r)} \Omega^{(r)}(n_{k_1}, \ldots, n_{k_r}) > 0.\end{aligned}$$

Hence $\Phi^{\Box r} \neq 0$. Since $r \to \infty$ as $j \to \infty$, it follows that Φ is not nilpotent. $\square$

CHAPTER 5

An Infinite C*-algebra With a Dense, Stably Finite *-subalgebra

5.1. Introduction

In this Chapter we use a construction based on semigroup algebras to solve an open problem in the theory of C*-algebras. The Chapter is based on [**56**].

Let A be a unital algebra. We say that A is *finite* (also called directly finite or Dedekind finite) if every left invertible element of A is right invertible, and we say that A is *infinite* otherwise. This notion originates in the seminal studies of projections in von Neumann algebras carried out by Murray and von Neumann in the 1930s. At the 22nd International Conference on Banach Algebras and Applications, held at the Fields Institute in Toronto in 2015, Yemon Choi raised the following questions:

Question 5.1.1.
(i) Let A be a unital, finite normed algebra. Must its completion be finite?
(ii) Let A be a unital, finite pre-C*-algebra. Must its completion be finite?

Choi also stated Question 5.1.1(i) in [**15**, Section 6].

A unital algebra A is said to be *stably finite* if the matrix algebra $M_n(A)$ is finite for each $n \in \mathbb{N}$. This stronger form of finiteness is particularly useful in the context of K-theory, and so it has become a household item in the Elliott classification program for C*-algebras. The notions of finiteness and stable finiteness differ even for C*-algebras, as was shown independently by Clarke [**17**] and Blackadar [**9**] (or see [**10**, Exercise 6.10.1]). A much deeper result is due to Rørdam [**70**, Corollary 7.2], who constructed a unital, simple C*-algebra which is finite (and separable and nuclear), but not stably finite.

We shall answer Question 5.1.1(i), and hence Question 5.1.1(ii), in the negative by proving the following result:

THEOREM 5.1.2. *There exists a unital, infinite C*-algebra which contains a dense, unital, stably finite *-subalgebra.*

Let A be a unital *-algebra. Then there is a natural variant of finiteness in this setting, namely we say that A is **-finite* if whenever we have $u \in A$ satisfying $u^*u = 1$, then $uu^* = 1$. However, it is known (see, e.g., [**69**, Lemma 5.1.2]) that a C*-algebra is finite if and only if it is *-finite, so we shall not need to refer to *-finiteness again.

The chapter is organised as follows. Section 5.2 contains some basic definitions and facts that we shall require throughout. In Section 5.3 we give a proof of a folklore result concerning free products of *-algebras for which there seems to be no self-contained proof in the literature. Then, in Section 5.4, we apply this folklore result to some examples that we shall need in the proof of our main result. The body of the proof will be given in Section 5.5.

5.2. Preliminaries

Our approach is based on semigroup algebras. Let S be a monoid, that is, a semigroup with an identity, which we shall usually denote by e. By an *involution* on S we mean a map from S to S, always denoted by $s \mapsto s^*$, satisfying $(st)^* = t^*s^*$ and $s^{**} = s$ $(s, t \in S)$. By a **-monoid* we shall mean a pair $(S, *)$, where S is a monoid, and $*$ is an involution on S. Given a *-monoid S, the algebra $\mathbb{C}S$ becomes a unital *-algebra simply by defining $\delta_s^* = \delta_{s^*}$ $(s \in S)$, and extending conjugate-linearly.

Next we shall recall some basic facts about free products of *-monoids, unital *-algebras, and their C*-representations.

Let S and T be monoids, and let A and B be unital algebras. Then we denote the free product (i.e. the coproduct) of S and T in the category of monoids by $S * T$, and similarly we denote the free product of the unital algebras A and B by $A * B$. It follows from the universal property satisfied by free products that, for monoids S and T, we have $\mathbb{C}(S * T) \cong (\mathbb{C}S) * (\mathbb{C}T)$.

