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Comparison between continuous and discrete methods of dynamical control

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ABSTRACT

A direct comparison between continuous and discrete forms of analysis of control and stability is investigated theoretically and numerically. We demonstrate that the continuous method provides a more energy-efficient means of controlling the switching of a periodically-driven class-B laser between its stable and unstable pulsing regimes. We provide insight into this result using the close correspondence that exists between the problems of energy-optimal control and the stability of a steady state.

Keywords: Dynamical control, control of maps, control of flows, fluctuations in lasers, control of lasers

1. INTRODUCTION

Investigation of the stability and control of a lasing mode addresses an important problem in the applied theory of lasers.[1] It is one that can also be mapped onto an analysis of bursting behavior arising in population dynamics.[2] In a more general context such an analysis describes a fundamental problem in the theory of nonlinear dynamical systems.[3,4] It is potentially of relevance in biology where switching takes place between distinct regimes of behaviour, e.g. in cardiac and cortical systems. While the topic is thus of broad interdisciplinary interest, laser systems can provide especially reliable and convincing tests of the new theoretical concepts. In general, the problem can be analyzed within either one of two theoretical and experimental frameworks: using either continuous or discrete time, with corresponding descriptions of the system dynamics in terms of either continuous flows or maps. Both methods have been extensively tested in application to laser systems. For example, a special protocol was been developed for the feedback control of steady generation in Nd lasers that have a tendency to self-pulsing.[5] To get over the uncertainty in switching, the methods of stochastic resonance together with an additional weak periodic modulation have been proposed.[6,7] The targeting of stable and unstable orbits has been discussed[8] and achieved experimentally by the use of a single large amplitude perturbation in a loss-modulated CO_2 laser[9,10] with special attention being paid to minimization of the duration of the transient processes.

However, a direct comparison between these two general methods for the analysis of stability and control of a steady state has not so far been considered, which is perhaps surprising given its broad interdisciplinary implications.

In this paper we consider, both theoretically and numerically, a direct comparison between the continuous and discrete forms of control of the lasing mode in class B lasers. In particular, we demonstrate that the two methods give estimates of the activation "energy" and of the energy of the control function that differ by an order of magnitude. We use the duality of the control and stability problems discussed in our earlier work[11,12] to provide insight into the origin of this difference.

In particular, we investigate the energy-optimal control of switching in a periodically-driven class-B laser between its stable and unstable pulsing regimes. Coexistence of nonstationary states can be realized experimentally in lasers e.g. by periodic modulation of intracavity loss[9,13] or of pumping rate.[14–19] Here we study the later case because it is more suitable for class B solid-state lasers including microchip lasers and semiconductor lasers.

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2. CONTINUOUS MODEL AND MAP

Our analysis is based on single-mode rate equations

$$\begin{cases} \frac{du}{dt} & = vu(y-1), \\ \frac{dy}{dt} & = q + k\cos(\omega t) - y - yu + f(t), \end{cases} \tag{1}$$

where u and y are proportional to the density of radiation and carrier inversion respectively, v is the ratio of the photon damping rate in the cavity to the rate of carrier inversion relaxation, the cavity loss is normalized to unity, and the pumping rate has a constant term q and a periodic component; k and ω are the amplitude and frequency of the external periodic modulation. The additive unconstrained control function is $f(t)$.

For class-B lasers the parameter v is large, $v \sim 10^3 - 10^4$, and provides for regimes of the spiking type under deep modulation of the pumping rate. Solutions can be obtained from the corresponding two-dimensional Poincaré map[20]:

$$\begin{cases} c_{i+1} = & q + G(c_i, \psi_i)e^{-T} + K\cos(\omega T + \psi_i) + f_i, \\ \varphi_{i+1} = & \varphi_i + \omega T, \mathrm{mod}\ 2\pi, \end{cases} \tag{2}$$

