C-SUPPLEMENTED SUBALGEBRAS OF LIE ALGEBRAS

DAVID A. TOWERS

Abstract

A subalgebra $B$ of a Lie algebra $L$ is $c$-supplemented in $L$ if there is a subalgebra $C$ of $L$ with $L = B + C$ and $B \cap C \leq B_L$, where $B_L$ is the core of $B$ in $L$. This is analogous to the corresponding concept of a c-supplemented subgroup in a finite group. We say that $L$ is $c$-supplemented if every subalgebra of $L$ is $c$-supplemented in $L$. We give here a complete characterisation of c-supplemented Lie algebras over a general field.

Mathematics Subject Classification 2000: 17B05, 17B20, 17B30, 17B50.

Key Words and Phrases: Lie algebras, c-supplemented subalgebras, completely factorisable algebras, Frattini ideal, subalgebras of codimension one.

1 Introduction

The concept of a c-supplemented subgroup of a finite group was introduced by Ballester-Bolinches, Wang and Xiuyun in [2] and has since been studied by a number of authors. The purpose of this paper is study the corresponding idea for Lie algebras. As we shall see, stronger results can be obtained in this context.

Throughout $L$ will denote a finite-dimensional Lie algebra over a field $F$. If $B$ is a subalgebra of $L$ we define $B_L$, the core (with respect to $L$) of $B$ to be the largest ideal of $L$ contained in $B$. We say that $B$ is core-free in $L$ if $B_L = 0$. A subalgebra $B$ of $L$ is $c$-supplemented in $L$ if there is a subalgebra $C$ of $L$ with $L = B + C$ and $B \cap C \leq B_L$. We say that $L$ is $c$-supplemented
if every subalgebra of $L$ is c-supplemented in $L$. We shall give a complete
characterisation of c-supplemented Lie algebras over a general field.

Following [4] we will say that $L$ is completely factorisable if for every
subalgebra $B$ of $L$ there is a subalgebra $C$ such that $L = B + C$ and $B \cap C = 0$. It turns out that c-supplemented Lie algebras are intimately related to
the completely factorisable ones, and our results generalise some of those
obtained in [4]. Incidentally, it is claimed in [4] that if $F$ has characteristic
zero then $L$ is completely factorisable if and only if the Frattini subalgebra
of every subalgebra of $L$ is trivial. We shall see that this is false.

If $A$ and $B$ are subalgebras of $L$ for which $L = A + B$ and $A \cap B = 0$
we will write $L = A \dot{+} B$; if, furthermore, $A, B$ are ideals of $L$ we write
$L = A \oplus B$. The notation $A \leq B$ will indicate that $A$ is a subalgebra of $B,$
and $A < B$ will mean that $A$ is a proper subalgebra of $B$.

\section{Preliminary results}

First we give some basic properties of c-supplemented subalgebras

\textbf{Lemma 2.1} \hspace{2mm} (i) If $B$ is c-supplemented in $L$ and $B \leq K \leq L$ then $B$ is
c-supplemented in $K$.

(ii) If $I$ is an ideal of $L$ and $I \leq B$ then $B$ is c-supplemented in $L$ if and
only if $B/I$ is c-supplemented in $L/I$.

(iii) If $X$ is the class of all c-supplemented Lie algebras then $X$ is subalgebra
and factor algebra closed.

\textit{Proof.}

(i) Suppose that $B$ is c-supplemented in $L$ and $B \leq K \leq L$. Then there
is a subalgebra $C$ of $L$ with $L = B + C$ and $B \cap C \leq B_L$. It follows
that $K = (B + C) \cap K = B + C \cap K$ and $B \cap C \cap K \leq B_L \cap K \leq B_K$, 
and so $B$ is c-supplemented in $K$.

(ii) Suppose first that $B/I$ is c-supplemented in $L/I$. Then there is a
subalgebra $C/I$ of $L/I$ such that $L/I = B/I + C/I$ and $(B/I) \cap
(C/I) \leq (B/I)_{L/I} = B_L/I$. It follows that $L = B + C$ and $B \cap C \leq B_L$,
whence $B$ is c-supplemented in $L$.
Suppose conversely that \( I \) is an ideal of \( L \) with \( I \leq B \) such that \( B \) is \( c \)-supplemented in \( L \). Then there is a subalgebra \( C \) of \( L \) such that \( L = B + C \) and \( B \cap C \leq B_L \). Now \( L/I = B/I + (C + I)/I \) and \((B/I) \cap (C + I)/I = (B \cap (C + I))/I = (I + B \cap C)/I \leq B_L/I = (B/I)_{L/I} \), and so \( B/I \) is \( c \)-supplemented in \( L/I \).

