A study of stochastic processes in Banach spaces

James Stuart Groves, MA

Submitted to the University of Lancaster in July 2000
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Abstract

The theory of 2-convex norms is applied to Banach space valued random vectors. Use is made of a norm on random vectors, introduced by Pisier, equal to the 2-absolutely summing norm on an associated space of operators.

For Q the variance of some centred Gaussian random vector in a separable Banach space it is shown that, necessarily, Q factors through l^2 as a product of 2-summing operators. This factorisation condition is sufficient when the Banach space is of Gaussian type 2. The stochastic integral of a family of operators with respect to a cylindrical Q-Wiener process is shown to exist under a Hölder continuity condition involving the 2-summing norm.

A Langevin equation

$$d\mathbf{Z}_t + \Lambda \mathbf{Z}_t dt = d\mathbf{B}_t$$

with values in a separable Banach space is studied. The operator Λ is closed and densely defined. A weak solution $(\mathbf{Z}_t, \mathbf{B}_t)$, where $\mathbf{Z}_t$ is centred, Gaussian and stationary while $\mathbf{B}_t$ is a Q-Wiener process, is given when $i\Lambda$ and $i\Lambda^*$ generate C_0 groups and the resolvent of Λ is uniformly bounded on the imaginary axis. Both $\mathbf{Z}_t$ and $\mathbf{B}_t$ are stochastic integrals with respect to a spectral Q-Wiener process.

The convolution of two arcsine probability densities is shown to be an elliptic integral.

Ensembles $(X_n)_{n\geq 1}$ of random Hermitian matrices are considered. Each X_n is n by n with distribution invariant under unitary conjugation and induced by a positive weight function on $\mathbb{R}$. New proofs are given of results, due to Boutet de Monvel, Pastur, Shcherbina and Sodin, on the behaviour of the empirical distribution of the eigenvalues of X_n as n tends to infinity.

Results in analytic function theory are proved. An H^∞ interpolating sequence in the disc $\mathbb{D}$ whose Horowitz product does not lie in the Bergman space $L^2_a(\mathbb{D})$ is exhibited. A condition satisfied by Banach spaces of non-trivial analytic Lusin cotype is obtained.

Contents

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Chapter 1

Introduction

This thesis considers various aspects of the theory of Banach space valued random vectors and stochastic processes. This topic has been extensively studied in recent years; of notable interest are the books [8] and [34].

1.1 Ornstein-Uhlenbeck processes

The thesis will study a Langevin equation for stochastic processes with values in a separable complex Banach space E. The one-dimensional Langevin equation is the Itô differential equation

$$dZ_t + \rho Z_t dt = db_t \tag{1.1}$$

for $t \in \mathbb{R}$, where the constant $\rho > 0$ describes a frictional resistance. We seek a pair of processes (b_t, Z_t), defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which solve equation (1.1); the process $(b_t)_{t\in\mathbb{R}}$ is required to be a complex Brownian motion on the line with $b_0 = 0$ and the process $(Z_t)_{t\in\mathbb{R}}$ is required to be a complex valued centred Gaussian stationary stochastic process which is adapted to the filtration induced by b_t and has almost surely Hölder continuous sample paths. The concept of a stationary stochastic process was introduced by Khinchin in [28].

It is well known, following Uhlenbeck and Ornstein's paper [49], that solutions (b_t, Z_t) of equation (1.1) exist. The stationary process Z_t is unique in distribution and called the

Ornstein-Uhlenbeck process with parameter ρ. We may write Z_t as a stochastic spectral integral

$$Z_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\rho + i\omega} \, d\tilde{b}_\omega, \tag{1.2}$$

where $(\tilde{b}_\omega)_{\omega \in \mathbb{R}}$ is a given complex Brownian motion on the line defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\tilde{b}_0 = 0$. The process b_t is given in terms of $\tilde{b}_\omega$ by the condition $b_0 = 0$ and the stochastic spectral integral, for $s < t$,

$$b_t - b_s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \, d\tilde{b}_\omega. \tag{1.3}$$

The formulae (1.2) and (1.3) were originally derived by mathematicians developing the theory of linear filters on stationary stochastic processes. Of note are Blanc-Lapierre and Fortet's paper [3], which discusses the basic properties of filters, and Kolmogorov's paper [29], which discusses spectral representations of solutions to linear constant coefficient stochastic differential equations — formula (24) of Kolmogorov's paper is a generalised form of (1.2). See chapter XI of [13] for a detailed description of the spectral theory of scalar valued stationary stochastic processes.

Adaptedness of the process Z_t to the filtration induced by b_t follows from the existence of a time domain integral

$$Z_t = \int_{-\infty}^{t} e^{-\rho(t-u)} \, db_u, \tag{1.4}$$

which expresses Z_t as a stochastic integral with respect to b_t. All stochastic integrals are interpreted in the Itô sense. The autocovariance of the process Z_t is

$$Cov(Z_s, Z_t) = \frac{e^{-\rho|s-t|}}{2\rho}. \tag{1.5}$$

The Ornstein-Uhlenbeck process is Gaussian, strongly Markovian and stationary with almost surely Hölder continuous sample paths.

For more information, particularly on the physical motivation for studying these processes, we refer the reader to [19] or [44].

In chapter 4 of this thesis we consider a generalisation of the Langevin equation to the Banach space valued case. We let E be a separable complex Banach space and consider the stochastic differential equation

$$d\mathbf{Z}_t + \Lambda \mathbf{Z}_t dt = d\mathbf{B}_t \tag{1.6}$$

for $t \in \mathbb{R}$, where Λ is a closed operator from a norm dense domain $\mathcal{D}(\Lambda) \subseteq E$ to E.

We seek a pair of processes $(\mathbf{B}_t, \mathbf{Z}_t)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which are a weak solution to equation (1.6); the concept of a weak solution follows that of Da Prato and Zabczyk, for which see [8], and is defined formally in chapter 4. The approach we adopt of seeking a pair of processes, neither of which is given in advance, is used by Øksendal in [40]. The process $(\mathbf{B}_t)_{t\in\mathbb{R}}$ is required to be an E valued cylindrical Q-Wiener process; our terminology for Wiener processes follows that used in [8]. The process $(\mathbf{Z}_t)_{t\in\mathbb{R}}$ is required to be an E valued, centred Gaussian, stationary stochastic process with almost surely Hölder continuous sample paths.

When generalising results on scalar valued random variables and stochastic processes to the Banach space valued case, several problems arise concerning how to describe concepts such as expectation, L^2 boundedness, covariance and stationarity in a wider setting. Chapter 2 of this thesis uses ideas developed by G. Pisier in the paper [43] to develop the theory of spaces of Banach space valued random vectors with bounded variance. Various weak forms of the L^2 norm are considered; these are contrasted with the more usual Bochner L^2 norm.

Particular use is made of a norm on spaces of random vectors, introduced by Pisier, which is equal to the 2-absolutely summing norm on an associated space of linear operators; this norm is denoted by π_2. Chapter 3 characterises Gaussian random vectors and cylindrical Q-Wiener processes in a separable Banach space E using this norm; it is shown that, necessarily, Q factors through l^2 as $A\overline{A^*}$, where A is an operator from l^2 to E with 2-summing adjoint. This factorisation condition is shown to be sufficient when E is of Gaussian type 2.

Chapter 3 also considers the theory of stochastic integration in a separable Banach space for deterministic integrands with respect to a cylindrical Q-Wiener process. The following theorem is proved.

Theorem 1.1.1 *For E a separable Banach space, let $\mathbf{B}_t$ be an E valued cylindrical $A\overline{A^*}$-Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for $s < t$, if $(T_u)_{s \le u \le t}$ is a non-random family of bounded linear operators on E such that $(A^* T_u^*)_{s \le u \le t}$ is Hölder continuous in the π_2 norm, the stochastic integral*

$$\int_s^t T_u \, d\mathbf{B}_u \tag{1.7}$$

exists in the Itô sense, as the L^2 limit of appropriate Riemann sums under refinement of partitions. Furthermore

$$\pi_2 \left(\int_s^t T_u \, d\mathbf{B}_u \right)^2 \le \int_s^t \pi_2 (A^* T_u^*)^2 \, du. \tag{1.8}$$

Having formalised the framework under which Banach space valued stochastic processes will be discussed we prove the existence in chapter 4, under certain boundedness conditions on Λ, of pairs of processes $(\mathbf{B}_t, \mathbf{Z}_t)$ which solve the Banach space valued Langevin equation in the weak sense.

Theorem 1.1.2 *Assume $i\Lambda$ and $i\Lambda^*$ generate C_0 groups of operators on E and the resolvent of Λ is uniformly bounded on the imaginary axis. Consider the stochastic spectral integral*

$$\mathbf{Z}_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega, \tag{1.9}$$

where $\tilde{\mathbf{B}}_\omega$ is a given E valued cylindrical Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\mathbf{B}_t$ subject to $\mathbf{B}_0 = 0$ and the stochastic spectral integral, for $s < t$,

$$\mathbf{B}_t - \mathbf{B}_s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \, d\tilde{\mathbf{B}}_\omega. \tag{1.10}$$

The processes $\mathbf{Z}_t$ and $\mathbf{B}_t$ converge in L^2 as Itô stochastic integrals and the pair $(\mathbf{B}_t, \mathbf{Z}_t)$ is a weak solution of equation (1.6).

The process $\mathbf{Z}_t$ is a generalisation to the E valued case of the classical Ornstein-Uhlenbeck process. Such a generalisation has been done previously, notably by Itô in his paper [26]; the difference in our case is that our solution $\mathbf{Z}_t$ is represented as a stochastic spectral integral, rather than an integral in the time domain.

Note we do not require that $\mathbf{Z}_t$ be adapted to the filtration induced by $\mathbf{B}_t$. We obtain adaptedness in the important case where $(-\Lambda)$ generates a C_0 semigroup $(e^{-\Lambda t})_{t\geq 0}$ of exponential norm decay, however, by demonstrating the existence of a time domain integral

$$\mathbf{Z}_t = \int_{-\infty}^{t} e^{-\Lambda(t-u)}\, d\mathbf{B}_u \tag{1.11}$$

which expresses $\mathbf{Z}_t$ as a stochastic integral with respect to $\mathbf{B}_t$.

Chapter 4 also considers some specific examples of E valued Langevin equations and their weak solutions. Each example corresponds to an operator Λ for which neither Λ nor $(-\Lambda)$ generate C_0 semigroups of exponential norm decay.

1.2 Random matrices

The thesis also concerns itself with the theory of random matrices. These arise in numerous areas of statistical and quantum physics; the book [39] provides an introduction to the subject.

Chapter 6 considers ensembles $(X_n)_{n\geq 1}$ of random Hermitian matrices. Each X_n is n by n and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution of X_n is invariant under conjugation by unitary maps and induced, via functional calculus, by a positive weight function on $\mathbb{R}$. Such matrices were studied, for example, in the papers [6] and [41].

We study a fundamental sequence of random measures associated to $(X_n)_{n\geq 1}$. For each X_n define the n-tuple $(\lambda_1(X_n), \ldots, \lambda_n(X_n))$ to be the eigenvalues of X_n arranged in decreasing order. Define, for $\omega \in \Omega$ and each n,

$$\nu_n(\omega) = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(X_n(\omega))}. \tag{1.12}$$

The random probability measure ν_n is referred to as the empirical distribution of the eigenvalues of X_n. It is also known as the spectral multiplicity measure.

The framework developed in chapter 2 for studying classes of random vectors is applied to these random measures; this enables us to define the expectation $\mathbb{E}\nu_n$ for each

n. The concept of L^2 norm convergence for such sequences is studied in this setting.

Chapter 6 will prove some results on the limiting behaviour of ν_n as n tends to infinity. We prove the following.

Theorem 1.2.1 *The sequence $(\nu_n - \mathbb{E}\nu_n)_{n\geq 1}$ tends to zero in L^2 norm as n tends to infinity. Furthermore if the weight function generating $(X_n)_{n\geq 1}$ is supported on $[-1, 1]$ and twice differentiable on $(-1, 1)$ with only finitely many zeros, all of finite order, then ν_n tends to the standard arcsine distribution in norm as n tends to infinity.*

These results are known — the first sentence above is due to Boutet de Monvel, Pastur and Shcherbina and stated, with proof, in their paper [6] while the second sentence is stated by Pastur in his paper [41] and attributed to Sodin. The proofs we give are new, however. Furthermore by stating and proving these results using the framework developed in chapter 2, we manage to simplify notation and avoid use of the Stieltjes transform.

The arcsine probability distribution appears prominently in the results of chapter 6. In chapter 5 we give background information on this distribution and show that the convolution of two standard arcsine probability densities may be expressed as a complete elliptic integral of the first kind.

1.3 Analytic function theory

The final part of the thesis obtains some results in analytic function theory.

Chapter 7 considers Bergman spaces and Horowitz products on the disc $\mathbb{D}$. Horowitz products were introduced in [25] and play a rôle in Bergman space theory analogous to the rôle played by Blaschke products in Hardy space theory. The Horowitz product of a Bergman space zero sequence (a_j) converges locally uniformly on $\mathbb{D}$ to an analytic function with zeros (a_j).

We exhibit a sequence (a_j) in $\mathbb{D}$ which is H^∞ interpolating, in Carleson's sense, yet the Horowitz product associated to (a_j) does not lie in the Bergman space $L^2_a(\mathbb{D})$.

Chapter 7 also considers the concept of analytic Lusin cotype for Banach spaces. It is shown that, if E is a Banach space of analytic Lusin cotype q, the Hardy space $H^q(\mathbb{D}; E)$ satisfies a so-called geometric radial lower q-estimate.

This part of the thesis is essentially separate from the parts preceding it. Analytic function theory is connected to the theory of stochastic processes, however, via the theory of spectral representations of stationary stochastic processes; the Ornstein-Uhlenbeck processes of chapter 4 are examples of such representations.

Masani and Wiener, in their papers [37] and [38], famously used analytic function theory to study spectral representations of stochastic processes in $\mathbb{C}^n$ and develop a prediction theory for such processes. It seems likely that, as such results are extended to the Banach space valued case, analytic function theory will continue play a crucial rôle; this was conjectured by Pisier at the end of his paper [43].

Chapter 2

Banach space valued random vectors

In this chapter we develop the formalism we need to adequately deal with the theory of Banach space valued random vectors and their covariances. The ideas largely derive from Pisier's paper [43]. The books [12], [34] and [42], together with the papers [35] and [36], have proved invaluable.

Following [43] we use positive sesquilinear forms to define norms on Banach space valued random vectors which equal certain 2-convex norms on equivalent operators from the dual of the Banach space to a Hilbert space. For details of 2-convexity the reader is directed to section 2 of [43]. One norm in particular is equal to the 2-absolutely summing norm; this will enable us to bound various stochastic integrals in chapters 3 and 4. The 2-convex norms yield a natural notion of covariance for pairs of random vectors.

2.1 Definitions

We introduce some notation. For a complex vector space V, $\overline{V}$ denotes V endowed with the conjugate scalar multiplication $(\lambda, v) \mapsto \overline{\lambda}v$. Write $\overline{v}$ for $v \in V$ viewed as an element of $\overline{V}$.

Throughout this thesis, all adjoints of linear operators are to be interpreted in the

Banach space sense.

A brief explanation is needed concerning relationships between finite rank operators and tensors. Let E and F be Banach spaces and let $(\xi_k)_k$, $(\xi_k^*)_k$ and $(\eta_k)_k$ denote sequences in E, E^* and F respectively. The space of finite rank operators $E \to F$ is to be identified with the space of tensors $E^* \otimes F$; we identify the finite rank operator

$$\xi \mapsto \sum_k \xi_k^*(\xi)\eta_k \tag{2.1}$$

in $\mathcal{B}(E, F)$ with the tensor

$$\sum_k \xi_k^* \otimes \eta_k \tag{2.2}$$

in $E^* \otimes F$. The space of finite rank $\sigma(E^*, E)$ continuous (that is to say weak-$*$ continuous) operators $E^* \to F$ is to be identified with the space of tensors $E \otimes F$; we identify the finite rank weak-$*$ continuous operator

$$\xi^* \mapsto \sum_k \xi^*(\xi_k)\eta_k \tag{2.3}$$

in $\mathcal{B}(E^*, F)$ with the tensor

$$\sum_k \xi_k \otimes \eta_k \tag{2.4}$$

in $E \otimes F$.

Given a space of algebraic tensors $E \otimes F$, we may impose norms on that space. A norm α on $E \otimes F$ is said to be tensorial (or a crossnorm) if

$$\alpha(\xi \otimes \eta) = \|\xi\|_E \|\eta\|_F \tag{2.5}$$

for all rank one tensors $\xi \otimes \eta$ in $E \otimes F$. A tensorial norm is said to be reasonable if

$$\alpha^*(\xi^* \otimes \eta^*) = \|\xi^*\|_{E^*} \|\eta^*\|_{F^*} \tag{2.6}$$

for all rank one tensors $\xi^* \otimes \eta^*$ in $(E \otimes F)^*$, where α^* denotes the dual norm to α. Note that we may replace equality with $\leq$ in the above equation and the property remains the same. The completion of $E \otimes F$ with respect to the tensorial norm α is denoted by

$E \otimes_\alpha F$. For more information consult [42] or chapter VIII of [12]; these concepts were originally introduced in [20].

The two most commonly used reasonable tensorial norms on a space of tensors $E \otimes F$ are the injective norm $\| \quad \|_\vee$, with completion the injective tensor product $E \check{\otimes} F$, and the projective norm $\| \quad \|_\wedge$, with completion the projective tensor product $E \hat{\otimes} F$. If $u \in E \otimes F$ then $\|u\|_\vee$ is the operator norm of u viewed as an element of $\mathcal{B}(E^*, F)$ while, writing $u = \sum_k \xi_k \otimes \eta_k$,

$$\|u\|_\wedge = \inf \left\{ \sum_k \|\xi_k\|_E \|\eta_k\|_F \right\} \tag{2.7}$$

where the infimum is over all representations of u. It is well-known that, given any $u \in E \otimes F$ and a reasonable tensorial norm α on $E \otimes F$,

$$\|u\|_\vee \leq \alpha(u) \leq \|u\|_\wedge. \tag{2.8}$$

We shall also consider certain classes of norms on spaces of operators from E to F. Let $\mathcal{A}(E, F)$ denote a subspace of $\mathcal{B}(E, F)$ equipped with a norm α under which $\mathcal{A}(E, F)$ is a Banach space. We say $\mathcal{A}(E, F)$ is a Banach operator ideal, and α is an operator ideal norm, if:

(i) for all $\xi^* \in E^*$ and $\eta \in F$, the rank one tensor $\xi^* \otimes \eta \in \mathcal{A}(E, F)$ and

$$\alpha(\xi^* \otimes \eta) = \|\xi^*\|_{E^*} \|\eta\|_F; \tag{2.9}$$

(ii) for all $u \in \mathcal{A}(E, F)$, $S \in \mathcal{B}(E)$ and $T \in \mathcal{B}(F)$, the product $TuS \in \mathcal{A}(E, F)$ and

$$\alpha(TuS) \leq \|T\| \alpha(u) \|S\|. \tag{2.10}$$

The most familiar example of a Banach operator ideal is $\mathcal{B}(E, F)$ equipped with the usual operator norm; this is also the largest operator ideal in the sense that, if α is an operator ideal norm on an operator ideal $\mathcal{A}(E, F)$ and $u \in \mathcal{A}(E, F)$, we have

$$\|u\| \leq \alpha(u). \tag{2.11}$$

Note that, for α an operator ideal norm on an operator ideal $\mathcal{A}(E, F)$, the restriction of α to the space $E^* \otimes F$ of algebraic tensors is a reasonable tensorial norm. For more information on operator ideals the reader is directed to [11].

We recall from, for example, [11] the definition of a 2-absolutely summing operator. For E and F Banach spaces, the operator $T : E \to F$ is 2-summing if

$$\pi_2(T) = \sup_{\{\xi_i\}_i} \left(\sum_i \|T\xi_i\|_F^2 \right)^{1/2} < \infty, \tag{2.12}$$

where the supremum is over all finite subsets $\{\xi_i\}_i$ of E satisfying

$$\sup \left\{ \sum_i |\xi^*(\xi_i)|^2 : \xi^* \in E^*, \|\xi^*\|_{E^*} \leq 1 \right\} \leq 1. \tag{2.13}$$

The constant $\pi_2(T)$ is the 2-summing norm of T; it is an operator ideal norm. We write $\Pi_2(E, F)$ for the space of all such T; it forms a Banach space with norm π_2 and so is a Banach operator ideal. In the case where E and F are both Hilbert spaces the space $\Pi_2(E, F)$ is the space of Hilbert-Schmidt operators from E to F.

Let E be a Banach space, H a Hilbert space and $\mathcal{A}(E, H)$ a Banach operator ideal. We denote by $\mathcal{S}_+(E^* \times E^*)$ the set of all positive sesquilinear forms on $E^* \times E^*$. Throughout this thesis all sesquilinear forms and inner products are understood to be linear in the first variable and conjugate linear in the second.

If φ is a weak-$*$ continuous element of $\mathcal{S}_+(E^* \times E^*)$ we define the action of φ on elements (u, v) of $\mathcal{A}(E, H) \times \mathcal{A}(E, H)$ as follows. Assume first that u and v are of finite rank; the spaces $Im\, u^*$ and $Im\, v^*$ are then finite dimensional subspaces of E^*, and the restriction of φ to $Im\, u^* \times Im\, v^*$ is of finite rank. We may write

$$\varphi|_{Im\, u^* \times Im\, v^*} = \sum_j \xi_j \otimes \overline{\eta_j} \tag{2.14}$$

for some sequences $(\xi_j)_j$ and $(\eta_j)_j$ in E. Define

$$\varphi(u, v) = \sum_j < u(\xi_j), v(\eta_j) >_H . \tag{2.15}$$

For u in $\mathcal{A}(E, H)$ not of finite rank we define

$$\varphi(u, u) = \sup_P \varphi(Pu, Pu) \tag{2.16}$$

where P is a finite rank orthogonal projection on H. Finally we calculate $\varphi(u, v)$ for distinct u and v in $\mathcal{A}(E, H)$ by the polarisation formula

$$\varphi(u, v) = \frac{1}{4} \sum_{j=0}^{3} i^j \varphi(u + i^j v, u + i^j v). \tag{2.17}$$

Type 2 is defined as follows. Let $(X_k)_k$ denote a sequence of independent real $N(0,1)$ random variables. A Banach space E is of (Gaussian) type 2 if there exists a finite positive constant C such that, for any finite sequence $(\xi_k)_k$ in E,

$$\left(\mathbb{E}\left\|\sum_k \xi_k X_k\right\|_E^2\right)^{1/2} \leq C\left(\sum_k \|\xi_k\|_E^2\right)^{1/2}. \tag{2.18}$$

We denote the infimum of all allowable constants C by $T_2(E)$, the type 2 constant of E. Note that both the Lebesgue spaces L^p and the Schatten-von Neumann spaces c^p are of type 2 for $2 \leq p < \infty$. For more information on the notion of type, and the related notion of cotype, see [11] or [42].

2.2 2-convex operator ideal norms

Let E be a complex Banach space and H a complex Hilbert space. Let $\mathcal{D}(E,H)$ be a Banach space of operators $E \to H$ equipped with a norm δ satisfying:

(D1) δ is an operator ideal norm and $\mathcal{D}(E,H)$ is a Banach operator ideal;

(D2) $\delta(u) = \sup_P \delta(Pu)$ for all $u \in \mathcal{D}(E,H)$, where P is a finite rank orthogonal projection on H;

(D3) if $u \in \mathcal{D}(E,H)$ then

$$\|u\| \leq \delta(u) \leq \pi_2(u). \tag{2.19}$$

It is straightforward to show that both $\mathcal{B}(E,H)$ and $\Pi_2(E,H)$ satisfy these properties.

If δ is such a norm satisfying (D1)–(D3) we say δ is 2-convex if

$$\delta\left(\sum_k P_k u\right)^2 \leq \sum_k \delta\left(P_k u\right)^2 \tag{2.20}$$

for all u and each finite set of mutually orthogonal projections (P_k) on H. Note that both operator norm $\| \quad \|$ and 2-summing norm π_2 are 2-convex.

The notion of 2-convexity will be used in chapters 3 and 4 to bound various sums of independent Banach space valued random vectors. For more information on 2-convexity we direct the reader to section 2 of [43].

We have the following result, which is proposition 2.1 in [43].

Theorem 2.2.1 *Assume (D1)–(D3). The following conditions are equivalent:*

(i) the norm δ is 2-convex;

(ii) there exists a family K of positive weak-$*$ continuous sesquilinear forms on $E^* \times E^*$, containing the rank one forms, of norm less than or equal to one and compact in the topology of pointwise convergence of sesquilinear forms, such that for all $u \in \mathcal{D}(E, H)$,

$$\delta(u) = \sup_{\varphi \in K} (\varphi(u, u))^{1/2}; \tag{2.21}$$

(iii) there exists a family $I(E)$ of finite subsets of E, satisfying $\sup_{J \in I(E)} \sum_{x \in J} |\xi^*(x)|^2 \leq 1$ for all $\xi^* \in E^*$, $\|\xi\| \leq 1$, such that for all $u \in \mathcal{D}(E, H)$,

$$\delta(u) = \sup_{J \in I(E)} \left(\sum_{x \in J} \|ux\|_H^2 \right)^{1/2}. \tag{2.22}$$

Proof See proposition 2.1 of [43], and the discussion which follows this proposition, for details.

□

If δ is operator norm $\| \quad \|$ then its associated family of sesquilinear forms is the set of all rank one sesquilinear forms on $E^* \times E^*$ of norm less than or equal to one, whereas if δ is 2-summing norm π_2 its associated family of sesquilinear forms is the set of all sesquilinear forms on $E^* \times E^*$ of norm less than or equal to one.

If δ is operator norm $\| \quad \|$ its associated family of subsets is the set of all finite $J \subset E$ such that $\sum_{x \in J} \|x\|^2 \leq 1$, whereas if δ is 2-summing norm π_2 its associated family is the set of all finite $J \subset E$ such that $\sum_{x \in J} |\xi^*(x)|^2 \leq 1$ for all $\xi^* \in E^*$ satisfying $\|\xi^*\| \leq 1$.

2.3 Weakly measurable random vectors

In this discussion $(\Omega, \mathcal{F}, \mathbb{P})$ will be a probability space, E will be a complex Banach space, X will be a function $\Omega \to E$ and K will be a collection of positive sesquilinear forms on $E \times E$, containing the rank one forms and contained in the set of all forms of norm less than or equal to one. Denote by $\sigma(\mathcal{N}_w(E))$ the cylindrical σ-algebra on E; this is the σ-algebra generated by the set $\mathcal{N}_w(E)$ of all weak neighbourhoods in E.