where $G(c_i, \psi_i) = c_i - g - q - K\cos\psi_i$, $K = k(1+\omega^2)^{-1/2}$, and $\psi_i = \varphi_i - \arctan(\omega)$. The control function f_i is now defined in discrete time. Functions $g = g(c_i)$ and $T = T(c_i, \varphi_i)$ are positive roots of the equations

$$g - c_i(1 - \exp(-g)) = 0,$$
$$(q-1)T + G(c_i, \psi_i)(1 - e^{-T}) + K\omega^{-1}[\sin(\omega T + \psi_i) - \sin\psi_i] = 0,$$

respectively. Variables c_i, φ_i correspond to the inversion of population $y(t_i)$ and to the phase of modulation $\varphi_i = \omega t_i, \mathrm{mod}\ 2\pi$ at the moments t_i of pulse onset when $u(t_i) = 1, \dot{u}(t_i) > 0$, $g(c_i)$ denotes the energy of the pulse, and $T(c_i, \varphi_i)$ gives the time interval between sequential pulses. The map has been derived by asymptotic integration of Eqs.(1) to an accuracy of $O(v^{-1})$. It is therefore valid for $q, k, \omega \ll v$ and $c_i > 1 + O(v^{-1})$.

For each iteration of the map one can find characteristics directly comparable with experimental measurements: the phase of the modulation signal at the moment of the spike and the interval between pulses, the energy of the spike and its maximal intensity given by $u_{max} = 1 + v[c_i - 1 - \ln(c_i)]$.

The fixed points of the map determine spiking solutions of the period multiple to the period of driving, $T_n = nT_M, n = 1, 2, ...$ where the period of driving is $T_M = 2\pi/\omega$. They are born through a saddle-node bifurcation at the modulation threshold level

$$k_{sn} = \sqrt{1+\omega^2}[q - C_n - g_n(e^{T_n} - 1)^{-1}]. \tag{3}$$

The stable cycles undergo a period-doubling bifurcation if the modulation level exceeds

$$k_{pd} = \frac{\sqrt{1+\omega^2}}{\omega}(q-1)\left[1 + 2\pi\left(\frac{qnT_n}{12}\right)^2 + O(T_n^4)\right]. \tag{4}$$

In this way we determine analytically the regions where generalized multistability is realized as the coexistence of a number of cycles, and we can approximate the location of saddles and stable cycles in the phase space.

3. CONTROL PROBLEM

We now consider the problem of controlled migration between the stable and saddle cycles that are related to the saddle-node bifurcation (3) and which are of the same amplitude and period. Specifically, we will study migration from the stable cycle C_3 to the saddle cycle S_3. As mentioned above, such a process can be proposed for the phase-coding information scheme because these cycles differ from each other only by virtue of their phase relationship to the modulation signal. Within the framework of the method proposed, switching between stable cycles with different periods and amplitudes can be also considered quite generally for an amplitude coding scheme as well as for a combination of phase and amplitude coding.

Two different forms of the control force $f(t)$ are considered: one is continuous $f_c(t)$ in time; and the other is a sequence of discrete impulses $f_d(t) = \sum_i A \int f_i \delta(t - \tau_i) dt$, $\tau_i \in [t_i + \tau_c]$, applied at the moments when the system crosses the Poincaré section $u(t_i) = 1, \dot{u}(t_i) > 0$, i.e. coinciding with the laser spikes. Here A is a constant defined experimentally to transform the map force f_i to the control force $f_d(t)$ for the continuous system (1); τ_c is the duration of an impulse of amplitudes Af_i. In other words, in the case of the discrete impulse force we first solve the control problem for the map (2) and then apply our results for the system (1) by determination of the constant A. We must note that for the selected Poincaré section $u(t_i) = 1, \dot{u}(t_i) > 0$ the duration of control pulses τ_c should be less than or comparable with the duration of the laser spikes $\sim v$; in the opposite case the map (2) with the force f_i is not valid.