Given *-monoids S and T, we can define an involution on $S * T$ by

$$(s_1 t_1 \cdots s_n t_n)^* = t_n^* s_n^* \cdots t_1^* s_1^*$$

for $n \in \mathbb{N}, s_1 \in S, s_2, \ldots, s_n \in S \backslash \{e\}, t_1, \ldots, t_{n-1} \in T \backslash \{e\}$, and $t_n \in T$. The resulting *-monoid, which we continue to denote by $S * T$, is the free product in the category of *-monoids. We can analogously define an involution on the free product of two unital *-algebras, and again the result is the free product in the category of unital *-algebras. We then find that $\mathbb{C}(S * T) \cong (\mathbb{C}S) * (\mathbb{C}T)$ as unital *-algebras.

We shall denote by S_∞ the free *-monoid on countably many generators; that is, as a monoid S_∞ is free on some countably-infinite generating set $\{t_n, s_n : n \in \mathbb{N}\}$, and the involution is determined by $t_n^* = s_n$ $(n \in \mathbb{N})$. For the rest of the text we shall simply write t_n^* in place of s_n. We define BC to be the bicyclic monoid $\langle p, q : pq = e \rangle$. This becomes a *-monoid when an involution is defined by $p^* = q$, and the corresponding *-algebra $\mathbb{C}BC$ is infinite because $\delta_p \delta_q = \delta_e$, but $\delta_q \delta_p = \delta_{qp} \neq \delta_e$.

Let A be a *-algebra. If there exists an injective *-homomorphism from A into some C*-algebra, then we say that A admits a *faithful C*-representation*. In this case, A admits a norm such that the completion of A in this norm is a C*-algebra, and we say that A admits a *C*-completion*. Our construction will be based on C*-completions of *-algebras of the form $\mathbb{C}S$, for S a *-monoid.

5.3. Free Products of *-algebras and Faithful States

In our main construction we shall want to take the free product of two unital *-algebras admitting faithful C*-representations and know that this free product again admits a faithful C*-representation. That this is true follows from a key folklore result in the theory of free products of *-algebras. The purpose of this section is to outline a proof of this result (Theorem 5.3.1 below).

By a *state* on a unital *-algebra A we mean a linear functional $\mu\colon A \to \mathbb{C}$ satisfying $\langle a^*a, \mu\rangle \geqslant 0$ $(a \in A)$ and $\langle 1, \mu\rangle = 1$. It can be shown that a state μ on a unital *-algebra A is automatically *-linear, that is $\langle a^*, \mu\rangle = \overline{\langle a, \mu\rangle}$ $(a \in A)$. We say that a state μ is *faithful* if $\langle a^*a, \mu\rangle > 0$ $(a \in A\backslash\{0\})$. A unital *-algebra with a faithful state admits a faithful C*-representation via the GNS representation associated with the state.

We can now state the main result of this section.

THEOREM 5.3.1. *Let A_1 and A_2 be unital *-algebras which admit faithful states. Then their free product $A_1 * A_2$ also admits a faithful state, and hence it has a faithful C*-representation.*

This result is folklore, and can be deduced from material in [**6**]; indeed this is where our argument originates. A more general result can be found in [**12**, Section 4]. Although the result is well known, we do not know of any source in the literature in which the proof is explicitly given, and so we provide a proof in this section for the convenience of the reader.

The free product of two unital *-algebras is best defined as another *-algebra satisfying a certain universal property. However, in order to show that such a *-algebra exists one must give an explicit construction of this object. This construction is briefly outlined in the proof of Theorem 5.3.1 below (see, for example, (5.2)), although we do not show that this object satisfies the universal property, which is standard; in the proof we always work with the explicit construction, rather than the universal property.

One may also define the free product in the category of unital C*-algebras, as in [**85**, Definition 1.4.1]. However, it is not clear *a priori* that the algebraic free-product of two unital pre-C*-algebras embeds into the C*-algebraic free product of their completions, so we cannot simply appeal to this result in the main construction of this chapter. Indeed, the special case in which both pre-C*-algebras are in fact C*-algebras, as well as related questions, seems to have caused some confusion [**1**].

This is in part our motivation for setting out the proof of Theorem 5.3.1 here. As it happens, it can be shown using Theorem 5.3.1, and Lemma 5.4.2 given below, that the algebraic free product of two unital pre-C*-algebras admits a faithful C*-representation, and hence embeds into the C*-algebraic free product, although we shall not give the details of the argument here.