We say $X : \Omega \to E$ is weakly (also known as Pettis) measurable if $\xi^* X : \Omega \to \mathbb{C}$ is Borel measurable for all $\xi^* \in E^*$; equivalently, X is weakly measurable if it is measurable with respect to $\sigma(\mathcal{N}_w(E))$. It is clear from the definition that the set of all weakly measurable functions from Ω to E forms a vector space; this is also known as the space of cylindrical random vectors.

By contrast we say $X : \Omega \to E$ is strongly (also known as Bochner) measurable if it is measurable with respect to the Borel σ-algebra on E and takes values almost surely in a separable subspace of E (we say it is almost surely separably valued). This implies in particular that $\|X\|_E : \Omega \to \mathbb{R}$ is measurable. Pettis' measurability theorem ([12], theorem II.2) states that X is strongly measurable if and only if it is weakly measurable and almost surely separably valued. A further theorem states that X is strongly measurable if and only if there exists a sequence $(X_n)_n$ of simple functions (i.e. each $X_n = \sum_j \xi_j \mathbf{1}_{A_j}$ for some sequence $(\xi_j)_j$ in E and some sequence $(A_j)_j$ of measurable sets in Ω) such that $\|X(\omega) - X_n(\omega)\|_E$ tends to zero as n tends to infinity for almost all ω. The set of all strongly measurable $X : \Omega \to E$ forms a vector space. More information is in [36].

Note by Pettis' measurability theorem that if E is separable then strong and weak notions of measurability coincide.

For $X : \Omega \to E$ weakly measurable and K a set of positive sesquilinear forms as above we may, following Pisier in [43], define a seminorm δ_K via the formula

$$\delta_K(X) = \sup_{\varphi \in K} \left(\int_\Omega \varphi(X(\omega), X(\omega)) \, \mathbb{P}(d\omega) \right)^{1/2}. \tag{2.23}$$

The only axiom of a seminorm which is unclear is the triangle inequality. However sesquilinearity and the Cauchy-Schwarz inequality enable us to prove $\delta_K(X_1 + X_2)^2 \leq [\delta_K(X_1) + \delta_K(X_2)]^2$. Denote the seminormed space of all weakly measurable $X : \Omega \to E$ with $\delta_K(X) < \infty$ by $\mathcal{L}^2_w(\Omega; E, \delta_K)$.

We may quotient $\mathcal{L}^2_w(\Omega; E, \delta_K)$ by the set of all weakly measurable $X : \Omega \to E$ with $\delta_K(X) = 0$, which we denote by N, to obtain a normed space, which we denote by $L^2_w(\Omega; E, \delta_K)$.

The null space N is, from the definition of δ_K, the set of all weakly measurable $X : \Omega \to E$ satisfying $\xi^* X = 0$ almost surely for all $\xi^* \in E^*$. A more natural choice of null space would be the set of all weakly measurable $X : \Omega \to E$ satisfying $X^{-1}(0) \in \mathcal{F}$ and, furthermore, $X = 0$ almost surely. Denote this by $\tilde{N}$.

Note that $\tilde{N} \subseteq N$. Also note the condition $X^{-1}(0) \in \mathcal{F}$ in the definition of $\tilde{N}$ is necessary since the set $\{0\}$ is not necessarily in $\sigma(\mathcal{N}_w(E))$. Further note that $\tilde{N}$ is not necessarily a vector space.

We may ask:

(i) when is $\tilde{N}$ a vector subspace of N?

(ii) when does $\tilde{N} = N$?

Proposition 2.3.1 *For N and $\tilde{N}$ as just defined:*

(i) if $\{0\} \in \sigma(\mathcal{N}_w(E))$ then $\tilde{N} = N$;

(ii) if $\{0\} \notin \sigma(\mathcal{N}_w(E))$ but $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\tilde{N}$ is a vector subspace of N — it may equal N;

(iii) if E is separable then $\{0\} \in \sigma(\mathcal{N}_w(E))$ and so $\tilde{N} = N$;

(iv) if E is separable then $\sigma(\mathcal{N}_w(E))$ coincides with the Borel σ-algebra on E.

Proof This is an exercise in technical measure theory; the reader is directed to the paper [36].

□

At times we shall also need to consider the more usual Bochner L^p spaces. If $1 \leq p < \infty$ and $X : \Omega \to E$ is a strongly measurable random vector then its Bochner L^p norm is given by

$$\|X\|_p = \left(\int_\Omega \|X(\omega)\|_E \mathbb{P}(d\omega) \right)^{1/p}. \tag{2.24}$$

Denote by $\mathcal{L}^p(\Omega; E)$ the set of all strongly measurable $X : \Omega \to E$ with finite Bochner L^p norm. This is a seminormed vector space whose null space is the subspace of all strongly measurable $X : \Omega \to E$ which are zero almost surely. Denote by $L^p(\Omega; E)$ the resulting quotient space; this is a Banach space.

We may define analogous $L^p(\Omega; E)$ in the cases $0 < p < 1$; the resulting spaces are complete quasinormed spaces.

Note that if δ_K is any of the weak L^2 norms introduced in this section and $X \in L^2(\Omega; E)$ then

$$\delta_K(X) \leq \|X\|_2. \tag{2.25}$$

The notion of the norms δ_K and the spaces $L^2_w(\Omega; E, \delta_K)$ will enable us in section 2.6 to develop the theory of covariance for Banach space valued random vectors.

We shall also need the theory of expectation. Following [12] we say a weakly measurable random vector $X : \Omega \to E$ is weakly (or Dunford) integrable if $\xi^* X \in L^1(\Omega)$ for all $\xi^* \in E^*$. This occurs if and only if

$$\sup_{\xi^* \in E^*, \|\xi^*\| \leq 1} \int_\Omega |\xi^* X(\omega)| \, \mathbb{P}(d\omega) < \infty. \tag{2.26}$$

For a weakly integrable $X : \Omega \to E$ and $A \in \mathcal{F}$ we see there exists

$$\int_A X(\omega) \, \mathbb{P}(d\omega) \in E^{**} \tag{2.27}$$

such that

$$\left(\int_A X(\omega) \, \mathbb{P}(d\omega) \right) (\xi^*) = \int_A \xi^* X(\omega) \, \mathbb{P}(d\omega) \tag{2.28}$$

for all $\xi^* \in E^*$. When $A = \Omega$ we refer to the integral as the weak expectation $\mathbb{E}X$ of X with respect to $\mathbb{P}$.

We note that if $X \in L^2_w(\Omega; E, \delta_K)$ then by the Cauchy-Schwarz inequality the above condition holds; thus the weak expectation exists and is finite as an element of E^{**}.

It may be of course that

$$\int_A X(\omega) \, \mathbb{P}(d\omega) \in E \tag{2.29}$$

for all $A \in \mathcal{F}$. In this case, following [12], we say X is Pettis integrable. In particular $\mathbb{E}X \in E$. Denote by $P^2(\Omega; E, \delta_K)$ the Pettis integrable elements of $L^2_w(\Omega; E, \delta_K)$.

Finally if X lies in the Bochner space $L^1(\Omega; E)$ we say it is Bochner (or strongly) integrable. Bochner integrability implies Pettis integrability.

Scholium For E^* a dual Banach space one may develop a theory of weak-$*$ measurable random vectors $\Omega \to E^*$, namely functions $X : \Omega \to E^*$ having the property that $X\xi : \Omega \to \mathbb{C}$ is Borel measurable for all $\xi \in E$. Norms δ_K are defined on weak-$*$ measurable random vectors $\Omega \to E^*$, with respect to sets K of weak-$*$ continuous sesquilinear forms on $E^* \times E^*$, in an analogous manner to the δ_K norms on weakly measurable random vectors.

Denote by $\mathcal{L}^2_{w^*}(\Omega; E^*, \delta_K)$ the seminormed space of all weak-$*$ measurable functions $X : \Omega \to E^*$ satisfying $\delta_K(X) < \infty$. As before we quotient to obtain a normed space $L^2_{w^*}(\Omega; E^*, \delta_K)$.

Following [12] we say a weak-$*$ measurable random vector $X : \Omega \to E^*$ is weak-$*$ (or Gel'fand) integrable if $X\xi \in L^1(\Omega)$ for all $\xi \in E$. Elements of $L^2_{w^*}(\Omega; E^*, \delta_K)$ are weak-$*$ integrable. For a weak-$*$ integrable $X : \Omega \to E^*$ and $A \in \mathcal{F}$ we see there exists

$$\int_A X(\omega)\, \mathbb{P}(d\omega) \in E^* \tag{2.30}$$

such that

$$\left(\int_A X(\omega)\, \mathbb{P}(d\omega)\right)(\xi) = \int_A X(\omega)\xi\, \mathbb{P}(d\omega) \tag{2.31}$$

for all $\xi \in E$. When $A = \Omega$ we refer to the integral as the weak-$*$ expectation $\mathbb{E}X$ of X with respect to $\mathbb{P}$.

In chapter 6 we will consider some classes of random measures on $\mathbb{R}$ (i.e. functions from some probability space Ω to the dual space $C_0(\mathbb{R})^*$ of Radon measures on $\mathbb{R}$), associated to certain ensembles of random matrices, which lie in $L^2_{w^*}(\Omega; C_0(\mathbb{R})^*, \delta_K)$ for a particular choice of δ_K.

2.4 An equivalence between random vectors and operators

We return to the notation of section 2.2 and consider the case where $H = L^2(\Omega)$, the Hilbert space of complex valued square-integrable functions on $(\Omega, \mathcal{F}, \mathbb{P})$, quotiented by functions which are zero almost surely. Denote by $\mathcal{L}^2(\Omega)$ the corresponding seminormed space of square-integrable functions. As before E is a complex Banach space.

Let δ_K be a 2-convex norm on operators $E^* \to L^2(\Omega)$ which satisifes conditions (D1)–(D3) and is associated with a set K of positive sesquilinear forms on $E \times E$ (i.e. weak-$*$ continuous positive sesquilinear forms on $E^{**} \times E^{**}$). Denote by $\mathcal{D}_K(E^*, L^2(\Omega))$ the associated Banach operator ideal of operators $u : E^* \to L^2(\Omega)$ satisfying $\delta_K(u) < \infty$. Denote by $\mathcal{D}_K^{w^*}(E^*, L^2(\Omega))$ the weak-$*$ continuous elements of $\mathcal{D}_K(E^*, L^2(\Omega))$.

Note that δ_K also induces a seminormed space $\mathcal{D}_K(E^*, \mathcal{L}^2(\Omega))$ in an analogous way. Furthermore when we quotient this space by the null space of operators u satisfying $\delta_K(u) = 0$ we obtain the Banach space $\mathcal{D}_K(E^*, L^2(\Omega))$; this follows because the null space comprises those operators u which satisfy $u(\xi^*) = 0$ almost surely, for all ξ^* in E^*, and this in turn is the space of operators u such that $u(\xi^*)$ lies in the null space of $\mathcal{L}^2(\Omega)$ for all ξ^* in E^*.

Denote by $\tilde{\delta}_K$, for the same K, the norm on weakly measurable functions $\Omega \to E$ described in section 2.3. The associated normed space is $L^2_w(\Omega; E, \tilde{\delta}_K)$.

We have the following result.

Proposition 2.4.1 *There is an isometric embedding*

$$\wedge : L^2_w(\Omega; E, \tilde{\delta}_K) \hookrightarrow \mathcal{D}_K(E^*, L^2(\Omega)) \tag{2.32}$$

given by the relation

$$\hat{X}(\xi^*)(\omega) = \xi^*(X(\omega)). \tag{2.33}$$

Furthermore we have isometric isomorphisms

$$\wedge : P^2(\Omega; E, \tilde{\delta}_K) \tilde{\to} \mathcal{D}_K^{w^*}(E^*, L^2(\Omega)) \tag{2.34}$$

and

$$\wedge : L^2_{w^*}(\Omega; E^{**}, \tilde{\delta}_K) \tilde{\to} \mathcal{D}_K(E^*, L^2(\Omega)) \tag{2.35}$$

given by relation (2.33) and the relation

$$\hat{X}(\xi^*)(\omega) = X(\omega)(\xi^*) \tag{2.36}$$

respectively.

Proof Referring to Theorem 2.2.1 (which is proposition 2.1 of [43]) we see the relations (2.33) and (2.36) define isometric embeddings $\wedge : \mathcal{L}^2_w(\Omega; E, \tilde{\delta}_K) \hookrightarrow \mathcal{D}_K(E^*, \mathcal{L}^2(\Omega))$ and $\wedge : \mathcal{L}^2_{w^*}(\Omega; E^{**}, \tilde{\delta}_K) \hookrightarrow \mathcal{D}_K(E^*, \mathcal{L}^2(\Omega))$ respectively between seminormed spaces.

We may quotient by the appropriate null spaces to obtain isometric embeddings between normed quotient spaces if, given a random vector X, $\tilde{\delta}_K(X) = 0$ implies $\delta_K(\hat{X}) = 0$. But this follows, since for the first embedding $\tilde{\delta}_K(X) = 0$ if and only if $\xi^*(X) = 0$ almost surely, for all ξ^* in E^*, which occurs if and only if $\delta_K(\hat{X}) = 0$. Similarly, for the second embedding $\tilde{\delta}_K(X) = 0$ if and only if $X(\xi^*) = 0$ almost surely, for all ξ^* in E^*, which occurs if and only if $\delta(\hat{X}) = 0$. Thus we have isometric embeddings between normed spaces as required.

Furthermore the embedding $\wedge : L^2_{w^*}(\Omega; E^{**}, \tilde{\delta}_K) \hookrightarrow \mathcal{D}_K(E^*, L^2(\Omega))$ is surjective; this follows since, given $\hat{X} \in \mathcal{D}_K(E^*, L^2(\Omega))$, the relationship $X(\omega)(\xi^*) = \hat{X}(\xi^*)(\omega)$ gives us an X in $L^2_{w^*}(\Omega; E^{**}, \tilde{\delta}_K)$ which maps to $\hat{X}$.

Turning to the case of spaces of Pettis integrable random vectors, we know that $P^2(\Omega; E, \tilde{\delta}_K)$ is a subspace of $L^2_w(\Omega; E, \tilde{\delta}_K)$ and so we have an isometric embedding $P^2(\Omega; E, \tilde{\delta}_K) \hookrightarrow \mathcal{D}_K(E^*, L^2(\Omega))$.

Now $\hat{X} \in \mathcal{D}_K(E^*, L^2(\Omega))$ (which is associated to X in $L^2_{w^*}(\Omega; E^{**}, \tilde{\delta}_K)$) is weak-$*$ continuous if and only if its adjoint $\hat{X}^* : L^2(\Omega) \to E^{**}$ takes values in E. This adjoint mapping is given by

$$\hat{X}^*(f) = \int_\Omega X(\omega)\overline{f(\omega)}\mathbb{P}(d\omega) \tag{2.37}$$

where the integral is a weak-$*$ integral.

It is clear this takes values in E for all choices of f in $L^2(\Omega)$ if and only if X is E valued and Pettis integrable. Thus the image of $P^2(\Omega; E, \tilde{\delta}_K)$ under $\wedge$ is precisely $\mathcal{D}^{w^*}_K(E^*, L^2(\Omega))$.

□

Henceforth we shall denote both norms by δ_K.

If K is the set of all positive rank one sesquilinear forms on $E \times E$ of norm less than or equal to one and $X \in L^2_w(\Omega; E, \delta_K)$ then $\delta_K(X) = \|\hat{X}\|$. Henceforth we shall denote $L^2_w(\Omega; E, \delta_K)$ by $L^2_w(\Omega; E, \| \quad \|)$ or just $L^2_w(\Omega; E)$. Thus $L^2_w(\Omega; E) \hookrightarrow \mathcal{B}(E^*, L^2(\Omega))$ isometrically.

If K is the set of all positive forms on $E \times E$ of norm less than or equal to one and $X \in L^2_w(\Omega; E, \delta_K)$ then $\delta_K(X) = \pi_2(\hat{X})$. Henceforth we shall denote $L^2_w(\Omega; E, \delta_K)$ by $L^2_w(\Omega; E, \pi_2)$. Thus $L^2_w(\Omega; E, \pi_2) \hookrightarrow \Pi_2(E^*, L^2(\Omega))$ isometrically.

2.5 Ideals of 2-factorable operators

Let E_1 and E_2 be Banach spaces. Following, for example, [42] or [11] we define $\Gamma_2(E_1, E_2^*)$ to be the set of all operators $u : E_1 \to E_2^*$ which factor through some Hilbert space H as $u = B^*A$ with $A \in \mathcal{B}(E_1, H)$ and $B \in \mathcal{B}(E_2, H)$. Its associated norm is

$$\gamma_2(u) = \inf\{\|A\|\|B\| : u = B^*A \text{ for some } A \in \mathcal{B}(E_1, H), B \in \mathcal{B}(E_2, H)\} \tag{2.38}$$

where the infimum runs over all such representations of u as B^*A. We also define $\Gamma_2^*(E_1, E_2^*)$ to be the set of all $u : E_1 \to E_2^*$ which factor through some Hilbert space H as $u = B^*A$ with $A \in \Pi_2(E_1, H)$ and $B \in \Pi_2(E_2, H)$. Its associated norm is

$$\gamma_2^*(u) = \inf\{\pi_2(A)\pi_2(B) : u = B^*A \text{ for some } A \in \Pi_2(E_1, H), B \in \Pi_2(E_2, H)\} \tag{2.39}$$

where the infimum runs over all such representations of u as B^*A. The set $\Gamma_2(E_1, E_2^*)$ is known as the space of 2-factorable operators from E_1 to E_2^* while the set $\Gamma_2^*(E_1, E_2^*)$ is known as the space of 2-dominated operators from E_1 to E_2^*. Both may be seen to be Banach operator ideals, and we note that Γ_2^* is the dual operator ideal to Γ_2; see chapter 7 of [11] for more information.

For H a Hilbert space consider a pair $\mathcal{D}_1(E_1, H)$ and $\mathcal{D}_2(E_2, H)$ of Banach operator ideals with 2-convex norms δ_1 and δ_2 satisfying conditions (D1)–(D3) of section 2, so $\Pi_2(E_i, H) \subseteq \mathcal{D}_i(E_i, H) \subseteq \mathcal{B}(E_i, H)$ for each i. Assume these ideals are defined for all possible Hilbert spaces H; we may then, following [43], define $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$ to be the set of all $u : E_1 \to E_2^*$ which factor through some Hilbert space H as $u = B^*A$ with $A \in \mathcal{D}_1(E_1, H)$ and $B \in \mathcal{D}_2(E_2, H)$. Its associated norm is

$$\gamma_{\delta_1,\delta_2}(u) = \inf\{\delta_1(A)\delta_2(B) : u = B^*A \text{ for some } A \in \mathcal{D}_1(E_1, H), B \in \mathcal{D}_2(E_2, H)\} \tag{2.40}$$

where the infimum runs over all such representations of u as B^*A. So

$$\Gamma_2^*(E_1, E_2^*) \subseteq \Gamma_{\delta_1,\delta_2}(E_1, E_2^*) \subseteq \Gamma_2(E_1, E_2^*) \tag{2.41}$$

and

$$\gamma_2(u) \leq \gamma_{\delta_1,\delta_2}(u) \leq \gamma_2^*(u) \tag{2.42}$$

for all pairs (δ_1, δ_2) and appropriate operators u. The space $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$ is a Banach operator ideal; see section 2 of [43] for more information.

For completeness we shall now demonstrate that the $\gamma_{\delta_1,\delta_2}$ norm is indeed a norm.

Proposition 2.5.1 *$\gamma_{\delta_1,\delta_2}$ is a norm on the space $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$.*

Proof The only axiom which is unclear is the triangle inequality. Let u and v be elements of $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$. Fix $\varepsilon > 0$. Assume u factors through some Hilbert space H_1 as $u = B^*A$ for $A : E_1 \to H_1$ and $B : E_2 \to H_1$, where H_1, A and B are chosen so that

$$\delta_1(A)\delta_2(B) \leq \gamma_{\delta_1,\delta_2}(u) + \varepsilon. \tag{2.43}$$

Polarising via

$$B^*A = \left(\sqrt{\frac{\delta_1(A)}{\delta_2(B)}}B\right)^* \left(\sqrt{\frac{\delta_2(B)}{\delta_1(A)}}A\right) \tag{2.44}$$

if necessary, we choose A and B so that $\delta_1(A) = \delta_2(B)$.

Further assume v factors through some Hilbert space H_2 as $v = D^*C$ for $C : E_1 \to H_2$ and $D : E_2 \to H_2$, where H_2, C and D are chosen so that

$$\delta_1(C)\delta_2(D) \leq \gamma_{\delta_1,\delta_2}(v) + \varepsilon. \tag{2.45}$$

Polarising via

$$D^*C = \left(\sqrt{\frac{\delta_1(C)}{\delta_2(D)}}D\right)^* \left(\sqrt{\frac{\delta_2(D)}{\delta_1(C)}}C\right) \tag{2.46}$$

if necessary, we choose C and D so that $\delta_1(C) = \delta_2(D)$.

Then $u + v$ factors through the Hilbert space $H_1 \oplus H_2$ as follows. Denote by $\iota_1 : H_1 \hookrightarrow H_1 \oplus H_2$ and $\iota_2 : H_2 \hookrightarrow H_1 \oplus H_2$ the natural isometric embeddings, and by

$\iota_1^* : H_1 \oplus H_2 \twoheadrightarrow H_1$ and $\iota_2^* : H_1 \oplus H_2 \twoheadrightarrow H_2$ the corresponding natural quotients, satisfying $\iota_1^*\iota_1 = id_{H_1}$ and $\iota_2^*\iota_2 = id_{H_2}$. We see that $\iota_1 A + \iota_2 C$ is a linear operator from E_1 to $H_1 \oplus H_2$, $\iota_1 B + \iota_2 D$ is a linear operator from E_2 to $H_1 \oplus H_2$ and

$$\begin{aligned}(\iota_1 B + \iota_2 D)^*(\iota_1 A + \iota_2 C) &= B^*\iota_1^*\iota_1 A + B^*\iota_1^*\iota_2 C + D^*\iota_2^*\iota_1 A + D^*\iota_2^*\iota_2 C && (2.47)\\ &= B^*A + D^*C && (2.48)\\ &= u + v. && (2.49)\end{aligned}$$

Now

$$\gamma_{\delta_1,\delta_2}(u+v) = \inf\{\delta_1(S)\delta_2(T)\} \quad (2.50)$$

where the infimum is over all factorisations of $u+v$ through some Hilbert space as T^*S. One such factorisation, through $H_1 \oplus H_2$, is $u + v = (\iota_1 B + \iota_2 D)^*(\iota_1 A + \iota_2 C)$. Thus

$$\begin{aligned}\gamma_{\delta_1,\delta_2}(u+v) &\leq \delta_1(\iota_1 A + \iota_2 C)\delta_2(\iota_1 B + \iota_2 D) && (2.51)\\ &\leq \left[\delta_1(A)^2 + \delta_1(C)^2\right]^{1/2}\left[\delta_2(B)^2 + \delta_2(D)^2\right]^{1/2} && (2.52)\end{aligned}$$

as δ_1 and δ_2 are 2-convex. We have chosen A, B, C and D so that $\delta_1(A) = \delta_2(B)$ and $\delta_1(C) = \delta_2(D)$; this gives

$$\begin{aligned}\gamma_{\delta_1,\delta_2}(u+v) &\leq \delta_1(A)\delta_2(B) + \delta_1(C)\delta_2(D) && (2.53)\\ &\leq \gamma_{\delta_1,\delta_2}(u) + \gamma_{\delta_1,\delta_2}(v) + 2\varepsilon && (2.54)\end{aligned}$$

by our choice of A, B, C and D. This inequality holds for all $\varepsilon > 0$, so

$$\gamma_{\delta_1,\delta_2}(u+v) \leq \gamma_{\delta_1,\delta_2}(u) + \gamma_{\delta_1,\delta_2}(v) \quad (2.55)$$

as required.

□

We now consider tensor products. We will, following the approach taken in section 1 of [43], impose $\gamma_{\delta_1,\delta_2}$ norms on elements of $E_1^* \otimes E_2^*$, viewing them as finite rank operators $E_1 \to E_2^*$, and form completions $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$.

Thus, our task is to impose a $\gamma_{\delta_1,\delta_2}$ norm on elements of $E_1^* \otimes E_2^*$. If $u \in E_1^* \otimes E_2^*$ is given by

$$u = \sum_{k=1}^{n} \xi_k^* \otimes \eta_k^* \tag{2.56}$$

then, regarding u as an operator $E_1 \to E_2^*$, we may factor u through an n-dimensional Hilbert space H with orthonormal basis (e_k) as

$$u = \left(\sum_{k=1}^{n} \eta_k^* \otimes e_k \right)^* \left(\sum_{k=1}^{n} \xi_k^* \otimes e_k \right) \tag{2.57}$$

where

$$\sum_{k=1}^{n} \xi_k^* \otimes e_k : E_1 \to H \tag{2.58}$$

and

$$\sum_{k=1}^{n} \eta_k^* \otimes e_k : E_2 \to H \tag{2.59}$$

are operators into H of rank n. Call these A and B respectively; we see $u = B^*A$.

Now consider a pair of norms (δ_1, δ_2). In view of the above factorisation of u we may impose a $\gamma_{\delta_1,\delta_2}$ norm in the usual way, yielding a norm on $E_1^* \otimes E_2^*$. To obtain a more explicit representation of the norm we apply Theorem 2.2.1 to assert the existence of families K_1 and K_2 of sesquilinear forms on $E_1 \times E_1$ and $E_2 \times E_2$ respectively, associated to δ_1 and δ_2 respectively, which are compact in the topology of pointwise convergence of sesquilinear forms, contain the rank one forms of norm less than or equal to one and are contained in the set of all forms of norm less than or equal to one. Then

$$\delta_1(A) = \sup_{\varphi \in K_1} \left(\sum_{k=1}^{n} \varphi(\xi_k^*, \xi_k^*) \right)^{1/2} \tag{2.60}$$

and

$$\delta_2(B) = \sup_{\psi \in K_2} \left(\sum_{k=1}^{n} \psi(\eta_k^*, \eta_k^*) \right)^{1/2}, \tag{2.61}$$

which we note are both finite as there are only finitely many ξ_k and η_k and the forms all have norm less than or equal to 1. Therefore we have an explicit representation of the

$\gamma_{\delta_1,\delta_2}$ norm on $E_1^* \otimes E_2^*$ in terms of the sesquilinear forms K_1 and K_2; this is

$$\gamma_{\delta_1,\delta_2}(u) = \inf \left\{ \sup_{\varphi \in K_1} \left(\sum_{k=1}^{n} \varphi(\xi_k^*, \xi_k^*) \right)^{1/2} \sup_{\psi \in K_2} \left(\sum_{k=1}^{n} \psi(\eta_k^*, \eta_k^*) \right)^{1/2} \right\}, \tag{2.62}$$

where the infimum runs over all possible expressions of u as $\sum_{k=1}^{n} \xi_k^* \otimes \eta_k^*$, for all finite values of n. The norm $\gamma_{\delta_1,\delta_2}$ is a reasonable tensorial norm.