The following energy-optimal control problem is considered: How can the system (1) with unconstrained control function $f_c(t)$ or $f_d(t)$ be steered between coexisting states such that the "cost" functional J_c or J_d

$$J_c = \inf_{f \in F} \frac{1}{2} \int_{t_0}^{t_1} f^2(t) dt, \qquad J_d = \inf_{f \in F} \frac{1}{2} \sum_{i=1}^{N} f_i^2 \tag{5}$$

is minimized? Here t_1 (or N) is unspecified and F is the set of control functions.

The solution of this problem is in general a very complicated task. It was shown in our earlier research,[11, 21] however, that if a solution of the control problem $(\bar{u}(t), \mathbf{q}(t))$ exists, then it can be identified with the solution of the corresponding problem of optimal fluctuational escape, and Pontryagin's Hamiltonian (see e.g.[22]) of the control theory can be identified with the Wentzel-Freidlin Hamiltonian[4] of the theory of fluctuations. This interrelationship is intuitively clear because, in thermal equilibrium ($D = 4\Gamma k_B T$), the probability of fluctuations is determined by the minimum work from the external source needed to produce the given change in the thermodynamic quantities $\rho \propto \exp(-R_{\min}/k_B T)$.[23] It was therefore suggested that the optimal control function $\bar{u}(t)$ can be found via statistical analysis of the optimal fluctuational force.[11, 21, 24, 25] Similar ideas are applicable to the analysis of the control problem in maps.[26] In this technique the control functions $f(t)$ and f_i in (1) and (2) are replaced by additive white Gaussian noise and the dynamics of the system is followed continuously. Several dynamical variables of the system and the random force are recorded simultaneously, and the statistics of all actual trajectories along which the system moves in a particular subspace of the coordinate space is then analyzed.[27–29] The so-called *prehistory probability distribution* of these trajectories that move the system from the equilibrium state to the remote state is sharply peaked about the optimal fluctuational path, thereby providing a solution to the control problem. We note, that this technique provides an experimental approach to the solution of the control problem, which is especially useful from the point of view of practical applications.

In the case of continuous control, it is useful to change variables

$$\begin{gathered} \epsilon = v^{-1/2}, \quad \tau = \epsilon^{-1} t, \quad \Omega = \epsilon \omega, \\ z = \epsilon^{-1}(y - 1), \quad x = \ln u. \end{gathered}$$

Following the Pontryagin theory of optimal control, we then reduce the energy-minimal migration task to boundary problems for the Hamilton equation (cf.[11, 21])

$$\begin{aligned} \dot{x} &= z, \\ \dot{z} &= q - 1 + k\cos(\Omega\tau) - e^x(1 + \epsilon z) - \epsilon z + p_2, \\ \dot{p}_1 &= p_2 e^x (1 + \epsilon z), \\ \dot{p}_2 &= -p_1 + p_2 \epsilon (1 + e^x), \end{aligned} \tag{6}$$

with the boundary conditions

$$x(\tau_s) = x_s, \quad z(\tau_s) = z_s, \quad p_1(\tau_s) = 0, \quad p_2(\tau_s) = 0, \tag{7}$$

$$x(\tau_e) = x_e, \quad z(\tau_e) = z_e, \quad p_1(\tau_e) = 0, \quad p_2(\tau_e) = 0, \tag{8}$$

where $\{x_s, z_s, \tau_s\}$ and $\{x_e, z_e, \tau_e\}$ are initial and final states correspondingly, with $\tau_s \to -\infty$ and $\tau_e \to \infty$.