The proof of Theorem 5.3.1 itself is not long, but relies heavily on some standard constructions in the theory of free products (see, e.g., [**85**, Chapter 1]), which we shall include in order to make the proof reasonably self-contained.

Proof of Theorem 5.3.1. We may suppose that A_1 and A_2 both have dimension at least 2 (because $A * \mathbb{C} \cong A \cong \mathbb{C} * A$). For $j \in \{1, 2\}$, take a faithful state μ_j on A_j, and let $\pi_j \colon A_j \to \mathcal{B}(H_j)$ be the GNS representation of A_j associated with μ_j, where H_j is the underlying Hilbert space. Since μ_j is faithful, this representation admits a separating unit vector $\xi_j \in H_j$. Let H_j° be the orthogonal complement of ξ_j in H_j, and let $A_j^\circ = \ker \mu_j$, so that H_j° and A_j° are closed subspaces of codimension 1 in H_j and A_j, respectively. (Note, however, that A_j° is not a subalgebra unless μ_j is multiplicative.) The GNS construction implies that

$$A_j^\circ = \{a \in A_j : \pi_j(a) \in H_j^\circ\}. \tag{5.1}$$

For $m \in \mathbb{N}$, let $A_j(m)$ be the tensor product of m factors alternating between A_1° and A_2°, beginning with the opposite index of j, and define $H_j(m)$ analogously using H_1° and H_2°, so that

$$A_j(m) = \underbrace{A_{\hat{j}}^\circ \otimes A_j^\circ \otimes \cdots \otimes A_{i(j,m)}^\circ}_{m \text{ factors}} \qquad \text{and} \qquad H_j(m) = \underbrace{H_{\hat{j}}^\circ \otimes H_j^\circ \otimes \cdots \otimes H_{i(j,m)}^\circ}_{m \text{ factors}},$$

where

$$\hat{j} = \begin{cases} 1 & \text{if } j = 2 \\ 2 & \text{if } j = 1 \end{cases} \qquad \text{and} \qquad i(j,m) = \begin{cases} j & \text{if } m \text{ is even} \\ \hat{j} & \text{if } m \text{ is odd.} \end{cases}$$

We can then state the standard construction of the free product of A_1 and A_2 as follows:

$$A_1 * A_2 = \mathbb{C}1 \oplus \bigoplus_{m \in \mathbb{N}} (A_1(m) \oplus A_2(m)), \tag{5.2}$$

where $\oplus$ denotes the direct sum in the category of vector spaces (so that elements are sequences with only finitely many non-zero terms). We identify $\mathbb{C}1$ and $A_j(m)$ (where $j \in \{1,2\}$ and $m \in \mathbb{N}$) with their natural images inside $A_1 * A_2$ and write P_0 and $P_{j,m}$ respectively for the canonical projections onto them. A key property of the multiplication on $A_1 * A_2$ is that

$$ab = a \otimes b \in A_j(m+n) \qquad (j \in \{1,2\},\ m,n \in \mathbb{N},\ a \in A_j(m),\ b \in A_{i(j,m)}(n)). \tag{5.3}$$

(Note that the condition that $b \in A_{i(j,m)}(n)$ ensures that the last tensor factor of a and the first tensor factor of b come from distinct subspaces A_k°, so that $a \otimes b$ belongs to $A_j(m+n)$, as stated.)

The free product $A_1 * A_2$ has a standard *-represention on the Hilbert space

$$H = \mathbb{C}\Omega \oplus \bigoplus_{m \in \mathbb{N}} (H_1(m) \oplus H_2(m)),$$

where Ω is a chosen unit vector (conventionally called the *vacuum vector*), and $\oplus$ denotes the direct sum in the category of Hilbert spaces (so that elements are sequences whose terms are square-summable in norm); again we identify $\mathbb{C}\Omega$ and $H_j(m)$ (where $j \in \{1,2\}$ and $m \in \mathbb{N}$) with their natural images inside H and write Q_0 and $Q_{j,m}$ for the canonical projections onto them.