If we complete $E_1^* \otimes E_2^*$ with respect to the $\gamma_{\delta_1,\delta_2}$ norm we form a Banach space $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$.

A question which arises at this point, motivated by the discussions on tensor products in [42], is whether the natural inclusion map $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^* \to E_1^* \check{\otimes} E_2^*$ of $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$ into the injective tensor product of E_1^* and E_2^* is injective. That is to say, is every element of $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$ which represents the zero operator necessarily the zero element? If (and only if) this is so, we may identify $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$ isometrically with a subspace of $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$ in a canonical way.

Proposition 2.5.2 *The natural map $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^* \to E_1^* \check{\otimes} E_2^*$ is injective.*

Proof We may regard the $\gamma_{\delta_1,\delta_2}$ norm on $E_1^* \otimes E_2^*$ as the restriction to the finite rank operators of the $\gamma_{\delta_1,\delta_2}$ norm on $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$. As any element of $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$ is the limit in $\gamma_{\delta_1,\delta_2}$ norm of elements of $E_1^* \otimes E_2^*$, we may regard elements of $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$ as the limit in $\gamma_{\delta_1,\delta_2}$ norm of finite rank elements of $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$. As $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$ is a Banach space, this limit in $\gamma_{\delta_1,\delta_2}$ norm exists as an element of $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$. We have realised $E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^*$ as a subspace of $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$ and, consequently, the map is injective.

□

Note the crucial part of the proof was realising the $\gamma_{\delta_1,\delta_2}$ norm on $E_1^* \otimes E_2^*$ as a restriction to a finite dimensional subspace of the $\gamma_{\delta_1,\delta_2}$ norm on $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$. This yields a model for the completion of $E_1^* \otimes E_2^*$ with respect to $\gamma_{\delta_1,\delta_2}$, namely a subspace of $\Gamma_{\delta_1,\delta_2}(E_1, E_2^*)$. We have inclusions

$$E_1^* \otimes_{\gamma_2} E_2^* \supseteq E_1^* \otimes_{\gamma_{\delta_1,\delta_2}} E_2^* \supseteq E_1^* \otimes_{\gamma_2^*} E_2^*. \tag{2.63}$$

If we consider the case of the projective tensor product $E_1^*\hat{\otimes}E_2^*$, for which the natural map $E_1^*\hat{\otimes}E_2^* \to E_1^*\check{\otimes}E_2^*$ is not injective in general, the above proof breaks down because it is not possible to realise the projective norm on $E_1^* \otimes E_2^*$ as the restriction to a finite dimensional subspace of some norm on a Banach space of operators $E_1 \to E_2^*$. Recall from, for example, [42] that given E_1, the natural map $E_1\hat{\otimes}E_2 \to E_1\check{\otimes}E_2$ is injective for all choices of E_2 if and only if the space E_1 has the approximation property; by definition this means the identity map on E_1 belongs to the closure of the finite rank operators on E_1 in the topology of uniform convergence on compact sets.

Note that we may impose $\gamma_{\delta_1,\delta_2}$ norms on tensor products of Banach spaces which are not dual spaces, via the explicit formulation (2.62). Specifically if $u \in E_1 \otimes E_2$ is given by

$$u = \sum_{k=1}^{n} \xi_k \otimes \eta_k, \tag{2.64}$$

and K_1 and K_2 are the collections of sesquilinear forms associated to δ_1 and δ_2 respectively, we have

$$\gamma_{\delta_1,\delta_2}(u) = \inf \left\{ \sup_{\varphi\in K_1} \left(\sum_{k=1}^{n} \varphi(\xi_k, \xi_k) \right)^{1/2} \sup_{\psi\in K_2} \left(\sum_{k=1}^{n} \psi(\eta_k, \eta_k) \right)^{1/2} \right\}, \tag{2.65}$$

where the infimum runs over all possible expressions of u as $\sum_{k=1}^{n} \xi_k \otimes \eta_k$, for all finite values of n. The norm $\gamma_{\delta_1,\delta_2}$ is a reasonable tensorial norm.

We may realise $E_1 \otimes E_2$ with the $\gamma_{\delta_1,\delta_2}$ norm as a subspace of a space of 2-factorable operators in the following way. View $(E_1 \otimes E_2, \gamma_{\delta_1,\delta_2})$ as the set of all finite rank elements u of $\Gamma_{\delta_1,\delta_2}(E_1^*, E_2^{**})$ satisfying the condition that, if u factors as B^*A, the operator $A : E_1^* \to H$ is continuous with respect to the $\sigma(E_1^*, E_1)$ weak-$*$ topology and the operator $B : E_2^* \to H$ is continuous with respect to the $\sigma(E_2^*, E_2)$ weak-$*$ topology.

For the same reasons as before the completion $E_1 \otimes_{\gamma_{\delta_1,\delta_2}} E_2$ is realisable as a subspace of the space of operators $\Gamma_{\delta_1,\delta_2}(E_1^*, E_2^{**})$. As before we have inclusions

$$E_1 \otimes_{\gamma_2} E_2 \supseteq E_1 \otimes_{\gamma_{\delta_1,\delta_2}} E_2 \supseteq E_1 \otimes_{\gamma_2^*} E_2. \tag{2.66}$$

2.6 Covariance

Let E_1 and E_2 be Banach spaces and let δ_1 and δ_2 be appropriate 2-convex norms. For $X_1 \in L^2_w(\Omega; E_1, \delta_1)$ and $X_2 \in L^2_w(\Omega; E_2, \delta_2)$ satisfying $\mathbb{E}X_1 = \mathbb{E}X_2 = 0$ (where expectation is defined in the weak sense; we say X_1 and X_2 are centred) we define the covariance of X_1 and X_2, $Cov(X_1, X_2)$, to be the sesquilinear form on $E_1^* \times E_2^*$ given by

$$Cov(X_1, X_2)(\xi_1^*, \xi_2^*) = \mathbb{E}\xi_1^*(X_1)\overline{\xi_2^*(X_2)}. \tag{2.67}$$

This definition of covariance is used in [35] and [34]. Frequently we view $Cov(X_1, X_2)$ as a linear operator $\bar{E}_2^* \to E_1^{**}$; in fact $Cov(X_1, X_2)$ is the operator $\hat{X}_1{}^*\overline{\hat{X}_2}$ where, as before, $\hat{X}_1 : E_1^* \to L^2(\Omega)$ and $\hat{X}_2 : E_2^* \to L^2(\Omega)$ are the operators associated to X_1 and X_2.

If E is a Banach space and X is a centred element of $L^2_w(\Omega; E, \delta)$, for some 2-convex norm δ, we define the variance of X, $Var(X)$, to be $Cov(X, X)$.

We note that, by the Cauchy-Schwarz inequality, if X_1 lies in $L^2_w(\Omega; E_1, \delta_1)$ and X_2 lies in $L^2_w(\Omega; E_2, \delta_2)$ then $Cov(X_1, X_2)$ is bounded as a sesquilinear form. In fact more is true.

Proposition 2.6.1 *Viewing $Cov(X_1, X_2)$ as an operator we have*

$$Cov(X_1, X_2) \in \Gamma_{\delta_1, \delta_2}(\bar{E}_1^*, E_2^{**}). \tag{2.68}$$

Proof We see from the factorisation $Cov(X_1, X_2) = \hat{X}_1{}^*\overline{\hat{X}_2}$ that

$$\|Cov(X_1, X_2)\|_{\gamma_{\delta_1, \delta_2}} \leq \delta_1(X_1)\delta_2(X_2). \tag{2.69}$$

□

In the case where $X_1 = X_2 = X$ we see

$$\|Var(X)\|_{\gamma_{\delta,\delta}} = \delta(X)^2. \tag{2.70}$$

Writing the covariance in the form

$$Cov(X_1, X_2)(\xi_1^*, \xi_2^*) = \mathbb{E}(X_1 \otimes \bar{X_2})(\xi_1^* \otimes \bar{\xi_2^*}) \tag{2.71}$$

we see from the inequality (2.69) that $Cov(X_1, X_2)$ may be viewed as the weak expectation of the $E_1 \otimes_{\gamma_{\delta_1,\delta_2}} \bar{E}_2$ valued random vector $X_1 \otimes \bar{X}_2$. We have a further proposition.

Proposition 2.6.2 *We have*

$$Cov(X_1, X_2) \in (E_1 \otimes_{\gamma_{\delta_1,\delta_2}} \bar{E}_2)^{**}. \tag{2.72}$$

Furthermore if X_1 and X_2 are both Bochner integrable then

$$Cov(X_1, X_2) \in E_1 \otimes_{\gamma_{\delta_1,\delta_2}} \bar{E}_2. \tag{2.73}$$

Proof We see

$$Cov(X_1, X_2) = \mathbb{E}(X_1 \otimes \bar{X}_2), \tag{2.74}$$

where we interpret expectation in the weak sense and view $X_1 \otimes \bar{X}_2$ as an $E_1 \otimes_{\gamma_{\delta_1,\delta_2}} \bar{E}_2$ valued random vector. As it is a weak expectation it follows that $Cov(X_1, X_2)$ lies in $(E_1 \otimes_{\gamma_{\delta_1,\delta_2}} \bar{E}_2)^{**}$.

If X_1 and X_2 are both Bochner integrable then by theorem II.8 of [12] the operators $\hat{X}_1$ and $\hat{X}_2$ are compact. This implies $Cov(X_1, X_2)$ is a compact weak-$*$ continuous operator from $\bar{E}_2^*$ to E_1 which, consequently, lies in $E_1 \otimes_{\gamma_{\delta_1,\delta_2}} \bar{E}_2$.

$\square$

A detailed discussion describing conditions for $Cov(X_1, X_2)$ to be compact may be found on pages 207–209 of [34]. In particular we will see in chapter 3 by applying the Gaussian isoperimetric inequality ([34], section 3.1) that if X_1 and X_2 are Gaussian, $Cov(X_1, X_2)$ is compact.

In the particular case $X_1 \in L^2_w(\Omega; E_1, \pi_2)$ and $X_2 \in L^2_w(\Omega; E_2, \pi_2)$ we see that $Cov(X_1, X_2)$ lies in $(E_1 \otimes_{\gamma_2^*} \bar{E}_2)^{**}$. Theorem 3.1 in [43] shows that when E_1 andE_2 are of type 2, $E_1 \otimes_{\gamma_2^*} \bar{E}_2$ is isomorphic to the projective tensor product $E_1 \hat{\otimes} \bar{E}_2$ with an equivalent norm; thus $Cov(X_1, X_2)$ lies in $(E_1 \hat{\otimes} \bar{E}_2)^{**}$. Note that when E_1 and E_2 are both Hilbert spaces, $E_1 \otimes_{\gamma_2^*} \bar{E}_2$ is the space c^1 of trace class operators.

Coda Throughout the rest of this thesis we shall, to avoid the measure theoretic complications discussed in this chapter and expanded on in [36], assume E is separable. Thus, in particular, strong (Bochner) and weak (Pettis) notions of measurability coincide. All the random vectors we consider in the rest of the thesis will be cylindrical; that is to say they will be measurable with respect to the cylindrical σ-algebra on E which, as E is separable, coincides with the Borel σ-algebra.

Chapter 3

Gaussian random vectors and Wiener processes

This chapter contains essential preliminary material for chapter 4. We study Gaussian random vectors, Wiener processes and Itô stochastic integrals (for deterministic integrands) with values in a separable complex Banach space E. We observe that, for all E valued cylindrical Q-Wiener processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Q factors through l^2 with 2-summing factors.

Background information on the topics covered here may be found in [32] or [34]. More recently the subject of Banach space valued stochastic integrals was discussed in [7].

3.1 Gaussian random vectors

This section will consider centred Gaussian random vectors taking values in the separable complex Banach space E.

Recall that a complex random variable X is said to be complex centred Gaussian with variance σ^2 (we say X is complex $N(0, \sigma^2)$) if

$$X = \frac{1}{\sqrt{2}}(X_{\mathcal{R}} + iX_{\mathcal{I}}) \tag{3.1}$$

where $X_{\mathcal{R}}$ and $X_{\mathcal{I}}$ are independent real $N(0, \sigma^2)$ random variables. It is straightforward to verify that if $(X_k)_k$ is a finite sequence of independent complex centred Gaussian random vectors on some probability space and $(z_k)_k$ is a finite sequence in $\mathbb{C}$ then $\sum_k z_k X_k$ is a complex centred Gaussian random vector.

Let X be an E valued cylindrical random vector defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Following [34] we say X is centred Gaussian if, for every $\xi^* \in E^*$, $\xi^*(X)$ is a complex centred Gaussian random variable.

Consider a sequence $(X_k)_{k \in \mathbb{Z}}$ of independent $N(0, \sigma^2)$ complex random variables. By Kolmogorov's consistency criterion ([46], page 129) $(X_k)_{k \in \mathbb{Z}}$ is a $\mathbb{C}^{\mathbb{Z}}$ valued random vector whose finite dimensional joint distributions are the joint distributions of the X_k.

The sequence $(X_k)_{k \in \mathbb{Z}}$ takes values lying almost surely outside l^∞. This follows from the following proposition.

Proposition 3.1.1 *Let $(X_k)_{k \in \mathbb{N}}$ be an independent sequence of real $N(0, \sigma_k^2)$ random variables. Then the following are equivalent:*
(i) $(X_k)_{k \in \mathbb{N}} \in l^\infty$ almost surely;
(ii) there exists $M > 0$ finite such that

$$\sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) < \infty \tag{3.2}$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, du. \tag{3.3}$$

Furthermore if (ii) holds then, denoting the infimum of all possible values of M by M',

$$\limsup_k |X_k| = M' \tag{3.4}$$

almost surely; otherwise

$$\limsup_k |X_k| = \infty \tag{3.5}$$

almost surely, so that $(X_k)_{k \in \mathbb{N}} \notin l^\infty$ almost surely. Note (ii) does not hold in particular whenever $\sigma_k^2 \not\to 0$. Finally if the X_k are not independent we still have $(ii) \Rightarrow (i)$ above.

Proof Fix $M > 0$ finite. Then

$$\left\{\limsup_k |X_k| \leq M\right\} = \bigcup_j \bigcap_{k\geq j} \{|X_k| \leq M\} \tag{3.6}$$

$$= \left[\bigcap_j \bigcup_{k\geq j} \{|X_k| > M\}\right]^c. \tag{3.7}$$

By the Borel-Cantelli lemmas ([53], sections 2.7 and 4.3),

$$\mathbb{P}\left(\bigcap_j \bigcup_{k\geq j} \{|X_k| > M\}\right) = \begin{cases} 0 & \text{if } \sum_k \mathbb{P}(|X_k| > M) < \infty \quad \text{(1st lemma);} \\ 1 & \text{if } \sum_k \mathbb{P}(|X_k| > M) = \infty \quad \text{(2nd lemma),} \end{cases} \tag{3.8}$$

so

$$\mathbb{P}\left(\limsup_k |X_k| \leq M\right) = \begin{cases} 1 & \text{if } \sum_k \mathbb{P}(|X_k| > M) < \infty; \\ 0 & \text{if } \sum_k \mathbb{P}(|X_k| > M) = \infty. \end{cases} \tag{3.9}$$

Now as $X_k \sim \sigma_k Z_k$, where each $Z_k \sim N(0,1)$, we see

$$\mathbb{P}(|X_k| > M) = 2\left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right). \tag{3.10}$$

Thus

$$\mathbb{P}\left(\limsup_k |X_k| \leq M\right) = \begin{cases} 1 & \text{if } \sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) < \infty; \\ 0 & \text{if } \sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) = \infty. \end{cases} \tag{3.11}$$

Next, we note that if $\sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right)$ converges, it converges for all $N > M$ and if $\sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right)$ diverges, it diverges for all $N < M$. So, noting that $M = 0$ yields divergence for all possible $(\sigma_k)_{k\in\mathbb{N}}$, we have two possibilities.

(i) For all $M > 0$, $\sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) = \infty$. In this case, for all $M > 0$,

$$\limsup_k |X_k| > M \tag{3.12}$$

almost surely, and so

$$\limsup_k |X_k| = \infty \tag{3.13}$$

almost surely. This implies $(X_k)_{k\in\mathbb{N}} \notin l^\infty$ almost surely.

(ii) There exists an $M > 0$ such that $\sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) < \infty$. Denote the infimum of all such M by M'. Then, for all $\varepsilon > 0$,

$$\limsup_k |X_k| > M' - \varepsilon \tag{3.14}$$

almost surely, and

$$\limsup_k |X_k| \leq M' + \varepsilon \tag{3.15}$$

almost surely. We deduce

$$\limsup_k |X_k| = M' \tag{3.16}$$

almost surely. This implies $(X_k)_{k\in\mathbb{N}} \in l^\infty$ almost surely.

If the X_k are not independent the first Borel-Cantelli lemma still applies and we have, for any $M > 0$,

$$\mathbb{P}\left(\limsup_k |X_k| \leq M\right) = 1 \text{ if } \sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) < \infty, \tag{3.17}$$

which yields $(ii) \Rightarrow (i)$.

□

As an aside, this has the following corollary.

Corollary 3.1.2 *We have equivalent conditions*

$$\sum_k \left(1 - \Phi\left(\frac{M}{\sigma_k}\right)\right) < \infty \tag{3.18}$$

and

$$\sum_k \sigma_k e^{-\frac{M^2}{2\sigma_k^2}} < \infty; \tag{3.19}$$

thus all instances of (3.2) in Proposition 3.1.1 may be replaced with (3.19).

Proof We note that

$$1 - \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{u^2}{2}}\, du. \tag{3.20}$$

Integrating by parts we have

$$\int_t^\infty e^{-\frac{u^2}{2}}\, du = \frac{1}{t} e^{-\frac{t^2}{2}} - \int_t^\infty \frac{1}{u^2} e^{-\frac{u^2}{2}}\, du \tag{3.21}$$

and

$$\int_t^\infty e^{-\frac{u^2}{2}}\, du = \frac{1}{t}\left(1 - \frac{1}{t^2}\right) e^{-\frac{t^2}{2}} + 3\int_t^\infty \frac{1}{u^4} e^{-\frac{u^2}{2}}\, du, \tag{3.22}$$

yielding

$$\frac{1}{t\sqrt{2\pi}}\left(1 - \frac{1}{t^2}\right) e^{-\frac{t^2}{2}} \leq 1 - \Phi(t) \leq \frac{1}{t\sqrt{2\pi}} e^{-\frac{t^2}{2}}. \tag{3.23}$$

In our specific case we have

$$\frac{\sigma_k}{M\sqrt{2\pi}}\left(1 - \frac{\sigma_k^2}{M^2}\right) e^{-\frac{M^2}{2\sigma_k^2}} \leq 1 - \Phi\left(\frac{M}{\sigma_k}\right) \leq \frac{\sigma_k}{M\sqrt{2\pi}} e^{-\frac{M^2}{2\sigma_k^2}} \tag{3.24}$$

which, as $\sigma_k \to 0$ is necessary for (3.2) or (3.19) to hold, yields the result.

$\square$

The following result is a combination of the Itô-Nisio theorem ([27]), a result on exponential integrability of Gaussian random vectors, due independently to Fernique ([16]) and Landau and Shepp ([33]), and the Karhunen-Loève representation of Gaussian measures on separable Banach spaces.

Proposition 3.1.3 *Let* $\mathbf{X} = (X_k)_{k\in\mathbb{Z}}$ *be a sequence of independent* $N(0, \sigma^2)$ *complex random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and let* $(\xi_k)_{k\in\mathbb{Z}}$ *be a sequence in a separable complex Banach space* E.

(a) The following are equivalent:

(i) $\sum_k \xi_k X_k$ *converges almost surely in* E*;*

(ii) $\sum_k \xi_k X_k$ *converges in* $L^p(\Omega; E)$ *for some (and hence for all)* $0 < p < \infty$*;*

(iii) $\sum_k \xi_k X_k$ *converges in probability.*

(b) If the sum $\sum_k \xi_k X_k$ converges in $L^2_w(\Omega; E, \pi_2)$ then $\sum_k \xi_k X_k$ is a centred Gaussian random vector and the equivalent conditions (i), (ii) and (iii) of part (a) hold.
(c) All cylindrical centred Gaussian random vectors with values in E are equal in distribution to some random vector of the form $\sum_k \xi_k X_k$, satisfying the equivalent conditions (i), (ii) and (iii) of part (a).

Proof (a) This is the Itô-Nisio theorem; see the original paper [27], or pages 29–36 of [32], for details.
(b) Assume $\sum_k \xi_k X_k$ converges in $L^2_w(\Omega; E, \pi_2)$; we shall show it is centred Gaussian. The map $(\xi^* \mapsto \xi^*(\sum_k \xi_k X_k))$ lies in $\Pi_2(E^*, L^2(\Omega))$ by definition of the π_2 norm. Thus, for each $\xi^* \in E^*$, we see that

$$\xi^*(\sum_k \xi_k X_k) = \sum_k \xi^*(\xi_k) X_k \tag{3.25}$$

is a sum, convergent in $L^2(\Omega)$, of scalar multiples of independent complex centred Gaussians; it is therefore a complex centred Gaussian random variable. We deduce that $\sum_k \xi_k X_k$ is a centred Gaussian random vector.

As $\sum_k \xi_k X_k$ is Gaussian we now apply the result, due to Fernique ([16]) and Landau and Shepp ([33]), that there exists $\alpha > 0$ such that

$$\mathbb{E} \exp \left\{ \alpha \left\| \sum_k \xi_k X_k \right\|_E^2 \right\} < \infty. \tag{3.26}$$

This implies condition (ii) (and hence conditions (i) and (iii)) of part (a) holds.
(c) The Karhunen-Loève representation of Gaussian measures, which is proposition 2.6.1 of [32] and proposition 3.6 of [34], shows all cylindrical centred Gaussian random vectors with values in E are equal in distribution to some random vector of the form $\sum_k \xi_k X_k$, satisfying condition (ii) (and hence conditions (i) and (iii)) of part (a).

□

Note that Fernique, Landau and Shepp's result is implied by (and, indeed, motivated the proof of) the Gaussian isoperimetric inequality, due independently to Borell ([5]) and Sudakov and Cirel'son ([47]). The Gaussian isoperimetric inequality is shown by

Borell to be the limiting case of the isoperimetric inequality for the rotation-invariant measure on the spheres in $\mathbb{R}^n$ as n tends to infinity. A proof of the Gaussian isoperimetric inequality is theorem 1.2 of [34]; for a proof of Proposition 3.1.3 (b) based on the Gaussian isoperimetric inequality, see lemma 3.1 of [34].

Proposition 3.1.3 has the following corollary.

Corollary 3.1.4 *Given a Gaussian random vector $\sum_k \xi_k X_k$ as in Proposition 3.1.3 we may define a bounded map $A : l^2 \to E$ by*

$$(x_k)_{k\in\mathbb{Z}} \mapsto \sum_k \xi_k x_k; \tag{3.27}$$

this has bounded adjoint $A^* \in \Pi_2(E^*, l^2)$.

Write $A\mathbf{X}$ for $\sum_k \xi_k X_k$; this converges to an almost surely E valued centred Gaussian random vector satisfying $\pi_2(A\mathbf{X}) = \sigma\pi_2(A^*)$ and $Var(A\mathbf{X}) = \sigma^2 A\overline{A^*}$.

Proof Writing $A\mathbf{X}$ for $\sum_k \xi_k X_k$ we know from Proposition 3.1.3 that $A\mathbf{X}$ is almost surely E valued and the map $(\xi^* \mapsto \xi^*(A\mathbf{X}))$ lies in $\Pi_2(E^*, L^2(\Omega))$. But, for a finite sequence $(\xi_j^*)_j$ in E^*,

$$\begin{aligned}
\sum_j \left\|\xi_j^*(A\mathbf{X})\right\|_{L^2(\Omega)}^2 &= \sum_j \mathbb{E}\left|\sum_k \xi_j^*(\xi_k) X_k\right|^2 && (3.28)\\
&= \sigma^2 \sum_{j,k} \left|\xi_j^*(\xi_k)\right|^2 && (3.29)\\
&= \sigma^2 \sum_j \left\|A^*(\xi_j^*)\right\|_{l^2}^2 && (3.30)
\end{aligned}$$

and so $A^* \in \Pi_2(E^*, l^2)$; furthermore $\sigma\pi_2(A^*) = \pi_2(A\mathbf{X})$. It follows that A is bounded. Finally

$$Var(A\mathbf{X})(\xi_1^*, \xi_2^*) = \sigma^2 < A^*(\xi_1^*), A^*(\xi_2^*) >_{l^2} \tag{3.31}$$

yielding $Var(A\mathbf{X}) = \sigma^2 A\overline{A^*}$ as required.

□

For definiteness we ensure $A\mathbf{X}$ is always E valued by defining $A\mathbf{X}$ to be $\sum_k \xi_k X_k$ at sample points where this sum converges, and zero on the null set where it diverges.

If E is of type 2 we may deduce more.

Corollary 3.1.5 *Under the additional hypothesis that E is of type 2, the sum $A\mathbf{X}$ converges to a centred Gaussian random vector if and only if $A^* \in \Pi_2(E^*, l^2)$.*

Proof If $A\mathbf{X}$ is centred Gaussian we know from Corollary 3.1.4 that $A^* \in \Pi_2(E^*, l^2)$. Conversely let us assume $A^* \in \Pi_2(E^*, l^2)$. Then by (0.6) on page 67 of [43], which is based on work in [17], we have

$$\left(\mathbb{E} \left\| \sum_k \xi_k X_k \right\|_E^2 \right)^{1/2} \leq \sigma T_2(E) \pi_2(A^*) < \infty. \tag{3.32}$$

We deduce $A\mathbf{X}$ lies in $L^2(\Omega; E)$ and so, by Proposition 3.1.3, is centred Gaussian.