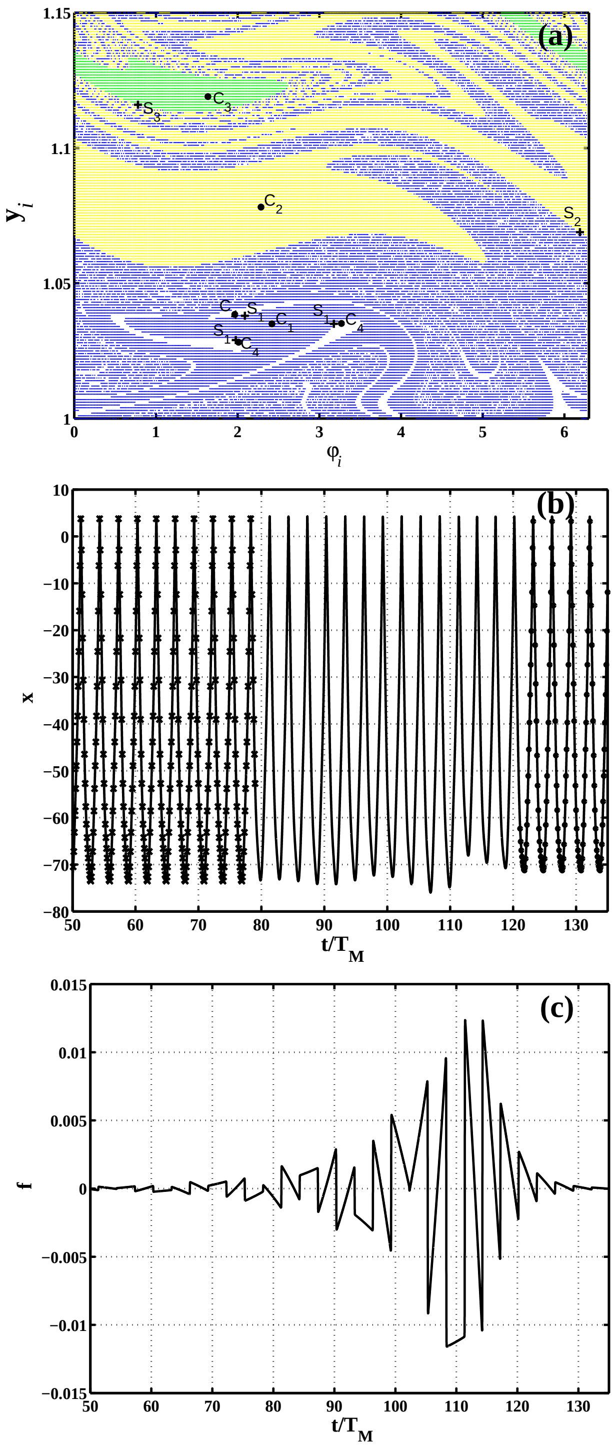

Figure 1. (a) Basins of attraction for the flow system (1). Time realizations of the coordinate $x(t)$ (b) and control force $f(t)$ (c) are shown during migration from C_3 to S_3. The stable cycle C_3 and saddle cycle S_3 are marked by "×" and "•" respectively.

The solution of the continuous control problem for transitions $C_3 \to S_3$ was found by application of a combination of the statistical approach discussed above and the numerical solution of the boundary problem for the Hamiltonian system (6). The Poincarè cross-section for the system (1) and corresponding solution of the control problem for transitions $C_3 \to S_3$ $(x(\tau), p_2(\tau))$ are shown in Fig. 1.

Numerical simulations confirm that the control function $f(\tau) = p_2(\tau)$ obtained induces migration from the cycle C_3 to the cycle S_3 in the optimal regime. Similar results are obtained for transition $C_2 \to S_2$.

In the case of discrete time control the Pontryagin theory of optimal control can be extended to obtain an area-preserving map:

$$
\begin{aligned}
&c_{i+1} = q + G(c_, \psi_i)e^{-T} + K\cos(\omega T + \psi_i) + p^c_{i+1} \\
&\varphi_{i+1} = \varphi_i + \omega T, \quad \mathrm{mod} 2\pi, \\
&\begin{pmatrix} p^c_{i+1} \\ p^\varphi_{i+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial c_{i+1}}{\partial c_i} & \frac{\partial c_{i+1}}{\partial \varphi_i} \\ \frac{\partial \varphi_{i+1}}{\partial c_i} & \frac{\partial \varphi_{i+1}}{\partial \varphi_i} \end{pmatrix}^{-1} \begin{pmatrix} p^c_i \\ p^\varphi_i \end{pmatrix}
\end{aligned}
\tag{9}
$$

