Supplementary Material for “Convergence of Regression Adjusted Approximate Bayesian Computation”

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1. NOTATIONS AND SET-UP

First some limit notations and conventions are given. For two sets \( A \) and \( B \), the sum of integrals \( \int_A f(x) \, dx + \int_B f(x) \, dx \) is written as \( (\int_A + \int_B) f(x) \, dx \). For a constant \( d \times p \) matrix \( A \), let the minimum and maximum eigenvalues of \( A^T A \) be \( \lambda_{\min}^2(A) \) and \( \lambda_{\max}^2(A) \) where \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) are non-negative. Obviously, for any \( p \)-dimension vector \( x \), \( \lambda_{\min}(A) \|x\| \leq \|Ax\| \leq \lambda_{\max}(B) \). For two matrices \( A \) and \( B \), we say \( A \) is bounded by \( B \) or \( A \leq B \) if \( \lambda_{\max}(A) \leq \lambda_{\min}(B) \). For a set of matrices \( \{A_i : i \in I\} \) for some index set \( I \), we say it is bounded if \( \max_{i \in I} \lambda_{\max}(A_i) \) are uniformly bounded in \( i \). Denote the identity matrix with dimension \( d \) by \( I_d \). Notations from the main text will also be used.

The following basic asymptotic results (Serfling, 2009) will be used throughout.

**Lemma 6.** (i) For a series of random variables \( Z_n \), if \( Z_n \rightarrow Z \) in distribution as \( n \rightarrow \infty \), \( Z_n = O_p(1) \). (ii) (Continuous mapping) For a series of continuous function \( g_n(x) \), if \( g_n(x) = O(1) \) almost everywhere, then \( g_n(Z_n) = O_p(1) \), and this also holds if \( O(1) \) and \( O_p(1) \) are replaced by \( \Theta(1) \) and \( \Theta_p(1) \).

Some notations regarding the posterior distribution of approximate Bayesian computation are given. For \( A \subset \mathbb{R}^p \) and a scalar function \( h(\theta, s) \), let

\[
\pi_A(h) = \int_A \int_{\mathbb{R}^d} h(\theta, s) \pi(\theta) f_n(s \mid \theta) K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\} \varepsilon_n^{-d} \, ds \, d\theta,
\]

and

\[
\pi_A(h) = \int_A \int_{\mathbb{R}^d} h(\theta, s) \pi_\delta(\theta) \tilde{f}_n(s \mid \theta) K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\} \varepsilon_n^{-d} \, ds \, d\theta.
\]

Then \( \Pi_\varepsilon(\theta \in A \mid s_{\text{obs}}) = \pi_A(1)/\pi(1) \) and its normal counterpart \( \tilde{\Pi}_\varepsilon(\theta \in A \mid s_{\text{obs}}) = \pi_A(1)/\tilde{\pi}(1) \).

The following results from Li & Fearnhead (2015) will be used throughout.

**Lemma 7.** Assume Conditions 1–4. Then as \( n \rightarrow \infty \),

(i) if Condition 5 also holds then, for any \( \delta < \delta_0 \), \( \pi_{B_\delta}(1) \) and \( \tilde{\pi}_{B_\delta}(1) \) are \( o_p(1) \), and \( O_p(e^{-a_n c_\delta c_\delta}) \) for some positive constants \( c_\delta \) and \( \alpha_\delta \) depending on \( \delta \).
(ii) \( \pi_{B_δ}(1) = \bar{\pi}_{B_δ}(1)\{1 + O_p(\alpha_n^{-1})\} \) and \( \sup_{A \in B_δ} |\pi_A(1) - \bar{\pi}_A(1)| / \bar{\pi}_{B_δ}(1) = O_p(\alpha_n^{-1}) \);

(iii) if \( \varepsilon_n = o(a_n^{-1/2}) \), \( \pi_{B_δ}(1) \) and \( \pi_{B_δ}(1) \) are \( \Theta_p(a_n^{-d-p}) \), and thus \( \bar{\pi}_p(1) \) and \( \bar{\pi}_p(1) \) are \( \Theta_p(a_n^{-d-p}) \);

(iv) if \( \varepsilon_n = o(a_n^{-1/2}) \) and Condition 5 holds, \( \theta_\varepsilon = \bar{\theta}_\varepsilon + o_p(a_n^{-1/2}) \). If \( \varepsilon_n = o(a_n^{-3/5}) \), \( \theta_\varepsilon = \bar{\theta}_\varepsilon + o_p(a_n^{-1}) \).

Proof. (i) is from Li & Fearnhead (2015, Lemma 3) and a trivial modification of its proof when Condition 5 does not hold; (ii) is from Li & Fearnhead (2015, equation 13 of supplements); (iii) is from Li & Fearnhead (2015, Lemma 5 and equation 13 of supplements); and (iv) is from Li & Fearnhead (2015, Lemma 3 and Lemma 6).

2. PROOF FOR RESULTS IN SECTION 3.1

Proof of Lemma 1. For any fixed \( v \in \mathbb{R}^d \), recall that \( \Pi(\theta \in A \mid s_{\text{obs}} + \varepsilon_n v) \) is the posterior distribution given \( s_{\text{obs}} + \varepsilon_n v \) with prior \( \pi_\delta(\theta) \) and the misspecified model \( f_n(\cdot \mid \theta) \). By Kleijn & van der Vaart (2012), if there exist \( \Delta_{n,0} \) and \( V_{\theta_0} \) such that,

(KV1) for any compact set \( K \subset t(B_δ) \),

\[
\sup_{t \in K} \left| \log \frac{\tilde{f}_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0 + a_n^{-1} t)}{f_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0)} - t^T V_{\theta_0} \Delta_{n,0} + \frac{1}{2} t^T V_{\theta_0} t \right| \to 0,
\]

in probability as \( n \to \infty \), and

(KV2) \( E(\Pi(a_n\|\theta - \theta_0\| \mid s_{\text{obs}} + \varepsilon_n v)) \to 0 \) as \( n \to \infty \) for any sequence of constants \( M_n \to \infty \),

then

\[
\sup_{A \in \partial B_p} \left| \Pi(a_n(\theta - \theta_0) \in A \mid s_{\text{obs}} + \varepsilon_n v) - \int_A N(t; \Delta_{n,0}, V_{\theta_0}^{-1}) dt \right| \to 0,
\]

in probability as \( n \to \infty \).