(iii) This follows immediately from (i) and (ii).

The Frattini ideal of \( L \), \( \phi(L) \), is the largest ideal of \( L \) contained in all maximal subalgebras of \( L \). We say that \( L \) is \( \phi \)-free if \( \phi(L) = 0 \). The next result shows that subalgebras of the Frattini ideal of a \( c \)-supplemented Lie algebra \( L \) are necessarily ideals of \( L \).

Proposition 2.2 Let \( B, D \) be subalgebras of \( L \) with \( B \leq \phi(D) \). If \( B \) is \( c \)-supplemented in \( L \) then \( B \) is an ideal of \( L \) and \( B \leq \phi(L) \).

Proof. Suppose that \( L = B + C \) and \( B \cap C \leq B_L \). Then \( D = D \cap L = D \cap (B + C) = B + D \cap C = D \cap C \) since \( B \leq \phi(D) \). Hence \( B \leq D \leq C \), giving \( B = B \cap C \leq B_L \) and \( B \) is an ideal of \( L \). It then follows from [6, Lemma 4.1] that \( B \leq \phi(L) \).

The Lie algebra \( L \) is called elementary if \( \phi(B) = 0 \) for every subalgebra \( B \) of \( L \); it is an \( E \)-algebra if \( \phi(B) \leq \phi(L) \) for all subalgebras \( B \) of \( L \). Then we have the following useful corollary.

Corollary 2.3 If \( L \) is \( c \)-supplemented then \( L \) is an \( E \)-algebra.

Proof. Simply put \( B = \phi(D) \) in Proposition 2.2.

It is clear that if \( L \) is completely factorisable then it is \( c \)-supplemented. However, the converse is false. Every completely factorisable Lie algebra must be \( \phi \)-free, whereas the same is not true for \( c \)-supplemented algebras. For example, the three-dimensional Heisenberg algebra is \( c \)-supplemented, as will be clear from the next result which gives the true relationship between these two classes of algebras.

Proposition 2.4 Let \( L \) be a Lie algebra. Then the following are equivalent:
(i) $L$ is c-supplemented.

(ii) $L/\phi(L)$ is completely factorisable and every subalgebra of $\phi(L)$ is an ideal of $L$.

Proof. (i) $\Rightarrow$ (ii): Suppose first that $L$ is $\phi$-free and c-supplemented, and let $B$ be a subalgebra of $L$. Then there is a subalgebra $C$ of $L$ such that $L = B + C$. Choose $D$ to be a subalgebra of $L$ minimal with respect to $L = B + D$. Then $B \cap D \leq \phi(D)$, by [6, Lemma 7.1], whence $B \cap D = 0$ since $L$ is elementary, by Corollary 2.3. Hence $L$ is completely factorisable, and (ii) follows from Lemma 2.1(iii) and Proposition 2.2.

(ii) $\Rightarrow$ (i): Suppose that (ii) holds and let $B$ be a subalgebra of $L$. Then there is a subalgebra $C/\phi(L)$ of $L/\phi(L)$ such that $L/\phi(L) = (B + \phi(L))/\phi(L) + (C/\phi(L))$ and $0 = ((B + \phi(L))/\phi(L)) \cap (C/\phi(L)) = (B \cap C + \phi(L))/\phi(L)$. Hence $L = B + C$ and $B \cap C \leq \phi(L)$, so $B \cap C$ is an ideal of $L$ and $B \cap C \leq B_L$; that is, $L$ is c-supplemented.

Note that if $L$ is the three-dimensional Heisenberg algebra, then condition (ii) in the above result holds, since $\phi(L) = L^2$ is one dimensional and $L/\phi(L)$ is abelian. Finally we shall need the following result concerning direct sums of completely factorisable Lie algebras.

Lemma 2.5 If $A$ and $B$ are completely factorisable, then so is $L = A \oplus B$.

Proof. Suppose that $A, B$ are completely factorisable and put $L = A \oplus B$. Let $U$ be a subalgebra of $L$. If $A \leq U$, then $U = A \oplus (B \cap U)$. Since $B$ is completely factorisable there is a subalgebra $C$ of $B$ such that $B = B \cap U + C$ and $U \cap C = B \cap U \cap C = 0$. Hence $L = U + C$.