□

Corollary 3.1.4 enables us to determine, for E a separable complex Banach space, a factorisation property for operators $Q : \overline{E^*} \to E$ which are variances of centred Gaussian random vectors in E. In the case where E is of type 2, Corollary 3.1.5 enables us to characterise such operators Q precisely.

Corollary 3.1.6 *Let E be a separable complex Banach space.*
(a) Let Q be the variance of some centred Gaussian random vector in E, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then Q factors as $A\overline{A^*}$ where the operator $A : l^2 \to E$ has 2-summing adjoint; furthermore Q lies in $E \otimes_{\gamma_2^*} \overline{E}$.
(b) If, furthermore, E is of type 2, then conversely any operator $A\overline{A^*}$, where $A : l^2 \to E$ has 2-summing adjoint, is the variance of some centred Gaussian random vector in E.

Proof (a) Let $Q = Var\,\mathbf{Z}$, where $\mathbf{Z}$ is a centred Gaussian random vector in E, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. By Proposition 3.1.3 (c) and Corollary 3.1.4, $\mathbf{Z}$ lies in $L^2(\Omega; E)$ and is equal in distribution to $A\mathbf{X}$, where $\mathbf{X}$ is a sequence of independent $N(0,1)$ complex random variables and $A : l^2 \to E$ is an operator with 2-summing adjoint. Corollary 3.1.4 now shows $Var\,(A\mathbf{X}) = A\overline{A^*}$. Thus $Q = A\overline{A^*}$ as required. By Proposition 2.6.2, as $A\mathbf{X}$ lies in $L^2(\Omega; E)$, $Var\,(A\mathbf{X})$ lies in $E \otimes_{\gamma_2^*} \overline{E}$; thus Q lies in $E \otimes_{\gamma_2^*} \overline{E}$.
(b) Taking A as given, by Corollary 3.1.5 the random vector $A\mathbf{X}$, for $\mathbf{X}$ a sequence of

independent $N(0,1)$ complex random variables, is a centred Gaussian random vector with variance $A\overline{A^*}$.

□

In [50] van Neerven states that, for E a separable Banach space, no necessary and sufficient conditions are known for an operator $\overline{E^*} \to E$ to be the variance of a cylindrical Gaussian measure on E. We see that Corollary 3.1.6 provides such conditions in the case where E is of type 2.

3.2 Wiener processes

We now consider E valued cylindrical Wiener processes where, as usual, E is a separable complex Banach space.

Following [8] or chapter 5 of [9] we say an E valued stochastic process $(\mathbf{B}_t)_{t\in\mathbb{R}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a cylindrical Q-Wiener process, where Q is the variance of some cylindrical centred Gaussian random vector in E, if:

(i) for each t, $\mathbf{B}_t$ is measurable with respect to the cylindrical σ-algebra on E;

(ii) the process $\mathbf{B}_t$ has almost surely continuous sample paths and $\mathbf{B}_0 = 0$;

(iii) the process $\mathbf{B}_t$ has independent increments;

(iv) for each $s < t$, $\mathbf{B}_t - \mathbf{B}_s$ is a cylindrical centred Gaussian random vector satisfying

$$Var(\mathbf{B}_t - \mathbf{B}_s) = Q(t - s). \tag{3.33}$$

Condition (i) ensures that, for all $\xi^* \in E^*$, the process $\xi^*(\mathbf{B}_t)$ is adapted to the filtration induced on $(\Omega, \mathcal{F}, \mathbb{P})$ by the process $\mathbf{B}_t$.

Proposition 3.1.3, Corollary 3.1.4 and Corollary 3.1.6 enable us to deduce various properties of a cylindrical Q-Wiener process.

Proposition 3.2.1 *Let E be a separable Banach space and let $\mathbf{B}_t$ be an E valued cylindrical Q-Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Q factor as $A\overline{A^*}$, where $A : l^2 \to E$ is some operator with 2-summing adjoint given by Corollary 3.1.6.*

For $s < t$:

(i) $\mathbf{B}_t - \mathbf{B}_s$ *lies in* $L^2_w(\Omega; E, \pi_2)$*;*

(ii) $\pi_2(\mathbf{B}_t - \mathbf{B}_s) = \pi_2(A^*)(t-s)^{1/2}$*;*

(iii) for T any bounded linear operator on E,

$$Var(T(\mathbf{B}_t - \mathbf{B}_s)) = TQ\overline{T^*}(t-s). \tag{3.34}$$

Proof We observe $\mathbf{B}_t - \mathbf{B}_s$ is a cylindrical centred Gaussian random vector satisfying $Var(\mathbf{B}_t - \mathbf{B}_s) = Q(t-s)$. Proposition 3.1.3 (c) and Corollary 3.1.4 show $\mathbf{B}_t - \mathbf{B}_s$ is equal in distribution to a random vector of the form $A\mathbf{X}^{(s,t)}$ where, for each s and t, $\mathbf{X}^{(s,t)} = (X_k^{(s,t)})_{k\in\mathbb{Z}}$ is a sequence of independent $N(0, t-s)$ complex random variables. Corollary 3.1.4 now gives the required results immediately.

□

Note that if E is of type 2 and $A : l^2 \to E$ is any operator with 2-summing adjoint, there always exists a cylindrical $A\overline{A^*}$-Wiener process $\mathbf{B}_t$. Namely take $\mathbf{B}_t = A\mathbf{b}_t$ where $\mathbf{b}_t = (b_t^{(k)})_{k\in\mathbb{Z}}$ is an independent sequence of complex Brownian motions on the line, with $b_0^{(k)} = 0$ for each k, defined on the canonical probability space of continuous paths $\mathbb{R} \to \mathbb{C}$ equipped with Wiener measure. Corollary 3.1.5 shows $(A\mathbf{b}_t)_{t\in\mathbb{R}}$ is an E valued cylindrical process with almost surely continuous sample paths and independent increments; for $s < t$, Corollary 3.1.5 shows $A(\mathbf{b}_t - \mathbf{b}_s)$ is centred Gaussian with

$$Var(A(\mathbf{b}_t - \mathbf{b}_s)) = A\overline{A^*}(t-s). \tag{3.35}$$

We wish to develop the theory of stochastic integration of a deterministic family of operators with respect to a Q-Wiener process. For $s < t$, let $(T_u)_{s\le u\le t}$ be a non-random family of bounded linear operators on E and let $\mathbf{B}_t$ be a cylindrical Q-Wiener process in E. Consider a sequence $(\mathcal{P}_n)_{n\ge1}$ of refining partitions of $[s,t]$. Thus, if

$$\mathcal{P}_n = \{s = u_0^{(n)} < u_1^{(n)} < \cdots < u_{r(n)-1}^{(n)} < u_{r(n)}^{(n)} = t\} \tag{3.36}$$

for each n, we assume that $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$ for all n and $\sup_j(u_{j+1}^{(n)} - u_j^{(n)}) \downarrow 0$ as n tends to infinity. We say the stochastic integral

$$\int_s^t T_u \, d\mathbf{B}_u \tag{3.37}$$

exists in the Itô sense as a limit in $L^2_w(\Omega; E, \pi_2)$ if the sequence of Riemann sums

$$\sum_{j=0}^{r(n)-1} T_{u_j^{(n)}} (\mathbf{B}_{u_{j+1}^{(n)}} - \mathbf{B}_{u_j^{(n)}}) \tag{3.38}$$

converges to a limit in $L^2_w(\Omega; E, \pi_2)$ as n tends to infinity, this limit being independent of the choice of partitions $(\mathcal{P}_n)_{n\geq 1}$.

We have the following theorem.

Theorem 3.2.2 *For E a separable Banach space, let $\mathbf{B}_t$ be an E valued cylindrical $A\overline{A^*}$-Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for $s < t$, if $(T_u)_{s\leq u\leq t}$ is a non-random family of bounded linear operators on E such that $(A^*T_u^*)_{s\leq u\leq t}$ is Hölder continuous in the π_2 norm, the stochastic integral*

$$\int_s^t T_u \, d\mathbf{B}_u \tag{3.39}$$

exists in the Itô sense as a limit in $L^2_w(\Omega; E, \pi_2)$. Furthermore

$$\pi_2 \left(\int_s^t T_u \, d\mathbf{B}_u \right)^2 \leq \int_s^t \pi_2(A^*T_u^*)^2 \, du. \tag{3.40}$$

Proof Consider a partition $\mathcal{P} = \{s = u_0 < u_1 < \cdots < u_r = t\}$ of the interval $[s, t]$. Put

$$\begin{aligned} I(\mathcal{P}) &= \sum_{j=0}^{r-1} T_{u_j} (\mathbf{B}_{u_{j+1}} - \mathbf{B}_{u_j}) & (3.41) \\ &= \sum_{j=0}^{r-1} I(u_j, u_{j+1}) & (3.42) \end{aligned}$$

say. When we refine the partition by adding a point $u_{j'}$ between u_j and u_{j+1}, the sum changes by

$$I(u_j, u_{j'}) + I(u_{j'}, u_{j+1}) - I(u_j, u_{j+1}) = (T_{u_{j'}} - T_{u_j})(\mathbf{B}_{u_{j+1}} - \mathbf{B}_{u_{j'}}). \tag{3.43}$$

By the classical theory of Riemann integration we can and do restrict ourselves to partition sequences of the following form. Let there be $2^n + 1$ elements in the partition $\mathcal{P}_n$

of $[s,t]$ at stage n; denote the Riemann sum at stage n by $I(n)$. At each stage we insert a new point between each old point of the partition such that

$$\sup_j (u_{j+1}^{(n)} - u_j^{(n)}) \leq M 2^{-n} \tag{3.44}$$

at stage n, for some constant $M > 0$.

Thus, writing u_j for $u_j^{(n)}$ to simplify our notation,

$$\begin{aligned} I(n+1) - I(n) &= \sum_{j=0}^{r-1} [I(u_j, u_{j'}) + I(u_{j'}, u_{j+1}) - I(u_j, u_{j+1})] && (3.45) \\ &= \sum_{j=0}^{r-1} (T_{u_{j'}} - T_{u_j})(\mathbf{B}_{u_{j+1}} - \mathbf{B}_{u_{j'}}). && (3.46) \end{aligned}$$

By assumption $(A^* T_u^*)_{s \leq u \leq t}$ is Hölder continuous in the π_2 norm; it follows that

$$\pi_2(A^*(T_v^* - T_u^*)) \leq C(v-u)^\alpha \tag{3.47}$$

for some fixed $C > 0$, $\alpha > 0$ and all $s \leq u < v \leq t$. Now the norm π_2 is 2-convex; see section 2.2 of this thesis or section 2 of [43] for details of this concept. This implies

$$\begin{aligned} \|I(n+1) - I(n)\|_{\pi_2}^2 &\leq \sum_{j=0}^{r-1} \pi_2(A^*(T_{u_{j'}}^* - T_{u_j}^*))^2 (u_{j+1} - u_{j'}) && (3.48) \\ &\leq C^2 \sum_{j=0}^{r-1} (u_{j'} - u_j)^{2\alpha} (u_{j+1} - u_{j'}) && (3.49) \\ &< C^2 M^{2\alpha} 2^{-2\alpha n} \sum_{j=0}^{r-1} (u_{j+1} - u_j) && (3.50) \\ &= C^2 M^{2\alpha} 2^{-2\alpha n} (t-s) && (3.51) \end{aligned}$$

and so

$$\|I(n+1) - I(n)\|_{\pi_2} < C M^\alpha 2^{-\alpha n} (t-s)^{1/2}. \tag{3.52}$$

This shows $(I(n))_{n\geq 1}$ is a Cauchy sequence in $L_w^2(\Omega; E, \pi_2)$; it therefore converges.

If $(\mathcal{P}_n)_{n\geq 1}$ and $(\mathcal{P}'_n)_{n\geq 1}$ are any two such sequences of partitions of $[s,t]$, it is clear the sequence of partitions $(\mathcal{P}_n \cup \mathcal{P}'_n)_{n\geq 1}$ also yields a convergent sequence of Riemann sums; furthermore the limits induced by $(\mathcal{P}_n)_{n\geq 1}$, $(\mathcal{P}'_n)_{n\geq 1}$ and $(\mathcal{P}_n \cup \mathcal{P}'_n)_{n\geq 1}$ must coincide.

Thus the limit of the sequence $(I(n))_{n\geq 1}$ in $L^2_w(\Omega; E, \pi_2)$ is independent of the choice of refining partitions. We deduce that the stochastic integral exists in the Itô sense as a limit in $L^2_w(\Omega; E, \pi_2)$.

Finally, by the 2-convexity of the norm π_2, if $s = u_0 < u_1 < \cdots < u_r = t$ is any partition of $[s,t]$ we have

$$\pi_2 \left(\sum_{j=0}^{r-1} T_{u_j}(\mathbf{B}_{u_{j+1}} - \mathbf{B}_{u_j}) \right)^2 \leq \sum_{j=0}^{r-1} \pi_2(A^* T^*_{u_j})^2 (u_{j+1} - u_j). \tag{3.53}$$

Passing to the $L^2_w(\Omega; E, \pi_2)$ limit gives the required result.

□

Note that, as it is the limit in $L^2_w(\Omega; E, \pi_2)$ of a sequence of centred Gaussian random vectors, the stochastic integral of Theorem 3.2.2 is itself a centred Gaussian random vector. Thus by Proposition 3.1.3 (c) it also converges in the Itô sense to a limit in the Bochner space $L^2(\Omega; E)$.

Theorem 3.2.2 will be used in the next chapter to prove the existence of a solution to the E valued Langevin equation.

Chapter 4

Ornstein-Uhlenbeck processes

In this chapter we study a Langevin equation for stochastic processes with values in a separable complex Banach space E. Section 4.1 defines the equation while section 4.2 proves the existence, under certain conditions, of solutions to it. Section 4.3 considers some explicit examples of such Langevin equations.

Background information on diffusion processes may be found in [44] or [46]. Information on semigroups of operators on Banach spaces may be found in [10] or [24]. General information on infinite dimensional stochastic differential equations and Wiener processes may be found in [8] or chapter 5 of [9].

Itô studied infinite dimensional Ornstein-Uhlenbeck processes in the time domain; see [26]. More recently Kolsrud studied such processes from the standpoint of Gaussian random fields in [30] and van Neerven contributed to this subject in [50].

4.1 The Langevin equation

Let E be a separable complex Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which we assume rich enough to support all the random vectors under consideration. Consider the E valued stochastic differential equation

$$d\mathbf{Z}_t + \Lambda \mathbf{Z}_t dt = d\mathbf{B}_t \tag{4.1}$$

for $t \in \mathbb{R}$, where:

(L1) the operator Λ is a closed operator from a norm dense domain $\mathcal{D}(\Lambda) \subseteq E$ to E.

We seek a pair of processes $(\mathbf{B}_t, \mathbf{Z}_t)$, each defined on $(\Omega, \mathcal{F}, \mathbb{P})$. To ensure the existence of L^2 bounded solutions to (4.1) we will impose some conditions on Λ. We assume the following:

(L2) $i\Lambda$ is the generator of a C_0 group $(e^{it\Lambda})_{t\in\mathbb{R}}$ of operators on E; by a corollary to the Hille-Yosida theorem, for which see section 12.3 of [24], this is equivalent to the resolvent of Λ satisfying

$$\|(\Lambda + i\omega I)^{-n}\| \leq \frac{C_\Lambda}{(|\omega| - \alpha_\Lambda)^n} \tag{4.2}$$

for some finite constants $C_\Lambda > 0$ and $\alpha_\Lambda \geq 0$, all $n \in \mathbb{N}$ and all real ω such that $|\omega| > \alpha_\Lambda$;

(L3) the resolvent of Λ satisfies

$$\|(\Lambda + i\omega I)^{-1}\| \leq K_\Lambda \tag{4.3}$$

for some finite constant $K_\Lambda > 0$ and all real ω;

(L4) $i\Lambda^*$ is the generator of a C_0 group $(e^{it\Lambda^*})_{t\in\mathbb{R}}$ of operators on E^* (if E is reflexive this follows from (L2)).

For more information on these conditions, consult [10] or chapters 11, 12 and 14 of [24]. Note that (L4), in the presence of (L2), is equivalent to the domain $\mathcal{D}(\Lambda^*)$ of Λ^*, the adjoint of Λ, being norm dense in E^*; without (L4) we only know it is weak-$*$ dense. For details see section 1.4 of [10] or chapter 14 of [24].

We interpret equation (4.1) in the following way. The process $(\mathbf{B}_t)_{t\in\mathbb{R}}$ is required to be an E valued cylindrical Q-Wiener process. The process $(\mathbf{Z}_t)_{t\in\mathbb{R}}$ is required to be an E valued, centred Gaussian, stationary stochastic process with almost surely Hölder continuous sample paths. As $\mathbf{Z}_t$ is stationary we may write $Cov(\mathbf{Z}_t, \mathbf{Z}_s) = \Psi_{\mathbf{Z}}(t-s)$ for some function $\Psi_{\mathbf{Z}}$, called the autocovariance function of the process. Finally we require that $\mathbf{B}_t$ and $\mathbf{Z}_t$ satisfy

$$\xi^*(\mathbf{Z}_t - \mathbf{Z}_s) + \int_s^t \Lambda^*(\xi^*)(\mathbf{Z}_u)\, du = \xi^*(\mathbf{B}_t - \mathbf{B}_s) \tag{4.4}$$

almost surely, for all $\xi^* \in \mathcal{D}(\Lambda^*)$ and $s < t$. Following [8] or chapter 5 of [9] we call the pair $(\mathbf{B}_t, \mathbf{Z}_t)$ a weak solution of the Langevin equation.

Condition (L4), together with the separability of E and the Hahn-Banach theorem, implies there is a countable subset of $\mathcal{D}(\Lambda^*)$ which separates the points of E. This ensures that, if we know $\xi^*(\mathbf{B}_t)$ and $\xi^*(\mathbf{Z}_t)$ almost surely for all $\xi^* \in \mathcal{D}(\Lambda^*)$, the processes $\mathbf{B}_t$ and $\mathbf{Z}_t$ are almost surely determined.

Note we do not require that $\mathbf{Z}_t$ be adapted to the filtration induced by $\mathbf{B}_t$; we will, however, consider important circumstances in which this is the case.

4.2 Spectral solutions of the Langevin equation

This section states and proves an existence theorem for weak solutions of equation (4.1). Consider

$$\mathbf{Z}_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega, \tag{4.5}$$

where $\tilde{\mathbf{B}}_\omega$ is a given E valued cylindrical Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. This formula is suggested by classical harmonic analysis. Consider also $\mathbf{B}_t$ defined by the condition $\mathbf{B}_0 = 0$ and, for $s < t$,

$$\mathbf{B}_t - \mathbf{B}_s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \, d\tilde{\mathbf{B}}_\omega. \tag{4.6}$$

All our stochastic integrals will be interpreted in the Itô sense within the framework of Theorem 3.2.2.

Theorem 4.2.1 *Assume conditions (L1)–(L4) hold. The expression $\mathbf{Z}_t$ above:*
(a) converges as an Itô stochastic integral for each $t \in \mathbb{R}$, defining an E valued centred Gaussian process on $(\Omega, \mathcal{F}, \mathbb{P})$;
(b) is a stationary process with bounded autocovariance $\Psi_{\mathbf{Z}}$ given by

$$\Psi_{\mathbf{Z}}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega u} (\Lambda + i\omega I)^{-1} Q[(\overline{\Lambda} - i\omega I)^{-1}]^* \, d\omega; \tag{4.7}$$

(c) has almost surely Hölder continuous sample paths, of exponent α for every $\alpha < 1/2$.

The expression $\mathbf{B}_t$ *above:*

(d) has increments $\mathbf{B}_t - \mathbf{B}_s$ *which converge as Itô stochastic integrals for each* $s < t$, *defining* E *valued centred Gaussian random vectors on* $(\Omega, \mathcal{F}, \mathbb{P})$*;*

(e) is an E *valued cylindrical* Q*-Wiener process defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

Furthermore:

(f) the pair $(\mathbf{B}_t, \mathbf{Z}_t)$ *is a weak solution of the* E *valued Langevin equation (4.1).*

Proof (a) Assume Q factors as $A\overline{A^*}$. Let us consider, for finite $a < b$,

$$\mathbf{Z}_t^{(a,b)} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{i\omega t} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega. \tag{4.8}$$

By Theorem 3.2.2 this will converge as an Itô stochastic integral if the family $(A^*[e^{i\omega t}(\Lambda + i\omega I)^{-1}]^*)_{a \leq \omega \leq b}$ is Hölder continuous in the π_2 norm. This in turn will follow if the family $(e^{i\omega t}(\Lambda + i\omega I)^{-1})_{a \leq \omega \leq b}$ is operator norm Hölder continuous, since A^* is 2-summing. But, for $a \leq p < q \leq b$,

$$\begin{aligned} & e^{iqt}(\Lambda + iqI)^{-1} - e^{ipt}(\Lambda + ipI)^{-1} \\ = \; & (e^{iqt} - e^{ipt})(\Lambda + ipI)^{-1} + e^{iqt}((\Lambda + iqI)^{-1} - (\Lambda + ipI)^{-1}) \quad (4.9) \\ = \; & (e^{iqt} - e^{ipt})(\Lambda + ipI)^{-1} - e^{iqt}(q - p)(\Lambda + ipI)^{-1}(\Lambda + iqI)^{-1} \quad (4.10) \end{aligned}$$

by the resolvent equation. We have $|e^{iqt} - e^{ipt}| \leq (q-p)|t|$; this and condition (L3) shows

$$\|e^{iqt}(\Lambda + iqI)^{-1} - e^{ipt}(\Lambda + ipI)^{-1}\| \leq K_\Lambda(|t| + K_\Lambda)(q - p) \tag{4.11}$$

which proves $(e^{i\omega t}(\Lambda + i\omega I)^{-1})_{a \leq \omega \leq b}$ is Hölder continuous in operator norm as required. We deduce that $\mathbf{Z}_t^{(a,b)}$ converges as an Itô stochastic integral.

Thus $\mathbf{Z}_t^{(a,b)} \in L^2_w(\Omega; E, \pi_2)$ for all finite $a < b$. The integral $\mathbf{Z}_t^{(a,b)}$ will converge to a limit in $L^2_w(\Omega; E, \pi_2)$ as $a \downarrow -\infty$ and $b \uparrow \infty$ if, for any $\varepsilon > 0$, there exists a positive finite $N(\varepsilon)$ such that $\|\mathbf{Z}_t^{(m,n)}\|_{\pi_2} < \varepsilon$ for all $n > m \geq N(\varepsilon)$. By Theorem 3.2.2 and condition

(L2) we have

$$\|\mathbf{Z}_t^{(m,n)}\|_{\pi_2}^2 = \left\| \frac{1}{\sqrt{2\pi}} \int_m^n e^{i\omega t} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega \right\|_{\pi_2}^2 \tag{4.12}$$

$$\leq \frac{1}{2\pi} \int_m^n \pi_2(((\Lambda + i\omega I)^{-1} A)^*)^2 \, d\omega \tag{4.13}$$

$$\leq \frac{\pi_2(A^*)^2}{2\pi} \int_m^n \|(\Lambda + i\omega I)^{-1}\|^2 \, d\omega \tag{4.14}$$

$$< \frac{C_\Lambda^2 \pi_2(A^*)^2}{2\pi(m - \alpha_\Lambda)} \tag{4.15}$$

for all $n > m > \alpha_\Lambda$, as required. We deduce that $\mathbf{Z}_t = \lim_{a\downarrow-\infty, b\uparrow\infty} \mathbf{Z}_t^{(a,b)}$ converges in $L_w^2(\Omega; E, \pi_2)$.

For each $t \in \mathbb{R}$ and $\xi^* \in E^*$ the random variable $\xi^*(\mathbf{Z}_t)$ is the limit in $L^2(\Omega)$ of a sequence of complex centred Gaussian random variables; it is therefore a complex centred Gaussian random variable. We deduce that, for each $t \in \mathbb{R}$, the stochastic integral $\mathbf{Z}_t$ is an E valued centred Gaussian random vector.

Hence, by Proposition 3.1.3 (c), the integral $\mathbf{Z}_t$ converges in the stronger norm of $L^2(\Omega; E)$.

(b) Fix $s < t$. For finite $a < b$, using the notation $\mathbf{Z}_t^{(a,b)}$ from part (a),

$$Cov(\mathbf{Z}_t^{(a,b)}, \mathbf{Z}_s^{(a,b)})$$

$$= \frac{1}{2\pi} Cov\left(\int_a^b e^{i\omega t} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega, \int_a^b e^{i\omega' s} (\Lambda + i\omega' I)^{-1} \, d\tilde{\mathbf{B}}_{\omega'} \right) \tag{4.16}$$

$$= \frac{1}{2\pi} \lim_{L^2} Cov \left(\sum_{j=0}^{r-1} e^{i\omega_j t} (\Lambda + i\omega_j I)^{-1} (\tilde{\mathbf{B}}_{\omega_{j+1}} - \tilde{\mathbf{B}}_{\omega_j}), \right.$$

$$\left. \sum_{k=0}^{r-1} e^{i\omega'_k s} (\Lambda + i\omega'_k I)^{-1} (\tilde{\mathbf{B}}_{\omega'_{k+1}} - \tilde{\mathbf{B}}_{\omega'_k}) \right) \tag{4.17}$$

$$= \frac{1}{2\pi} \lim_{L^2} \sum_{j=0}^{r-1} e^{i\omega_j (t-s)} Var\left((\Lambda + i\omega_j I)^{-1} (\tilde{\mathbf{B}}_{\omega_{j+1}} - \tilde{\mathbf{B}}_{\omega_j}) \right), \tag{4.18}$$

where the L^2 limit is taken to mean the limit in $L_w^2(\Omega; E, \pi_2)$ over refinements of appropriate partitions $a = \omega_0 = \omega'_0 < \omega_1 = \omega'_1 < \cdots < \omega_r = \omega'_r = b$ of $[a, b]$. By Proposition

3.2.1 (iii) we see

$$\begin{aligned}
& Cov(\mathbf{Z}_t^{(a,b)}, \mathbf{Z}_s^{(a,b)}) \\
= \; & \frac{1}{2\pi} \lim_{L^2} \sum_{j=0}^{r-1} e^{i\omega_j(t-s)} (\Lambda + i\omega_j I)^{-1} Q[(\overline{\Lambda} - i\omega_j I)^{-1}]^* (\omega_{j+1} - \omega_j) \qquad (4.19) \\
= \; & \frac{1}{2\pi} \int_a^b e^{i\omega(t-s)} (\Lambda + i\omega I)^{-1} Q[(\overline{\Lambda} - i\omega I)^{-1}]^* \, d\omega. \qquad (4.20)
\end{aligned}$$

Letting $a \downarrow -\infty$ and $b \uparrow \infty$ yields the required formula for the autocovariance; it is clear $\mathbf{Z}_t$ is stationary.