For (KV1), by the definition of \( \tilde{f}_n(s \mid \theta) \),

\[
\log \frac{\tilde{f}_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0 + a_n^{-1} t)}{f_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0)} = \log \frac{N(s_{\text{obs}} + \varepsilon_n v; s(\theta_0 + a_n^{-1} t), a_n^{-2} A(\theta_0 + a_n^{-1} t))}{N(s_{\text{obs}} + \varepsilon_n v; s(\theta_0), a_n^{-2} A(\theta_0))}.
\]

As \( x^T A x - y^T B y = x^T (A - B) x + (x - y)^T B (x + y) \), for vectors \( x \) and \( y \) and matrices \( A \) and \( B \), by applying a Taylor expansion on \( s(\theta_0 + x t) \) and \( A(\theta_0 + x t) \) around \( x = 0 \), the right hand side of above equation equals

\[
\{D s(\theta_0 + e_n^{(1)} t) t_j^T A(\theta_0)^{-1} \zeta_n(v, t) - \frac{a_n^{-1}}{2} \zeta_n(v, t)^T \left\{ \sum_{i=1}^p D \theta_i A^{-1}(\theta_0 + e_n^{(1)} t) t_i \right\} \zeta_n(v, t) + \frac{a_n^{-1}}{2} \left\{ D \log A(\theta_0 + e_n^{(3)} t) \right\}^T t,\]

where \( \zeta_n(v, t) = A(\theta_0)^{1/2} W_{\text{obs}} + a_n \varepsilon_n v - \frac{1}{2} D s(\theta_0 + e_n^{(1)} t) t \) and for \( j = 1, 2, 3 \), \( e_n^{(j)} \) is a function of \( t \) satisfying \( |e_n^{(j)}| \leq a_n^{-1} \) which is from the remainder of the Taylor expansions. Since
with the transformation 

\[ \beta \tilde{v} \]

seen that the expectation under 

\[ o \]

Then since for any 

\[ t \]

where 

\[ \pi \]

For (KV2), let 

\[ r_n(s \mid \theta_0) = \alpha_n f_n(s \mid \theta_0) - \tilde{f}_n(s \mid \theta_0) \]

Since \( r_n(s \mid \theta_0) \) is bounded by a function integrable in \( \mathbb{R}^d \) by Condition 4,

\[
E \{ \bar{\Pi}(a_n \| \theta - \theta_0 \| > M_n \mid s_{obs} + \varepsilon_n v) \} - \int_{\mathbb{R}^d} \bar{\Pi}(a_n \| \theta - \theta_0 \| > M_n \mid s + \varepsilon_n v) \tilde{f}_n(s \mid \theta_0) \, ds \\
\leq \alpha_n^{-1} \int_{\mathbb{R}^d} \| r_n(s \mid \theta_0) \| \, ds = o(1).
\]

Then it is sufficient for the expectation under \( \tilde{f}_n(s \mid \theta_0) \) to be \( o(1) \). For any constant \( M > 0 \), with the transformation \( \bar{v} = a_n \{ s - s(\theta_0) \} \),

\[
\int_{\mathbb{R}^d} \bar{\Pi}(a_n \| \theta - \theta_0 \| > M_n \mid s + \varepsilon_n v) \tilde{f}_n(s \mid \theta_0) \, ds \\
\leq \int \frac{\int_{\| u \| > M} \pi(t, \bar{v} \mid v) \, dt}{\int_{B_\delta} \pi(t, \bar{v} \mid v) \, dt} N \{ \bar{v}; 0, A(\theta_0) \} \, d\bar{v} + \int_{\| u \| > M} N \{ \bar{v}; 0, A(\theta_0) \} \, d\bar{v},
\]

where \( \bar{\pi}(t, \bar{v} \mid v) = \pi_\delta(\theta_0 + a_n^{-1} t) \tilde{f}_n \{ s(\theta_0) + a_n^{-1} \bar{v} + \varepsilon_n v \mid \theta_0 + a_n^{-1} t \} \). For the first term in the above upper bound, it is bounded by a series which does not depend on \( M \) and is \( o(1) \) as \( M_n \to \infty \), as shown below. Obviously \( \int_{t(B_\delta)} \bar{\pi}(t, \bar{v} \mid v) \, dt \) can be lower bounded for some constant \( m_\delta > 0 \). Choose \( \delta \) small enough such that \( D_s(\theta) \) and \( A(\theta)^{1/2} \) are bounded for \( \theta \in B_\delta \). Let \( \lambda_{\min} \) and \( \lambda_{\max} \) be their common bounds. When \( \| \bar{v} \| < M \) and \( M_n \) is large enough,

\[
\{ t : \| t \| > M_n \} \subset \left\{ t : \sup_{\theta \in B_\delta} \| D_s(\theta) t \| \geq \| a_n \varepsilon_n v + \bar{v} \| \right\}.
\]