Now $A \leq A + U$ so, by the above, there is a subalgebra $C$ of $B$ with $L = A + U + C$ and $(A + U) \cap C = 0$. Moreover, since $A$ is completely factorisable, there is a subalgebra $D$ of $A$ such that $A = A \cap U + D$ and $U \cap D = A \cap U \cap D = 0$. It follows that $L = U + (D \oplus C)$ and $U \cap (D + C) \leq U \cap [(A + U) \cap (D + C)] = U \cap [D + (A + U) \cap C] = U \cap D = 0$. It follows that $L$ is completely factorisable.

Note that the corresponding result for c-supplemented Lie algebras is false. For, let $L_1 = Fx + Fy + Fz$ with $[x, y] = -[y, x] = y + z$, $[x, z] = -[z, x] = z$ and all others products equal to zero. Then it is straightforward to check that $\phi(L_1) = Fz$ and that $L_1$ is c-supplemented. Now take $L$ to be
a direct sum of two copies of $L_1$: say, $L = A \oplus B$ where $A = Fx + Fy + Fz$, $B = Fa + Fb + Fc$, $[x, y] = -[y, x] = y + z, [x, z] = -[z, x] = z, [a, b] = -[b, a] = b + c, [a, c] = -[c, a] = c$ and all others products equal to zero. Suppose that $F(z + c)$ is c-supplemented in $L$. Then there is a subalgebra $M$ of $L$ with $L = F(z + c) + M$ and $F(z + c) \cap M \leq (F(z + c))_L$. If $z + c \notin M$ then $M$ is a maximal subalgebra of $L$, contradicting the fact that $z + c \in (\phi(A) \oplus \phi(B)) = \phi(L)$, by [6, Theorem 4.8]. It follows that $z + c \in M$, whence $F(z + c)$ is an ideal of $L$. But $[x, z + c] = z \notin F(z + c)$, a contradiction. Thus $L$ is not c-supplemented in $L$.

3 The structure theorems

We can now give the main structure theorems for c-supplemented Lie algebras. First we determine the solvable ones.

**Theorem 3.1** Let $L$ be a solvable Lie algebra. Then the following are equivalent:

(i) $L$ is c-supplemented.

(ii) $L$ is supersolvable and every subalgebra of $\phi(L)$ is an ideal of $L$.

**Proof.** (i) $\Rightarrow$ (ii): We have that every subalgebra of $\phi(L)$ is an ideal of $L$ by Proposition 2.4, so we have only to show that $L$ is supersolvable. Let $L$ be a minimal counter-example. Then all proper subalgebras and factor algebras of $L$ are supersolvable, by Lemma 2.1(iii). If we can show that all maximal subalgebras have codimension one in $L$, we shall have the desired contradiction, by [3, Theorem 7]; so let $M$ be any maximal subalgebra of $L$. Since the result is clear if $M_L \neq 0$, we may assume that $M_L = 0$.

Pick a minimal ideal $A$ of $L$. Then $L = A \oplus M$ and $A$ is the unique minimal ideal of $L$, by [7, Lemma 1.4]. Let $a \in A$. Then $Fa$ is c-supplemented in $L$, and so there is a subalgebra $B$ of $L$ such that $L = Fa + B$ and $Fa \cap B \leq (Fa)_L$. If $a \in B$ then $Fa$ is an ideal of $L$, whence $A = Fa$ and $M$ has codimension one in $L$.

So suppose that $L = Fa + B$. Since $A \not\leq B$ we have $B_L = 0$. But then $L = A + B$ by [7, Lemma 1.4] again. It follows that $\dim A = 1$ and $M$ has codimension one in $L$. 