(c) We have, for $s < t$,

$$\mathbf{Z}_t - \mathbf{Z}_s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{i\omega t} - e^{i\omega s})(\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega, \qquad (4.21)$$

which, applying Theorem 3.2.2, yields

$$\|\mathbf{Z}_t - \mathbf{Z}_s\|_{\pi_2}^2 \leq \frac{\pi_2(A^*)^2}{2\pi} \int_{-\infty}^{\infty} \left| e^{i\omega t} - e^{i\omega s} \right|^2 \left\| (\Lambda + i\omega I)^{-1} \right\|^2 d\omega. \qquad (4.22)$$

We now split the integral into separate parts with ranges $|\omega| < 2\alpha_\Lambda$ and $|\omega| \geq 2\alpha_\Lambda$. Note that $\left| e^{i\omega t} - e^{i\omega s} \right| \leq \min(2, |\omega|(t-s)) \leq \sqrt{2}|\omega|^{1/2}(t-s)^{1/2}$ and, for $|\omega| \geq 2\alpha_\Lambda$, that $(|\omega| - \alpha_\Lambda)^{-1} \leq 2|\omega|^{-1}$. Applying conditions (L2) and (L3) gives

$$\begin{aligned}
& \|\mathbf{Z}_t - \mathbf{Z}_s\|_{\pi_2}^2 \\
\leq \; & \frac{K_\Lambda^2 \pi_2(A^*)^2 (t-s)}{\pi} \int_{|\omega| < 2\alpha_\Lambda} |\omega| \, d\omega \\
+ \; & \frac{4C_\Lambda^2 \pi_2(A^*)^2}{2\pi} \int_{|\omega| \geq 2\alpha_\Lambda} \left| \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right|^2 d\omega \qquad (4.23) \\
= \; & 4 \left(\frac{\alpha_\Lambda^2 K_\Lambda^2}{\pi} + C_\Lambda^2 \right) \pi_2(A^*)^2 (t-s) \qquad (4.24)
\end{aligned}$$

by Plancherel's theorem applied to the indicator function of $[s,t]$, as required. This expression shows the map $t \mapsto \mathbf{Z}_t$ is Hölder continuous as a function $\mathbb{R} \to L_w^2(\Omega; E, \pi_2)$ with exponent $1/2$. The fact that $\mathbf{Z}_t$ is Gaussian enables us to now apply Kolmogorov's continuity lemma in its vector valued form to deduce the existence of a version of $\mathbf{Z}_t$ with almost surely Hölder continuous sample paths of exponent α for every $\alpha < 1/2$; see pages 59–61 of [44] for details.

(d) Fix $s < t$. Keeping the notation of part (a) we define $\mathbf{B}_t^{(a,b)} - \mathbf{B}_s^{(a,b)}$, for finite $a < b$, in the same manner as we defined $\mathbf{Z}_t^{(a,b)}$. By Theorem 3.2.2 this will converge in $L^2_w(\Omega; E, \pi_2)$ as an Itô stochastic integral if the family $\left(\frac{e^{i\omega t} - e^{i\omega s}}{i\omega}\right)_{a \le \omega \le b}$ is Hölder continuous. But, for $a \le p < q \le b$,

$$\begin{aligned} \left| \frac{e^{iqt} - e^{iqs}}{iq} - \frac{e^{ipt} - e^{ips}}{ip} \right| &= \left| \int_s^t (e^{iqu} - e^{ipu})\, du \right| &(4.25)\\ &\le \left(\int_s^t |u|\, du \right)(q-p), &(4.26)\end{aligned}$$

and so we have Hölder continuity as required. Thus the stochastic integral $\mathbf{B}_t^{(a,b)} - \mathbf{B}_s^{(a,b)}$ converges in $L^2_w(\Omega; E, \pi_2)$.

To consider the case when $a \downarrow -\infty$ and $b \uparrow \infty$ we note by Theorem 3.2.2 that

$$\|\mathbf{B}_t^{(a,b)} - \mathbf{B}_s^{(a,b)}\|_{\pi_2}^2 \le \frac{\pi_2(A^*)^2}{2\pi} \int_a^b \left| \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right|^2 d\omega. \tag{4.27}$$

By Plancherel's theorem, applied to the indicator function of $[s,t]$, this integral increases to $\pi_2(A^*)^2(t-s)$ as $a \downarrow -\infty$ and $b \uparrow \infty$. It follows that, for any $\varepsilon > 0$, there exists a positive finite $N(\varepsilon)$ such that $\|\mathbf{B}_t^{(m,n)} - \mathbf{B}_s^{(m,n)}\|_{\pi_2} < \varepsilon$ for all $n > m \ge N(\varepsilon)$; consequently the stochastic integral $\mathbf{B}_t - \mathbf{B}_s = \lim_{a\downarrow -\infty, b\uparrow\infty}(\mathbf{B}_t^{(a,b)} - \mathbf{B}_s^{(a,b)})$ converges in $L^2_w(\Omega; E, \pi_2)$.

For $s < t$ and $\xi^* \in E^*$ it is clear the random variable $\xi^*(\mathbf{B}_t - \mathbf{B}_s)$ is a complex centred Gaussian random variable, as it is obtained as the limit in $L^2(\Omega)$ of complex centred Gaussian random variables. Consequently, for $s < t$, the stochastic integral $\mathbf{B}_t - \mathbf{B}_s$ is an E valued centred Gaussian random vector.

Hence, by Proposition 3.1.3 (c), the integral $\mathbf{B}_t - \mathbf{B}_s$ converges in the stronger norm of $L^2(\Omega; E)$.

(e) For $s < t$, by part (d) above $\|\mathbf{B}_t - \mathbf{B}_s\|_{\pi_2}^2 \le \pi_2(A^*)^2(t-s)$; we may therefore apply Kolmogorov's continuity lemma in the same manner as part (c) to deduce almost sure sample path continuity.

By a calculation similar to that of part (b),

$$\begin{aligned} Var(\mathbf{B}_t - \mathbf{B}_s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right|^2 d\omega\, Q &(4.28)\\ &= (t-s)Q &(4.29)\end{aligned}$$

by Plancherel's theorem applied to the indicator function of $[s,t]$; also, for $s < t < u < v$,

$$\begin{aligned} Var(\mathbf{B}_t - \mathbf{B}_s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(e^{i\omega t} - e^{i\omega s})(e^{-i\omega v} - e^{-i\omega u})}{|i\omega|^2} \, d\omega \, Q && (4.30) \\ &= 0 && (4.31) \end{aligned}$$

again by Plancherel's theorem, this time applied to the indicator functions of the disjoint intervals $[s,t]$ and $[u,v]$. We deduce $\mathbf{B}_t$ is a cylindrical Q-Wiener process as required.

(f) For $\xi^* \in \mathcal{D}(\Lambda^*)$ and $s < t$ we calculate

$$\begin{aligned} & \xi^*(\mathbf{Z}_t - \mathbf{Z}_s) + \int_s^t \Lambda^*(\xi^*)(\mathbf{Z}_u) \, du \\ = \; & \frac{1}{\sqrt{2\pi}} \xi^* \left(\int_{-\infty}^{\infty} (e^{i\omega t} - e^{i\omega s})(\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega \right) \\ + \; & \frac{1}{\sqrt{2\pi}} \int_s^t \Lambda^*(\xi^*) \left(\int_{-\infty}^{\infty} e^{i\omega u} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega \right) du && (4.32) \\ = \; & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right) i\omega \xi^* (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega \\ + \; & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_s^t e^{i\omega u} \, du \right) \Lambda^*(\xi^*)(\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega && (4.33) \end{aligned}$$

where the change in the order of integration in the second integral is justified by the almost sure Hölder continuity of the sample paths of $\mathbf{Z}_u$ and the fact that, for each u, $\mathbf{Z}_u$ is the limit in $L^2(\Omega; E)$ of a sequence of finite sums of elements of $L^2(\Omega; E)$. This gives

$$\begin{aligned} & \xi^*(\mathbf{Z}_t - \mathbf{Z}_s) + \int_s^t \Lambda^*(\xi^*)(\mathbf{Z}_u) \, du \\ = \; & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right) (i\omega \xi^* + \Lambda^*(\xi^*))(\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega && (4.34) \\ = \; & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right) (i\omega \xi^* + \xi^* \Lambda)(\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega; && (4.35) \end{aligned}$$

we know that $\Lambda^*(\xi^*) = \xi^* \Lambda$ because the range of the resolvent satisfies $(\Lambda + i\omega I)^{-1}(E) \subseteq \mathcal{D}(\Lambda)$ for all $\omega \in \mathbb{R}$. Thus

$$\begin{aligned} & \xi^*(\mathbf{Z}_t - \mathbf{Z}_s) + \int_s^t \Lambda^*(\xi^*)(\mathbf{Z}_u) \, du \\ = \; & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \right) \xi^*(d\tilde{\mathbf{B}}_\omega) && (4.36) \\ = \; & \frac{1}{\sqrt{2\pi}} \xi^* \left(\int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} \, d\tilde{\mathbf{B}}_\omega \right) && (4.37) \\ = \; & \xi^*(\mathbf{B}_t - \mathbf{B}_s) && (4.38) \end{aligned}$$

which is justified by the convergence in $L^2(\Omega; E)$ of the stochastic integral $\mathbf{B}_t - \mathbf{B}_s$. We deduce $(\mathbf{B}_t, \mathbf{Z}_t)$ is a weak solution of the E valued Langevin equation (4.1).

□

The proof of Theorem 4.2.1 (f) naturally generalises the corresponding proof in the scalar valued case; for details of this, see chapter XI, §10 of [13].

We now consider a condition on Λ under which $\mathbf{Z}_t$ is adapted to the filtration induced by $\mathbf{B}_t$, and $\mathbf{Z}_t$ is unique in distribution. The following corollary mirrors results in Itô's paper [26].

Corollary 4.2.2 *Let $(\mathbf{B}_t, \mathbf{Z}_t)$ be a weak solution of the E valued Langevin equation (4.1). Assume, as well as (L1)–(L4), that the operator $(-\Lambda)$ generates a C_0 semigroup $(e^{-\Lambda t})_{t\geq 0}$ of exponential norm decay. Then the process $\mathbf{Z}_t$ is adapted to the filtration induced by $\mathbf{B}_t$; furthermore if $(\mathbf{B}'_t, \mathbf{Z}'_t)$ is also a weak solution then $\mathbf{Z}'_t$ is identical in distribution to $\mathbf{Z}_t$.*

Proof Firstly let $(\mathbf{B}_t, \mathbf{Z}_t)$ and $(\mathbf{B}_t, \mathbf{Z}'_t)$ be weak solutions of the E valued Langevin equation. Setting $\Theta_t = \mathbf{Z}'_t - \mathbf{Z}_t$ we see Θ_t is an E valued, centred Gaussian, stationary stochastic process with almost surely Hölder continuous sample paths defined on $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies

$$\xi^*(\Theta_t - \Theta_s) + \int_s^t \Lambda^*(\xi^*)(\Theta_u)\, du = 0 \tag{4.39}$$

almost surely, for all $\xi^* \in \mathcal{D}(\Lambda^*)$ and $s < t$. Itô's paper [26] now shows, for $s < t$,

$$\Theta_t = e^{-\Lambda(t-s)}\Theta_s. \tag{4.40}$$

The condition that $(e^{-\Lambda t})_{t\geq 0}$ is of exponential norm decay, together with the requirement that Θ_t be stationary, now implies $\Theta_t = 0$ almost surely — we simply let t tend to infinity in (4.40).

Now by [26], if $\mathbf{B}_t$ is any E valued cylindrical Q-Wiener process then the process

$$\mathbf{Z}_t = \int_{-\infty}^t e^{-\Lambda(t-u)}\, d\mathbf{B}_u \tag{4.41}$$

is such that $(\mathbf{B}_t, \mathbf{Z}_t)$ is a weak solution of the E valued Langevin equation.

Combining these results we see that, given $\mathbf{B}_t$, the process $\mathbf{Z}_t$ is almost surely unique and is given almost surely by the stochastic integral (4.41).

The stochastic integral (4.41) shows $\mathbf{Z}_t$ is adapted to the filtration induced by $\mathbf{B}_t$. Furthermore we see that, whenever $(\mathbf{B}_t, \mathbf{Z}_t)$ and $(\mathbf{B}'_t, \mathbf{Z}'_t)$ are weak solutions, $\mathbf{Z}_t$ must be expressible almost surely in the form (4.41) as a stochastic integral with respect to $\mathbf{B}_t$ and $\mathbf{Z}'_t$ must be expressible almost surely in the form (4.41) as a stochastic integral with respect to $\mathbf{B}'_t$. We deduce, as $\mathbf{B}_t$ and $\mathbf{B}'_t$ have the same distribution, that $\mathbf{Z}_t$ and $\mathbf{Z}'_t$ have the same distribution.

□

4.3 Examples

Let $1 < p < \infty$ and $\varepsilon > 0$. Define

$$E = \left\{ f \in L^p(\mathbb{R}) : \hat{f}(\zeta) = 0 \text{ for all } |\zeta| \leq \varepsilon \right\}, \tag{4.42}$$

where $\hat{f}$ denotes the Fourier transform of f. The continuity of the Riesz projection on $L^p(\mathbb{R})$ ([31], section V.B) shows E is a closed complemented subspace of $L^p(\mathbb{R})$; it is therefore a reflexive Banach space. We consider various possibilities for Λ, each defined via Fourier multipliers:

(i) $\widehat{(e^{it\Lambda} f)}(\zeta) = e^{it\zeta} \hat{f}(\zeta)$, so $(e^{it\Lambda} f)(x) = f(x+t)$ for all $x \in \mathbb{R}$ and $(e^{it\Lambda})_{t\in\mathbb{R}}$ is a C_0 group of translation operators on $L^p(\mathbb{R})$ with $\Lambda = -i\frac{d}{dx}$;

(ii) $\widehat{(e^{it\Lambda} f)}(\zeta) = e^{it|\zeta|} \hat{f}(\zeta)$, so $(e^{it\Lambda} f)(x) = (\mathcal{R}_+ f)(x+t) + (\mathcal{R}_- f)(x-t)$ for all $x \in \mathbb{R}$, where $\mathcal{R}_+$ and $\mathcal{R}_-$ denote the positive and negative Riesz projections;

(iii) $\widehat{(e^{it\Lambda} f)}(\zeta) = e^{it\log|\zeta|} \hat{f}(\zeta)$, implying $e^{it\Lambda} = \Delta^{it}$ where $\Delta = -\frac{d^2}{dx^2}$ is the Laplace operator. The Laplacian is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ (see chapter 4 of [10] for details) which enables us to define its imaginary powers by Fourier multipliers as described. These imaginary powers $(\Delta^{it})_{t\in\mathbb{R}}$ form a C_0 group on $L^p(\mathbb{R})$ of polynomial growth; see [21] for details.

Each of the $(e^{it\Lambda})_{t\in\mathbb{R}}$ considered here restrict to E, yielding C_0 groups. Furthermore the condition on each $f \in E$ that $\hat{f}(\zeta) = 0$ for all $|\zeta| \leq \varepsilon$ implies, in each case, that the resolvent of Λ is bounded on the imaginary axis; conditions (L2)-(L4) are therefore satisfied. In addition neither Λ nor $-\Lambda$ generate C_0 semigroups of exponential norm decay.

Thus, given any such Λ and any Q which is the variance of some centred Gaussian random vector in E, the E valued Langevin equation (4.1) associated to Λ and Q has a weak solution as described in Theorem 4.2.1.

Chapter 5

Some probability distribution theory

This chapter recalls two classes of compactly supported probability distributions on $\mathbb{R}$ and proves a result on the convolution of one of these. Both these distributions arise in the theory of random matrices; these are studied in chapter 6.

5.1 The arcsine and semicircle distributions

The arcsine law with mean μ and variance σ^2 is the probability measure on $\mathbb{R}$ given by

$$\mathbb{P}_{as}^{\mu,\sigma^2}(dx) = \begin{cases} \frac{1}{\pi\sqrt{2\sigma^2-(x-\mu)^2}}\,dx & \mu-\sqrt{2}\sigma \le x \le \mu+\sqrt{2}\sigma; \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

It is straightforward to verify that this does indeed give a probability measure on $\mathbb{R}$ with mean μ and variance σ^2. Usually we consider the case where $\mu = 0$ and $\sigma^2 = 1/2$. This is known as the standard arcsine law and is given by

$$\mathbb{P}_{as}(dx) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}}\,dx & -1 \le x \le 1; \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

$$= \frac{d\theta}{2\pi} \tag{5.3}$$

where $d\theta/2\pi$ is normalised arc length on the unit circle $\mathbb{T}$. So if X is a random variable uniformly distributed on $\mathbb{T}$, its x co-ordinate is distributed according to the standard arcsine law.

The arcsine distribution arises extensively in the theory of random walks; see chapter III of [15] for more information.

An important related probability distribution is the following. The semicircle law with mean μ and variance σ^2 is the probability measure on $\mathbb{R}$ given by

$$\mathbb{P}_{ss}^{\mu,\sigma^2}(dx) = \begin{cases} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-\mu)^2}\, dx & \mu - 2\sigma \leq x \leq \mu + 2\sigma; \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Again it is straightforward to verify this gives a probability measure on $\mathbb{R}$ with mean μ and variance σ^2. Usually we consider the case where $\mu = 0$ and $\sigma^2 = 1/4$. This is known as the standard semicircle law and is given by

$$\mathbb{P}_{ss}(dx) = \begin{cases} \frac{2}{\pi}\sqrt{1 - x^2}\, dx & -1 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Note that, for f a continuous bounded function on $\mathbb{R}$,

$$\int_{-1}^{1} f(x)\mathbb{P}_{ss}(dx) = \iint_{\mathbb{D}} f(x) A(dx, dy) \tag{5.6}$$

where $A(dx, dy)$ is normalised area measure on the unit disc $\mathbb{D}$. So if X is a random variable uniformly distributed on $\mathbb{D}$, its x co-ordinate is distributed according to the standard semicircle law.

The semicircle distribution arises extensively in the theory of free random variables; see the book [51] for more information.

5.2 Convolutions of the arcsine law

In this section we shall denote by f_1 the standard arcsine density function. We shall denote by f_n, for $n \in \mathbb{N}$, the convolution of n standard arcsine densities. Thus if $(X_j)_{1 \leq j \leq n}$ is a sequence of n independent random variables, each following the standard arcsine distribution, the random variable $\sum_{j=1}^{n} X_j$ follows a distribution with density f_n.

Such convolutions were studied recently in [4]. It is clear, as f_1 is a probability density function, that $f_1 \in L^1(\mathbb{R})$ and therefore $f_n \in L^1(\mathbb{R})$.

Define the Bessel function of order zero, J_0, to be

$$J_0(\xi) = \frac{1}{\pi}\int_0^{\pi} e^{i\xi\cos\theta}\, d\theta \tag{5.7}$$

which, on putting $t = \cos\theta$, becomes

$$J_0(\xi) = \frac{1}{\pi}\int_{-1}^{1} e^{i\xi t}\frac{1}{\sqrt{1-t^2}}dt. \tag{5.8}$$

We see J_0 is the characteristic function of the standard arcsine density f_1. We deduce from the properties of characteristic functions that the characteristic function of f_n is J_0^n. Furthermore by Lévy's inversion formula, for which see chapter 16 of [53], we may recover f_n from J_0^n via the formula, for $s < t$,

$$\int_s^t f_n(u)\, du = \lim_{M\uparrow\infty}\int_{-M}^{M}\frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} J_0(\xi)^n \frac{d\xi}{2\pi}. \tag{5.9}$$

Now, note the following identity.

Proposition 5.2.1

$$J_0(\xi)J_0(\eta) = \int_0^{2\pi} J_0((\xi^2 + \eta^2 + 2\xi\eta\cos\varphi)^{1/2})\frac{d\varphi}{2\pi} \tag{5.10}$$

which holds for all ξ and η in $\mathbb{R}$.

Proof See §11.41 of [52] for details.

□

If $\xi = \eta$ this formula simplifies to

$$J_0(\xi)^2 = \int_0^{2\pi} J_0\left(2\xi\left|\cos\frac{\varphi}{2}\right|\right)\frac{d\varphi}{2\pi} \tag{5.11}$$

which we will use to evaluate the convolute f_2. Recall that, for $-1 < u < 1$, the complete elliptic integral of the first kind $K(u)$ is defined to be

$$K(u) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - u^2\sin^2\psi}}\, d\psi. \tag{5.12}$$

We now come to the principal result of this chapter.

Theorem 5.2.2 *The function f_2 is given by*

$$f_2(u) = \begin{cases} \frac{1}{\pi^2} K\left(\sqrt{1-\left(\frac{u}{2}\right)^2}\right) & -2 \le u \le 2; \\ 0 & otherwise. \end{cases} \tag{5.13}$$

Proof We know by Lévy's inversion formula that, for $s < t$,

$$\int_s^t f_2(u)\, du = \lim_{M\uparrow\infty} \int_{-M}^{M} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} J_0(\xi)^2 \frac{d\xi}{2\pi}. \tag{5.14}$$

Thus

$$\begin{aligned} \int_s^t f_2(u)\, du &= \lim_{M\uparrow\infty} \int_{-M}^{M} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} \left(\int_0^{2\pi} J_0\left(2\xi\left|\cos\frac{\varphi}{2}\right|\right)\frac{d\varphi}{2\pi}\right)\frac{d\xi}{2\pi} && (5.15) \\ &= \lim_{M\uparrow\infty} \int_0^{2\pi} \left(\int_{-M}^{M} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} J_0\left(2\xi\left|\cos\frac{\varphi}{2}\right|\right)\frac{d\xi}{2\pi}\right)\frac{d\varphi}{2\pi} && (5.16) \\ &= \lim_{M\uparrow\infty} \int_0^{2\pi} g_M(\varphi)\frac{d\varphi}{2\pi} && (5.17) \end{aligned}$$

say, where

$$g_M(\varphi) = \int_{-M}^{M} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} J_0\left(2\xi\left|\cos\frac{\varphi}{2}\right|\right)\frac{d\xi}{2\pi}. \tag{5.18}$$

The asymptotic result

$$\sqrt{x}J_0(x) - \sqrt{\frac{2}{\pi}}\cos\left(x - \frac{\pi}{4}\right) \to 0 \tag{5.19}$$

as $x \to \infty$, which is detailed in §7.1 of [52], implies

$$|J_0(x)| \le \min\left(1, \frac{C}{\sqrt{x}}\right) \tag{5.20}$$

for some constant $C > 0$, which in turn gives, for $\varphi \ne \pi$,

$$|g_M(\varphi)| \le \alpha + \frac{\beta}{\sqrt{\left|\cos\frac{\varphi}{2}\right|}} \tag{5.21}$$

for some constants $\alpha > 0$, $\beta > 0$; the right hand side of this inequality is integrable as a function of $\varphi \in [0, 2\pi]$. Now, for $\varphi \ne \pi$,

$$\begin{aligned} \lim_{M\uparrow\infty} g_M(\varphi) &= \lim_{M\uparrow\infty} \int_{-M}^{M} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} J_0\left(2\xi\left|\cos\frac{\varphi}{2}\right|\right)\frac{d\xi}{2\pi} && (5.22) \\ &= \int_{-\infty}^{\infty} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi} J_0\left(2\xi\left|\cos\frac{\varphi}{2}\right|\right)\frac{d\xi}{2\pi} && (5.23) \end{aligned}$$

as this converges as a Lebesgue integral via the asymptotic result for J_0 above. Setting $\nu = 2\xi \left|\cos\frac{\varphi}{2}\right|$ gives, as J_0 is even,

$$\lim_{M\uparrow\infty} g_M(\varphi) = \int_{-\infty}^{\infty} \frac{e^{-i\nu\left(\frac{s}{2\left|\cos\frac{\varphi}{2}\right|}\right)} - e^{-i\nu\left(\frac{t}{2\left|\cos\frac{\varphi}{2}\right|}\right)}}{i\nu} J_0(\nu)\frac{d\nu}{2\pi} \tag{5.24}$$

$$= \int_{\frac{s}{2\left|\cos\frac{\varphi}{2}\right|}}^{\frac{t}{2\left|\cos\frac{\varphi}{2}\right|}} f_1(u)\,du \tag{5.25}$$

by Lévy's inversion formula. For $\varphi = \pi$,

$$\lim_{M\uparrow\infty} g_M(\pi) = \lim_{M\uparrow\infty}\int_{-M}^{M} \frac{e^{-i\xi s} - e^{-i\xi t}}{i\xi}\frac{d\xi}{2\pi} \tag{5.26}$$

$$= \begin{cases} 0 & s < t < 0 \text{ or } 0 < s < t; \\ \frac{1}{2} & s < t = 0 \text{ or } 0 = s < t; \\ 1 & s < 0 < t \end{cases} \tag{5.27}$$

$$= \lim_{\varphi\to\pi}\int_{\frac{s}{2\left|\cos\frac{\varphi}{2}\right|}}^{\frac{t}{2\left|\cos\frac{\varphi}{2}\right|}} f_1(u)\,du \tag{5.28}$$

$$= \lim_{\varphi\to\pi}\left(\lim_{M\uparrow\infty} g_M(\varphi)\right). \tag{5.29}$$

We have shown $\lim_{M\uparrow\infty} g_M(\varphi)$ exists, and is bounded and continuous, for all φ and that $|g_M(\varphi)| \le \alpha + \frac{\beta}{\sqrt{\left|\cos\frac{\varphi}{2}\right|}}$ for some $\alpha > 0$ and $\beta > 0$ and all $\varphi \ne \pi$. Further we have noted that the function $\varphi \mapsto \alpha + \frac{\beta}{\sqrt{\left|\cos\frac{\varphi}{2}\right|}}$ is integrable on $[0, 2\pi]$. We may therefore apply the dominated convergence theorem to give

$$\int_s^t f_2(u)\,du = \lim_{M\uparrow\infty}\int_0^{2\pi} g_M(\varphi)\frac{d\varphi}{2\pi} \tag{5.30}$$

$$= \int_0^{2\pi} \lim_{M\uparrow\infty} g_M(\varphi)\frac{d\varphi}{2\pi}. \tag{5.31}$$