Then since for any \( \bar{v} \) satisfying \( \| \bar{v} \| < M \), by a Taylor expansion,

\[
\tilde{f}_n \{ s(\theta_0) + a_n^{-1} \bar{v} + \varepsilon_n v \mid \theta_0 + a_n^{-1} t \} = a_n^d N \{ D_s(\theta_0 + \varepsilon_n(1) t) ; \bar{v} + a_n \varepsilon_n v, A(\theta_0 + a_n^{-1} t) \},
\]

\[
\bar{\pi}(t, \bar{v} \mid v) \leq c N(\lambda_{\max}^{-1} \lambda_{\min} \| t \| / 2; 0, 1),
\]

where \( c \) is some positive constant, for \( t \) in the right hand side of (1). Then

\[
\int \frac{\int_{\| u \| > M} \pi(t, \bar{v} \mid v) \, dt}{\int_{B_\delta} \pi(t, \bar{v} \mid v) \, dt} N \{ \bar{v}; 0, A(\theta_0) \} \, d\bar{v} \leq m_\delta^{-1} c \int_{\| u \| > M_n} N(\lambda_{\max}^{-1} \lambda_{\min} \| t \| / 2; 0, 1) \, dt,
\]

the right hand side of which is \( o(1) \) when \( M_n \to \infty \). Meanwhile by letting \( M \to \infty \), it can be seen that the expectation under \( \tilde{f}_n(s \mid \theta_0) \) is \( o(1) \). Therefore (KV2) holds and the lemma holds. \( \square \)

The following lemma is used for equations

\[
\int_{\mathbb{R}^p} g_n(t, v) \, dt = |A(\theta)|^{-1/2} G_n(v)
\]

and

\[
\int_{\mathbb{R}^p} g(t, v) \, dt = |A(\theta)|^{-1/2} G(v).
\]

**Lemma 8.** For a rank-\( p \) \( d \times p \) matrix \( A \), a rank-\( d \) \( d \times d \) matrix \( B \) and a \( d \)-dimension vector \( c \),

\[
N(At; Bv + c, I_d) = N \{ t; (A^T A)^{-1} A^T (c + Bv), (A^T A)^{-1} \} g(v; A, B, c),
\]

(2)
where $P = A^T A$, and
\[
g(v; A, B, c) = \frac{1}{(2\pi)^{(d-p)/2}} \exp \left\{ -\frac{1}{2} (c + Bv)^T (I - A(A^T A)^{-1} A^T)(c + Bv) \right\}.
\]

**Proof.** This can be verified easily by matrix algebra. \qed

The following lemma regarding the continuity of a certain form of integral will be helpful when applying the continuous mapping theorem.

**Lemma 9.** Let $l_1, l_1', l_2, l_2'$ and $l_3$ be positive integers satisfying $l_1' \leq l_1$ and $l_2' \leq l_2$. Let $A$ and $B$ be $l_1 \times l_1'$ and $l_2 \times l_2'$ matrices, respectively, satisfying that $A^T A$ and $B^T B$ are positive definite. Let $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$ be functions in $\mathbb{R}^{l_1}$, $\mathbb{R}^{l_2}$ and $\mathbb{R}^{l_3}$, respectively, that are integrable and continuous almost everywhere. Assume:

(i) $g_1(\cdot)$ is bounded in $\mathbb{R}^{l_1}$ for $j = 1, 2$;

(ii) $g_j(w)$ depends on $w$ only through $\|w\|$ and is a decreasing function of $\|w\|$, for $j = 1, 2$; and

(iii) there exists a non-negative integer $l$ such that $\int_{\mathbb{R}^{l_3}} \prod_{k=1}^{l_1+l_2+l} w_i g_3(w) \, dw < \infty$ for any coordinates $(w_1, \ldots, w_{l_1' + l_2' + l})$ of $w$.

Then the function,
\[
\int \int \int P_l(w_1, w_2, w_3) |g_1(Aw_1 + x_1 w_2 + x_2 w_3 + x_3) - g_1(Aw_1)| g_2(Bw_2 + x_4 w_3 + x_5) g_3(w_3) \, dw_3 \, dw_2 \, dw_1,
\]

where $x_1 \in \mathbb{R}^{l_1 \times l_2}$, $x_2 \in \mathbb{R}^{l_1 \times l_3}$, $x_4 \in \mathbb{R}^{l_2 \times l_3}$, $x_3 \in \mathbb{R}^{l_1}$ and $x_5 \in \mathbb{R}^{l_2}$, is continuous almost everywhere.

**Proof.** Let $m_A$ and $m_B$ be the lower bound of $A$ and $B$ respectively. For any $(x_0, \ldots, x_0) \in \mathbb{R}^{l_1 \times l_2} \times \mathbb{R}^{l_1 \times l_3} \times \mathbb{R}^{l_2 \times l_3} \times \mathbb{R}^{l_1} \times \mathbb{R}^{l_2}$ such that the integrand in the target integral is continuous, consider any sequence $(x_n, \ldots, x_n)$ converging to $(x_0, \ldots, x_0)$. It is sufficient to show the convergence of the target function at $(x_0, \ldots, x_0)$. Let $V_A = \{ w_1 : \|Aw_1\|/2 \geq \sup(x_{n1}x_{n2}x_{n3}) \|x_{n1}w_2 + x_{n2}w_3 + x_{n3}\| \}$, $V_B = \{ w_2 : \|Bw_2\|/2 \geq \sup(x_{n4}x_{n5}) \|x_{n4}w_3 + x_{n5}\| \}$, $U_A = \{ w_1 : \|w_1\| \leq 4m_A^{-1}(\|x_{01}w_2\| + \|x_{02}w_3\| + \|x_{03}\|) \}$ and $U_B = \{ w_2 : \|w_2\| \leq 4m_B^{-1}(\|x_{04}w_3\| + \|x_{05}\|) \}$. We have $V_A \subset U_A$ and $V_B \subset U_B$. Then according to the following upper bounds and condition (iii),
\[
|g_1(Aw_1 + x_1 w_2 + x_2 w_3 + x_3) - g_1(Aw_1)| \leq g_1(Aw_1 + x_1 w_2 + x_2 w_3 + x_3) + g_1(Aw_1),
\]
\[
g_1(Aw_1 + x_1 w_2 + x_2 w_3 + x_3) \leq \bar{g}_1(m_A \|w_1\|/2) \mathbb{1}_{\{w_1 \in U_A\}} + \sup_{w \in \mathbb{R}^{l_1}} g_1(w) \mathbb{1}_{\{w \in U_A\}},
\]
\[
g_2(Bw_2 + x_4 w_3 + x_5) \leq \bar{g}_2(m_B \|w_2\|/2) \mathbb{1}_{\{w_2 \in U_B\}} + \sup_{w \in \mathbb{R}^{l_2}} g_2(w) \mathbb{1}_{\{w \in U_B\}},
\]