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(ii) ⇒ (i): By Proposition 2.4, it suffices to show that if \( L \) is supersolvable and \( \phi \)-free then it is completely factorisable. Let \( L \) be a minimal counterexample. Then \( L \) is elementary, by [5, Theorem 1], and so every proper subalgebra of \( L \) is completely factorisable. Also \( L = A + B \) where \( A = Fa_1 \oplus \ldots \oplus Fa_n \) is the abelian socle of \( L \) and \( B \) is abelian, by [7, Theorem 7.3]. Let \( U \) be a subalgebra of \( L \). If \( A \leq U \) it is clear that there is a subalgebra \( C \) of \( L \) such that \( L = U + C \) and \( U \cap C = 0 \). So suppose that \( a_i \notin U \) for some \( 1 \leq i \leq n \); we may as well assume that \( i = 1 \). Then \( L/Fa_1 \cong (Fa_2 \oplus \ldots \oplus Fa_n) + B \), which is a proper subalgebra of \( L \) and so is completely factorisable. Hence there is a subalgebra \( C \) of \( L \) such that \( L/Fa_1 = ((U + Fa_1)/Fa_1) + (C/Fa_1) \) and \( Fa_1 = (U + Fa_1) \cap C = U \cap C + Fa_1 \). It follows that \( L = U + C \) and \( U \cap C \leq Fa_1 \). But \( a_1 \notin U \cap C \) so \( U \cap C = 0 \) and \( L \) is completely factorisable, a contradiction.

We shall need the following classification of Lie algebras with core-free subalgebras of codimension one which is given by Amayo in [1].

**Theorem 3.2** ([1, Theorem 3.1]) Let \( L \) have a core-free subalgebra of codimension one. Then either (i) \( \dim L \leq 2 \), or else (ii) \( L \cong L_m(\Gamma) \) for some \( m \) and \( \Gamma \) satisfying certain conditions (see [1] for details).

We shall also need the following properties of \( L_m(\Gamma) \) which are given by Amayo in [1].

**Theorem 3.3** ([1, Theorem 3.2])

(i) If \( m > 1 \) and \( m \) is odd, then \( L_m(\Gamma) \) is simple and has only one subalgebra of codimension one.

(ii) If \( m > 1 \) and \( m \) is even, then \( L_m(\Gamma) \) has a unique proper ideal of codimension one, which is simple, and precisely one other subalgebra of codimension one.

(iii) \( L_1(\Gamma) \) has a basis \( \{u_{-1}, u_0, u_1\} \) with multiplication \( [u_{-1}, u_0] = u_{-1} + \gamma_0 u_1 \) (\( \gamma_0 \in F, \gamma_0 = 0 \) if \( \Gamma = \{0\} \)), \( [u_{-1}, u_1] = u_0, [u_0, u_1] = u_1 \).

(iv) If \( F \) has characteristic different from two then \( L_1(\Gamma) \cong L_1(0) \cong \mathfrak{sl}_2(F) \).
(v) If $F$ has characteristic two then $L_1(\Gamma) \cong L_1(0)$ if and only if $\gamma_0$ is a square in $F$.

The above properties enable us to determine which of the algebras $L_m(\Gamma)$ are c-supplemented.

**Proposition 3.4** If $L \cong L_m(\Gamma)$ then $L$ is c-supplemented if and only if $L \cong L_1(0)$ and $F$ has characteristic different from two.

**Proof.** Suppose that $L \cong L_m(\Gamma)$ and $L$ is c-supplemented, and let $x \in L$. Then there is a subalgebra $M_1$ of $L$ such that $L = Fx + M_1$, and $Fx \cap M_1 \leq (Fx)_L = 0$, since $L_m(\Gamma)$ has no one-dimensional ideals. Choose $y \in M_1$. Then, similarly, there is a subalgebra $M_2$ of codimension one in $L$ such that $L = Fy + M_2$ and $M_1 \neq M_2$. Since $L = M_1 + M_2$ we have that $M_1 \cap M_2 \neq 0$. Let $z \in M_1 \cap M_2$. Then there is a subalgebra $M_3$ of codimension one in $L$ such that $L = Fz + M_3$, so $L$ has at least three subalgebras of codimension one in $L$. It follows from Theorem 3.3 that $m = 1$.

Suppose that $L \not\cong L_1(0)$. Then $F$ has characteristic two and $\gamma_0$ is not a square in $F$. Since $L$ is completely factorisable there is a two-dimensional subalgebra $M$ of $L$ such that $L = Fu_1 + M$. It follows that $M = F(u_{-1} + \alpha u_1) + F(u_0 + \beta u_1)$ for some $\alpha, \beta \in F$. But then $[u_{-1} + \alpha u_1, u_0 + \beta u_1] \in M$ shows that $\gamma_0 = \beta^2$, a contradiction. A further straightforward calculation shows that if $L \cong L_1(0)$ and $F$ has characteristic two, then $Fu_1$ is contained in every maximal subalgebra of $L$, and so has no c-supplement in $L$.