Thus

$$\int_s^t f_2(u)\,du = \int_0^{2\pi}\left(\int_{\frac{s}{2\left|\cos\frac{\varphi}{2}\right|}}^{\frac{t}{2\left|\cos\frac{\varphi}{2}\right|}} f_1(u)\,du\right)\frac{d\varphi}{2\pi} \tag{5.32}$$

$$= \int_{-\pi}^{\pi}\left(\int_{\frac{s}{2\left|\cos\frac{\varphi}{2}\right|}}^{\frac{t}{2\left|\cos\frac{\varphi}{2}\right|}} f_1(u)\,du\right)\frac{d\varphi}{2\pi} \tag{5.33}$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{\frac{s}{2\left|\cos\frac{\varphi}{2}\right|}}^{\frac{t}{2\left|\cos\frac{\varphi}{2}\right|}} \mathbf{1}_{[-1,1]}(u)\frac{1}{\sqrt{1-u^2}}\,du\right)\frac{d\varphi}{2\pi} \tag{5.34}$$

and setting $v = 2u\left|\cos\frac{\varphi}{2}\right|$ gives

$$\begin{aligned}
\int_s^t f_2(u)\,du &= \int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_s^t \mathbf{1}_{[-1,1]}\left(\frac{v}{2\left|\cos\frac{\varphi}{2}\right|}\right)\frac{1}{\sqrt{1-\frac{v^2}{4\cos^2\frac{\varphi}{2}}}}\frac{dv}{2\left|\cos\frac{\varphi}{2}\right|}\right)\frac{d\varphi}{2\pi} && (5.35)\\
&= \int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_s^t \mathbf{1}_{[-1,1]}\left(\frac{v}{2\left|\cos\frac{\varphi}{2}\right|}\right)\frac{1}{\sqrt{4\cos^2\frac{\varphi}{2}-v^2}}\,dv\right)\frac{d\varphi}{2\pi} && (5.36)\\
&= \int_s^t\left(\frac{1}{\pi}\int_{-\pi}^{\pi} \mathbf{1}_{[-1,1]}\left(\frac{v}{2\left|\cos\frac{\varphi}{2}\right|}\right)\frac{1}{\sqrt{4\cos^2\frac{\varphi}{2}-v^2}}\frac{d\varphi}{2\pi}\right)dv && (5.37)
\end{aligned}$$

by Fubini's theorem, as the integrand is positive. Now we need to know for which v and φ we have

$$-1 \le \frac{v}{2\left|\cos\frac{\varphi}{2}\right|} \le 1. \tag{5.38}$$

This has no solutions when $|v| > 2$. If $|v| \le 2$ the condition holds precisely when

$$-2\cos^{-1}\frac{|v|}{2} \le \varphi \le 2\cos^{-1}\frac{|v|}{2}. \tag{5.39}$$

Consequently

$$\begin{aligned}
\int_s^t f_2(u)\,du &= \int_{\max(s,-2)}^{\min(t,2)}\left(\frac{1}{\pi}\int_{-2\cos^{-1}\frac{|v|}{2}}^{2\cos^{-1}\frac{|v|}{2}}\frac{1}{\sqrt{4\cos^2\frac{\varphi}{2}-v^2}}\frac{d\varphi}{2\pi}\right)dv && (5.40)\\
&= \int_{\max(s,-2)}^{\min(t,2)}\left(\frac{1}{\pi^2}\int_0^{2\cos^{-1}\frac{|v|}{2}}\frac{1}{\sqrt{4\cos^2\frac{\varphi}{2}-v^2}}\,d\varphi\right)dv. && (5.41)
\end{aligned}$$

For fixed v, we replace the variable φ with a new variable ψ via the substitution

$$\sin\psi = \frac{\sin\frac{\varphi}{2}}{\sqrt{1-\left(\frac{v}{2}\right)^2}} \tag{5.42}$$

which gives

$$\begin{aligned}
\int_s^t f_2(u)\,du &= \int_{\max(s,-2)}^{\min(t,2)}\left(\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-\left(1-\left(\frac{v}{2}\right)^2\right)\sin^2\psi}}\,d\psi\right)dv && (5.43)\\
&= \int_{\max(s,-2)}^{\min(t,2)}\frac{1}{\pi^2}K\left(\sqrt{1-\left(\frac{v}{2}\right)^2}\right)dv && (5.44)
\end{aligned}$$

for K a complete elliptic integral of the first kind. As s and t were arbitrarily chosen with $s < t$, this completes the proof.

□

Chapter 6

Random matrices

This chapter defines and develops some results for ensembles of Hermitian random matrices and certain random measures on $\mathbb{R}$ associated to these.

The key reference on random matrix theory is [39]; this includes a detailed description of the physical motivation for studying such ensembles. Among more recent publications the paper [6] is particularly noteworthy.

6.1 Physical motivation

Since the pioneering work of Wigner in the 1950s, the theory of random matrices has been developed and used in such diverse areas as statistical mechanics, nuclear physics, quantum field theory (for example [2]) and scattering problems for one dimensional Schrödinger operators (for example [45]). Until the late 1970s the great majority of the random matrices considered were real and symmetric; however recent developments in areas such as quantum chromodynamics, string theory and two-dimensional gravity have led to a surge of interest in complex Hermitian random matrices.

The basic motivation for studying random matrices is as follows; our explanation is taken from chapter 1 of [39].

According to quantum mechanics, the energy levels of a physical system are described by the eigenvalues of an operator H (symmetric or Hermitian according to whether the system is real or complex) called the Hamiltonian, acting on a certain infinite dimensional Hilbert space Ψ. The spectrum of H, $\sigma(H)$, generally has both discrete and continuous components; it is the discrete part which attracts most interest.

To avoid having to work in infinite dimensions, we restrict the action of our Hamiltonian H to a finite (but generally large) dimensional subspace of Ψ, namely the direct sum of a finite number of the eigenspaces associated to the discrete part of $\sigma(H)$. Thus our Hamiltonian becomes a large symmetric or Hermitian matrix.

We do not, of course, have accurate knowledge of all the properties and constants associated to the system; consequently we make statistical hypotheses and view these quantities as random variables. In particular this means our Hamiltonian H is treated as a random quantity.

We therefore study the properties of large random matrices. Often we seek limiting properties of such matrices as their order approaches infinity; this corresponds to our model encompassing more and more of the eigenspaces associated to the discrete part of $\sigma(H)$.

6.2 Matrices with unitarily invariant distributions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and, for $1 \leq p < \infty$, let $L^p(\Omega)$ denote the usual space of Borel measurable functions $\Omega \to \mathbb{C}$ having finite L^p norm, quotiented by functions which are zero almost surely. For $n \in \mathbb{N}$ let $M_n(\mathbb{C})$ denote the space of n by n complex matrices and let $M_n^{sa}(\mathbb{C})$ denote the space of n by n Hermitian complex matrices (the self-adjoint part of $M_n(\mathbb{C})$).

Denote by Σ^n the simplex $\{(\lambda_1, \dots, \lambda_n) \subseteq \mathbb{R}^n : \lambda_1 \geq \dots \geq \lambda_n\}$. Denote by $\mathcal{U}_n$ the Lie group of n by n unitary matrices equipped with Haar probability measure, which we denote by $d\tau$. Denote by $\Lambda(\lambda_1, \dots, \lambda_n)$ the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$ in sequence. Denote by dX Lebesgue measure on $M_n^{sa}(\mathbb{C})$, i.e.

$$dX = \prod_j d\,[X]_{jj} \prod_{j<k} d\,Re[X]_{jk} d\,Im[X]_{jk}. \tag{6.1}$$

For each $n \in \mathbb{N}$ define $\mathcal{A}_n$ to be the algebra $M_n(\mathbb{C}) \otimes \bigcap_{p<\infty} L^p(\Omega)$ of n by n matrices with entries in $\bigcap_{p<\infty} L^p(\Omega)$. We equip $\mathcal{A}_n$ with the normalised trace $\frac{1}{n}\mathbb{E}tr$. Define $\mathcal{A}_n^{sa}$ to be $M_n^{sa}(\mathbb{C}) \otimes \bigcap_{p<\infty} L^p(\Omega)$, the self-adjoint part of $\mathcal{A}_n$.

If X_n is an element of $\mathcal{A}_n^{sa}$ we refer to X_n as a random Hermitian matrix; this definition of random matrices is taken from [51]. By construction the entries of X_n are random variables having finite moments of all orders. The random matrix X_n induces a Radon probability measure μ^{X_n} on $M_n^{sa}(\mathbb{C})$, the law of X_n, given by

$$\mu^{X_n}(\mathcal{S}) = \mathbb{P}(\{\omega \in \Omega : X_n(\omega) \in \mathcal{S}\}) \tag{6.2}$$

for $\mathcal{S}$ a Borel subset of $M_n^{sa}(\mathbb{C})$.

This chapter considers ensembles of random matrices, that is sequences $(X_n)_{n\geq 1}$ where X_n lies in $\mathcal{A}_n^{sa}$ for each n. In particular we shall consider ensembles $(X_n)_{n\geq 1}$ in which each X_n has distribution invariant under unitary conjugation; that is to say

$$\mu^{X_n}(\{USU^* : S \in \mathcal{S}\}) = \mu^{X_n}(\mathcal{S}) \tag{6.3}$$

for all unitary U in $\mathcal{U}_n$ and Borel subsets $\mathcal{S}$ of $M_n^{sa}(\mathbb{C})$. The law of X_n is said to be unitarily invariant. The importance and physical motivation for unitary invariance is detailed in [6] and [39].

We have the following invariance property.

Proposition 6.2.1 *Lebesgue measure dX is a unitarily invariant measure on $M_n^{sa}(\mathbb{C})$.*

Proof See section A.17 of [39].

□

Consider the map $\gamma : \mathcal{U}_n \times \Sigma^n \to M_n^{sa}(\mathbb{C})$ given by

$$\gamma(U, \lambda_1, \ldots, \lambda_n) = U\Lambda(\lambda_1, \ldots, \lambda_n)U^*. \tag{6.4}$$

Proposition 6.2.2 *The map γ is continuous, surjective and almost surely injective, with the only violations of injectivity occurring on the boundary of Σ^n.*

Proof This follows from the spectral theorem for Hermitian operators on $\mathbb{C}^n$.

□

Denote by $\gamma^*(dX)$ the pullback measure on $\mathcal{U}_n \times \Sigma^n$ defined by the formula

$$\int_{\gamma(\mathcal{S})} dX = \int_{\mathcal{S}} \gamma^*(dX) \tag{6.5}$$

for $\mathcal{S}$ all Borel subsets of $\mathcal{U}_n \times \Sigma^n$.

Proposition 6.2.3 *The measure $\gamma^*(dX)$ is absolutely continuous with respect to the natural product measure $d\tau\, d\lambda_1 \ldots d\lambda_n$ on $\mathcal{U}_n \times \Sigma^n$, according to the formula*

$$\gamma^*(dX) = d\tau\, d\zeta \tag{6.6}$$

where $d\zeta$ denotes the measure

$$d\zeta = C_n^{-1} \prod_{j<k} (\lambda_j - \lambda_k)^2\, d\lambda_1 \ldots d\lambda_n \tag{6.7}$$

on Σ^n, where C_n is a normalising constant.

Proof The argument in chapter 3 of [39] gives us

$$\gamma^*(dX) = p(U)\, d\tau\, d\zeta \tag{6.8}$$

for some probability density function p on $\mathcal{U}_n$. The left invariance of $d\tau$ and the unitary invariance of dX now tell us that $p(UV) = p(V)$ for all U and V in $\mathbb{U}_n$; this implies p is identically equal to 1. We deduce the required formula.

□

The values of the constants C_n are known.

Proposition 6.2.4 *The value of C_n is*

$$C_n = \frac{\pi^{n(n-1)/2}}{\prod_{j=1}^n j!}. \tag{6.9}$$

Proof See appendix 2 of [2]. The method is due to Mehta; see section 5.2 of [39].

□

We seek representations of those laws of random Hermitian matrices which are absolutely continuous with respect to Lebesgue measure on $M_n^{sa}(\mathbb{C})$.

Let $\varphi_n(X)\,dX$ be a Radon probability measure on $M_n^{sa}(\mathbb{C})$ which is absolutely continuous with respect to Lebesgue measure dX. Here $\varphi_n : M_n^{sa}(\mathbb{C}) \to \mathbb{R}$ is an integrable function. We have a pullback probability measure $\gamma^*(\varphi_n(X)\,dX)$ on $\mathcal{U}_n \times \Sigma^n$ via the formula

$$\gamma^*(\varphi_n(X)\,dX) = \gamma^*(\varphi_n)(U, \lambda_1, \dots, \lambda_n)\gamma^*(dX) \tag{6.10}$$

where $\gamma^*(\varphi_n)$ denotes the composition $\varphi_n \circ \gamma$ and $X = U\Lambda(\lambda_1, \dots, \lambda_n)U^*$.

We observe a condition for unitary invariance.

Proposition 6.2.5 *The measure $\varphi_n(X)\,dX$ is unitarily invariant if and only if*

$$\varphi_n(UXU^*) = \varphi_n(X) \tag{6.11}$$

for all X in $M_n^{sa}(\mathbb{C})$ and U in $\mathcal{U}_n$; equivalently, $\varphi_n(X)\,dX$ is unitarily invariant if and only if

$$\gamma^*(\varphi_n)(U, \lambda_1, \dots, \lambda_n) = \gamma^*(\varphi_n)(I, \lambda_1, \dots, \lambda_n) \tag{6.12}$$

for all $U \in \mathcal{U}_n$ and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$.

Proof The representation (6.10), together with the left invariance of $d\tau$ and the unitary invariance of dX, inform us that

$$\gamma^*(\varphi_n)(UV, \lambda_1, \dots, \lambda_n) = \gamma^*(\varphi_n)(V, \lambda_1, \dots, \lambda_n) \tag{6.13}$$

for all U and V in $\mathcal{U}_n$ and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$. This gives the required conclusions.

□

In such circumstances we say the function φ_n is itself unitarily invariant; we may (with a slight abuse of notation) regard $\gamma^*(\varphi_n)$ as a function from Σ^n to $\mathbb{R}$.

It is often difficult working on the simplex Σ^n. The following proposition allows us to work with the space $\mathcal{U}_n \times \mathbb{R}^n$ rather than $\mathcal{U}_n \times \Sigma^n$. Consider the map

$$\pi : \mathbb{R}^n \to \Sigma^n \tag{6.14}$$

which takes $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ to its decreasing rearrangement in Σ^n.

Proposition 6.2.6 *The map π is continuous, surjective and almost surely $n!$ to 1, with the only points for which it is not $n!$ to 1 occurring on the preimage of the boundary of Σ^n. If ζ is any probability measure on $\mathcal{U}_n \times \Sigma^n$ there is a pullback measure $\pi^*(\zeta)$ of total mass $n!$; it follows that $\frac{1}{n!}\pi^*(\zeta)$ is a probability measure on $\mathcal{U}_n \times \mathbb{R}^n$.*

Proof This is immediate from the definitions.

□

Thus, given any Radon probability measure $\varphi_n(X)\,dX$ on $M_n^{sa}(\mathbb{C})$, there is a pullback probability measure on $\mathcal{U}_n \times \mathbb{R}^n$ given by

$$\frac{1}{n!}\pi^*(\gamma^*(\varphi_n(X)\,dX)) = \frac{1}{n!}\pi^*(\gamma^*(\varphi_n))(U, \lambda_1, \dots, \lambda_n)\,d\tau\,d\zeta \tag{6.15}$$

where $\pi^*(\gamma^*(\varphi_n))$ denotes the composition $\varphi_n \circ \gamma \circ \pi$ and $X = U\Lambda(\lambda_1, \dots, \lambda_n)U^*$.

Let us summarise what we know for a random n by n Hermitian matrix X_n with unitarily invariant law μ^{X_n} given by $\varphi_n(X)\,dX$. We see that (abusing notation slightly) the composition $\pi^*(\gamma^*(\varphi_n))$ may be viewed as a positive integrable function $\mathbb{R}^n \to \mathbb{R}$.

If $F : M_n^{sa}(\mathbb{C}) \to \mathbb{C}$ is a locally integrable function which is unitarily invariant, so

$$F(UXU^*) = F(X) \tag{6.16}$$

for all X in $M_n^{sa}(\mathbb{C})$ and U in $\mathcal{U}_n$, we see that (again abusing notation slightly) the composition $\pi^*(\gamma^*(F))$ may be viewed as a locally integrable function $\mathbb{R}^n \to \mathbb{C}$. We observe the formula

$$\mathbb{E}F(X_n) = \frac{1}{n!}\int_{\mathbb{R}^n} \pi^*(\gamma^*(F))(\lambda_1, \dots, \lambda_n)\pi^*(\gamma^*(\varphi_n))(\lambda_1, \dots, \lambda_n)\,d\zeta. \tag{6.17}$$

In practice the most common use of this formula occurs when F is of the form $tr\, f$, where f is a function from $M_n^{sa}(\mathbb{C})$ to itself.

Note that, as φ_n is unitarily invariant,

$$\pi^*(\gamma^*(\varphi_n))(\lambda_1, \ldots, \lambda_n) = \varphi_n(\Lambda(\lambda_1, \ldots, \lambda_n)) \tag{6.18}$$

and is a positive symmetric function of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ lying in $L^1(\mathbb{R}^n)$. Similarly, as F is unitarily invariant we see $\pi^*(\gamma^*(F))$ is a symmetric function of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ lying in $L^1_{loc}(\mathbb{R}^n)$.

For example if $F(X) = \|X\|$, which is a unitarily invariant function, we observe that $\gamma^*(F)(\lambda_1, \ldots, \lambda_n) = \max(|\lambda_1|, |\lambda_n|)$ and $\pi^*(\gamma^*(F))(\lambda_1, \ldots, \lambda_n) = \max_{1 \leq j \leq n} |\lambda_j|$.

6.3 The level spacing distributions

Given a random Hermitian n by n matrix X_n with unitarily invariant law $\varphi_n(X)\, dX$, define the n-tuple $(\lambda_1(X_n), \ldots, \lambda_n(X_n))$ to be the ordered eigenvalues of X_n, where $\lambda_1(X_n) \geq \cdots \geq \lambda_n(X_n)$. Note this is an element of Σ^n. We have the following proposition, whose proof is immediate.

Proposition 6.3.1 *Define a probability density function ρ_n on $\mathbb{R}^n$ via the formula*

$$\rho_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{n! C_n} \prod_{j<k} (\lambda_j - \lambda_k)^2 \pi^*(\gamma^*(\varphi_n))(\lambda_1, \ldots, \lambda_n). \tag{6.19}$$

Then

$$\mathbb{P}((\lambda_1(X_n), \ldots, \lambda_n(X_n)) \in \mathcal{S}) = \int_{\pi^{-1}(\mathcal{S})} \rho_n(\lambda_1, \ldots, \lambda_n)\, d\lambda_1 \ldots d\lambda_n \tag{6.20}$$

for all Borel subsets $\mathcal{S}$ of Σ^n.

□

We refer to the density ρ_n as the joint density of the eigenvalues of X_n.

We may define, for $r = 1, \ldots, n-1$, a probability density function ρ_n^r on $\mathbb{R}^r$ by the formula

$$\rho_n^r(\lambda_1, \ldots, \lambda_r) = \frac{1}{n! C_n} \int_{\mathbb{R}^{n-r}} \prod_{j<k} (\lambda_j - \lambda_k)^2 \pi^*(\gamma^*(\varphi_n))(\lambda_1, \ldots, \lambda_n)\, d\lambda_{r+1} \ldots d\lambda_n. \tag{6.21}$$

For completeness set $\rho_n^n = \rho_n$. The family of probability density functions $\rho_n^1, \ldots, \rho_n^n$ is known as the family of level spacing distributions; ρ_n^1 is particularly important and may be viewed as the probability density of a randomly selected eigenvalue.

Note that, if f is a bounded continuous function from $\mathbb{R}$ to $\mathbb{C}$ then, via functional calculus, for any n we may extend f to a map from $M_n^{sa}(\mathbb{C})$ to $M_n(\mathbb{C})$. It therefore makes sense to consider, for X_n an n by n random matrix,

$$\begin{aligned} \frac{1}{n}\mathbb{E}tr\ f(X_n) &= \frac{1}{n!}\int_{\mathbb{R}^n}\left(\frac{1}{n}\sum_{j=1}^{n} f(\lambda_n)\right)\pi^*(\gamma^*(\varphi_n))(\lambda_1, \ldots, \lambda_n)\, d\zeta &\quad (6.22)\\ &= \int_{\mathbb{R}} f(t)\rho_n^1(t)\, dt, &\quad (6.23)\end{aligned}$$

which is a useful result.

In studying random matrices it is common to seek ensembles $(X_n)_{n\geq 1}$ such that $(\rho_n^1)_{n\geq 1}$ tends weakly to a limiting probability density as n tends to infinity. This limit, when it exists, is referred to as the integrated density of states (IDS) by physicists; see [6] for example.

6.4 The empirical distribution of the eigenvalues

We next define an important class of random measures. Write $\mathcal{M}(\mathbb{R})$ for the vector space of all Radon measures on $\mathbb{R}$. By the Riesz representation theorem this may be viewed as $C_0(\mathbb{R})^*$, where $C_0(\mathbb{R})$ is the space of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ which vanish at infinity; we equip $\mathcal{M}(\mathbb{R})$ with the dual norm. Denote by $\mathcal{M}_1(\mathbb{R})$ the closed subset of $\mathcal{M}(\mathbb{R})$ comprising all Radon probability measures on $\mathbb{R}$. Let $(X_n)_{n\geq 1}$ be an ensemble of random Hermitian matrices defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define, for $\omega \in \Omega$ and each n,

$$\nu_n(\omega) = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(X_n(\omega))}. \quad (6.24)$$

It is clear that, for each n, ν_n is a function from Ω to $\mathcal{M}_1(\mathbb{R})$; it is a random probability measure on $\mathbb{R}$. We refer to ν_n as the empirical distribution of the eigenvalues of X_n.

Note that, for $f \in C_0(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\nu_n(dt) = \frac{1}{n}\sum_{j=1}^{n} f(\lambda_j(X_n)) \tag{6.25}$$

$$= \frac{1}{n} tr\, f(X_n); \tag{6.26}$$

for this reason ν_n is often described as the spectral multiplicity measure of X_n. We see that, viewing ν_n as a map $\Omega \to C_0(\mathbb{R})^*$, ν_n is a weak-$*$ measurable random vector. Furthermore, viewing ν_n as an operator from $C_0(\mathbb{R})$ to $L^2(\Omega)$,

$$\|\nu_n\|_{\mathcal{B}(C_0(\mathbb{R}),L^2(\Omega))} = \sup_{f\in C_0(\mathbb{R}):\|f\|\leq 1} \left(\mathbb{E}\left|\int_{\mathbb{R}} f(t)\nu_n(dt)\right|^2\right)^{1/2} \leq 1 \tag{6.27}$$

and so ν_n lies in $L^2_{w^*}(\Omega; C_0(\mathbb{R})^*)$. It follows that the weak-$*$ expectation $\mathbb{E}\nu_n$ exists. For $f \in C_0(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\,(\mathbb{E}\nu_n)\,(dt) = \mathbb{E}\int_{\mathbb{R}} f(t)\nu_n(dt) \tag{6.28}$$

$$= \frac{1}{n}\mathbb{E} tr\; f(X_n) \tag{6.29}$$

$$= \int_{\mathbb{R}} f(t)\rho_n^1(t)\,dt, \tag{6.30}$$

which shows us that

$$(\mathbb{E}\nu_n)\,(dt) = \rho_n^1(t)\,dt \tag{6.31}$$

for each n.

Note that by a corollary to Grothendieck's inequality, detailed in theorem 3.5 of [11] (the result was originally proved in [20]), $\mathcal{B}(C_0(\mathbb{R}), L^2(\Omega))$ equals $\Pi_2(C_0(\mathbb{R}), L^2(\Omega))$ with

$$\|\nu\|_{op} \leq \pi_2(\nu) \leq \kappa_G^{\mathbb{C}}\|\nu\|_{op} \tag{6.32}$$

for all elements ν of $\mathcal{B}(C_0(\mathbb{R}), L^2(\Omega))$; here $\kappa_G^{\mathbb{C}}$ denotes Grothendieck's constant in $\mathbb{C}$.

Consequently the spaces $L^2_{w^*}(\Omega; C_0(\mathbb{R})^*)$ and $L^2_{w^*}(\Omega; C_0(\mathbb{R})^*, \pi_2)$ coincide; it follows that ν_n lies in $L^2_{w^*}(\Omega; C_0(\mathbb{R})^*, \pi_2)$ for each n.

Viewing ν_n as an operator once more, an identical argument to (6.27) shows that ν_n lies in the space $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$ for each n. Here $C_b(\mathbb{R})$ denotes the bounded continuous functions from $\mathbb{R}$ to $\mathbb{C}$.

This chapter will consider circumstances under which $\|\nu_n - \mathbb{E}\nu_n\|_{\mathcal{B}(C_b(\mathbb{R}),L^2(\Omega))}$ tends to zero as n tends to infinity, and furthermore $\rho_n^1 = \mathbb{E}\nu_n$ tends weakly to some limit ρ^1 (the integrated density of states) as n tends to infinity; it will then follow that $\nu_n \to \rho^1$ in the Banach space $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$ as n tends to infinity. This will imply $\nu_n \to \rho^1$ in the Banach space $\mathcal{B}(C_0(\mathbb{R}), L^2(\Omega)) \cong L^2_{w*}(\Omega; \mathcal{M}(\mathbb{R}))$ as n tends to infinity.