where $g_1(w) = \bar{g}_1(\|w\|)$ and $g_2(w) = \bar{g}_2(\|w\|)$, by applying the dominated convergence theorem, the target function at $(x_0, \ldots, x_0)$ converges to its value at $(x_0, \ldots, x_0)$. \qed

**Proof of Lemma 2.** The first part holds according to Lemma 5 of Li & Fearnhead (2015). For the second part, when $c_\varepsilon = \infty$, by the transformation $v' = v'(v, t)$,
\[
\int_{\mathbb{R}^d} \int_{t(B_\delta)} P_1(v) g_n(t, v) \, dt dv = \int_{\mathbb{R}^d} \int_{t(B_\delta)} P_1 \left\{ D_{\theta_0} t + \frac{1}{a_n \varepsilon_n} v' - \frac{1}{a_n \varepsilon_n} A(\theta_0)^{1/2} W_{obs} \right\} g'(t, v') \, dt dv'.
\]
By applying Lemma 9 and the continuous mapping theorem in Lemma 6 to the right hand side of the above when \( c_\varepsilon = \infty \), and to \( \int_{B_0} \int_{(B_0)} P_t(v) g_n(t, v) \, dt \, dv \) when \( c_\varepsilon < \infty \), and using \( \int_{B_0} g(t, v) \, dt = |A(\theta_0)|^{-1/2} G(v) \), the lemma holds. \( \square \)

**Proof of Lemma 3.** (a), (b) and the first part of (c) hold immediately by Lemma 7. The second part of (c) is stated in the proof of Theorem 1 of Li & Fearnowhead (2015). \( \square \)

**Lemma 10.** Assume conditions 1–5.

(i) If \( c_\varepsilon \in (0, \infty) \) then \( \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} \) and \( \tilde{\Pi}_\varepsilon \{ a_n(\theta - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \} \) have the same limit in distribution.

(ii) If \( c_\varepsilon = 0 \) or \( c_\varepsilon = 0 \infty \) then

\[
\sup_{A \in \mathcal{B}_0} \left| \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} - \tilde{\Pi}_\varepsilon \{ a_n(\theta - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \} \right| = o_p(1).
\]

(iii) If Condition 6 holds then

\[
\sup_{A \in \mathcal{B}_0} \left| \Pi_\varepsilon \{ a_n(\theta^* - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} - \tilde{\Pi}_\varepsilon \{ a_n(\theta^* - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \} \right| = o_p(1).
\]

**Proof.** Let \( \lambda_n = a_n(\theta_\varepsilon - \tilde{\theta}_\varepsilon) \), and by Lemma 3(c), \( \lambda_n = o_p(1) \). When \( c_\varepsilon \in (0, \infty) \), for any \( A \in \mathcal{B}_0 \), decompose \( \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} \) into the following three terms,

\[
\begin{align*}
\left[ \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} - \tilde{\Pi}_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} \right] \\
+ \left[ \tilde{\Pi}_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A + \lambda_n \mid s_{\text{obs}} \} - \tilde{\Pi}_\varepsilon \{ a_n(\theta - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \} \right] \\
+ \tilde{\Pi}_\varepsilon \{ a_n(\theta - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \}.
\end{align*}
\]

For (i) to hold, it is sufficient that the first two terms in the above are \( o_p(1) \). The first term is \( o_p(1) \) by Lemma 3. For the second term to be \( o_p(1) \), given the leading term of \( \tilde{\Pi}_\varepsilon \{ a_n(\theta - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \} \) stated in the proof of Proposition 1 in the main text, it is sufficient that

\[
\sup_{v \in \mathcal{B}_0} \left( \int_{A + \lambda_n} - \int_A \right) N\{t; \mu_n(v), I(\theta_0)^{-1}\} \, dt = o_p(1).
\]

This holds by noting that the left hand side of the above is bounded by \( (\int_{A + \lambda_n} - \int_A) c \, dt \) for some constant \( c \) and this upper bound is \( o_p(1) \) since \( \lambda_n = o_p(1) \). Therefore (i) holds.