Conversely, suppose that $L \cong L_1(0)$ and $F$ has characteristic different from two. Then $L \cong sl_2(F)$, by Theorem 3.3 (iv) and it is easy to check that $L$ is c-supplemented.

We can now determine the simple and semisimple c-supplemented Lie algebras.

**Corollary 3.5** If $L$ is simple then $L$ is c-supplemented if and only if $L \cong L_1(0)$ and $F$ has characteristic different from two.

**Proof.** Let $L$ be simple and c-supplemented. Then $L$ has a core-free maximal subalgebra of codimension one in $L$ and so $L \cong L_m(\Gamma)$, by Theorem 3.2. The result now follows from Proposition 3.4.
Notice, in particular, that \( \mathfrak{sl}_2(F) \) is the only simple completely factorisable Lie algebra over any field. However, this is not the only simple elementary Lie algebra, even over a field of characteristic zero: over the real field every compact simple Lie algebra, and \( \mathfrak{so}(n, 1) \) for \( n > 3 \), for example, are elementary, as is shown in [8, Theorem 5.1]. This justifies the assertion made at the end of the third paragraph of the introduction.

**Proposition 3.6** Let \( L \) be a semisimple Lie algebra over a field \( F \). Then the following are equivalent:

(i) \( L \) is c-supplemented.

(ii) \( L = S_1 \oplus \ldots \oplus S_n \) where \( S_i \cong \mathfrak{sl}_2(F) \) for \( 1 \leq i \leq n \) and \( F \) has characteristic different from two.

**Proof.** (i) \( \Rightarrow \) (ii): Let \( L \) be semisimple and c-supplemented and suppose the result holds for all such algebras of dimension less than \( \dim L \). Then \( \phi(L) = 0 \), since \( \phi(L) \) is nilpotent, and so \( L \) is completely factorisable. Let \( A \) be a minimal ideal of \( L \) and pick \( a \in A \). Let \( M \) be a subalgebra of \( L \) such that \( L = Fa + M \) and put \( B = A + M_L \). Then \( M_L < B \) and \( A \cap M_L = 0 \), since \( a \notin M_L \). If \( \dim L/M_L \leq 2 \) then \( A \) is abelian, contradicting the fact that \( L \) is semisimple. It follows from Theorem 3.2 and Proposition 3.4 that \( L/M_L \cong L_1(0) \), whence \( B = L \) and \( L = A \oplus M_L \). Since \( A, M_L \) are semisimple and c-supplemented the result follows.

(ii) \( \Rightarrow \) (i): The converse follows from Corollary 3.5 and Lemma 2.5.

Finally we have the main classification theorem.

**Theorem 3.7** Let \( L \) be Lie algebra. Then the following are equivalent:

(i) \( L \) is c-supplemented.

(ii) \( L/\phi(L) = R \oplus S \) where \( R \) is supersolvable and \( \phi \)-free, \( S \) is given by Proposition 3.6, and every subalgebra of \( \phi(L) \) is an ideal of \( L \).

**Proof.** (i) \( \Rightarrow \) (ii): Factor out \( \phi(L) \) so that \( L \) is \( \phi \)-free and c-supplemented and hence completely factorisable, by Proposition 2.4. Then \( L = R \oplus S \) where \( R \) is the radical of \( L \) and \( S \) is semisimple. It suffices to show that \( SR = 0 \); the rest follows from Lemma 2.1, Corollary 2.3, Proposition 2.4, Theorem
3.1 and Proposition 3.6. Suppose there is $0 \neq x \in L^{(3)} \cap R$. Then there is a subalgebra $M$ of $L$ such that $L = Fx + M$ and $L/M_L$ is given by Theorem 3.2. If $L/M_L \cong L_m(\Gamma)$ then $L/M_L$ is simple, by Proposition 3.4, and $M_L < R + M_L$. But then $L/M_L$ is solvable, a contradiction. It follows that $\dim L/M_L \leq 2$, whence $x \in L^{(3)} \cap R \leq L^{(3)} \leq M_L \leq M$, a contradiction. Hence $L^{(3)} \cap R = 0$. But $SR = S^2R \leq S(SR) = S^2(SR) \leq L^{(3)} \cap R = 0$, as required.

(ii) $\Rightarrow$ (i): This follows from Proposition 2.4, Lemma 2.5, Theorem 3.1 and Proposition 3.6.

References