with the boundary conditions:

$$
\begin{cases} c_s = & c^s, \quad \varphi_s = \varphi^s, \quad p^c_s = 0, \quad p^\varphi_s = 0, \\ c_e = & c^e, \quad \varphi_e = \varphi^e, \quad p^c_e = 0, \quad p^\varphi_e = 0, \end{cases} \tag{10}
$$

where $s = -\infty$ and $e = \infty$ are initial and final time moments.

The corresponding boundary value problem (9), (10) for transitions $C_3 \to S_3$ can be solved using either the statistical approach or a shooting method. However, because of the complexity of the map, the accuracy of both methods is limited and only allows us to identify a nearly optimal discrete control function.

The results of such an analysis for the transition $C_3 \to S_3$ in the map (2) are shown in Fig. 2. The statistical approach gives superior results, the corresponding energy $J_{stat} \approx 8.1 \times 10^{-3}$, being less than the energy found by the shooting method, $J_{shoot} \approx 9.3 \times 10^{-3}$ (and twice smaller then the energy of a single control pulse $J_{single} = 1.6 \times 10^{-2}$). Similar results are obtained from analysis of the optimal transition $C_2 \to S_2$.

In the next step we defined experimentally a value of the constant A for all the types of pulsed control function considered, in order to apply the map results to the continuous system (1). The results are summarized in the Table 1.

Numerical analysis has also revealed the dependence of the energy of the discrete control function $f_d(t)$ on the duration of the control pulses τ_c. It was found that there is an optimal duration τ_c^{opt} for which the total energy of the control function has a minimum. Note that the optimal duration is still less than the duration of the laser spike, so that the map (2) is applicable.

4. DISCUSSION: DISCRETE PULSES VERSUS CONTINUOUS CONTROL

A direct comparison between continuous and different discrete methods of control is given in Table 1. It follows from our analysis (see Table 1) that continuous control is considerably more efficient energetically then the discrete form of control. In turn a discrete control function consisting of a sequence of pulses (multi-pulsed force) is more efficient then single pulse control function. That can be explained by the small duration τ of impulses of discrete force. As can be seen in Table 1 the energies of the discrete control functions can be decreased by several order of magnitude by optimization of impulse duration and the location of the impulse. But the optimized energy of the discrete control function still exceeds the energy of the continuous force by three orders of magnitude.

Thus we see that, although the continuous and discrete models of a laser describe very well the system dynamics on larger time scales and provide quantitatively similar basins of attractions for the stable limit cycles, estimates of stability and of the energies of optimal control functions may differ by orders of magnitude. One of possible reason for this lies in the particular form of control function f_i in the map (2): we choose the force that is additive in the map. So we can expect another conclusion for other form of control function f_i. The