When \( c_\varepsilon = 0 \) or \( \infty \), \( \sup_{A \in \mathcal{B}_0} \left| \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} - \tilde{\Pi}_\varepsilon \{ a_n(\theta - \tilde{\theta}_\varepsilon) \in A \mid s_{\text{obs}} \} \right| \) is bounded by

\[
\begin{align*}
\sup_{A \in \mathcal{B}_0} \left| \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} - \tilde{\Pi}_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} \right| \\
\quad + \sup_{A \in \mathcal{B}_0} \left| \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A + \lambda_n \mid s_{\text{obs}} \} - \int_{A + \lambda_n} \psi(t) \, dt \right| \\
\quad + \sup_{A \in \mathcal{B}_0} \left| \Pi_\varepsilon \{ a_n(\theta - \theta_\varepsilon) \in A \mid s_{\text{obs}} \} - \int_A \psi(t) \, dt \right| \\
\quad + \sup_{A \in \mathcal{B}_0} \left| \int_{A + \lambda_n} \psi(t) \, dt - \int_A \psi(t) \, dt \right| \quad (3).
\end{align*}
\]

With similar arguments as before, the first three terms are \( o_p(1) \). For the fourth term, by transforming \( t \) to \( t + \lambda_n \), it is upper bounded by \( \int_{\mathcal{B}_0} |\psi(t - \lambda_n) - \psi(t)| \, dt \) which is \( o_p(1) \) by the continuous mapping theorem. Therefore (ii) holds.
For (iii), the left hand side of the equation has the decomposed upper bound similar to (3), with \( \theta, \theta_\varepsilon, \theta_\varepsilon \) and \( \psi(t) \) replaced by \( \theta^*, \theta_\varepsilon^*, \theta_\varepsilon^* \) and \( N\{t; 0, I(\theta_0)^{-1}\} \). Then by Lemma 5, using the leading term of \( \Pi_\varepsilon \{ a_n(\theta^* - \theta_\varepsilon^*) \in A \mid s_{\text{obs}} \} \) stated in the proof of Theorem 1, and similar arguments to those used for the fourth term of (3), it can be seen that this upper bound is \( o_p(1) \). Therefore (iii) holds.

\[ \square \]

3. PROOF FOR RESULTS IN SECTION 3.2

To prove Lemmas 4 and 5, some notation regarding the regression adjusted approximate Bayesian computation posterior, similar to those defined previously, are needed. Consider transformations \( t = t(\theta) \) and \( v = v(s) \). For \( A \subset \mathbb{R}^p \) and the scalar function \( h(t, v) \) in \( \mathbb{R}^p \times \mathbb{R}^d \), let

\[
\tilde{\pi}_{A, tv}(h) = \int_{\Omega(A)} \pi_{A, tv}(t, v) \, dv.
\]

**Proof of Lemma 4.** Since \( \beta_\varepsilon = \text{cov} \pi_{\varepsilon} \text{var}_\varepsilon(s)^{-1} \), to evaluate the covariance matrices, we need to evaluate \( \pi_{\beta_\varepsilon} \{ (\theta - \theta_0)^{k_1} (s - s_{\text{obs}})^{k_2} \} / \pi_{\beta_\varepsilon}(1) \) for \( (k_1, k_2) = (0, 0), (1, 0), (1, 1), (0, 1) \) and \( (0, 2) \).

First of all, we show that \( \pi_{\beta_\varepsilon} \{ (\theta - \theta_0)^{k_1} (s - s_{\text{obs}})^{k_2} \} \) is ignorable for any \( \delta < \delta_0 \) by showing that it is \( O_p(e^{-a_{\beta_\varepsilon} c_\delta}) \) for some positive constants \( c_\delta \) and \( a_\delta \). By dividing \( \mathbb{R}^d \) into \( \{ v : \|\varepsilon_n v\| \leq \delta'/3 \} \) and its complement,

\[
\sup_{\varepsilon_n \in B_\delta} \int_{\mathbb{R}^d} (s - s_{\text{obs}})^{k_2} f_n(s \mid \theta) K \left( \frac{s - s_{\text{obs}}}{\varepsilon_n} \right) \varepsilon_n^{-d} ds 
\]

\[
\leq \sup_{\varepsilon_n \in B_\delta} \left\{ \sup_{\|s - s_{\text{obs}}\| \leq \delta'/3} f_n(s \mid \theta) \int_{\mathbb{R}^d} (s - s_{\text{obs}})^{k_2} K \left( \frac{s - s_{\text{obs}}}{\varepsilon_n} \right) \varepsilon_n^{-d} ds \right\} 
\]

\[
+ \mathcal{K} \left( \lambda_{\min} A \varepsilon_n^{-1} \delta'/3 \right) \varepsilon_n^{-d} \int_{\mathbb{R}^d} (s - s_{\text{obs}})^{k_2} f_n(s \mid \theta) ds.
\]

By Condition 2(ii), Condition 6 and following the arguments in the proof of Lemma 3 of Li & Fearnhead (2015), the right hand side of (4) is \( O_p(e^{-a_{\beta_\varepsilon} c_\delta}) \), which is sufficient for \( \pi_{\beta_\varepsilon} \{ (\theta - \theta_0)^{k_1} (s - s_{\text{obs}})^{k_2} \} \) to be \( O_p(e^{-a_{\beta_\varepsilon} c_\delta}) \).

For the integration over \( B_\delta \), by Lemma 7 (ii),

\[
\pi_{\beta_\varepsilon} \{ (\theta - \theta_0)^{k_1} (s - s_{\text{obs}})^{k_2} \} \bigg/ \pi_{\beta_\varepsilon}(1) = a_{n, \varepsilon, c_\delta} \left\{ \frac{\tilde{\pi}_{\beta, tv}(t^{k_1} v^{k_2})}{\tilde{\pi}_{\beta, tv}(1)} + \frac{1}{\pi_{\beta_\varepsilon}(1)} \right\}
\]

where \( r_n(s \mid \theta) \) is the scaled remainder \( \alpha_n \{ f_n(s \mid \theta) - \tilde{f}_n(s \mid \theta) \} \). In the above, the second term in the first brackets is \( O_p(\alpha_n^{-1}) \) by the proof of Lemma 6 of Li & Fearnhead (2015). Then

\[
\frac{\pi_{\beta_\varepsilon} \{ (\theta - \theta_0)^{k_1} (s - s_{\text{obs}})^{k_2} \} \bigg/ \pi_{\beta_\varepsilon}(1)} = a_{n, \varepsilon, c_\delta} \left\{ \frac{\tilde{\pi}_{\beta, tv}(t^{k_1} v^{k_2})}{\tilde{\pi}_{\beta, tv}(1)} + O_p(\alpha_n^{-1}) \right\} ,
\]