To define the above-mentioned *-representation of $A_1 * A_2$ on H, we require a pair of unitary operators $V_j \colon H_j \otimes H(j) \to H$, where $j \in \{1,2\}$ and $H(j)$ denotes the subspace $\mathbb{C}\Omega \oplus \bigoplus_{m \in \mathbb{N}} H_j(m)$ of H. It suffices to say how V_j acts on elementary tensors of the form $x \otimes y$ with $x \in H_j$ and $y \in H(j)$. In turn, we need only consider the case where x belongs to one of the two direct summands $\mathbb{C}\xi_j$ and H_j° of H_j and

y to one of the summands $\mathbb{C}\Omega$ and $H_j(m)$, $m \in \mathbb{N}$, of $H(j)$. These cases are specified by:

(1) $V_j(\xi_j \otimes \Omega) = \Omega$;

(2) $V_j(\xi_j \otimes y) = y$ for $y \in H_j(m)$;

(3) $V_j(x \otimes \Omega) = x$ for $x \in H_j^\circ = H_{\hat{j}}(1)$;

(4) $V_j(x \otimes y) = x \otimes y \in H_j^\circ \otimes H_j(m) = H_{\hat{j}}(m+1)$ for $x \in H_j^\circ$ and $y \in H_j(m)$.

We then obtain a unital *-homomorphism $\lambda_j \colon A_j \to \mathcal{B}(H)$ by the definition

$$\lambda_j(a) = V_j(\pi_j(a) \otimes I_{H(j)})V_j^* \qquad (a \in A_j,\ j \in \{1,2\}),$$

and finally the universal property of the free product implies that there is a unique unital *-homomorphism $\pi = \lambda_1 * \lambda_2 \colon A_1 * A_2 \to \mathcal{B}(H)$ such that $\pi|_{A_j} = \lambda_j$ for $j \in \{1,2\}$.

We are now ready to embark on the actual proof of Theorem 5.3.1. Our aim is to show that the vacuum vector Ω is separating for the *-representation π. Once we have shown that, it follows that the map $a \mapsto \langle \pi(a)\Omega, \Omega\rangle_H$, where $\langle\cdot,\cdot\rangle_H$ denotes the inner product on H, is a faithful state on $A_1 * A_2$ and that the *-representation π is faithful.

Observe that by applying (5.1) together with the above definitions we may deduce that, for $j \in \{1,2\}$ and $a \in A_j^\circ$,

$$\pi(a)\Omega = \pi_j(a)\xi_j \in H_j^\circ = H_{\hat{j}}(1), \tag{5.4}$$

$$\pi(a)y = \pi_j(a)\xi_j \otimes y \in H_{\hat{j}}(m+1) \qquad (m \in \mathbb{N},\ y \in H_j(m)). \tag{5.5}$$

We then use these identities to prove that

$$\pi(a)\Omega = \pi_{\hat{j}}(a_1)\xi_{\hat{j}} \otimes \pi_j(a_2)\xi_j \otimes \cdots \otimes \pi_{i(j,m)}(a_m)\xi_{i(j,m)} \in H_j(m) \tag{5.6}$$

for $j \in \{1,2\}$, $m \in \mathbb{N}$ and $a = a_1 \otimes \cdots \otimes a_m \in A_j(m)$. The proof is by induction on m, with the base case ($m = 1$) already established by (5.4). Now let $m \geqslant 2$, and assume inductively that the result holds for $m-1$. Equation (5.3) implies that

$a = a_1(a_2 \otimes \cdots \otimes a_m)$, so that

$$\pi(a)\Omega = \pi(a_1)\pi(a_2 \otimes \cdots \otimes a_m)\Omega = \pi(a_1)(\pi_j(a_2)\xi_j \otimes \cdots \otimes \pi_{i(\widehat{j},m-1)}(a_m)\xi_{i(\widehat{j},m-1)})$$
$$= \pi_{\widehat{j}}(a_1)\xi_{\widehat{j}} \otimes \pi_j(a_2)\xi_j \otimes \cdots \otimes \pi_{i(j,m)}(a_m)\xi_{i(j,m)}$$

by the multiplicativity of π, the fact that $i(\widehat{j}, m-1) = i(j,m)$, the induction hypothesis, and (5.5). Thus the induction continues.