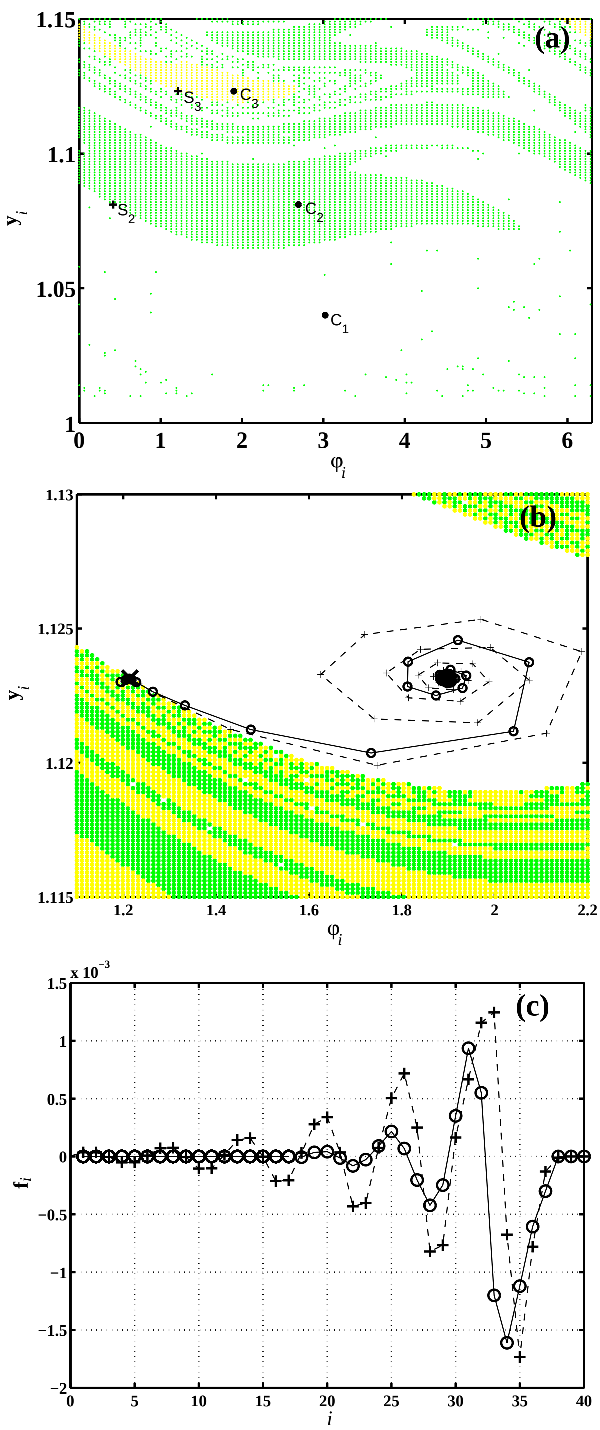

Figure 2. (a) Basins of attraction of stable limit cycles of the map (2). (b) Migration trajectory of map (2), and (c) realizations of the control force inducing migration from the cycle C_3 to the cycle S_3. Dashed lines and marker "+" correspond to solution of boundary problem, the full line and "○"s indicate the solution based on fluctuational prehistory analysis.

Transition	J_c	J_d^H	J_d^p	J_i	J_d^{opt}	J_i^{opt}	J_d^{pnew}	J_i^{new}
$C_3 \to S_3$	0.00004	0.1338	0.08275	0.1607	0.0505	0.00275	0.000077	0.000103

Table 1. The energy of the optimal control function for controlling migration between cycles in the continuous system (1) (in dimensionless units). J_c corresponds to the continuous function obtained by solution of the boundary problem (7)-(8) for the system (6). The energies J_d^H and J_d^p correspond to multi-pulsed control functions determined by solving the boundary problem (10) for the map (2) and by prehistory approach respectively. The energy J_i corresponds to a single pulse function. The energies J_d^H, J_d^p and J_i were determined for very short-duration control pulses $\tau_c \ll T_M$. Here T_M is the period of the external pumping. The ultra-short time duration τ_c is used to approximate a δ-function. The energies J_d^{opt} and J_i^{opt} correspond to the multi-pulsed and single-pulse control functions, correspondingly, with an optimal duration τ_c^{opt}. The energies J_d^{pnew} and J_i^{new} were determined for the map (11) and correspond to a multi-pulsed and single-pulse control functions, respectively.