Lemma 6.4.1 *For $f \in C_b(\mathbb{R})$ and all n,*

$$\begin{aligned}
&\mathbb{E}\left|\int_{\mathbb{R}} f(t)(\nu_n - \mathbb{E}\nu_n)(dt)\right|^2 \\
= \; &\frac{1}{n}\int_{\mathbb{R}} |f(\lambda_1)|^2 \rho_n^1(\lambda_1)\, d\lambda_1 + \\
&\iint\limits_{\mathbb{R}^2} f(\lambda_1)\overline{f(\lambda_2)}\left[\left(1 - \frac{1}{n}\right)\rho_n^2(\lambda_1,\lambda_2) - \rho_n^1(\lambda_1)\rho_n^1(\lambda_2)\right] d\lambda_1\, d\lambda_2. \qquad (6.33)
\end{aligned}$$

Proof A calculation shows

$$\begin{aligned}
&\mathbb{E}\left|\int_{\mathbb{R}} f(t)(\nu_n - \mathbb{E}\nu_n)(dt)\right|^2 \\
= \; &\mathbb{E}\left|\frac{1}{n}\sum_{j=1}^n f(\lambda_j) - \frac{1}{n}\mathbb{E}\sum_{j=1}^n f(\lambda_j)\right|^2 \qquad (6.34)\\
= \; &\frac{1}{n^2}\mathbb{E}\sum_{j=1}^n |f(\lambda_j)|^2 + \frac{1}{n^2}\mathbb{E}\sum_{j\neq k} f(\lambda_j)\overline{f(\lambda_k)} - \\
&\frac{1}{n^2}\left(\mathbb{E}\sum_{j=1}^n f(\lambda_j)\right)\overline{\left(\mathbb{E}\sum_{k=1}^n f(\lambda_k)\right)} \qquad (6.35)\\
= \; &\frac{1}{n}\int_{\mathbb{R}} |f(\lambda_1)|^2\rho_n^1(\lambda_1)\, d\lambda_1 + \\
&\left(1 - \frac{1}{n}\right)\iint\limits_{\mathbb{R}^2} f(\lambda_1)\overline{f(\lambda_2)}\rho_n^2(\lambda_1,\lambda_2)\, d\lambda_1\, d\lambda_2 - \\
&\left(\int_{\mathbb{R}} f(\lambda_1)\rho_n^1(\lambda_1)\, d\lambda_1\right)\overline{\left(\int_{\mathbb{R}} f(\lambda_2)\rho_n^1(\lambda_2)\, d\lambda_2\right)} \qquad (6.36)\\
= \; &\frac{1}{n}\int_{\mathbb{R}} |f(\lambda_1)|^2\rho_n^1(\lambda_1)\, d\lambda_1 + \\
&\iint\limits_{\mathbb{R}^2} f(\lambda_1)\overline{f(\lambda_2)}\left[\left(1 - \frac{1}{n}\right)\rho_n^2(\lambda_1,\lambda_2) - \rho_n^1(\lambda_1)\rho_n^1(\lambda_2)\right] d\lambda_1\, d\lambda_2 \qquad (6.37)
\end{aligned}$$

as required.

□

To obtain more effective bounds on the sequence $(\nu_n - \mathbb{E}\nu_n)_{n\geq 1}$ we need to make further hypotheses on the laws of the $(X_n)_{n\geq 1}$.

6.5 Matrices generated by weights

Specific ensembles $(X_n)_{n\geq 1}$ of random Hermitian matrices with unitarily invariant laws may be generated according to the following procedure.

Let $w : \mathbb{R} \to [0, \infty)$ be an integrable function which has finite moments of all orders and induces a probability measure $w(x)\,dx$ on $\mathbb{R}$. Such a function is known as a weight function or a probability density function. Write $supp\, w$ for the support of w. We may define a function $v : supp\, w \to \mathbb{R}$ via

$$w(x) = \begin{cases} e^{-v(x)} & x \in supp\, w; \\ 0 & \text{otherwise.} \end{cases} \tag{6.38}$$

The function v is often referred to as the potential.

Consider as before an ensemble of random Hermitian matrices $(X_n)_{n\geq 1}$ where, for each n, X_n has unitarily invariant law given by $\varphi_n(X)\,dX$. We know that $\pi^*(\gamma^*(\varphi_n))$ is a non-negative symmetric function on $\mathbb{R}^n$. This section will consider the special case under which, for each n, φ_n is generated from w via the formula

$$\pi^*(\gamma^*(\varphi_n))(\lambda_1, \dots, \lambda_n) = K_n^{-1} \prod_{j=1}^{n} w(\lambda_j) \tag{6.39}$$

where K_n is a normalising constant; we know this to be finite since w has finite moments of all orders. Note that, because w has finite moments of all orders, each element of the ensemble $(X_n)_{n\geq 1}$ lies in $\mathcal{A}_n^{sa}$. Some ensembles of this type, and the physical motivation for studying them, are discussed in [45].

We may also define $\varphi_n(X)\,dX$ via functional calculus as follows:

$$\varphi_n(X)\,dX = \begin{cases} K_n^{-1} e^{-tr\, v(X)}\,dX & \sigma(X) \subseteq supp\, w; \\ 0 & \text{otherwise,} \end{cases} \tag{6.40}$$

where $\sigma(X)$ denotes the spectrum of X.

Note that the entries of the random matrices generated in this way are not in general independent. The most important case where the entries are independent occurs with Gaussian random matrices, i.e. ensembles with $v(x)$ a convex quadratic polynomial and *supp* w the entire real line. There is an extensive theory of Gaussian random matrices and random matrices with independent (or weakly independent) entries. See, for example, [22] or [39].

Now, let w be a weight function and let $P_0, P_1, \ldots$ be the sequence of orthonormal polynomials associated to it; see [48] for background details of this subject. Let $(X_n)_{n\geq 1}$ be the ensemble of random matrices generated from the weight function w in the above way. Then the following result holds.

Lemma 6.5.1 *For each n and r the level spacing distributions ρ_n^r are given by the formula*

$$\rho_n^r(\lambda_1, \ldots, \lambda_r) = \frac{(n-r)!}{n!} \det \left[\sum_{j=0}^{n-1} P_j(\lambda_s) P_j(\lambda_t) \right]_{s,t=1}^{r} \prod_{j=1}^{r} w(\lambda_j). \tag{6.41}$$

In particular, for the case $r = 1$,

$$\rho_n^1(\lambda_1) = \frac{1}{n} \left(\sum_{j=0}^{n-1} P_j(\lambda_1)^2 \right) w(\lambda_1), \tag{6.42}$$

and for the case $r = 2$,

$$\begin{aligned} &\rho_n^2(\lambda_1, \lambda_2) \\ = \; &\frac{n}{n-1} \left(\rho_n^1(\lambda_1) \rho_n^1(\lambda_2) - \frac{1}{n^2} \left(\sum_{j=0}^{n-1} P_j(\lambda_1) P_j(\lambda_2) \right)^2 w(\lambda_1) w(\lambda_2) \right). \end{aligned} \tag{6.43}$$

Proof These are quoted without proof in [6], (2.28)–(2.30) and proved in [39], sections 5.2, 6.1 and A.13.

□

The following result was proved, under a different framework and using different notation, by Boutet de Monvel, Pastur and Shcherbina in section 2 of [6]. Our proof differs from theirs; in particular it avoids use of the Stieltjes transform.

Theorem 6.5.2 *For $f \in C_b(\mathbb{R})$ and all n,*

$$\mathbb{E}\left|\int_{\mathbb{R}} f(t)(\nu_n - \mathbb{E}\nu_n)(dt)\right|^2 \leq \frac{1}{n}\int_{\mathbb{R}} |f(t)|^2(\mathbb{E}\nu_n)(dt), \tag{6.44}$$

and thus

$$\|\nu_n - \mathbb{E}\nu_n\|_{\mathcal{B}(C_b(\mathbb{R}),L^2(\Omega))} \leq \frac{1}{\sqrt{n}}, \tag{6.45}$$

implying that $(\nu_n - \mathbb{E}\nu_n)_{n\geq 1}$ converges to zero in $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$.

Proof We have, by Lemma 6.4.1 and Lemma 6.5.1,

$$\begin{aligned}
& \mathbb{E}\left|\int_{\mathbb{R}} f(t)(\nu_n - \mathbb{E}\nu_n)(dt)\right|^2 \\
= \; & \frac{1}{n}\int_{\mathbb{R}} |f(\lambda_1)|^2 \rho_n^1(\lambda_1)\, d\lambda_1 - \\
& \frac{1}{n^2}\int_{\mathbb{R}^2} f(\lambda_1)\overline{f(\lambda_2)}\left(\sum_{j=0}^{n-1} P_j(\lambda_1)P_j(\lambda_2)\right)^2 w(\lambda_1)w(\lambda_2)\, d\lambda_1\, d\lambda_2 && (6.46) \\
= \; & \frac{1}{n}\int_{\mathbb{R}} |f(\lambda_1)|^2 \rho_n^1(\lambda_1)\, d\lambda_1 - \\
& \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\int_{\mathbb{R}} f(\lambda_1)P_j(\lambda_1)P_k(\lambda_1)w(\lambda_1)\, d\lambda_1\right|^2 && (6.47) \\
\leq \; & \frac{1}{n}\int_{\mathbb{R}} |f(\lambda_1)|^2 \rho_n^1(\lambda_1)\, d\lambda_1 && (6.48)
\end{aligned}$$

as required.

□

The next section examines the case in which *supp* w is a compact interval; we will show that, provided w satisfies certain smoothness conditions, ρ_n^1 tends weakly to an arcsine distribution as n tends to infinity. It follows that the sequence $(\nu_n)_{n\geq 1}$ converges in the Banach space $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$ to an arcsine distribution.

Scholium One may, using the same weight function w, define a slightly different ensemble of random Hermitian matrices $(Y_n)_{n\geq 1}$. The matrix Y_n has unitarily invariant law $\psi_n(Y)\, dY$ where

$$\pi^*(\gamma^*(\psi_n))(\lambda_1, \ldots, \lambda_n) = \tilde{K}_n^{-1}\left(\prod_{j=1}^{n} w(\lambda_j)\right)^n \tag{6.49}$$

for $\tilde{K}_n$ a normalising constant; we know this to be finite since w has finite moments of all orders. Consequently each Y_n lies in $\mathcal{A}_n^{sa}$.

We may also, as before, define $\psi_n(Y)\,dY$ via functional calculus as follows:

$$\psi_n(Y)\,dY = \begin{cases} \tilde{K}_n^{-1} e^{-n\,tr\,v(Y)}\,dY & \sigma(Y) \subseteq supp\,w; \\ 0 & \text{otherwise,} \end{cases} \tag{6.50}$$

where $\sigma(Y)$ denotes the spectrum of Y.

Ensembles of this type have been studied extensively, notably in [6] where the physical motivation for studying such ensembles is discussed. In this paper Boutet de Monvel, Pastur and Shcherbina show that, provided the potential v satisfies a certain growth condition, the density ρ_n^1 tends weakly to an absolutely continuous limit ρ^1, a formula for which is given as a function of w. A result identical in statement to Theorem 6.5.2, namely that

$$\|\nu_n - \mathbb{E}\nu_n\|_{\mathcal{B}(C_b(\mathbb{R}),L^2(\Omega))} = O\left(\frac{1}{\sqrt{n}}\right) \tag{6.51}$$

as n tends to infinity, is stated and proved. It is deduced that the sequence $(\nu_n)_{n\geq 1}$ converges to ρ^1 in the Banach space $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$. Note that Boutet de Monvel, Pastur and Shcherbina express their results in a different form, involving the Stieltjes transform, and that our Theorem 6.5.2 provides an alternative proof to their result (6.51).

When one applies Boutet de Monvel, Pastur and Shcherbina's theorem to Gaussian random matrices ($supp\,w = \mathbb{R}$ and v a convex quadratic polynomial) we obtain the famous semicircle law of Wigner. The limit distribution is a semicircle distribution; see [22] or sections 4.2, 5.4 and A.8 of [39] for more information. Note, though, that in the Gaussian case we actually obtain almost sure convergence of the empirical distributions, not just convergence in $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$; this follows from the fact that, in the Gaussian case,

$$\|\nu_n - \mathbb{E}\nu_n\|_{\mathcal{B}(C_b(\mathbb{R}),L^2(\Omega))} = O\left(\frac{1}{n}\right) \tag{6.52}$$

as n tends to infinity, not just $O\left(\frac{1}{\sqrt{n}}\right)$.

6.6 Matrices generated by compactly supported weights

We now assume *supp* $w = [-1, 1]$ and w is twice differentiable with only finitely many zeros on $(-1, 1)$, all of finite order. Such ensembles were discussed recently in [45].

Then we have the following result due to Szegö.

Lemma 6.6.1 *For all* $t \in (-1, 1)$,

$$(1-t^2)^{1/4} w(t)^{1/2} P_n(t) = \left(\frac{2}{\pi}\right)^{1/2} \cos(n \cos^{-1} t + \gamma(t)) + \varepsilon_n(t) \tag{6.53}$$

where γ *depends on* w *but not* n *and* $\varepsilon_n(t)$ *tends to zero uniformly in* t *as* n *tends to infinity.*

Proof See theorem 12.1.6 of [48].

□

In fact the function γ is given by

$$\gamma(\cos\theta) = \mathcal{H} \log\{w(\cos\theta)|\sin\theta|\} \tag{6.54}$$

where $\mathcal{H}$ denotes the Hilbert transform. See section 10.2 of [48] for details.

We now use this lemma, together with Lemma 6.5.1, to obtain an analogous result for the density ρ_n^1.

Lemma 6.6.2 *For all* $t \in (-1, 1)$,

$$(1-t^2)^{1/2} \rho_n^1(t) = \frac{1}{\pi} + \frac{\cos((n-1)\cos^{-1} t + 2\gamma(t)) \sin(n \cos^{-1} t)}{n\pi(1-t^2)^{1/2}} + \eta_n(t) \tag{6.55}$$

where $\eta_n(t)$ *tends to zero uniformly in* t *as* n *tends to infinity.*

Proof Applying Lemma 6.5.1 and Lemma 6.6.1, and writing θ for $\cos^{-1} t$, we have

$$(1-t^2)^{1/2} \rho_n^1(t) = \frac{1}{n}(1-t^2)^{1/2} \left(\sum_{j=0}^{n-1} P_j(t)^2\right) w(t) \tag{6.56}$$

$$= \frac{2}{n\pi} \sum_{j=0}^{n-1} \cos^2(j\theta + \gamma(t)) + \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_j(t)^2 \tag{6.57}$$

$$= \frac{1}{\pi} + \frac{1}{n\pi} \sum_{j=0}^{n-1} \cos 2(j\theta + \gamma(t)) + \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_j(t)^2. \tag{6.58}$$

Of the three summands on the right hand side, the third clearly tends to zero uniformly in t as n tends to infinity; denote this by $\eta_n(t)$. Considering the second, we have

$$\frac{1}{n\pi}\sum_{j=0}^{n-1}\cos 2(j\theta+\gamma(t)) = \frac{1}{n\pi}\Re\left\{e^{2i\gamma(t)}\frac{1-e^{2in\theta}}{1-e^{2i\theta}}\right\} \tag{6.59}$$

$$= \frac{1}{n\pi}\Re\left\{e^{i((n-1)\theta+2\gamma(t))}\frac{\sin n\theta}{\sin\theta}\right\} \tag{6.60}$$

$$= \frac{\cos((n-1)\theta+2\gamma(t))\sin n\theta}{n\pi\sin\theta} \tag{6.61}$$

which gives the desired-for expression.

□

These lemmas yield the following result, which is stated without proof by Pastur at the end of his paper [41] and attributed to a remark by M. Sodin.

Theorem 6.6.3 *The density ρ_n^1 tends weakly to the standard arcsine density as n tends to infinity.*

Proof Applying Lemma 6.6.2 we require that

$$\int_{-1}^{1}\left|\frac{\cos((n-1)\cos^{-1}t+2\gamma(t))\sin(n\cos^{-1}t)}{n\pi(1-t^2)}\right| dt \tag{6.62}$$

tend to zero as n tends to infinity. Making the substitution $t=\cos\theta$ gives

$$\frac{1}{n}\int_0^{\pi}\left|\frac{\cos((n-1)\theta+2\gamma(\cos\theta))\sin n\theta}{\sin\theta}\right|\frac{d\theta}{\pi}$$

$$\leq \frac{1}{n}\left(\int_0^{\pi}\cos^2((n-1)\theta+2\gamma(\cos\theta))\frac{d\theta}{\pi}\right)^{1/2}\left(\int_0^{\pi}\left(\frac{\sin n\theta}{\sin\theta}\right)^2\frac{d\theta}{\pi}\right)^{1/2} \tag{6.63}$$

by the Cauchy-Schwarz inequality. The left hand integral is clearly less than or equal to 1 since $|\cos\theta|\leq 1$ for all θ. For the right hand integral we note the integrand is a Fejér kernel and recall from Fourier analysis (see, for example, page 35 of [14]) that

$$\int_0^{\pi}\left(\frac{\sin n\theta}{\sin\theta}\right)^2\frac{d\theta}{\pi} = n. \tag{6.64}$$

We conclude that the overall product is less than or equal to $\frac{1}{\sqrt{n}}$. We are finished.

□

We deduce that the sequence $(\nu_n)_{n\geq 1}$ of empirical distributions of the eigenvalues of the ensemble $(X_n)_{n\geq 1}$ converges in the Banach space $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$ to the standard arcsine distribution.

Chapter 7

Some analytic function theory

This chapter studies Bergman spaces of analytic functions and various notions of cotype, characterised in terms of analytic functions, for Banach spaces.

The books [18] and [31] provide an introduction to analytic function theory, while more specific information on Bergman spaces may be found in the papers [1] and [23]. Background details on the various forms of cotype considered may be found in [54].

7.1 A result on Horowitz products

We start with definitions. We have the unit disc

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}, \tag{7.1}$$

the pseudohyperbolic metric for the disc

$$\rho(z, w) = \left| \frac{w - z}{1 - \overline{w}z} \right|, \tag{7.2}$$

where $z, w \in \mathbb{D}$, and the Blaschke factor

$$B_a(z) = \frac{\overline{a}}{|a|} \frac{a - z}{1 - \overline{a}z} \tag{7.3}$$

for $z \in \mathbb{D}$.

Recall that a sequence $(a_j) \subset \mathbb{D}$ is said to be H^∞ interpolating if, given any bounded sequence $(z_j) \subset \mathbb{C}$, there exists a function $f \in H^\infty(\mathbb{D})$ satisfying $f(a_j) = z_j$ for each j.

Here $H^\infty(\mathbb{D})$ denotes the Hardy space of bounded analytic functions on $\mathbb{D}$. For more information consult chapter VII of [18].

We say a sequence $(a_j) \subset \mathbb{D}$ satisfies the Carleson separation condition with constant δ, for some $\delta > 0$, if for all k,

$$\prod_{j \neq k} \rho(a_j, a_k) > \delta. \tag{7.4}$$

By Carleson's theorem, for which see theorem VII.1.1 of [18] for example, a sequence in $\mathbb{D}$ is H^∞ interpolating if and only if it satisfies the Carleson separation condition for some δ. We call δ the Carleson interpolation constant for (a_j).

If $(a_j) \subset \mathbb{D}$ we define its Horowitz product to be

$$P_{(a_j)}(z) = \prod_j B_{a_j}(z)(2 - B_{a_j}(z)). \tag{7.5}$$

It may easily be shown this product converges locally uniformly to an analytic function on $\mathbb{D}$, with zeros (a_j), if and only if $\sum_j (1 - |a_j|)^2 < \infty$. See [25] for details. In particular the Horowitz product is a well-defined analytic function for H^∞ interpolating sequences.

Consider the Bergman space $L^2_a(\mathbb{D})$ of analytic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$\|f\|^2_{L^2_a(\mathbb{D})} = \iint_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \tag{7.6}$$

where $dA(z)$ denotes normalised area measure on $\mathbb{D}$.

The zeros (a_j) of a function f lying in $L^2_a(\mathbb{D})$ satisfy $\sum_j (1 - |a_j|)^2 < \infty$; thus their Horowitz product $P_{(a_j)}$ defines an analytic function on $\mathbb{D}$ with zeros that of f. Furthermore ([23], page 114) $P_{(a_j)}$ is a contractive divisor on $L^2_a(\mathbb{D})$ in the sense that $f/P_{(a_j)}$ lies in $L^2_a(\mathbb{D})$ and satisfies $\|f/P_{(a_j)}\|_{L^2_a(\mathbb{D})} \leq \|f\|_{L^2_a(\mathbb{D})}$.

We note that any H^∞ interpolating sequence (a_j) is a Bergman space zero sequence as the Blaschke product for (a_j) is trivially in $H^\infty(\mathbb{D})$ and thus $L^2_a(\mathbb{D})$; for details on the theory of Blaschke products consult section II.2 of [18] or section IV.A of [31]. The Horowitz product for (a_j) may grow dramatically near the boundary of $\mathbb{D}$, however, and is not necessarily an element of $L^2_a(\mathbb{D})$. We have the following result.

Theorem 7.1.1 *There exists an H^∞ interpolating sequence $(a_j) \subset \mathbb{D}$ for which the corresponding Horowitz product $P_{(a_j)}$ does not lie in the Bergman space $L^2_a(\mathbb{D})$. That is*

$$\iint\limits_{\mathbb{D}} \left| \prod_j B_{a_j}(z)(2 - B_{a_j}(z)) \right|^2 dA(z) = \infty. \tag{7.7}$$

Before proving this, we will prove the following.

Lemma 7.1.2 *Suppose we have a sequence $(a_j) \subset \mathbb{D}$ satisfying the Carleson separation condition with constant δ. Fix a_k and $\eta \in (0, \frac{\delta}{2-\delta})$. Then for any $z \in \mathbb{D}$ satisfying*

$$\rho(z, a_k) < \eta \tag{7.8}$$

we have

$$\prod_{j \neq k} \rho(z, a_j) > \delta^{\frac{1}{1-\delta} \log\left(\frac{1-\eta}{\delta - 2\eta + \delta\eta}\right)}. \tag{7.9}$$

In particular if $\eta \leq \delta/2$ then

$$\prod_{j \neq k} \rho(z, a_j) > \delta^{\frac{1}{1-\delta} \log\left(\frac{2-\delta}{\delta^2}\right)} > 0 \tag{7.10}$$

which implies the cruder inequality

$$\prod_{j \neq k} \rho(z, a_j) > e^{-\frac{3}{1-\delta}\left(\log\left(\frac{1}{\delta}\right)\right)^2} > 0. \tag{7.11}$$

Proof of Lemma 7.1.2 For any $u, v, w \in \mathbb{D}$ we have the standard result for the pseudohyperbolic metric ([18], lemma I.1.4) that

$$\rho(u, v) \geq \frac{|\rho(u, w) - \rho(v, w)|}{1 - \rho(u, w)\rho(v, w)}. \tag{7.12}$$

Thus

$$\begin{aligned} \prod_{j \neq k} \rho(z, a_j) &\geq \prod_{j \neq k} \frac{\rho(a_j, a_k) - \rho(z, a_k)}{1 - \rho(a_j, a_k)\rho(z, a_k)} && (7.13) \\ &= \prod_{j \neq k} \left[1 - \frac{(1 + \rho(z, a_k))(1 - \rho(a_j, a_k))}{1 - \rho(a_j, a_k)\rho(z, a_k)}\right] && (7.14) \\ &> \prod_{j \neq k} \left[1 - \left(\frac{1+\eta}{1-\eta}\right)(1 - \rho(a_j, a_k))\right]. && (7.15) \end{aligned}$$

Now on the interval $[0, K]$ for $0 < K < 1$,

$$e^{-x} \geq 1 - x \geq e^{-\left(\frac{-\log(1-K)}{K}\right)x}, \tag{7.16}$$

and so as

$$0 < \left(\frac{1+\eta}{1-\eta}\right)(1 - \rho(a_j, a_k)) < \left(\frac{1+\eta}{1-\eta}\right)(1 - \delta), \tag{7.17}$$

and the right hand side of this expression is less than 1 precisely when $\eta < \frac{\delta}{2-\delta}$, we see

$$\begin{aligned}
1 - \left(\frac{1+\eta}{1-\eta}\right)(1 - \rho(a_j, a_k)) &\geq e^{-\left[\frac{-\log\left(1-\left(\frac{1+\eta}{1-\eta}\right)(1-\delta)\right)}{\left(\frac{1+\eta}{1-\eta}\right)(1-\delta)}\right]\left(\frac{1+\eta}{1-\eta}\right)(1-\rho(a_j,a_k))} && (7.18)\\
&= e^{-\left[\frac{-\log\left(\frac{\delta-2\eta+\delta\eta}{1-\eta}\right)}{1-\delta}\right](1-\rho(a_j,a_k))} && (7.19)\\
&= \left[e^{-(1-\rho(a_j,a_k))}\right]^{\left(\frac{-\log\left(\frac{\delta-2\eta+\delta\eta}{1-\eta}\right)}{1-\delta}\right)} && (7.20)\\
&\geq \rho(a_j, a_k)^{\left(\frac{-\log\left(\frac{\delta-2\eta+\delta\eta}{1-\eta}\right)}{1-\delta}\right)} && (7.21)\\
&= \rho(a_j, a_k)^{\frac{1}{1-\delta}\log\left(\frac{1-\eta}{\delta-2\eta+\delta\eta}\right)}. && (7.22)
\end{aligned}$$

So

$$\begin{aligned}
\prod_{j \neq k} \rho(z, a_j) &> \left(\prod_{j \neq k} \rho(a_j, a_k)\right)^{\frac{1}{1-\delta}\log\left(\frac{1-\eta}{\delta-2\eta+\delta\eta}\right)} && (7.23)\\
&> \delta^{\frac{1}{1-\delta}\log\left(\frac{1-\eta}{\delta-2\eta+\delta\eta}\right)}. && (7.24)
\end{aligned}$$

Now, suppose $\eta \leq \delta/2$. Then as

$$\frac{1-\eta}{\delta - 2\eta + \delta\eta} = \frac{1}{2-\delta}\left[1 + \frac{2(1-\delta)}{\delta - (2-\delta)\eta}\right] \tag{7.25}$$

we see that this quantity is increasing as η increases from 0. Thus, for any $\eta \leq \delta/2$, this quantity is less than or equal to its value at $\eta = \delta/2$. So

$$\begin{aligned}
\frac{1-\eta}{\delta - 2\eta + \delta\eta} &\leq \left.\frac{1-\eta}{\delta - 2\eta + \delta\eta}\right|_{\eta=\delta/2} && (7.26)\\
&= \frac{2-\delta}{\delta^2}. && (7.27)
\end{aligned}$$

Thus, as $\delta < 1$ and log is increasing,

$$\prod_{j \neq k} \rho(z, a_j) > \delta^{\frac{1}{1-\delta} \log\left(\frac{2-\delta}{\delta^2}\right)} > 0. \tag{7.28}$$

Now

$$\begin{aligned} \log\left(\frac{2-\delta}{\delta^2}\right) &< \log\left(\frac{2}{\delta^2}\right) && (7.29)\\ &= \log 2 + 2\log\left(\frac{1}{\delta}\right) && (7.30)\\ &< (\log 2 + 2)\log\left(\frac{1}{\delta}\right) && (7.31)\\ &< 3\log\left(\frac{1}{\delta}\right) && (7.32) \end{aligned}$$

as $\log\left(\frac{1}{\delta}\right) < 1$. So, as $\delta < 1$,

$$\begin{aligned} \prod_{j \neq k} \rho(z, a_j) &> \delta^{\frac{3}{1-\delta} \log\left(\frac{1}{\delta}\right)} && (7.33)\\ &= e^{-\frac{3}{1-\delta}\left(\log\left(\frac{1}{\delta}\right)\right)^2} > 0. && (7.34) \end{aligned}$$

□

It will also be useful later to use the following result.