Our next step is to show that, for each $a \in A_1 * A_2$,

$$Q_0(\pi(a)\Omega) = \pi(P_0 a)\Omega \quad \text{and} \quad Q_{j,m}(\pi(a)\Omega) = \pi(P_{j,m}a)\Omega \quad (j \in \{1,2\},\ m \in \mathbb{N}). \tag{5.7}$$

Take $M \in \mathbb{N}$ such that $a = P_0 a + \sum_{j=1}^{2}\sum_{m=1}^{M} P_{j,m}a$. The linearity of π implies that

$$\pi(a)\Omega = \pi(P_0 a)\Omega + \sum_{j=1}^{2}\sum_{m=1}^{M} \pi(P_{j,m}a)\Omega,$$

where $\pi(P_0 a)\Omega \in \mathbb{C}\Omega$ because π is unital, while (5.6) shows that $\pi(P_{j,m}a)\Omega \in H_j(m)$ for each j and m. Hence (5.7) follows from the definitions of Q_0 and $Q_{j,m}$.

We can now verify that Ω is a separating vector for π. Suppose that $\pi(a)\Omega = 0$ for some $a \in A_1 * A_2$. We must prove that $a = 0$, that is, $P_0 a = 0$ and $P_{j,m}a = 0$ for each $j \in \{1,2\}$ and $m \in \mathbb{N}$. The first of these identities is easy: taking $\alpha \in \mathbb{C}$ such that $P_0 a = \alpha 1$, we have

$$\alpha\Omega = \pi(P_0 a)\Omega = Q_0(\pi(a)\Omega) = 0,$$

so that $\alpha = 0$ and therefore $P_0 a = 0$. To establish the other identity, let $j \in \{1,2\}$ and $m \in \mathbb{N}$, and write

$$P_{j,m}a = \sum_{k=1}^{n} a_{1,k} \otimes a_{2,k} \otimes \cdots \otimes a_{m,k},$$

where $n \in \mathbb{N}$ and $a_{i,k} \in A_{\hat{j}}^{\circ}$ for i odd and $a_{i,k} \in A_j^{\circ}$ for i even. Equations (5.7) and (5.6) imply that

$$
\begin{aligned}
(5.8) \qquad 0 &= Q_{j,m}(\pi(a)\Omega) = \pi(P_{j,m}a)\Omega \\
&= \sum_{k=1}^{n} \pi_{\hat{j}}(a_{1,k})\xi_{\hat{j}} \otimes \pi_j(a_{2,k})\xi_j \otimes \cdots \otimes \pi_{i(j,m)}(a_{m,k})\xi_{i(j,m)} \\
&= (\pi_{\hat{j}} \otimes \pi_j \otimes \cdots \otimes \pi_{i(j,m)})(P_{j,m}a)(\xi_{\hat{j}} \otimes \xi_j \otimes \cdots \otimes \xi_{i(j,m)}),
\end{aligned}
$$

where $\pi_{\hat{j}} \otimes \pi_j \otimes \cdots \otimes \pi_{i(j,m)}$ is the unique *-homomorphism from the m-fold alternating tensor product $A_{\hat{j}} \otimes A_j \otimes \cdots \otimes A_{i(j,m)}$ into $\mathcal{B}(H_{\hat{j}} \otimes H_j \otimes \cdots \otimes H_{i(j,m)})$ such that

$$
\pi_{\hat{j}} \otimes \pi_j \otimes \cdots \otimes \pi_{i(j,m)}(a) = \pi_{\hat{j}}(a_1) \otimes \pi_j(a_2) \otimes \cdots \otimes \pi_{i(j,m)}(a_m)
$$

for each $a = a_1 \otimes a_2 \otimes \cdots \otimes a_m \in A_{\hat{j}} \otimes A_j \otimes \cdots \otimes A_{i(j,m)}$. The vector $\xi_{\hat{j}} \otimes \xi_j \otimes \cdots \otimes \xi_{i(j,m)}$ is separating for this *-representation because ξ_1 and ξ_2 are separating for π_1 and π_2, respectively, and therefore (5.8) implies that $P_{j,m}a = 0$, as required. ☐

5.4. Applications of Theorem 5.3.1 to Some Examples

In this section we prove that the *-algebras important to the proof our main theorem have faithful C*-completions.