and the moments \( \tilde{\pi}_{\beta, tv}(t^{k_1} v^{k_2}) / \tilde{\pi}_{\beta, tv}(1) \) need to be evaluated. Theorem 1 of Li & Fearnhead (2015) gives the value of \( \tilde{\pi}_{\beta, tv}(t) / \tilde{\pi}_{\beta, tv}(1) \), and this is obtained by substituting the leading term of \( \tilde{\pi}_{\varepsilon, tv}(t, v) \), that is \( \pi(\theta_0) g_\mu(t, v) \) as stated in Lemma 2, into the integrands. The other
moments can be evaluated similarly, and give
\[
\tilde{\pi}_{B_k,t}(k_1;v_{k_2}) = \begin{cases} 
  b_n^{-1} \beta_0 \{ A(\theta_0)^{1/2} W_{obs} + a_n \epsilon_n E_{G_n}(v) \}, & (k_1, k_2) = (1, 0), \\
  b_n^{-1} \beta_0 \{ A(\theta_0)^{1/2} W_{obs} E_{G_n}(v) + a_n \epsilon_n E_{G_n}(vv^T) \}, & (k_1, k_2) = (1, 1), \\
  E_{G_n}(v), & (k_1, k_2) = (0, 1), \\
  E_{G_n}(vv^T), & (k_1, k_2) = (0, 2), \\
+ O_p(a_n^{-1}) + O_p(a_n^2 \epsilon_n^4),
\end{cases}
\]  
(5)
where \( b_n = 1 \) when \( \epsilon_n < \infty \), and \( a_n \epsilon_n \) when \( \epsilon_n = \infty \). By Lemma 2, \( E_{G_n}(vv^T) = \Theta_p(1) \). Since
\[
a_n^{-1} = o(a_n^{-2/5}), \quad \text{cov}_\epsilon(\theta, s) = \epsilon_n^2 \beta_0 \text{var}_{\epsilon_n}(v) + o_p(a_n^{-2/5} \epsilon_n^2) \quad \text{and} \quad \text{var}_\epsilon(s) = \epsilon_n^2 \text{var}_{\epsilon_n}(v)\{1 + o_p(a_n^{-2/5})\}.
\]
Thus
\[
\beta_\epsilon = \beta_0 + o_p(a_n^{-2/5}),
\]  
(6)
and the lemma holds.

For \( A \subset \mathbb{R}^p \) and \( B \subset \mathbb{R}^d \), let \( \pi(A, B) = \int_A \int_B \pi(\theta) f_n(s \mid \theta) K\{ \epsilon_n^{-1}(s - s_{obs}) \} \epsilon_n^{-d} ds d\theta \) and
\[
\tilde{\pi}(A, B) = \int_A \int_B \pi(\theta) \tilde{f}_n(s \mid \theta) K\{ \epsilon_n^{-1}(s - s_{obs}) \} \epsilon_n^{-d} ds d\theta.
\]
Denote the marginal mean values of \( s \) for \( \pi(\theta, s \mid \text{obs}) \) and \( \tilde{\pi}(\theta, s \mid \text{obs}) \) by \( \tilde{s}_\epsilon \) and \( s_\epsilon \) respectively.

**Proof of Lemma 5.** For (a), write \( \Pi_\epsilon(\theta^* \in B_\delta^c \mid \text{obs}) \) as \( \pi[\mathbb{R}^p, \{ s : \theta^*(\theta, s) \in B_\delta^c \}] / \pi(\mathbb{R}^p, \mathbb{R}^d) \). By Lemma 7, \( \pi(\mathbb{R}^p, \mathbb{R}^d) = \theta_p(1) = \Theta_p(a_n^{d-p}) \). By the triangle inequality,
\[
\pi[\mathbb{R}^p, \{ s : \theta^*(\theta, s) \in B_\delta^c \}] \leq \pi(B_\delta^c, \mathbb{R}^d) + \pi[B_\delta^c, \{ s : \| \beta_\epsilon(s - s_{obs}) \| \geq \delta/2 \}],
\]  
(7)
and it is sufficient that the right hand side of the above inequality is \( o_p(1) \). Since its first term is \( \pi_{B_\delta^c}(1) \), by Lemma 7 the first term is \( o_p(1) \).

When \( \epsilon_n = \Omega(a_n^{-7/5}) \) or \( \Theta(a_n^{-7/5}) \), by (6), \( \beta_\epsilon - \beta_0 = o_p(1) \) and so \( \beta_\epsilon \) is bounded in probability. For any constant \( \beta_{\text{sup}} > 0 \) and \( \beta \in \mathbb{R}^{p \times d} \) satisfying \( \beta \leq \beta_{\text{sup}} \),
\[
\pi[B_\delta^c, \{ s : \| \beta_\epsilon(s - s_{obs}) \| \geq \delta/2 \}] \leq K \left( \epsilon_n^{-1} \frac{\delta}{2 \beta_{\text{sup}}} \right) \epsilon_n^{-d},
\]
and by Condition 2(iv), the second term in (7) is \( o_p(1) \).