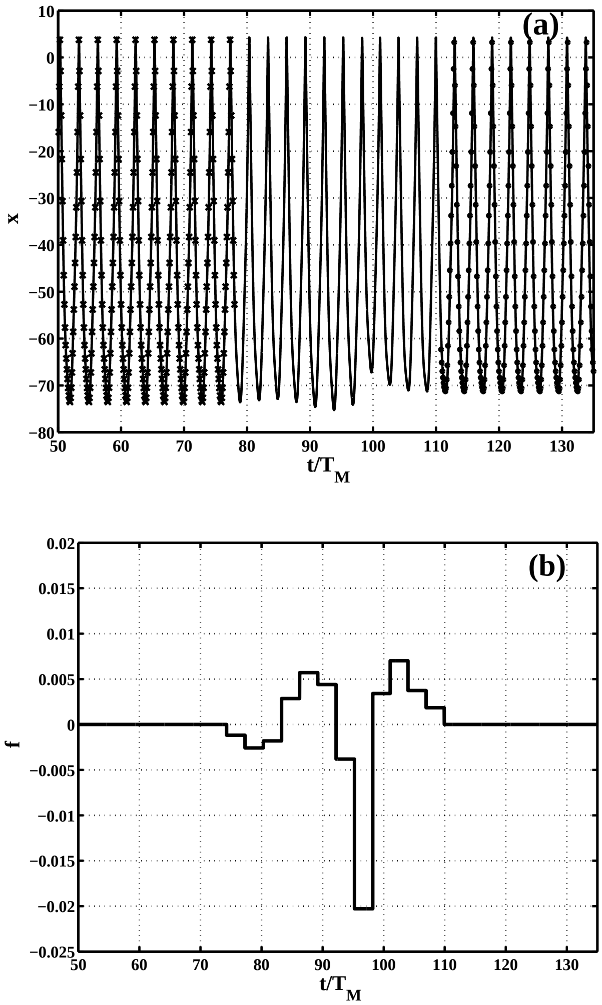

Figure 3. Time realizations of coordinate $x(t)$ (a) and control force $f(t)$ (b) are shown during migration from C_3 to S_3. The control function was obtained by prehistory analysis of the map (11). The stable cycle C_3 and saddle cycle S_3 are marked by "$\times$" and "$\bullet$" correspondingly.

alternative form can be identified through a detailed inspection of the continuous function $f(t)$ in Fig. 1(c). It is seen that the force $f(t)$ has a rather complex structure, but that it can be fitted by a sequence of pulses. The duration of these pulses is close to the period of the cycle C_2 or to the time interval between laser spikes, i.e. it is $\sim 3T_M$, T_M is the period of external driving. Further we note that external driving changes the value of pump, i.e. the parameter q. This observation allows us to suggest a new form of control for the map:

$$\begin{cases} c_{i+1} = & q_i + G(c_i, \psi_i, q_i)e^{-T} + K\cos(\omega T + \psi_i) + f_i, \\ \varphi_{i+1} = & \varphi_i + \omega T, \bmod 2\pi, \\ q_{i+1} = & q_i + f_i, \end{cases} \tag{11}$$

here the value of q is constant between the Poincaré sections and it changes at the moment of the cross-section. For this map (11) we formulate the same boundary problem as for the map (2), we can use the same method to determine the control function. The results of a prehistory analysis (Fig. 3) and the result for single pulse in the map 11) are presented in two last columns of Table 1. The energy of the control function is significantly decreased but it is still nearly twice as large as the energy of the continuous function.

Summarizing, we have found the energy-optimal control function for effecting migration of a B-class laser from its stable limit cycle to a saddle cycle, for both continuous and discrete forms of control. This allowed us to compare directly the efficiency of the two technique and to show that continuous control provides a more energy-efficient form of control as compared to discrete control, and that it provides more accurate estimates of the stability of quasi-stable states. Moreover analysis of the shape of the continuous function allows us to suggest the discrete control function that approaches continuous efficiency. We note that specific targeting of the periodic orbits has been achieved experimentally by single-shot short-lived perturbation of intracavity losses in a CO_2 laser[30, 31] that can be described by the continuous and discrete laser equations used in this paper. The results obtained can be directly verified in experiment, therefore, and applied e.g. to the phase coding information scheme.

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