LEMMA 5.4.1. *The following unital *-algebras admit faithful C*-representations:*

(i) $\mathbb{C}(BC)$,

(ii) $\mathbb{C}(S_\infty)$.

PROOF. (i) Since BC is an inverse semigroup, this follows from [**8**, Theorem 2.3].

(ii) By [**7**, Theorem 3.4] $\mathbb{C}S_2$ admits a faithful C*-representation, where S_2 denotes the free monoid on two generators $S_2 = \langle a, b \rangle$, endowed with the involution determined by $a^* = b$. There is a *-monomorphism $S_\infty \hookrightarrow S_2$ defined by $t_n \mapsto a(a^*)^n a$ $(n \in \mathbb{N})$ and this induces a *-monomorphism $\mathbb{C}S_\infty \hookrightarrow \mathbb{C}S_2$. The result follows. ☐

Next we prove a lemma which is probably well known to experts in the theory of C*-algebras, but, much like Theorem 5.3.1, does not seem to have an appropriate reference available. We record a short proof.

LEMMA 5.4.2. *Any separable C*-algebra admits a faithful state.*

PROOF. Let A be a separable C*-algebra. Note that the unit ball of A' with the weak*-topology is a compact metric space, and hence also separable. It follows that the set of states $S(A)$ is weak*-separable. Taking $\{\rho_n : n \in \mathbb{N}\}$ to be a dense subset of $S(A)$, we then define $\rho = \sum_{n=1}^{\infty} 2^{-n}\rho_n$, which is easily seen to be a faithful state on A. □

LEMMA 5.4.3. *The unital *-algebra $\mathbb{C}(BC * S_\infty)$ admits a faithful C*-representation.*

PROOF. By Lemma 5.4.1, both $\mathbb{C}(BC)$ and $\mathbb{C}(S_\infty)$ admit C*-completions. Since both of these algebras have countable dimension, their C*-completions are separable, and, as such, each admits a faithful state by Lemma 5.4.2, which we may then restrict to obtain faithful states on $\mathbb{C}BC$ and $\mathbb{C}S_\infty$. By Theorem 5.3.1, $(\mathbb{C}BC) * (\mathbb{C}S_\infty) \cong \mathbb{C}(BC * S_\infty)$ admits a faithful C*-representation. □

5.5. Proof of Theorem 5.1.2

The main idea of the proof is to embed $\mathbb{C}S_\infty$, which is finite, as a dense *-subalgebra of some C*-completion of $\mathbb{C}(BC * S_\infty)$, which will necessarily be infinite. In fact we have the following:

LEMMA 5.5.1. *The *-algebra $\mathbb{C}S_\infty$ is stably finite.*

PROOF. As we remarked in the proof of Lemma 5.4.1, $\mathbb{C}S_\infty$ embeds into $\mathbb{C}S_2$. It is also clear that, as an algebra, $\mathbb{C}S_2$ embeds into $\mathbb{C}F_2$, where F_2 denotes the free group on two generators. Hence $\mathbb{C}S_\infty$ embeds into $\mathrm{vN}(F_2)$, the group von Neumann algebra of F_2, which is stably finite since it is a C*-algebra with a faithful tracial state. It follows that $\mathbb{C}S_\infty$ is stably finite as well. □

We shall next define a notion of length for elements of $BC * S_\infty$. Indeed, each $u \in (BC * S_\infty)\backslash\{e\}$ has a unique expression of the form $w_1 w_2 \cdots w_n$, for some $n \in \mathbb{N}$ and some $w_1, \ldots, w_n \in (BC\backslash\{e\}) \cup \{t_j, t_j^* : j \in \mathbb{N}\}$, satisfying $w_{i+1} \in \{t_j, t_j^* : j \in \mathbb{N}\}$ whenever $w_i \in BC\backslash\{e\}$ $(i = 1, \ldots, n-1)$. We then define $\operatorname{len} u = n$ for this value of n, and set $\operatorname{len} e = 0$. This also gives a definition of length for elements of S_∞ by considering S_∞ as a submonoid of $BC * S_\infty$ in the natural way. For $m \in \mathbb{N}_0$ we set