When \( \epsilon_n = o(a_n^{-7/5}) \), \( \beta_\epsilon \) is unbounded and the above argument does not apply. Let \( \delta_1 \) be a constant less than \( \delta_0 \) such that \( \inf_{\theta \in B_{\delta_1/2}} \lambda_{\text{min}} \{ A(\theta)^{-1/2} \} \geq m \) and \( \inf_{\theta \in B_{\delta_1/2}} \lambda_{\text{min}} \{ D\sigma(\theta) \} \geq m \) for some positive constant \( m \). In this case, it is sufficient to consider \( \delta < \delta_1 \). By Condition 4,
\[
r_n(s \mid \theta) \leq a_n^d |A(\theta)|^{1/2} r_{\text{max}}(a_n|A(\theta)|^{-1/2} \{ s - s(\theta) \}).
\]
Using the transformation \( t = \theta(s) \) and \( v = v(s) \), \( f_n(s \mid \theta) = \tilde{f}_n(s \mid \theta) + \alpha_n^{-1} r_n(s \mid \theta) \) and applying the Taylor expansion of \( s(\theta_0 + xt) \) around \( x = 0 \),
\[
\pi[B_{\delta/2}, \{ s : \| \beta_\epsilon(s - s_{obs}) \| \geq \delta/2 \}] \leq \int_{\{B_{\delta/2}\}} \int_{\| \beta_\epsilon(s) \| \geq \delta/2} N[A(\theta_0 + a_n^{-1} t)^{-1/2} \{ Ds(\theta_0 + e_n^1 t) t - A(\theta_0)^{1/2} W_{obs} - a_n \epsilon_n v \}] : 0, I_d \]  
(\epsilon_n^2) K(v) dv dt
\]
\[
+ c \int_{\{B_{\delta/2}\}} \int_{\| \beta_\epsilon(s) \| \geq \delta/2} r_{\text{max}}(A(\theta_0 + a_n^{-1} t)^{-1/2} \{ Ds(\theta_0 + e_n^1 t) t - A(\theta_0)^{1/2} W_{obs} - a_n \epsilon_n v \}) K(v) dv dt,
\]
\[
+ c \int_{\{B_{\delta/2}\}} \int_{\| \beta_\epsilon(s) \| \geq \delta/2} \]  
(\epsilon_n^2) K(v) dv dt
\]
for some positive constant $c$. To show that the right hand side of the above inequality is $o_p(1)$, consider a function $g_4(\cdot)$ in $\mathbb{R}^d$ satisfying $g_4(v) = \bar{g}_4(||v||)$ and $g_4(\cdot)$ is decreasing. Let $A_n(t) = A(\theta_0 + a_n^{-1}t)^{-1/2}$, $C_n(t) = D_0s(\theta_0 + \xi_1)$ and $c = A(\theta_0)^{-1/2} W_{obs}$. For each $n$ divide $\mathbb{R}^p$ into $V_n = \{ t : ||C_n(t)||/2 \geq c + a_n e_n v \}$ and $V_n^c$. In $V_n$, $\|A_n(t)\|C_n(t)t - c - a_n e_n v\| \geq m^2/||t||/2$ and in $V_n^c$, $||t|| \leq 2m^{-1} ||c + a_n e_n v||$. Then

$$\int_{t(B_{\beta}/2)} \int_{\|\beta \|e_n v \| \geq \delta/2} g_4[A_n(t)\{C_n(t)t - c - a_n e_n v\}K(v)\,dvdt\leq \int_{\|\beta \|e_n v \| \geq \delta/2} \left\{ \int_{\mathbb{R}^p} \bar{g}_4(m^2/||t||/2)\,dt + \sup_{v \in \mathbb{R}^p} g_4(v)\int_{V_n^c}^d 1\,dt \right\} K(v)\,dv,$$

where $\int_{V_n^c}^d 1\,dt$ is the volume of $V_n^c$ in $\mathbb{R}^p$. Then since $\beta \|e_n v \| = o_p(1)$, $a_n e_n v = o_p(1)$ and $\int_{V_n^c}^d 1\,dt$ is proportional to $||c + a_n e_n v||^p$, the right hand side of the above inequality is $o_p(1)$. This implies $\pi(B_{\beta}/2, \{ s : ||\beta(s - s_{obs})|| \geq \delta/2 \}) = o_p(1)$.

Therefore in both cases $\Pi_{\epsilon}(\theta^* \in B_{\delta}^c | s_{obs}) = o_p(1)$. For $\Pi_{\epsilon}(\theta^* \in B_{\delta}^c | s_{obs})$, since the support of its prior is $B_{\delta}$, there is no probability mass outside $B_{\delta}$, i.e., $\Pi_{\epsilon}(\theta^* \in B_{\delta}^c | s_{obs}) = 0$. Therefore (a) holds.

For (b),

$$\sup_{A \in \mathbb{R}^p} \Pi_{\epsilon}(\theta^* \in A (\theta_0 \cap B_\delta | s_{obs}) - \Pi_{\epsilon}(\theta^* \in A_0(\theta_0 \cap B_\delta) = o_p(1)$$

where $\pi(B_{\beta}/2, \{ s : ||\beta(s - s_{obs})|| \geq \delta/2 \}) = o_p(1)$. Then by the proof of Lemma 6 of Li & Fearnhead (2015), (b) holds.

For (c) to begin with, $a_n(\theta - \hat{\theta}) = a_n(\theta_0 - \hat{\theta}) - a_n(\beta(s_e - \tilde{s}_e)).$ By Lemma 7, $a_n(\beta - a_n(\theta - \hat{\theta})) = o_p(1)$. For $a_n(\beta(s_e - \tilde{s}_e)$, similar to the arguments of the proof of Lemma 4,

$$s_e - s_{obs} = \epsilon_n \{ \frac{\pi_{B_{\delta},v}(v)}{\pi_{B_{\delta},v}(1)} + O_p(\alpha^{-1}) \} 1 + O_p(\alpha^{-1}), \quad s_e - s_{obs} = \epsilon_n \frac{\pi_{B_{\delta},v}(v)}{\pi_{B_{\delta},v}(1)} (1 + O_p(\alpha^{-1})).$$

Then $a_n(\beta(s_e - \tilde{s}_e) = O_p(\alpha^{-1} a_n e_n v)$ which is $o_p(1)$ if $\epsilon_n = o(a_n^{-3/5})$. Therefore the first part of (c) holds. Since $\hat{\theta}_c = \theta_c - \beta(s_e - s_{obs})$, by the expansion of $\hat{\theta}_c$ in Lemma 3(c), the above expansion of $s_e - s_{obs}$ and (5), the second part of (c) holds.