Lemma 7.1.3 *There exists a sequence $(\lambda_j) \subset \mathbb{D}$, lying on the positive real axis and satisfying $0 \leq \lambda_1 < \lambda_2 < \cdots$, which is H^∞ interpolating in Carleson's sense and yet*

$$\sum_j (2-\delta)^{2(j-1)}(1-\lambda_j)^2 = \infty \tag{7.35}$$

where δ is the Carleson interpolation constant for (λ_j).

Proof of Lemma 7.1.3 Consider the sequence

$$\lambda_j = 1 - \alpha^{j-1} \tag{7.36}$$

for $j = 1, 2, \cdots$, where $0 < \alpha < 1$. We shall first show the sequence (7.36) is H^∞ interpolating.

By Carleson's theorem ([18], theroem VII.1.1) a sequence $(a_j) \subset \mathbb{D}$ is H^∞ interpolating if and only if firstly it is separated, in the sense that for all $j \neq k$,

$$\rho(a_j, a_k) > \eta \tag{7.37}$$

for some constant $\eta > 0$, and secondly the measure

$$\sum_j (1 - |a_j|)\, \delta_{a_j} \tag{7.38}$$

is a Carleson measure. Now for our sequence (7.36) we have from above that, for $j < k$,

$$\begin{aligned} \rho(1-\alpha^{j-1}, 1-\alpha^{k-1}) &= \frac{1-\alpha^{k-j}}{1+\alpha^{k-j}-\alpha^{k-1}} && (7.39)\\ &> \frac{1-\alpha^{k-j}}{1+\alpha^{k-j}} && (7.40)\\ &\geq \frac{1-\alpha}{1+\alpha} && (7.41) \end{aligned}$$

as, for fixed j, the quantity on the second-to-last line increases with k. Thus, for any $\alpha \in (0,1)$, our sequence is separated with $\eta = (1-\alpha)/(1+\alpha)$.

We now need to show the measure $\sum_j \alpha^{j-1} \delta_{(1-\alpha^{j-1})}$ is Carleson. As our sequence lies along the positive real axis, we see this is Carleson if and only if there is a constant $C > 0$ such that, for every $\varepsilon > 0$,

$$\sum_{j:\alpha^{j-1} \leq \varepsilon} \alpha^{j-1} \leq C\varepsilon. \tag{7.42}$$

But

$$\begin{aligned} \sum_{j:\alpha^{j-1}\leq\varepsilon} \alpha^{j-1} &= \sum_{j:j\geq 1+(\log\varepsilon/\log\alpha)} \alpha^{j-1} && (7.43)\\ &= \frac{\alpha^{\left\lceil \frac{\log\varepsilon}{\log\alpha} \right\rceil}}{1-\alpha} && (7.44)\\ &\leq \frac{\alpha^{\left(\frac{\log\varepsilon}{\log\alpha}\right)}}{1-\alpha} && (7.45)\\ &= \frac{\varepsilon}{1-\alpha}; && (7.46) \end{aligned}$$

thus, for any $\alpha \in (0,1)$, our measure is Carleson with $C = 1/(1-\alpha)$.

Thus for all $\alpha \in (0,1)$ the sequence (7.36) is H^∞ interpolating. Assume, for fixed α, that (7.36) is H^∞ interpolating with constant δ. For all $j < k$ we have

$$\begin{aligned} \delta &< \rho(\lambda_j, \lambda_k) && (7.47)\\ &= \frac{\alpha^{j-1} - \alpha^{k-1}}{1 - (1-\alpha^{j-1})(1-\alpha^{k-1})} && (7.48)\\ &= \frac{1-\alpha^{k-j}}{1+\alpha^{k-j} - \alpha^{k-1}}; && (7.49) \end{aligned}$$

which leads, putting $k = j+1$, to

$$\delta < \frac{1-\alpha}{1+\alpha-\alpha^j} \qquad (7.50)$$

for each j. As the term on the right hand side decreases and α^j tends to zero as j increases we see

$$\delta \leq \frac{1-\alpha}{1+\alpha}. \qquad (7.51)$$

Now

$$\sum_j (2-\delta)^{2(j-1)}(1-\lambda_j)^2 = \sum_j [(2-\delta)\alpha]^{2(j-1)}, \qquad (7.52)$$

which tends to infinity provided $(2-\delta)\alpha > 1$. As

$$(2-\delta)\alpha \geq \left(2 - \left(\frac{1-\alpha}{1+\alpha}\right)\right)\alpha = \frac{\alpha(1+3\alpha)}{1+\alpha}, \qquad (7.53)$$

we see the sum (7.35) tends to infinity if

$$\frac{\alpha(1+3\alpha)}{1+\alpha} > 1. \qquad (7.54)$$

But this holds if and only if $\alpha(1+3\alpha) > 1+\alpha$, which in turn is equivalent to $\alpha^2 > 1/3$. As $\alpha > 0$, we see the sum (7.35) tends to infinity if $\alpha > 1/\sqrt{3}$.

We have thus shown that, for any $\alpha \in (1/\sqrt{3}, 1)$, the sequence (7.36) satisfies the requirements of the lemma.

□

We may now finally prove our main result.

Proof of Theorem 7.1.1 Let $(a_j) \subset \mathbb{D}$ be an H^∞ interpolating sequence with constant δ, so for all k,

$$\prod_{j \neq k} \rho(a_j, a_k) > \delta. \tag{7.55}$$

For any $w \in \mathbb{D}$ and $\eta > 0$, let

$$H_\eta(w) = \{z \in \mathbb{C} \mid \rho(w, z) < \eta\} \tag{7.56}$$

denote the pseudohyperbolic ball of radius η about w. Consider the balls $H_{\delta/2}(a_j)$. Note that, because of the Carleson separation condition, these balls are disjoint. Fix k. Then for any $z \in H_{\delta/2}(a_k)$,

$$\prod_j B_{a_j}(z)(2 - B_{a_j}(z)) = B_{a_k}(z) \prod_{j \neq k} B_{a_j}(z) \prod_{j<k} (2 - B_{a_j}(z)) \prod_{j \geq k} (2 - B_{a_j}(z)), \tag{7.57}$$

which gives

$$\left| P_{(a_j)}(z) \right| = |B_{a_k}(z)| \prod_{j \neq k} \left| B_{a_j}(z) \right| \prod_{j<k} \left| 2 - B_{a_j}(z) \right| \prod_{j \geq k} \left| 2 - B_{a_j}(z) \right|. \tag{7.58}$$

Now

$$\left| 2 - B_{a_j}(z) \right| \geq 1 \tag{7.59}$$

for all $j \geq k$ and

$$\prod_{j \neq k} \left| B_{a_j}(z) \right| = \prod_{j \neq k} \rho(z, a_j) > e^{-\frac{3}{1-\delta}\left(\log\left(\frac{1}{\delta}\right)\right)^2} > 0 \tag{7.60}$$

by Lemma 7.1.2, so

$$\left| P_{(a_j)}(z) \right| > e^{-\frac{3}{1-\delta}\left(\log\left(\frac{1}{\delta}\right)\right)^2} |B_{a_k}(z)| \prod_{j<k} \left| 2 - B_{a_j}(z) \right|. \tag{7.61}$$

Consider $\left| 2 - B_{a_j}(z) \right|$ for $j < k$. Certainly these are all greater than or equal to 1. But can we improve on this? Note that

$$2 - B_{a_j}(z) = 2 - B_{a_j}(a_k) + B_{a_j}(a_k) - B_{a_j}(z) \tag{7.62}$$

and therefore

$$\left|2 - B_{a_j}(z)\right| \geq \left|2 - B_{a_j}(a_k)\right| - \left|B_{a_j}(z) - B_{a_j}(a_k)\right|. \tag{7.63}$$

Now, the generalised form of Schwarz's lemma ([18], lemma I.1.2) states that analytic functions $f : \mathbb{D} \to \mathbb{D}$ are contractions for the pseudohyperbolic metric, that is

$$\rho(f(z), f(w)) \leq \rho(z, w) \tag{7.64}$$

for all $z, w \in \mathbb{D}$. Thus, as Blaschke factors are analytic functions $\mathbb{D} \to \mathbb{D}$, we have

$$\rho(B_{a_j}(z), B_{a_j}(a_k)) \leq \rho(z, a_k) < \delta/2, \tag{7.65}$$

which implies

$$\left|B_{a_j}(z) - B_{a_j}(a_k)\right| < (\delta/2)\left|1 - \overline{B_{a_j}(a_k)}B_{a_j}(z)\right| \leq \delta. \tag{7.66}$$

So

$$\left|2 - B_{a_j}(z)\right| > \left|2 - B_{a_j}(a_k)\right| - \delta. \tag{7.67}$$

Henceforth we will assume the (a_j) lie along the positive real axis, with $0 \leq a_1 < a_2 < \cdots$. This implies that $B_{a_j}(a_k) \leq 0$ for all $j < k$, which in turn implies $\left|2 - B_{a_j}(a_k)\right| \geq 2$ for all $j < k$. Thus

$$\left|2 - B_{a_j}(z)\right| > 2 - \delta \tag{7.68}$$

for all $j < k$. So

$$\left|P_{(a_j)}(z)\right| > e^{-\frac{3}{1-\delta}\left(\log\left(\frac{1}{\delta}\right)\right)^2}(2-\delta)^{k-1}\left|B_{a_k}(z)\right|, \tag{7.69}$$

and thus

$$\iint\limits_{H_{\delta/2}(a_k)} \left|P_{(a_j)}(z)\right|^2 dA(z) > e^{-\frac{6}{1-\delta}\left(\log\left(\frac{1}{\delta}\right)\right)^2}(2-\delta)^{2(k-1)} \iint\limits_{H_{\delta/2}(a_k)} \left|B_{a_k}(z)\right|^2 dA(z). \tag{7.70}$$

Let us consider this last integral. Put

$$w = \frac{z - a_k}{1 - \overline{a_k}z}, \tag{7.71}$$

then

$$z = \frac{w + a_k}{1 + \overline{a_k} w}, \tag{7.72}$$

which yields, upon calculating the Jacobian,

$$\begin{aligned}
dA(z) &= \frac{(1 - |a_k|^2)^2}{(1 + a_k \overline{w})^2 (1 + \overline{a_k} w)^2} \, dA(w) && (7.73) \\
&\geq \frac{(1 - |a_k|^2)^2}{(1 + |a_k| \, |w|)^2} \, dA(w) && (7.74) \\
&\geq \frac{(1 - |a_k|^2)^2}{(1 + |a_k|)^2} \, dA(w) && (7.75) \\
&= (1 - |a_k|)^2 \, dA(w) && (7.76)
\end{aligned}$$

where we interpret all inequalities in the sense of positive measures on $\mathbb{D}$. Thus

$$\begin{aligned}
\iint\limits_{H_{\delta/2}(a_k)} |B_{a_k}(z)|^2 \, dA(z) &\geq \iint\limits_{|w|<\delta/2} |w|^2 \, (1 - |a_k|)^2 \, dA(w) && (7.77) \\
&= \frac{(1 - |a_k|)^2}{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{\delta/2} r^3 \, dr \, d\theta && (7.78) \\
&= 2(1 - |a_k|)^2 \int_0^{\delta/2} r^3 \, dr && (7.79) \\
&= 2(1 - |a_k|)^2 \frac{(\delta/2)^4}{4} && (7.80) \\
&= \frac{\delta^4}{32} (1 - |a_k|)^2. && (7.81)
\end{aligned}$$

So

$$\iint\limits_{H_{\delta/2}(a_k)} \left| P_{(a_j)}(z) \right|^2 \, dA(z) > \frac{\delta^4}{32} e^{-\frac{6}{1-\delta} \left(\log\left(\frac{1}{\delta}\right)\right)^2} (2 - \delta)^{2(k-1)} (1 - |a_k|)^2 \tag{7.82}$$

and hence, as the $H_{\delta/2}(a_k)$ are disjoint,

$$\begin{aligned}
\iint\limits_{\mathbb{D}} \left| P_{(a_j)}(z) \right|^2 \, dA(z) &> \sum_k \iint\limits_{H_{\delta/2}(a_k)} \left| P_{(a_j)}(z) \right|^2 \, dA(z) && (7.83) \\
&> \frac{\delta^4}{32} e^{-\frac{6}{1-\delta} \left(\log\left(\frac{1}{\delta}\right)\right)^2} \sum_k (2 - \delta)^{2(k-1)} (1 - |a_k|)^2. && (7.84)
\end{aligned}$$

Thus our theorem will be proved if we can exhibit an interpolating sequence (a_j) with constant δ, lying along the positive real axis, satisfying

$$\sum_k (2 - \delta)^{2(k-1)} (1 - |a_k|)^2 = \infty. \tag{7.85}$$

We observe that Lemma 7.1.3 provides such a sequence, namely $(1-\alpha^{j-1})$ with α lying in $(1/\sqrt{3},1)$. The proof is complete.

□

7.2 On analytic Lusin cotype

Let E be a Banach space, let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$, let $\mathbb{T}^{\mathbb{N}}$ denote the infinite torus equipped with normalised Haar measure, let $2 \leq q < \infty$, let $L^q(\mathbb{T}^{\mathbb{N}}; E)$ denote the Bochner L^q space of functions from $\mathbb{T}^{\mathbb{N}}$ to E and let $H^q(\mathbb{D}; E)$ denote the Hardy space of analytic functions $f : \mathbb{D} \to E$ satisfying

$$\|f\|_q = \sup_{r<1} \left(\int_0^{2\pi} \|f(re^{i\theta})\|_E^q \frac{d\theta}{2\pi} \right)^{1/q} < \infty. \tag{7.86}$$

Let $\tilde{H}^q(\mathbb{D}; E)$ denote the closure of the E valued polynomials in the space $H^q(\mathbb{D}; E)$.

We say E is of analytic Lusin cotype q if there exists $C > 0$ such that for all $f \in H^q(\mathbb{D}; E)$, equivalently for all $f \in \tilde{H}^q(\mathbb{D}; E)$,

$$\left(\int_0^1 (1-r)^{q-1} \left\| f_r' \right\|_q^q \, dr \right)^{1/q} \leq C \left\| f \right\|_q \tag{7.87}$$

where $f_r(e^{i\theta}) = f(re^{i\theta})$. Note that we may replace the left hand side above with any of the following, which are all equivalent up to constants:

$$\left\| G_q(f) \right\|_q \text{ where } G_q(f)(z) = \left(\int_0^1 (1-r)^{q-1} \left\| f'(rz) \right\|_E^q \, dr \right)^{1/q}, \tag{7.88}$$

which is a variant of the Littlewood-Paley g-function;

$$\left\| S_q(f) \right\|_q \text{ where } S_q(f)(z) = \left(\int_{\Gamma(z)} (1-|\zeta|)^{q-2} \left\| f'(\zeta) \right\|_E^q \, dA(\zeta) \right)^{1/q}, \tag{7.89}$$

which is a variant of the Lusin area integral; and

$$\left(\sum_{n \geq 0} 2^{-nq} \left\| f_{r_n}' \right\|_q^q \right)^{1/q}. \tag{7.90}$$

Here $z \in \mathbb{T}$ and $\Gamma(z) = \{\zeta \in \mathbb{D} : |\arg z - \arg \zeta| < 1 - |\zeta|\}$.

We say the space $H^q(\mathbb{D};E)$ satisfies a radial lower q-estimate if there exists $C \geq 1$ such that, for all sequences $0 \leq r_0 < r_1 < \cdots < r_N < 1$ and $f \in H^q(\mathbb{D};E)$,

$$\left(\sum_{n\geq 0} \|f_{r_n} - f_{r_{n-1}}\|_q^q\right)^{1/q} \leq C \|f\|_q , \tag{7.91}$$

where we set $f_{r_{-1}} = 0$. Notice that the property remains the same when we only require that $f \in \tilde{H}^q(\mathbb{D};E)$.

We say E is of Hardy (respectively analytic) martingale cotype q if there exists $C > 0$ such that, for all Hardy (respectively analytic) martingales $(M_n)_{n\geq 0}$ with M_n lying in $L^q(\mathbb{T}^{\mathbb{N}};E)$ for each n,

$$\left(\sum_{n\geq 0} \|M_n - M_{n-1}\|_q^q\right)^{1/q} \leq C \sup_{n\geq 0} \|M_n\|_q , \tag{7.92}$$

where we set $M_{-1} = 0$. Definitions and basic properties of Hardy and analytic martingales are in [54]. In [43] Pisier shows that a Banach space E has Hardy martingale cotype q if and only if $H^q(\mathbb{D};E)$ satisfies a radial lower q-estimate (theorem 7.8).

In [54] Xu shows that, for a Banach space E, E has Hardy martingale cotype q implies E is of analytic Lusin cotype q (theorem 5.1(i)) and, furthermore, E is of analytic Lusin cotype q implies E has analytic martingale cotype q (theorem 5.1(ii)). Xu conjectures that analytic martingale cotype and Hardy martingale cotype are identical properties.

We see that, if it could be proved that E of analytic Lusin cotype q implies $H^q(\mathbb{D};E)$ satisfies a radial lower q-estimate, it would follow that Hardy martingale cotype and analytic Lusin cotype are identical properties.

We will prove a somewhat weaker result.

Theorem 7.2.1 *If E is a Banach space of analytic Lusin cotype q then $H^q(\mathbb{D};E)$ satisfies a geometric radial lower q-estimate; that is to say, given any K satisfying $0 < K < 1$, the radial lower q-estimate condition (7.91) holds, with constant C depending on K, for all sequences $0 \leq r_0 < r_1 < \cdots < r_N < 1$ satisfying*

$$\frac{1 - r_n}{1 - r_{n-1}} \geq K \tag{7.93}$$

for all n.

Proof Let $f \in \tilde{H}^q(\mathbb{D}; E)$ and $0 < K < 1$. It suffices to show there is a finite C, depending on K but independent of f, such that

$$\sum_{n\geq 1} \left\| f_{r_n} - f_{r_{n-1}} \right\|_q^q \leq C \int_0^1 (1-r)^{q-1} \left\| f_r' \right\|_q^q \, dr \tag{7.94}$$

for all sequences $0 \leq r_0 < r_1 < \cdots < r_N < 1$ satisfying (7.93). Now, we have

$$\begin{aligned} \left\| f_{r_n} - f_{r_{n-1}} \right\|_q^q &= \int_0^{2\pi} \left\| f(r_n e^{i\theta}) - f(r_{n-1} e^{i\theta}) \right\|_E^q \frac{d\theta}{2\pi} && (7.95) \\ &= \int_0^{2\pi} \left\| \int_{r_{n-1}}^{r_n} f'(re^{i\theta}) \, dr \right\|_E^q \frac{d\theta}{2\pi}. && (7.96) \end{aligned}$$

Notice this last step employs the analyticity of f.

We now use the result that, if $\mathbb{P}$ is any Radon probability measure on $[r_{n-1}, r_n]$ and X is any element of $L^q([r_{n-1}, r_n], \mathbb{P}; E)$ then, by Hölder's inequality, $\|\mathbb{E}X\|_E \leq \left(\mathbb{E}\|X\|_E^q\right)^{1/q}$. Take $\mathbb{P}$ to be the probability measure $\frac{dr}{1-r}\left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{-1}$ and X to be the function $r \mapsto (1-r) f'(re^{i\theta})$. Then

$$\begin{aligned} & \left\| \int_{r_{n-1}}^{r_n} f'(re^{i\theta}) \, dr \right\|_E^q \\ = & \left\| \int_{r_{n-1}}^{r_n} (1-r) f'(re^{i\theta}) \cdot \frac{dr}{1-r} \left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{-1} \right\|_E^q \left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^q && (7.97) \\ \leq & \int_{r_{n-1}}^{r_n} (1-r)^q \left\| f'(re^{i\theta}) \right\|_E^q \cdot \frac{dr}{1-r} \left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{q-1} && (7.98) \\ = & \left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{q-1} \int_{r_{n-1}}^{r_n} (1-r)^{q-1} \left\| f'(re^{i\theta}) \right\|_E^q \, dr. && (7.99) \end{aligned}$$

Thus we have

$$\begin{aligned} & \sum_{n\geq 1} \left\| f_{r_n} - f_{r_{n-1}} \right\|_q^q \\ \leq & \sum_{n\geq 1} \left[\int_0^{2\pi} \left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{q-1} \int_{r_{n-1}}^{r_n} (1-r)^{q-1} \left\| f'(re^{i\theta}) \right\|_E^q \, dr \frac{d\theta}{2\pi} \right] && (7.100) \\ = & \sum_{n\geq 1} \left[\left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{q-1} \int_{r_{n-1}}^{r_n} (1-r)^{q-1} \left\| f_r' \right\|_q^q \, dr \right] && (7.101) \\ \leq & \sup_{n\geq 1} \left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{q-1} \cdot \int_0^1 (1-r)^{q-1} \left\| f_r' \right\|_q^q \, dr. && (7.102) \end{aligned}$$

As $q \geq 2$ and log is an increasing function,

$$\left[\log\left(\frac{1-r_{n-1}}{1-r_n}\right)\right]^{q-1} \leq \left[\log\left(\frac{1}{K}\right)\right]^{q-1} < \infty \tag{7.103}$$

for all n. Thus we may take $C = \left[\log\left(\frac{1}{K}\right)\right]^{q-1}$; the proof is complete.

□

Chapter 8

Unresolved questions

This last chapter will detail some possible avenues for future research.

1) On stochastic integration with predictable integrands

Let E be a separable Banach space. Theorem 3.2.2 considered the stochastic integral in E of a deterministic integrand with respect to a Q-Wiener process. Under what circumstances may we generalise this, and define an Itô stochastic integral

$$\int_s^t T_u \, d\mathbf{B}_u, \tag{8.1}$$

for $(T_u)_{s \leq u \leq t}$ a random family of bounded linear operators on E which is predictable with respect to the filtration induced by the Q-Wiener process $\mathbf{B}_u$? Under what circumstances may we define the integral in the case where $\mathbf{B}_u$ is a more general stochastic process in E with independent increments? These questions were considered recently in the paper [7].

2) On uniqueness and adaptedness of the Ornstein-Uhlenbeck process

Within the framework of Theorem 4.2.1, under what precise circumstances is the Banach space valued Ornstein-Uhlenbeck process

$$\mathbf{Z}_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} (\Lambda + i\omega I)^{-1} \, d\tilde{\mathbf{B}}_\omega \tag{8.2}$$

unique in distribution? Under what precise circumstances is $\mathbf{Z}_t$ adapted to the filtration induced by the Q-Wiener process $\mathbf{B}_t$? Corollary 4.2.2 gives a sufficient condition for both uniqueness and adaptedness.

3) On the almost sure convergence of $(\nu_n - \mathbb{E}\nu_n)_{n\geq 1}$

Within the framework of Theorem 6.5.2, under what precise circumstances do we have almost sure convergence of the sequence $(\nu_n - \mathbb{E}\nu_n)_{n\geq 1}$ to zero? This occurs in the case of Gaussian random matrices; see [22] for more details.

4) On the zeros of an orthogonal polynomial

Within the framework of sections 6.5 and 6.6, let w be a weight function on $\mathbb{R}$ and let $(P_n)_{n\geq 0}$ denote the sequence of orthonormal polynomials associated to w. For each $n \geq 1$ let $(\alpha_j(P_n))_{j=1}^n$ denote the set of zeros of P_n, where $\alpha_1 \geq \cdots \geq \alpha_n$. Define a probability measure κ_n on $\mathbb{R}$ for each $n \geq 1$ via

$$\kappa_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\alpha_j(P_n)}. \tag{8.3}$$

The probability measure κ_n is known as the empirical distribution of the zeros of P_n; it is natural to ask how this behaves as n tends to infinity.

Under certain circumstances the measures $(\kappa_n)_{n\geq 1}$ and the random measures $(\nu_n)_{n\geq 1}$ associated to the ensemble $(X_n)_{n\geq 1}$ converge to the same limit as n tends to infinity; this limit is the integrated density of states ρ^1. The measures $(\kappa_n)_{n\geq 1}$ converge weakly while the random measures $(\nu_n)_{n\geq 1}$ converge in $\mathcal{B}(C_b(\mathbb{R}), L^2(\Omega))$.

In [41] Pastur proved this holds in the special case where $supp\, w$ is the whole of $\mathbb{R}$ and $w(x) = \exp\{-|x|^r\}$ for some $r > 1$.

This result also holds when the weight function w has compact support; this was proved by Szegö in [48]. In his paper [45] Shirai used Szegö's results to calculate the limiting value of $\frac{1}{n}\log \mathbb{E}\det(\lambda - X_n)^{-1}$, for complex λ and compactly supported w, as n tends to infinity.

Under what precise circumstances do $(\kappa_n)_{n\geq 1}$ and $(\nu_n)_{n\geq 1}$ converge to the same limit as n tends to infinity?

5) On spectral representations and analytic functions

The Ornstein-Uhlenbeck process (8.2) provides an example of a spectral representation for a Banach space valued weakly stationary stochastic process. What other Banach space valued weakly stationary processes may be expressed as a stochastic integral in this way and under precisely which circumstances may we represent the autocovariance function of a Banach space valued weakly stationary process as a spectral integral?

In other words, given a weakly stationary process $\mathbf{X}_t$ in a Banach space E, under what circumstance may we write

$$\mathbf{X}_t = \int_{-\infty}^{\infty} e^{i\omega t}\, d\mathbf{U}_\omega \tag{8.4}$$

where $\mathbf{U}_\omega$ is a spectral stochastic process with independent increments? If $\mathbf{X}_t$ has autocovariance function Ψ then under what circumstances may we write

$$\Psi(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, \Upsilon(d\omega) \tag{8.5}$$

where Υ is a measure on $\mathbb{R}$ taking values in a space of covariance operators? If we do have such a measure Υ, under what circumstances does Υ factor as $\zeta\overline{\zeta^*}$, where ζ is an analytic function on the upper half plane? This problem was considered extensively in [43].

When considering such factorisations, should we view our analytic functions as elements of vector valued Hardy spaces, as was done in the finite dimensional case by Masani and Wiener ([37], [38]) and more recently in the general case by Pisier ([43]), or should we be viewing them as elements of more general spaces, such as vector valued Bergman spaces?

Such a factorisation theory is likely to have major applications to the prediction theory of Banach space valued stochastic processes; such a conjecture was made by Pisier at the end of [43].

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