$$L_m(BC * S_\infty) = \{u \in BC * S_\infty : \operatorname{len} u \leqslant m\};$$

$$L_m(S_\infty) = \{u \in S_\infty : \operatorname{len} u \leqslant m\}.$$

We now describe our embedding of $\mathbb{C}S_\infty$ into $\mathbb{C}(BC * S_\infty)$. By Lemma 5.4.3, $\mathbb{C}(BC * S_\infty)$ has a C*-completion $(A, \|\cdot\|)$. Let $\gamma_n = (n\|\delta_{t_n}\|)^{-1}$ $(n \in \mathbb{N})$ and define elements a_n in $\mathbb{C}(BC * S_\infty)$ by $a_n = \delta_p + \gamma_n \delta_{t_n}$ $(n \in \mathbb{N})$, so that $a_n \to \delta_p$ as $n \to \infty$. Using the universal property of S_∞ we may define a unital *-homomorphism $\varphi\colon \mathbb{C}S_\infty \to \mathbb{C}(BC * S_\infty)$ by setting $\varphi(\delta_{t_n}) = a_n$ $(n \in \mathbb{N})$ and extending to $\mathbb{C}S_\infty$. In what follows, given a monoid S and $s \in S$, δ'_s will denote the linear functional on $\mathbb{C}S$ defined by $\langle \delta_t, \delta'_s \rangle = \mathbb{1}_{s,t}$ $(t \in S)$, where $\mathbb{1}_{s,t}$ is the Kronecker delta, as defined in (1.1).

Lemma 5.5.2. *Let $w \in S_\infty$ with $\operatorname{len} w = m$. Then*

(i) $\varphi(\delta_w) \in \operatorname{span}\{\delta_u : u \in L_m(BC * S_\infty)\}$;

(ii) *for each $y \in L_m(S_\infty)$ we have*

$$\langle \varphi(\delta_y), \delta'_w \rangle \neq 0 \Leftrightarrow y = w.$$

Proof. We proceed by induction on m. When $m = 0$, w is forced to be e and hence, as φ is unital, $\varphi(\delta_e) = \delta_e$, so that (i) is satisfied. In (ii), y is also equal to e, so that (ii) is trivially satisfied as well.

Assume $m \geqslant 1$ and that (i) and (ii) hold for all elements of $L_{m-1}(S_\infty)$. We can write w as $w = vx$ for some $v \in S_\infty$ with $\operatorname{len} v = m-1$ and some $x \in \{t_j, t_j^* : j \in \mathbb{N}\}$.

First consider (i). By the induction hypothesis, we can write $\varphi(\delta_v) = \sum_{u\in E} \alpha_u \delta_u$, for some finite set $E \subset L_{m-1}(BC * S_\infty)$ and some scalars $\alpha_u \in \mathbb{C}$ $(u \in E)$. Suppose that $x = t_j$ for some $j \in \mathbb{N}$. Then

$$\varphi(\delta_w) = \varphi(\delta_v)\varphi(\delta_{t_j}) = \left(\sum_{u\in E} \alpha_u \delta_u\right)(\delta_p + \gamma_j \delta_{t_j}) = \sum_{u\in E} \alpha_u \delta_{up} + \alpha_u \gamma_j \delta_{ut_j},$$

which belongs to $\operatorname{span}\{\delta_u : u \in L_m(BC * S_\infty)\}$ because

$$\operatorname{len}(up) \leqslant \operatorname{len}(u) + 1 \leqslant m \quad \text{and} \quad \operatorname{len}(ut_j) = \operatorname{len}(u) + 1 \leqslant m$$

for each $u \in L_{m-1}(BC * S_\infty)$. The case $x = t_j^*$ is established analogously.

Next consider (ii). Let $y \in L_m(S_\infty)$. If $\operatorname{len} y \leqslant m - 1$ then, by (i), we know that $\varphi(\delta_y) \in \operatorname{span}\{\delta_u : u \in L_{m-1}(BC * S_\infty)\} \subset \ker \delta'_w$. Hence in this case $y \neq w$ and $\langle \varphi(\delta_y), \delta'_w \rangle = 0$.