### 4. Proof for Results in Section 3.3

**Proof of Theorem 2.** The integrand of $p_{acc,q}$ is similar to that of $\pi_{\mathbb{R}^p}(1)$. The expansion of $\pi_{\mathbb{R}^p}(1)$ is given in Lemma 7(ii), and following the same reasoning, $p_{acc,q}$ can be expanded as $\epsilon_n^d \int_{B_{\delta}} \int_{B_{\delta}^c} q_n(\theta) \tilde{f}(s_{obs} + \epsilon_n v) \, K(v)\,dvd(1 + o_p(1))$. With transformation $t = \theta$, plugging the expression of $q_n(\theta)$ and $\tilde{f}(s_{obs}, \theta)$ gives that

$$p_{acc,q} = (a_n \epsilon_n v)^d \int_{t(B_{\delta})} (r_n,c)^p q(r_n,c) \frac{\pi_{\epsilon,t,v}(t,v)}{\pi(\theta_0 + a_n \epsilon t)} \, dtd(1 + o_p(1),$$
where $r_{n,\varepsilon} = \sigma_n/a_{n,\varepsilon}$ and $c_{\mu, n} = \sigma_n(\mu_n - \theta_0)$. By the assumption of $\mu_n$, denote the limit of $c_{\mu, n}$ by $c_\mu$. Then by Lemma 2, $p_{acc,q}$ can be expanded as

$$p_{acc,q} = (a_{n,\varepsilon,n})^d \int_{t(B\delta) \times \mathbb{R}^d} (r_{n,\varepsilon})^{p-1} q(r_{n,\varepsilon} t + c_{\mu, n}) g_n(t, v) \, dv \, dt \{1 + o_p(1)\}. \quad (8)$$

Denote the leading term of the above by $Q_{n,\varepsilon}$.

For (1), when $c_\varepsilon = 0$, since $\sup_{t \in \mathbb{R}^p} g_n(t, v) \leq c_1 K(v)$ for some positive constant $c_1$, $Q_{n,\varepsilon}$ is upper bounded by $(a_{n,\varepsilon,n})^d c_1$ almost surely. Therefore $p_{acc,q} \to 0$ almost surely as $n \to \infty$.

When $r_{n,\varepsilon} \to \infty$, since $q(\cdot)$ is bounded in $\mathbb{R}^p$ by some positive constant $c_2$, $Q_{n,\varepsilon}$ is upper bounded by $(r_{n,\varepsilon})^{-p} c_2 (a_{n,\varepsilon,n})^d \int_{\mathbb{R}^p \times \mathbb{R}^d} g_n(t, v) \, dv \, dt$. Therefore $p_{acc,q} \to 0$ in probability as $n \to \infty$ since $\int_{\mathbb{R}^p \times \mathbb{R}^d} g_n(t, v) \, dv \, dt = \Theta_p(1)$ by Lemma 2.

For (2), let $\tilde{t}(\theta) = r_{n,\varepsilon}^{-1}(\theta - c_{\mu, n})$ and $\tilde{t}(A)$ be the set $\{\phi : \phi = \tilde{t}(\theta) \text{ for some } \theta \in A\}$. Since $\tilde{t} = \sigma_n^{-1}(\theta - \theta_0) - c_{\mu, n}$ and $\sigma_n^{-1} \to 0$, $\tilde{t}(B\delta)$ converges to $\mathbb{R}^p$ in probability as $n \to \infty$. With the transformation $\tilde{t} = \tilde{t}(\theta)$,

$$Q_{n,\varepsilon} = \begin{cases} (a_{n,\varepsilon,n})^d \int_{\tilde{t}(B\delta) \times \mathbb{R}^d} q(\tilde{t}) g_n(r_{n,\varepsilon}(\tilde{t} + c_{\mu, n}), v) \, d\tilde{t} \, dv, & c_\varepsilon < \infty, \\ \int_{\tilde{t}(B\delta) \times \mathbb{R}^d} q(\tilde{t}) g_n(r_{n,\varepsilon}(\tilde{t} + c_{\mu, n}), v') \, d\tilde{t} \, dv', & c_\varepsilon = \infty. \end{cases}$$

By Lemma 9 and the continuous mapping theorem,

$$Q_{n,\varepsilon} \to \begin{cases} c_\varepsilon^d \int_{\mathbb{R}^p \times \mathbb{R}^d} q(\tilde{t}) g \{r_1(\tilde{t} + c_{\mu, n}), v\} \, d\tilde{t} \, dv, & c_\varepsilon < \infty, \\ \int_{\mathbb{R}^p \times \mathbb{R}^d} q(\tilde{t}) g \{r_1(\tilde{t} + c_{\mu, n}), v\} \, d\tilde{t} \, dv, & c_\varepsilon = \infty, \end{cases}$$

in distribution as $n \to \infty$. Since the limits above are $\Theta_p(1)$, $p_{acc,q} = \Theta_p(1)$.

For (3), when $c_\varepsilon = \infty$ and $r_1 = 0$, in the above, the limit of $Q_{n,\varepsilon}$ in distribution is $\int_{\mathbb{R}^p \times \mathbb{R}^d} q(\tilde{t}) g(0, v) \, d\tilde{t} \, dv = 1$. Therefore $p_{acc,q}$ converges to 1 in probability as $n \to \infty$. \hfill \Box

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