Now suppose instead that $\operatorname{len} y = m$, and write $y = uz$ for some $u \in L_{m-1}(S_\infty)$ and $z \in \{t_j, t_j^* : j \in \mathbb{N}\}$. By (i) we may write $\varphi(\delta_u) = \sum_{s\in F} \beta_s \delta_s$ for some finite subset $F \subset L_{m-1}(BC * S_\infty)$ and some scalars $\beta_s \in \mathbb{C}$ $(s \in F)$, and we may assume that $v \in F$ (possibly with $\beta_v = 0$). We prove the result in the case that $z = t_j$ for some $j \in \mathbb{N}$, with the argument for the case $z = t_j^*$ being almost identical. We have $\varphi(\delta_z) = \delta_p + \gamma_j \delta_{t_j}$ and it follows that

$$\varphi(\delta_y) = \varphi(\delta_u)\varphi(\delta_z) = \sum_{s\in F} \beta_s \delta_{sp} + \beta_s \gamma_j \delta_{st_j}.$$

Observe that $sp \neq w$ for each $s \in F$. This is because we either have $\operatorname{len}(sp) < m = \operatorname{len}(w)$, or else sp ends in p when considered as a word over the alphabet $\{p, p^*\} \cup \{t_j, t_j^* : j \in \mathbb{N}\}$, whereas $w \in S_\infty$. Moreover, given $s \in F$, $st_j = w = vx$ if and only if $s = v$ and $t_j = x$. Hence

$$\langle \varphi(\delta_y), \delta'_w \rangle = \beta_v \gamma_j \mathbb{1}_{t_j,x} = \langle \varphi(\delta_u), \delta'_v \rangle \gamma_j \mathbb{1}_{t_j,x}.$$

As $\gamma_j > 0$, this implies that $\langle \varphi(\delta_y), \delta'_w \rangle \neq 0$ if and only if $\langle \varphi(\delta_u), \delta'_v \rangle \neq 0$ and $t_j = x$, which, by the induction hypothesis, occurs if and only if $u = v$ and $t_j = x$. This final statement is equivalent to $y = w$. □

COROLLARY 5.5.3. *The map φ is injective.*

PROOF. Assume towards a contradiction that $\sum_{u \in F} \alpha_u \delta_u \in \ker \varphi$ for some non-empty finite set $F \subset S_\infty$ and $\alpha_u \in \mathbb{C} \backslash \{0\}$ $(u \in F)$. Take $w \in F$ of maximal length. Then

$$0 = \left\langle \varphi\left(\sum_{u \in F} \alpha_u \delta_u\right), \delta'_w \right\rangle = \sum_{u \in F} \alpha_u \langle \varphi(\delta_u), \delta'_w \rangle = \alpha_w \langle \varphi(\delta_w), \delta'_w \rangle,$$

where the final equality follows from Lemma 5.5.2(ii). That lemma also tells us that $\langle \varphi(\delta_w), \delta'_w \rangle \neq 0$, forcing $\alpha_w = 0$, a contradiction. □

We can now prove our main theorem.

PROOF OF THEOREM 5.1.2. Recall that $(A, \| \cdot \|)$ denotes a C*-completion of $\mathbb{C}(BC * S_\infty)$, which exists by Lemma 5.4.3, and A is infinite since $\delta_p, \delta_q \in A$. Let $A_0 \subset A$ be the image of φ. Corollary 5.5.3 implies that $A_0 \cong \mathbb{C}S_\infty$, which is stably finite by Lemma 5.5.1. Moreover, $\varphi(\delta_{t_n}) = a_n \to \delta_p$ as $n \to \infty$, so that $\delta_p \in \overline{A_0}$, and we see also that $\delta_{t_n} = \frac{1}{\gamma_n}(a_n - \delta_p) \in \overline{A_0}$ $(n \in \mathbb{N})$. The elements δ_p and δ_{t_n} $(n \in \mathbb{N})$ generate A as a C*-algebra, and since $\overline{A_0}$ is a C*-subalgebra containing them, we must have $A = \overline{A_0}$, which completes the proof. □

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